# 1. 3rd laboratory

## Bifurcation theory

Let:

$$\cdot \quad \begin{array}{rcl} f : \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (a,x) & \longmapsto & x^3 - 3x^2 + (5-a)x - 2 + a \end{array}$$

$\cdot\ \forall\ a \in \mathbb{R}$ :

$$\begin{array}{rcl} f_a : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^3 - 3x^2 + (5-a)x - 2 + a \end{array}$$

Study:

$\cdot$ Bifurcations of $(\mathbb{R}, \mathbb{N}, \{f_a\}_{a\in\mathbb{R}})$

Start:

Fixed points:

$$f_a(x) = x \leftrightarrow x^3 - 3x^2 + (4-a)x - 2 + a = 0 \leftrightarrow (x-1)(x^2 - 2x + 2 - a) = 0$$

$$x^2 - 2x + 2 - a = 0 \leftrightarrow x = \pm\sqrt{a-1}$$

$\forall\ a \in \mathbb{R}$ " $a \le 1$ :

$$\mathrm{Fix}(f_a) = \{1\}$$

$\forall\ a \in \mathbb{R}$ " $a > 1$ :

$$\mathrm{Fix}(f_a) = \{1, \pm\sqrt{a-1}\}$$

Stability:

$$\partial_x f_a(x) = 3x^2 - 6x + 5 - a$$

$$\partial_{x^2} f_a(x) = 6x - 6$$

$$\partial_{x^3} f_a(x) = 6$$

$|\partial_x f_a(1)| < 1 \leftrightarrow |2 - a| < 1 \leftrightarrow a \in (1, 3)$

$\partial_{x^2} f_1(1) = 0, \partial_{x^3} f_1(1) > 0$

$\partial_{x^2} f_3(1) = 0, \partial_{x^3} f_3(1) > 0$

$\forall\ a \in \mathbb{R}\ \text{״}\ a \leq 1\ \vee\ a \geq 3\ :$

1 repulsive

$\forall\ a \in \mathbb{R}\ \text{״}\ a \in (1, 3)\ :$

1 attractive

$\forall\ a \in \mathbb{R}\ \text{״}\ a > 1\ :$

$\left|\partial_x f_a(\pm\sqrt{a-1})\right| = |2a - 1| > 1$

$\pm\sqrt{a-1}$ repulsive

Pitchfork bifurcation at $1$ :

$\partial_a f(1, 1) = 1 - 1 = 0$

$\partial_{x^2} f(1, 1) = 6 - 6 = 0$

$\partial_{ax} f(1, 1) = -1 \neq 0$

$\partial_{x^3} f(1, 1) = 6 \neq 0$

Period-doubling bifurcation at $3$ :

$\partial_a f^2(3, 1) = \partial_a f(f(3, 1))\partial_a f(3, 1) = 0$

$\partial_{x^2} f^2(3, 1) = \partial_a f(f(3, 1))\partial_a f(3, 1) = 0$

$\partial_{ax} f^2(3, 1) = \partial_{ax} f(f(3, 1))\partial_a f(3, 1) \neq 0$

$\partial_{x^3} f^2(3, 1) = \partial_{x^3} f(f(3, 1))\partial_{x^3} f(3, 1) \neq 0$

Python program:

Graphical result