

Complex Analysis

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Contents

Block I

Definitions

1. The field of complex numbers

introduction

The field of complex numbers

Let:

$$\begin{aligned} \cdot \quad & + : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ & ((a, b), (c, d)) \longmapsto (a + c, b + d) \\ \cdot \quad & \cdot : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ & ((a, b), (c, d)) \longmapsto (ac - bd, ad + bd) \end{aligned}$$

We name the field of complex numbers to:

$$\cdot (\mathbb{R}^2, +, \cdot)$$

We denote:

$$\cdot (\mathbb{R}^2, +, \cdot) : \mathbb{C}$$

$$\cdot (0, 1) \in \mathbb{C} : i$$

$$\cdot (a, b) \in \mathbb{C} : a + bi$$

$$\cdot \pi_1(\mathbb{C}) : Re$$

$$\cdot \pi_2(\mathbb{C}) : Im$$

Conjugation

We name complex conjugation to:

$$\begin{aligned} \cdot \quad & f : \mathbb{C} \longrightarrow \mathbb{C} \\ & (a, b) \longmapsto (a, -b) \end{aligned}$$

We denote:

$$\cdot f((a, b)) : \overline{(a, b)}$$

Norm

We name complex norm to:

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{R} \\ (a, b) &\longmapsto \sqrt{a^2 + b^2} \end{aligned}$$

We denote:

$$\cdot f((a, b)) : |(a, b)|$$

Polar transformation

We name polar transformation to:

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{R}^+ \times [0, 2\pi) \\ (a, b) &\longmapsto (\sqrt{|(a, b)|}, \arctan(\frac{b}{a})) \end{aligned}$$

We denote:

$$\cdot f((a, b)) : (r, \theta)$$

Unit sphere projection

We name unit sphere projection to:

$$\begin{aligned} f : \mathbb{C} &\longrightarrow S^1 \\ z &\longmapsto \frac{z}{|z|} \end{aligned}$$

We denote:

$$\cdot f(z) : \pi(z)$$

Roots of unity

Let:

$$\cdot z \in \mathbb{C}$$

Then, z is a root of unity if:

$$\cdot \exists n \in \mathbb{N} :$$

$$z^n = 1$$

We denote:

$$\cdot \{z \in \mathbb{C} \mid z \text{ root of unity} \} : S^1$$

Disk

Let:

$$\cdot p \in \mathbb{C}$$

$$\cdot r \in \mathbb{R}^+ \setminus \{0\}$$

We name Disk centered in p and radius r to:

$$\cdot \{z \in \mathbb{C} \mid |z - p| < r\}$$

We denote:

$$\cdot \{z \in \mathbb{C} \mid |z - p| < r\} : D^1$$

Component decomposition

Let:

$$f : \mathbb{C} \rightarrow \mathbb{C}$$

We name real component of f to:

$$\begin{aligned} f_{Re} : \mathbb{C} &\longrightarrow \mathbb{R} \\ z &\longmapsto Re(f(z)) \end{aligned}$$

We name imaginary component of f to:

$$\begin{aligned} f_{Im} : \mathbb{C} &\longrightarrow \mathbb{R} \\ z &\longmapsto Im(f(z)) \end{aligned}$$

We name component decomposition of f to:

$$\begin{aligned} f_{\mathbb{R}^2} : \mathbb{C} &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (f_{Re}(x + yi), f_{Im}(x + yi)) \end{aligned}$$

2. Holomorphic functions**Incremental quotient**

Let:

$$\cdot \mathcal{U} \subset \mathbb{C} \text{ open}$$

$$\cdot f : \mathcal{U} \rightarrow \mathbb{C}$$

$$\cdot p \in \mathcal{U}$$

We name incremental quotient of f in p to:

$$\cdot \lim_{z \rightarrow p} \frac{f(z) - f(p)}{z - p}$$

We denote:

$$\cdot f'(p)$$

Holomorphic function

Let:

$$\cdot \mathcal{U} \subset \mathbb{C} \text{ open}$$

$$\cdot f : \mathcal{U} \rightarrow \mathbb{C}$$

$$\cdot p \in \mathcal{U}$$

Then, f is holomorphic over p if:

$$\cdot \exists f'(p)$$

Then, f is holomorphic over U if:

$$\cdot \forall p \in \mathcal{U} :$$

$$\exists f'(p)$$

We denote:

$$\cdot \{f : \mathcal{U} \rightarrow \mathbb{C} \mid f \text{ holomorphic over } \mathcal{U}\} : \mathcal{H}(\mathcal{U})$$

$$\cdot f \in \mathcal{H}(\mathbb{C}) : f \text{ entire}$$

Cauchy-Riemman equations

Let:

$$\cdot u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\cdot \begin{array}{ccc} f : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (u((x, y)), v((x, y))) \end{array}$$

