I. Linear maps

Real sequence of order 2

Let:

- $p, q \in \mathbb{R}$
- $\forall\ a, b \in \mathbb{R}$:

  $x_0 \ := \ a$

  $x_1 \ := \ b$

  $\forall\ n \in \mathbb{N}$:

  $x_{n+2} \ := \ px_{n+1} + q_{x_n}$

Study:

- $\lim\limits_n \dfrac{x_n}{x_{n+1}}$

Start:

Consider:

$$A = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$$

$$A \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}$$

Eigenvalue analysis:

$$\chi_A(t) = t^2 - \mathrm{tr}(A)\, t + \det(A) = t^2 - pt - q$$

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \chi_A(\lambda) = 0\} = \{\tfrac{p}{2}\}$$

$$\lambda \ := \ \tfrac{p}{2}$$

$$E_\lambda(A) = \mathrm{Ker}(A - \lambda \mathbb{1}) = \{(x, y) \in \mathbb{R}^2 \mid x = \lambda y\}$$

$$\dim(A) = 2 \ \wedge \ \sigma(A) = \{\lambda\} \ \wedge \ \gamma_A(\lambda) = 1 \rightarrow A \text{ no diagonalizable}$$

Jordan form:

$$J := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

$$v_2 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_1 := (A - \lambda \mathbb{1})v_2 = \begin{pmatrix} p - \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$$

$$C := (v_1|v_2) = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}$$

$$C^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}$$

$$A = CJC^{-1} \rightarrow A^n = CJ^nC^{-1} = \begin{pmatrix} (n+1)\lambda^n & -n\lambda^{n+1} \\ n\lambda^{n-1} & (1-n)\lambda^n \end{pmatrix}$$

Sequence analysis:

$$\forall \ a, b \in \mathbb{R} :$$

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = A^n \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} -n\lambda^{n+1}a + (n+1)\lambda^n b \\ (1-n)\lambda^n a + n\lambda^{n-1} b \end{pmatrix}$$

$$\lim_n \frac{x_n}{x_{n+1}} = \lim_n \frac{\lambda^{n-1}(bn + (1-n)\lambda a)}{\lambda^{n-1}(-n\lambda^2 a + (n+1)\lambda b)} =$$

$$\lim_n \frac{(b - \lambda a)n + \lambda a}{(\lambda b - \lambda^2 a)n + \lambda b} = \frac{1}{\lambda} = \frac{2}{p}$$

Graphical interpretation:

Let be l the line that passes through the origin and $\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}$

The slope of l is $\frac{x_{n+1}}{x_n}$

$$\lim_n \frac{x_{n+1}}{x_n} = \lim_n \left(\frac{x_n}{x_{n+1}}\right)^{-1} = \lambda$$

the slope of l approaches the slope of $E_\lambda(A)$ as n increases

The orbit of $\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}$ is a curve tangent to the eigenvectors line