Homework 5

(1) $p_1(x) = x^2 - 1 \qquad p_2(x) = x^2 + x + 1 \qquad p_3(x) = x^2 + x$

(a) Show that $\{p_1, p_2, p_3\}$ is basis for $\mathbb{R}_2[x]$

Lin. ind:

$$\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = 0$$

$$\begin{array}{c} \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \end{array} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow P_1 = P_2 = P_3 = 0$$

$N(P) = \emptyset$

$\Rightarrow p_1, p_2, p_3$ are lin. ind.

$\dim \mathbb{R}_2[x] = 3$ and $\dim(\{p_1, p_2, p_3\}) = 3$

$\Rightarrow \{p_1\ p_2\ p_3\}$ are basis for $\mathbb{R}_2[x]$

(b) Find the dual basis for $\{p_1\ p_2\ p_3\}$

β_1:

$[a_1'\ a_2'\ a_3'] \cdot P_1 = 1$

$[a_1'\ a_2'\ a_3'] \cdot P_2 = 0$

$[a_1'\ a_2'\ a_3'] \cdot P_3 = 0$

$$\begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & -1 & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$a_3' = 0 \qquad -1a_2' - \cancel{2a_3'} = 1 \qquad a_2' = -1 \qquad a_1' = 1 \qquad \boxed{\beta_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x - y}$

$$\beta_2 \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & -1 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$a_3^2 = 1$

$-a_2^2 - 2 = -1$

$a_2^2 = -1$

$a_1^2 = 1$

$\boxed{\beta_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x - y + z}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$a_3^3 = -1$

$-a_2^3 = -2$

$a_2^3 = 2$

$a_1^3 = -1$

$\boxed{\beta\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = -x + 2y - z}$

(2) $f_1(p) = \int_{-1}^{1} p(x)\,dx \quad f_2(p) = \int_0^1 p(x)\,dx \quad f_3(p) = \int_0^2 p(x)\,dx$

(a) Prove that $\{f_1 f_2 f_3\}$ form a basis for $(\mathbb{R}_2[x])^*$

$p(x) = ax^2 + bx + c$

$\int p(x)\,dx = \int ax^2+bx+c\,dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$

$f_1(p(x)) = \frac{a}{3} + \cancel{\frac{b}{2}} + c - \left(-\frac{a}{3} + \cancel{\frac{b}{2}} - c\right) = \frac{2a}{3} + 2c$

$f_2(p) = \frac{a}{3} + \frac{b}{2} + c$

$f_3(p) = \frac{8a}{3} + \frac{4b}{2} + 2c$

$\alpha_1(f_1) + \alpha_2(f_2) + \alpha_3(f_3) = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$ (if lin. ind.)

$$\begin{array}{c} f_1 \\ \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{8}{3} \\ 0 & \frac{1}{2} & 2 \\ 2 & 1 & 2 \end{bmatrix} \end{array} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of this mtx is 3, so the columns are lin. independant

(b) Find basis B such that $B^* = \{f_1\ f_2\ f_3\}$

f_1: $\left[\frac{2}{3}\ 0\ 2\right] b_1 = 1 \quad \left[\frac{2}{3}\ 0\ 2\right] b_2 = 0 \quad \left[\frac{2}{3}\ 0\ 2\right] b_3 = 0$

f_2: $\left[\frac{1}{3}\ \frac{1}{2}\ 1\right] b_1 = 0 \quad [\ f_2\ ]\, b_2 = 1 \quad [\ f_2\ ]\, b_3 = 0$

f_3: $\left[\frac{8}{3}\ 2\ 2\right] b_1 = 0 \quad [\ f_3\ ]\, b_2 = 0 \quad [\ f_3\ ]\, b_3 = 1$

$\downarrow$

b_1 $\left[\begin{array}{ccc|c} \frac{2}{3} & 0 & 2 & 1 \\ 1/3 & 1/2 & 1 & 0 \\ 8/3 & 2 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1/3 & 1/2 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 6 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} & & & \\ & & & \\ 0 & 0 & -6 & -2 \end{array}\right]$ $\quad \frac{1}{3}x - \frac{1}{2} + \frac{1}{3} = 0 \quad x = \frac{1}{2}$, $Y = -1$, $Z = \frac{1}{3}$

b_2 $\left[\begin{array}{ccc|c} 1/3 & 1/2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 6 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} & & & \\ & & & \\ 0 & 0 & +6 & 2 \end{array}\right]$ $\quad Z = \frac{2}{3}$, $Y = 2$, $x = -2$

b_3 $\left[\begin{array}{ccc|c} 1/3 & 1/2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 6 & -1 \end{array}\right] \Rightarrow$ $Z = -1/6$, $Y = 0$, $x = 1/2$

$$B = \left\{ \begin{bmatrix} 1/2 \\ -1 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ -1/6 \end{bmatrix} \right\}$$

