

Dynamical systems

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Contents

Block I

Definitions

1. Discrete dynamical systems
Dynamical system

Let:

- M manifold
- T monoid
- $\phi : M \times T \rightarrow M$

Then, (M, T, ϕ) is a dynamical system if:

- $\forall x \in X :$

$$\phi(x, 0) = x$$

$$\forall t_1, t_2 \in T :$$

$$\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$$

Dimension

Let:

- (M, T, ϕ) dynamical system

We name dimension of (M, T, ϕ) to:

- $\dim(M)$

We denote:

- $\dim(M) = n : (M, T, \phi)$ n-D

Discrete & Continuous

Let:

· (M, T, ϕ) dynamical system

Then, (M, T, ϕ) is discrete if:

· $T \simeq \mathbb{N}$

Then, (M, T, ϕ) is continuous if:

· $T \subset \mathbb{R}$

· T open

Defined by a function

Let:

· (M, T, ϕ) dynamical system

· $f : M \rightarrow M$

Then, (M, T, ϕ) is a dynamical system defined by f if:

· $T = \mathbb{N}$

·
$$\begin{array}{ccc} \phi : & M \times \mathbb{N} & \longrightarrow & M \\ & (x, n) & \longmapsto & f^n(x) \end{array}$$

We denote:

· (M, T, ϕ) dynamical system defined by $f : (M, \mathbb{N}, f)$

· $f \in \mathcal{C}^n(M) : (M, \mathbb{N}, f) \in \mathcal{C}^n$

Orbit

Let:

- (M, \mathbb{N}, f) functional dynamical system
- $x \in M$

We name orbit of x to:

- $\{f^n(x)\}_{n \in \mathbb{N}}$

We denote:

- $o(x)$

Periodicity

Let:

- (M, \mathbb{N}, f) dynamical system
- $x \in M$
- $m \in \mathbb{N}$

Then, x is a m -periodic point if:

- $f^m(x) = x$

We denote:

- $\{x \in M \mid f(x) = x\} : \text{Fix}(f)$

Stability

Let:

$$\cdot f : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\cdot p \in \mathbb{R}^n \text{ m-periodic point}$$

Then, p is stable if:

$$\cdot \forall \varepsilon \in \mathbb{R}^+ :$$

$$\exists \delta \in \mathbb{R}^+ :$$

$$\forall x \in B(p, \delta) :$$

$$\forall n \in \mathbb{N} :$$

$$f^{nm}(x) \in B(p, \varepsilon)$$

Then, p is unstable if:

$$\cdot p \text{ not stable}$$

Attractive & Repulsive

Let:

$$\cdot f : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\cdot p \in \mathbb{R}^n \text{ m-periodic point}$$

Then, p is attractive if:

$$\cdot p \text{ stable}$$

$$\cdot \exists \varepsilon \in \mathbb{R}^+ :$$

$$\forall x \in B(p, \varepsilon) :$$

$$f^{nm}(x) \xrightarrow{n} p$$

Then, p is repulsive if:

$$\cdot p \text{ attractive by } f^{-1}$$

$$\cdot \forall \mathcal{U} \subset M \quad \mathcal{U} \text{ open} \wedge x \in \mathcal{U} :$$

$$\forall x' \in \mathcal{U} \quad x' \neq x :$$

$$\exists N \in \mathbb{N} :$$

$$\forall n \in \mathbb{N} \quad n \geq N :$$

$$f^{nm}(x') \notin \mathcal{U}$$

Fixed point character

Let:

$\cdot (M, \mathbb{N}, f)$ functional dynamical system

We name Fixed point character to:

$$f : \text{Fix}(f) \longrightarrow \{-1, 0, 1\}$$

$$\cdot \quad x \quad \longmapsto \begin{cases} +1 & x \text{ repulsive} \\ -1 & x \text{ attractive} \\ 0 & x \text{ no attractive no repulsive} \end{cases}$$

We denote:

$$\cdot f : \chi_f$$

Attraction set

Let:

