

# Dynamical systems

Martin Azpillaga

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# Block I

## Definitions

**1. One-dimensional discrete dynamical systems**
**Dynamical system**

Let:

- $M$  manifold
- $T$  monoid
- $\phi : M \times T \rightarrow M$

Then,  $(M, T, \phi)$  is a dynamical system if:

- $\forall x \in X:$

$$\phi(x, 0) = x$$

$$\forall t_1, t_2 \in T:$$

$$\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$$

**Dimension**

Let:

- $(M, T, \phi)$  dynamical system

We name dimension of  $(M, T, \phi)$  to:

- $\dim(M)$

We denote:

- $\dim(M) = n : (M, T, \phi)$  n-D dynamical system

### Discrete & Continuous

Let:

·  $(M, T, \phi)$  dynamical system

Then,  $(M, T, \phi)$  is discrete if:

·  $T \lesssim \mathbb{N}$

Then,  $(M, T, \phi)$  is continuous if:

·  $T \subset \mathbb{R}$   $T$  open

### Defined by a function

Let:

·  $(M, T, \phi)$  dynamical system

·  $f : M \rightarrow M$

Then,  $(M, T, \phi)$  is a dynamical system defined by  $f$  if:

·  $T = \mathbb{N}$

· 
$$\begin{array}{ccc} \phi : M \times \mathbb{N} & \longrightarrow & M \\ (x, n) & \longmapsto & f^n(x) \end{array}$$

We denote:

·  $(M, T, \phi)$  dynamical system defined by  $f : (M, \mathbb{N}, f)$

**Orbit**

Let:

·  $(M, \mathbb{N}, f)$  functional dynamical system

·  $x \in M$

We name orbit of  $x$  to:

·  $\{f^n(x)\}_{n \in \mathbb{N}}$

We denote:

·  $o(x)$

**n-periodic point**

Let:

·  $(M, \mathbb{N}, f)$  functional dynamical system

·  $x \in M$

·  $n \in \mathbb{N}$

Then,  $x$  is a n-periodic point if:

·  $f^n(x) = x$

·  $\forall n' \in \mathbb{N} \quad n' < n:$

$$f^{n'}(x) \neq x$$

We denote:

·  $n = 1 : x$  fixed point

**Attractive & Repulsive**

Let:

- $(M, \mathbb{N}, f)$  metrical dynamical system
- $x \in M$  m-periodic point

Then,  $x$  is attractive if:

- $\exists \mathcal{U} \subset M$  open :

$$\forall y \in \mathcal{U}:$$

$$\exists N \in \mathbb{N}:$$

$$\forall n \in \mathbb{N} \quad n \geq N:$$

$$f^{nm}(y) \in \mathcal{U}$$

Then,  $x$  is repulsive if:

- $\forall \mathcal{U} \subset M \quad x \in \mathcal{U}:$

$$\forall y \in \mathcal{U}:$$

$$\exists N \in \mathbb{N}:$$

$$\forall n \in \mathbb{N} \quad n \geq N:$$

$$f^{nm}(x) \notin \mathcal{U}$$



**Attraction set**

Let:

- $(M, \mathbb{N}, f)$  dynamical system
- $x \in M$  attractive m-periodic point
- $o(x)$  orbit of  $x$

We name attraction set of  $x$  to:

$$\cdot \{y \in M \mid \exists x' \in o(x) : f^{nm}(y) \xrightarrow{n} x'\}$$

We denote:

$$\cdot A(x)$$

**Neutral point**

Let:

- $(M, \mathbb{N}, f)$  differentiable dynamical system
- $x \in M$

Then,  $x$  is a neutral point if:

$$\cdot f'(x) \in \{-1, 1\}$$

**Feeble attractive & repulsive points**

Let:

·  $(M, \mathbb{N}, f)$   $\mathcal{C}^3$  dynamical system

·  $x \in M$

Then,  $x$  is feeble attractive point if:

·  $f'(x) = 1$

·  $f''(x) = 0$

·  $f'''(x) > 0$

Then,  $x$  is feeble repulsive point if:

·  $f'(x) = 1$

·  $f''(x) = 0$

·  $f'''(x) < 0$

**Multiplier**

Let:

·  $(M, \mathbb{N}, f)$   $\mathcal{C}^1$  dynamical system

·  $x \in M$

We name multiplier of  $x$  to:

·  $f'(x)$

**Logistic**

Let:

$$\cdot f : \mathbb{R} \rightarrow \mathbb{R}$$

$$\cdot (M, T, \phi) \text{ dynamical system defined by } f$$

Then,  $(M, T, \phi)$  is a logistic dynamical system if:

$$\cdot \exists a \in \mathbb{R}:$$

$$\begin{array}{ccc} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & ax(1-x) \end{array}$$