Then, f is satisfies the Cauchy-Riemman equations if:

$$\cdot \exists u_x, u_y, v_x, v_y$$

$$\cdot u_x = v_y$$

$$\cdot u_y = -v_x$$

We denote:

$$\cdot u_x + iv_x : f_x$$

$$\cdot u_y + iv_y : f_y$$

Conformal

Let:

$$\cdot \mathcal{U} \subset \mathbb{C} \text{ open}$$

$$\cdot f : \mathcal{U} \rightarrow \mathbb{C}$$

$$\cdot z \in \mathcal{U}$$

Then, f is conformal in z if:

$$\cdot \exists c \in \mathbb{C} :$$

$$\forall I \subset \mathbb{R} \quad 0 \in I :$$

$$\forall \gamma : I \rightarrow \mathbb{R}^2 \quad \gamma \text{ differentiable} \wedge \gamma(0) = z \wedge \gamma'(0) \neq$$

$0 :$

$$\frac{(f \circ \gamma)'(0)}{\gamma'(0)} = c$$

Then, f is conformal if:

$$\cdot \forall z \in \mathcal{U} :$$

$$f \text{ conformal in } z$$

Power series

Let:

$$\cdot \sum_{n \geq 0} a_n f_n \text{ complex valued sequence}$$

Then, $\sum_{n \geq 0} a_n f_n$ is a power series if:

$$\cdot \forall n \in \mathbb{N} :$$

$$\begin{array}{ccc} f_n : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & (z - a)^n \end{array}$$

Convergence radius

Let:

$$\cdot \text{statements}$$

.

Then, *item* is a/an entity if:

$$\cdot \text{conditions}$$

.

We denote:

$$\cdot \text{property} : \text{notation}$$

.

Absolutely convergent

Let:

· $\sum c_n n[0]$ series

Then, $\sum c_n n[0]$ is absolutely convergent if:

· $\sum |c_n| n[0]$ convergent

Numeric series

Let:

· $(c_k)_{k \in \mathbb{N}}$

Then, *item* is a/an entity if:

· *conditions*

.

We denote:

· *property : notation*

.

Block II

Propositions

1. The field of complex numbers

introduction

go

2. Holomorphic functions**Cauchy-Riemman**

Let:

$$\cdot f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ satisfies CR}$$

Then, holds:

$$\cdot f_x = f'(z)$$

$$\cdot f_y = -if_y$$

$$\cdot \partial_{f(z)} z = 0$$

Demonstration:

$$f_x = u_x + iv_x = a + bi = f'(z)$$

$$f_y = i(a + ib) = if'(z)$$

$$f'(z) = -if_y$$

notation

tangent vector

Let:

γ differentiable plane arc $\quad \forall t \in I :$

$$\gamma'(t) \neq 0$$

Then, holds:

$\gamma'(t)$ tangent to γ

Demonstration:

demonstration

arc images

Let:

- $f \in \mathcal{H}(\mathcal{U})$
- γ differentiable plane arc $\gamma \subset \mathcal{U}$
- $\sigma = f(\gamma)$
- $z_0 = \gamma(0)$

Then, holds:

- $\sigma' = f'(\gamma)\gamma'$
- $\gamma'(0) \neq 0 \rightarrow f'(z_0) \neq 0$
- $\sigma'(0) = f'(z_0)\gamma'(0)$
- $|\sigma'(0)| = |f'(z_0)||\gamma'(0)|$
- $\arg \sigma'(0) = \arg \gamma'(0) + \arg f'(z_0)$
- f aplica una homotecia mas una rotacion constante a todos los

vectores tangentes que salen de z_0

Demonstration:

obvio

Holomorphic functions are conform

Let:

$$f : \mathcal{U} \rightarrow \mathbb{C}$$

.

Then, holds:

$$f \text{ holomorph in } z \Leftrightarrow f'(z) \neq 0 \leftrightarrow f \text{ conform}$$

Demonstration:

\rightarrow):

already seen

\leftarrow):

too hard

Convergence of complex series

Let:

$$\cdot \sum_{n \geq 0} c_n \text{ complex series}$$

Then, holds:

$$\cdot \sum_{n \geq 0} c_n \text{ convergent} \leftrightarrow \sum_{n \geq 0} \text{Re} c_n \text{ convergent} \wedge \sum_{n \geq 0} \text{Im} c_n \text{ convergent}$$

vergent

Demonstration:

demonstration

Absolutely convergent are convergent

Let:

· $\sum c_n n[0]$ absolutely convergent

Then, holds:

· $\sum c_n n[0]$ convergent

Demonstration:

$$S_k := \sum c_n n[0][k]$$

$$\forall m \in \mathbb{N} \quad m < k :$$

$$|S_k - S_m| = |\sum c_n n[m+1][k]| \leq \sum |c_n| n[m+1][k]$$

$$\leq \sum |c_n| n[m+1] \xrightarrow{n} 0$$

$$|S_k - S_m| \xrightarrow{n} 0 \rightarrow (S_k)_k \text{ convergent} \rightarrow \sum c_n n[0] \text{ convergent}$$

gent

Series and norm

Let:

$$\cdot \sum c_n n[0] \text{ convergent}$$

Then, holds:

$$\cdot |c_n| \xrightarrow{n} 0$$

Demonstration:

$$\sum c_n n[0] \text{ convergent} \leftrightarrow (S_n)_n \text{ convergent}$$

$$\rightarrow \text{Cauchy } |S_n - S_m| \xrightarrow{n} 0 \text{ por n y m} \rightarrow |S_n - S_{n-1}| \xrightarrow{n} 0$$

$$\rightarrow |c_n| \xrightarrow{n} 0$$

Root test

Let:

$$\begin{aligned} & \cdot \sum_{n \geq 0} c_n \text{ real series} \\ & \cdot l \in \mathbb{R} \quad \text{,,} \quad \overline{\lim}_k |c_k|^{\frac{1}{k}} = l \end{aligned}$$

Then, holds:

$$\begin{aligned} & \cdot l > 1 \rightarrow \sum_{n \geq 0} c_n \notin \mathbb{R} \\ & \cdot l < 1 \rightarrow \sum_{n \geq 0} c_n \in \mathbb{R} \end{aligned}$$

Demonstration:

demonstration

Quotient test

Let:

$$\begin{aligned} & \cdot \sum_{n \geq 0} c_n \text{ real series} \\ & \cdot l \in \mathbb{R} \quad \text{„} \end{aligned}$$

Then, holds:

$$\begin{aligned} & \cdot \exists l \in \mathbb{R} : \\ & \quad \lim_k \frac{c_{k+1}}{c_k} = l \\ & \cdot \overline{\lim}_{c_k} |c_k|^{\frac{1}{k}} k = l \end{aligned}$$

Power series theorem

Let:

$$\cdot \sum_{n \geq 0} a_n c^n \text{ power series}$$

Then, holds:

$$\cdot |z - a| < R \rightarrow \text{absolutely convergent}$$

$$\cdot |z - a| > R \rightarrow \text{divergent}$$

$$\cdot \text{convergent in } D(a, R)$$

$$\cdot \begin{array}{ccc} f : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \sum_{n \geq 0} c_n (z - a)^n \in \mathcal{H}(D(a, R)) \end{array}$$

$$\cdot \forall z \in \mathbb{C} :$$

$$f'(z) = \sum_{n \geq 0} n c_n (z - a)^{n-1} \text{ convergent}$$

$$\cdot \text{convergence radius of } f' = \text{convergence radius of } f$$

Demonstration:

$$\forall z \in \mathbb{C} \quad \text{,, } |z - a| < R :$$

$$\begin{array}{l} \text{Root test over } \sum_{n \geq 0} |c_n| |z - a|^n \lim_n (|c_n| |z - a|^n)^{\frac{1}{n}} = |z - a| \lim_n |c_n|^{\frac{1}{n}} = \frac{|z - a|}{R} < 1 \text{ Root test} \rightarrow \text{absolutely convergent} \end{array}$$

$$\forall z \in \mathbb{C} \quad \text{,, } |z - a| < R :$$

$$\forall \rho \in \mathbb{R} \quad \text{,, } |z - a| < \rho < R :$$

$$\frac{1}{\rho} < \frac{1}{R}$$

$$\lim_n |c_n|^{1/n} = \frac{1}{R} \rightarrow \text{exists partial of } |c_n|^{1/n}$$

$$|c_n| |z - a|^n > \frac{|z - a|^n}{\rho^n} \text{ no } \xrightarrow{n} 0$$

General term test \rightarrow divergent

Block III

Examples

1. Holomorphic functions

introduction

go

2. Holomorphic functions**Conjugation**

Let:

$$\begin{array}{ccc} \bar{a} : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \bar{z} \end{array}$$

Then, \bar{a} is not holomorphic :

$$u_x = 1$$

$$u_y = 0$$

$$v_x = 0$$

$$v_y = -1$$

$$\forall z \in \mathbb{C} :$$

$$-1 \neq 1 \rightarrow f \text{ not holomorphic in } z$$

Quadratic norm

Let:

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto |z|^2 \end{aligned}$$

$\cdot f_{\mathbb{R}^2}$ component decomposition of f

Then, f is holomorphic in 0:

f differentiable in \mathbb{R}^2 polynomial

$\forall z \in \mathbb{C} :$

$$u_x(x, y) = 2x$$

$$u_y(x, y) = 2y$$

$$v_x(x, y) = 0$$

$$v_y(x, y) = 0$$

$$u_x = v_y \leftrightarrow x = 0$$

$$u_y = -v_x \leftrightarrow y = 0$$

f holomorphic function in $z \leftrightarrow z = 0$

Non preserving angles function

Let:

$$\cdot f(z) = z^2$$

Then, f is conform in $\mathbb{R} \setminus \{0\}$:

$$f(\{(x, 0) \in \mathbb{C} \mid x > 0\}) = \{(x, 0) \in \mathbb{C} \mid x > 0\}$$

$$f(\{(x, 0) \in \mathbb{C} \mid x < 0\}) = \{(x, 0) \in \mathbb{C} \mid x > 0\}$$

$$\text{ang}(A, B) = \pi \neq 0 = \text{ang}(f(A), f(B))$$

Exponential

Let:

$$\cdot a : 0$$

$$\cdot c_n : \frac{1}{n}$$

Then, $\sum_{n \geq 0} c_n (z - a)^n$ is convergent in D1 :

$$\lim_n \frac{|c_n|}{|c_{n+1}|} = \lim_n \frac{n+1}{n} = 1 \rightarrow R = 1$$

$\mathbb{CH} \rightarrow D(0, 1)$ convergent

$\mathbb{C} \setminus D(0, 1)$ divergent

$$f' = f$$

Block IV

Problems

PROBLEMES D'ANÀLISI COMPLEXA
2n quadrimestre del curs 2013-2014.

Llista 1: Els nombres complexos

B.1. Expressen en la forma $a + ib$ els següents nombres:

- | | | | |
|-----------------------|---------------------|-----------------|------------------|
| (a) $(2 + 3i)(4 + i)$ | (c) $\frac{1}{4+i}$ | (e) \sqrt{i} | (g) $\sqrt{9i}$ |
| (b) $(4 + 2i)^2$ | (d) $\frac{i}{4+i}$ | (f) $\sqrt{-i}$ | (h) $\sqrt{1+i}$ |

B.2. Si $z = x + iy$ trobeu les parts real i imaginària de les expressions següents:

- | | | | |
|-----------|----------------|-------------------|---------------------|
| (a) z^2 | (b) $z(z + 1)$ | (c) $\frac{1}{z}$ | (d) $\frac{1}{z-3}$ |
|-----------|----------------|-------------------|---------------------|

B.3. És cert que

- a) $\operatorname{Re}(z + w) = \operatorname{Re} z + \operatorname{Re} w$?
- b) $\operatorname{Re}(zw) = (\operatorname{Re} z)(\operatorname{Re} w)$?
- c) $\operatorname{Re}\left(\frac{z}{w}\right) = \frac{\operatorname{Re} z}{\operatorname{Re} w}$?

B.4. Trobeu la forma polar dels nombres següents i dibuixeu-los.

- | | | | |
|------------------------|----------------------|---------------|--------------|
| (a) $3(1 + \sqrt{3}i)$ | (b) $2\sqrt{3} - 2i$ | (c) $-2 + 2i$ | (d) $-1 - i$ |
|------------------------|----------------------|---------------|--------------|

B.5. Sigui $(x + iy)/(x - iy) = a + ib$. Proveu que $a^2 + b^2 = 1$.

B.6. Proveu que si $p(z)$ és un polinomi amb coeficients reals i z és un zero de p llavors \bar{z} també ho és.

B.7. Descriu els conjunts del pla que satisfan (recordeu que $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.)

- | | | |
|---------------------------------------------------------------|-------------------------------------|-------------------------|
| (a) $\operatorname{Im} \frac{z-a}{z} = 0, a \in \mathbb{C}^*$ | (b) $ z = \operatorname{Re} z + 1$ | (c) $ z - 2 > z - 3 $ |
|---------------------------------------------------------------|-------------------------------------|-------------------------|

SOL. B.1. a) $5 + 14i$; b) $12 + 16i$; c) $4/17 - i/17$; d) $1/17 + 4i/17$; e) $\pm\sqrt{2}/2(1 + i)$; f) $\pm\sqrt{2}/2(1 - i)$; g) $\pm 3\sqrt{2}/2(1 + i)$; h) $\pm 2^{1/4}(\cos(\pi/8) + i \sin(\pi/8))$.