$\cdot (M, \mathbb{N}, f)$ dynamical system

$\cdot x \in M$ attractive m-periodic point

$\cdot o(x)$ orbit of x

We name attraction set of x to:

$$\cdot \{y \in M \mid \exists x' \in o(x) : f^{nm}(y) \xrightarrow{n} x'\}$$

We denote:

$$\cdot A(x)$$

Multiplier

Let:

· (M, \mathbb{N}, f) \mathcal{C}^1 dynamical system

· $x \in M$

We name multiplier of x to:

· $f'(x)$

We denote:

· $m(x)$

· $|m(x)| = 1$: x neutral point

Feeble point

Let:

· (M, \mathbb{N}, f) \mathcal{C}^3 dynamical system

· $x \in M$

Then, x is feeble point if:

· x neutral point

· $f''(x) = 0$

Sarkovskii's order

We name Sarkovskii's order to:

$$\begin{aligned} & \cdot a = 2^n a', b = 2^m b' \\ & \cdot a <_s b \leftrightarrow \begin{cases} m < n & a' = b' = 1 \\ & a' = 1, b' \neq 1 \\ a' < b' & a' = b' \neq 1 \\ n < m & 1 \neq a' \neq b' \end{cases} \end{aligned}$$

Chaos

Let:

$$\cdot (\mathbb{R}, \mathbb{N}, f) \text{ dynamical system}$$

Then, $(\mathbb{R}, \mathbb{N}, f)$ is chaotic if:

$$\cdot \text{Fix}(f) \text{ dense in } \mathbb{R}$$

$$\cdot \exists x \in \mathbb{R} :$$

$$o(x) \text{ dense in } \mathbb{R}$$

$$\cdot \forall x \in \mathbb{R} :$$

$$\exists \varepsilon \in \mathbb{R}^+ :$$

$$\forall \delta \in \mathbb{R}^+ :$$

$$\exists \tilde{x} \in B(x, \delta) :$$

$$\lim_n o(\tilde{x}) \notin B(x, \varepsilon)$$

Topological equivalence

Let:

$$\cdot (M, \mathbb{N}, f_1), (M, \mathbb{N}, f'_2) \text{ dynamical systems}$$

Then, (M, \mathbb{N}, f_1) is topologically equivalent to (M, \mathbb{N}, f_2) if:

$$\cdot \text{Fix}(f) = \text{Fix}(f')$$

$$\cdot \forall x \in \text{Fix}(f) :$$

$$\chi_{f_1}(x) = \chi_{f_2}(x)$$

We denote:

$$\cdot (M, \mathbb{N}, f) \sim (M, \mathbb{N}, f')$$

Bifurcation

Let:

$$\cdot \Lambda \subset M$$

$$\cdot (M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda} \text{ dynamical systems parametrized by } \Lambda$$

$$\cdot \lambda_0 \in \Lambda$$

Then, λ_0 is a bifurcation value if:

$$\cdot \forall \varepsilon \in \mathbb{R}^+ :$$

$$\exists \lambda_1, \lambda_2 \in B(\lambda_0, \varepsilon) :$$

$$(M, \mathbb{N}, f_{\lambda_1}) \not\sim (M, \mathbb{N}, f_{\lambda_2})$$

Saddle-node bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is a saddle-node bifurcation value at x_0 if:

- $x_0 \in \text{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0) \neq 0$
- $\partial_{xx} f_\lambda(x_0) \neq 0$

Pitchfork bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is Pitchfork bifurcation value at x_0 if:

- $x_0 \in \text{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{x^2} f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{\lambda x} f_\lambda(x_0, \lambda_0) \neq 0$
- $\partial_{x^3} f_\lambda(x_0, \lambda_0) \neq 0$

