THOUGHTS FROM THE PCS 2023 MEETING

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A measure of a meeting are the thoughts it stimulated, and so PCS 2023 was useful for me. Among my thoughts are raised questions that will be answered here. This includes the comment that "The Borda Count is a method; can it be treated as a measure?" Then, a new voting method was evaluated with numerous simulations; can this be done in a simpler analytic way? I promised some of you that I would send my thoughts on these and related topics; this also is being sent to others who may be curious.

What follows are typical hour exam questions for my first year Econ graduate students (when I taught this course many years ago). If you can quickly answer all of them, it probably is not worth reading what follows. Otherwise answers, involving simple algebra, follow.

1. Positional methods
   - (a) For the following three-candidate profile, find the pairwise outcomes and all possible positional method rankings. For each possible ranking, specify all positional methods that have that election ranking.
     - 7 prefer $A \succ C \succ B$, 1 prefers $C \succ A \succ B$, 2 prefer $C \succ B \succ A$
     - 6 prefer $B \succ C \succ A$, and 1 prefers $B \succ A \succ C$.
   - (b) What is the maximum number of positional outcomes that a three-candidate profile can have? Characterize this behavior.
2. More about positional methods
   - (a) Find a profile where the Borda outcome is $A \succ B \succ C$, the plurality outcome is $C \succ A \succ B$ and the pairwise outcome is the cycle $C \succ B, B \succ A.A \succ C$.
   - (b) For the above question, characterize all possible profiles that have these outcomes.
   - (c) Characterize all possible profiles where the Borda outcome is $B \succ A \succ C$ and the plurality and the pairwise outcomes agree with $A \succ C \succ B$.
3. What *must* be true about a profile if the Borda and Condorcet winner differ? Explain why.
4. State and provide an analytic proof of Sen's Theorem. (Not a "proof by example" as normally done, but an analytic proof.)

1. Geometry of Profiles

To explain what follows, an element from the space of mappings $\{F : D \to R\}$ is given by specifying the mapping, the range, and the domain. Much of social choice followed the

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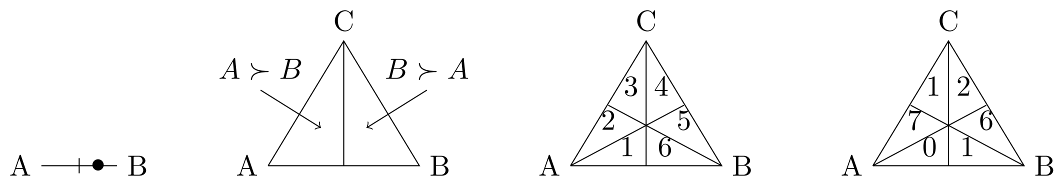
lead of Arrow by characterizing properties of mappings. Duncan Luce characterized the range R, which means that the structure of the mapping is a consequence. Luce's work, which led to a Nobel Prize in Econ (McFadden for logit models), is widely used in Econ and Cog Science, but overlooked in Social Choice.

What remains is the domain D. What I did, and will briefly outline some of it below for $n = 3$ candidates, is to characterize the space of profiles in a manner where many questions can be answered with just elementary algebra. The idea is to create a basis for the domain where each basis vector influences the outcomes of specific voting rules. Elementary geometry is a main tool: here, when describing the outcome of a pair, it is standard to plot it as a point on an interval as given in Fig. a. As the point is on the B side of midpoint, the outcome is $B \succ A$.



a. A pair **b.** Equilateral triangle **c.** Ranking triangle **d.** A profile

A line does not suffice for three candidates; trying to put the points on the line prejudices the profile choice by limiting the number of possible outcomes. To resolve this neutrality concern, replace the line segment with an equilateral triangle as in Fig. b. The vertical line is equal distance from A and B, so it represents the ranking $A \sim B$; a point to the left of the line is $A \succ B$ while one to the right is $B \succ A$.

