Block I

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1. nth root determinations of a function

Relationship between nth root determinations

Let:

- X connected topological space
- $f : X \to \mathbb{C} \setminus \{0\}$ continuous
- g, h nth root determinations of f

Then, holds:

- $\exists\, \zeta \in \mu_n(\mathbb{C})$:
  - $h = \zeta g$

Demonstration:

h, g continuous , $g \neq 0 \to h/g$ continuous

$\forall\, x \in X$:

$h(x)^n = f(x),\ \ g(x)^n = f(x)$

$\left(\frac{h(x)}{g(x)}\right)^n = \frac{h(x)^n}{g(x)^n} = \frac{f(x)}{f(x)} = 1$

$\frac{h(x)}{g(x)} \in \mu_n(\mathbb{C})$

$\mathrm{Im}(h/g) = \mu_n(\mathbb{C})$ finite

h/g constant over connected components

X connected $\to h/g$ constant

$\exists\, \zeta \in \mu_n(\mathbb{C})$:

$h = \zeta g$

Cubic root determinations

Let:

- h_0, h_1, h_2 cubic root determinations over $\mathbb{C} \setminus (\mathbb{R}^- \times \{0\})$ with
- $h_0(1) = 1$
- $h_1(1) = \exp(\frac{2\pi i}{3})$
- $h_2(1) = \exp(\frac{4\pi i}{3})$

Study:

- $\mathrm{Im}(h_0)$,$\mathrm{Im}(h_1)$,$\mathrm{Im}(h_2)$
- Relationship with Log and Arg

Demonstration:

$\forall\ z \in \mathbb{C} \setminus (\mathbb{R}^- \times \{0\})$:

$\sqrt[3]{z} = \sqrt[3]{|\ z\ |}\exp(\frac{i(Arg(z)+2k\pi)}{3})$

$\forall\ k \in \dot{3}$:

$\sqrt[3]{z} = \sqrt[3]{|\ z\ |}\exp(\frac{iArg(z)}{3})$

$arg(z) = \frac{Arg(z)}{3}$

$Arg(z) \in (-\pi, \pi) \to arg(z) \in (\frac{-\pi}{3}, \frac{\pi}{3})$

$\Omega_0 := \{z \in \mathbb{C} \mid Arg(z) \in (\frac{-\pi}{3}, \frac{\pi}{3})\}$

$\forall\ k \in \dot{3} + 1$:

$\sqrt[3]{z} = \sqrt[3]{|\ z\ |}\exp(\frac{i(Arg(z)+2\pi)}{3})$

$arg(z) = \frac{Arg(z)}{3} + \frac{2\pi}{3}$

$Arg(z) \in (-\pi, \pi) \to arg(z) \in (\frac{\pi}{3}, \pi)$

$\Omega_1 := \{z \in \mathbb{C} \mid Arg(z) \in (\frac{\pi}{3}, \pi)\}$

$\forall\ k \in \dot{3} + 2$:

$\sqrt[3]{z} = \sqrt[3]{|\ z\ |}\exp(\frac{iArg(z)}{3})$

$arg(z) = \frac{Arg(z)}{3} + \frac{4\pi}{3}$

$Arg(z) \in (-\pi, \pi) \to arg(z) \in (\pi, \frac{5\pi}{3})\Omega_2 := \{z \in \mathbb{C} \mid Arg(z) \in (\pi, \frac{5\pi}{3})\}$

$h_0(1) = 1 \to \mathrm{Im}(h_0) = \Omega_0$

$h_1(1) = \exp(\frac{2\pi i}{3}) \to \mathrm{Im}(h_1) = \Omega_1$

$h_2(1) = \exp(\frac{4\pi i}{3}) \to \mathrm{Im}(h_2) = \Omega_2$

In particular:

$h_0(i) = \sqrt[3]{|\ i\ |}\exp(\frac{iArg(i)}{3}) = \exp(\frac{\pi}{6}i)$

$h_0(i) = \sqrt[3]{|\ i\ |}\exp(\frac{iArg(i)+2\pi}{3}) = \exp(\frac{5\pi}{6}i)$

$h_0(i) = \sqrt[3]{|\ i\ |}\exp(\frac{i(Arg(i)+4\pi)}{3}) = \exp(\frac{9\pi}{6}i)$