# Homework 3

## ③ Analytic center

$Ax \le b$ ... has $m$ inequalities

$\Phi(x)$ ... contains all feasible solutions

$s(x) = b - Ax$

$I \subseteq \{1, \dots, n\}$ s.that $s(x)_i > 0$ or $(b - Ax)_i > 0$

The unique vector $x \in \Phi$ which maximizes $\prod_{i \in I} s(x)_i$ is analytic c.

3. If $Ax \le b$ has a feasible solution then $\Phi$ is not empty. In such case we set $I$ to be a set of all indices $i$ s. that $b_i - (Ax)_i > 0$. Then we have proven the statement holds.

If $Ax \le b$ does not have a feasible solution then $\Phi$ is empty and there does not exist $x \in \Phi$ s.that the statement holds.

## ④. Analytic center has "cost" function $\max \prod s(x)_i$.

$\max \prod s(x)_i = \max \prod (b_i - (Ax)_i) = \min(-1 \cdot \prod (b_i - (Ax_i))) = \max(\prod b_i - A_i^T x)$

to make it easier we will transform it with log. Since log is convex and non-decreasing $\log(f)$ is convex iff $f$ is conv

$$\log \prod (b_i - A_i^T x) = \sum \ln(b_i - A_i^T x)$$

$$\frac{\partial J}{\partial x} = \sum \frac{1}{(b_i - A_i^T x)} \cdot A_i^T$$

$$\frac{\partial J}{\partial x^2} = \sum -1 \cdot \frac{1}{(b_i - A_i^T x)^2} \cdot A_i \cdot A_i^T = -1 \cdot \sum \frac{A_i A_i^T}{(b_i - A_i^T x)^2}$$

positive sem. def. mtx

always positive scalar

makes it negative sem. def.

$$\vec{x}^T A_i A_i^T \vec{x} = (A_i^T \vec{x})^T \cdot (A_i^T \vec{x}) = (A_i^T \vec{x})^2 \ge 0$$

So if max is NSD then min is PSD so it is a convex function.

Since matrix A can be converted to full rank we can also say that the Hessian is neg. def hense we have strictly concave func.

5. Since we showed that analytic center is strictly convex, then it follows that it has at most one solution so it is unique

6. Find AC of

$-x_1 \le 0$

$-x_2 \le 0$

$ax_1 + x_2 \le 1$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ a & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\max\ (0+x_1)(0+x_2)(1-ax_1-x_2) = x_1x_2(1-ax_1-x_2) =$$

$$= x_1x_2 - ax_1^2x_2 - x_1x_2^2$$

$$\frac{\partial}{\partial x_1} = x_2 - 2ax_1x_2 - x_2^2 = 0 \qquad x_1 = \frac{x_2 - x_2^2}{2ax_2} = \frac{1-x_2}{2a}$$

$$\frac{\partial}{\partial x_2} = x_1 - ax_1^2 - 2x_1x_2 = 0$$

$$\frac{1-x_2}{2a} - \frac{\cancel{a}(1-x_2)^2}{4a^{\cancel{2}}} - \frac{\cancel{2}(1-x_2)x_2}{\cancel{2}a} =$$

$$\frac{2(1-x_2)-(1-x_2)^2-4(1-x_2)x_2}{4a} = \frac{(1-x_2)(2-1+x_2-4x_2)}{4a} = 0$$

This gives 0 so it's not cost

$1-x_2 = 0 \quad \cancel{\boxed{x_2 = 1 \mid x_1 = 0}}$

$-3x_2+1=0 \quad \boxed{x_2 = \frac{1}{3}}$

$\boxed{x_1 = \frac{1}{3a}}$

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$-x_1 \le 0$ $\quad f(x) = (0+x_1)(0+x_2)(1-x_1-x_2) = x_1x_2(1-x_1-x_2) = x_1x_2 - x_1^2x_2 - x_1x_2^2$

$-x_2 \le 0$ $\quad \frac{\partial f}{\partial x_1} = x_2 - 2x_1x_2 - x_2^2 = 0$ $\quad x_1^2 - x_2^2 = 0 \quad x_1^2 = x_2^2 \quad x_1 = \pm x_2$

$x_1 + x_2 \le 1$

$x_1 + x_2 \le 1$ $\quad \frac{\partial f}{\partial x_2} = x_1 - 2x_1x_2 - x_1^2 = 0$

I: $x_1 = x_2$

$x_2 - 2x_2^2 - x_2^2 = 0$

$x_2 - 3x_2^2 = 0$

$x_2(1-3x_2) = 0$

$x_2 = 0 \quad x_1 = 0$

$\boxed{x_2 = \frac{1}{3}} \quad \boxed{x_1 = \frac{1}{3}}$

II $x_1 = -x_2$

$-x_2 + 2x_2^2 - x_2^2 = 0$

$x_2^2 - x_2 = 0$

$x_2(x_2 - 1) = 0$

$x_2 = 0 \quad x_1 = 0$

$x_2 = 1 \quad x_1 = -1$

$f(0,0) = 0$

$f(-1,1) = -1 \cdot 1 \cdot 1 = -1$

$\boxed{f(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}}$ ← cost is max so this is the solution