B.2 a) $x^2 - y^2 + 2ixy$; b) $x^2 - y^2 + x + i(y + 2xy)$; c) $(x - iy)/(x^2 + y^2)$; d) $(x - 3 - iy)/((x - 3)^2 + y^2)$.

B.3 a) si. b) no. c) no.

B.4 a) $6(\cos(\pi/3) + i \sin(\pi/3))$; b) $4(\cos(\pi/6) - i \sin(\pi/6))$; c) $2\sqrt{2}(\cos(\pi/4) + i \sin(\pi/4))$; d) $\sqrt{2}(\cos(3\pi/4) - i \sin(3\pi/4))$.

B.6 Conjugueu tot el polinomi.

B.7 a) Recta que passa per 0 i a ; b) Paràbola horitzontal $x = (1/2)(y^2 - 1)$; c) $\{\operatorname{Re} z > 3/2\}$.

1. Expressen en la forma $a + ib$ els següents nombres:

- | | | | |
|-----------------------|----------------------------------------|---------------------------------------|------------------------------|
| (a) $\frac{1}{i}$ | (c) $\frac{1}{2+i} + \frac{1}{2-i}$ | (e) $\left(\frac{2+i}{3-2i}\right)^2$ | (g) $\sqrt[4]{-i}$ |
| (b) $\frac{1+i}{1-i}$ | (d) $\frac{1}{2+i} + \frac{4-2i}{3+i}$ | (f) $(1+i)^{100} + (1-i)^{100}$ | (h) $(3 + 4i)^{\frac{1}{2}}$ |

2. Si $z = x + iy$ on $x, y \in \mathbb{R}$, trobeu les parts real i imaginària de:

PROBLEMES D'ANLISI COMPLEXA
2n quadrimestre del curs 2013-2014

Llista 2: Funcions de variable complexa i equacions de Cauchy-Riemann

B.1. Trobeu els punts on la funció f és derivable (en el sentit complex), en els següents casos, i calculeu la derivada.

(a) $\cos |z|^2$

(c) e^{iz}

(e) $\frac{1}{(z-1)^2(z^2+2)}$

(b) $|z|^4$

(d) $z + \frac{1}{z}$

(f) $\frac{1}{(z+\frac{1}{z})^2}$

Solució: (a) \emptyset ; (b) \emptyset ; (c) \mathbb{C} ; $f'(z) = ie^{iz}$; (d) $\mathbb{C} \setminus \{0\}$; $f'(z) = 1 - \frac{1}{z^2}$; (e) $\mathbb{C} \setminus \{1, \pm\sqrt{2}i\}$; (f) \mathbb{C} .

B.2. Determineu si aquestes funcions poden ser la part real d'una funció holomorfa, i en cas que ho siguin calculeu la part imaginària.

(a) $e^x \cos y$

(b) $x^3 + 6xy^2$

(c) $\log(x^2 + y^2)$

Solució: (a) $e^x \sin y$; $f(z) = e^z$; (b) No ho és; (c) $2 \arctan(y/x)$; $f(z) = \log(z^2)$.

B.3. Sigui f una funció holomorfa en un obert $\Omega \subset \mathbb{C}$ i $z_0 \in \Omega$ tal que $f'(z_0) \neq 0$. Quin angle formen les corbes $\operatorname{Re} f(z) = \operatorname{Re} f(z_0)$ i $\operatorname{Im} f(z) = \operatorname{Im} f(z_0)$ en un punt z_0 ?

Solució: $\pi/2$.

1. Trobeu els punts on la funció f és derivable (en el sentit complex), en els següents casos:

(a) $f(z) = |z|$

(d) $f(z) = z + z\bar{z}$

(b) $\cosh x \cos y + i \sinh x \sin y$

(c) $f(z) = \operatorname{Re} z$

(e) $f(z) = \operatorname{Im} e^{\bar{z}} + i \operatorname{Re} e^z$

2. Sigui $\Omega \subset \mathbb{C}$ un obert, $z_0 \in \Omega$ i $f : \Omega \rightarrow \mathbb{C}$ una funció.

a) Identificant \mathbb{R}^2 amb \mathbb{C} de la forma habitual, demostreu que si f és diferenciable en z_0 , llavors

$$Df(z_0)(z) = \frac{\partial f}{\partial z}(z_0) \cdot z + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot \bar{z} \quad (z \in \mathbb{C}),$$

on

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

b) Proveu que f és holomorfa en Ω si, i només si, f és diferenciable i $\frac{\partial f}{\partial \bar{z}} = 0$ en Ω . En tal cas, $f' = \frac{\partial f}{\partial z}$.