What is done for one pair can be done for all pairs leading to Fig. c. All 13 transitive rankings are represented in this figure. (Indeed, the intersection of the large right triangles correspond to the definition of transitivity; e.g., if $A \succ B$ and $B \succ C$, then transitivity requires $A \succ C$; with geometry, this follows from the intersection of the $A \succ B$ and $B \succ C$ right triangles which is the region $A \succ B \succ C$.) By construction, the six open triangles are strict rankings, the six short line segments have a pairwise tie, and the center dot is the complete tie $A \sim B \sim C$. Using Fig. b, the associated rankings are immediate. Namely, the triangle with a 2 is closest to A, next closest to C, and farthest from B for the $A \succ C \succ B$ ranking. The short line segment between regions 4 and 5 is a $B \sim C$ tie where A is farthest away to represent the $B \sim C \succ A$ ranking.

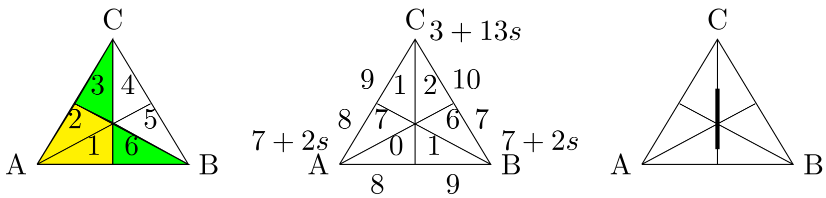
To explain the numbers in Fig. c, a profile can be expressed as a vector

$$(ABC, ACB, CAB, CBA, BCA, BAC), \tag{1}$$

So, in a clockwise manner, the number in Fig. c is the coordinate position of the ranking in Eq. 1. Figure d is the profile for the above question 1a.

2. Tallying Ballots

The next step is to tally ballots. Pairwise is simple; e.g., all of the preferences preferring $A \succ B$ are on the left side of the $A \sim B$ line (Fig. b), so the sum of a profile entries to the left of $A \sim B$ is A's tally in this majority vote election. The same is true for each pair where the dividing line divides the rankings of the pair. This greatly simplifies computing the tallies because it separates the pairwise rankings into appropriate lists; e.g., the pairwise tallies for the 1a question are given below each line segment in Fig. f; the pairwise ranking is $C \succ B \succ A$.



e. A's positional tallies **f.** A profile **g.** Procedure line

To tally positional methods, normalize the three-candidate methods to $(1, s, 0)$. Here one point is given to a ballot's top ranked candidate, s points to the second ranked candidate, and zero points to the bottom ranked candidate. So, to obtain all of A's positional tallies from a profile, assign one point to the sum of the profile's entries in regions 1 and 2 of Fig. e (colored yellow), and s points to the sum of the profile's entries in regions 3 and 6 (colored green). In Fig. f, A's tallies over all positional methods is $7(1) + s(2)$. As Fig. e shows, there is an easy geometric representation; i.e., , to find X's tally, one point is given to each profile entry in the regions with X as a vertex, and s points to each of the two adjacent regions. The Fig. f positional tallies are given next to the appropriate vertex.

All sorts of results follow from Fig. e; e.g., it shows the different information being used to determine A's pairwise and positional tallies, which provides opportunities to create profiles with all sorts of different outcomes. Indeed, creating examples now is easy; just take a triangle, insert numbers in the ranking regions, quickly tally the outcomes to see if the behavior is what you want, and, if not, it should be clear from the triangle's structure and Fig. e how to modify everything.

The positional tallies for Question 1a (Fig. f) are

$$(A, B, C) = (7 + 2s, 7 + 2s, 3 + 13s), \tag{2}$$

which is the equation of a straight line segment. Displaying a distinct lack of imagination, I call it a "procedure line." A line can be represented by two points; where the obvious choices are the endpoints of $s = 0$ (the plurality tally) and $s = 1$ (the "vote for two," or anti-plurality tally). So, just by computing and plotting two points, we find all possible positional outcomes!! The Eq. 1 procedure line is the thick Fig. g line; the bottom point is the plurality outcome and the top is the anti-plurality outcome. As the line shows, this profile admits three positional rankings: $A \sim B \succ C, A \sim B \sim C, C \succ A \sim B$. To determine which positional methods deliver which rankings, use simple algebra on the ties; e.g., a complete $A \sim B \sim C$ tie is where $7 + 2s = 3 + 13s$ or $s = \frac{4}{11}$. Thus, all methods with $s < \frac{4}{11}$ have the $A \sim B \succ C$ ranking, those with $s > \frac{4}{11}$ have $C \succ A \sim B$.

