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An anytime tree search algorithm for the 2018 ROADEF/EURO challenge glass cutting problem

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ABSTRACT

In this article, we present the anytime tree search algorithm we designed for the 2018 ROADEF/EURO challenge glass cutting problem proposed by the French company Saint-Gobain. The resulting program was ranked first among 64 participants. Its key components are: a new search algorithm called Iterative Memory Bounded A* (IMBA*) with guide functions, a symmetry breaking strategy, and a pseudo-dominance rule. We perform a comprehensive study of these components showing that each of them contributes to the algorithm global performances. In addition, we designed a second tree search algorithm fully based on the pseudo-dominance rule and dedicated to some of the challenge instances with strong precedence constraints. On these instances, it finds the best-known solutions very quickly.

1. Introduction

In sub-fields of Artificial Intelligence (AI) such as automated planning and scheduling communities (we use in this article the term “AI planning” to refer to these communities), resolution methods often involve exploring a search tree. These methods usually perform an advanced greedy procedure. They explore the search tree starting with the parts that are evaluated *a priori* most promising and continue the exploration (so long as computational time is provided). Such methods are called *anytime tree search algorithms* because they can be stopped at any time and provide good solutions relatively to the allowed computation time. Thus, they share the same purpose as classical meta-heuristics, but are less common in Operations Research. Still, one called Beam Search has shown a relative popularity in the *Cutting & Packing* literature (Akeb & Hifi, 2013; Baldi, Crainic, Perboli, & Tadei, 2014; Bennell, Cabo, & Martínez-Sykora, 2018). They usually perform little inference within each node, and thus can open millions of nodes per second.

On the other hand, branch & bound algorithms are ubiquitous in Operations Research. Such methods are usually designed to prove optimality, thus relying on strong bound computations (such as Lagrangian relaxations) and advanced pruning rules (dominances, symmetries, etc.) to reduce the size of the search tree as much as possible. Even though these components usually improve the quality of the solutions found by branch & bound heuristics executed within the nodes, branch & bound algorithms usually remain non-competitive on large instances for which the tree search is unable to terminate compared to meta-heuristics (Gardi, Benoist, Darlay, Estellon, & Megel, 2014).

The constructive nature of both anytime tree search algorithms and branch & bounds suggests that it could be possible to incorporate their respective advantages in a common approach. Indeed, some branch & bound components may be (relatively) inexpensive to compute while still greatly reducing the search space. Using them within an anytime tree search algorithm would allow getting the best of both methods. It would provide a constructive method that is designed to find good solutions fast while taking advantage of the search space reductions from branch & bounds.

With this in mind, we decided to develop such an algorithm for the 2018 ROADEF/EURO challenge glass cutting problem. We may note that this is unusual as almost all top-ranked methods in previous editions of the challenge mainly rely on local search or mathematical programming techniques.

We propose an anytime tree search with some simple bounds, pseudo-dominance properties, and symmetry breaking rules. We introduce some new guidance strategy that allows the algorithm to perform significantly better than if it was guided by a bound as in classical branch & bound methods.

The search strategy can be roughly described as follows. It is a restarting strategy that starts its first iteration by performing very aggressive heuristic prunings. At the second iteration, it performs less aggressive heuristic prunings, taking more time than the previous iteration, but finding better solutions. If the algorithm runs long enough, some iteration may perform no heuristic pruning,

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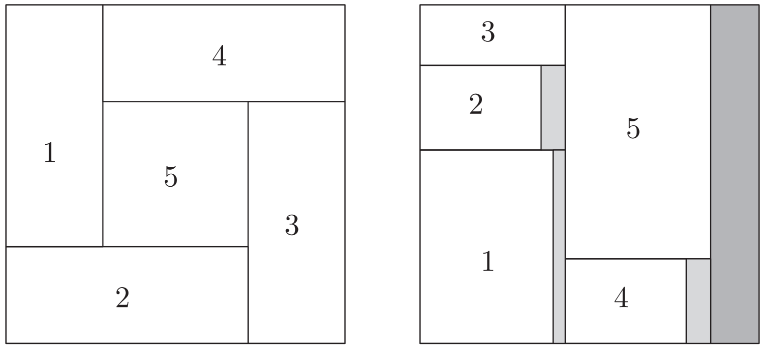


(a) Non-guillotine pattern (b) Guillotine pattern

Fig. 1. Illustration of a non-guillotine pattern (a) and a guillotine one (b).

thus the method will be able to guarantee optimality. The resulting method obtained the best results compared to the other submitted approaches during the final phase. We named it *Iterative Memory Bounded A** as it performs a series of A* (also called *best first search* and presented in Section 6) with heuristic prunings which guarantee no-more than a given amount of nodes active at the same time.

We also highlight a general methodology that can be applied to other complex problems (and with other tree search algorithms). Indeed, the method can be divided into two parts: the *Branching Scheme*, usually problem-specific, which is a definition of the implicit search tree (*i.e.* root node, how to generate children of a given node, lower bounds, dominance rules, *etc.*); and a strategy, usually generic, to explore the tree. This decomposition allows rapid prototyping of both search tree definitions and tree search algorithms as many generic parts can be reused within other algorithms. It also helps to draw insights about the contribution of each component to the resulting search algorithm.

This paper is structured as follows. In Section 2, we state the problem constraints and objective. Section 3 contains a literature review. In Section 4, we give some notations and definitions. In Section 5, we describe the branching scheme and in Section 6, the tree search algorithm we designed. Finally, in Section 7, we show the numerical results we obtained.

2. Problem description

The 2018 ROADEF/EURO challenge was dedicated to an industrial cutting problem from the French company *Saint-Gobain*. The challenge consists in packing rectangular glass items into standardized bins of dimensions $W \times H$ (6m $\times$ 3.21m).

The cutting plan needs to satisfy the following constraints:

- All items need to be produced
- Item rotation is allowed
- Cuts must be of guillotine type. Fig. 1 illustrates two examples of non-guillotine and guillotine patterns. Furthermore, the number of stages is limited to four, with only one 4-cut allowed on a sub-plate obtained after 3-cuts. This configuration is close to classical three-staged non-exact guillotine patterns, but differs in that a sub-plate obtained after 3-cuts may contain two items as illustrated in Fig. 2.
- Items are subject to chain precedence constraints. The extraction order is as follows: rightmost first level sub-plates first; within a first level sub-plate, bottommost second level sub-plates first; within a second level sub-plate, rightmost items first; and within a third level sub-plate, bottommost item first. Fig. 3 illustrates the precedence constraints and the extraction order. Most instances have a dozen chains, three instances have 2 chains and five instances are not subject to precedence constraints.
- Bins contain defects (between 0 and 8 rectangles about a few centimeters high and wide). Items must be defect-free and it is forbidden to cut through a defect. Even if the bins have the same dimensions, the presence of defects makes the set of bins heterogenous. It is important to note that bins must be used in the order they are given.
- Depending on their level, sub-plates are subject to minimum and maximum size constraints. The width of first level sub-plates must lie between $w^1_{\min} = 100$ and $w^1_{\max} = 3500$, except for wastes. The height of second-level sub-plates must be at least $w^2_{\min} = 100$, except for wastes. Finally, the width and the height of any waste area must be at least $w_{\min} = 20$. This last constraint has an unusual consequence as illustrated in Fig. 4.

