1. Fixed points cardinality

Martin Azpillaga

Let:

$\cdot\ f\ :\ [0,1] \to [0,1] \in \mathcal{C}^2([0,1])$

$\cdot\ f(1) < 1$

$\cdot\ f'' > 0 \in [0,1]$

Show that:

$\cdot\ \#\ \{x \in [0,1] \mid f(x) = x\} = 1$

Demonstration:

$\#\ \{x \in [0,1] \mid f(x) = x\} \geq 1$:

Case $f(0) = 0$:

0 fixed point

Case $f(0) > 0$:

$$\begin{array}{rcl} g\ : [0,1] & \longrightarrow & [-1,1] \\ x & \longmapsto & f(x) - x \end{array} \in \mathcal{C}^2([0,1])$$

$g(0) = f(0) - 0 > 0$

$g(1) = f(1) - 1 < 0$

Bolzano's theorem:

$\exists\ x \in (0,1)$:

$g(x) = 0$

$f(x) = x$

$\# \{x \in [0,1] \mid f(x) = x\} \leq 1$:

$f'' > 0$ over $[0,1]$

Rolle's theorem:

$\# \{x \in (0,1) \mid f'(x) = 0\} \leq 1$

$\# \{x \in (0,1) \mid f(x) = 0\} \leq 2$

$f'' > 0$ over $[0,1]$

Monotonicity test:

f' increasing in $[0,1]$

$\forall\ a < b \in [0,1)$ „ $f(a) = a, f(b) = b$:

Mean Value Theorem:

$\exists\ c \in (a,b)$:

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 1$$

$\exists\ d \in (b,1)$:

$$f'(d) = \frac{f(1) - f(b)}{1 - b} < 1$$

f' increasing $\to f'(c) < f'(b) < f'(d)$

$1 < f'(b) < 1$ absurd

$\therefore)\ \# \{x \in [0,1] \mid f(x) = x\} = 1$