To answer another of the above questions, it follows from the geometry that a procedure line can cross at most 7 regions (meeting four open regions (strict rankings) and three line segments (that have a pairwise tie). This means there can be no more than 7 different positional rankings, where four of them are strict rankings and three have a pairwise tie.

For the procedure line to admit this behavior, the end points, the plurality and anti-plurality rankings, must be in open triangles (strict transitive rankings) that are opposite of each other. Plus the procedure line cannot pass through the completely tied point. To find which methods have which rankings, follow the above to find the s values where there are tie votes. Answers follow from elementary algebra.

3. Decomposition of profile space

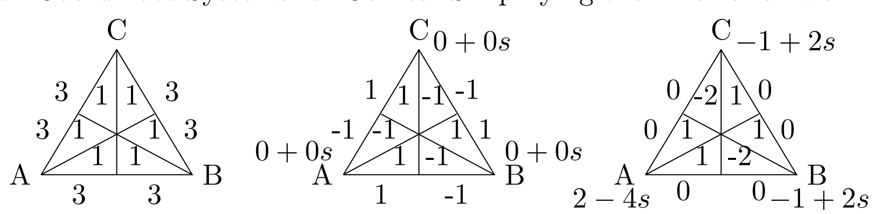
The key for decomposing the profile space is to exploit natural symmetries. An interesting problem is that the symmetries can be so obvious that often they are ignored. To demonstrate this point with an example from game theory, we know that the following two games are identical.

$$\mathcal{E} = \begin{array}{|cc|cc|}\hline -5 & 7 & 3 & 3 \\ \hline -9 & -9 & 11 & -1 \\ \hline \end{array}, \quad \mathcal{O} = \begin{array}{|cc|cc|}\hline -9 & -9 & 11 & -1 \\ \hline -5 & 7 & 3 & 3 \\ \hline \end{array}$$

But *the symmetry, where the top and bottom rows can be exchanged, has profound consequences!* It permits decomposing a game into a component that has all and only information about all possible Nash behavior, a Behavioral component that has all and only information about all cooperation and coordination effects, etc. For instance, the Nash and Behavioral components for $\mathcal{E}$ are

$$\mathcal{N} = \begin{array}{|cc|cc|}\hline 2 & 2 & -4 & -2 \\ \hline -2 & -4 & 4 & 4 \\ \hline \end{array}, \mathcal{B} = \begin{array}{|cc|cc|}\hline -7 & 5 & 7 & 5 \\ \hline -7 & -5 & 7 & -5 \\ \hline \end{array}$$

where, trivially from $\mathcal{N}$, BR and TL are the pure Nash equilibria and the mixed Nash equilibrium is $(p, q) = (\frac{2}{3}, \frac{2}{3})$. "Cooperation" from $\mathcal{B}$ follows from its Pareto superior cell of TR, where the players must coordinate to get this outcome. Notice; the coordination for the group to attain the best $\mathcal{B}$ outcome runs counter to an individual's Nash incentives given by $\mathcal{N}$; this captures the inherent conflict between individual (me) and group (we) efforts. All of this is based on a mathematical topic called "Representation Theory." (The game theory consequences, minus the heavy mathematics, is in my recent book with Dan Jessie "Coordinate Systems for Games: Simplifying the "me" and "we" interactions.")



h. The $\mathcal{K}$ component **i.** Component $\mathcal{C}$ **j.** "A" reversal, $\mathcal{R}_A$

O mention Representation Theory because it also is central to the decomposition of profiles. Fortunately, it is possible to hide the mathematics and get to the substance by asking, "What are the symmetries of a triangle?" An obvious symmetric profile is where each ranking has the same number of voters. The associated basis vector has "1" placed in each ranking region as in Fig. h; call it $\mathcal{K}$ for kernel. Here, the pairwise and all positional methods are complete ties. The number of voters is given by $z\mathcal{K}$, which is six times the multiple of z. This means that z can be a mixed number where the fractional part has denominator equal to 6; e.g., for 32 voters, $z = \frac{32}{6} = 5\frac{2}{6}$.

