Homework 3

③ Analytic center

$Ax \leq b$ ... has m inequalities

$\Phi(x)$ ... contains all feasible solutions

$s(x) = b - Ax$

$I \subseteq \{1, \dots, n\}$ s. that $s(x)_i > 0$ or $(b - Ax)_i > 0$

The unique vector $x \in \Phi$ which maximizes $\prod_{i \in I} s(x)_i$ is analytic c.

3. If $Ax \leq b$ has a feasible solution then Φ is not empty. In such case we set I to be a set of all indices i s. that $b_i - (Ax)_i > 0$. Then we have proven the statement holds.

If $Ax \leq b$ does not have a feasible solution then Φ is empty and there does not exist $x \in \Phi$ s. that the statement holds.

④ Analytic center has "cost" function $\max \prod s(x)_i$.

$\max \prod s(x)_i = \max \prod (b_i - (Ax)_i) = \min(-1 \cdot \prod (b_i - (Ax_i))) = \max (\prod b_i - A_i^T x)$

to make it easier we will transform it with log. Since log is convex and non-decreasing $\log(f)$ is convex iff f is convex

$$\log \prod (b_i - A_i^T x) = \sum \ln(b_i - A_i^T x)$$

$$\frac{\partial J}{\partial x} = \sum \frac{1}{(b_i - A_i^T x)} \cdot A_i^T$$

$$\frac{\partial J}{\partial x^2} = \sum -1 \cdot \frac{1}{(b_i - A_i^T x)^2} \cdot A_i \cdot A_i^T = -1 \cdot \sum \frac{A_i A_i^T}{(b_i - A_i^T x)^2}$$

positive semi. def. mtx

always positive scalar

makes it negative semi. def.

$$\vec{x}^T A_i A_i^T \vec{x} = (A_i^T \vec{x})^T \cdot (A_i^T \vec{x}) = (A_i^T \vec{x})^2 \geq 0$$

So if max is NSD then min is PSD so it is a convex function.

Since matrix A can be converted to full rank we can also say that the Hessian is neg. def hense we have strictly concave func.

5. Since we showed that analytic center is strictly convex, then it follows that it has at most one solution so it is unique

6. Find AC of

$-x_1 \leq 0$

$-x_2 \leq 0$

$ax_1 + x_2 \leq 1$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ a & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\max (0 + x_1)(0 + x_2)(1 - ax_1 - x_2) = x_1 x_2 (1 - ax_1 - x_2) =$$

$$= x_1 x_2 - a x_1^2 x_2 - x_1 x_2^2$$

$$\frac{\partial}{\partial x_1} = x_2 - 2a x_1 x_2 - x_2^2 = 0 \qquad x_1 = \frac{x_2 - x_2^2}{2a x_2} = \frac{1 - x_2}{2a}$$

$$\frac{\partial}{\partial x_2} = x_1 - a x_1^2 - 2 x_1 x_2 = 0$$

$$\frac{1 - x_2}{2a} - \frac{\cancel{a}(1 - x_2)^2}{4a^{\cancel{2}}} - \frac{\cancel{2}(1 - x_2) x_2}{2a} =$$

$$\frac{2(1 - x_2) - (1 - x_2)^2 - 4(1 - x_2) x_2}{4a} = \frac{(1 - x_2)(2 - 1 + x_2 - 4x_2)}{4a} = 0$$

This gives cost 0 so it's not

$1 - x_2 = 0 \quad \cancel{\boxed{x_2 = 1 \mid x_1 = 0}}$

$-3x_2 + 1 = 0 \quad \boxed{x_2 = \frac{1}{3}}$

$\boxed{x_1 = \frac{1}{3a}}$

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$-x_1 \le 0$ $\quad f(x) = (0+x_1)(0+x_2)(1-x_1-x_2) = x_1x_2(1-x_1-x_2) = x_1x_2 - x_1^2x_2 - x_1x_2^2$

$-x_2 \le 0$ $\quad \frac{\partial f}{\partial x_1} = x_2 - 2x_1x_2 - x_2^2 = 0$ $\quad x_1^2 - x_2^2 = 0 \quad x_1^2 = x_2^2 \quad x_1 = \pm x_2$

$x_1 + x_2 \le 1$

$x_1 + x_2 \le 1$ $\quad \frac{\partial f}{\partial x_2} = x_1 - 2x_1x_2 - x_1^2 = 0$

I: $x_1 = x_2$

$x_2 - 2x_2^2 - x_2^2 = 0$

$x_2 - 3x_2^2 = 0$

$x_2(1-3x_2) = 0$

$x_2 = 0 \quad x_1 = 0$

$\boxed{x_2 = \frac{1}{3}} \quad \boxed{x_1 = \frac{1}{3}}$

II $x_1 = -x_2$

$-x_2 + 2x_2^2 - x_2^2 = 0$

$x_2^2 - x_2 = 0$

$x_2(x_2 - 1) = 0$

$x_2 = 0 \quad x_1 = 0$

$x_2 = 1 \quad x_1 = -1$

$f(0,0) = 0$

$f(-1,1) = -1 \cdot 1 \cdot 1 = -1$

$\boxed{f(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}}$ ← cost is max so this is the solution

```
The new mu is: 1.8100532473005348e-119 and the value objective gap is [[3.25809598e-118]]
The new mu is: 1.4207944844402046e-119 and the value objective gap is [[2.55743016e-118]]
The new mu is: 1.1152472834853219e-119 and the value objective gap is [[2.007445e-118]]
The new mu is: 8.754091580046074e-120 and the value objective gap is [[1.57573672e-118]]
The new mu is: 6.87149124025122e-120 and the value objective gap is [[1.23686802e-118]]
The new mu is: 5.3937511885842904e-120 and the value objective gap is [[9.70875366e-119]]
The new mu is: 4.2338046964156256e-120 and the value objective gap is [[7.62084813e-119]]
B set contains: [1, 5, 8, 9, 10, 11, 12, 13]
N set contains: [0, 2, 3, 4, 6, 7, 14, 15]
N complement set contains: [1, 5, 8, 9, 10, 11, 12, 13]
Optimal solution is:
       potatoes : 0.0
          bread : 7.4
           milk : 0.0
           eggs : 0.0
        yoghurt : 0.0
        veg oil : 0.46
           beef : 0.0
    strawberies : 0.0
```

My solver

```
Cost of all these items is: 171.9889502762431
Daily needs (CH,PR,FT,EN):
 [[ 355.35911602   81.43646409   90.         2400.        ]]
------Commercial solver solution----------
Optimal solution is:
       potatoes : 0.0
          bread : 6.48
           milk : 0.0
           eggs : 0.0
        yoghurt : 0.0
        veg oil : 0.51
           beef : 0.0
    strawberies : 0.0
```

Commercial solver

```
Cost of all these items is: 152.83609606712864
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