Understanding the Central Limit Theorem through the exponential distribution

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1 Synopsis

In this report, we examine the central limit theorem by doing many simulations of random samplings of the exponential distribution. We measure the distribution of the means of these observations and show that it is approximately normally distributed as $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$.

2 Setup

For the rest of this project, we set the rate parameter, lambda, to 0.2, the number of simulations, N, to 1000, and the number of drawings for each simulation, n, to 40.

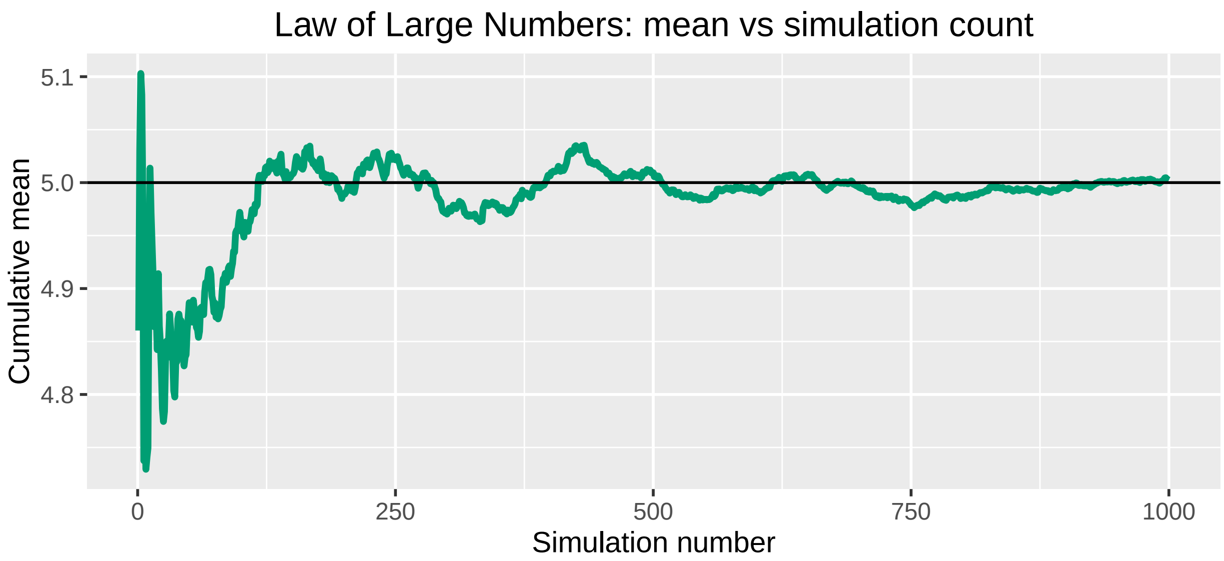
```
library(ggplot2)
lambda <- 0.2
N <- 1000
n <- 40
mean_theory <- 1/lambda
var_theory <- 1/(lambda*lambda)
var_theory_means <- var_theory/n
```

Let's run 1000 simulations of the exponential distribution. Each time we do a simulation, we save the mean to a vector of means. We also compute the mean of the sample and the variance. We also set the seed of each drawing so we can be sure that we can exactly reproduce the results.

```
sim_means <- NULL
for (i in 1 : N){
    set.seed(i)
    sim_means <- c(sim_means, mean(rexp(n,lambda)))
}
mean_sim_means <- mean(sim_means)
var_sim_means <- var(sim_means)
```