A next symmetry is to rotate the triangle; e.g., $A \to B \to C \to A$. The goal is to find a profile that is invariant with respect to this rotation; that is the rotation has no impact because the profile remains unchanged. This is given by the 1's and the -1's in Fig. i. As can be seen by playing with Fig. i, the rotation maps each 1 to a different 1 and each -1 is mapped to another -1. Call this the $\mathcal{C}$ (for Condorcet) component. As Fig. i shows,

> *1. The $\mathcal{C}$ component has no impact on positional method outcomes. This component affects only the majority vote tallies of pairs where it gives a twist or cyclic effect to the outcome.*

The sum of the components in $\mathcal{C}$ equals zero. This means that $\mathcal{C}$ and $\mathcal{K}$ are orthogonal, so neither has an impact on the responsibilities of the other; e.g., $\mathcal{C}$ does not impact on the number of voters while $\mathcal{K}$ does not have any impact on pairwise comparisons. Remember, $\mathcal{K}$ is an initial, neutral arrangement of the voters. So, treat the -1 components as being where a voter with that ranking in $\mathcal{K}$ is being moved to a ranking where there is a +1. To illustrate with Fig. i, a voter from the $\mathcal{K}$ ranking in, say, region $B \succ A \succ C$ is moved to one of the regions with a +1.

Another triangle symmetry is to rotate the triangle about one of the three indifference lines. For instance, if vertices B and C are interchanged, the rotation is about the $B \sim C$ line, which identifies regions {1, 2}, {3, 6}, and {4, 5}. But, this also captures a different symmetry where a ranking and its reversal are identified. That is, the pairs $\{A \succ B \succ C, C \succ B \succ A\}, \{A \succ C \succ B, B \succ C \succ A\}$, and $\{C \succ A \succ B, B \succ A \succ C\}$. To capture their impact, emphasize a top ranked candidate. For instance, emphasizing A are the two pairs $\{A \succ B \succ C, C \succ B \succ A\}, \{A \succ C \succ B, B \succ C \succ A\}$; give each of these rankings (Fig. j) one voter. To ensure the this term does not affect the number of voters, place a -2 in the two remaining ranking regions.

The B and C reversal vectors, $\mathcal{R}_B$ and $\mathcal{R}_C$, are found in the same way–place 1's in the two regions where the designated candidate is top-ranked and in the two regions where the candidate is bottom ranked. Then place -2 in the two remaining regions. Because $\mathcal{R}_A + \mathcal{R}_B + \mathcal{R}_C = 0$, only two of these vectors are needed for a basis of this Reversal subspace. Also, each reversal vector is orthogonal to $\mathcal{K}$ and to $\mathcal{C}$. The tallies are given in Fig. j, which lead to

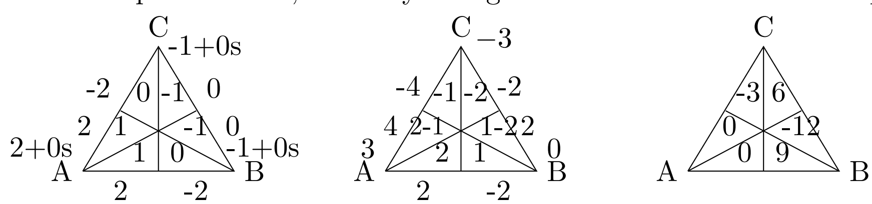
> *2. If X is one of the three-candidates, $\mathcal{R}_X$ has no impact on the majority vote of pairs. It affects positional outcomes in that X has a tally of $2 - 4s$ with the two other candidates are tied with tallies of $-1 + 2s$. This is the source of all possible differences of a positional method's ranking of a profile! The outcome for the Borda Count is a complete tie.*