Period doubling bifurcation

Let:

$$\cdot \Lambda \subset M$$

$$\cdot (M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda} \text{ dynamical systems parametrized by } \Lambda$$

$$\cdot \lambda_0 \in \Lambda$$

$$\cdot x_0 \in M$$

Then, λ_0 is Period doubling bifurcation value at x_0 if:

$$\cdot \lambda_0 \text{ Pitchfork bifurcation value at } x_0 \text{ of } f^2$$

Invariant curve

Let:

$$\cdot \gamma \text{ differentiable curve}$$

$$\cdot p \in \mathbb{R}^n$$

Then, γ is invariant if:

$$\cdot \forall x \in \gamma^* :$$

$$o(x) \subset \gamma^*$$

Then, γ is converges to p if:

$$\cdot \forall x \in \gamma^* :$$

$$o(x) \xrightarrow{n} p$$

2. 2-D linear dynamical systems

Linear system

Let:

· (M, \mathbb{N}, f) functional dynamical system

Then, (M, \mathbb{N}, f) is linear if:

$$\begin{aligned} \cdot \exists A \in \mathcal{M}_{n \times n}(\mathbb{R}) : \\ f : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto Ax \end{aligned}$$

Multiplier

Let:

· (M, \mathbb{N}, f) functional dynamical system

· $x \in M$

We name multiplier of x to:

· $Df(p)$

Block II

Propositions

1. Discrete dynamical systems

introduction

Fixed points theorem

Let:

- $I \subset \mathbb{R}$ open
- $f : I \rightarrow I$ differentiable
- $x \in I$

Then, holds:

- $|f'(x)| < 1 \rightarrow x$ attractive
- $|f'(x)| > 1 \rightarrow x$ repulsive

Demonstration:

demonstration

Attractiveness of periodic points does not involve the chosen point

Let:

- (M, \mathbb{N}, f) functional dynamical system
- $x \in M$ n-periodic point
- $\{x_i\}_{i=1}^r$ orbit of x

Then, holds:

- x attractive $\leftrightarrow \forall x' \in o(x) :$
 x' attractive

Demonstration:

$$\forall x' \in o(x) :$$

$$f^{n'}(x') = \prod_{i=1}^r f'(x_i) = f^{n'}(x)$$

Partition of attraction set

Let:

- (M, \mathbb{N}, f) functional dynamical system
- x n-periodic point
- $o(x)$ orbit of x

Then, holds:

- $\forall x' \in o(x) :$

$$\exists \mathcal{U} \subset M \text{ open} :$$

$$\forall y \in \mathcal{U} :$$

$$f^n(y) \xrightarrow{n} x'$$

Demonstration:

demonstration

Homeomorphisms are monotonous

Let:

$\cdot f : \mathbb{R} \rightarrow \mathbb{R}$ homeomorphism

Then, holds:

$\cdot f$ monotonous

Demonstration:

no demonstration

Homeomorphisms and n-periodic points

Let:

$\cdot f : \mathbb{R} \rightarrow \mathbb{R}$ homeomorphism (M, T, ϕ) dynamical system defined by f

Then, holds:

$\cdot \forall n \in \mathbb{N} :$

$\exists x \in M \text{ „ } x \text{ n-periodic point}$

Demonstration:

graphically

Sarkovskii's theorem

Let:

$$\cdot f : I \rightarrow I$$

$$\cdot (M, \mathbb{N}, f) \text{ dynamical system}$$

Then, holds:

$$\cdot \exists x \in M :$$

$$o(x) \text{ k-period}$$

$$\cdot \rightarrow \forall l \in \mathbb{N} \quad \text{,, } l > k :$$

$$\exists x' \in M :$$

$$x' \text{ l-period}$$

Invariance of stability over periods

Let:

- $(\mathbb{R}^n, \mathbb{N}, f)$ n-D dynamical system
- $p \in \mathbb{R}^n$ k-periodic point
- χ character of periodic points

Then, holds:

- $\exists \sigma \in \text{Im}(\chi) :$

$$\forall x \in o(p) :$$

$$\chi(x) = \sigma$$

Demonstration:

i will

2. 2-D linear dynamical systems

Invariance of stability over orbits

Let:

· (M, \mathbb{N}, f) functional dynamical system

· $x \in M$

Then, holds:

· $\forall x' \in o(x) :$

$$\chi(x') = \chi(x)$$

Demonstration:

Follow 2 steps

Step 1 : *falta* :

rows

Step 2 : *attractiveness* :

$$\chi(x) = -1$$

$$\exists \varepsilon \in \mathbb{R}^+ :$$

$$x \in B_\varepsilon(x) \rightarrow f^{2n}(x) \xrightarrow{n} x$$

$$f \in \mathcal{C}^0(M) \rightarrow \exists \varepsilon_1 \in \mathbb{R}^+ :$$

$$f(B_{\varepsilon_1}(x_1)) \subset B_{\varepsilon}(x)$$

$$x \in B_{\varepsilon_1}(x_1) \rightarrow f(x) \in B_{\varepsilon}(x) \rightarrow f^{2n-1}(f(x)) \xrightarrow{n} x$$

falta

Linear property

Let:

$\cdot (M, \mathbb{N}, f)$ linear dynamical system

Then, holds:

$\cdot \forall a, b \in \mathbb{R} :$

$\forall x, y \in M :$

$$f(ax + by) = af(x) + bf(y)$$

Demonstration:

matrius

Fixed points of linear applications

Let:

$\cdot (M, \mathbb{N}, f)$ linear dynamical system

Then, holds:

$\cdot 0 \in \text{Fix}(f)$

Demonstration:

demonstration

Jordan form of 2-D real linear maps

Let:

$$\cdot A \in \mathcal{M}_{2 \times 2}(\mathbb{R})$$

$$\cdot \chi_A(t) \text{ characteristic polynomial of } A$$

Then, holds:

$$\cdot \exists \beta \text{ base of } K : \begin{cases} A = \lambda, 0, 0, \mu & \#Z(\chi_A(t)) = 2 \\ A = \lambda, 1, 0, \lambda & \#Z(\chi_A(t)) = 1 \\ A = \alpha, \beta, -\beta, \alpha & \#Z(\chi_A(t)) = 0 \end{cases}$$

Demonstration:

demonstration

Topology of 2-D real linear maps

Let:

- $A \in \mathcal{M}_{2 \times 2}(\mathbb{R})$
- $\lambda \neq \mu$ eigenvalues of A

Then, holds:

- $|\lambda|, |\mu| < 1 \rightarrow (0,0)$ attractive
- $|\lambda| > |\mu| \rightarrow$ tangent to $y = 0$
- $|\mu| > |\lambda| \rightarrow$ tangent to $x = 0$
- $|\mu| = |\lambda| \rightarrow$ only invariant lines
-
- $|\lambda|, |\mu| > 1 \rightarrow (0,0)$ repulsive
- equivalent to other case

Demonstration:

demonstration

Block III

Examples

1. One-dimensional discrete dynamical systems
--

Examples of what are and what are not one-dimensional dynamical systems

logistic function

Let:

$\cdot (M, T, \phi)$ logistical dynamical system defined by f

Then, holds:

$$\cdot Fix(f) = \{0, \frac{a-1}{a}\}$$

$$\cdot Per_2(f) =$$

Demonstration:

demonstration

Quadratic function

Let:

$$\cdot \begin{array}{ccc} f : & \mathbb{R} & \longrightarrow \mathbb{R} \\ & x & \longmapsto a - x^2 \end{array}$$

$\cdot (M, T, f_c)$ dynamical system family

Then, f is bifurcates in $-1/4$:

$$f_{-\frac{1}{4}}(x) = x \leftrightarrow x = -\frac{1}{2}$$

$$f'_{-\frac{1}{4}}(x) = -2x$$

$$f'_{-\frac{1}{4}}(-\frac{1}{2}) = 1$$

$$\partial_a f = 1 \neq 0$$

$$\partial_{x^2} f = -2 \neq 0$$

$$sgn(1 * -2) = - \rightarrow -\frac{1}{2} \text{ SN}$$

Henon's application

Let:

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (-x^2 + 0.4y, x) \end{aligned}$$

Study:

- Fixed points of f

Demonstration:

$$(0, 0), (-0$$

$$6, -0$$

6) fixed points

Block IV

Problems

MODELS I SISTEMES DINÀMICS

Llista 1: Aplicacions unidimensionals

B.1. Trobeu els punts fixos i les òrbites de període 2 de les següents funcions. En el cas que apareixin paràmetres, feu-ho en funció d'aquests.

- | | |
|--|---|
| (a) $* f(x) = 2x(1-x)$, on $x \in \mathbb{R}$. | (c) $f(x) = x^2 + 1$, on $x \in \mathbb{R}$. |
| (b) $* f_c(x) = x^2 + c$, on $x, c \in \mathbb{R}$ (només punts fixos). | (d) $f_{a,b}(x) = ax + b$, on $a, b, x \in \mathbb{R}$. |
| | (e) $f(x) = 2x^2 - 5x$, on $x \in \mathbb{R}$. |

B.2. Fent servir anàlisi gràfic, dibuixeu el retrat de fases de

- | | |
|--|--|
| (a) $f(x) = x^2$, $x \in \mathbb{R}$. | (c) $f_a(x) = ax$, $x \in \mathbb{R}$, pels diferents valors de $a \in \mathbb{R}$. |
| (b) $f(x) = x(1-x)$, $x \in \mathbb{R}$. | |

B.3. * Trobeu els punts fixos atractors i les seves conques d'atracció per a la funció $f(x) = \frac{3x-x^3}{2}$, per $|x| \leq \sqrt{3}$.

B.4. Per a la funció logística $f_a(x) = ax(1-x)$, calculeu els punts fixos i els cicles de període 2 en funció del paràmetre, i determineu-ne l'estabilitat.

1. Estudieu el comportament asimptòtic de la successió $\{x_n\}_{n \in \mathbb{N}}$, pels diferents valors de x_0 indicats.

- | | |
|---|--|
| (a) $* x_{n+1} = \frac{\sqrt{x_n}}{2}$, $x_0 \geq 0$. | (b) $x_{n+1} = \frac{x_n}{2} + \frac{2}{x_n}$, $x_0 \geq 2$. |
|---|--|

2. Donada la successió $x_{n+1} = \frac{x_n+2}{x_n+1}$,

- (a) Trobeu el límit $L = \lim_{n \rightarrow \infty} x_n$ per a $x_0 \geq 0$.
- (b) Descriu el conjunt dels $x_0 < 0$ pels quals el límit $\lim_{n \rightarrow \infty} x_n$ existeix i no és igual a L , o bé no existeix. (Per exemple $x_0 = -1$).

3. (**Examen 2011**) Considereu el sistema dinàmic real definit per $x_{n+1} = \frac{x_n}{4} + x_n^3$. Trobeu el comportament asimptòtic de les òrbites per a tota condició inicial $x_0 \in \mathbb{R}$. Justifiqueu rigorosament les vostres afirmacions.

4. Demostreu rigurosament que $f(x) = \sin(x)$ té $x = 0$ com atractor global.

5. Demostreu que si $f: \mathbb{R} \rightarrow \mathbb{R}$ és derivable, x_0 és un punt fix i $|f'(x_0)| > 1$ llavors x_0 és un punt fix repulsor.

6. Signi $f: \mathbb{R} \rightarrow \mathbb{R}$ de classe \mathcal{C}^∞ i sigui x_0 un punt fix tal que $f'(x_0) = 1$. Doneu criteris sobre les derivades d'ordre superior, per determinar el retrat de fase local al voltant de x_0 . Apliqueu-ho a determinar l'estabilitat dels punts fixos de $x^3 - x$.

