# Homework 7

(1) $f_1(p) = \int_{-1}^{1} p(x)\,dx \qquad f_2(p) = \int_0^1 p(x)\,dx \qquad f(p) = \int_0^2 p(x)\,dx$

(a) $B = \{f_1, f_2, f_3\}$ is basis for $(\mathbb{R}_2[x])^*$

(b) Find dual basis for $\{f_1, f_2, f_3\}$ in $(V^*)^*$

$$f_1(ax^2+bx+c) = \int_{-1}^{1} ax^2+bx+c\,dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + \frac{cx}{1}\right]_{-1}^{1} = \frac{2a}{3} + 2c = \begin{bmatrix}\frac{2}{3} & 0 & 2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$f_2(ax^2+bx+c) = \frac{a}{3} + \frac{b}{2} + c = \begin{bmatrix}\frac{1}{3} & \frac{1}{2} & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$f_3(ax^2+bx+c) = \frac{8a}{3} + \frac{4b}{2} + 2c = \begin{bmatrix}\frac{8}{3} & 2 & 2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$$

$V^{**}: V^* \to \mathbb{R}$

$\beta^1 : \alpha \to [\frac{1}{2}\ -1\ \frac{1}{3}]\alpha$

$$\left[\begin{array}{ccc|ccc} 2/3 & 0 & 2 & 1 & 0 & 0\\ 1/3 & 1/2 & 1 & 0 & 1 & 0\\ 8/3 & 2 & 2 & 0 & 0 & 1\end{array}\right] \xrightarrow{\text{HW5}} \left[\begin{array}{ccc|ccc} 1 & & 0 & 1/2 & -2 & 1/2\\ & 1 & & -1 & 2 & 0\\ 0 & & 1 & 1/3 & 2/3 & -1/6\end{array}\right] \quad B = \left(\begin{bmatrix}1/2\\-1\\1/3\end{bmatrix}, \begin{bmatrix}-2\\2\\2/3\end{bmatrix}, \begin{bmatrix}1/2\\0\\-1/6\end{bmatrix}\right)$$

(c) $g(p,q) = \int_{-1}^{1} p(x)\,q(x)\,dx$

So far we have 3 linear forms: $\{f_1 f_2 f_3\}$ which are the dual space for $B = \{p_1 p_2 p_3\}$. Next we want to find the reciprocal basis for $B$ w.r.t. $g$. Find $B^g$! $B^g = \{p^1\ p^2\ p^3\}$ from $B$

$g(p^i, p_j) = \delta^i_j \qquad g(p^i, p_j) = [v^1 \ldots v^n]_B\, G \begin{bmatrix}w_1\\ \vdots\\ w_n\end{bmatrix}_B \qquad p^i = \sum v^i \cdot p_i$

$p_j = \sum_i w_i \cdot p_i$

$$I = \underbrace{\begin{bmatrix} - p_1 - \\ - p_2 - \\ - p_3 - \end{bmatrix}}_{\text{reciprocal basis}} \quad G_{\text{std basis}} \begin{array}{c} \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \begin{bmatrix} 1/2 & -2 & 1/2\\ -1 & 2 & 0\\ 1/3 & 2/3 & -1/6\end{bmatrix} \end{array} \text{ in std basis} \qquad G = \begin{bmatrix} 2/5 & 0 & 2/3\\ 0 & 2/3 & 0\\ 2/3 & 0 & 2\end{bmatrix} \Rightarrow$$