So far, we have 4 vectors for the six-dimensional space of profiles; two more are needed. The last two also are consequences of rotating the triangle about indifference lines. Let's just cut to the chase with the Basic vector $\mathcal{B}_A$ given in Fig. k. Here, for candidate X, $\mathcal{B}_X$ assigns one point in each of the two regions where X is top-ranked, -1 in the two regions where X is bottom ranked, and zero in the remaining regions. A simple exercise shows that

$\mathcal{B}_X$ is orthogonal to both $\mathcal{R}$'s, $\mathcal{C}$ and $\mathcal{K}$. As $\mathcal{B}_A + \mathcal{B}_B + \mathcal{B}_C = 0$, any two of these vectors form a basis for this Basic subspace. From the Fig. k tallies, we have that

> *3. If X is one of the three candidates, all positional methods have the exact same tally with $\mathcal{B}_X$. So, if the tally for some positional method is known with a Basic profile, all positional methods have the same outcome. Moreover the pairwise rankings are the same as the positional methods where the tallies can be computed from the common paositional tally. Similarly, if the pairwise tallies for a Basic profile are known, then all positional methods have the same ranking and their tallies can be determined.*

Some algebra is needed to get the tallies of the different positional methods. They all agree with a Basic profile alone, but they change somewhat when the $\mathcal{K}$ component is included.



k. The $\mathcal{B}_A$ Basic **l.** Creating an example **m.** Example

4. Consequences

We are done! The six basis vectors that describe the profile space are as specified above. They can be used in the same manner as done with the basis of any vector space to derive consequences. A small sample follows.

(1) The Borda Count ranking is strictly determined by the Basic vectors. In fact, the Borda Count is the *only* positional method where its outcome and that of the pairs are completely determined by the Basic component of a profile. Indeed, with Basic vectors, the pairwise and the Borda rankings are the same. So, a profile's Borda outcome actually identifies a profile's central Basic component along with its properties. (This statement extends to all $n \geq 3$ candidate settings.)

(2) According to the above, the majority vote ranking of pairs is strictly determined by the Basic Profile and the Condorcet $\mathcal{C}$. To answer question 3, suppose the Borda and Condorcet winners differ. As the Borda outcome and its associated pairwise outcomes are strictly determined by the Basic component of a profile, if the Condorcet winner differs then *the profile must have a Condorcet $\mathcal{C}$ component.*

Indeed, most profiles have a $\mathcal{C}$ component, which must be expected from vector analysis; if a profile does not, it belongs to a lower dimensional subspace, which has probability zero. The property holds even the unanimity profile. To see this, profile $\mathbf{u} = (1, 0, 0, 0, 0, 0)$ corresponds to a unanimous $A \succ B \succ C$ profile (from Fig. c). To find how much of $\mathcal{C}$ is in this direction, notice that a normal vector for $\mathcal{C}$ (Fig. i) is $\mathbf{c} = (\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}})$. So, the amount of $\mathbf{u}$ that is going in the $\mathbf{c}$ direction is given by the dot product $(\mathbf{u}, \mathbf{c})\mathbf{c} = \frac{1}{\sqrt{6}}\mathbf{c} = \frac{1}{6}\mathcal{C}$.

As a manifestation of this $\mathcal{C}$ term, the pairwise outcome of the unanimity $A \succ B \succ C$ is $A \succ B$ by 1:0, and $A \succ C$ by 1:0, which does not indicate the significance A has over the other candidates in this profile. The Borda outcome is $A \succ B \succ C$ with 2:1:0, which reflects the Basic Vector, and more accurately represents profile $\mathbf{u}$. The twist of Condorcet component $\frac{1}{6}\mathcal{C}$ pulls the A, C tally closer.

It is of such importance that it is worth repeating: *a profile can experience a cycle even if it does not explicitly contain a Condorcet triplet!* But, the cycle still is caused by $\mathcal{C}$ terms. Just compute the dot product of the profile with $\mathbf{c}$, which is trivial; this registers the impact of the Condorcet direction.

