Real Analysis

Martin Azpillaga

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Contents

Block I

Definitions

1. The Field of Real Numberss

The field of real numbers is the transcendent extension of the rational numbers that represent all the possible values that can be obtained once a reference is set in a line. This set and its properties have been used since the very beggining of the mathematics but its formalization didn't come off till the 20th century. As we will discuss in this unit, the field of real numbers is the defined by a field that accomplishes some special properties. The point is that exists one unique field (over isomorphisms) that has this properties and thats why its calles The field of real numbers. Lets start From the very beggining:

Ordered Conmutative Field

Let:

$\cdot$ $(K, +, \cdot)$ commutative field

$\cdot$ R total order over K

Then, $(K, +, \cdot, R)$ is an ordered commutative field if:

$\cdot$ $\forall\ x, y, z \in K$ " $(x, y) \in R$:

$(x + z, y + z) \in R$

$\cdot$ $\forall\ x, y \in K$ " $(0, x) \in R \wedge (0, y) \in R$:

$(0, x \cdot y) \in R$

We denote:

$\cdot$ $\{x \in \mathbb{K} \mid (0, x) \in R\} \setminus \{0\}\ :\ \mathbb{K}^+$

$\cdot$ $\{x \in \mathbb{K} \mid (x, 0) \in R\} \setminus \{0\}\ :\ \mathbb{K}^-$

Arquimedian

Let:

$\cdot$ $\mathbb{K}$ ordered commutative field

Then, $\mathbb{K}$ is arquimedian if:

$\cdot$ $\forall\ \phi\ :\ \mathbb{N} \to \mathbb{K}$ " ϕ inyective :

$\phi(\mathbb{N})$ not bounded

Convergent Succession

Let:

· A set

· $(a_n)_{n\in\mathbb{N}}$

Then, $(a_n)_{n\in\mathbb{N}}$ is convergent if:

· $\exists\ l \in K \ \ \shortparallel\ \ \forall\ \varepsilon \in \mathbb{K}^+$:

$\exists\ n_0 \in \mathbb{N} \ \ \shortparallel\ \ \forall\ n \in \mathbb{N} \ \ \shortparallel\ n \geq n_0$:

$|a_n - l| < \varepsilon$

We denote:

· $l \ : \ \lim\limits_n \ a_n$

· a_n convergent $\wedge \lim\limits_n \ a_n = l \ : \ a_n \to l$

Cauchy Succession

Let:

· $\mathbb{K}$ ordered commutative field

Then, $(a_n)_{n\in\mathbb{N}} \subset \mathbb{K}$ is a Cauchy Succession if:

· $\forall\ \varepsilon \in K^+$:

$\exists\ n_0 \in \mathbb{N} \ \ \shortparallel\ \ \forall\ n, m \in \mathbb{N} \ \ \shortparallel\ n, m \geq n_0$:

$|a_n - a_m| < \varepsilon$

Complete

Let:

- $\mathbb{K}$ ordered commutative field

Then, $\mathbb{K}$ is complete if:

- $\forall\ (a_n)_{n\in\mathbb{N}} \subset \mathbb{K} \ _{\shortparallel}\ (a_n)_{n\in\mathbb{N}}$ Cauchy :

  $(a_n)_{n\in\mathbb{N}}$ convergent

Field of Real Numbers

Let:

- A set

Then, A is the field of real numbers if:

- A ordered commutative field
- A arquimedian
- A complete

We denote:

- $A\ :\ \mathbb{R}$

2. Metric Topology

The field of real numbers is the transcendent extension of the rational numbers that represent all the possible values that can be obtained once a reference is set in a line. This set and its properties have been used since the very beggining of the mathematics but its formalization didn't come off till the 20th century. As we will discuss in this unit, the field of real numbers is the defined by a field that accomplishes some special properties. The point is that exists one unique field (over isomorphisms) that has this properties and thats why its calles The field of real numbers. Lets start From the very beggining:

Metric Space

Let:

- A set
- $d : A \times A \to \mathbb{R}$

Then, (A, d) is a metric space if:

- $\forall\ x, y \in A$:

  $d(x, y) \geq 0$

- $\forall\ x, y \in A$:

  $d(x, y) = 0 \leftrightarrow x = y$

- $\forall\ x, y \in A$:

  $d(x, y) = d(y, x)$

- $\forall\ x, y, z \in A$:

  $d(x, z) \leq d(x, y) + d(y, z)$

We denote:

- d : distance over A

Ball

Let:

$\cdot$ (A, d) metric space

$\cdot$ $x \in A$

$\cdot$ $r \in \mathbb{R}^+$

Then, $B(x, r) \subset A$ is the ball centered in x of radius r if:

$\cdot$ $B(x, r) = \{a \in A \mid d(x, a) < r\}$

Inner point

Let:

$\cdot$ (A, d) metric space

$\cdot$ $B \subset A$

$\cdot$ $x \in B$

Then, x is an inner point of B if:

$\cdot$ $\exists\, r \in \mathbb{R}^+ \;\; /\!\!/ \;\; B(x, r) \subset B$

We denote:

$\cdot$ $\{x \in B \mid x \text{ inner point of } B\} \;:\; \mathring{B}$

Adherent point

Let:

- (A, d) metric space
- $B \subset A$
- $x \in A$

Then, x is an adherent point of B if:

- $\forall\ r \in \mathbb{R}^+$:

  $B(x,r) \cap B \neq \varnothing$

We denote:

- $\{x \in B \mid x \text{ adherent point of } B\}\ :\ \overline{B}$

Accumulation point

Let:

- (A, d) metric space
- $B \subset A$
- $x \in B$

Then, x is an accumulation point of B if:

- $\forall\ r \in \mathbb{R}^+$:

  $B(x,r) \setminus \{x\} \cap B \neq \varnothing$

We denote:

- $\{x \in B \mid x \text{ accumulation point of } B\}\ :\ B'$
- $x \in \overline{B}\ \wedge\ x \notin B'\ :\ x$ aislated point of B

Compact

Let:

· (A, d) metric space

· $K \subset A$

Then, K is compact if:

· $\forall \{U_i\}_{i\in I} \subset \mathcal{P}(A) \;\text{"}\; K \subset \bigcup_{i\in I} U_i$:

$\exists \{U_j\}_{j=1}^n \subset \{U_i\}_{i\in I} \;\text{"}\; K \subset \bigcup_{j=1}^{n} U_j$

Connex

Let:

· (A, d) metric space

· $B \subset A$

Then, B is connex if:

· $\forall U_1, U_2 \subset A \;\text{"}\; U_1 = \mathring{U}_1 \wedge U_2 = \mathring{U}_2 \wedge B = (B \cap U_1) \cup (B \cap U_2)$:

$(B \cap U_1) = \varnothing \vee (B \cap U_2) = \varnothing$

3. Metric Continuity

The field of real numbers is the transcendent extension of the rational numbers that represent all the possible values that can be obtained once a reference is set in a line. This set and its properties have been used since the very beggining of the mathematics but its formalization didn't come off till the 20th century. As we will discuss in this unit, the field of real numbers is the defined by a field that accomplishes some special properties. The point is that exists one unique field (over isomorphisms) that has this properties and thats why its calles The field of real numbers. Lets start From the very beggining:

Limit

Let:

- $(A_1, d_1), (A_2, d_2)$ metric spaces
- $f : A_1 \to A_2$
- $B \subset A_1$
- $x \in B'$
- $l \in A_1$

Then, l is the limit of f in x if:

- $\forall\, \varepsilon \in \mathbb{R}^+$:

  $\exists\, \delta \in \mathbb{R}^+ \;\; " \;\; \forall\, y \in B \;\; " \; d_1(x, y) < \delta$:

  $d_2(l, f(y)) < \varepsilon$

We denote:

- $l : \lim\limits_{t \to x} f(t)$

Continuity

Let:

- $(A_1, d_1), (A_2, d_2)$ metric spaces
- $B \subset A_1$
- $f : B \to A_2$
- $x \in B'$

Then, f is continuous in x if:

- $f(x) = \lim_{t \to x} f(t)$

Then, f is continuous in B if:

- $\forall\, x \in B$:

  f continuous in x

We denote:

- f continuous in A : $f \in \mathcal{C}(A_1)$

Uniform Continuity

Let:

- $(A_1, d_1), (A_2, d_2)$ metric spaces
- $B \subset A$
- $f : B \to A_2$

Then, f is uniformly continuous in B if:

- $\forall\, \varepsilon \in \mathbb{R}^+$:

  $\exists\, \delta \in \mathbb{R}^+$ „ $\forall\, x, y \in B$ „ $d_1(x, y) < \delta$:

  $d_2(f(x), f(y)) < \varepsilon$

4. Function Succesions

asdf

Function Succesion

Let:

· A, B sets

We call function succesion over the sets A, B to:

$$\cdot \quad \begin{array}{rcl} \Phi : \mathbb{N} & \longrightarrow & \mathcal{F}(A,B) \\ n & \longmapsto & f_n \end{array}$$

We denote:

· $Im\Phi \;:\; (f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(A,B)$

· A, B metric spaces $:$ $(f_n)_{n\in\mathbb{N}}$ metric function succesion

· A metric space $\wedge\, B = \mathbb{R}$ $:$ $(f_n)_{n\in\mathbb{N}}$ real function succesion

Punctually convergent

Let:

· $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(A_1, A_2)$ metric function succesion

Then, $(f_n)_{n\in\mathbb{N}}$ is punctually convergent if:

· $\exists\, f \in \mathcal{F}(A_1, A_2)$ „ $\forall\, x \in A_1$:

$$f(x) = \lim_{n} f_n(x)$$

We denote:

· f $:$ punctual limit of $(f_n)_{n\in\mathbb{N}}$

· f punctual limit of $(f_n)_{n\in\mathbb{N}}$ $:$ $f_n \xrightarrow{n} f$

Uniformly convergent

Let:

$\cdot\ (f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(A_1, A_2)$ metric function succesion

Then, $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent if:

$\cdot\ \exists\ f \in \mathcal{F}(A_1, A_2)\ \ _{"}\ \ \forall\ \varepsilon \in A_1$:

$\exists\ n_0 \in \mathbb{N}\ \ _{"}\ \ \forall\ n \in \mathbb{N}\ \ _{"}\ n \geq n_0$:

$f_n - f < \varepsilon$

We denote:

$\cdot\ f\ :$ uniform limit of $(f_n)_{n\in\mathbb{N}}$

$\cdot\ f$ uniform limit of $(f_n)_{n\in\mathbb{N}}\ :\ f_n \overset{n}{\rightrightarrows} f$

Uniformly Cauchy

Let:

$\cdot\ (A_1, d_1), (A_2, d_2)$ metric spaces

$\cdot\ B \subset A_1$

$\cdot\ (f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(B, A_2)$

Then, $(f_n)_{n\in\mathbb{N}}$ is Uniformly Cauchy if:

$\cdot\ \ \forall\ \varepsilon \in \mathbb{R}^+$:

$\exists\ n_0 \in \mathbb{N}\ \ _{"}\ \ \forall\ n, m \in \mathbb{N}\ \ _{"}\ n, m \geq n_0$:

$|f_m - f_n| < \varepsilon$

5. Power Series

introduction

Power Series

Let:

$\cdot\ (a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$

$\cdot\ (f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(\mathbb{R},\mathbb{R})$

Then, $\sum\limits_{n\geq 1} a_n f_n$ is a power series if:

$\cdot\ \exists\ a \in \mathbb{R}$ „ $\forall\ n \in \mathbb{N}$:

$\forall\ x \in \mathbb{R}$:

$f_n(x) = (x-a)^n$

Domain of Convergence

Let:

- $\sum\limits_{n\geq 1} a_n f_n$ power series

We call domain of convergence of $\sum\limits_{n\geq 1} a_n f_n$ to:

- $\{x \in \mathbb{R} \mid \sum\limits_{n\geq 1} a_n f_n \text{ convergent in } x\}$

We denote:

- D

Radius of Convergence

Let:

- $\sum\limits_{n\geq 1} a_n f_n$ power series

We call radius of convergence of $\sum\limits_{n\geq 1} a_n f_n$ to:

- $\dfrac{1}{\overline{\lim\limits_{n}} \sqrt[n]{|a_n|}}$

We denote:

- $\dfrac{1}{\overline{\lim\limits_{n}} \sqrt[n]{|a_n|}} : R$

Associated function

Let:

- $\sum\limits_{n\geq 1} a_n f_n$ power series
- D convergency domain of $\sum\limits_{n\geq 1} a_n f_n$

We call associated function of $\sum\limits_{n\geq 1} a_n f_n$ to:

- $\begin{array}{rcl} f : D & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \sum\limits_{n\geq 1} a_n f_n(x) \end{array}$

Exponential Function

We call exponential function to:

6. Fourier Series

Fourier Series

go

Annex 1. Set Theory

introduction

Order

Let:

- A set
- $R \subset A \times A$

Then, R is an order over A if:

- $\forall\, x \in A$:

  $(x, x) \in R$

- $\forall\, x, y \in A \;\shortparallel\; (x, y) \in R \;\wedge\; (y, x) \in R$:

  $x = y$

- $\forall\, x, y, z \in A \;\shortparallel\; (x, y) \in R \;\wedge\; (y, z) \in R$:

  $(x, z) \in R$

We denote:

- $(x, y) \in R \;:\; x \leq y$
- $(x, y) \in R \;\wedge\; x \neq y \;:\; x < y$

Total Order

Let:

- A set
- R order over A

Then, R is a total order over A if:

- $\forall\, x, y \in A$:

  $(x, y) \in R \;\vee\; (y, x) \in R$

Bounds

Let:

- $\mathbb{K}$ ordered commutative field
- $A \subset K$

Then, $x \in K$ is an upper bound of A if:

- $\forall\ a \in A$:

  $a \leq x$

Then, $x \in K$ is a lower bound of A if:

- $\forall\ a \in A$:

  $x \leq a$

We denote:

- A superiorly bounded : $\exists\ x \in K$ „ x upper bound of A
- A inferiorly bounded : $\exists\ x \in K$ „ x upper bound of A
- A bounded : A inferiorly bounded and superiorly bounded

Supremum and Infimum

Let:

· $\mathbb{K}$ ordered commutative field

· $A \subset K$

Then, $x \in K$ is the supremum of A if:

· x upper bound of A

· $\forall\ y \in K$ $_{\shortparallel}$ y upper bound of A:

$x \leq y$

Then, $x \in K$ is the infimum of A if:

· $\forall\ y \in K$ $_{\shortparallel}$ y lower bound of A:

$y \leq x$

We denote:

· supremum of A : $supA$

· infimum of A : $infA$

· $supA \in A$: $maxA$

· $infA \in A$: $minA$

·

Bounded function

Let:

- $(A_1, d_1), (A_2, d_2)$ metric spaces
- $B \subset A$
- $f : B \to A_2$

Then, f is bounded if:

- $f(B)$ bounded

Succession

Let:

- A set

Then, $(a_n)_{n \in N} \subset A$ is sucesion of elements of A if:

- $\exists\, \phi : \mathbb{N} \to A \;\shortparallel\; Im\phi = \{a_n\}_{n \in N}$

We denote:

- $(a_n)_{n \in N} : (a_n)_{n \in \mathbb{N}}$

Annex 2. Algebraic Structures

introduction

Commutative Field

Let:

- A set
- $\star_1, \star_2 \;:\; A \times A \to A$

Then, (A, ϕ_1, ϕ_2) is a commutative field if:

- $(A, \star_1)$ abelian group
- $(A, \star_2)$ abelian group
- $\forall\, x, y, z \in A$:

$$x \star_1 (y \star_2 z) = (x \star_1 y) \star_2 (x \star_1 z)$$

We denote:

- $\star_1 \;:\; +$
- $\star_2 \;:\; \cdot$
- $(A, +, \cdot)$ commutative field $\;:\; \mathbb{K}$

Annex 3. Riemann Integral

introduction

Partition

Let:

- $a, b \in \mathbb{R}$
- $\{x_i\}_{i=1}^n \subset [a, b]$

Then, $\{x_i\}_{i=1}^n$ is a partition of $[a, b]$ if:

- $x_1 = a$
- $x_n = b$
- $\forall\ x_i, x_j \in \{x_i\}_{i=1}^n$ " $i < j$:

  $x_i < x_j$

Superior/Inferior sum of partitions

Let:

- $a, b \in \mathbb{R}$
- $f \in \mathcal{F}([a, b], \mathbb{R})$
- $P = \{x_i\}_{i=1}^n$ partition of $[a, b]$

We call superior sum of the partition P over f to:

- $\displaystyle\sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1})$

We call superior sum of the partition P over f to:

- $\displaystyle\sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1})$

We denote:

- the superior sum of the partition P over f : $S(f, P)$
- the inferior sum of the partition P over f : $s(f, P)$

Superior/Inferior integral

Let:

- $a, b \in \mathbb{R}$
- $f \in \mathcal{F}([a,b], \mathbb{R})$
- $I \in \mathbb{R}$

Then, I is the superior integral of f over $[a,b]$ if:

- $I = \inf_{P \in \mathcal{P}_f([a,b])} S(f,P)$

Then, I is the inferior integral of f over $[a,b]$ if:

- $I = \sup_{P \in \mathcal{P}_f([a,b])} s(f,P)$

We denote:

- the superior integral of f : $\overline{\int_a^b} f(x)dx$
- the inferior integral of f : $\underline{\int_a^b} f(x)dx$

Riemann Integrability

Let:

- $a, b \in \mathbb{R}$
- $f \in \mathcal{F}([a,b], \mathbb{R})$

Then, f is Riemann integrable if:

- $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$

We denote:

- $\{f \in \mathcal{F}([a,b], \mathbb{R}) \mid f \text{ Riemman integrable } \}$: $\mathcal{R}([a,b])$
- $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$: $\int_a^b f(x)dx$

Block II

Propositions

1. The Field of Real Numbers

4 propositions and 2 theorems.

Field Properties

Let:

- $\mathbb{K}$ field

Then, holds:

- $\forall\ x \in \mathbb{K}$:

  $x \cdot 0 = 0$

- $\forall\ x, y \in K \ \ " \ x \cdot y = 0$:

  $\exists\ z \in \{x, y\} \ \ " \ z = 0$

Demonstration:

$x \cdot 0 = x \cdot (0 + 0) = (x \cdot 0) + (x \cdot 0)$

$(x \cdot 0) + [-(x \cdot 0)] = (x \cdot 0) + (x \cdot 0) + [-(x \cdot 0)]$

$0 = x \cdot 0$

Case $x = 0$:

$x = 0$Case $x \neq 0$:

$x^{-1} \cdot x \cdot y = x^{-1} \cdot 0 = 0$

$y = 0 \exists\ z \in \{x, y\} \ \ " \ z = 0$

Order Properties

Let:

· $\mathbb{K}$ ordered commutative field

Then, holds:

· $\forall\, x, y, u, v \in \mathbb{K}$ „ $x \leq y \,\wedge\, u \leq v$:

$x + u \leq y + v$

· $\forall\, x, y \in \mathbb{K}$ „ $x \leq 0 \,\wedge\, y \leq 0$:

$xy \geq 0$

Demonstration:

$x \leq y \rightarrow x + u \leq y + u$

$u \leq v \rightarrow u + y \leq v + y$

$\therefore)\; x + u \leq y + u \leq v + y$

$(-1)x + x = x(-1+1) = 0 \rightarrow (-1)x = -x$

In particular, $(-1)(-1) = -(-1) = 1$

$x \leq 0 \rightarrow x + (-x) \leq 0 + (-x) \rightarrow -x \geq 0$

In particular, $y \leq 0 \rightarrow -y \geq 0$

$$\left\{ \begin{array}{c} -x \geq 0 \\ -y \geq 0 \end{array} \right\} \rightarrow (-x)(-y) \geq 0 \rightarrow (-1)(-1)xy \geq 0 \rightarrow xy \geq 0$$

Inyection of the rationals in commutative ordered fields

Let:

$\cdot$ $\mathbb{K}$ ordered commutative field

Then, holds:

$\cdot$ $\exists\ \phi\ :\ \mathbb{Q} \to \mathbb{K}\ \ "\ \phi$ inyective

Demonstration:

Follow 2 steps:

Step 1: Existence:

$$\begin{aligned} f\ :\mathbb{N}\ &\longrightarrow\ \mathbb{K}^{+} \\ n\ &\longmapsto\ \sum_{i=1}^{n} 1_{\mathbb{K}} \\ \phi\ :\mathbb{Q}\ &\longrightarrow\ \mathbb{K} \\ \frac{n}{m}\ &\longmapsto\ \begin{cases} f(n)\cdot f(m)^{-1}, n \geq 0 \\ -f(n)\cdot f(m)^{-1}, n < 0 \end{cases} \end{aligned}$$

Step 2: Inyectivity:

$\forall\ \frac{n}{m}, \frac{p}{q} \in \mathbb{Q}\ \ "\ \phi(\frac{n}{m}) = \phi(\frac{p}{q})$:

$$f(n)f(m)^{-1} = f(p)f(q)^{-1}$$

$$f(n)f(q) = f(p)f(m)$$

$$\sum_{i=1}^{n} 1_{\mathbb{K}} \cdot \sum_{i=1}^{q} 1_{\mathbb{K}} = \sum_{i=1}^{p} 1_{\mathbb{K}} \cdot \sum_{i=1}^{m} 1_{\mathbb{K}}$$

$$\sum_{i=1}^{nq} 1_{\mathbb{K}} = \sum_{i=1}^{pm} 1_{\mathbb{K}}$$

$$\sum_{i=1}^{nq-pm} 1_{\mathbb{K}} = 0$$

$$nq - pm = 0$$

$$\frac{n}{m} = \frac{p}{q}$$

Inclusion of Convergent succesions in Cauchy succesions

Let:

$\cdot\, (a_n)_{n\in\mathbb{N}}$ convergent succesion over $\mathbb{K}^{+}$

Then, holds:

$\cdot\, (a_n)_{n\in\mathbb{N}}$ Cauchy succesion

Demonstration:

$\forall\, \varepsilon \in \mathbb{K}^{+}$:

$\exists\, n_0 \in \mathbb{N}$ „ $\forall\, n \in \mathbb{N}$ „ $n \geq n_0$:

$|a_n - l| < \varepsilon\ \forall\, n, m \in \mathbb{N}$ „ $n, m \geq n_0$:

$|a_n - a_m| \leq |a_n - l| + |a_m - l| < 2\varepsilon \therefore)$ $(a_n)_{n\in\mathbb{N}}$ Cauchy succesion

Supremum Theorem

Let:

$\cdot\ A \subset \mathbb{R}$ ‚‚ superiorly bounded $\wedge\ A \neq \varnothing$

Then, holds:

$\cdot\ \exists\ s \in \mathbb{R}$ ‚‚ $s = supA$

Demonstration:

Follow 3 steps

Step 1: Find s:

A superiorly bounded $\rightarrow \exists\ b_0 \in \mathbb{R}$ ‚‚ b_0 upper bound of A

$A \neq \varnothing \rightarrow \exists\ a_0 \in \mathbb{R}$ ‚‚ $a_0 \in A$

$\forall\ x, y \in \mathbb{R}$:

$[x, y]' \subset \mathbb{R}\ :=\ [x, \frac{x+y}{2}]$

$[x, y]'' \subset \mathbb{R}\ :=\ [\frac{x+y}{2}, y]$

$\forall\ n \in \mathbb{N}$:

$[a_n, b_n] \subset \mathbb{R}\ :=\ I \in \{[a_{n-1}, b_{n-1}]', [a_{n-1}, b_{n-1}]''\}[I \cap A \neq \varnothing]$

$[a_n, b_n] \neq \varnothing \rightarrow \exists\ (c_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ ‚‚ $\forall\ n \in \mathbb{N}$:

$c_n \in [a_n, b_n] \quad \forall\ \varepsilon \in \mathbb{R}^+$:

$\exists\ n_0 \in \mathbb{N}$ ‚‚ $\forall\ n \in \mathbb{N}$ ‚‚ $n \geq n_0$:

$b_n - a_n < \varepsilon$

$\forall\ n, m \in \mathbb{N}$ ‚‚ $n, m \geq n_0\ \wedge\ n \leq m$:

$|c_n - c_m| < b_n - a_n < \varepsilon\ \therefore)\ (c_n)_{n \in \mathbb{N}}$ Cauchy succesion

$\mathbb{R}$ complete $\rightarrow (c_n)_{n \in \mathbb{N}}$ convergent succesion $\rightarrow \exists\ s \in$

$\mathbb{R}$ ״ $c_n \xrightarrow{n} s$

Step 2: s upper bound of A:

$\forall\ n \in \mathbb{N}$:

$$a_n \leq c_n \leq b_n \to 0 \leq b_n - s \leq b_n - a_n$$

$$\left\{\begin{array}{l} 0 \xrightarrow{n} 0 \\ b_n - a_n \xrightarrow{n} 0 \end{array}\right\} \text{Sandwitch Theorem} \to b_n - s \xrightarrow{n} 0 \to$$

$b_n \xrightarrow{n} s$

$\forall\ d \in \mathbb{R}$ ״ $d > s$:

$$\left\{\begin{array}{l} b_n \xrightarrow{n} s \\ \varepsilon = d - s \end{array}\right\} \to \exists\ n_0 \in \mathbb{N} \text{ ״ } \forall\ n \in \mathbb{N} \text{ ״ } n \geq n_0:$$

$$b_n - s < d - s \to b_n < d$$

b_n upper bound of $A \to d \notin A$

$\therefore)$ s upper bound of AStep 3: s minimum upper bound of A:

$\forall\ n \in \mathbb{N}$:

$$a_n \leq c_n \leq b_n \to a_n - b_n \leq a_n - s \leq 0$$

$$\left\{\begin{array}{l} 0 \xrightarrow{n} 0 \\ a_n - b_n \xrightarrow{n} 0 \end{array}\right\} \text{Sandwitch Theorem} \to a_n - s \xrightarrow{n} 0 \to$$

$a_n \xrightarrow{n} s$

$\forall\ e \in \mathbb{R}$ ״ $e < s$:

$$\left\{\begin{array}{l} a_n \xrightarrow{n} s \\ \varepsilon = s - e \end{array}\right\} \to \exists\ n_0 \in \mathbb{N} \text{ ״ } \forall\ n \in \mathbb{N} \text{ ״ } n \geq n_0:$$

$$|a_n - s| = s - a_n < s - e \to a_n > e$$

$I_n \cap A \neq \varnothing \to \exists\ x \in A$ ״ $x > e \to e$ not upper bound of A

$\therefore)$ s minimum upper bound of A

2. Metric Topology

metric topology

Inclusion of Compacts in Bounded Closed sets

Let:

$\cdot$ (A, d) metric space

$\cdot$ $K \subset A$ compact

Then, holds:

$\cdot$ K bounded

$\cdot$ $K = \overline{A}$

Demonstration:

Follow 2 steps

Step 1: K bounded:

$$K \subset \cup_{x \in K} B(x, 1) \to \exists \ \{x_i\}_{i=1}^{n} \subset K \ \ " \ K \subset \bigcup_{i=1}^{n} B(x_i, 1)$$

$\forall \ x_0 \in A$:

$$K \subset \bigcup_{i=1}^{n} B(x_0, d(x_0, x_i) + 1)$$

$$m \in \mathbb{R} \ := \ max\{d(x_0, x_i) + 1\}_{i=1}^{n}$$

$$K \subset B(x_0, m)$$

K bounded Step 2: $K = \overline{K}$:

$$K = \overline{K} \leftrightarrow K^c = \mathring{K}^c$$

$\forall \ p \in K^c$:

$\forall \ x \in K$:

$$d_x \in \mathbb{R}^+ \ := \ d(x, p)$$

$$U_x \subset A \ := \ B(x, \tfrac{d_x}{2})$$

$$V_x \subset A \ := \ B(p, \tfrac{d_x}{2})$$

$$K \subset \cup_{x \in K} U_x$$

$$K \text{ compact} \to \exists\, \{x_i\}_{i=1}^{n} \subset K \quad " \; K \subset \bigcup_{i=1}^{n} U_{x_i}$$

$$U := \bigcup_{i=1}^{n} U_{x_i}$$

$$V := \bigcap_{i=1}^{n} V_{x_i}$$

$$U \cap V \neq \varnothing \to V \subset K^c$$

$$V = \mathring{V}$$

$$p \in V \quad p \in \mathring{K}^c$$

$$K^c = \mathring{K}^c \to K = \overline{K}$$

Finite intersection property

Let:

· (A, d) metric space

· $\{K_i\}_{i\in I} \subset \mathcal{P}(A)$ compacts „ $\forall\ \{K_j\}_{j=1}^{n} \subset \{K_i\}_{i\in I}$:

$\bigcap\limits_{j=1}^{n} K_j \neq \varnothing$

Then, holds:

· $\bigcap\limits_{i\in I} K_i \neq \varnothing$

Demonstration:

Counter-reciprocal: $\bigcap\limits_{i\in I} K_i = \varnothing$

$\forall\ K \in \{K_i\}_{i\in I}$:

$\forall\ x \in K$:

$x \notin \bigcap\limits_{i\in I} K_i \to x \in (\bigcap\limits_{i\in I} K_i)^c \to x \in \bigcup\limits_{i\in I} K_i^c\ K \subset \bigcup\limits_{i\in I} K_i^c$

K compact $\to \exists\ \{K_j\}_{j=1}^{n} \subset \{K_i\}_{i\in I}$ „ $K \subset \bigcup\limits_{j=1}^{n} K_j^c$

$K \subset (\bigcap\limits_{j=1}^{n} K_j)^c$

$\bigcap\limits_{j=1}^{n} K_j \cap K = \varnothing$

Inheritance of compacticity by closed subsets

Let:

- (A, d) metric space
- $K \subset A$ compact
- $F \subset K$ closed
-

Then, holds:

- F compact

Demonstration:

$$\forall \ \{U_i = \mathring{U}_i\}_{i\in I} \subset \mathcal{P}(A) \ \text{"} \ F \subset \bigcup_{i\in I} U_i:$$

$$K \subset \bigcup_{i\in I} U_i \cup F^c$$

$$K \text{ compact} \to \exists \ \{U_j\}_{j=1}^{n} \subset \{U_i\}_{i\in I} \ \text{"} \ K \subset \bigcup_{j=1}^{n} U_j \cup F^c$$

$$F \subset K \to F \subset \bigcup_{j=1}^{n} U_j \cup F^c \to F \subset \bigcup_{j=1}^{n} U_j$$

F compact

Characterization of real compacts

Let:

$\cdot\, K \subset \mathbb{R}$

Then, holds:

$\cdot\, K$ compact $\leftrightarrow K$ closed $\wedge\, K$ bounded

Demonstration:

$\to$):

Already seen$\leftarrow$):

Follow 2 steps

Step 1: $\forall a, b \in \mathbb{R} : [a, b]$ compact:

Counter-reciprocal: $[a, b]$ no compact

$\exists\, \{U_i = \mathring{U}_i\}_{i \in I} \subset \mathcal{P}(\mathbb{R})$ ¨ $[a, b] \subset \bigcup\limits_{i \in I} U_i \wedge\ \forall\, \{U_j\}_{j=1}^{n} \subset \{U_i\}_{i \in I}$:

$$[a, b] \not\subset \bigcup_{j=1}^{n} U_j$$

$I_0 := [a, b]$

$\forall\, n \in \mathbb{N}$:

$I_n := I \in \{I'_{n-1}, I''_{n-1}\}[I$ no compact $]$ Nested intervals theorem $\to \exists\, x_0 \in \mathbb{R}$ ¨ $\bigcap\limits_{n \in N} I_n = \{x_0\}$

$\exists\, U_{i_0} \in \{U_i\}_{i \in I}$ ¨ $x_0 \in U_{i_0}$

$U_{i_0} = \mathring{U}_{i_0} \to \exists\, \delta \in \mathbb{R}^+$ ¨ $(x - \delta, x + \delta) \subset U_{i_0}$

$\exists\, n \in \mathbb{N}$ ¨ $\frac{b-a}{2^n} < \delta \to I_n \subset (x - \delta, x + \delta) \subset U_{i_0}$

I_n compact

Step 2: A compact:

A bounded $\to \exists\, a, b \in \mathbb{R}$ ¨ $A \subset [a, b]$

$$\left.\begin{cases} A \text{ closed} \\ A \subset [a,b] \text{ compact} \end{cases}\right\} \to A \text{ compact}$$

Characterization of real connex sets

Let:

$\cdot\ A \subset \mathbb{R}$

Then, holds:

$\cdot\ A$ connex $\leftrightarrow \exists\ a, b \in \mathbb{R}\ \shortparallel\ A = |a, b|$

Demonstration:

$\rightarrow$):

Counter-reciprocal: $\nexists a, b \in \mathbb{R}\ \shortparallel\ A = |a, b|$

$\exists\ x \in (infA, supA)\ \shortparallel\ x \notin A$

$A = (-\infty, x) \cup (x, \infty)$

A no connex $\leftarrow$):

$$\begin{array}{rcl} arctg : \mathbb{R} & \longrightarrow & (-\frac{\pi}{2}, \frac{\pi}{2}) \\ x & \longmapsto & arctg(x) \end{array}$$

$$\left.\begin{cases} \mathbb{R} \text{ connex} \\ arctg \text{ continuous} \end{cases}\right\} \rightarrow (-\frac{\pi}{2}, \frac{pi}{2}) \text{ connex}$$

$\forall\ a, b \in \mathbb{R}$:

$$\begin{array}{rcl} \phi : (-\frac{\pi}{2}, \frac{pi}{2}) & \longrightarrow & (a, b) \\ x & \longmapsto & a + (b - a)\frac{(x + \frac{\pi}{2})}{\pi} \end{array}$$

(a, b) connex $|a, b| \subset \overline{(a, b)} \rightarrow |a, b|$ connex $\ A$ connex

3. Metric Continuity

Continuity in Metric Spaces

Characterization of limits by succesions

Let:

$\cdot$ $(A_1, d_1), (A_2, d_2)$ metric spaces

$\cdot$ $B \subset A_1$

$\cdot$ $f : B \to A_2$

$\cdot$ $p \in B'$

$\cdot$ $l \in A_2$

Then, holds:

$\cdot$ $l = \lim_{x \to p} f(x) \leftrightarrow \forall\ (p_n)_{n \in \mathbb{N}} \subset B \smallsetminus \{0\}$ „ $p_n \xrightarrow{n} p$: $(f(p_n))_{n \in \mathbb{N}} \xrightarrow{n} l$

Demonstration:

$\to$):

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\exists\ \delta \in \mathbb{R}^+$ „ $\forall\ x \in A_1$ „ $d(x, p) < \delta$:

$d(f(x), l) < \varepsilon\ \ \forall\ (p_n)_{n \in \mathbb{N}} \subset B \smallsetminus \{p\}$ „ $p_n \xrightarrow{n} p$:

$\left\{ \begin{array}{l} p_n \xrightarrow{n} p \\ \varepsilon = \delta \end{array} \right\} \to \exists\ n_0 \in \mathbb{N}$ „ $\forall\ n \in \mathbb{N}$ „ $n \geq n_0$:

$d(p_n, p) < \delta \to d(f(p_n), l) < \varepsilon$$\leftarrow$):

Counter-reciprocal: $\lim_{x \to p} f(x) \neq l$

$\exists\ \varepsilon \in \mathbb{R}^+$ „ $\forall\ n \in \mathbb{N}$:

$\exists\ x_n \in B$ „ $d_1(x_n, p) < \frac{1}{n}\ \wedge\ d_2(f(x_n), l) > \varepsilon$ In particular, $x_n \neq p \to x \in B \smallsetminus \{p\}$

$\exists\ (x_n)_{n \in \mathbb{N}} \subset B \smallsetminus \{p\}$ „ $\lim_n x_n = p\ \wedge\ \lim_n f(x_n) \neq l$

Compacts are closed by limit

Let:

· (A, d) metric space

· $K \subset A$ compact

· $(a_n)_{n\in\mathbb{N}} \subset K$ " $a_n \xrightarrow{n} a$

Then, holds:

· $a \in K$

Demonstration:

Counter-reciprocal: $a \notin K$

K compact $\rightarrow K$ closed $\rightarrow K^c$ open

$a \in K^c \rightarrow \exists\ \delta \in \mathbb{R}^+$ " $B(a, \delta) \subset K^c$

$\forall\ (a_n)_{n\in\mathbb{N}} \subset A$ " $a_n \xrightarrow{n} a$:

$\exists\ n_0 \in \mathbb{N}$ " $\forall\ n \in \mathbb{N}$ " $n \geq n_0$:

$d(a_n, a) < \delta \rightarrow a_n \in B(a, \delta) \rightarrow a_n \in K^c \nexists (a_n)_{n\in\mathbb{N}} \subset K$ " $a_n \xrightarrow{n} a$

Continuous funtions are closed by composition

Let:

$\cdot\, (A_1, d_1), (A_2, d_2), (A_3, d_3)$ metric spaces

$\cdot\, B \subset A_1$

$\cdot\, f \,:\, B \to A_2$

$\cdot\, g \,:\, f(B) \to A_3$

$\cdot\, p \in B'$

$\cdot\, f$ continuous in p

$\cdot\, f$ continuous in $f(p)$

Then, holds:

$\cdot\, g \circ f$ continuous in p

Demonstration:

g continuous in $f(p) \to \forall\ \varepsilon \in \mathbb{R}^+$:

$\exists\ \delta \in \mathbb{R}^+$ " $\forall\ y \in f(B)$ " $d(f(p), y) < \delta$:

$d(g \circ f(p), g(y)) < \varepsilon f$ continuous in $p \to \exists\ \mu \in \mathbb{R}^+$ "

$\forall\ x \in B$ " $d(p, x) < \mu$:

$d(f(p), f(x)) < \delta \to d(g \circ f(p)), g \circ f(x)) < \varepsilon$

Characterization of continuous functions by open sets

Let:

$\cdot\ (A_1, d_1), (A_2, d_2)$ metric spaces

$\cdot\ B \subset A_1$

$\cdot\ f\ :\ B \to A_2$

Then, holds:

$\cdot\ f$ continuous in $B \leftrightarrow \forall\ U_i \subset A_2\ \text{„}\ U = \mathring{U}$:

$f^{-1}(U) = \overset{\circ}{f^{-1}(U)}$

Demonstration:

$\to$):

$\forall\ U \subset A_2\ \text{„}\ U = \mathring{U}$:

$\forall\ p \in f^{-1}(U)$:

$f(p) \in U = \mathring{U} \to \exists\ \varepsilon \in \mathbb{R}^+\ \text{„}\ B(f(p), \varepsilon) \subset A_2$

f continious in $p \to \exists\ \delta \in \mathbb{R}^+\ \text{„}\ B(p, \delta) \subset f^{-1}(U)$

$p \in \overset{\circ}{f^{-1}(U)}\ f^{-1}(U) = \overset{\circ}{f^{-1}(U)} \leftarrow$):

$\forall\ p \in A_1$:

$\forall\ \varepsilon \in \mathbb{R}^+$:

$U \subset A_2\ :=\ B(f(p), \varepsilon)$

$U = \mathring{U} \to f^{-1}(U) = \overset{\circ}{f^{-1}(U)}$

$p\ \in\ \overset{\circ}{f^{-1}(U)}\ \to\ \exists\ \delta\ \in\ \mathbb{R}^+\ \text{„}\ B(p, \delta)\ \subset\ f^{-1}(U)\ f$ continuous in p f continuous in B

Invariancy of compacts by continuous functions

Let:

- $(A_1, d_1), (A_2, d_2)$ metric spaces
- $K \subset A_1$ compact
- $f : K \to A_2$ continuous

Then, holds:

- $f(K) \subset A_2$ compact

Demonstration:

$$\forall \ \{U_i \mid U_i = \mathring{U}_i\}_{i \in I} \subset \mathcal{P}(A_2) \ \text{\guillemotright}\ f(K) \subset \bigcup_{i \in I} U_i:$$

$\forall \ i \in I$:

$$V_i := f^{-1}(U_i)$$

f continuous $\to V_i = \mathring{V}_i \ K \subset \bigcup_{i \in I} V_i \to \exists \ \{V_j\}_{j=1}^n \subset \{V_i\}_{i \in I} \ \text{\guillemotright}\ K \subset \bigcup_{j=1}^n V_j$

$$f(K) \subset f\Big(\bigcup_{j=1}^n V_j\Big) = \bigcup_{j=1}^n f(V_j) = \bigcup_{j=1}^n U_j$$

$f(K)$ compact

Invariancy of connex sets by continuous functions

Let:

- $(A_1, d_1), (A_2, d_2)$ metric spaces
- $B \subset A_1$ connex
- $f\ :\ A \to A_2$ continuous

Then, holds:

- $f(B)$ connex

Demonstration:

Counter-reciprocal: $f(B)$ no connex$\to$ B no connex :

$$\exists\, U, V \subset A_2 \;\;/\!\!/\;\; \begin{cases} U = \mathring{U} \\ V = \mathring{V} \\ U \cap f(B) \neq \varnothing \\ V \cap f(B) \neq \varnothing \\ B = (U \cap B) \mathbin{\dot\cup} (V \cap B) \end{cases}$$

$$f^{-1}(f(B)) = f^{-1}(U \cap f(B) \mathbin{\dot\cup} V \cap f(B))$$

$$B \subset f^{-1}(f(B))$$

$$B \subset f^{-1}(U \cap f(B) \mathbin{\dot\cup} V \cap f(B))$$

$$B \subset (f^{-1}(U) \cap f^{-1}(f(B))) \mathbin{\dot\cup} (f^{-1}(V) \cap f^{-1}(f(B)))$$

$$B = (f^{-1}(U) \cap f^{-1}(f(B)) \cap B) \mathbin{\dot\cup} (f^{-1}(V) \cap f^{-1}(f(B)) \cap B)$$

$$B = (f^{-1}(U) \cap B) \mathbin{\dot\cup} (f^{-1}(V) \cap B)$$

$$\left.\begin{cases} f^{-1}(U) = \mathring{f^{-1}(U)} \\ f^{-1}(V) = \mathring{f^{-1}(V)} \\ f^{-1}(U) \cap B \neq \varnothing \\ f^{-1}(V) \cap B \neq \varnothing \\ B = (f^{-1}(U) \cap B) \mathbin{\dot\cup} (f^{-1}(V) \cap B) \end{cases}\right\} \to B \text{ no connex}$$

Characterization of Uniformly continuous functions over compacts by continuous functions

Let:

$\cdot\ (A_1, d_1), (A_2, d_2)$ metric space $\;\shortparallel\; A_1$ compact

Then, holds:

$\cdot\ f$ uniformly continuous over $A_1 \leftrightarrow f$ continuous over A_1

Demonstration:

f continuous over $A_1 \rightarrow$

$\forall\ p \in A_1$:

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\exists\ \delta_p \in \mathbb{R}^+ \;\shortparallel\; \forall\ x \in A_1 \;\shortparallel\; d(x,p) < \delta_p$:

$d(f(x), f(p)) < \frac{\varepsilon}{2}$ $\left\{\begin{array}{l} A_1 \text{ compact} \\ A_1 \subset \displaystyle\bigcup_{p \in A_1} B(p, \frac{\delta_p}{2}) \end{array}\right\} \rightarrow \exists\ \{p_i\}_{i=1}^n \subset$ $A_1 \;\shortparallel\; A_1 \subset \displaystyle\bigcup_{i=1}^n p_i$

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\delta := min\{\frac{\delta_{p_i}}{2}\}_{i=1}^n$

$\forall\ x, y \in A_1 \;\shortparallel\; d(x,y) < \delta$:

$\exists\ p \in \{p_i\}_{i=1}^n \;\shortparallel\; x \in B(p, \frac{\delta_p}{2}$

$d(x,p) < \frac{\delta_p}{2} \rightarrow d(f(x), f(p)) < \frac{\varepsilon}{2}$

$d(y,p) \leq d(y,x) + d(x,p) < \delta + \frac{\delta_p}{2} \leq \delta_p$

$d(y,p) < \delta_p \rightarrow d(f(y), f(p)) < \frac{\varepsilon}{2}$

$d(f(x), f(y)) \leq d(f(x), f(p)) + d(f(p), f(y)) < \varepsilon$ f uniformly continuous over A_1

Weierstrass' theorem

Let:

$\cdot$ (A, d) metric space

$\cdot$ $K \subset A$ compact

$\cdot$ $f : K \to \mathbb{R}$ continuous over K

Then, holds:

$\cdot$ $f(K)$ compact

Demonstration:

Follow 2 steps

Step 1: $f(K)$ bounded:

K compact $\to$ K bounded

f continuous over $K \to f$ bounded $\to f(K)$ bounded

Step 2: $f(K)$ closed:

f bounded $\to \exists\ s, i \in \mathbb{R}$ " $s = supf(K) \wedge i = inff(K)$

$\forall\ n \in \mathbb{N}$:

$\exists\ x_n \in K$ " $s - \frac{1}{n} < f(x_n) \leq s \left\{ \begin{array}{l} (f(x_n))_{n\in\mathbb{N}} \subset f(K) \\ f(x_n) \xrightarrow{n} s \end{array} \right\} \to$ $s \in f(K)$

Similarly:

$i \in f(K)$

$f(K)$ closed

Intermediate Value Theorem

Let:

- $a, b \in \mathbb{R}$
- $f\ :\ [a, b] \to \mathbb{R}$ continuous over $[a, b]$

Then, holds:

- $\forall\ y \in < f(a), f(b) >:$

  $\exists\ x \in [a, b]\ \text{"}\ f(x) = y$

Demonstration:

$$\left.\begin{cases} [a, b] \text{ connex} \\ f \text{ continuous over } [a, b] \end{cases}\right\} \to f([a, b]) \text{ connex}$$

$\exists\ c, d \in \mathbb{R}\ \text{"}\ f([a, b]) = |c, d|$

$f(a), f(b) \in f([a, b]) \to < f(a), f(b) > \subset f([a, b])$

$\forall\ y \in < f(a), f(b) >:$

$y \in f([a, b]) \to \exists\ x \in [a, b]\ \text{"}\ f(x) = y$

4. Metric Function Succesions

The punctual limit of functions does not inherit the properties of the original functions such as continuity, differentiability and integrability. Thats why an uniform limit is defined.

Inclusion of uniformly convergent mfs' in punctually convergent mfs'

Let:

- $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}(A_1, A_2)$ uniformly convergent
- $f \in \mathcal{F}(A_1, A_2)$ ꞉ $f_n \overset{n}{\rightrightarrows} f$

Then, holds:

- $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}(A_1, A_2)$ punctually convergent

Demonstration:

$\forall\ x \in A_1$:

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\exists\ n_0 \in \mathbb{N}$ ꞉ $\forall\ n \in \mathbb{N}$ ꞉ $n \geq n_0$:

$f_n(x) - f(x) < \varepsilon \rightarrow \lim\limits_{n} f_n(x) = f(x) \quad f_n \xrightarrow{n} f$

Characterization of uniformly convergent real function succesions by uniformly Cauchy function succesions

Let:

$\cdot\ (f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(A_1, \mathbb{R})$ real function succesion

Then, holds:

$\cdot\ f_n \overset{n}{\rightrightarrows} f \leftrightarrow f_n$ uniformly Cauchy

Demonstration:

$\rightarrow$):

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\forall\ x \in A_1$:

$\exists\ n_0 \in \mathbb{N}$ " $\forall\ n \in \mathbb{N}$ " $n \geq n_0$:

$|f_n(x) - f(x)| < \frac{\varepsilon}{2}$. $\forall\ n, m \in \mathbb{N}$ " $n, m \geq n_0$:

$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| <$

ε

$(f_n)_{n\in\mathbb{N}}$ uniformly Cauchy $\leftarrow$):

Follow 2 steps

Step 1: found f:

$\forall\ x \in A_1$:

$(f_n(x))_{n\in\mathbb{N}}$ Cauchy

$(f_n(x))_{n\in\mathbb{N}} \subset \mathbb{R} \to (f_n(x))_{n\in\mathbb{N}}$ convergent

$$\begin{array}{rcl} f\ :A_1 & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \lim\limits_{b} f_n(x) \end{array}$$