(3) $p, q \in \mathbb{R}_3[x]$

$$g(p,q) = \int_{-1}^{1} p(x)q(x)\,dx \qquad h(p,q) = \int_0^1 p(x)q(x)\,dx$$

(a) Proove g and h are inner product

$B = \{1, x, x^2, x^3\}$

$g(1,1) = \int_{-1}^{1} 1\,dx = x\Big|_{-1}^{1} = 1-(-1) = 2$ $\qquad g(x,x^2) = \int_{-1}^{1} x^3\,dx = 0$

$g(1,x) = \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1}{2} - (+\frac{1}{2}) = 0$ $\qquad g(x,x^3) = \int x^4\,dx = \frac{x^5}{5}\Big|_{-1}^{1} = \frac{2}{5}$

$g(1,x^2) = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}$ $\qquad g(x^2,x^3) = \int x^5\,dx = 0$

$g(1,x^3) = \frac{x^4}{4}\Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$ $\qquad g(x,x) = \frac{2}{3} \quad g(x^2,x^2) = \frac{2}{5} \quad g(x^3,x^3) = \frac{2}{7}$

$$G = \begin{bmatrix} 2 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 2/5 \\ 2/3 & 0 & 2/5 & 0 \\ 0 & 2/5 & 0 & 2/7 \end{bmatrix}$$ is PD so g is inner product

Is g bilinear form? $g(\alpha_1 p_1 + \alpha_2 p_2, q) = \int_{-1}^{1} \alpha_1 p_1(x)q(x)\,dx + \int_{-1}^{1} \alpha_2 p_2(x) q(x)\,dx =$

Same proof for bilinearity $\qquad = \alpha_1 g(p_1,q) + \alpha_2 g(p_2,q)$

holds for h!

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$$ is PD so h is inner product

(b) Polynomial pairs like $(1,x), (1,x^3), (x,x^2), (x^2,x^3)$ are orthogonal in g but not in h.

(c) Let $V = \mathcal{L}\{\overset{v_1}{1}, \overset{v_2}{x}, \overset{v_3}{x^2}\}$. Find the $\overset{\text{orthonormal}}{\text{normal}}$ basis for V w.r.t. g.

$u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad e_1 = \frac{u_1}{\sqrt{g(u_1,u_1)}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad g(e_1,e_1) = 1$ ✓

$u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{g(x,1)}{g(1,1)}}^{\,0} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad e_2 = \frac{u_2}{\sqrt{g(u_2,u_2)}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \sqrt{\frac{3}{2}} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{3/2} \\ 0 \end{bmatrix}$ ✓

$u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{g(x^2,1)}{g(1,1)} - \cancel{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \frac{g(x,x^2)}{g(x,x)}}^{\,0} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{\frac{2}{3}}{\frac{2}{1}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{3} \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/3 \end{bmatrix} \cdot \sqrt{\frac{45}{8}}$ ✓

(d) Find the orthogonal complement $V^\perp$ w.r.t. g.

$V = \mathcal{L}\{1, x, x^2\}$

For orth. compl. $v \in V^\perp$ it has to hold that $g(v, w) = 0 \quad w \in V$

So $g(v, 1) = 0$, $g(v, x) = 0$, $g(v, x^2) = 0$ $\qquad v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = ax^3 + bx^2 + cx + d$

$$\int_{-1}^{1} ax^3 + bx^2 + cx + d = 0 = \left(\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + \frac{dx}{1}\right)\Big|_{-1}^{1} = \frac{2b}{3} + 2d = 0$$

$$\int_{-1}^{1} ax^4 + bx^3 + cx^2 + dx = 0 = \left(\frac{ax^5}{5} + \frac{bx^4}{4} + \frac{cx^3}{3} + \frac{dx^2}{2}\right)\Big|_{-1}^{1} = \frac{2a}{5} + \frac{2c}{3} = 0$$

$$\int_{-1}^{1} ax^5 + bx^4 + cx^3 + dx^2 = 0 = \frac{2b}{5} + \frac{2d}{2} = 0$$

$$\begin{cases} 2b + 6d = 0 & \Rightarrow b + 3d = 0 & b = -3d \\ 6a + 10c = 0 & \Rightarrow 3a + 5c = 0 & a = \frac{-5c}{3} \\ 4b + 10d = 0 & 2b + 5d = 0 & -6d + 5d = 0 \rightarrow b = d = 0 \end{cases}$$

$$V^\perp = \mathcal{L}\left\{ \begin{bmatrix} -5c/3 \\ 0 \\ c \\ 0 \end{bmatrix}, c \in \mathbb{R} \right\}$$