1 theorem

$f : X \to Y$ has inverse iff it is bijective.

1.1 lemma

For two functions $f : X \to Y, g : Y \to X$ whenever $gf = e_X$, then f is injective (into, no one image has more than one pre-image) and g is surjective (onto, all of Y is mapped onto X).

(e_M is an identity automorphism on set M.)

1.2 proof of the lemma

Consider an element x in X such that $g(f(x)) = e_X(x) = x$. For any two $x, x' \in X$ if $f(x) = f(x') = y$, then $g(y) = x = x'$, thus $x = x'$.

Definition: a function $f : X \to Y$ is injective, or into, if $\forall x, x' \in X : f(x) = f(x') \Rightarrow x = x'$.

That is to say, there are no such element of range,$y \in Y$, that has two different pre-images $x, x' \in X : y = f(x) = f(x')$.

Definition: a function $g : X \to Y$ is surjective, or onto, if $\forall y \in Y \,|\exists x \in X : g(y) = x$.

That is to say, all elements of the range have pre-images.

1.3 proof of the theorem

$\textsc{Bijective}(f) \Leftarrow \exists f^{-1}$:

Bijective means that $\forall x \in X \exists! y \in Y : f(x) = y$ Since y is unique, we can begin to construct $f^{-1} : Y \to X$ with (y, x). There is such a pair of every $x \in X$ and every $x \in Y$ and all such pairs are unique by bijectivity.

This is a mess, not a proof.
What do I need to do so that the theorem is properly shown to be true? Unfold the definitions and show that the pieces align accoring to the rules of derivation. So.

$\textsc{Bijective}(f) \triangleq \textsc{Injective}(f) \wedge \textsc{Surjective}(f)$
Expanding further,
$\textsc{Bijective}(f) = (\forall y \exists x : y = f(x)) \wedge (\forall x, x' : f(x) = f(x') \Rightarrow x = x')$

$\exists f^{-1} \Leftarrow \textsc{Bijective}(f)$:

So far so good. Now, what does it mean for a function to have inverse? It means that the function is one-to-one *and* that it covers the whole range.

Injectivity and surjectivity are not symmetric with respect to domain and range: function is always considered defined on all of the domain, but we specifically call /injective/ those functions for which all of their range is used. Any non-injective function can be narrowed down to the used part of its range to make an injective function. More formally: $\forall f, f : X \to Y, \exists f' : X \to Y' where Y' = \{y \in Y : \exists x \in X : f(x) = y\}$.

In contrast, non-surjective functions can't be narrowed... no, wait, they can! their /domain/ can be narrowed so that for any surjective $f : X \to Y$ we can construct $f' : X' \to Y$ where $X' = X\{z \in X, \exists x \in X, x \neq z, f(x) = f(z)$. Nope that formalization excludes all non-unique pre-images, but I want to retain one. Actually, the formalization works: what remains /is/ a surjective function, a minimal one in a large family. Obviously, we often have additional properties we want our surjective narrowing to have, say, analiticity. This seems to be amenable to formalization via an ordering on X, where we simply retain the smallest (or the largest) element, and throw away the rest.

So surjectivity says something about the *domain*: none of its elements have matching images, the function is one-to-one

Injectivity is a statement about the range: the whole of the range is covered.

When both are true, the function is nicely one-to-one and there are no holes in its range.

2 Lamport-style structured proof

THEOREM: When a composition of two functions is an identity, the functions are inverses of each other and are bijective.

What follows as an attempt to prove the previous statement in line with Leslie Lamport's article "How to write a 21-century proof" using his `pf2.sty`. It suffices to prove that bijectivity is equivalent to the existence of inverse, and ... Dammit, this is trivial but unclear how to prove.

⟨1⟩1. BIJECTIVE$(f) \Leftrightarrow \exists f^{-1}$
  ⟨2⟩1. $g \circ f = id_X \Rightarrow$ INJECTIVE$(f) \wedge$ SURJECTIVE(g)
    ⟨3⟩1. $g \circ f = id_X \Rightarrow$ SURJECTIVE(g)
      PROOF:by $(g \circ f = id_X \Rightarrow \forall x \exists f(x) \in Y : x = id_X(x) = g \circ f(x) = g(f(x)))$ and definition of Surjectivity. □
    ⟨3⟩2. $g \circ f = id_X \Rightarrow$ INJECTIVE(f)
      PROOF:by definition of Injectivity. □
    ⟨3⟩3. Q.E.D.