1. New

Let:

- $|\lambda| < 1 < |\mu|$

Then, holds:

- $A^n = \lambda^n, \mu^n$
- $\forall\ (x,y) \in \mathbb{R}^2$ „ $y \neq 0$:

$$o((x,y)) \xrightarrow{n} (0,\infty)$$

$\forall\ (x,y) \in \mathbb{R}^2$ „ $y = 0$:

$$o((x,y)) \xrightarrow{n} (0,0)$$

Demonstration:

$$x_n = \lambda^n x \xrightarrow{n} 0$$

$$y_n = \mu^n y \xrightarrow{n} \begin{cases} 0 & y = 0 \\ \infty & y \neq 0 \end{cases}$$

Stable & Unstable subspaces

Let:

- $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$

We name stable subspace of A to:

We name unstable subspace of A to:

We denote:

- E^s, E^u

abstract

Let:

- $A = \lambda, 1\lambda, 0$

Then, holds:

- $\mid \lambda \mid < 1 \to o(x, y) \xrightarrow{n} (0, 0)$
- $\mid \lambda \mid > 1 \to o(x, y) \xrightarrow{n} (\infty, \infty)$

Demonstration:

$$A^n = \lambda^{n-1}(\lambda, n, 0, \lambda)$$

Origin stability theorem

Let:

- $A \in \mathcal{M}_{n\times n}(\mathbb{R})$
- $$\begin{array}{rccc} f: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \longmapsto & Ax \end{array}$$
- $\sigma(A)$ eigenvalues of A

Then, holds:

- $\forall\ i \in [1,n]_{\mathbb{N}}$:

  $|\ \lambda_i\ | < 1 \quad \rightarrow$ origin is attractive

  $\forall\ i \in [1,n]_{\mathbb{N}}$:

  $|\ \lambda_i\ | > 1 \quad \rightarrow$ origin is repulsive

  $\exists\ \lambda_1, \lambda_2 \in \sigma(A)$:

  $|\ \lambda_1\ | > 1\ \wedge\ |\ \lambda_2\ | < 1 \quad \rightarrow \mathbb{R}^n = E^s + E^u$

Demonstration:

$$demonstration$$

Local stability of fixed points of non-linear applications

Let:

- $f \ : \ \mathbb{R}^n \to \mathbb{R}^n$
- $x \in \mathrm{Fix}(f)$
- $\mathcal{U} \subset \mathbb{R}^n$ open " $x \in \mathcal{U} f \ : \ \mathcal{U} \to \mathbb{R}^n$ differentiable in x

Then, holds:

- $\rho(Df(p)) < 1 \to x$ attractive
- $\forall \ \lambda \in \sigma(A) \ :$

  $| \ \lambda \ | > 1 \quad \to x$ repulsive

Demonstration:

Suppose $p = 0$

$A \ := \ Df(0)$

Consider:

$|| - || \ : \ \mathbb{R}^n \to \mathbb{R}^n$ norm " $||A|| < 1$

$\exists \ \mathcal{U} \subset \mathbb{R}^n$ open :

$f \in \mathcal{C}^1(\mathcal{U}) \quad \forall \ v \in \mathbb{R}^n \ :$

$$f(v) = Av + \mu(v) \quad " \quad \lim_{v \to 0} \frac{\mu(v)}{||v||} = 0$$

$\forall \ \eta \in \mathbb{R}^+ \ :$

$\exists \ r \in \mathbb{R}^+ \ :$

$$||v|| < r \to ||\mu(v)|| \leq \eta ||v|| \qquad \eta \ := \ a + \eta < 1$$

$\forall\ n \in \mathbb{N}$:

$\forall\ v \in \mathbb{D}(0, r)$:

$f^n(v) \in \mathbb{D}(0, r)$

$\|f^n(v)\| \leq (a + \eta)^n \|v\|$

Induction over n:

$n = 0$ ok

Suppose true for n

$\|f^{n+1}(v)\| = \|f(f^n(v))\| \leq \|Af^n(v)\| + \|\mu(f^n(v))\|$

$\leq a\|f^n(v)\| + \eta\|f^n(v)\| = (a + \eta)\|f^n(v)\| \leq (a + \eta)^{n+1}\|v\|$

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\delta := \min(\varepsilon, r)$

$\forall\ v \in \mathbb{R}^n$ " $\|v\| < \delta$:

$\|f^n(v)\| \leq (a + \eta)^n \|v\| \leq \|v\| < \varepsilon$

$\|f^n(v)\| \xrightarrow{n} 0$

repulsive:

$D(f^{-1})(x) = (Df(x))^{-1}$

$A := D(f^{-1})(x)$

$A^{-1} = (Df(x))^{-1}$

$\sigma(A^{-1}) = \sigma(A)^{-1}$

x attractive by $f^{-1} \rightarrow x$ repulsive by f

Local stability of periodic points

Let:

- $f : \mathbb{R}^n \to \mathbb{R}^n$
- $x \in \mathbb{R}^n$ period k point
- $o(x)$ orbit of x

Then, holds:

- $\forall\ p_i \in o(x)$:

$$A_i := Df^k(p_i) = Df(p_{i-1}) - - - Df(p_k) - - - Df(p_i)$$

$$\sigma(A_i) = \sigma(A_{i'})$$

Demonstration:

regla de la cadena da producto de matrices que no es conmutativo

Invariant stable & unstable manifold

Let:

- $f : \mathbb{R}^n \to \mathbb{R}^n$

We name invariant stable manifold of x to:

$$\{v \in \mathbb{R}^n \mid f^n \xrightarrow{n} 0\}$$

We name invariant unstable manifold of x to:

$$\{v \in \mathbb{R}^n \mid f^{-n} \xrightarrow{n} 0\}$$

We denote:

- stable invariant manifold : W^s
- unstable invariant manifold : W^u

Stable & unstable invariant manifolds

Let:

$$\cdot \quad \begin{array}{cccc} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & (\frac{x}{2}, 2y - 7x^2) \end{array}$$

Study:

· stability of $\text{Fix}(f)$

Start:

$$Df(x,y) = \begin{pmatrix} 1/2 & 0 \\ -4x & 2 \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\sigma(Df(0,0)) = \{1/2, 2\} \to (0,0) \text{ sella point}$$

$$\forall\ y \in \mathbb{R}:$$

$$f(0,y) = (0,2y) \to f^n(0,y) = (0,2^n) \xrightarrow{n} \infty$$

$$\{(x,y) \in \mathbb{R}^2 \mid x = 0\} \subset W^u(0,0)$$

$$C := \{(x,y) \in \mathbb{R}^2 \mid y = 4x^2\}$$

$$\forall\ (x,y) \in C:$$

$$f(x,y) = f(x,4x) = (x/2, x^2) \in C$$

$$f^{-1}(x,y) = (2x, 16x^2) \in C$$

$$f^n(x,4x) = (x/2^n, x^2/2^{2n-2}) \xrightarrow{n} (0,0)$$

$$W^s(0,0) = C$$