3. Demostreu que si f és diferenciable en un obert de \mathbb{C} , llavors

$$\frac{\partial \bar{f}}{\partial z} = \frac{\partial \bar{f}}{\partial \bar{z}} \quad \text{i} \quad \frac{\partial \bar{f}}{\partial \bar{z}} = \frac{\partial \bar{f}}{\partial z}.$$

1. The field of complex numbers

introduction

entity

Let:

· *statements*

·

Then, *item* is a/an entity if:

· *conditions*

·

We denote:

· *property : notation*

·

2. Holomorphic functions

3. Cauchy-Riemann

Let:

$$\cdot f \in \mathcal{H}(\mathbb{C}) \quad \text{,,} \quad \operatorname{Re} f + i \operatorname{Im} f = c_a$$

Show that:

$$\cdot \exists a' \in \mathbb{C} :$$

$$f = c_{a'}$$

Demonstration:

u, v real components of f

$$u(x, y) + v(x, y) = a$$

differentiate respect x and y

$$u_x + v_x = 0$$

$$u_y + v_y = 0$$

f holomorphic $\rightarrow f$ CR

$$u_x - u_y = 0$$

$$u_y + u_x = 0$$

$$u_x, u_y, v_x, v_y = 0$$

$$\exists a_1 \in \mathbb{R} :$$

$$u = c_{a_1}$$

$$\exists a_2 \in \mathbb{R} :$$

$$v = c_{a_2}$$

$$f = c_{(a_1, a_2)}$$

B.2 a)

Let:

$$\begin{aligned} u : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto \exp(y) \cos(x) \end{aligned}$$

Show that:

$$\exists f \in \mathcal{H}(\mathbb{C}) :$$

u real component of f

Demonstration:

$$\text{lab } 1 \rightarrow u_{xx} + v_{yy} = 0$$

$$u_x = \exp(x) \cos(y)$$

$$u_{xx} = \exp(x) \cos(y)$$

$$u_y = -\exp(x) \sin(y)$$

$$u_{yy} = -\exp(x) \cos(y)$$

ok

Calculate v using CR

$$v_y = u_x = \exp(x) \cos(y)$$

$$v(x, y) = \int_{\mathbb{C}} \exp(x) \sin(y) dy = \exp(x) \sin(y) + \phi(x)$$

$$v_x = \partial_v x = \exp(x) \sin(y) + \phi'(x)$$

$$-u_y = \exp(x) \sin(y) + \phi'(x)$$

$$\text{CR} \rightarrow \phi'(x) = 0$$

$$\forall c \in \mathbb{R} :$$

$$\phi(x) = c \text{ ok}$$

$$v(x, y) = \exp(x) \sin(y)$$

Preservation of angles

Let:

$$\cdot \gamma_1, \gamma_2 \text{ plane arcs} \quad \parallel \quad \gamma_1(0) = \gamma_2(0)$$

Then, holds:

$$\cdot \text{angle of } \gamma_1'(0) \text{ and } \gamma_2'(0) = \text{angle } \sigma_1'(0), \sigma_2'(0)$$

Demonstration:

rotations and homotecies let angles invariant

Block V

Tasks

1. 1st laboratory
Existence of holomorphic functions

Let:

$$\cdot f \in \mathcal{H}(\mathbb{D})$$

Study:

$$\cdot \exists f \in \mathcal{H}(\mathbb{D}) :$$

$$\forall n \in \mathbb{N} \quad n \geq 2 :$$

$$a) f\left(\pm \frac{1}{n}\right) = \frac{1}{2n+1}$$

$$b) f\left(\pm \frac{1}{n}\right) = \frac{1}{n^2}$$

$$c) \left|f\left(\frac{1}{n}\right)\right| = \frac{1}{\log(n+1)}$$

$$d) \left|f\left(\frac{1}{n}\right)\right| = \frac{n}{n+1}$$

Demonstration:

a):

$$E_1 := \left\{ +\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N} \right\}$$

$$E_2 := \left\{ -\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N} \right\}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} = \lim_n \frac{f\left(\frac{1}{n}\right) - f(0)}{\frac{1}{n}} = \frac{f(0)}{\frac{1}{n}} = \frac{1}{2} - \lim_n \frac{f(0)}{\frac{1}{n}}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} \begin{cases} = \frac{1}{2} & f(0) = 0 \\ \notin \mathbb{C} & f(0) \neq 0 \end{cases}$$

Case $f(0) = 0$:

$$\lim_{E_2} \frac{f(z) - f(0)}{z - 0} = \lim_n \frac{f\left(-\frac{1}{n}\right) - f(0)}{-\frac{1}{n}} = -\frac{1}{2} \neq \lim_{E_1} \frac{f(z) - f(0)}{z - 0}$$

$$\nexists f \in \mathcal{H}(0) \quad \text{„} f \text{ satisfies } a)$$

In particular:

$$\nexists f \in \mathcal{H}(\mathbb{D}) \quad \text{„} f \text{ satisfies } a)$$

b):