3 Comparing the sample mean to the theoretical mean

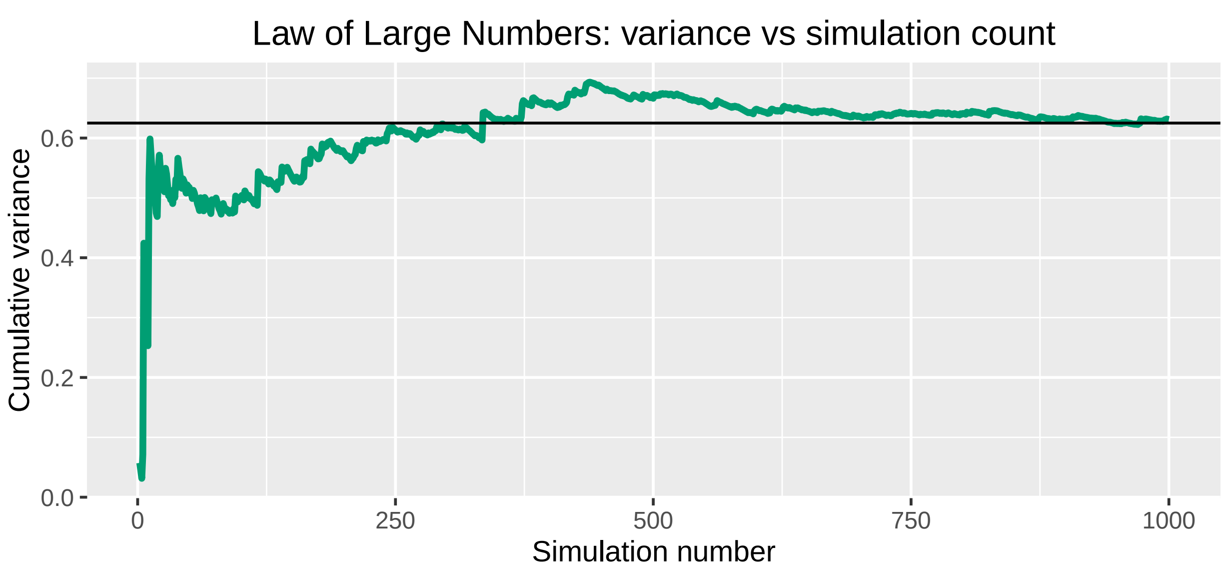
The sample mean is 5.002327 and the theoretical mean is 5. The Law of Large Numbers states that as we increase the number of simulations, the sample mean will converge to the theoretical mean. This is shown in the following figure.



4 Comparing the sample variance to the theoretical variance and

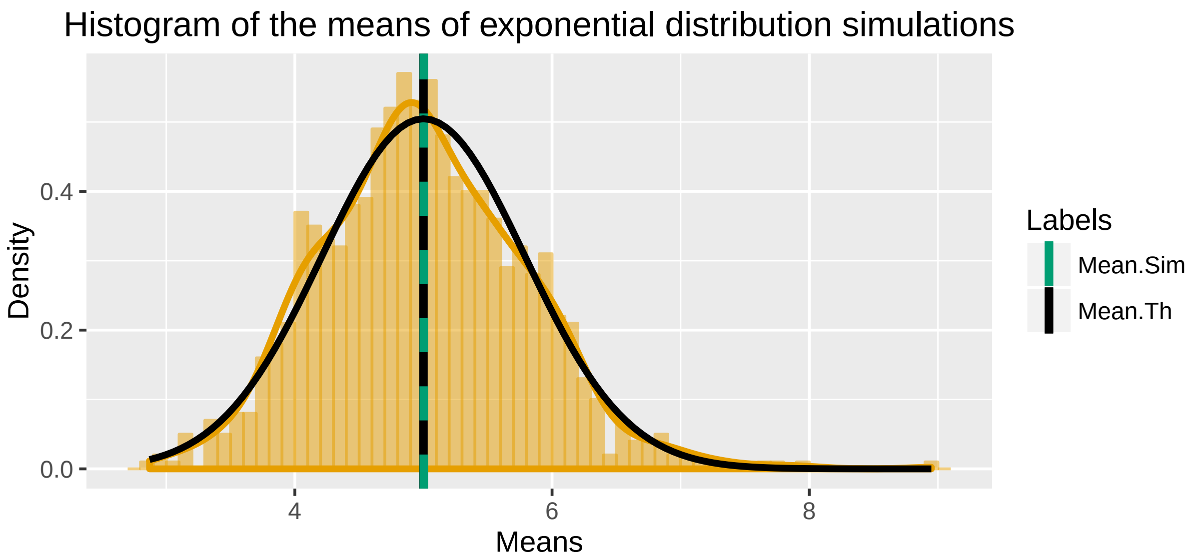
The sample variance is 0.6308244 and the theoretical variance is 0.625. The Law of Large Numbers states that as we increase the number of simulations, the sample variance will converge to the theoretical variance. This is shown in the following figure.

```
## Warning: Removed 1 rows containing missing values (geom_path).
```



