Dynamical systems

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Contents

Block I

Definitions

1. Discrete dynamical systems

Dynamical system

Let:

$\cdot$ M manifold

$\cdot$ T monoid

$\cdot$ $\phi : M \times T \to M$

Then, (M, T, ϕ) is a dynamical system if:

$\cdot$ $\forall\ x \in X$:

$$\phi(x, 0) = 0$$

$\forall\ t_1, t_2 \in T$:

$$\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$$

Dimension

Let:

$\cdot$ (M, T, ϕ) dynamical system

We name dimension of (M, T, ϕ) to:

$\cdot$ $\dim(M)$

We denote:

$\cdot$ $dim(M) = n : (M, T, \phi)$ n-D

Discrete & Continuous

Let:

- (M, T, ϕ) dynamical system

Then, (M, T, ϕ) is discrete if:

- $T \underset{\sim}{\subset} \mathbb{N}$

Then, (M, T, ϕ) is continuous if:

- $T \subset \mathbb{R}$
- T open

Defined by a function

Let:

- (M, T, ϕ) dynamical system
- $f : M \to M$

Then, (M, T, ϕ) is a dynamical system defined by f if:

- $T = \mathbb{N}$
- $\begin{array}{rccc} \phi : & M \times \mathbb{N} & \longrightarrow & M \\ & (x, n) & \longmapsto & f^n(x) \end{array}$

We denote:

- (M, T, ϕ) dynamical system defined by f : $(M, \mathbb{N}, f)$
- $f \in \mathcal{C}^n(M)$: $(M, \mathbb{N}, f)$ $\mathcal{C}^n$

Orbit

Let:

· $(M, \mathbb{N}, f)$ functional dynamical system

· $x \in M$

We name orbit of x to:

· $\{f^n(x)\}_{n \in \mathbb{N}}$

We denote:

· $o(x)$

Periodicity

Let:

· $(M, \mathbb{N}, f)$ dynamical system

· $x \in M$

· $m \in \mathbb{N}$

Then, x is a m-periodic point if:

· $f^m(x) = x$

We denote:

· $\{x \in M \mid f(x) = x\}$: $\text{Fix}(f)$

Stability

Let:

- $f : \mathbb{R}^n \to \mathbb{R}^n$
- $p \in \mathbb{R}^n$ m-periodic point

Then, p is stable if:

- $\forall\ \varepsilon \in \mathbb{R}^+ :$

  $\exists\ \delta \in \mathbb{R}^+ :$

  $\forall\ x \in B(p, \delta) :$

  $\forall\ n \in \mathbb{N} :$

  $f^{nm}(x) \in B(p, \varepsilon)$

Then, p is unstable if:

- p not stable

Attractive & Repulsive

Let:

$\cdot\, f \,:\, \mathbb{R}^n \to \mathbb{R}^n$

$\cdot\, p \in \mathbb{R}^n$ m-periodic point

Then, p is attractive if:

$\cdot\, p$ stable

$\cdot\ \exists\, \varepsilon \in \mathbb{R}^+ \,:$

$\forall\, x \in B(p, \varepsilon) \,:$

$f^{nm}(x) \xrightarrow{n} p$

Then, p is repulsive if:

$\cdot\, p$ attractive by f^{-1}

$\cdot\, \forall\, \mathcal{U} \subset M$ „ $\mathcal{U}$ open $\wedge\, x \in \mathcal{U} \,:$

$\forall\, x' \in \mathcal{U}$ „ $x' \neq x \,:$

$\exists\, N \in \mathbb{N} \,:$

$\forall\, n \in \mathbb{N}$ „ $n \geq N \,:$

$f^{nm}(x') \notin \mathcal{U}$

Fixed point character

Let:

· $(M, \mathbb{N}, f)$ functional dynamical system

We name Fixed point character to:

$$\cdot \quad f : \begin{array}{ccc} \mathrm{Fix}(f) & \longrightarrow & \{-1, 0, 1\} \\ x & \longmapsto & \begin{cases} +1 & x \text{ repulsive} \\ -1 & x \text{ attractive} \\ 0 & x \text{ no attractive no repulsive} \end{cases} \end{array}$$

We denote:

· $f : \chi_f$

Attraction set

Let:

· $(M, \mathbb{N}, f)$ dynamical system

· $x \in M$ attractive m-periodic point

· $o(x)$ orbit of x

We name attraction set of x to:

· $\{y \in M \mid \exists\, x' \in o(x) : f^{nm}(y) \xrightarrow{n} x'\}$

We denote:

· $A(x)$

Multiplier

Let:

· $(M, \mathbb{N}, f)$ $\mathcal{C}^1$ dynamical system

· $x \in M$

We name multiplier of x to:

· $f'(x)$

We denote:

· $m(x)$

· $|m(x)| = 1$: x neutral point

Feeble point

Let:

· $(M, \mathbb{N}, f)$ $\mathcal{C}^3$ dynamical system

· $x \in M$

Then, x is feeble point if:

· x neutral point

· $f''(x) = 0$

Sarkovskii's order

We name Sarkovskii's order to:

$\cdot\ a = 2^n a', b = 2^m b'$

$$\cdot\ a <_s b \leftrightarrow \begin{cases} m < n & a' = b' = 1 \\ & a' = 1, b' \neq 1 \\ a' < b' & a' = b' \neq 1 \\ n < m & 1 \neq a' \neq b' \end{cases}$$

Chaos

Let:

$\cdot\ (\mathbb{R}, \mathbb{N}, f)$ dynamical system

Then, $(\mathbb{R}, \mathbb{N}, f)$ is chaotic if:

$\cdot$ $\text{Fix}(f)$ dense in $\mathbb{R}$

$\cdot$ $\exists\ x \in \mathbb{R}$:

$o(x)$ dense in $\mathbb{R}$

$\cdot\ \forall\ x \in \mathbb{R}$:

$\exists\ \varepsilon \in \mathbb{R}^+$:

$\forall\ \delta \in \mathbb{R}^+$:

$\exists\ \tilde{x} \in B(x, \delta)$:

$$\lim_n\ o(\tilde{x}) \notin B(x, \varepsilon)$$

Topological equivalence

Let:

- $(M, \mathbb{N}, f_1), (M, \mathbb{N}, f_2')$ dynamical systems

Then, $(M, \mathbb{N}, f_1)$ is topologically equivalent to $(M, \mathbb{N}, f_2)$ if:

- $\text{Fix}(f) = \text{Fix}(f')$
- $\forall\ x \in \text{Fix}(f)$:

  $\chi_{f_1}(x) = \chi_{f_2}(x)$

We denote:

- $(M, \mathbb{N}, f) \sim (M, \mathbb{N}, f')$

Bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$

Then, λ_0 is a bifurcation value if:

- $\forall\ \varepsilon \in \mathbb{R}^+$:

  $\exists\ \lambda_1, \lambda_2 \in B(\lambda_0, \varepsilon)$:

  $(M, \mathbb{N}, f_{\lambda_1}) \nsim (M, \mathbb{N}, f_{\lambda_2})$

Saddle-node bifurcation

Let:

· $\Lambda \subset M$

· $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ

· $\lambda_0 \in \Lambda$

· $x_0 \in M$

Then, λ_0 is a saddle-node bifurcation value at x_0 if:

· $x_0 \in \mathrm{Fix}(f_{\lambda_0})$

· x_0 neutral point of f_{λ_0}

· $\partial_\lambda f_\lambda(x_0) \neq 0$

· $\partial_{xx} f_\lambda(x_0) \neq 0$

Pitchfork bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is Pitchfork bifurcation value at x_0 if:

- $x_0 \in \mathrm{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{x^2} f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{\lambda x} f_\lambda(x_0, \lambda_0) \neq 0$
- $\partial_{x^3} f_\lambda(x_0, \lambda_0) \neq 0$

Period doubling bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is Period doubling bifurcation value at x_0 if:

- λ_0 Pitchfork bifurcation value at x_0 of f^2

Invariant curve

Let:

- γ differentiable curve
- $p \in \mathbb{R}^n$

Then, γ is invariant if:

- $\forall\ x \in \gamma* :$

  $o(x) \subset \gamma*$

Then, γ is converges to $p$if:

- $\forall\ x \in \gamma* :$

  $o(x) \xrightarrow{n} p$

2. 2-D linear dynamical systems

Linear system

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system

Then, $(M, \mathbb{N}, f)$ is linear if:

- $\exists\ A \in \mathcal{M}_{n\times n}(\mathbb{R})$:

$$\begin{array}{rccc} f : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \longmapsto & Ax \end{array}$$

Multiplier

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$

We namemultiplier of x to:

- $Df(p)$

Block II

Propositions

1. Discrete dynamical systems

introduction

Fixed points theorem

Let:

- $I \subset \mathbb{R}$ open
- $f : I \to I$ differentiable
- $x \in I$

Then, holds:

- $|f'(x)| < 1 \to x$ attractive
- $|f'(x)| > 1 \to x$ repulsive

Demonstration:

demonstration

Attractiveness of periodic points does not involve the chosen point

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$ n-periodic point
- $\{x_i\}_{i=1}^{r}$ orbit of x

Then, holds:

- x attractive $\leftrightarrow \forall\ x' \in o(x)$:

  x' attractive

Demonstration:

$\forall\ x' \in o(x)$:

$$f^{n'}(x') = \prod_{i=1}^{r} f'(x_i) = f^{n'}(x)$$

Partition of attraction set

Let:

$\cdot$ $(M, \mathbb{N}, f)$ functional dynamical system

$\cdot$ x n-periodic point

$\cdot$ $o(x)$ orbit of x

Then, holds:

$\cdot$ $\forall\ x' \in o(x)$:

$\exists\, \mathcal{U} \subset M$ open :

$\forall\ y \in \mathcal{U}$:

$$f^n(y) \xrightarrow{n} x'$$

Demonstration:

demonstration

Homeomorphisms are monotonous

Let:

· $f : \mathbb{R} \to \mathbb{R}$ homeomorphism

Then, holds:

· f monotonous

Demonstration:

no demonstration

Homeomorphisms and n-periodic points

Let:

$\cdot\ f\ :\ \mathbb{R} \to \mathbb{R}$ homeomorphism (M, T, ϕ) dynamical system defined by f

Then, holds:

$\cdot\ \forall\ n \in \mathbb{N}\ :$

$\nexists\ x \in M$ 〃 x n-periodic point

Demonstration:

graphically

Sarkovskii's theorem

Let:

- $f : I \to I$
- $(M, \mathbb{N}, f)$ dynamical system

Then, holds:

- $\exists\, x \in M$:

  $o(x)$ k-period

- $\to \forall\, l \in \mathbb{N}$ " $l > k$:

  $\exists\, x' \in M$:

  x' l-period

Invariance of stability over periods

Let:

- $(\mathbb{R}^n, \mathbb{N}, f)$ n-D dynamical system
- $p \in \mathbb{R}^n$ k-periodic point
- χ character of periodic points

Then, holds:

- $\exists\ \sigma \in \mathrm{Im}(\chi)$:

  $\forall\ x \in o(p)$:

  $\chi(x) = \sigma$

Demonstration:

i will

2. 2-D linear dynamical systems

Invariance of stability over orbits

Let:

· $(M, \mathbb{N}, f)$ functional dynamical system

· $x \in M$

Then, holds:

· $\forall\ x' \in o(x)$:

$$\chi(x') = \chi(x)$$

Demonstration:

Follow 2 steps

Step 1 : $falta$:

$rows$

Step 2 : $attractiveness$:

$$\chi(x) = -1$$

$$\exists\ \varepsilon \in \mathbb{R}^+ :$$

$$x \in B_\varepsilon(x) \to f^{2n}(x) \xrightarrow{n} x$$

$f \in \mathcal{C}^0(M) \to \exists\, \varepsilon_1 \in \mathbb{R}^+ :$

$$f(B_{\varepsilon_1}(x_1)) \subset B_\varepsilon(x)$$

$x \in B_{\varepsilon_1}(x_1) \to f(x) \in B_\varepsilon(x) \to f^{2n-1}(f(x)) \xrightarrow{n} x$

falta

Linear property

Let:

· $(M, \mathbb{N}, f)$ linear dynamical system

Then, holds:

· $\forall\ a, b \in \mathbb{R}$:

$\forall\ x, y \in M$:

$$f(ax + by) = af(x) + bf(y)$$

Demonstration:

matrius

Fixed points of linear applications

Let:

- $(M, \mathbb{N}, f)$ linear dynamical system

Then, holds:

- $0 \in \text{Fix}(f)$

Demonstration:

demonstration

Jordan form of 2-D real linear maps

Let:

$\cdot\ A \in \mathcal{M}_{2\times 2}(\mathbb{R})$

$\cdot\ \chi_A(t)$ characteristic polynomial of A

Then, holds:

$\cdot\ \exists\ \beta$ base of K :

$$\begin{cases} A = \lambda, 0, 0, \mu & \#Z(\chi_A(t)) = 2 \\ A = \lambda, 1, 0, \lambda & \#Z(\chi_A(t)) = 1 \\ A = \alpha, \beta, -\beta, \alpha & \#Z(\chi_A(t)) = 0 \end{cases}$$

Demonstration:

demonstration

Topology of 2-D real linear maps

Let:

- $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$

- $\lambda \neq \mu$ eigenvalues of A

Then, holds:

- $|\lambda|, |\mu| < 1 \rightarrow (0,0)$ attractive

- $|\lambda| > |\mu| \rightarrow$ tangent to y = 0

- $|\mu| > |\lambda| \rightarrow$ tangent to x = 0

- $|\mu| = |\lambda| \rightarrow$ only invariant lines

·

- $|\lambda|, |\mu| > 1 \rightarrow (0,0)$ repulsive

- equivalent to other case

Demonstration:

demonstration

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Block III

Examples

1. One-dimensional discrete dynamical systems

$Examplesofwhatareandwhatarenotone-dimensionaldynamicalsystems$

logistic function

Let:

$\cdot$ (M, T, ϕ) logistical dynamical system defined by f

Then, holds:

$\cdot$ $Fix(f) = \{0, \frac{a-1}{a}\}$

$\cdot$ $Per_2(f) =$

Demonstration:

demonstration

Quadratic function

Let:

$$\cdot \quad \begin{array}{rrcl} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & a - x^2 \end{array}$$

$\cdot$ (M, T, f_c) dynamical system family

Then, f is bifurcates in -1/4 :

$f_{-\frac{1}{4}}(x) = x \leftrightarrow x = -\frac{1}{2}$

$f'_{-\frac{1}{4}}(x) = -2x$

$f'_{-\frac{1}{4}}(-\frac{1}{2}) = 1$

$\partial_a f = 1 \neq 0$

$\partial_{x^2} f = -2 \neq 0$

$sgn(1 * -2) = - \rightarrow -\frac{1}{2}$ SN

Henon's application

Let:

$$\cdot \quad f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (-x^2+0.4y,x) \end{array}$$

Study:

$\cdot$ Fixed points of f

Demonstration:

$(0,0),(-0$

$6,-0$

$6)$ fixed points

Block IV

Problems

MODELS I SISTEMES DINÀMICS

Llista 1: Aplicacions unidimensionals

B.1. Trobeu els punts fixos i les òrbites de període 2 de les següents funcions. En el cas que apareixin paràmetres, feu-ho en funció d'aquests.

(a) * $f(x) = 2x(1-x)$, on $x \in \mathbb{R}$.

(b) * $f_c(x) = x^2 + c$, on $x, c \in \mathbb{R}$ (només punts fixos).

(c) $f(x) = x^2 + 1$, on $x \in \mathbb{R}$.

(d) $f_{a,b}(x) = ax + b$, on $a, b, x \in \mathbb{R}$.

(e) $f(x) = 2x^2 - 5x$, on $x \in \mathbb{R}$.

B.2. Fent servir anàlisi gràfic, dibuixeu el retrat de fases de

(a) $f(x) = x^2, \quad x \in \mathbb{R}$.

(b) $f(x) = x(1-x), \quad x \in \mathbb{R}$.

(c) $f_a(x) = ax, \quad x \in \mathbb{R}$, pels diferents valors de $a \in \mathbb{R}$.

B.3. * Trobeu els punts fixos atractors i les seves conques d'atracció per a la funció $f(x) = \frac{3x - x^3}{2}$, per $|x| \leq \sqrt{3}$.

B.4. Per a la funció logística $f_a(x) = ax(1-x)$, calculeu els punts fixos i els cicles de període 2 en funció del paràmetre, i determineu-ne l'estabilitat.

1. Estudieu el comportament asimptòtic de la successió $\{x_n\}_{n\in\mathbb{N}}$, pels diferents valors de x_0 indicats.

(a) * $x_{n+1} = \frac{\sqrt{x_n}}{2}$, $x_0 \geq 0$.

(b) $x_{n+1} = \frac{x_n}{2} + \frac{2}{x_n}$, $x_0 \geq 2$.

2. Donada la successió $x_{n+1} = \frac{x_n + 2}{x_n + 1}$,

(a) Trobeu el límit $L = \lim_{n\to\infty} x_n$ per a $x_0 \geq 0$.

