The Transition to Turbulence

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Plane Couette flow and pressure-driven pipe flow are two examples of flows where turbulence sets in while the laminar profile is still linearly stable. Experiments and numerical studies have shown that the transition has features compatible with the formation of a strange saddle rather than an attractor. The turbulent state is not initially present, but an expected lifetime to turbulence is exponential. Embedded within the turbulent dynamics are coherent structures, which transiently show up in the temporal evolution of the turbulent flow.

note: This is a summary of Eckhardt's papers (2008,2007,1999) with comparisons to other papers.

I. INTRODUCTION

The transition to turbulence is straightforward to discuss in cases where the laminar profile becomes unstable when the driving of the flow is increased (Chandrasekhar 1961; Drazin & Reid 1981). The transitions occur at well-defined values of the control parameter (varied from paper to paper, but was typically the perturbation amplitude and Reynolds number), and these values can be easily experimentally verified. The flows between two parallel plates moving relative to each other or pressure-driven flow down a pipe of circular cross section behave differently. However, their linear and parabolic laminar profiles are linearly stable, and the simple geometry of plane Couette and pipe flow make them easy to study.

Since many common flows can be approximated by Couette and pipe flows, understanding the dynamics of these two flows can contribute to the understanding of other cases.

II. PLANE COUETTE FLOW

A. FORMULATION

In plane Couette flow, the fluid is bounded by two infinitely extended parallel plates which are free to move streamwise. There is no pressure gradient.

For an incompressible flow, where the velocities of the plates are $\pm U_0$, the finite disturbance $\vec{u} = (u, v, w)$ superimposed on laminar flow evolves according to

$$\frac{\partial \vec{u}}{\partial t} = -(\vec{u}\cdot\nabla)\vec{u} - z\frac{\partial \vec{u}}{\partial x} - w\vec{e_x} - \nabla p + \frac{\nabla^2 \vec{u}}{Re}, \quad (1)$$

where $\vec{U_0} = z \cdot \vec{e_x}$, $Re = \frac{2U_0 d}{\nu}$ with d as the plate separation and ν as the kinematic viscosity, and p is the thermodynamic pressure.

The flow field is incompressible, so $\vec{u}$ is non divergent and shear stress forces u to be 0 at the plates.

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Since no closed form of Navier-Stokes exists, numerical approximations for specific conditions are the only way to solve turbulence. For the numerical representation of the flow field, a Fourier expansion is used for the flow direction, and normalized Legendre polynomials $\phi_p(z)$ are used for the plane-normal direction.

$$\vec{u}(\vec{x}, t) = \sum_{n_x, n_y, p} \tilde{u_{k,p}}(t) \exp\left(i\left(\frac{2\pi n_x}{L_x} + \frac{2\pi n_y}{L_y}\right)\right)\phi_p(x), \quad (2)$$

where L_x and L_y are the limits of integration along the flow and spawn of the flow respectively, n_x and n_y are the wavevectors constrained by $|n_x| + |n_y| \le N$. For the papers simulations, they set the maximum order of Legendre polynomial to be 9.

Two kinds of initial conditions were used: 1) a sudden perturbation introduced by instantenously adding it to the laminar flow, and 2) adiabatic switching of a perturbation. The sudden perturbation has a transient period in which it adjusts to the flow. The adiabatic one is free of this initial transient and gives information about the approach to the turbulent state.

B. SUDDEN PERTURBATIONS

The suddenly introduced perturbations consist of poloidal vortex rings that are superimposed on the linear profile to form the initial flow field. The vortex rings are given by

$$\vec{u_0} = A\nabla\times\nabla\times\exp\left(-\sigma^2(x^2+y^2+z^2)\right)\vec{e_j}, \quad (3)$$

where A is the amplitude and $\vec{e_j}$ is a unit vector pointing in the direction of the axis of the vortex. Eckhardt *et al.* used a dimensionless constant of $\sigma = 8$ so that the vortex is strongly localized in directions perpendicular to e_j. Daviaud *et al.* used $j = z$ so that the vortex pointed in the wall-normal direction.

The initial evolution of the vortex rings is governed by the shear and stretch of the laminar flow; vorticity spreads out very quickly and the dynamics become less obvious. They used two scalar variables: 1) the energy in the perturbation and 2) the mean shear.