**Chaos**

Let:

$$\cdot f : \mathbb{R} \rightarrow \mathbb{R}$$

$$\cdot (M, T, \phi) \text{ dynamical system defined by } f$$

Then,  $(M, T, \phi)$  is chaotic if:

$$\cdot \text{Fix}(f) \text{ dense in } \mathbb{R}$$

$$\cdot \exists x \in \mathbb{R}:$$

$$o(x) \text{ dense in } \mathbb{R}$$

$$\cdot f \text{ sensibility of } x_0$$

**Sarkovskii's order**

We name Sarkovskii's order to:

· *naming*

·

We denote:

· *property : notation*

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# Block II

# Propositions

<b>1. One-dimensional discrete dynamical systems</b>
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*introduction*

**Fixed points theorem**

Let:

- $I \subset \mathbb{R}$  open
- $f : I \rightarrow I$  differentiable
- $x \in I$

Then, holds:

- $|f'(x)| < 1 \rightarrow x$  attractive
- $|f'(x)| > 1 \rightarrow x$  repulsive

Demonstration:

*demonstration*

## Attractiveness of periodic points does not involve the chosen point

Let:

- $(M, \mathbb{N}, f)$  functional dynamical system
- $x \in M$  n-periodic point
- $\{x_i\}_{i=1}^r$  orbit of  $x$

Then, holds:

- $x$  attractive  $\leftrightarrow \forall x' \in o(x)$ :
- $x'$  attractive

Demonstration:

$$\forall x' \in o(x):$$

$$f^{n'}(x') = \prod_{i=1}^r f'(x_i) = f^{n'}(x)$$



**Partition of attraction set**

Let:

- $(M, \mathbb{N}, f)$  functional dynamical system
- $x$  n-periodic point
- $o(x)$  orbit of  $x$

Then, holds:

- $\forall x' \in o(x):$

$\exists \mathcal{U} \subset M$  open :

$\forall y \in \mathcal{U}:$

$$f^n(y) \xrightarrow{n} x'$$

Demonstration:

*demonstration*

**Homeomorphisms are monotonous**

Let:

$\cdot f : \mathbb{R} \rightarrow \mathbb{R}$  homeomorphism

Then, holds:

$\cdot f$  monotonous

Demonstration:

no demonstration

### Homeomorphisms and n-periodic points

Let:

·  $f : \mathbb{R} \rightarrow \mathbb{R}$  homeomorphism  $(M, T, \phi)$  dynamical system defined by  $f$

Then, holds:

·  $\forall n \in \mathbb{N}$ :

$\exists x \in M$  „  $x$  n-periodic point

Demonstration:

graphically

**Sarkovskii's theorem**

Let:

$$\cdot f : I \rightarrow I$$

$$\cdot (M, T, \phi) \text{ dynamical system}$$

Then, holds:

$$\cdot \exists x \in M:$$

$$o(x) \text{ k-period}$$

$$\cdot \rightarrow \forall l \in \mathbb{N} \quad \text{" } l > k:$$

$$\exists x' \in M:$$

$$x' \text{ l-period}$$

# Block III

# Examples

<b>1. One-dimensional discrete dynamical systems</b>
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*Examples of what are and what are not one-dimensional dynamical systems*

**Analysis of logistic dynamical systems**

Let:

·  $(M, T, \phi)$  logistical dynamical system defined by  $f$

Then, holds:

$$\cdot \operatorname{Fix}(f) = \{0, \frac{a-1}{a}\}$$

$$\cdot \operatorname{Per}_2(f) =$$

Demonstration:

*demonstration*





# Block IV

# Problems

## MODELS I SISTEMES DINÀMICS

### Llista 1: Aplicacions unidimensionals

**B.1.** Trobeu els punts fixos i les òrbites de període 2 de les següents funcions. En el cas que apareixin paràmetres, feu-ho en funció d'aquests.