The objective is to minimize the total waste area. It differs from classical Bin Packing Problems in that the remaining part of the last bin is not counted as waste. This objective is known in the packing literature as Bin Packing with Leftovers. It can be formulated as:

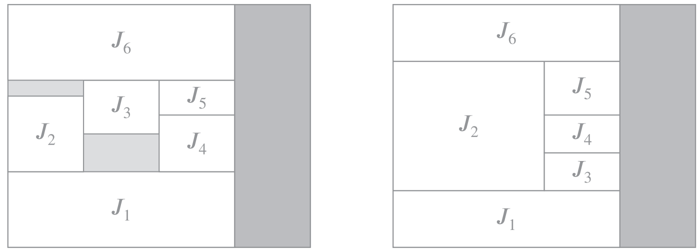
$$\min \quad nHW - Hw - \sum_{i \in \mathcal{I}} w_i h_i$$

where n is the number of bins used; W and H are respectively the standardized width and height of the bins; w is the position of the last 1-cut; $\mathcal{I}$ is the set of produced items; and w_i and h_i are respectively the width and the height of item $i \in \mathcal{I}$.

3. Literature review

Guillotine cutting and packing have been largely considered in the literature and a large variety of variants have been considered during the last decades.

The objective considered here is a combination of bin packing and strip packing objectives, known as Bin Packing with Leftovers. Only Clautiaux, Sadykov, Vanderbeck, and Viaud (2019) considered it before. Note that another objective with a similar name known as Bin Packing with Usable Leftovers also exists and has been for example studied by Andrade, Birgin, and Morabito (2016), but it is different: it first minimizes the number of bins and then the number of waste areas with a minimum size. The most efficient methods from the literature to solve bin packing problems are all based on column generation (Clautiaux et al., 2019; Cui, Yao, & Zhang, 2018; Pisinger & Sigurd, 2005; 2007; Puchinger & Raidl, 2007). For the problem considered in this paper, these methods seem difficult to apply because of the many remaining difficult constraints in the pricing problem but also and more importantly because of the precedence constraints. For strip packing problems, excluding



(a) (b)

Fig. 2. Only one 4-cut is allowed. Therefore, pattern (a) is feasible but pattern (b) is not.

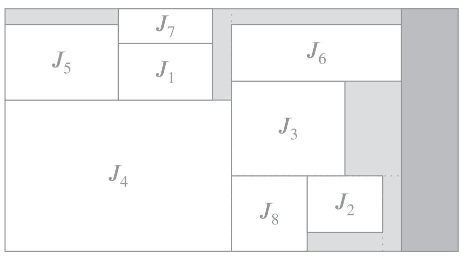


Fig. 3. An instance with 8 items included in 3 precedence chains: $J_1 \rightarrow J_2 \rightarrow J_3$, $J_4 \rightarrow J_5 \rightarrow J_6$ and $J_7 \rightarrow J_8$, meaning that J_1 must be extracted before J_2, J_2 before J_3... but no extraction order is imposed between J_1 and J_4 or J_5. In this solution, the extraction order is $J_4 \rightarrow J_5 \rightarrow J_1 \rightarrow J_7 \rightarrow J_8 \rightarrow J_2 \rightarrow J_3 \rightarrow J_6$, and therefore, the precedence constraints are satisfied..

Fig. 4. Optimal solution of the case containing the following three items with the chain precedence constraint $J_1 \rightarrow J_2 \rightarrow J_3$. Additional waste must be added before the first 1-cut. Otherwise either the waste area to the right of J_1 or the waste area to the right of J_2 would violate the minimum waste constraint.

the two-staged non-oriented case, the most recent proposed approaches are a sequential grouping and value correction procedure proposed by Cui, Yang, and Chen (2013) and a block-based layer building approach proposed by Wei, Tian, Zhu, and Lim (2014).

Regarding the constraints, restricted stage number is a common setting in the literature on guillotine cutting problems. For instance, three-staged exact (Puchinger & Raidl, 2007), three-staged non-exact (Cui, Cui, Tang, & Hu, 2015), four-staged (Clautiaux, Sadykov, Vanderbeck, & Viaud, 2018; 2019). Here, the required patterns are more restricted than four-staged patterns but more permissive than three-staged non-exact pattern since a third-level sub-plate can contain two items. We did not find such patterns in the literature.

Handling defects has always been a struggle when solving packing problems. In the literature on cutting problems with defects, only the knapsack objective with only one bin has been actually considered. Hahn (1968) and Scheithauer and Terno (1988) proposed an approach based on dynamic programming. Beasley (1985) proposed an integer programming formulation with its lagrangian relaxation used in a tree search procedure and later Beasley (2004) used it within a genetic algorithm. Martin, Hokama, Morabito, and Munari (2020) proposed a Bender's decomposition approach. Finally, for the unbounded knapsack variant, Afsharian, Niknejad, and Wäscher (2014) proposed a dynamic programming approach and Gonçalves and Wäscher (2020) adapted the maximal space placement scheme for non-guillotine case.

Precedence constraints have not been considered in the packing literature yet. A related constraint found in the non-guillotine packing literature is the unloading constraint (da Silveira, Miyazawa, & Xavier, 2013; da Silveira, Xavier, & Miyazawa, 2011a; 2011b; 2014; Wei, Wang, Cheng, & Huang, 2019). Another related constraint is the presence of due dates for items (Bennell, Soon Lee, & Potts, 2013).

The maximum distance between first-level cuts constraint has been recently considered by Long, Zheng, Gao, Pardalos, and Hu (2020). However, we did not find any article related to neither the minimum distance between first and second-level cuts constraints nor the minimum waste constraint.

Finally, we note that because of the defects, the bins are heterogenous. Heterogenous bins are an element of variable sized bin packing problems where the objective is minimize the area of all bin selected to pack all the items (Hong, Zhang, Lau, Zeng, & Si, 2014; Wei, Oon, Zhu, & Lim, 2013). The main difference here is that the bin order is imposed.

The conclusion of the literature review is that most proposed approaches seem difficult to adapt for the problem; which is actually the reason why it was proposed as a challenge.

4. Definitions and notations

We use the following vocabulary: a k-cut is a cut performed in the kth stage. Cuts separate bins or sub-plates in kth level sub-plates. For example, 1-cuts separate the bin in several first level sub-plates. S denotes a solution or a node in the search tree.