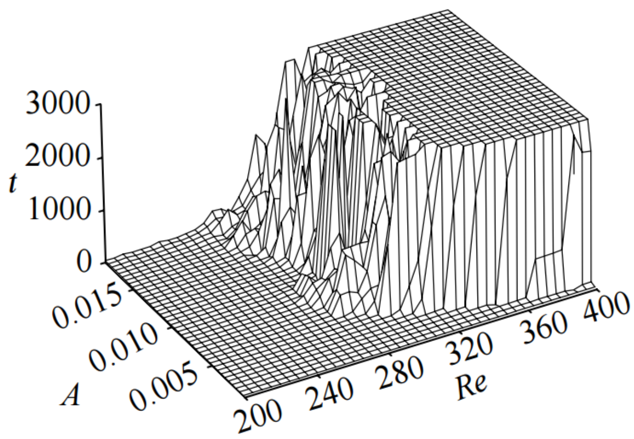

FIG. 1: Lifetimes of perturbations in plane Couette flow as a function of the Reynolds number and amplitude of perturbations for a vortex ring with axis pointing in the plane-normal direction.The lifetimes are calculated on a 40 by 40 grid.

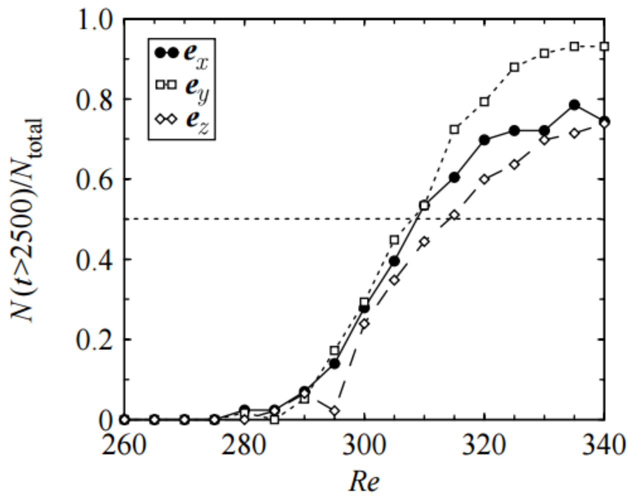

FIG. 2: Statistics of lifetimes in plane Couette flow. Fraction of initial conditions that have a lifetime in excess of 2500 time units. The dotted line near 0.5 can be used to read off the Reynolds numbers above which more than half the initial conditions lead to lifetimes exceeding 2500. The values are 310 and 320, depending on the type of the perturbation. The data basis is the same which underlies Figure 1.

The energy density in the perturbation E divided by the energy density of the laminar profile is given by

$$E = \frac{1}{V}\int \vec{u}\cdot\vec{u*}dV, \tag{4}$$

where V is the volume and $\vec{u*}$ is the complex conjugate of $\vec{u}$. The shear rate is calculated from the downstream component $\vec{u}$ of the velocity field and the area A of one plate,

$$M = \frac{1}{AS_0}\int \frac{\partial\vec{u}}{\partial z}_{z=1} dA, \tag{5}$$

normalized by the laminar shear rate $S_0 = \frac{2U_0}{d}$.

Fig. 1 shows the perturbations die off quickly when Re are small and when combined with a small amplitude. The region of small Re and small amplitude are governed by the linearized equation. For larger amplitude and Reynolds numbers, the perturbations show nonlinearly sustained oscillations for some time before finally decaying. For even higher values of the Reynolds number and a sufficiently large initial amplitude, the perturbation does not decay within the observation time.

It could be that some of the initial conditions in this regime decay anytime after the maximal integration time, which here was typically limited to $t_{max} = 3000$ dimensionless units. While earlier studies suggested that there indeed was a Reynolds number beyond which no trajectories would ever decay (Bottin and Chate 1998), more recent studies give evidence that a decay will always be possible (see Hof et al. (2006)).

In order to extract the trend in the lifetime, a collection of results for different amplitudes but the same Re can be used. In the linear regime, the dynamics should be independent of the amplitude of the perturbation so that the median and the maximum of the lifetimes coincide.

An important feature of the probabilities in Fig. 2. is their curvature and S-shape: as was emphasized by Hof et al. (2006), a divergence of the lifetimes at a finite Re would imply that these curves reach a maximal value at a Reynolds number that is independent of the observation time. However, all curves bend towards increasing Re, suggesting that there is always a small but non-negligible fraction of initial conditions which decays. The analysis of the characteristic times then shows that they seem to increase exponentially with Reynolds number.

C. ADIABATIC PERTURBATIONS

The second class of perturbations is driven by a slowly switched body force that allows one to adiabatically introduce the perturbation on top of the laminar profile. It allows one to explore a wider range of intermediate energies and to study the approach to the turbulent flow.