Step 2: $f_n \overset{n}{\rightrightarrows} f$:

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\forall\ x \in A_1$:

$$\exists\, n_0 \in \mathbb{N} \;\; „ \;\; \forall\, n, m \in \mathbb{N} \;\; „\; n, m \geq n_0:$$

$$|f_n(x) - f_m(x)| < \varepsilon$$

$$\lim_{m} |f_n(x) - f_m(x)| \leq \varepsilon$$

$$|f_n(x) - f(x)| \leq \varepsilon$$

WeierStrass' M Test

Let:

- $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(A_1, \mathbb{R})$ real function succesion $\;_{''}\;$ $\forall\ n \in \mathbb{N}$:

  $\exists\ M_n \in \mathbb{R}\ _{''}\ \sup_{A_1} f_n < M_n$

Then, holds:

- $\sum_{n\geq 1} M_n$ convergent $\to \sum_{n\geq 1} f_n$ uniformly convergent

Demonstration:

$\forall\ n \in \mathbb{N}$:

$|f_n(x)| < M_n$

$\sum_{n\geq 1} f_n$ absolutely punctually convergent

$\exists\ f \in \mathcal{F}(A_1, \mathbb{R})\ _{''}\ f_n \xrightarrow{n} f$

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\exists\ n_0 \in \mathbb{N}\ _{''}\ \forall\ k \in \mathbb{N}\ _{''}\ k \geq n_0$:

$$\sum_{n\geq k} M_n < \varepsilon$$

$$|f(x) - \sum_{n=1}^{k-1} f_n(x)| = |\sum_{n\geq k} f_n(x)| \leq \sum_{n\geq k} |f_n(x)| \leq$$
$$\leq \sum_{n\geq k} M_n < \varepsilon$$

Inheritance of Continuity by uniform continuity

Let:

$\cdot\ A_1, A_2$ metric spaces

$\cdot\ a \in A_1$

$\cdot\ (f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(A_1, A_2)$ metric function succesion $\;\;_{''}\;\; \forall\, n \in \mathbb{N}$:

f_n continuous in a

$\cdot\ f \in \mathcal{F}(A_1, A_2) \;\;_{''}\; f_n \overset{n}{\rightrightarrows} f$

Then, holds:

$\cdot\ f$ continuous in a

Demonstration:

$\forall\, \varepsilon \in \mathbb{R}^+$:

$\exists\, n_0 \in \mathbb{N} \;\;_{''}\;\; \forall\, n \in \mathbb{N} \;\;_{''}\; n \geq n_0$:

$\forall\, x \in A_1$:

$d(f_n(x), f(x)) < \frac{\varepsilon}{3}$

$\exists\, \delta \in \mathbb{R}^+ \;\;_{''}\;\; \forall\, x \in A_1 \;\;_{''}\; d(x, a) < \delta$:

$d(f_{n_0}(x), f_{n_0}(a)) < \frac{\varepsilon}{3}$

$d(f(x), f(a)) \leq$

$\leq d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(a)) + d(f_{n_0}(a), f(a)) <$

ε

Inheritance of Integrability by uniform continuity

Let:

- $(f_n)_{n\in\mathbb{N}} \subset \mathcal{R}(()[a,b])$
- $f \in \mathcal{F}(\mathbb{R},\mathbb{R})$,, $f_n \overset{n}{\rightrightarrows} f$

Then, holds:

- $f \in \mathcal{R}(()[a,b])$
- $\lim\limits_n \displaystyle\int_a^b f_n(x)dx = \int_a^b f(x)dx$

Demonstration:

$\forall\ n \in \mathbb{N}$:

$\varepsilon_n \ := \ \sup_{x\in[a,b]} |f_n(x) - f(x)| \in \mathbb{R}$

$\varepsilon_n \xrightarrow{n} 0$

$\forall\ n \in \mathbb{N}$:

$f_n - \varepsilon_n \le f \le f_n + \varepsilon_n$

$f_n - \varepsilon_n, f_n + \varepsilon_n \in \mathcal{R}([a,b])$

$\int_a^b f_n - \varepsilon_n dx = \underline{\int_a^b f_n - \varepsilon_n dx} \le \underline{\int_a^b f dx} \le \overline{\int_a^b f dx} \le$

$\le \overline{\int_a^b f_n + \varepsilon_n dx} \le \int_a^b f_n + \varepsilon_n dx$

$\int_a^b f_n dx - \varepsilon_n(b-a) \le \underline{\int_a^b f dx} \le \overline{\int_a^b f dx} \le \int_a^b f_n dx + \varepsilon_n(b-a)$

$|\overline{\int_a^b f dx} - \underline{\int_a^b f dx}| \le 2\varepsilon_n(b-a) \xrightarrow{n} 0$

$\overline{\int_a^b f dx} = \underline{\int_a^b f dx} \to f \in \mathcal{R}([a,b])$

Step 2: Existency of $\lim\limits_n \displaystyle\int_a^b f dx$:

$* \int_a^b f_n dx - \varepsilon_n(b-a) \le \int_a^b f dx \le \int_a^b f_n dx + \varepsilon_n(b-a)$

$\int_a^b f dx - \varepsilon_n(b-a) \le \int_a^b f_n dx \le \int_a^b f dx + \varepsilon_n(b-a)$

Sandwitch theorem$\to \exists \lim\limits_n \displaystyle\int_a^b f_n dx = \int_a^b f dx$

Uniform Continuity and Derivation

Let:

$\cdot\ (f_n)_{n\in\mathbb{N}} \subset C^1([a,b])$

$\cdot\ x_0 \in [a,b]$ 〃 $(f_n(x_0))_{n\in\mathbb{N}}$ convergent

$\cdot\ (f_n')_{n\in\mathbb{N}}$ uniformly convergent

Then, holds:

$\cdot\ \exists\ f \in C^1([a,b])$ 〃 $f_n \overset{n}{\rightrightarrows} f$

$\cdot\ f_n' \xrightarrow{n} f'$

Demonstration:

Follow 3 steps

Step 1: Apply Medium Value Theorem :

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\exists\ n_0 \in \mathbb{N}$ 〃 $\forall\ n, m \in \mathbb{N}$ 〃 $n \geq n_0$:

$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$

$\exists\ n_1 \in \mathbb{N}$ 〃 $\forall\ n, m \in \mathbb{N}$ 〃 $n, m \geq n_1$:

$\forall\ t \in [a,b]$:

$|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)}$

Medium Value Theorem→ $\exists \zeta \in (a,b)$ 〃

$|(f_n - f_m)(x) - (f_n - f_m)(t)| = |(f_n - f_m)'(\zeta)||t - x| \leq \frac{\varepsilon}{2}$

$|f_n(x) - f_m(x)| \leq |(f_n - f_m)(x) - (f_n - f_m)(x_0)| + |(f_n - f_m)(x_0) - (f_n - f_m)(x)| < \varepsilon$

$(f_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ uniformly Cauchy → uniformly convergent

$\exists\ f \in \mathcal{F}([a,b], \mathbb{R})$ 〃 $f_n \overset{n}{\rightrightarrows} f$

Step 2: Define ϕ_n and ϕ:

$$\begin{aligned} \phi_n : [a,b] &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} \frac{f_n(t)-f_n(x)}{t-x}, t \neq x \\ f_n'(t), t = x \end{cases} \end{aligned}$$

$$\begin{aligned} \phi : [a,b] &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} \frac{f_n(t)-f_n(x)}{t-x}, t \neq x \\ \lim\limits_{n} f_n'(t), t = x \end{cases} \end{aligned}$$

$\forall\ n \in \mathbb{N}$:

ϕ_n continuous over $[a,b]$

Step 3: ϕ_n uniformly Cauchy :

Separate 2 cases:

Case $t \neq x$:

$$|\phi_n(x) - \phi_m(x)| = |\frac{|(f_n(t)-f_m(x))-(f_n(t)-f_m(x))|}{|t-x|} \leq \frac{\varepsilon}{2(b-a)}$$

Case $t = x$:

$$|\phi_n(x) - \phi_m(x)| = |f_n'(x) - f_m'(x)|$$

f_n' uniformly convergent $\to \leq \frac{\varepsilon}{2(b-a)}$

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\exists\ n_2 \in \mathbb{N}$ „ $\forall\ n, m \in \mathbb{N}$:

$|\phi_n(t) - \phi_m(t)| < \varepsilon$

$(\phi_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ uniformly Cauchy $\to$ uniformly convergent

$\lim\limits_{n} \phi_n(t) = \phi(t) \to \phi_n \overset{n}{\rightrightarrows} \phi$

ϕ continuous over $[a,b] \to f$ differentiable over $[a,b]$

$f_n'(x) = \lim\limits_{n} f_n'(x)$

Inheritance of boundness by uniform continuity

Let:

$\cdot\ A \subset \mathbb{R} (f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(A,\mathbb{R})$ metric function succesion "

$\forall\ n \in \mathbb{N}$:

f_n bounded

$\cdot\ f \in \mathcal{F}(A,\mathbb{R})$ " $f_n \overset{n}{\rightrightarrows} f$

Then, holds:

$\cdot\ f$ bounded

Demonstration:

$\exists\ (M_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ " $\forall\ n \in \mathbb{N}$:

$|f_n(x)| \leq M_n$

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\exists\ n_0 \in \mathbb{N}$ " $\forall\ n \in \mathbb{N}$ " $n \geq n_0$:

$|f_n(x) - f(x)| < \varepsilon$

$|f(x)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x)| < \varepsilon + M_{n_0}$

Criterion where punctually convergent function succesions are uniformly convergent

Let:

- A metric space
- K compact over A
- $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(K,\mathbb{R})$ real function succesion " $\forall\ n \in \mathbb{N}$:

  f_n continuous over K

  $f_{n+1} < f_n$
- $f \in \mathcal{F}(K,\mathbb{R})$ " f continuous over $K \ \wedge\ f_n \xrightarrow{n} f$

Then, holds:

- $f_n \overset{n}{\rightrightarrows} f$

Demonstration:

$\forall\ n \in \mathbb{N}$:

$g_n := f_n - f \in \mathcal{F}(K,\mathbb{R})$

g_n continuous over K

$g_n \xrightarrow{n} 0$

$g_{n+1} \leq g_n$

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\forall\ n \in \mathbb{N}$:

$K_n := \{x \in K \mid g_n(x) \geq 0\}$

$K_n = g_n^{-1}([\varepsilon,\infty)) \to K_n$ closed

$K_n \subset K \to K_n$ compact

$\forall\ x \in \bigcap_{n\geq 0} K_n$:

$g_n(x) \geq \varepsilon$

$$\lim_{n} g_n(x) \geq \varepsilon > 0!!!$$

$$\bigcap_{n\geq 0} K_n = \varnothing$$

Finite intersection property→

$$\to \exists \, \{K_m\}_{m=1}^{r} \subset \{K_n\}_{n\geq 0} \;\text{„}\; \bigcap_{m=1}^{r} K_m = \varnothing$$

$$K_{n+1} \subset K_n \to K_r = \varnothing$$

$$\forall \, x \in K:$$

$$g_r(x) < \varepsilon$$

$$\exists \, r \in \mathbb{N} \;\text{„}\; \forall \, r' \in \mathbb{N} \;\text{„}\; r' \geq r:$$

$$|g_{r'}| < \varepsilon$$

$$g_n \overset{n}{\rightrightarrows} 0 \to f_n \overset{n}{\rightrightarrows} f$$

Weierstrass' Aproximation Theorem

Let:

$\cdot\ a, b \in \mathbb{R}$

$\cdot\ f\mathcal{C}([a,b])$

Then, holds:

$\cdot\ \exists\ (P_n)_{n\in\mathbb{N}} \subset \mathcal{F}([a,b],\mathbb{R})$ polynomials $\shortparallel\ P_n \overset{n}{\rightrightarrows} f$

Demonstration:

Follow 5 steps

Step 1: Reduce to [0,1] :

$$\begin{array}{rcl} \phi\ :[0,1] & \longrightarrow & [a,b] \\ x & \longmapsto & a+(b-a)x \end{array}$$

Step 2: Reduce to $f(0)=0$ and $f(1)=0$:

$$\begin{array}{rcl} g\ :[0,1] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x)-f(0)-x[f(1)-f(0)] \end{array}$$

g continuous over $[0,1]$

$g(0)=0\ \wedge\ g(1)=0$

$f-g$ polynomial

$P_n \overset{n}{\rightrightarrows} g \rightarrow P_n+(f-g) \overset{n}{\rightrightarrows} f$

Step 3: extend f to uniformly continuous over $\mathbb{R}$:

$$\begin{array}{rcl} f\ :\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \begin{cases} f(x), x\in[0,1] \\ 0, x\notin[0,1] \end{cases} \end{array}$$

f continuous over $(0,1)\subset[-5,5] \rightarrow f$ uniformly continuous over $[-5,5]$

$f=0$ over $\mathbb{R}\setminus[-5,5] \rightarrow f$ uniformly continuous over $\mathbb{R}\setminus[-5,5]$

f uniformly continuous over $\mathbb{R}$

Step 4: Define polynomial P_n:

$\forall\ n \in \mathbb{N}$:

$P_n\ :=\ \int_{-1}^{1} f(x+t)Q_n(t)dt$

$\forall\ x \in [0,1]$:

$x+t \in [0,1] \to t \in [-x, 1-x]$

$u\ :=\ x+t \in [0,1] P_n(x) = \int_0^1 f(u)Q_n(u-x)du =$

$= c_n \int_0^1 f(u)(1-(u-x)^2)^n du =$

$= Rc_n \int_0^1 f(u)a_j(u)du =$

$= Rc_n C$ polynomial

Step 5: $P_n \overset{n}{\rightrightarrows} f$:

$|P_n(x) - f(x)| =$

$= |\int_{-1}^{1} f(x+t)Q_n(t)dt - f(x)| =$

$\int_{-1}^{1} Q_n(t)dt = 1 \to = |\int_{-1}^{1}(f(x+t) - f(x))Q_n(t)dt| \leq$

$Q_n > 0 \to \leq \int_{-1}^{1} |f(x+t) - f(x)|Q_n(t)dt$

$\forall\ \delta \in (0,1)$:

$I_{\delta,n}\ :=\ \int_{-1}^{-\delta} |f(x+t) - f(x)|Q_n(t)dt$

$II_{\delta,n}\ :=\ \int_{-\delta}^{\delta} |f(x+t) - f(x)|Q_n(t)dt$

$III_{\delta,n}\ :=\ \int_{\delta}^{1} |f(x+t) - f(x)|Q_n(t)dt$

f continuous over $[-5,5] \to f$ bounded over $[-5,5]$

$f = 0$ over $\mathbb{R} \setminus [-5,5] \to f$ bounded over $\mathbb{R} \setminus [-5,5]$

f bounded over $\mathbb{R} \to \exists\ M \in \mathbb{R}$ " $|f| \leq M$

$I_{\delta,n} \leq 2M \int_{-1}^{\delta} Q_n(t)dt$

$\lim\limits_{n}\ I_{\delta,n} = 2M \lim\limits_{n} \displaystyle\int_{-1}^{\delta} Q_n(t)dt$

UCI theorem$\to \lim\limits_{n}\ I_{\delta,n} = \displaystyle\int_{-1}^{\delta} \lim\limits_{n}\ Q_n(t)dt = 0$

$\exists\ n_0 \in \mathbb{N}$ " $\forall\ n \in \mathbb{N}$ " $n \geq n_0$:

$$I_{\delta,n} < \frac{\varepsilon}{3}$$

$$III_{\delta,n} < \frac{\varepsilon}{3}$$

$$II_{\delta,n} \leq \frac{\varepsilon}{3} \int_{-\delta}^{\delta} Q_n(t)dt \leq \frac{\varepsilon}{3} \int_{-1}^{1} Q_n(t)dt \leq \frac{\varepsilon}{3}$$

$$\int_{-1}^{1} |f(x+t) - f(x)| Q_n(t)dt < \varepsilon$$

$$P_n \overset{n}{\rightrightarrows} f$$

Dirichlet's Criterion

Let:

- $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(\mathbb{R},\mathbb{R})$ " $\sup_x \sup_N \sum_{n=1}^{N} f_n(x) \in \mathbb{R}$
- $(g_n)_{n\in\mathbb{N}} \subset \mathcal{F}(\mathbb{R},\mathbb{R})$ " $g_n \overset{n}{\rightrightarrows} 0 \land \ \forall\ n \in \mathbb{N}$:

  $g_{n+1} < g_n$

Then, holds:

- $\sum_{n\geq 1} f_n g_n$ uniformly convergent

Demonstration:

$\forall\ k \in \mathbb{N}$:

$$F_k := \sum_{n=1}^{k} f_n$$

$$|\sum_{k=n}^{m} f_k(x)g_k(x)| = |\sum_{k=n}^{m} (F_k - F_{k-1}(x)g_k(x)| =$$

$$= |\sum_{k=n}^{m} F_k g_k(x) - \sum_{k=n-1}^{m-1} F_k(x)g_{k+1}(x)| =$$

$$= |\sum_{k=n}^{m-1} F_k(x)(g_k(x) - g_{k+1}(x)) + F_m(x)g_m(x) - F_{n-1}(x)g_n(x)|$$

bound f $\rightarrow \leq K(\sum_{k=n}^{m-1} |g_k(x) - g_{k+1}(x)| + |g_m(x)| + |g_n(x)|)$

$g_n \xrightarrow{n} 0 \rightarrow \leq K(\sum_{k\geq n} g_k(x) - g_{k+1}(x) + g_m(x) + g_n(x)) =$

$$= 2K g_n(x) \xrightarrow{n} 0$$

5. Power Series

Power Series

Cauchy-Hadamard theorem

Let:

- $\sum_{n\geq 0} a_n f_n$ power series
- D domain of convergence of $\sum_{n\geq 0} a_n x^n$
- R radius of convergence of $\sum_{n\geq 0} a_n f_n$

Then, holds:

- $\forall\ x \in \mathbb{R}\ \ _{"}\ |x| < R$:

  $x \in D$

- $\forall\ x \in \mathbb{R}\ \ _{"}\ |x| > R$:

  $x \in D^c$

- $\forall\ r \in \mathbb{R}\ \ _{"}\ |r| < R$:

  $\sum_{n\geq 0} a_n x^n$ uniformly convergent over $[-r, r]$

Demonstration:

Follow 2 steps

Step 1: Apply the root criterion :

$\forall\ x \in \mathbb{R}$:

$\sqrt[n]{|a_n x^n|} = |x| \sqrt[n]{|a_n|} = \frac{|x|}{R}$

Separate 2 cases:

Case $|x| < R$:

$\sqrt[n]{|a_n x^n|} < 1$

root criterion $\rightarrow x \in D$Case $|x| > R$:

$\sqrt[n]{|a_n x^n|} > 1$

root criterion $\rightarrow x \in D^c$

Step 2: Apply Weierstrass' M Criterion :

$\forall\ r \in \mathbb{R}$ „ $r < R$:

$\forall\ n \in \mathbb{N}$:

$f_n(x) \ := \ a_n x^n$

$M_n \ := \ a_n r^n$

$|f_n| \leq M_n$

step 1→ M_n convergent

M Criterion→ $\sum\limits_{n\geq 0} a_n x^n$ uniformly convergent over $[-r, r]$

Abel's Theorem

Let:

$\cdot \sum_{n\geq 0} a_n x^n$ power series

$\cdot\, R$ radius of convergency of $\sum_{n\geq 0} a_n x^n$ power series

Then, holds:

$\cdot \sum_{n\geq 0} a_n R^n$ convergent $\rightarrow \sum_{n\geq 0} a_n x^n$ uniformly convergent over $[0, R]$

$\cdot \sum_{n\geq 0} a_n (-R)^n$ convergent $\rightarrow \sum_{n\geq 0} a_n x^n$ uniformly convergent over $[-R, 0]$

Demonstration:

$$\forall\, n \in \mathbb{N}:$$

$$S_n := \sum_{k\geq n} a_k R^k \in \mathbb{R}$$

$$\forall\, m \in \mathbb{N} \;\;"\;\; m \geq n:$$

$$S_n^m := \sum_{k=n}^{m} a_k x^k$$

$$\forall\, x \in [0, R]:$$

$$\exists\, \lambda \in [0, 1] \;\;"\;\; x = \lambda R$$

$$\sum_{n\geq 0} a_n R^n \text{ convergent} \rightarrow \forall\, \varepsilon \in \mathbb{R}^+:$$

$$\exists\, n_0 \in \mathbb{N} \;\;"\;\; \forall\, n \in \mathbb{N} \;\;"\;\; n \geq n_0:$$

$$|S_n| < \tfrac{\varepsilon}{2} |S_n^m| = |\sum_{k=n}^{m} a_k \lambda^k R^k| =$$

$$= |\sum_{k=n}^{m} (S_k - S_{k+1}) \lambda^k| =$$

$$= |S_n \lambda^n + \sum_{k=n+1}^{m} [S_k(\lambda^k - \lambda^{k-1}) + S_{m+1}\lambda^m \leq$$

$$\leq \tfrac{\varepsilon}{2}[\lambda^n + \sum_{k=n+1}^{m} |\lambda^k - \lambda^{k-1}| + \lambda^m] \leq$$

$$\leq \tfrac{\varepsilon}{2}[\lambda^n + \sum_{k=n+1}^{m} (\lambda^{k-1} - \lambda^k) + \lambda^m] \leq \lambda^n \varepsilon \leq \varepsilon$$

Superior Limit Lemma

Let:

- $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ „ $\overline{\lim_n}\, a_n = a$
- $(b_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ „ $\lim_n\, b_n = b > 0$

Then, holds:

- $\overline{\lim_n}\, a_n b_n = ab$

Demonstration:

Sandwitch $\overline{\lim_n}\, a_n b_n$ with ab :

$\geq$):

$\exists\ (a_{n_k})_{k\in\mathbb{N}} \vdash (a_n)_{n\in\mathbb{N}}$ „ $\lim_k\, a_{n_k} = a$

$\lim_k\, a_{n_k} b_{n_k} = ab$

$ab \leq \overline{\lim_n}\, a_n b_n$

$\leq$):

$c := \overline{\lim_n}\, a_n b_n \in \overline{\mathbb{R}}$

$\exists\ (a_{n_k} b_{n_k})_{k\in\mathbb{N}} \vdash (a_n b_n)_{n\in\mathbb{N}}$ „ $\lim_k\, a_{n_k} b_{n_k} = c$

$$\lim_k\, a_{n_k} = \frac{\lim_k\, a_{n_k} b_{n_k}}{\lim_k\, b_{n_k}} = \frac{c}{b}$$

$\frac{c}{b} \leq a \to c \leq ab \to \overline{\lim_n}\, a_n b_n \leq ab$

Diferentiability of Power Series

Let:

- $\sum_{n\geq 0} a_n x^n$ power series
- R radius of convergence of $\sum_{n\geq 0} a_n x^n$
- f associated function of $\sum_{n\geq 0} a_n x^n$

Then, holds:

- f differentiable over $((-R, R))$
- $\forall\ x \in (-R, R)$:

$$f'(x) = \sum_{n\geq 0} a_n n x^{n-1}$$

Demonstration:

Follow 2 steps

Step 1: Calculate radius of convergence :

$$\frac{1}{R'} = \overline{\lim_{n}}\ \sqrt[n]{|a_n n|} = \sqrt[n]{|n|}\overline{\lim_{n}}\ \sqrt[n]{|a_n|} = \frac{1}{R}$$

$$R' = R$$

Step 2: Apply UCD theorem :

$\forall\ r \in \mathbb{R}$ " $|r| < R$:

$$\sum_{n\geq 0} a_n n x^{n-1} \text{ uc over } [-r, r]$$

UCD→ f derivable $\wedge$ $\forall\ x \in (-R, R)$:

$$f'(x) = \sum_{n\geq 0} \frac{\partial}{\partial x} a_n x^n$$

$$f'(x) = \sum_{n\geq 0} a_n n x^{n-1}$$

Integrability of Power Series

Let:

- $\sum_{n\geq 0} a_n x^n$ power series
- f associated function of $\sum_{n\geq 0} a_n x^n$
- R_f radius of convergence of $\sum_{n\geq 0} a_n x^n$

Then, holds:

- $f \in \mathcal{R}((-R,R))$
- $\forall\ x \in (-R,R)$:

  $$\int_0^x f(x)dx = \sum_{n\geq 0} \frac{a_n}{n+1} x^{n+1}$$

Demonstration:

Follow 2 steps

Step 1: Calculate radius of convergence :

$$R' := \text{ radius of convergence of } \sum_{n\geq 0} \frac{a_n}{n+1} x^{n+1} \in \overline{\mathbb{R}}$$

$$\tfrac{1}{R'} = \overline{\lim_n} \sqrt[n]{\frac{|a_n|}{n+1}} = \sqrt[n]{\frac{1}{n+1}} \overline{\lim_n} \sqrt[n]{|a_n|} = \frac{1}{R}$$

$$R' = R$$

Step 2: Apply UCI theorem :

$\forall\ r \in \mathbb{R}$ „ $|r| < R$:

$$\sum_{n\geq 0} \frac{a_n}{n+1} x^{n+1} \text{ uc over } [-r,r]$$

UCI$\rightarrow$ $f \in \mathcal{R}((-R,R)) \wedge\ \forall\ x \in (-R,R)$:

$$\int_0^x f(x)dx = \sum_{n\geq 0} \int_0^x f(x)dx a_n x^n$$

$$\int_0^x f(x)dx = \sum_{n\geq 0} \frac{a_n}{n+1} x^{n+1}$$

Exponential Properties

Then, holds:

- $exp \in C^{\infty}(\mathbb{R})$
- $exp' = exp$
- $exp(a+b) = exp(a) + exp(b)$
- $exp > 0$
- exp increasing
- $exp(a) = exp(1)^a$
- $\forall\ n \in \mathbb{N}$:

$$exp(-x)x^n \xrightarrow{n} 0$$

Demonstration:

Follow 7 steps

Step 1: $exp \in C^{\infty}(\mathbb{R})$:

$$R = \lim_n \frac{(n+1)!}{n!} = \lim_n\ n+1 = \infty$$

Step 2: $exp' = exp$:

UCD theorem→ $\forall\ x \in \mathbb{R}$:

$$exp'(x) = \sum_{n\geq 1} \frac{n}{n!}x^{n-1} =$$

$$= \sum_{n\geq 1} \frac{1}{(n-1)!}x^{n-1} \sum_{n\geq 0} \frac{1}{n!}x^n = exp(x)$$

Step 3: $exp(a+b) = exp(a)exp(b)$:

Exponential lemma→ $exp(a)exp(b) = \sum_{n\geq 0} \sum_{k=0}^{n} \frac{1}{k!}a^k \frac{1}{(n-k)!}b^{n-k} =$

$$= \sum_{n\geq 0} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}a^k b^{n-k} =$$

$$= \sum_{n\geq 0} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} =$$

$$= \sum_{n\geq 0} \frac{1}{n!}(a+b)^n = exp(a+b)$$

Step 4: $exp > 0$:

$exp(0) = 1$

$\forall\ x \in \mathbb{R}$ „ $x \geq 0$:

$exp(x) > 0$

$exp(-x)exp(x) = exp(0) = 1$

$exp(-x) = \frac{1}{exp(x)} > 0$

Step 5: exp increasing :

$exp' = exp > 0$

Step 6: $exp(a) = exp(1)^a$:

$\forall\ n \in \mathbb{N}$:

step 3$\rightarrow$ $exp(n) = exp(1)^n$

$\forall\ z \in \mathbb{Z}$ „ $z < 0$:

$\exists\ n \in \mathbb{N}$ „ $z = -n$

$exp(z) = exp(-n) = \frac{1}{exp(n)} = exp(1)^{-n} = exp(1)^z$

$\forall\ q \in \mathbb{Q}$ „ $q \geq 0$:

$\exists\ n, m \in \mathbb{N}$ „ $q = \frac{n}{m}$

$n = qm$

$exp(n) = exp(qm) = exp(\sum_{i=1}^{m} q) = exp(q)^m$

$exp(q) = exp(n)^{\frac{1}{m}} = exp(1)^{\frac{n}{m}} = exp(1)^q$

$\forall\ r \in \mathbb{R}$:

exp continuous over $\mathbb{R}$ $\rightarrow$ $exp(r) = \lim_{q \to r} exp(1)^q = exp(1)^r$

Step 7: $exp(-x)x^n \xrightarrow{n} 0$:

$\forall\ n \in \mathbb{N}$:

$\forall\ x \in \mathbb{R}\ _{''}\ x \geq 0$:

$exp(x) > \frac{x^{n+1}}{(n+1)!}$

$0 < exp(-x)x^n < \frac{(n+1)!}{x^{n+1}}$

Sandwitch theorem$\rightarrow$ $exp(-x)x^n \xrightarrow{n} 0$