We call the last first level sub-plate, the rightmost one containing an item; the last second level sub-plate, the topmost one containing an item in the last first level sub-plate; and the last third level sub-plate the rightmost one containing an item in the last second level sub-plate. $x_1^{\text{prev}}(S)$ and $x_1^{\text{curr}}(S)$ are the left and right coordinates of the last first level sub-plate; $y_2^{\text{prev}}(S)$ and $y_2^{\text{curr}}(S)$ are the bottom and top coordinates of the last second level sub-plate; and $x_3^{\text{prev}}(S)$ and $x_3^{\text{curr}}(S)$ are the left and right coordinates of the last third level sub-plate. Fig. 5 presents a usage example of these definitions. We define the area and the waste of a solution S as follows:

To compute $\text{area}(S)$ we distinguish two cases

- if S contains all items:

$$\text{area}(S) = x_1^{\text{curr}}(S)h$$

- and otherwise:

$$\text{area}(S) = \begin{aligned} & A + x_1^{\text{prev}}(S)h \\ & +(x_1^{\text{curr}}(S) - x_1^{\text{prev}}(S))y_2^{\text{prev}}(S) \\ & +(x_3^{\text{curr}}(S) - x_1^{\text{prev}}(S))(y_2^{\text{curr}}(S) - y_2^{\text{prev}}(S)) \end{aligned}$$

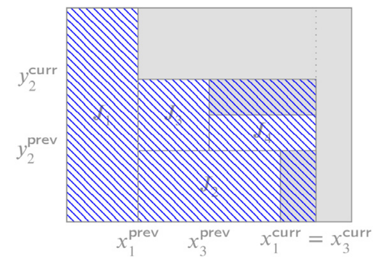


Fig. 5. Last bin of a solution which does not contain all items. The area is the whole hatched part and the waste in the grey hatched part.

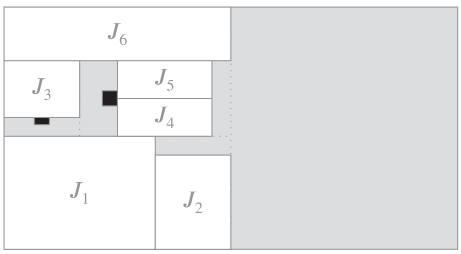


Fig. 6. Illustration of third level sub-plate possible configurations. Black rectangles are defects.

We compute the waste of a partial solution as follows:

$$\text{waste}(S) = \text{area}(S) - \text{item\_area}(S)$$

with A the sum of the areas of all but the last bin, h the height of the last bin and item_area(S) the sum of the area of the items of S. Area and waste are illustrated in Fig. 5.

5. Branching scheme

5.1. General scheme

Two kinds of packing strategies are used in the packing literature: item-based and block-based. In item based strategies, only one item is inserted at each step, whereas in block-based strategies, multiple items are inserted. Although several researchers highlighted the benefits of block-based approaches (Bortfeldt & Jungmann, 2012; Lodi, Monaci, & Pietrobuoni, 2017; Wei et al., 2014), we chose an item-based strategy. Two reasons support this choice. First, the problem has more constraints than classical packing problems from the literature. Thus, generating feasible solutions is already challenging and block-based approaches add even more complexity. Second, the benefits of the block-based approaches might be compensated by allowing more time to the tree search algorithm (in our case *Iterative Memory Bounded A**).

However, our strategy is not purely item-based: instead of packing one item at each step, we pack the next third level sub-plate. This comes from the observation that because only one 4-cut is allowed in a third level sub-plate, a third level sub-plate has only five possible configurations; it may contain

1. exactly one item, without waste
2. exactly one item with some waste above
3. exactly one item with some waste below
4. exactly two items, without waste
5. no item, only waste

These configurations are illustrated in Fig. 6. The sub-plates containing J_1 and J_2 respectively follow configurations 1 and 2. These are the "standard" configurations. Placing an item on top of the sub-plate as in configuration 3 may be necessary if it would contain a defect otherwise as J_3. Similarly, inserting only waste (configuration 5) may also be necessary if the region contains a defect as the sub-plate containing the second defect. We do not allow directly inserting only waste in a region containing no defects. Such sub-plate may appear in a solution, as the third-level sub-plate to the right of J_4 and J_5, but it is implicitly generated when J_6 is inserted. Finally, the sub-plate containing items J_4 and J_5 corresponds to configuration 4.

Third level sub-plates are inserted in the order they are extracted. In Fig. 6, this follows the numbering of the items. This ensures to never violate the precedence constraints. All items are candidates if their insertion does not lead to a precedence constraint violation.

Then, a third level sub-plate can be inserted at several depths

- depth 0: in a new bin
- depth 1: in a new first level sub-plate to the right of the current one
- depth 2: in a new second-level sub-plate above the current one
- depth 3: in the current second-level sub-plate already, to the right of the last third-level sub-plate

To reduce the size of the tree, we apply some simple pruning rules:

- if a third-level sub-plate can be inserted in the current bin, we do not consider insertions in a new bin; and if a third level sub-plate can be inserted in the current first (resp. second) level sub-plate without increasing the position of its left 1-cut (resp. top 2-cut), we do not consider insertions in a new first (resp. second) level sub-plate;
- If the last insertion is an empty sub-plate at depth d, then the next insertion must also happen at depth d;
- If the last insertion is a 2-item insertion at depth $d \neq 3$, then the next insertion must be at depth 3.

With this branching scheme, item rotation and minimum and maximum distances between cuts constraints are easy to take into account.

5.2. Pseudo-dominance rule

In this section, we describe a more sophisticated heuristic dominance rule. For a (partial) solution, we define its "front" as the polygonal chain

$$((x_1^{\text{curr}}, 0), (x_1^{\text{curr}}, y_2^{\text{prev}}), (x_3^{\text{curr}}, y_2^{\text{prev}}),$$
$$(x_3^{\text{curr}}, y_2^{\text{curr}}), (x_1^{\text{curr}}, y_2^{\text{curr}}), (x_1^{\text{curr}}, h))$$

Fig. 7 shows two examples of solution fronts.

Then we say that solution S_1 dominates solution S_2 iff they contain the same items and the front of S_1 is "before" the front of S_2, *i.e.* for each point (x_1, y) for the front of S_1, there exists a point (x_2, y) with $x_2 \geq x_1$ in the front of S_2 (see Fig. 8).

If the number of possible subsets of items is small, then for a given subset, we can memorize the best front currently seen during the search and prune any new dominated node encountered. This situation occurs in instances with strong precedence constraints (*i.e.* two chains) and this is the strategy of the Dynamic Programming A* (DPA*) algorithm presented afterward. However, for most instances, the number of possible subsets is too large and we only use the pseudo-dominance rule among the children of a node. To compensate, an additional symmetry breaking strategy is introduced.

