1. New

Power series theorem

Let:

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Then, holds:

$\cdot \, Part II, III, IV$

Demonstration:

Follow 3 steps

Step 1 : Uniform convergence in compacts of $D(a,R)$:

$$\begin{array}{rcl} g_n : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & c_n(z-a)^n \end{array}$$

$\forall \ \rho \in \mathbb{R} \ \ " \ \rho < R$:

$\forall \ z \in \overline{D(a,\rho)}$:

$$|g_n(z)| = |c_n||z-a|^n \leq |c_n|\rho^n$$

$M_n \ := \ |c_n|\rho^n$

$\lim_n \left(|c_n|\rho^n\right)^{1/n} = \rho \lim_n |c_n|^{1/n} = \frac{\rho}{R} < 1$

Root test→ $\sum_{n\geq 0} M_n$ convergent

M-Weierstrass→ $\sum_{n\geq 0} c_n(z-a)^n$ uniformly convergent over compacts of $D(a,\rho)$

$$\sum_{n\geq 0} c_n(z-a)^n \text{ uniformly convergent over compacts of } D(a,r)$$

$$f(z) := \sum_{n\geq 0} g_n(z)$$

g uniformly convergent $\rightarrow f$ continuous

Step 2 : IV. R' :

$$\tilde{f}(z) := \sum_{n\geq 1} nc_n(z-a)^{n-1}$$

$$\tilde{f}(z) = \sum_{n\geq 0}(n+1)c_{n+1}(z-a)^n$$

$\frac{1}{R'} = \lim_n (n+1)|c_{n+1}|^{1/n} = \lim_n (n+1)^{1/n}|c_{n+1}|^{1/n} = \lim_n \left(|c_{n+1}|^{1/n+1}\right)^{\frac{n+1}{n}} =$

$$\frac{1}{R}$$

$$R' = R$$

Step 3 : III :

$\tilde{f}$well defined in $D(a,R)$

$\forall\ z_0 \in D(a,R)$:

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - \tilde{f}(z_0)\right| \xrightarrow{n} 0?$$

$\forall\ n \in \mathbb{N}$:

$$S_n(z) := \sum_{k=0}^{n} f_k(z)$$

$$R_n(z) := \sum_{k\geq n+1} f_k(z)$$

$$f(z) = S_n(z) + R_n(z)$$

$$\tilde{f}(z) = \tilde{S_n}(z) + \tilde{R_n}(z)$$

$$R_n(z), \tilde{R_n}(z) \xrightarrow{n} 0$$

$\forall\ \varepsilon \in \mathbb{R}^+$:

$$\left|\tfrac{f(z)-f(z_0)}{z-z_0}\right| = \left|\tfrac{S_n(z)-S_n(z_0)}{z-z_0} + \tfrac{R_n(z)-R_n(z_0)}{z-z_0} - \tilde{S_n}(z_0) - \tilde{R_n}(z_0)\right|$$

$\forall\ \rho \in \mathbb{R}$ 〃 $|z_0 - a| < \rho < R$:

$$\left|\tilde{R_n}(z_0)\right| \leq \sum_{k \geq n+1} k|c_k|\rho^{k-1} < \frac{\varepsilon}{3}(n \geq n_1)$$

$$\left|\tfrac{R_n(z)-R_n(z_0)}{z-z_0}\right| \leq \sum_{k \geq n+1} |c_k|\left|\frac{(z-a)^k - (z_0-a)^k}{z-z_0}\right|$$

$$\tfrac{a^k-b^k}{a-b} = a^{k-1} - a^{k-2}b + - - - + b^{k-1}$$

$$\leq \sum_{k \geq n+1} |c_k|(|z-a|^{k-1} + |z-a|^{k-2}|z_0-a| + - - - +$$

$|z_0 - a|^{k-1}$

$$|z-a|, |z_0| < \rho$$

$$\leq \sum_{k \geq n+1} |c_k| k\rho^{k-1} < \frac{\varepsilon}{3}(n \geq n_1)$$

$$\left|\tfrac{S_n(z)-S_n(z_0)}{z-z_0} - \tilde{S_n}(z_0)\right| < \tfrac{\varepsilon}{3}(S'_n = \tilde{S_n})(n \geq n_2)$$

$\forall\ n \in \mathbb{N}$ 〃 $n \geq \max(n_1, n_2)$:

$$\left|\tfrac{f(z)-f(z_0)}{z-z_0}\right| < \varepsilon$$

functions associated to series are $\mathcal{C}^\infty$

Let:

$\cdot\ f(z) = \sum c_n (z-a)^n n[0]$ series

$\cdot\ R$ radius of convergence of f

Then, holds:

$\cdot\ f \in \mathcal{C}^\infty$ over $D(a,R)$

$\cdot\ \forall\ n \in \mathbb{N}$:

$$f^{n)} \in \mathcal{H}(D(a,R))$$

$\cdot\ c_k = \frac{f^{k)}}{k!}$

$\cdot$ series associated to f is unique

Demonstration:

$$f^{k)}(z) = \sum n(n-1) - - - (n-k+1) c_n (z-a)^{n-k} n[k]$$

$$f^{k)}(a) = k! c_k$$

$$c_k = \frac{f^{k)}(a)}{k!}$$

Geometric series

Let:

$\cdot\, a\ :\ 0$

$\cdot\, c_n : 0$

Then, $\sum z^n n[0]$ is convergent in $\mathbb{D}$:

$R = \frac{c_n}{c_{n+1}} = 1$ Then, holds:

$\cdot\ \sum z^n n[0] = \frac{1}{1-z}$

$\cdot\ \sum n z^{n-1} n[0] = \frac{1}{(1-z)^2}$

$\cdot\ \sum \frac{z^{n+1}}{n+1} n[0] = -\log(1-z)$

Demonstration:

$\forall\ z \in \mathbb{D}$:

$\sum z^n n[0]$ geometric series

$\sum z^n n[0] = \frac{1}{1-z}$

II differentiating

III integrating

Series not centered in 0

Let:

- a : i
- c_n : $\frac{n+1}{5^{n+1}}$

Then, *item* is a/an entity :

$$\sum \frac{n(z-i)^{n-1}}{5^n} n[1]$$

$$= \frac{1}{5} \sum n \frac{z-i}{5}^{n-1} n[1] = \frac{1}{5} \sum n u^{n-1} n[1]$$

$$S(u) = \tilde{S}'(u)$$

$$\tilde{S}(u) = \frac{1}{5} \sum u^n n[1] = \frac{u}{5(1-u)}$$

$$S(u) = \frac{1}{5(1-u)^2}$$

$$S(z) = \frac{5}{(5+i-z)^2} \text{ over } D(i,5)$$

Radius of convergence without quotient test

Let:

- $\sum \frac{(-1)^n}{n(n+1)} (z-2)^{n(n+1)} n[1]$

Then, R is a/an entity :

$$\lim_{c_{n+1}} c_n \nexists$$

$$\lim_n \frac{1}{n(n+1)}^{\frac{1}{n(n+1)}} = 1$$

ignore zeros