1. Discrete dynamical systems

Dynamical system

Let:

- $\cdot$ M manifold
- $\cdot$ T monoid
- $\cdot$ $\phi \,:\, M \times T \to M$

Then, (M, T, ϕ) is a dynamical system if:

$\cdot$ $\forall\ x \in X\ :$

$\phi(x, 0) = 0$

$\forall\ t_1, t_2 \in T\ :$

$\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$

Dimension

Let:

- $\cdot$ (M, T, ϕ) dynamical system

We name dimension of (M, T, ϕ) to:

- $\cdot$ $\dim(M)$

We denote:

- $\cdot$ $dim(M) = n\ :\ (M, T, \phi)$ n-D

Discrete & Continuous

Let:

- (M, T, ϕ) dynamical system

Then, (M, T, ϕ) is discrete if:

- $T \underset{\sim}{\subset} \mathbb{N}$

Then, (M, T, ϕ) is continuous if:

- $T \subset \mathbb{R}$

- T open

Defined by a function

Let:

- (M, T, ϕ) dynamical system

- $f \ : \ M \to M$

Then, (M, T, ϕ) is a dynamical system defined by f if:

- $T = \mathbb{N}$

- $\begin{aligned} \phi : M \times \mathbb{N} &\longrightarrow M \\ (x, n) &\longmapsto f^n(x) \end{aligned}$

We denote:

- (M, T, ϕ) dynamical system defined by $f \ : \ (M, \mathbb{N}, f)$

- $f \in \mathcal{C}^n(M) \ : \ (M, \mathbb{N}, f) \ \ \mathcal{C}^n$

Orbit

Let:

· $(M, \mathbb{N}, f)$ functional dynamical system

· $x \in M$

We name orbit of x to:

· $\{f^n(x)\}_{n \in \mathbb{N}}$

We denote:

· $o(x)$

Periodicity

Let:

· $(M, \mathbb{N}, f)$ dynamical system

· $x \in M$

· $m \in \mathbb{N}$

Then, x is a m-periodic point if:

· $f^m(x) = x$

We denote:

· $\{x \in M \mid f(x) = x\}$: $\mathrm{Fix}(f)$

Stability

Let:

$\cdot\ f\ :\ \mathbb{R}^n \to \mathbb{R}^n$

$\cdot\ p \in \mathbb{R}^n$ m-periodic point

Then, p is stable if:

$\cdot\ \forall\ \varepsilon \in \mathbb{R}^+\ :$

$\exists\ \delta \in \mathbb{R}^+\ :$

$\forall\ x \in B(p,\delta)\ :$

$\forall\ n \in \mathbb{N}\ :$

$f^{nm}(x) \in B(p,\varepsilon)$

Then, p is unstable if:

$\cdot\ p$ not stable

Attractive & Repulsive

Let:

- $f : \mathbb{R}^n \to \mathbb{R}^n$
- $p \in \mathbb{R}^n$ m-periodic point

Then, p is attractive if:

- p stable
- $\exists\, \varepsilon \in \mathbb{R}^+ :$

  $\forall\, x \in B(p, \varepsilon) :$

  $f^{nm}(x) \xrightarrow{n} p$

Then, p is repulsive if:

- p attractive by f^{-1}
- $\forall\, \mathcal{U} \subset M \;{}_{\shortparallel}\; \mathcal{U} \text{ open } \wedge x \in \mathcal{U} :$

  $\forall\, x' \in \mathcal{U} \;{}_{\shortparallel}\; x' \neq x :$

  $\exists\, N \in \mathbb{N} :$

  $\forall\, n \in \mathbb{N} \;{}_{\shortparallel}\; n \geq N :$

  $f^{nm}(x') \notin \mathcal{U}$

Fixed point character

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system

We name Fixed point character to:

$$\cdot \begin{aligned} f \, : \mathrm{Fix}(f) &\longrightarrow \{-1, 0, 1\} \\ x &\longmapsto \begin{cases} +1 & x \text{ repulsive} \\ -1 & x \text{ attractive} \\ 0 & x \text{ no attractive no repulsive} \end{cases} \end{aligned}$$

We denote:

- $f \, : \, \chi_f$

Attraction set

Let:

- $(M, \mathbb{N}, f)$ dynamical system
- $x \in M$ attractive m-periodic point
- $o(x)$ orbit of x

We name attraction set of x to:

- $\{y \in M \mid \exists\, x' \in o(x) \, : \, f^{nm}(y) \xrightarrow{n} x'\}$

We denote:

- $A(x)$

Multiplier

Let:

$\cdot\ (M, \mathbb{N}, f)\ \mathcal{C}^1$ dynamical system

$\cdot\ x \in M$

We name multiplier of x to:

$\cdot\ f'(x)$

We denote:

$\cdot\ m(x)$

$\cdot\ |m(x)| = 1\ :\ x$ neutral point

Feeble point

Let:

$\cdot\ (M, \mathbb{N}, f)\ \mathcal{C}^3$ dynamical system

$\cdot\ x \in M$

Then, x is feeble point if:

$\cdot\ x$ neutral point

$\cdot\ f''(x) = 0$

Sarkovskii's order

We name Sarkovskii's order to:

$\cdot\ a = 2^n a', b = 2^m b'$

$$\cdot\ a <_s b \leftrightarrow \begin{cases} m < n & a' = b' = 1 \\ & a' = 1, b' \neq 1 \\ a' < b' & a' = b' \neq 1 \\ n < m & 1 \neq a' \neq b' \end{cases}$$

Chaos

Let:

$\cdot\ (\mathbb{R}, \mathbb{N}, f)$ dynamical system

Then, $(\mathbb{R}, \mathbb{N}, f)$ is chaotic if:

$\cdot\ \mathrm{Fix}(f)$ dense in $\mathbb{R}$

$\cdot\ \exists\ x \in \mathbb{R}$:

$o(x)$ dense in $\mathbb{R}$

$\cdot\ \forall\ x \in \mathbb{R}$:

$\exists\ \varepsilon \in \mathbb{R}^+$:

$\forall\ \delta \in \mathbb{R}^+$:

$\exists\ \tilde{x} \in B(x, \delta)$:

$$\lim_n\ o(\tilde{x}) \notin B(x, \varepsilon)$$

Topological equivalence

Let:

- $(M, \mathbb{N}, f_1), (M, \mathbb{N}, f_2')$ dynamical systems

Then, $(M, \mathbb{N}, f_1)$ is topologically equivalent to $(M, \mathbb{N}, f_2)$ if:

- $\mathrm{Fix}(f) = \mathrm{Fix}(f')$
- $\forall\ x \in \mathrm{Fix}(f)$:

$$\chi_{f_1}(x) = \chi_{f_2}(x)$$

We denote:

- $(M, \mathbb{N}, f) \sim (M, \mathbb{N}, f')$

Bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$

Then, λ_0 is a bifurcation value if:

- $\forall\ \varepsilon \in \mathbb{R}^+$:

$$\exists\ \lambda_1, \lambda_2 \in B(\lambda_0, \varepsilon) :$$

$$(M, \mathbb{N}, f_{\lambda_1}) \nsim (M, \mathbb{N}, f_{\lambda_2})$$

Saddle-node bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is a saddle-node bifurcation value at x_0 if:

- $x_0 \in \mathrm{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0) \neq 0$
- $\partial_{xx} f_\lambda(x_0) \neq 0$

Pitchfork bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is Pitchfork bifurcation value at x_0 if:

- $x_0 \in \mathrm{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{x^2} f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{\lambda x} f_\lambda(x_0, \lambda_0) \neq 0$
- $\partial_{x^3} f_\lambda(x_0, \lambda_0) \neq 0$

Period doubling bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is Period doubling bifurcation value at x_0 if:

- λ_0 Pitchfork bifurcation value at x_0 of f^2

Invariant curve

Let:

- γ differentiable curve
- $p \in \mathbb{R}^n$

Then, γ is invariant if:

- $\forall\ x \in \gamma* :$

  $$o(x) \subset \gamma*$$

Then, γ is converges to $p$if:

- $\forall\ x \in \gamma* :$

  $$o(x) \xrightarrow{n} p$$