5.3. Symmetry breaking strategy

We designed the following symmetry breaking strategy: if they do not contain defects and can be exchanged without violating the

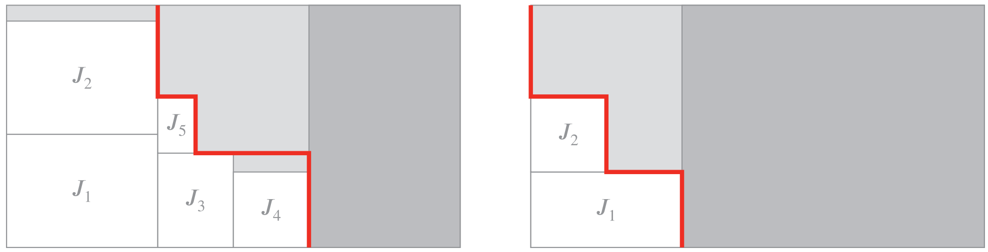


Fig. 7. Illustration of the front of two partial solutions.

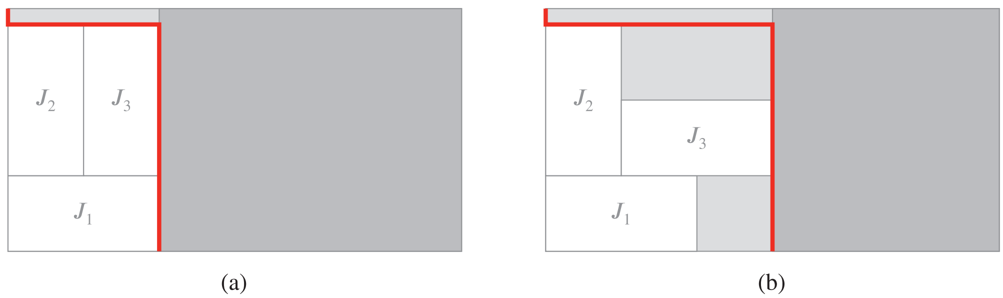


Fig. 8. Solution (a) dominates solution (b).

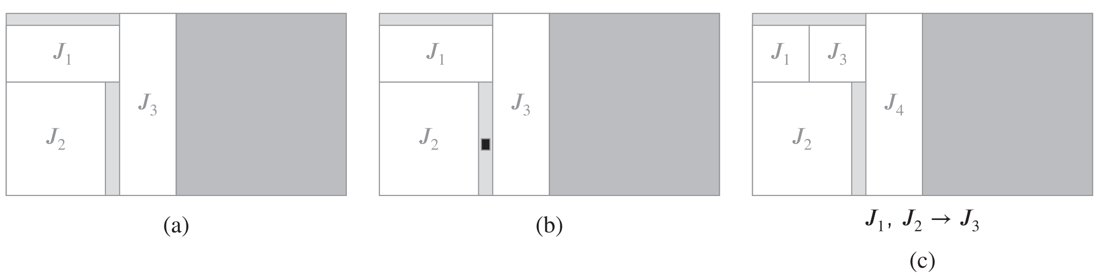


Fig. 9. Illustration of the symmetry breaking strategy: pattern (a) is forbidden because the second-level sub-plates containing J_1 and J_2 can be exchanged without a feasibility issue. However, pattern (b) is allowed because of the defect and pattern (c) is also allowed because if the second-level sub-plates are exchanged, then the precedence constraint between J_2 and J_3 is violated.

precedence constraints, a k-level sub-plate is forbidden to contain an item with a smaller index than the previous k level sub-plate of the same $(k-1)$-level sub-plate.

Preliminary experiments showed that applying the strategy for $k=2$ and $k=3$ yield the best results. The symmetry breaking strategy is illustrated in Fig. 9.

It should be noted that the branching scheme is not dominant, *i.e.* for some instances, it may not contain an optimal solution. Likewise, the pseudo-dominance rule considers that solution S_1 dominates solution S_2 whereas no optimal solution can be reached from S_1 but one can be from S_2. More details about this are given by Fontan (2019).

6. Tree search

During our initial work on the challenge, we first explored the classical *"Operations Research"* optimization algorithms (local-search, evolutionary algorithms and branch and bounds). However, it seemed difficult for us to find efficient local-search or evolutionary moves, while it felt relatively natural to design constructive methods. We implemented several classical constructive algorithms: a greedy algorithm quickly providing solutions but with limited quality; a Best First (A*) algorithm returning the "optimal" one (relatively to the branching scheme) on small instances; and a Depth First struggling to improve the greedy solution.

In the AI/planning communities, many tree search algorithms were designed to find good solutions quickly on large instances (called anytime algorithms). These algorithms are usually variations of Depth First Search (DFS), Breadth First Search (BrFS) or A* (also known as *best first search* in operations research) algorithms and aim to improve their behavior on large instances. We remind the pseudo-code of the A* algorithm in Algorithm 1 . At each iteration, the "best" node is extracted from the opened nodes set (called "open") and its children are added to the opened nodes set. For instance, Beam Search "improves" BrFS by restricting the number of nodes opened at each level so the search focuses only on the *a priori* more interesting parts of the tree. Limited Discrepancy

Algorithm 1: A* (best bound first).

open $\leftarrow$ {root};
while open $\neq \emptyset$ *and* time $<$ time limit **do**
  $n \leftarrow$ *extractBest*(open);
  open $\leftarrow$ open $\setminus \{n\}$;
  forall the $v \in$ *neighbours*(n) **do**
    open $\leftarrow$ open $\cup \{v\}$;
  end
end

Search “improves” DFS by allowing it to recover from bad decisions taken close to the root. In this section, we present a (simple) way to alter the behavior of the A* to find good solutions fast. Other variations of A* have been proposed, for instance, SMA* (Russell, 1992) and ANA* (Van Den Berg, Shah, Huang, & Goldberg, 2011). Some other algorithms combine and make anytime two other search algorithms. For instance Beam Stack Search (Zhou & Hansen, 2005) combines Beam Search and DFS.

As written above, our implementation of A* was able to find the optimal solutions on very small instances but was quickly running out of memory on larger ones because of the size of the opened nodes set. Therefore, we decided to heuristically prune nodes to bound the required memory. This “heuristic” algorithm performed beyond expectations and provided excellent solutions. However, it depended on the amount of memory allowed for the opened nodes set. If this parameter is too small, the search ends quickly and does not benefit from the remaining available time. If too big, the search takes more time and does not provide any solution within the time limit. To get rid of this parameter, we chose to use a restart strategy where we geometrically increase the allowed memory at each restart. The new parameter to calibrate becomes the growth factor, but we found that any value between 1.25 and 3 provided similar results. This simple approach provided good solutions.