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z^2 \end{aligned}$$

$$\forall n \in \mathbb{N} \quad \text{„} n \geq 2 :$$

$$f\left(\pm \frac{1}{n}\right) = \frac{1}{n^2}$$

f satisfies b)

$$\begin{aligned} \bar{f} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (u(x, y), v(x, y)) = (x^2 - y^2, 2xy) \\ \bar{f} \in \text{Pol}(\mathbb{R}^2) &\rightarrow \bar{f} \text{ differentiable in } \mathbb{R}^2 \end{aligned}$$

$$\forall (x, y) \in \mathbb{R}^2 :$$

$$\partial_x u(x, y) = 2x = \partial_y v(x, y)$$

$$\partial_y u(x, y) = -2y = -\partial_x v(x, y)$$

f satisfies CR

$$\therefore f \in \mathcal{H}(\mathbb{R}^2)$$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

c):

Suppose $\exists f \in \mathcal{H}(\mathbb{D}) \quad \text{„} f \text{ satisfies } c)$

$$f \in \mathcal{C}^0(\mathbb{D}) \rightarrow f(0) = f\left(\lim_n \frac{1}{n}\right) = \lim_n f\left(\frac{1}{n}\right) = 0$$

$$\left| \lim_{E_1} \frac{f(z) - f(0)}{z - 0} \right| = \lim_n \frac{\left| f\left(\frac{1}{n}\right) \right|}{\frac{1}{n}} \notin \mathbb{C}$$

$f \notin \mathcal{H}(0)$ absurd

d):

$$\begin{array}{ccc} f : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \frac{1}{z+1} \end{array}$$

$\forall n \in \mathbb{N} \quad n \geq 2 :$

$$\left| f\left(\frac{1}{n}\right) \right| = \frac{1}{\frac{1}{n}+1} = \frac{n}{n+1}$$

f satisfies d)

$$\begin{array}{ccc} \bar{f} : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (u(x, y), v(x, y)) = \left(\frac{x+1}{(x+1)^2+y^2}, \frac{-y}{(x+1)^2+y^2} \right) \end{array}$$

$\bar{f} \in \text{Rat}(\mathbb{R}^2) \wedge \forall (x, y) \in \mathbb{R}^2 :$

$$(x+1)^2 + y^2 \neq 0$$

\bar{f} differentiable in \mathbb{R}^2

$\forall (x, y) \in \mathbb{R}^2 :$

$$\partial_x u(x, y) = \frac{y^2 - (x+1)^2}{((x+1)^2 + y^2)^2} = \partial_y v(x, y)$$

$$\partial_y u(x, y) = \frac{-2y(x+1)}{((x+1)^2 + y^2)^2} = -\partial_x v(x, y)$$

f satisfies CR

$\therefore f \in \mathcal{H}(\mathbb{R}^2)$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

Constant tests

Let:

$$\cdot \Omega \subset \mathbb{C} \text{ region}$$

$$\cdot f \in \mathcal{H}(\Omega)$$

Then, holds:

$$\cdot f_{Re} = 0 \vee f_{Im} = 0 \rightarrow f \in \text{Cst}(\Omega)$$

$$\cdot |f| \in \text{Cst}(\Omega) \rightarrow f \in \text{Cst}(\Omega)$$

$$\cdot \text{Im} f \text{ circumference} \rightarrow f \in \text{Cst}$$

Demonstration:

$$f_{Re} = 0 \vee f_{Im} = 0:$$

$$u := f_{Re}$$

$$v := f_{Im}$$

$$f \in \mathcal{H}(\Omega) \rightarrow f \text{ satisfies CR in } \Omega$$

$$\partial_x u = \partial_y v = 0$$

$$\partial_y u = -\partial_x v = 0$$

Null diferencial test:

$$\Omega \text{ connex} \rightarrow u, v \in \text{Cst}$$

$$u, v \in \text{Cst} \rightarrow f \in \text{Cst}$$

$$|f| \in \text{Cst}(\Omega):$$

$$\begin{aligned} |f| : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \sqrt{u(x, y)^2 + v(x, y)^2} \end{aligned}$$