It is modeled by adding a time dependent volume force to the linear laminar flow equation Eq. (1):

$$\frac{\partial\vec{u}}{\partial t} = \mathcal{N}(\vec{u}) + f(t)\vec{F}(\vec{x}), \tag{6}$$

where $\mathcal{N}$ is the right hand side of the Navier-Stokes equation (1), an amplitude function of

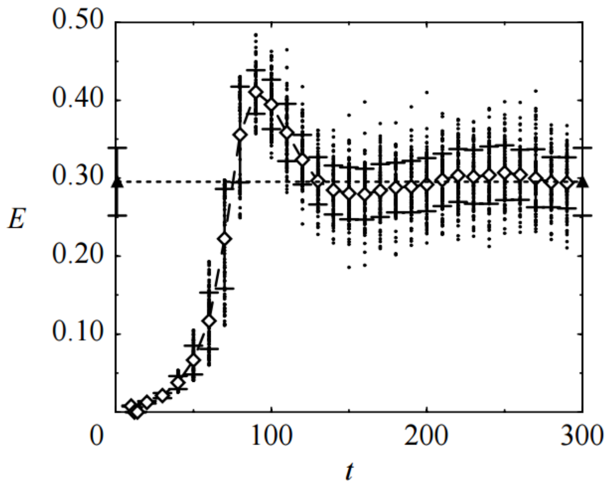

FIG. 3: Approaching the turbulent state with adiabatically switched perturbations. Time evolution for an ensemble with larger perturbation amplitudes $A = 0.035 \pm 005$. The line at approximately $E = 0.3$ indicates the time average of a turbulent signal at $Re = 400$. Also indicated are the mean energy and the variance at each Reynolds number. They are connected by straight line segments to highlight the oscillatory relaxation to the turbulent state.

$$f(t) = A_0 \left[\sin\left(\frac{2\pi t}{t_{dist}} - \frac{\pi}{2} + 1 \right) \right], \tag{7}$$

with maximum amplitude A_0, impulse duration t_{dist}, and finally a a volume force with inflow and outflow that is asymmetric with respect to the mirror symmetry in the spanwise direction, y/Ky and v/Kv, $(\vec{F})$is used. The analytical form of the z -component is

$$F_z = \sqrt{\pi} \exp\left[-\sigma(x^2 + y^2) - 20(z+1) \right] \int_0^{\sqrt{\sigma}x} e^{-\tilde{y}^2} d\tilde{y}, \tag{8}$$

with an inverse width $\sigma = 32$, and the other components are also non-zero after projecting onto the divergencefree subspace.

Time traces for a weak perturbation at $Re = 400$ are shown in Fig. 3.. The perturbation ends at $t_{dist} = 10$, before the first data point in the figure. The traces show that there is a gradual increase in energy until a value of approximately 0.05 is reached. Then, the energy quickly builds up to the value of the fully developed turbulent state, approximately 0.3. The collection of time traces for perturbations with slightly different amplitudes shows that the slope of the initial increase can differ widely and is mapped to a wide range of times after which the perturbations swing up to the turbulent state.

III. PIPE FLOW

A. FORMULATION

The laminar profile for flow down an infinitely long pipe is parabolic:

$$u_z(r) = u_c \left(1 - \frac{r^2}{R^2} \right) \vec{e_z}, \tag{9}$$

where u_c is the centreline velocity and R is the radius. A Reynolds number may be based on the maximum velocity u_c at the centre and the radius, or the mean velocity $\vec{u}$ and the diameter:

$$Re = \frac{u_c R}{\nu} = \frac{\vec{u}(2R)}{\nu}, \tag{10}$$

where ν is the dynamic viscosity of the fluid.

While the Reynolds numbers are the same, the characteristic time scales vary by a factor of 4 since $\vec{t} = 2R/\vec{u} = 4R/u_c = 4t_c$.

For the numerical representation of the Navier-Stokes equation, a Fourier-Legendre collation method in cylindrical coordiantes with Lagrange multipliers to account for the no-slip boundary condition at the wall and the constraints that the flow field is solenoidal, analytical and regular in the centre for $r = 0$.

B. COMPARISON TO PLANE COUETTE

The laminar profile is stable for all Reynolds numbers, but experiments start showing turbulence for Reynolds numbers of approximately 1700. Probing the phase space with initial conditions as in the case of plane Couette flow reveals similar characteristics: for low Reynolds number, all initial conditions decay; and for increasing Reynolds number, more and more initial conditions show turbulence and the turbulent lifetimes increase until eventually it becomes virtually impossible to find an initial condition that decays.

Averaging over initial conditions at a fixed Reynolds number, the distribution of lifetimes can be measured, and once again it follows an exponential law that can be characterized by the time over which the probability drops to 1/e. Following earlier observations in plane Couette flow, it was suspected that this time would diverge at a critical Reynolds number somewhere between 1750 and 2250(Faisst and Eckhardt 2004; Mullin and Peixinho 2006a,b; Peixinho and Mullin 2006; Willis and Kerswell 2007).