Motivated by this simple but yet efficient algorithm, we investigated other anytime tree search algorithms such as beam search (OW & MORTON, 1988) and beam stack search (Zhou & Hansen, 2005). We implemented and compared them on the challenge problem. To our surprise, they did not perform as well as the previously described approach. To the best of our knowledge, this approach has not been used in the Operations Research literature before. In the next subsections, we present the algorithm we designed for the challenge. Section 6.1 presents the search strategy (iterative memory bounded A*), Section 6.2 presents the guidance strategy we used. In this subsection, we show that using only the bound, as commonly done in many branch-and-bounds leads to solutions with poor quality and can be corrected by guiding using another function. Finally, in Section 7.2, we present an integration of dynamic programming principles within our branch-and-bound to solve optimally – with respect to the branching scheme - constrained instances (two stacks or less).

6.1. Iterative Memory Bounded A* (IMBA*)

A* minimizes the cost estimate on nodes it opens (see Algorithm 1). However, it suffers from a large memory requirement since it has to store a large number of nodes in the opened nodes set. We propose a simple but yet powerful heuristic variant of A* that cuts less promising nodes if the size of the opened nodes set goes over a parameter D. We call this tree search algorithm *Iterative Memory Bounded A* (IMBA*)*. If $D = 1$, it behaves like a greedy algorithm and if $D = \infty$, it behaves like A*.

The pseudo-code of IMBA* is given in Algorithm 2 . The opened nodes set maximum size D is initialized to D_0 (line 1) and at each iteration, it grows geometrically according to a given growth factor (line 15). An iteration of IMBA* starts with a opened nodes set containing only the root node (line 3). At each iteration, the “best” node is extracted from the opened nodes set (lines 5 and 6) and its children are added to the opened nodes set (lines 7 to 9). If the opened nodes set size goes over the maximum size D, the “worst” nodes are discarded (lines 10 to 12).

We may note that there exist other “memory bounded A*” algorithms in the literature (Russell, 1992; Zhou & Hansen, 2002). These algorithms are based on different principles than the Iterative Memory Bounded A* we propose. Such algorithms usually maintain a *closed list mechanism* (*i.e.* storing all visited nodes, usually using a hash table) and back-propagate g-values as the search goes.

Algorithm 2: Iterative Memory Bounded A* (IMBA*).

```
1  D ← D_init;
2  while time < timelimit do
3      open ← {root};
4      while open ≠ ∅ and time < timelimit do
5          n ← extractBest(open);
6          open ← open \ {n};
7          forall the v ∈ neighbours(n) do
8              open ← open ∪ {v};
9          end
10         while |open| > D do
11             n ← extractWorst(open);
12             open ← open \ {n};
13         end
14     end
15     D ← D × growth factor;
16 end
```

6.2. Guide functions

Tree search methods are dependent on well-crafted guide functions (we later refer to “guide functions” as “guides” in the text) that define the meaning of “best” and “worst” nodes. Using a lower bound is common in the tree search literature. Indeed, if the objective is to prove optimality, the lower bound can be used to prune nodes dominated by the best-known solution, thus proving optimality faster. It is also usually considered as an efficient heuristic in tree search algorithms (for instance beam search) to identify the most promising nodes (as the selected nodes locally optimize the criteria to be optimized). Therefore, we first tried this approach and used the waste (*i.e. lower bound*) as a guide function. We noticed that the resulting solutions packed small items on the first plates and big items on the last ones, thus generating little waste in the beginning but a lot in the end. Globally, the solution quality was not satisfactory as illustrated in Fig. 10a. Notice that at the beginning of the solution, small items are omnipresent whereas in later plates, only large items are present, thus globally generates more waste.

Taking this into account, we designed new guides to balance the cost of inserting small items at the beginning of the solutions.

Let $\mathcal{I}(S)$ be the set of items packed in S; and $\text{area}(S)$ and $\text{waste}(S)$ defined as in Section 4. We define the following guides:

- waste percentage:

$$g_1(S) = \frac{\text{waste}(S)}{\text{area}(S)}$$

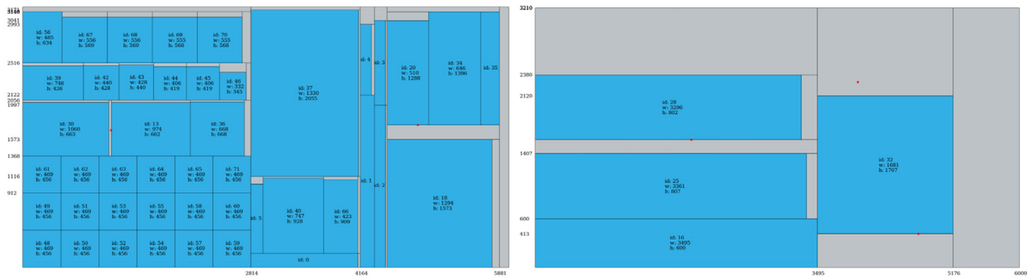
- waste percentage over mean item area:

$$g_2(S) = \frac{g_1(S)|\mathcal{I}(S)|}{\sum_{j\in\mathcal{I}(S)}(w_j h_j)}$$

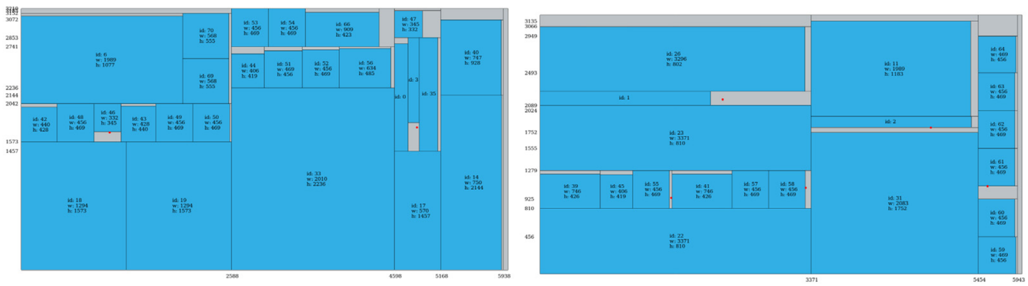
As discussed after, both guides provide better guidance than the “waste only” guide. The benefit of these guides is illustrated in Fig. 10b. We observe that small and large items are better mixed and significantly less waste is generated at the end of the solution, reducing the waste on some instances by more than 30%.

6.3. Dynamic Programming A* (DPA*): solving instances with strong precedence constraints

Three instances of the challenge contain only 2 precedence chains. If we denote by n_1 (resp. n_2) the length of the first



(a) waste only guide – first and last plate (total solution waste: 14.918.729)



(b) waste percentage regularized by the “waste percentage over mean item area” guide – first and last plate (total solution waste: 10.537.079)

Fig. 10. 10a shows a solution obtained using the waste as guide function. 10b shows the effect of the guide biased by the item average size on a solution of the same instance.