(b) Descriviu el conjunt dels $x_0 < 0$ pels quals el límit $\lim_{n\to\infty} x_n$ existeix i no és igual a L, o bé no existeix. (Per exemple $x_0 = -1$).

3. **(Examen 2011)** Considereu el sistema dinàmic real definit per $x_{n+1} = \frac{x_n}{4} + x_n^3$. Trobeu el comportament asimptòtic de les òrbites per a tota condició inicial $x_0 \in \mathbb{R}$. Justifiqueu rigorosament les vostres afirmacions.

4. Demostreu rigurosament que $f(x) = \sin(x)$ té $x = 0$ com atractor global.

5. Demostreu que si $f : \mathbb{R} \to \mathbb{R}$ és derivable, x_0 és un punt fix i $|f'(x_0)| > 1$ llavors x_0 és un punt fix repulsor.

6. Sigui $f : \mathbb{R} \to \mathbb{R}$ de classe $\mathcal{C}^\infty$ i sigui x_0 un punt fix tal que $f'(x_0) = 1$. Doneu criteris sobre les derivades d'ordre superior, per determinar el retrat de fase local al voltant de x_0. Apliqueu-ho a determinar l'estabilitat dels punts fixos de $x^3 - x$.

1. One-dimensional discrete dynamical system

introduction

Decreasing function orbits

Let:

$\cdot$ *declarations*

$\cdot$

Show that:

$\cdot$ *statements*

$\cdot$

Demonstration:

f corta en un punto

f decreasing $\rightarrow f^2$ increasing

$f^{2n} \xrightarrow{n}$ fixed point of f

9. Periodic points

Let:

$$\cdot \quad \begin{array}{rccc} f: & \mathbb{R}\times\mathbb{R}^{+} & \longrightarrow & \mathbb{R} \\ & (x,r) & \longmapsto & r\frac{x}{1+x^2} \end{array}$$

Study:

· Periodic points of f

Demonstration:

Graphical analysis :

f odd

f has 2 extrema in ± 1

$f \xrightarrow{n} 0$

Fixed points :

$f(x) = x \leftrightarrow x = \pm\sqrt{r-1}$

$f'(\pm\sqrt{r-1}) = \frac{2-r}{r}$

n-periodic points :

$f^n(x) = x$

10. Global orbit analysis

Let:

- $f : \mathbb{R}^+ \to \mathbb{R}^+ \in \mathbb{C}^\infty$
- $f(0) = 0$
- $p \in \mathbb{R}^+ \setminus \{0\}$,, $f'(p) \geq 0$
- f' decreasing

Show that:

- $\forall\, x \in \mathbb{R}^+ \setminus \{0\}$:

$$f^n(x) \xrightarrow{n} p$$

Demonstration:

f' decreasing $\to f'' < 0 \to f$ concave

f positive $\to f$ has no extrema $\to f' > 0 \to f$ increasing

f has only one fixed point

Suppose 2 fixed points : p, p'

$IVT \to \exists\, c \in (0, p')$:

$$f'(c) = 1$$

$f'(p) < 1 \to p$ attractive $IVT \to$ dont exist more fixed points $\to f'(c') = 1 \nleq 1$

$\forall\, x \in (0, p)$:

$$f(x) > x$$

$\forall\, x \in \mathbb{R}$,, $x > p$:

$$f(x) < x$$

f increasing $\to f([0, p]) = [0, p]$

Block V

Laboratory

1. Fixed points cardinality

II. Martin Azpillaga

Let:

$\cdot\, f\ :\ [0,1] \to [0,1] \in \mathcal{C}^2([0,1])$

$\cdot\, f(1) < 1$

$\cdot\, f'' > 0 \in [0,1]$

Show that:

$\cdot\, \#\ \{x \in [0,1] \mid f(x) = x\} = 1$

Demonstration:

$\#\ \{x \in [0,1] \mid f(x) = x\} \geq 1$:

Case $f(0) = 0$:

0 fixed point

Case $f(0) > 0$:

$$g\ :\ \begin{array}{ccc} [0,1] & \longrightarrow & [-1,1] \\ x & \longmapsto & f(x) - x \end{array} \in \mathcal{C}^2([0,1])$$

$g(0) = f(0) - 0 > 0$

$g(1) = f(1) - 1 < 0$

Bolzano's theorem:

$\exists\ x \in (0,1)$:

$g(x) = 0$

$f(x) = x$

$\#\ \{x \in [0,1] \mid f(x) = x\} \leq 1$:

$g'' > 0$ over $[0,1]$

Rolle's theorem:

$\#\ \{x \in (0,1) \mid g'(x) = 0\} \leq 1$

$\#\ \{x \in (0,1) \mid g(x) = 0\} \leq 2$

$\#\ \{x \in (0,1) \mid f(x) = x\} \leq 2$

$f'' > 0$ over $[0,1]$

Monotonicity test:

f' increasing in $[0,1]$

$\forall\ a < b \in [0,1)\ \ {}_{''}\ \ f(a) = a, f(b) = b$:

Mean Value Theorem:

$\exists\ c \in (a,b)$:

$f'(c) = \frac{f(b)-f(a)}{b-a} = 1$

$\exists\ d \in (b,1)$:

$$f'(d) = \frac{f(1)-f(b)}{1-b} < 1$$
$$f' \text{ increasing} \to f'(c) < f'(b) < f'(d)$$
$$1 < f'(b) < 1 \text{ absurd}$$
$\therefore)\ \#\{x \in [0,1] \mid f(x) = x\} = 1$

$$\boxed{\textbf{III. Bifurcation Theory}}$$

Bifurcation diagram

Let:

$\cdot \begin{array}{rccc} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (a,x) & \longmapsto & x^3 - 3x^2 + (5-a)x - 2 + a \end{array}$

$\cdot\ \forall\ a \in \mathbb{R}$:

$$\begin{array}{rccc} f_a: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & x^3 - 3x^2 + (5-a)x - 2 + a \end{array}$$

Study:

$\cdot$ Bifurcations of $(\mathbb{R}, \mathbb{N}, \{f_a\}_{a\in\mathbb{R}})$

Start:

Fixed points:

$f_a(x) = x \leftrightarrow x^3 - 3x^2 + (4-a)x - 2 + a = 0 \leftrightarrow (x-1)(x^2-2x+2-a) = 0$

$\leftrightarrow x = 1 \ \vee\ x^2 - 2x + 2 - a = 0 \leftrightarrow x \in \{1, 1 \pm \sqrt{a-1}\}$

$\forall\ a \in \mathbb{R}$ „ $a \leq 1$:

$$\mathrm{Fix}(f_a) = \{1\}$$

$\forall\ a \in \mathbb{R}$ „ $a > 1$:

$$\mathrm{Fix}(f_a) = \{1, 1 \pm \sqrt{a-1}\}$$

Stability:

$\partial_x f(a,x) = 3x^2 - 6x + 5 - a$

$\partial_{x^2} f(a,x) = 6x - 6$

$\partial_{x^3} f(a,x) = 6$

$|\ \partial_x f(a,1)\ | < 1 \leftrightarrow |\ 2 - a\ | < 1 \leftrightarrow a \in (1,3)$

$\partial_{x^2} f(1,1) = 0, \ \ \partial_{x^3} f(1,1) > 0$

$\partial_{x^2} f(3,1) = 0, \ \ \partial_{x^3} f(3,1) > 0$

$\forall\ a \in (1,3)\ :$

  1 attractive

$\forall\ a \in \mathbb{R} \setminus (1,3)\ :$

  1 repulsive

$\forall\ a \in \mathbb{R}\ \ {}_{''}\ \ a > 1\ :$

  $\left|\ \partial_x f(a, 1 \pm \sqrt{a-1})\ \right| = |\ 2a - 1\ | > 1$

  $1 \pm \sqrt{a-1}$ repulsive

Pitchfork bifurcation at 1 :

$\partial_a f(1,1) = 1 - 1 = 0$

$\partial_{x^2} f(1,1) = 6 - 6 = 0$

$\partial_{ax} f(1,1) = -1 \neq 0$

$\partial_{x^3} f(1,1) = 6 \neq 0$

Period-doubling bifurcation at 3 :