1. One-dimensional discrete dynamical system

introduction

Decreasing function orbits

Let:

· *declarations*

·

Show that:

· *statements*

·

Demonstration:

f corta en un punto

f decreasing $\rightarrow f^2$ increasing

$f^{2n} \xrightarrow{n}$ fixed point of f

9. Periodic points

Let:

$$\begin{aligned} f : \mathbb{R} \times \mathbb{R}^+ &\longrightarrow \mathbb{R} \\ (x, r) &\longmapsto r \frac{x}{1+x^2} \end{aligned}$$

Study:

- Periodic points of f

Demonstration:

Graphical analysis :

f odd

f has 2 extrema in ± 1

$f \xrightarrow{n} 0$

Fixed points :

$$f(x) = x \leftrightarrow x = \pm\sqrt{r-1}$$

$$f'(\pm\sqrt{r-1}) = \frac{2-r}{r}$$

n-periodic points :

$$f^n(x) = x$$

10. Global orbit analysis

Let:

- $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \in \mathbb{C}^\infty$
- $f(0) = 0$
- $p \in \mathbb{R}^+ \setminus \{0\} \quad \text{,} \quad f'(p) \geq 0$
- f' decreasing

Show that:

- $\forall x \in \mathbb{R}^+ \setminus \{0\} :$

$$f^n(x) \xrightarrow{n} p$$

Demonstration:

$$f' \text{ decreasing} \rightarrow f'' < 0 \rightarrow f \text{ concave}$$

$$f \text{ positive} \rightarrow f \text{ has no extrema} \rightarrow f' > 0 \rightarrow f \text{ increasing}$$

$$f \text{ has only one fixed point}$$

$$\text{Suppose 2 fixed points : } p, p'$$

$$IVT \rightarrow \exists c \in (0, p') :$$

$$f'(c) = 1$$

$$f'(p) < 1 \rightarrow p \text{ attractive } IVT \rightarrow \text{don't exist more fixed points}$$

$$\rightarrow f'(c') = 1 \not\leq 1$$

$$\forall x \in (0, p) :$$

$$f(x) > x$$

$$\forall x \in \mathbb{R} \quad \text{,} \quad x > p :$$

$$f(x) < x$$

$$f \text{ increasing} \rightarrow f([0, p]) = [0, p]$$

Block V

Laboratory

1. Fixed points cardinality

II. Martin Azpillaga

Let:

- $f : [0, 1] \rightarrow [0, 1] \in \mathcal{C}^2([0, 1])$
- $f(1) < 1$
- $f'' > 0 \in [0, 1]$

Show that:

- $\# \{x \in [0, 1] \mid f(x) = x\} = 1$

Demonstration:

- $\# \{x \in [0, 1] \mid f(x) = x\} \geq 1:$

Case $f(0) = 0$:

0 fixed point

Case $f(0) > 0$:

$$\begin{array}{ccc} g : [0, 1] & \longrightarrow & [-1, 1] \\ x & \longmapsto & f(x) - x \end{array} \in \mathcal{C}^2([0, 1])$$

$$g(0) = f(0) - 0 > 0$$

$$g(1) = f(1) - 1 < 0$$

Bolzano's theorem:

$$\exists x \in (0, 1) :$$

$$g(x) = 0$$

$$f(x) = x$$

- $\# \{x \in [0, 1] \mid f(x) = x\} \leq 1:$

$$g'' > 0 \text{ over } [0, 1]$$

Rolle's theorem:

$$\# \{x \in (0, 1) \mid g'(x) = 0\} \leq 1$$

$$\# \{x \in (0, 1) \mid g(x) = 0\} \leq 2$$

$$\# \{x \in (0, 1) \mid f(x) = x\} \leq 2$$

$$f'' > 0 \text{ over } [0, 1]$$

Monotonicity test:

$$f' \text{ increasing in } [0, 1]$$

$$\forall a < b \in [0, 1] \quad f(a) = a, f(b) = b :$$

Mean Value Theorem:

$$\exists c \in (a, b) :$$

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 1$$

$$\exists d \in (b, 1) :$$

$$\begin{aligned}
 f'(d) &= \frac{f(1)-f(b)}{1-b} < 1 \\
 f' \text{ increasing} &\rightarrow f'(c) < f'(b) < f'(d) \\
 1 < f'(b) < 1 &\text{ absurd} \\
 \therefore \# \{x \in [0, 1] \mid f(x) = x\} &= 1
 \end{aligned}$$