(resp. second) chain, then the number of possible subsets of items packed in a partial solution of the branching scheme becomes $n_1 n_2$. Since the number of items in an instance is less than 700 (this information was given in the challenge description), it becomes possible to store the non-dominated fronts encountered for each possible subset without overcoming the memory limitation, compare the front of each opened node with the non-dominated fronts from all the previously encountered nodes and prune the dominated ones.

Therefore, we developed a dedicated algorithm for these instances named Dynamic Programming A* (DPA*) which pseudo-code is given in Algorithm 3 . DPA* is an A* algorithm implementing the scheme described in the previous paragraph. The front of each encountered node is stored in a history and a node is discarded if its front is dominated by a front from the history. DPA* does not bound the size of the open nodes as IMBA* does, and uses the waste as a guide function. Therefore, if it terminates, it returns the “optimal” solution (relatively to the branching scheme and the pseudo-dominance rule).

Algorithm 3: DPA*.

```
open nodes ← {root};
while opennodes ≠ ∅ and time < timelimit do
    n ← extractBest(opennodes);
    opennodes ← opennodes \ {n};
    forall the v ∈ neighbours(n) do
        if the front of v is dominated by a front from the history
        then
            continue
        end
        add the front of v to the history;
        opennodes ← opennodes ∪ {v};
    end
end
```

Table 1
Comparison over all possible algorithms using the proposed algorithmic components on the 50 instances of the competition (datasets A,B and X). “# best” indicate the number of times the algorithm was able to find the best solution on given instances compared to the 11 other algorithms. “# only best” indicates the number of times the algorithm was the only one to find the best solution.

Combination	# best	# only best
BS+no-sym+w	2	0
BS+no-sym+p	2	0
BS+no-sym+a	2	0
BS+sym+w	5	1
BS+sym+p	9	4
BS+sym+a	9	4
IMBA*+no-sym+w	2	0
IMBA*+no-sym+p	3	0
IMBA*+no-sym+a	4	0
IMBA*+sym+w	3	0
IMBA*+sym+p	23	17
IMBA*+sym+a	22	18

We may note that for a given subset of items, there could be an exponential number of non-dominated fronts, which could degrade DPA* performances. This is at least not an issue for the concerned instances from the challenge.

6.4. Global algorithm

For the competition, we distinguished the case where the instance has two chains or less. In this case, we run DPA*.

If it has strictly more than two chains, we do not use DPA* since it would overcome the memory limitation. Since the processor used to evaluate the participant submissions had 4 physical cores, we run 4 threads, each one running a restarting IMBA* with a given growth factor and a given guide function. Each IMBA* is initially executed with a open nodes maximal size of 2, and each time one terminates, it is restarted with a maximal open nodes size multiplied by its growth factor. If the growth factor is 2, the maximal size doubles at each iteration. All the threads share the information of the best solution found. If one finds a better solution, the others take advantage of it to perform more cuts and globally perform better together than alone. The threads run the following algorithms:

- IMBA*, waste percentage guide, growth factor 1.33
- IMBA*, waste percentage guide, growth factor 1.5
- IMBA*, waste percentage / average size guide, growth factor 1.33
- IMBA*, waste percentage / average size guide, growth factor 1.5

Table 2
DPA* vs IMBA*.

Instance	IMBA* (4 threads, 3600s)	DPA*
B5	88 590 815	72 155 615 (2.06s)
X8	24 875 331	22 265 601 (59.12s)

7. Numerical results

In this section, we first evaluate the contribution of the components we described in the previous sections in the main algorithm. Then, we show the benefits of using DPA* on instances with only two precedence chains. Finally, we provide computational results

Table 3
Analysis of the contribution of each introduced algorithmic component.