- |  |   |
|--|---|
| (a) $* f(x) = 2x(1-x)$ , on $x \in \mathbb{R}$ .                         | (c) $f(x) = x^2 + 1$ , on $x \in \mathbb{R}$ .            |
| (b) $* f_c(x) = x^2 + c$ , on $x, c \in \mathbb{R}$ (només punts fixos). | (d) $f_{a,b}(x) = ax + b$ , on $a, b, x \in \mathbb{R}$ . |
|  | (e) $f(x) = 2x^2 - 5x$ , on $x \in \mathbb{R}$ .          |

**B.2.** Fent servir anàlisi gràfic, dibuixeu el retrat de fases de

- |  |  |
|--|--|
| (a) $f(x) = x^2$ , $x \in \mathbb{R}$ .    | (c) $f_a(x) = ax$ , $x \in \mathbb{R}$ , pels diferents valors de $a \in \mathbb{R}$ . |
| (b) $f(x) = x(1-x)$ , $x \in \mathbb{R}$ . |  |

**B.3.** \* Trobeu els punts fixos atractors i les seves conques d'atracció per a la funció  $f(x) = \frac{3x-x^3}{2}$ , per  $|x| \leq \sqrt{3}$ .

**B.4.** Per a la funció logística  $f_a(x) = ax(1-x)$ , calculeu els punts fixos i els cicles de període 2 en funció del paràmetre, i determineu-ne l'estabilitat.

1. Estudieu el comportament asimptòtic de la successió  $\{x_n\}_{n \in \mathbb{N}}$ , pels diferents valors de  $x_0$  indicats.

- |   |  |
|---|--|
| (a) $* x_{n+1} = \frac{\sqrt{x_n}}{2}$ , $x_0 \geq 0$ . | (b) $x_{n+1} = \frac{x_n}{2} + \frac{2}{x_n}$ , $x_0 \geq 2$ . |
|---|--|

2. Donada la successió  $x_{n+1} = \frac{x_n+2}{x_n+1}$ ,

- (a) Trobeu el límit  $L = \lim_{n \rightarrow \infty} x_n$  per a  $x_0 \geq 0$ .
- (b) Descriu el conjunt dels  $x_0 < 0$  pels quals el límit  $\lim_{n \rightarrow \infty} x_n$  existeix i no és igual a  $L$ , o bé no existeix. (Per exemple  $x_0 = -1$ ).

3. (**Examen 2011**) Considereu el sistema dinàmic real definit per  $x_{n+1} = \frac{x_n}{4} + x_n^3$ . Trobeu el comportament asimptòtic de les òrbites per a tota condició inicial  $x_0 \in \mathbb{R}$ . Justifiqueu rigorosament les vostres afirmacions.

4. Demostreu rigurosament que  $f(x) = \sin(x)$  té  $x = 0$  com atractor global.

5. Demostreu que si  $f: \mathbb{R} \rightarrow \mathbb{R}$  és derivable,  $x_0$  és un punt fix i  $|f'(x_0)| > 1$  llavors  $x_0$  és un punt fix repulsor.

6. Signi  $f: \mathbb{R} \rightarrow \mathbb{R}$  de classe  $\mathcal{C}^\infty$  i sigui  $x_0$  un punt fix tal que  $f'(x_0) = 1$ . Doneu criteris sobre les derivades d'ordre superior, per determinar el retrat de fase local al voltant de  $x_0$ . Apliqueu-ho a determinar l'estabilitat dels punts fixos de  $x^3 - x$ .



<b>1. One-dimensional discrete dynamical system</b>
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*introduction*

### Decreasing function orbits

Let:

· *declarations*

·

Show that:

· *statements*

·

Demonstration:

$f$  corta en un punto

$f$  decreasing  $\rightarrow f^2$  increasing

$f^{2n} \xrightarrow{n}$  fixed point of  $f$

10.

Let:

$$\cdot f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \in \mathbb{C}^\infty$$

$$\cdot f(0) = 0$$

$$\cdot p \in \mathbb{R}^+ \setminus \{0\} \quad \text{,, } f'(p) \geq 0$$

$$\cdot f' \text{ decreasing}$$

Show that:

$$\cdot \forall p \in \mathbb{R}^+ \setminus \{0\}:$$

$$f^n(x) \xrightarrow{n} p$$

Demonstration:

$$f' \text{ decreasing} \rightarrow f'' < 0 \rightarrow f \text{ concave}$$

$$f \text{ positive} \rightarrow f \text{ has no extrema} \rightarrow f' > 0 \rightarrow f \text{ increasing}$$

$$f \text{ has only one fixed point}$$

$$\text{Suppose 2 fixed points : } p, p'$$

$$IVT \rightarrow \exists c \in (0, p'):$$

$$f'(c) = 1$$

$$f'(p) < 1 \rightarrow p \text{ attractive } IVT \rightarrow \text{dont exist more fixed points}$$

$$\rightarrow f'(c') = 1 \not\leq 1$$

$$\forall x \in (0, p):$$

$$f(x) > x$$

$$\forall x \in \mathbb{R} \quad \text{,, } x > p:$$

$$f(x) < x$$

$$f \text{ increasing} \rightarrow f([0, p]) = [0, p]$$