$$|f| \in \text{Cst} \rightarrow \exists a \in \mathbb{R} :$$

$$\sqrt{u(x, y)^2 + v(x, y)^2} = a$$

$$u(x, y)^2 + v(x, y)^2 = a^2$$

$$2\partial_x u(x, y) + 2\partial_x v(x, y) = 0$$

$$2\partial_y u(x, y) + 2\partial_y v(x, y) = 0$$

$$f \in \mathcal{H}(\Omega) \rightarrow f \text{ satisfies CR in } \Omega$$

$$2\partial_y v(x, y) + 2\partial_x v(x, y) = 0$$

$$-2\partial_x v(x, y) + 2\partial_y v(x, y) = 0$$

$$+ : 4\partial_y v(x, y) = 0 \rightarrow \partial_y v(x, y) = 0$$

$$- : 4\partial_x v(x, y) = 0 \rightarrow \partial_x v(x, y) = 0$$

Null differential test:

$$\Omega \text{ connex} \rightarrow u, v \in \text{Cst}$$

$$u, v \in \text{Cst} \rightarrow f \in \text{Cst}$$

$\text{Im}(f)$ circumference :

$$\exists (x_0, y_0) \in \mathbb{R}^2, r \in \mathbb{R}^+ :$$

$$\text{Im}(f) = C_r(x_0, y_0)$$

$$\begin{aligned} \bar{f} : \mathbb{R}^2 &\longrightarrow C_r(x_0, y_0) \\ (x, y) &\longmapsto (r \cos(x - x_0), r \sin(y - y_0)) \end{aligned}$$

$$\forall (x, y) \in \Omega :$$

$$|\bar{f}|(x, y) = \sqrt{r^2(\cos^2(x - x_0) + \sin^2(y - y_0))} = r$$

$$|f| \in \text{Cst} \rightarrow f \in \text{Cst}$$

Real part of holomorphic functions

Let:

$$\cdot \Omega \subset \mathbb{R}^2 \text{ region}$$

$$\cdot u \in \mathcal{C}^2(\Omega) \quad \text{,,} \quad \exists f \in \mathcal{H}(\Omega) :$$

$$f_{Re} = u$$

Show that:

$$\cdot \partial_{xx}u + \partial_{yy}u = 0$$

Study:

$$\cdot \exists f \in \mathcal{H}(\Omega) :$$

$$a) f_{Re}(x, y) = x^2 + y^2$$

$$b) f_{Re}(x, y) = x(x+1) - y^2$$

$$c) \forall \alpha \in \mathbb{R} :$$

$$f_{Re} = y^3 + \alpha x^2 y \wedge \Omega = \mathbb{C}$$

Demonstration:

$$\partial_{xx}u + \partial_{yy}u = 0:$$

$$u := f_{Re}, v := f_{Im}$$

$$f \in \mathcal{H}(\Omega) \rightarrow f \text{ satisfies CR in } \Omega$$

$$\partial_x u = \partial_y v \rightarrow \partial_{xx}u = \partial_{xy}v$$

$$\partial_y u = -\partial_x v \rightarrow \partial_{yy}u = -\partial_{xy}v$$

$$\therefore \partial_{xx}u + \partial_{yy}u = 0$$

$$f_{Re}(x, y) = x^2 + y^2:$$

$$\partial_{xx}u + \partial_{yy}u = 4 \neq 0$$

$$\nexists f \in \mathcal{H}(\Omega) \quad \text{,} \quad f_{Re}(x, y) = x^2 + y^2$$

$$f_{Re}(x, y) = x(x+1) - y^2:$$

$$\bar{f} : \Omega \longrightarrow \mathbb{R}^2$$

$$(x, y) \longmapsto (u(x, y), v(x, y)) = (x(x+1) - y^2, 2xy + y)$$

$$\bar{f} \in \text{Pol} \rightarrow \bar{f} \text{ differentiable in } \Omega$$

$$\forall (x, y) \in \Omega :$$

$$\partial_x u(x, y) = 2x + 1 = \partial_y v(x, y)$$

$$\partial_y u(x, y) = -2y = -\partial_x v(x, y)$$

$$f \text{ satisfies CR in } \Omega$$

$$f \in \mathcal{H}(\Omega) \wedge f_{Re} = u$$

$$f_{Re}(x, y) = y^3 + \alpha x^2 y:$$

$$f \text{ has to satisfy CR in } \mathbb{C}:$$

$$\forall (x, y) \in \mathbb{R}^2 :$$

$$\partial_y v(x, y) = \partial_x u(x, y) = 2\alpha xy$$

$$\partial_x v(x, y) = -\partial_y u(x, y) = -3y^2 - \alpha x^2$$

$$v(x, y) = \alpha xy^2 + c(x)$$

$$v(x, y) = -3xy^2 - \frac{\alpha}{3}x^3 + c(y)$$

$$\alpha = -3, \quad c(x) = x^3, \quad c(y) = 0$$

$$v(x, y) = -3xy^2 + x^3$$