Instance	best IBS	best IMBA*	best no sym	best with sym	best w	best p	best a
A1	**425 486**	**425 486**	**425 486**	**425 486**	**425 486**	**425 486**	**425 486**
A2	10 514 609	**9 676 799**	10 537 079	**9 676 799**	10 659 059	10 418 309	**9 676 799**
A3	**2 651 880**	**2 651 880**	3 441 540	**2 651 880**	3 056 340	**2 651 880**	**2 651 880**
A4	3 242 520	**3 220 050**	3 306 720	**3 220 050**	3 505 740	**3 220 050**	3 306 720
A5	**3 033 273**	3 566 133	4 856 553	**3 033 273**	3 736 263	**3 033 273**	3 566 133
A6	**3 225 930**	3 572 610	3 652 860	**3 225 930**	3 800 520	**3 225 930**	3 460 260
A7	5 063 280	**4 800 060**	5 194 890	**4 800 060**	5 933 190	**4 800 060**	4 938 090
A8	**9 187 874**	10 077 044	12 218 114	**9 187 874**	12 568 004	10 077 044	**9 187 874**
A9	**2 930 706**	2 985 276	3 550 236	**2 930 706**	3 929 016	**2 930 706**	2 985 276
A10	4 097 221	**4 084 381**	5 272 081	**4 084 381**	4 797 001	**4 084 381**	4 122 901
A11	**4 718 449**	4 978 459	6 076 279	**4 718 449**	6 711 859	5 251 309	**4 718 449**
A12	**2 050 084**	2 245 894	2 342 194	**2 050 084**	**2 050 084**	2 104 654	2 194 534
A13	15 096 453	**12 133 623**	14 865 333	**12 133 623**	15 099 663	12 197 823	**12 133 623**
A14	14 363 778	**12 097 518**	14 793 918	**12 097 518**	14 363 778	**12 097 518**	13 490 658
A15	15 277 961	**13 185 041**	15 168 821	**13 185 041**	16 029 101	**13 185 041**	15 014 741
A16	**3 380 333**	**3 380 333**	**3 380 333**	**3 380 333**	**3 380 333**	**3 380 333**	**3 380 333**
A17	**3 617 251**	**3 617 251**	**3 617 251**	**3 617 251**	**3 617 251**	**3 617 251**	**3 617 251**
A18	5 898 468	**5 535 738**	5 763 648	**5 535 738**	7 737 798	5 596 728	**5 535 738**
A19	**3 323 744**	3 654 374	4 187 234	**3 323 744**	4 620 584	**3 323 744**	3 965 744
A20	**1 467 925**	**1 467 925**	1 493 605	**1 467 925**	**1 467 925**	**1 467 925**	**1 467 925**
B1	4 173 228	**3 633 948**	4 150 758	**3 633 948**	4 012 728	4 324 098	**3 633 948**
B2	15 715 685	**15 359 375**	18 466 655	**15 359 375**	20 155 115	**15 359 375**	15 715 685
B3	32 668 193	**21 253 433**	23 365 613	**21 253 433**	41 315 933	24 890 363	**21 253 433**
B4	8 885 365	**8 862 895**	11 238 295	**8 862 895**	9 222 415	**8 862 895**	8 920 675
B5	92 433 185	**88 590 815**	**88 590 815**	**88 590 815**	103 719 545	**88 590 815**	**88 590 815**
B6	**13 371 637**	13 480 777	15 653 947	**13 371 637**	17 509 327	14 113 147	**13 371 637**
B7	14 576 799	**11 434 209**	12 801 669	**11 434 209**	14 319 999	**11 434 209**	12 801 669
B8	24 490 999	**19 512 289**	24 121 849	**19 512 289**	24 490 999	**19 512 289**	20 048 359
B9	20 511 607	**20 046 157**	25 085 857	**20 046 157**	46 721 257	39 071 827	**20 046 157**
B10	28 012 013	**27 344 333**	29 225 393	**27 344 333**	35 815 523	31 055 093	**27 344 333**
B11	38 143 250	**29 113 520**	34 175 690	**29 113 520**	41 523 380	32 589 950	**29 113 520**
B12	18 122 077	**16 086 937**	19 929 307	**16 086 937**	18 122 077	**16 086 937**	16 314 847
B13	31 138 545	**29 674 785**	33 716 175	**29 674 785**	31 138 545	32 213 895	**29 674 785**
B14	10 482 820	**10 043 050**	11 619 160	**10 043 050**	12 046 090	10 434 670	**10 043 050**
B15	41 533 241	**28 372 241**	34 143 821	**28 372 241**	41 533 241	**28 372 241**	31 466 681
X1	21 022 877	**17 299 277**	17 970 167	**17 299 277**	29 911 367	17 803 247	**17 299 277**
X2	11 459 837	**8 583 677**	8 923 937	**8 583 677**	9 318 767	9 206 417	**8 583 677**
X3	9 424 756	**8 712 136**	9 842 056	**8 712 136**	9 578 836	**8 712 136**	8 927 206
X4	19 035 422	**15 976 292**	19 305 062	**15 976 292**	19 035 422	**15 976 292**	16 772 372
X5	**5 383 037**	5 620 577	7 029 767	**5 383 037**	6 728 027	5 623 787	**5 383 037**
X6	14 443 523	**12 167 633**	14 488 463	**12 167 633**	14 443 523	13 024 703	**12 167 633**
X7	30 327 120	**26 170 170**	27 146 010	**26 170 170**	31 328 640	29 161 890	**26 170 170**
X8	27 693 711	**27 109 491**	27 693 711	**27 109 491**	27 693 711	27 494 691	**27 109 491**
X9	33 431 655	**23 599 425**	33 919 575	**23 599 425**	33 370 665	**23 599 425**	26 716 335
X10	23 400 722	**19 901 822**	23 522 702	**19 901 822**	23 673 572	23 975 312	**19 901 822**
X11	14 349 972	**14 247 252**	16 102 632	**14 247 252**	14 349 972	**14 247 252**	14 921 352
X12	14 775 805	**12 422 875**	14 576 785	**12 422 875**	14 775 805	**12 422 875**	12 589 795
X13	19 208 322	**14 624 442**	18 900 162	**14 624 442**	20 007 612	16 271 172	**14 624 442**
X14	11 075 552	**9 730 562**	11 916 572	**9 730 562**	11 004 932	**9 730 562**	10 128 602
X15	16 301 394	**13 540 794**	17 261 184	**13 540 794**	16 461 894	13 990 194	**13 540 794**
total waste	779 159 574	679 871 064	779 027 964	676 914 654	870 817 914	725 241 204	693 016 014
nb best	14	41	4	50	5	27	28
nb only best	9	36	0	46	1	21	22

Table 4
Computational experiments comparing the proposed approach compared to other contestants.

Instance	Comments	Final phase best 180s	IMBA*/DPA* 180s	Final phase best 3600s	IMBA*/DPA* 3600s	Best known
A1	Trivial	–	425 486	–	425 486	425 486
A2	No prec	–	9 506 669	–	4 383 509	4 383 509
A3		–	2 651 880	–	2 651 880	2 651 880
A4		–	3 024 240	–	2 924 730	2 924 730
A5		–	2 924 730	–	3 283 653	3 017 223
A6		–	3 389 640	–	3 225 930	3 188 646
A7		–	4 703 760	–	4 334 610	3 920 520
A8		–	9 691 844	–	8 378 954	8 378 954
A9		–	2 664 276	–	2 664 276	2 664 276
A10		–	4 084 381	–	4 084 381	4 084 381
A11		–	4 660 669	–	4 622 149	4 358 929
A12		–	2 056 504	–	1 879 954	1 879 954
A13		–	10 226 883	–	9 440 433	9 331 293
A14		–	11 686 638	–	10 383 378	10 383 378
A15		–	12 918 611	–	11 108 171	10 828 901
A16	Trivial	–	3 380 333	–	3 380 333	3 380 333
A17	2 chains	–	3 617 251	–	3 617 251	3 617 251
A18		–	5 596 728	–	4 983 618	4 983 618
A19		–	3 654 374	–	3 323 744	3 323 744
A20	Trivial	–	1 467 925	–	1 467 925	1 467 925
B1	No prec	3 232 698	3 765 558	***2 661 318**	3 136 398	2 661 318
B2		*15 635 435	14 312 915	*13 674 125	13 398 065	11 931 095
B3		20 540 813	19 786 463	18 191 093	17 093 273	15 786 803
B4		*8 269 045	8 323 615	*8 269 045	7 973 725	7 315 675
B5	2 chains	72 155 615	72 155 615	**72 155 615**	**72 155 615**	72 155 615
B6		*12 116 527	12 488 887	*11 195 257	11 089 327	10 800 427
B7	No prec	9 601 299	9 177 579	*8 355 819	7 678 509	6 628 839
B8		*17 865 559	17 152 939	16 067 959	15 840 049	14 398 759
B9		18 502 147	19 969 117	17 484 577	17 474 947	16 495 897
B10		26 012 183	26 904 563	***21 951 533**	23 065 403	21 951 533
B11		25 251 890	27 312 710	22 584 380	23 820 230	20 626 280
B12		*15 868 657	13 734 007	*13 958 707	13 120 897	12 514 207
B13		*28 349 055	27 360 375	*24 471 375	23 078 235	22 657 725
B14		*9 346 480	9 442 780	*8 656 330	8 377 060	8 023 960
B15		*27 794 441	24 568 391	*24 517 031	23 088 581	22 619 921
X1		*15 508 097	15 302 657	*14 127 797	14 127 797	13 720 127
X2	No prec	6 034 937	6 083 087	*5 434 667	4 879 337	4 795 877
X3		*8 285 206	7 649 626	*7 473 076	7 180 966	6 837 496
X4		12 182 072	15 488 372	**11 405 252**	13 366 562	11 405 252
X5		5 081 297	4 988 207	4 712 147	4 715 357	4 522 757
X6		12 565 673	11 031 293	*10 363 613	9 496 913	9 365 303
X7		*22 443 360	22 876 710	21 127 260	21 191 460	20 568 720
X8	2 chains	*24 788 661	22 265 601	*24 788 661	**22 265 601**	22 265 601
X9		*22 251 225	22 312 215	20 167 935	20 479 305	20 039 535
X10		*20 110 472	18 778 322	*17 824 952	17 186 162	16 865 162
X11		*13 489 692	12 802 752	*12 417 552	11 676 042	11 011 572
X12		*11 963 845	12 358 675	*10 583 545	10 503 295	10 246 495
X13		15 950 172	14 345 172	*13 533 042	13 125 372	12 130 272
X14		*8 889 542	8 591 012	*8 013 212	7 644 062	7 422 572
X15		13 990 194	13 710 924	11 682 204	11 682 204	10 882 914

