

1. 1st laboratory
Existence of holomorphic functions

Let:

$$\cdot f \in \mathcal{H}(\mathbb{D})$$

Study:

$$\cdot \exists f \in \mathcal{H}(\mathbb{D}) :$$

$$\forall n \in \mathbb{N} \quad n \geq 2 :$$

$$a) f\left(\pm \frac{1}{n}\right) = \frac{1}{2n+1}$$

$$b) f\left(\pm \frac{1}{n}\right) = \frac{1}{n^2}$$

$$c) \left|f\left(\frac{1}{n}\right)\right| = \frac{1}{\log(n+1)}$$

$$d) \left|f\left(\frac{1}{n}\right)\right| = \frac{n}{n+1}$$

Demonstration:

a):

$$E_1 := \left\{ +\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N} \right\}$$

$$E_2 := \left\{ -\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N} \right\}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} = \lim_n \frac{f\left(\frac{1}{n}\right) - f(0)}{\frac{1}{n} - 0} = \frac{f(0) - f(0)}{\frac{1}{n} - 0} = \frac{0}{\frac{1}{n} - 0} = 0$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} \begin{cases} = \frac{1}{2} & f(0) = 0 \\ \notin \mathbb{C} & f(0) \neq 0 \end{cases}$$

Case $f(0) = 0$:

$$\lim_{E_2} \frac{f(z) - f(0)}{z - 0} = \lim_n \frac{f\left(-\frac{1}{n}\right) - f(0)}{-\frac{1}{n} - 0} = \frac{f(0) - f(0)}{-\frac{1}{n} - 0} = \frac{0}{-\frac{1}{n} - 0} = 0$$

$$\nexists f \in \mathcal{H}(0) \quad \text{„} f \text{ satisfies } a)$$

In particular:

$$\nexists f \in \mathcal{H}(\mathbb{D}) \quad \text{„} f \text{ satisfies } a)$$

b):

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z^2 \end{aligned}$$

$$\forall n \in \mathbb{N} \quad \text{„} n \geq 2 :$$

$$f\left(\pm \frac{1}{n}\right) = \frac{1}{n^2}$$

f satisfies b)

$$\begin{aligned} \bar{f} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (u(x, y), v(x, y)) = (x^2 - y^2, 2xy) \end{aligned}$$

$$\bar{f} \in \text{Pol}(\mathbb{R}^2) \rightarrow \bar{f} \text{ differentiable in } \mathbb{R}^2$$

$$\forall (x, y) \in \mathbb{R}^2 :$$

$$\partial_x u(x, y) = 2x = \partial_y v(x, y)$$

$$\partial_y u(x, y) = -2y = -\partial_x v(x, y)$$

f satisfies CR

$$\therefore f \in \mathcal{H}(\mathbb{R}^2)$$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

c):

$$\text{Suppose } \exists f \in \mathcal{H}(\mathbb{D}) \quad \text{„} f \text{ satisfies } c)$$

$$f \in \mathcal{C}^0(\mathbb{D}) \rightarrow f(0) = f\left(\lim_n \frac{1}{n}\right) = \lim_n f\left(\frac{1}{n}\right) = 0$$

$$\left| \lim_{E_1} \frac{f(z) - f(0)}{z - 0} \right| = \lim_n \frac{\left| f\left(\frac{1}{n}\right) \right|}{\frac{1}{n}} \notin \mathbb{C}$$

$f \notin \mathcal{H}(0)$ absurd

d):

$$\begin{array}{ccc} f : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \frac{1}{z+1} \end{array}$$

$\forall n \in \mathbb{N} \quad n \geq 2 :$

$$\left| f\left(\frac{1}{n}\right) \right| = \frac{1}{\frac{1}{n}+1} = \frac{n}{n+1}$$

f satisfies d)

$$\begin{array}{ccc} \bar{f} : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (u(x, y), v(x, y)) = \left(\frac{x+1}{(x+1)^2+y^2}, \frac{-y}{(x+1)^2+y^2} \right) \end{array}$$

$\bar{f} \in \text{Rat}(\mathbb{R}^2) \wedge \forall (x, y) \in \mathbb{R}^2 :$

$$(x+1)^2 + y^2 \neq 0$$

\bar{f} differentiable in \mathbb{R}^2

$\forall (x, y) \in \mathbb{R}^2 :$

$$\partial_x u(x, y) = \frac{y^2 - (x+1)^2}{((x+1)^2 + y^2)^2} = \partial_y v(x, y)$$

$$\partial_y u(x, y) = \frac{-2y(x+1)}{((x+1)^2 + y^2)^2} = -\partial_x v(x, y)$$

f satisfies CR

$\therefore f \in \mathcal{H}(\mathbb{R}^2)$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

Constant tests

Let:

$$\cdot \Omega \subset \mathbb{C} \text{ region}$$

$$\cdot f \in \mathcal{H}(\Omega)$$

Then, holds:

$$\cdot f_{Re} = 0 \vee f_{Im} = 0 \rightarrow f \in \text{Cst}(\Omega)$$

$$\cdot |f| \in \text{Cst}(\Omega) \rightarrow f \in \text{Cst}(\Omega)$$

$$\cdot \text{Im} f \text{ circumference} \rightarrow f \in \text{Cst}(\Omega)$$

Demonstration:

$$f_{Re} = 0 \vee f_{Im} = 0:$$

$$u := f_{Re}$$

$$v := f_{Im}$$

$$f \in \mathcal{H}(\Omega) \rightarrow f \text{ satisfies CR in } \Omega$$

$$\partial_x u = \partial_y v = 0$$

$$\partial_y u = -\partial_x v = 0$$

Null differential test:

$$\Omega \text{ connex} \rightarrow u, v \in \text{Cst}(\Omega)$$

$$u, v \in \text{Cst}(\Omega) \rightarrow f \in \text{Cst}(\Omega)$$

$$|f| \in \text{Cst}(\Omega):$$

$$\begin{aligned} |f| : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \sqrt{u(x, y)^2 + v(x, y)^2} \end{aligned}$$

$$|f| \in \text{Cst}(\Omega) \rightarrow \exists a \in \mathbb{R} :$$

$$\sqrt{u(x, y)^2 + v(x, y)^2} = a$$

$$u(x, y)^2 + v(x, y)^2 = a^2$$

$$2\partial_x u(x, y) + 2\partial_x v(x, y) = 0$$

$$2\partial_y u(x, y) + 2\partial_y v(x, y) = 0$$

$$f \in \mathcal{H}(\Omega) \rightarrow f \text{ satisfies CR in } \Omega$$

$$2\partial_y v(x, y) + 2\partial_x v(x, y) = 0$$

$$-2\partial_x v(x, y) + 2\partial_y v(x, y) = 0$$

$$+ : 4\partial_y v(x, y) = 0 \rightarrow \partial_y v(x, y) = 0$$

$$- : 4\partial_x v(x, y) = 0 \rightarrow \partial_x v(x, y) = 0$$

Null differential test:

$$\Omega \text{ connex} \rightarrow u, v \in \text{Cst}(\Omega)$$

$$u, v \in \text{Cst}(\Omega) \rightarrow f \in \text{Cst}(\Omega)$$

$\text{Im}(f)$ circumference :

$$\exists (x_0, y_0) \in \mathbb{R}^2, r \in \mathbb{R}^+ :$$

$$\text{Im}(f) = C_r(x_0, y_0)$$

$$\begin{aligned} \bar{f} : \mathbb{R}^2 &\longrightarrow C_r(x_0, y_0) \\ (x, y) &\longmapsto (r \cos(x - x_0), r \sin(y - y_0)) \end{aligned}$$

$$\forall (x, y) \in \Omega :$$

$$|\bar{f}|(x, y) = \sqrt{r^2(\cos^2(x - x_0) + \sin^2(y - y_0))} = r$$

$$|f| \in \text{Cst}(\Omega) \rightarrow f \in \text{Cst}(\Omega)$$

Real part of holomorphic functions

Let:

$$\cdot \Omega \subset \mathbb{R}^2 \text{ region}$$

$$\cdot u \in \mathcal{C}^2(\Omega) \quad \text{,,} \quad \exists f \in \mathcal{H}(\Omega) :$$

$$f_{Re} = u$$

Show that:

$$\cdot \partial_{xx}u + \partial_{yy}u = 0$$

Study:

$$\cdot \exists f \in \mathcal{H}(\Omega) :$$

$$a) f_{Re}(x, y) = x^2 + y^2$$

$$b) f_{Re}(x, y) = x(x+1) - y^2$$

$$c) \forall \alpha \in \mathbb{R} :$$

$$f_{Re} = y^3 + \alpha x^2 y \wedge \Omega = \mathbb{C}$$

Demonstration:

$$\partial_{xx}u + \partial_{yy}u = 0:$$

$$u := f_{Re}, v := f_{Im}$$

$$f \in \mathcal{H}(\Omega) \rightarrow f \text{ satisfies CR in } \Omega$$

$$\partial_x u = \partial_y v \rightarrow \partial_{xx}u = \partial_{xy}v$$

$$\partial_y u = -\partial_x v \rightarrow \partial_{yy}u = -\partial_{xy}v$$

$$\therefore \partial_{xx}u + \partial_{yy}u = 0$$

$$f_{Re}(x, y) = x^2 + y^2:$$

$$\partial_{xx}u + \partial_{yy}u = 4 \neq 0$$

$$\nexists f \in \mathcal{H}(\Omega) \quad \text{,} \quad f_{Re}(x, y) = x^2 + y^2$$

$$f_{Re}(x, y) = x(x+1) - y^2:$$

$$\begin{aligned} \bar{f} : \Omega &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (u(x, y), v(x, y)) = (x(x+1) - y^2, 2xy + y) \end{aligned}$$

$$\bar{f} \in \text{Pol}(\Omega) \rightarrow \bar{f} \text{ differentiable in } \Omega$$

$$\forall (x, y) \in \Omega :$$

$$\partial_x u(x, y) = 2x + 1 = \partial_y v(x, y)$$

$$\partial_y u(x, y) = -2y = -\partial_x v(x, y)$$

$$f \text{ satisfies CR in } \Omega$$

$$f \in \mathcal{H}(\Omega) \wedge f_{Re} = u$$

$$f_{Re}(x, y) = y^3 + \alpha x^2 y:$$

$$f \text{ has to satisfy CR in } \mathbb{C}:$$

$$\forall (x, y) \in \mathbb{R}^2 :$$

$$\partial_y v(x, y) = \partial_x u(x, y) = 2\alpha xy$$

$$\partial_x v(x, y) = -\partial_y u(x, y) = -3y^2 - \alpha x^2$$

$$v(x, y) = \alpha xy^2 + c(x)$$

$$v(x, y) = -3xy^2 - \frac{\alpha}{3}x^3 + c(y)$$

$$\alpha = -3, \quad c(x) = x^3, \quad c(y) = 0$$

$$v(x, y) = -3xy^2 + x^3$$