5 Normality of the distribution

As shown in this figure, with the expected normal distribution contour in black, the distribution of the means has a shape close to a normal distribution $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) = N(5, 0.625)$.


Histogram of the means of exponential distribution simulations
Density
0.0
0.2
0.4
4
6
8
Means
Labels
Mean.Sim
Mean.Th

6 Appendix

The version history of this document can be found at the GitHub repository page.

```r
## ------------------------------------------------------------------------
library(ggplot2)
lambda <- 0.2
N <- 1000
n <- 40
mean_theory <- 1/lambda
var_theory <- 1/(lambda*lambda)
var_theory_means <- var_theory/n

## ------------------------------------------------------------------------
sim_means <- NULL
for (i in 1 : N){
    set.seed(i)
    sim_means <- c(sim_means, mean(rexp(n,lambda)))
}
mean_sim_means <- mean(sim_means)
var_sim_means <- var(sim_means)

## ----echo = FALSE--------------------------------------------------------
cum_mean <- cumsum(sim_means)/(1:N)
ggplot() +
    aes(x = 1:N, y = cum_mean) +
    geom_line(colour="#009E73", lwd=1.2) +
    geom_hline(aes(yintercept=mean_theory)) +
    labs(x = "Simulation number",
         y = "Cumulative mean",
         title = "Law of Large Numbers: mean vs simulation count")

## ----echo = FALSE--------------------------------------------------------
sim_vars <- NULL
for (i in 1 : N){ sim_vars <- c(sim_vars, var(sim_means[1:i]))}
ggplot() +
    aes(x = 1:N, y = sim_vars) +
    geom_line(colour="#009E73", lwd=1.2) +
    geom_hline(aes(yintercept=var_theory_means)) +
    labs(x = "Simulation number",
         y = "Cumulative variance",
         title = "Law of Large Numbers: variance vs simulation count")

## ----echo = FALSE--------------------------------------------------------
ggplot() +
    aes(sim_means) +
    geom_histogram(aes(y=..density..),binwidth=0.1, colour='#E69F00', fill = '#E69F00', alpha = 0.5) +
    geom_density(lwd=1.2,colour='#E69F00') +
    geom_vline(aes(xintercept=mean_theory,
                   colour="Mean.Th",
                   linetype = "Mean.Th"),
               size = 1.5) +
    geom_vline(aes(xintercept=mean_sim_means,
                   colour="Mean.Sim",
```

```
                   linetype = "Mean.Sim"),
               size = 1.5,
               show.legend=TRUE) +
    scale_colour_manual(name="Labels",
                        values=c(Mean.Sim="#009E73",Mean.Th="black")) +
    scale_linetype_manual(name="Labels",
                          values=c(Mean.Sim="dashed",Mean.Th="solid"),
                          guide=FALSE)+
    stat_function(fun = dnorm,
                  size = 1.2,
                  args = list(mean = mean_theory, sd = sqrt(var_theory_means))) +
    labs(x = "Means",
         y = "Density",
         title = "Histogram of the means of exponential distribution simulations")

## ----code=readLines(knitr::purl('./expo_clt.Rmd', documentation = 1)), eval = FALSE----
## NA
```