with the challenge setting. Instances generally have between 300 and 600 items and 10 to 15 chains. They are available online[2].

7.1. Contribution of the components

Computational experiments have been performed on a personal computer with an Intel(R) Core(TM) i5-3470 CPU @ 3.20GHz with 16GB RAM.

We consider the 12 possible combinations of the components we designed, namely IMBA* to be compared with an Iterative Beam Search Zhang (1998); with or without the symmetry breaking strategy; and with waste (w), waste percentage (p), or waste percentage/average size (a) guide. We run each pair of instance-algorithm for 100 seconds.

In Table 1, we show a summary of the performances of each variant. The goal is to help us select the best combinations to use. IMBA* with the symmetry breaking strategy and guided by the waste percentage (*IMBA*+sym+p*) and IMBA* with the symmetry breaking strategy and guided by the waste percentage / average size (*IMBA*+sym+a*) clearly outperform all other combinations. That is why these are the two combinations that we use. Since the processor used to evaluate participant submissions has 4 physical cores, we dedicate two threads for each combination with different growth factor (1.33 and 1.5) to add some robustness.

Table 3 presents an analysis of the contribution of each component individually. Each column corresponds to the best result per instance obtained by a subset of algorithms that uses a given component. For instance *best IBS* corresponds to a subset of algorithms using Iterative Beam Search, thus excluding IMBA* (6 algorithms). IMBA* variants outperform the Iterative Beam Search variants (producing 12% less waste). It finds 41/50 best solutions and 36 best solutions that the Beam Search variants were not able to obtain. The algorithms using the symmetry breaking strategy clearly produce better results than the one without (13% less waste and 46

[2] https://www.roadef.org/challenge/2018/en/instances.php.

best solutions not attainable by the variants without the symmetry breaking strategy) showing that integrating state-space reductions can greatly benefit to anytime tree search algorithms and probably even to constructive meta-heuristics. Finally, as expected, the waste (lower bound) guide provides the worst results among the 3 considered guides (16% more waste than the waste percentage guide and 20% more waste than the waste percentage / average size guide and only 5 best solutions on 50 instances). However, the waste percentage guide and the waste percentage / average size guide provided similar results (with a slight advantage on the latest as it produces 5% less waste and finds one best solution more). These results are interesting as they show that both guides are complementary. Indeed, they produce 21 (resp. 22) best solutions where they are the only one to obtain them. Thus it is worth using both.

7.2. DPA*

Table 2 shows the benefits of using DPA* on instances with only two precedence chains. There is one such instance in each dataset, but the one in dataset A is trivial to solve, therefore we only consider instances B5 and X8. Unlike IMBA*, DPA* only returns a solution when it finishes. Thus it requires the instance to be relatively small (*i.e. 2 stacks or less*). Otherwise, it is likely to not return anything within the 3600 seconds time limit. However, if it can find one, this solution is generally significantly better than the ones IMBA* returns.

7.3. Final results

Table 4 sums up the challenge final results. Computational experiments have been performed on a computer with an Intel Core i7-4790 CPU @ 3.60 GHz $\times$ 8 processor with 31.3 Gb of RAM. This configuration is similar to the one of the challenge. Since the challenge, a few adjustments have been made. Therefore, the results presented here slightly differ from the results obtained during the final phase. Compared to the challenge version, the current version performs better: the total waste on dataset B and X decreases from 493,600,549 for the challenge version to 469,910,749 for the current one, respectively. Columns *Final phase best 180s* and *Final phase best 3600s* contain the best solutions found during the final phase. Results annotated with a star indicate that this solution was found by our algorithm during the final phase of the challenge. The *Best known* column contains the best solution up to our knowledge. They may have been found during the development of the algorithm, with execution times exceeding 3600 seconds or by other teams. Finally, even if it is not indicated in the table, on most of the instances, if the algorithm is run longer, for example, 2 hours, the solution may still be improved.

8. Conclusion and perspectives

In this article, we presented a new anytime tree search algorithm called IMBA* for the 2018 ROADEF/EURO challenge glass cutting problem. It performs successive iterations, restarting when its heuristic search tree exploration is completed. During the first iterations, it performs aggressive prunings and behaves like a greedy algorithm. As iterations go, the algorithm performs fewer heuristic prunings, and thus gets access to better solutions (at the cost of an increase of the computation time of each iteration). If enough time and memory are available, the algorithm ends up performing an iteration with no heuristic pruning, finding the best solution regarding the branching scheme.

We proposed two guides (waste percentage, and waste percentage / average item size). These guides can find significantly better solutions than using a lower bound (the waste), which is what is usually used in branch and bounds. We also presented a symmetry breaking strategy and showed that it significantly improves the quality of the solutions returned by the algorithm.

Also, we designed another algorithm, DPA*, for instances with only two precedence chains. This algorithm returns the best-knowns on these instances within short times.

This result shows that anytime tree search algorithms from the AI planning community, and branch and bounds from the Operations Research community can benefit from each other, leading to algorithms competitive with classical meta-heuristics. We believe that the representation of anytime tree search algorithms in the Operations Research literature does not reflect the benefits of applying such methods on complex optimization problems. To the best of our knowledge, many of them remain unexplored such as Beam Stack Search (Zhou & Hansen, 2005) which aims to hybridize beam search with a depth first search algorithm, allowing beam search to continue exploring the tree, SMA* (Russell, 1992) which allows A* to only open one child at a time compared to other tree search algorithms that open all children at once, or Anytime Column Search (Cohen et al., 2018) which can be seen as an hybridization between cyclic best first search and a beam search algrithm. We refer the reader to Russel's and Novig's book (Russell & Norvig, 2002) for more details on AI/planning tree search algorithms.

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