III. Bifurcation Theory

Bifurcation diagram

Let:

$$\begin{aligned}
 f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\
 (a, x) &\longmapsto x^3 - 3x^2 + (5-a)x - 2 + a \\
 \forall a \in \mathbb{R} :
 \end{aligned}$$

$$\begin{aligned}
 f_a : \mathbb{R} &\longrightarrow \mathbb{R} \\
 x &\longmapsto x^3 - 3x^2 + (5-a)x - 2 + a
 \end{aligned}$$

Study:

$$\cdot \text{Bifurcations of } (\mathbb{R}, \mathbb{N}, \{f_a\}_{a \in \mathbb{R}})$$

Start:

Fixed points:

$$f_a(x) = x \Leftrightarrow x^3 - 3x^2 + (4-a)x - 2 + a = 0 \Leftrightarrow (x-1)(x^2 - 2x + 2 - a) = 0$$

$$\Leftrightarrow x = 1 \vee x^2 - 2x + 2 - a = 0 \Leftrightarrow x \in \{1, 1 \pm \sqrt{a-1}\}$$

$$\forall a \in \mathbb{R} \quad \text{,} \quad a \leq 1 :$$

$$\text{Fix}(f_a) = \{1\}$$

$$\forall a \in \mathbb{R} \quad \text{,} \quad a > 1 :$$

$$\text{Fix}(f_a) = \{1, 1 \pm \sqrt{a-1}\}$$

Stability:

$$\partial_x f(a, x) = 3x^2 - 6x + 5 - a$$

$$\partial_{x^2} f(a, x) = 6x - 6$$

$$\partial_{x^3} f(a, x) = 6$$

$$|\partial_x f(a, 1)| < 1 \leftrightarrow |2 - a| < 1 \leftrightarrow a \in (1, 3)$$

$$\partial_{x^2} f(1, 1) = 0, \quad \partial_{x^3} f(1, 1) > 0$$

$$\partial_{x^2} f(3, 1) = 0, \quad \partial_{x^3} f(3, 1) > 0$$

$$\forall a \in (1, 3) :$$

1 attractive

$$\forall a \in \mathbb{R} \setminus (1, 3) :$$

1 repulsive

$$\forall a \in \mathbb{R} \quad a > 1 :$$

$$|\partial_x f(a, 1 \pm \sqrt{a-1})| = |2a-1| > 1$$

$1 \pm \sqrt{a-1}$ repulsive

Pitchfork bifurcation at 1 :

$$\partial_a f(1, 1) = 1 - 1 = 0$$

$$\partial_{x^2} f(1, 1) = 6 - 6 = 0$$

$$\partial_{ax} f(1, 1) = -1 \neq 0$$

$$\partial_{x^3} f(1, 1) = 6 \neq 0$$

Period-doubling bifurcation at 3 :

$$\partial_a f(3, 1) = 0 \rightarrow \partial_a f^2(3, 1) = 0$$

$$\partial_{x^2} f(3, 1), \partial_{x^2} f(3, f(3, 1)) = 0 \rightarrow \partial_a f^2(3, 1) = 0$$