$\partial_a f(3,1) = 0 \rightarrow \partial_a f^2(3,1) = 0$

$\partial_{x^2} f(3,1), \partial_{x^2} f(3, f(3,1)) = 0 \rightarrow \partial_a f^2(3,1) = 0$

$$\partial_{ax} f^2(3,1) = 2 \neq 0$$

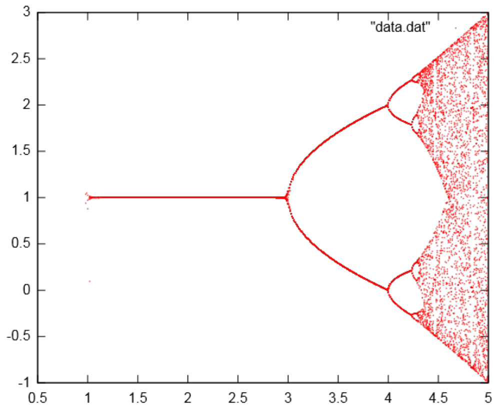
$$\partial_{x^3} f^2(3,1) = -12 \neq 0$$

Source Code

```
#include "stdio.h"
#include "stdlib.h"
#include "math.h"
#include "string.h"

void plot( char *input_file, char *output_file )
{
  FILE *gnuplot;
  gnuplot = popen("gnuplot", "w");
  if( output_file )
  {
    fprintf(gnuplot, "set term svg\n");
    fprintf(gnuplot, "set out \"%s\"\n", output_file );
  }
  fprintf(gnuplot, "plot \"%s\" with dots\n", input_file);
  fflush(gnuplot);
  fclose(gnuplot);
}
double example_function( double param, double point )
{
  return pow(point,3) - 3*pow(point,2) + (5-param)*point - 2 + param;
}

void bifurcation_diagram( int param_min, int param_max, double param_step,
 int point_min, int point_max, int num_points,
 double (*f)(double,double), int num_iter, int tolerancy)
{
  FILE* file;
  double param, point;
  int i,j;

  srand(time(NULL));
  file = fopen("data.dat","w");

  for ( param = param_min; param < param_max; param += param_step )
  {
    for ( i = 0; i < num_points; i++ )
    {
      point = point_min + ((double) rand() / (double) RAND_MAX) * (point_max-
          point_min);

      for ( j = 0; j < num_iter && abs(point) < tolerancy; j++ )
      {
        point = (*f)(param,point);
      }

      if(abs(point) < tolerancy)
      {
        fprintf(file,"%lf %lf\n",param, point);
      }
    }
  }
  plot( "data.dat", "graph.svg" );
}
```

```
int main(int argc, char const *argv[])
{
  bifurcation_diagram( 0, 5, 10e-3, 0, 5, 100, &example_function, 100, 10e1);

  return 0;
}
```

Bifurcation Diagram



IV. Linear maps

Real sequence of order 2

Let:

- $p, q \in \mathbb{R}$
- $\forall\ a, b \in \mathbb{R}$:

$$x_0 \ := \ a$$

$$x_1 \ := \ b$$

$$\forall\ n \in \mathbb{N} :$$

$$x_{n+2} \ := \ px_{n+1} + q_{x_n}$$

Study:

- $\lim\limits_{n} \dfrac{x_n}{x_{n+1}}$

Start:

Consider:

$$A = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$$

$$A \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}$$

Eigenvalue analysis:

$$\chi_A(t) = t^2 - \mathrm{tr}(A)\, t + \det(A) = t^2 - pt - q$$

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \chi_A(\lambda) = 0\} = \{\tfrac{p}{2}\}$$

$$\lambda \ := \ \tfrac{p}{2}$$

$$E_\lambda(A) = \mathrm{Ker}(A - \lambda \mathbb{1}) = \{(x, y) \in \mathbb{R}^2 \mid \text{−−−−−−−−−−−}\}$$

$\dim(A) = 2 \ \wedge \ \sigma(A) = \{\lambda\} \ \wedge \ \gamma_A(\lambda) = 1 \rightarrow A$ no diagonalizable

Jordan form:

$$J \ := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$
$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$
$$v_2 \ := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_1 \ := \ (A - \lambda \mathbb{1})v_2 = \begin{pmatrix} p - \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$$
$$C \ := \ (v_1 | v_2) = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}$$
$$C^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$A = CJC^{-1} \rightarrow A^n = CJ^nC^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}$$

Sequence analysis:

$\forall \ a, b \in \mathbb{R}$:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = A^n \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} 99 \\ 99 \end{pmatrix}$$
$$\lim_n \ \frac{x_n}{x_{n+1}} = \lambda = \frac{p}{2}$$

Graphical interpretation:

$asdfghjkl$