1. Orbit analysis

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Let:

$$\begin{array}{rccl} f & :\mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & x^3 + \frac{1}{4}x \end{array}.$$

Study:

· Orbit behavior of the real dynamical system defined by f

Demonstration:

Formalization :

Consider (M, T, ϕ) where:

$M = \mathbb{R}$

$T = \mathbb{N}$

$$\begin{array}{rccl} \phi & :\mathbb{R} \times \mathbb{N} & \longrightarrow & \mathbb{R} \\ & (x, n) & \longmapsto & f^n(x) \end{array}$$

Study the orbits of $(\mathbb{R}, \mathbb{N}, \phi)$

We will denote $f^n(x)$ as x_n

Fixed points :

$\forall\ x \in \mathbb{R}$:

$$f(x) = x \leftrightarrow x^3 + \frac{1}{4}x - x = 0 \leftrightarrow x^3 - \frac{3}{4}x = 0$$

$$\leftrightarrow x = 0 \ \vee \ x^2 - \frac{3}{4} = 0$$

$$x \text{ fixed point} \leftrightarrow x \in \{0, \pm\frac{\sqrt{3}}{2}\}$$

Graphic analysis :

Parity:

$\forall\ x \in \mathbb{R}$:

$$f(-x) = (-x)^3 + \frac{(-x)}{4} = -(x^3 + \frac{x}{4}) = -f(x)$$

f is odd

Monotonicity:

$\forall\ x \in \mathbb{R}$:

$$f'(x) = 3x^2 + \frac{1}{4} > 0$$

f is increasing over $\mathbb{R}$

Convexity:

$\forall\ x \in \mathbb{R}^-$:

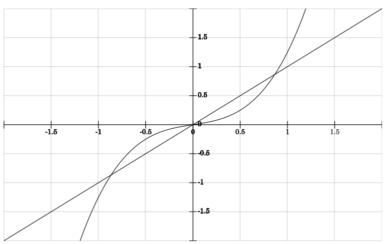
$$f''(x) = 6x \leq 0$$

$\forall\ x \in \mathbb{R}^+$:

$$f''(x) = 6x \geq 0$$

f is concave over $\mathbb{R}^-$ and convex over $\mathbb{R}^+$

Graphic representation :



$\underline{\text{I}}$ $\forall\, x \in (-\infty, -\frac{\sqrt{3}}{2})$:

Induction over n:

f incresing $\rightarrow f(x_n) < f(-\frac{\sqrt{3}}{2})$

$x_{n+1} \in (-\infty, -\frac{\sqrt{3}}{2})$

$\therefore)$ $o(x)$ is enclosed in $(-\infty, -\frac{\sqrt{3}}{2})$

Induction over n:

$x_n^2 > \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) > 0$

$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) < 0$

$\therefore)$ $o(x)$ decreasing

$\nexists\, x < -\frac{\sqrt{3}}{2}$ „ x fixed point $\rightarrow o(x) \xrightarrow{n} -\infty$

$\underline{\text{II}}$ $\forall\, x \in (-\frac{\sqrt{3}}{2}, 0)$:

Induction over n:

f increasing $\rightarrow f(-\frac{\sqrt{3}}{2}) < f(x_n) < f(0)$

$x_{n+1} \in (-\frac{\sqrt{3}}{2}, 0)$

$\therefore)$ $o(x)$ is enclosed in $(-\frac{\sqrt{3}}{2}, 0)$

Induction over n:

$x_n^2 < \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) < 0$

$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) > 0$

$\therefore)$ $o(x)$ increasing

$o(x)$ convergent $\wedge$ 0 fixed point $\rightarrow o(x) \xrightarrow{n} 0$

$\mathrm{I\underline{I}I}$ $\forall\, x \in (0, \frac{\sqrt{3}}{2})$:

Induction over n:

$-x_n \in (-\frac{\sqrt{3}}{2}, 0)$

$\mathrm{I\underline{I}} \to f(-x_n) \in (-\frac{\sqrt{3}}{2}, 0) \;\wedge\; f(-x_n) > -x_n$

f odd $\to f(x_n) = -f(-x_n) \in (0, \frac{\sqrt{3}}{2})$

f odd $\to f(x_n) = -f(-x_n) < x_n$

$\therefore)$ $o(x)$ is enclosed in $(0, \frac{\sqrt{3}}{2})$ $\wedge$ $o(x)$ decreasing

$o(x)$ convergent $\wedge\, 0$ fixed point $\to o(x) \xrightarrow{n} 0$

$\mathrm{I\underline{V}}$ $\forall\, x \in \mathbb{R}$ „ $x > \frac{\sqrt{3}}{2}$:

Induction over n:

$-x_n \in (\frac{\sqrt{3}}{2}, \infty)$

$\underline{\mathrm{I}} \to f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty) \;\wedge\; f(-x_n) < -x_n$

f odd $\to f(x_n) = -f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty)$

f odd $\to f(x_n) = -f(-x_n) > x_n$

$\therefore)$ $o(x)$ is inf bounded by in $\frac{\sqrt{3}}{2}$ $\wedge$ $o(x)$ increasing

$o(x)$ convergent

$\nexists\, x > \frac{\sqrt{3}}{2}$ „ x fixed point $\to o(x) \xrightarrow{n} +\infty$