$$\partial_{ax}f^2(3,1) = 2 \neq 0$$

$$\partial_{x^3}f^2(3,1) = -12 \neq 0$$

Source Code

```

#include "stdio.h"
#include "stdlib.h"
#include "math.h"
#include "string.h"

void plot( char *input_file , char *output_file )
{
    FILE *gnuplot;
    gnuplot = popen("gnuplot", "w");
    if( output_file )
    {
        fprintf(gnuplot, "set term svg\n");
        fprintf(gnuplot, "set out \"%s\" \n", output_file );
    }
    fprintf(gnuplot, "plot \"%s\" with dots\n", input_file);
    fflush(gnuplot);
    fclose(gnuplot);
}

double example_function( double param, double point )
{
    return pow(point,3) - 3*pow(point,2) + (5-param)*point - 2 + param;
}

void bifurcation_diagram( int param_min, int param_max, double param_step,
    int point_min, int point_max, int num_points,
    double (*f)(double,double), int num_iter, int tolerancy)
{
    FILE* file;
    double param, point;
    int i,j;

    srand(time(NULL));
    file = fopen("data.dat", "w");

    for ( param = param_min; param < param_max; param += param_step )
    {
        for ( i = 0; i < num_points; i++ )
        {
            point = point_min + ((double) rand() / (double) RAND_MAX) * (point_max -
                point_min);

            for ( j = 0; j < num_iter && abs(point) < tolerancy; j++ )
            {
                point = (*f)(param, point);
            }

            if(abs(point) < tolerancy)
            {
                fprintf(file, "%lf %lf\n", param, point);
            }
        }
    }
    plot( "data.dat", "graph.svg" );
}

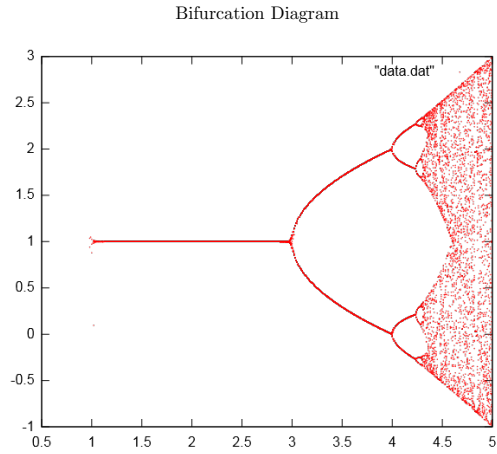
```

```

int main(int argc, char const *argv[])
{
    bifurcation_diagram( 0, 5, 10e-3, 0, 5, 100, &example_function, 100, 10e1);

    return 0;
}

```



IV. Linear maps

Real sequence of order 2

Let:

$$\cdot p, q \in \mathbb{R}$$

$$\cdot \forall a, b \in \mathbb{R} :$$

$$x_0 := a$$

$$x_1 := b$$

$$\forall n \in \mathbb{N} :$$

$$x_{n+2} := px_{n+1} + qx_n$$

Study:

$$\cdot \lim_n \frac{x_n}{x_{n+1}}$$

Start:

Consider:

$$A = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$$

$$A \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}$$

Eigenvalue analysis:

$$\chi_A(t) = t^2 - \text{tr}(A)t + \det(A) = t^2 - pt - q$$

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \chi_A(\lambda) = 0\} = \left\{\frac{p}{2}\right\}$$

$$\lambda := \frac{p}{2}$$

$$E_\lambda(A) = \text{Ker}(A - \lambda \mathbb{1}) = \{(x, y) \in \mathbb{R}^2 \mid \text{-----}\}$$

$\dim(A) = 2 \wedge \sigma(A) = \{\lambda\} \wedge \gamma_A(\lambda) = 1 \rightarrow A$ no diagonalizable

Jordan form:

$$J := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

$$v_2 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_1 := (A - \lambda \mathbb{1})v_2 = \begin{pmatrix} p - \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$$

$$C := (v_1 | v_2) = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}$$

$$C^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A = CJC^{-1} \rightarrow A^n = CJ^nC^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}$$

Sequence analysis:

$\forall a, b \in \mathbb{R} :$

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = A^n \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} 99 \\ 99 \end{pmatrix}$$

$$\lim_n \frac{x_n}{x_{n+1}} = \lambda = \frac{p}{2}$$

Graphical interpretation:

asdfghjkl