(3) The outcomes of all positional and pairwise methods for a profile are consequences of perturbations from the Basic component of a profile; that is, the Condorcet terms cause cyclic effects in majority votes over pairs and, similarly, the reversal terms change all positional tallies. In this manner, the Borda Outcome becomes more than a tally for a particular system; it provides a measure of comparison for what is happening because of the other perturbing profile components.

(4) While it was not done explicitly, one of the talks at the conference explored consequences of the weight s in the positional method (1, s, 0) via simulations. But *all possible effects of the choice of s* are captured and caused by the $\{\mathcal{R}_X\}_{X=A,B,C}$ components of a profile, so such an analysis can be carried out analytically.

(5) To answer question 2a, start with the Borda outcome, which can be $2\mathcal{B}_A + \mathcal{B}_B$ as given in Fig. l. The $C \succ A \succ B$ plurality outcome requires adding a $\mathcal{R}_C$ component to what is in Fig. l. Adding $x\mathcal{R}_C$ will change C's plurality tally from -3 to -3+2x. The plurality tallies for A and B are 3-x and 0-x. All that is needed is for $-3 + 2x > 3 - x$, which holds for any $x > 2$, say $x = 3$.

To have a cycle of pairs, a multiple of $\mathcal{C}$ is needed. Using $y\mathcal{C}$ (Fig. i) changes the Fig. l pairwise tallies to (for (A, B)) $2 + y : -2 - y$, for (B, C) to $2 + y : -2 - y$, and for (A, C) to $4 - y : -4 + y$. The cycle requires (for $C \succ B$) $-2 - y > 2 + y$, (for $A \succ C$) $4 - y > -4 + y$, and (for $B \succ A$) $-2 - y > 2 + y$. So, any $y < -4$ suffices, say $y = -5$ This leads to the profile in Fig. m.

Now a $z\mathcal{K}$ needs to be added. Select z so that all entries of the profile are non-negative; here $z = 12$ is fine; that is, add 12 to each of the six entries of Fig. m.

(6) To answer 2b, the dot product of the profile $\mathbf{p}$ with the normalized versions of $\mathcal{B}_A$ and $\mathbf{B}_B$ must have a larger component for $\mathcal{B}_A$ over $\mathcal{B}_B$ over $\mathcal{B}_C$. Similarly, the dot product of $\mathbf{p}$ with $\mathcal{R}_X$ must have the component size satisfying $\mathcal{R}_C$ over $\mathcal{R}_A$ over $\mathcal{R}_B$ where the coefficients must satisfy the obvious algebraic relationship with the coefficients from the Basic vector. A similar statement holds for $\mathcal{C}$.

(7) To answer 2c, the Basic profile is of the form $b\mathcal{B}_B + a\mathcal{B}_A + c\mathcal{B}_C = (b-c)\mathcal{B}_B + (a-c)\mathcal{B}_B$ where $b > a > c$. Ignore c by replacing b-c and a-c, respectively, with b and a.

The pairwise outcomes are (for A:B) 2a-2b:2b-2a, (for (B:C) 2b:-2b, and (for A:C) 2a:-2a. Adding $z\mathcal{C}$ has (for A:B) 2a-2b+z:2b-2a-z, (for (B:C) 2b+z:-2b-z, and (for A:C) 2a-z:-2a+z. To obtain the desired pairwise rankings of $A \succ C \succ B$,

(a) For $A \succ C$, it must be that $2a - z > -2a + z$ or $2a > z$.

(b) To get $A \succ B$, it must be that $2a - 2b + z > 2b - 2a - z$ or $z > 2b - 2a > 0$.

   (c) Finally, to get $C \succ B$, it must be that $-2b - z > 2b + z$ or $0 > -2b > z$.
   But line (b) requires z to be positive, and line (c) requires z to be negative. This contradiction means that no such profile exists. (Of course, the example wants Borda to rank the Condorcet loser over the Condorcet winner, which is impossible.) This is a simple algebraic way to find properties of voting methods.

(8) The question about Sen's theorem was a question I put on an hour test for one of my first year Econ grad classes. To encourage the students, I took the best answer, polished it up a bit, and published it with the student. If I remember to do so, I will attach the paper.