

# Block I

# Laboratory

**1. Orbit analysis**

Martin Azpillaga

Let:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^3 + \frac{1}{4}x \end{aligned}$$

Study:

- Orbit behavior of the real dynamical system defined by  $f$

Demonstration:

Formalization :

Consider  $(M, T, \phi)$  where:

$$M = \mathbb{R}$$

$$T = \mathbb{N}$$

$$\begin{aligned} \phi : \mathbb{R} \times \mathbb{N} &\longrightarrow \mathbb{R} \\ (x, n) &\longmapsto f^n(x) \end{aligned}$$

Study the orbits of  $(\mathbb{R}, \mathbb{N}, \phi)$

We will denote  $f^n(x)$  as  $x_n$

Fixed points :

$$\forall x \in \mathbb{R}:$$

$$f(x) = x \leftrightarrow x^3 + \frac{1}{4}x - x = 0 \leftrightarrow x^3 - \frac{3}{4}x = 0$$

$$\leftrightarrow x = 0 \vee x^2 - \frac{3}{4} = 0$$

$$x \text{ fixed point} \leftrightarrow x \in \{0, \pm \frac{\sqrt{3}}{2}\}$$

Graphic analysis :

Parity:

$$\forall x \in \mathbb{R}:$$

$$f(-x) = (-x)^3 + \frac{(-x)}{4} = -(x^3 + \frac{x}{4}) = -f(x)$$

$f$  is odd

Monotonicity:

$$\forall x \in \mathbb{R}:$$

$$f'(x) = 3x^2 + \frac{1}{4} > 0$$

$f$  is increasing over  $\mathbb{R}$

Convexity:

$$\forall x \in \mathbb{R}^-:$$

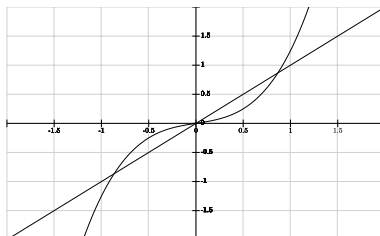
$$f''(x) = 6x \leq 0$$

$$\forall x \in \mathbb{R}^+:$$

$$f''(x) = 6x \geq 0$$

$f$  is concave over  $\mathbb{R}^-$  and convex over  $\mathbb{R}^+$

Graphic representation :



I  $\forall x \in (-\infty, -\frac{\sqrt{3}}{2})$ :

Induction over  $n$ :

$$f \text{ increasing} \rightarrow f(x_n) < f(-\frac{\sqrt{3}}{2})$$

$$x_{n+1} \in (-\infty, -\frac{\sqrt{3}}{2})$$

$$\therefore o(x) \text{ is enclosed in } (-\infty, -\frac{\sqrt{3}}{2})$$

Induction over  $n$ :

$$x_n^2 > \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) > 0$$

$$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) < 0$$

$$\therefore o(x) \text{ decreasing}$$

$$\nexists x < -\frac{\sqrt{3}}{2} \quad \text{|| } x \text{ fixed point} \rightarrow o(x) \xrightarrow{n} -\infty$$

II  $\forall x \in (-\frac{\sqrt{3}}{2}, 0)$ :

Induction over  $n$ :

$$f \text{ increasing} \rightarrow f(-\frac{\sqrt{3}}{2}) < f(x_n) < f(0)$$

$$x_{n+1} \in (-\frac{\sqrt{3}}{2}, 0)$$

$$\therefore o(x) \text{ is enclosed in } (-\frac{\sqrt{3}}{2}, 0)$$

Induction over  $n$ :

$$x_n^2 < \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) < 0$$

$$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) > 0$$

$$\therefore o(x) \text{ increasing}$$

$$o(x) \text{ convergent} \wedge 0 \text{ fixed point} \rightarrow o(x) \xrightarrow{n} 0$$

III  $\forall x \in (0, \frac{\sqrt{3}}{2})$ :

Induction over  $n$ :

$$-x_n \in (-\frac{\sqrt{3}}{2}, 0)$$

$$\text{II} \rightarrow f(-x_n) \in (-\frac{\sqrt{3}}{2}, 0) \wedge f(-x_n) > -x_n$$

$$f \text{ odd} \rightarrow f(x_n) = -f(-x_n) \in (0, \frac{\sqrt{3}}{2})$$

$$f \text{ odd} \rightarrow f(x_n) = -f(-x_n) < x_n$$

$$\therefore o(x) \text{ is enclosed in } (0, \frac{\sqrt{3}}{2}) \wedge o(x) \text{ decreasing}$$

$$o(x) \text{ convergent} \wedge 0 \text{ fixed point} \rightarrow o(x) \xrightarrow{n} 0$$

IV  $\forall x \in \mathbb{R} \quad \text{,, } x > \frac{\sqrt{3}}{2}$ :

Induction over  $n$ :

$$-x_n \in (\frac{\sqrt{3}}{2}, \infty)$$

$$\text{I} \rightarrow f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty) \wedge f(-x_n) < -x_n$$

$$f \text{ odd} \rightarrow f(x_n) = -f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty)$$

$$f \text{ odd} \rightarrow f(x_n) = -f(-x_n) > x_n$$

$$\therefore o(x) \text{ is inf bounded by in } \frac{\sqrt{3}}{2} \wedge o(x) \text{ increasing}$$

$$o(x) \text{ convergent}$$

$$\nexists x > \frac{\sqrt{3}}{2} \quad \text{,, } x \text{ fixed point} \rightarrow o(x) \xrightarrow{n} +\infty$$

## 2. Fixed points cardinality

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Let:

$$\cdot f : [0, 1] \rightarrow [0, 1] \in \mathcal{C}^2([0, 1])$$

$$\cdot f(1) < 1$$

$$\cdot f'' > 0 \in [0, 1]$$

Show that:

$$\cdot \# \{x \in [0, 1] \mid f(x) = x\} = 1$$

Demonstration:

$$\# \{x \in [0, 1] \mid f(x) = x\} \geq 1:$$

Case  $f(0) = 0$ :

0 fixed point

Case  $f(0) > 0$ :

$$\begin{array}{ccc} g : [0, 1] & \longrightarrow & [-1, 1] \\ x & \longmapsto & f(x) - x \end{array} \in \mathcal{C}^2([0, 1])$$

$$g(0) = f(0) - 0 > 0$$

$$g(1) = f(1) - 1 < 0$$

Bolzano's theorem:

$$\exists x \in (0, 1):$$

$$g(x) = 0$$

$$f(x) = x$$

$$\# \{x \in [0, 1] \mid f(x) = x\} \leq 1:$$

$$f'' > 0 \in [0, 1]$$

Rolle's theorem:

$$\# \{x \in (0, 1) \mid f'(x) = 0\} \leq 1$$

$$\# \{x \in (0, 1) \mid f(x) = 0\} \leq 2$$

$$f'' > 0 \in [0, 1]$$

Monotonicity test:

$$f' \text{ increasing in } [0, 1]$$

$$\forall a < b \in [0, 1] \quad f(a) = a, f(b) = b:$$

Medium Value Theorem:

$$\exists c \in (a, b):$$

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 1$$

$$\exists d \in (b, 1):$$

$$f'(d) = \frac{f(1) - f(b)}{1 - b} < 1$$

$$f' \text{ increasing} \rightarrow f'(c) < f'(b) < f'(d)$$

$$1 < f'(b) < 1 \text{ absurd}$$

$$\therefore \# \{x \in [0, 1] \mid f(x) = x\} = 1$$