However, experiments in a very long pipe suggest that there is no divergence, but that the characteristic time increases rapidly. There is no direct theoretical reason for any specific behaviour (Eckhardt and Faisst 2004), but the analysis by Hof et al. (2006) shows that the

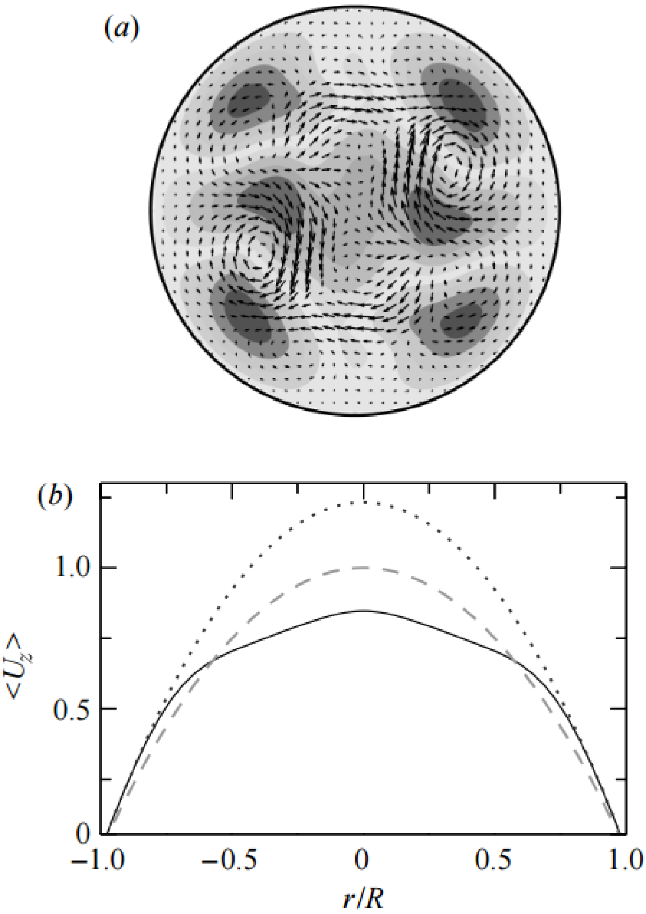

FIG. 4: (a) Cross sections of a coherent state in pipe flow with two vortex pairs at its point of bifurcation, Re=1350 (bifurcation). The differences in the downstream component between the coherent state and a parabolic profile with the same mean speed are indicated by grey scales. The perpendicular velocity components are shown as arrows. (b) Mean profiles for the upper and lower branches of the four-vortex states at Re=1350 (bifurcation) The continuous line is the mean downstream profile. The dashed line shows the parabolic profile with the same mean velocity and the dotted line shows the parabolic profile with the same friction coefficient, which agrees with the slope near the wall.

data are compatible with an increase that is exponential in Reynolds number. This suggests that (i) there is no transition to an attractor, (ii) the turbulent dynamics and the laminar one are always dynamically connected without barrier, and (iii) there should be new possibilities for turbulence control.

C. TRAVELLING WAVES

As in the case of plane Couette flow, there exist in pipe flow coherent structures around which the chaotic dynamics develops. They were found by adiabatically following the states that bifurcate from downstream vortices superimposed on the parabolic flow (Faisst and Eckhardt 2003; Wedin and Kerswell 2004). The initial vortices are translation invariant downstream and are sustained by a suitable body force which will eventually be reduced to zero. The details of the body force are not important, as long as the initial state undergoes a bifurcation that breaks the translational symmetry, since it can be shown that a translationally invariant flow must decay

The coherent states in pipe flow have one free parameter, the downstream wavelength. At the point of bifurcation, only a single length is possible, but for higher Reynolds numbers a whole interval of wavelengths becomes accessible. Over this interval, the properties of the states change. The phase speed of the travelling waves is higher than the mean velocity, but slower than the maximal velocity at the centre of a parabolic profile with the same mean speed. The variations in friction factor are small and tend to lower values as the Reynolds number increases.

IV. CONCLUSION

The dynamical system approach to the transition to turbulence has helped to characterize the transient nature of the turbulent state in the transition region and explains why a diversity of critical Reynolds numbers is quoted in the literature. It links the transition to turbulence to the appearance of three-dimensional states, which in their simplest form are stationary states or traveling wave solutions. These states typically are saddles and do not persist forever, but they can appear transiently during the evolution of the flow. Dynamical system theory suggests that a complete classification of all periodic states and their statistical weights could be used to estimate the statistical properties, but we are far from a sufficient understanding of the organization of these states.

In addition to plane Couette flow and pipe flow as discussed here, dynamical systems ideas can also be applied to other transitional flows (Itano and Toh 2001; Toh and Itano 2003) and boundary layers (Holmes et al. 1996). All these efforts demonstrate the value of dynamical systems ideas in the transitional regime and suggest many lines of continuation

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