Gram matrix from HW4 calculated on std basis

$$I = P \cdot \begin{bmatrix} 2/5 & 0 & 2/3\\ 0 & 2/3 & 0\\ 2/3 & 0 & 2\end{bmatrix}\begin{bmatrix} 1/2 & -2 & 1/2\\ -1 & 2 & 0\\ 1/3 & 2/3 & -1/6\end{bmatrix} = P \cdot \underbrace{\begin{bmatrix} 19/45 & -16/45 & 4/45\\ -2/3 & 4/3 & 0\\ 1 & 0 & 0\end{bmatrix}}_{\text{Find inverse of this}}$$

$$\begin{bmatrix} 19/45 & -16/45 & 4/45\\ -2/3 & 4/3 & 0\\ 1 & 0 & 0\end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1\\ 0 & 3/4 & 1/2\\ 45/4 & 3 & -11/4\end{bmatrix} \begin{array}{l} \to p^1 \\ \to p^2 \\ \to p^3 \end{array}$$

$p^1, p^2, p^3$: reciprocal basis

calculation was done by hand! :)

(2) Show $Bil(V \times V)$ is isomorphic to the set of all lin. trans. from $V \to V^*$ by explicitly constructing the isomorphism.

$$V^* \otimes V^* \cong V \to V^* \quad \text{or} \quad Bil(V \times V) \cong (V \to V^*)$$

$$\varphi : Bil(V \times V) \to (V \to V^*)$$

Intuitively this map $\varphi$ takes a bilinear form and then maps it to a linear map that takes a vector from $V$ and returns a linear form. So we can use the bilinear form from the input, insert one vector from $V$ and we end up with linear form.

$$\varphi : V^* \otimes V^* \to (V \to V^*)$$

$$\phi \to \alpha_\phi \qquad \alpha_\phi : \vec{v} \to \phi(\vec{v}, \cdot)$$

$\boxed{1}$ $\varphi$ is linear

$$\varphi(\gamma_1 \phi_1 + \gamma_2 \phi_2) = \alpha_{\gamma_1\phi_1 + \gamma_2\phi_2}(\vec{v}) =$$

$$= (\gamma_1\phi_1 + \gamma_2\phi_2)(\vec{v}, \cdot) = \gamma_1 \phi_1(\vec{v}, \cdot) + \gamma_2 \phi_2(\vec{v}, \cdot) = \gamma_1 \varphi(\phi_1) + \gamma_2 \varphi(\phi_2)$$

$\boxed{2}$ $\varphi$ is injective

$\ker(\varphi)$ is trivial : $\phi \in \ker(\varphi)$

$$\varphi(\phi) = \alpha_\phi(\vec{v}) = 0$$

$$\phi(\vec{v}, \cdot) = 0 \quad \text{for all } \vec{v}$$

this lin form is zero lin form for all $\vec{v}$ only if the bilinear form $\phi$ is zero.

$\boxed{3}$ $\dim(Bil(V \times V)) = n \times n$

$\dim(V \to V^*) = n \times n$ (takes vector $n$ returns lin form of dim $n$)

$\Rightarrow$ $\varphi$ is bijective and $Bil(V \times V) \cong V \to V^*$

(3) $S = \{e_1, e_2\}$ std basis for $R_2$ and $S^* = \{\varepsilon^1, \varepsilon^2\}$ the dual. $T$ is $(1,2)$ tensor on $R^2$ given by:

$$V^* \times V \times V \to R \qquad T(\varepsilon_i, e_j, e_k) = \begin{cases} j-i & \text{if } k=1 \\ i-j & \text{if } k=2 \end{cases}$$

ABC = 365. Compute $T(\alpha, v, w)$ where $\alpha = \varepsilon^1 + \varepsilon^2$, $v = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $w = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$v = 3e_1 + 6e_2$

$w = 5e_1 + 0e_2$

$T(\varepsilon^1 + \varepsilon^2, 3e_1 + 6e_2, 5e_1) =$

$= T(\varepsilon^1, 3e_1, 5e_1) + T(\varepsilon^1, 6e_2, 5e_1) + T(\varepsilon^2, 3e_1, 5e_1) + T(\varepsilon^2, 6e_2, 5e_1) =$

$= 15 \cdot 0 + 30 \cdot \underline{1} + 15 \cdot (-1) + 5 \cdot 0 = 30 - 15 = 15$