Block I

Definitions

1. Discrete dynamical systems

Dynamicalsystem Let:
- M manifold
- T monoid
- $\phi : M \times T \to M$

Then, (M, T, ϕ) is a dynamical system if:
- $\forall\ x \in X$:
  - $\phi(x, 0) = 0$
  - $\forall\ t_1, t_2 \in T$:
    - $\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$

Dimension Let:
- (M, T, ϕ) dynamical system

We name dimension of (M, T, ϕ) to:

$\dim(M)$

We denote:
- $dim(M) = n : (M, T, \phi)$ n-D

Discrete&Continuous Let:
- (M, T, ϕ) dynamical system

Then, (M, T, ϕ) is discrete if:
- $T \overset{\subseteq}{\sim} \mathbb{N}$

Then, (M, T, ϕ) is continuous if:
- $T \subset \mathbb{R}$
- T open

Definedbyafunction Let:
- (M, T, ϕ) dynamical system
- $f : M \to M$

Then, (M, T, ϕ) is a dynamical system defined by f if:
- $T = \mathbb{N}$
- $\begin{array}{rccc} \phi : & M \times \mathbb{N} & \longrightarrow & M \\ & (x, n) & \longmapsto & f^n(x) \end{array}$

We denote:
- (M, T, ϕ) dynamical system defined by $f : (M, \mathbb{N}, f)$
- $f \in \mathcal{C}^n(M) : (M, \mathbb{N}, f)\ \mathcal{C}^n$

Orbit Let:
- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$

We name orbit of x to:

$\{f^n(x)\}_{n \in \mathbb{N}}$

We denote:

- $o(x)$

Periodicity Let:

- $(M, \mathbb{N}, f)$ dynamical system
- $x \in M$
- $m \in \mathbb{N}$

Then, x is a m-periodic point if:

- $f^m(x) = x$

We denote:

- $\{x \in M \mid f(x) = x\} \ : \ \mathrm{Fix}(f)$

Stability Let:

- $f \ : \ \mathbb{R}^n \to \mathbb{R}^n$
- $p \in \mathbb{R}^n$ m-periodic point

Then, p is stable if:

- $\forall\ \varepsilon \in \mathbb{R}^+$:
  - $\exists\ \delta \in \mathbb{R}^+$:
    - $\forall\ x \in B(p, \delta)$:
      - $\forall\ n \in \mathbb{N}$:
        - $f^{nm}(x) \in B(p, \varepsilon)$

Then, p is unstable if:

- p not stable

Attractive&Repulsive Let:

- $f \ : \ \mathbb{R}^n \to \mathbb{R}^n$
- $p \in \mathbb{R}^n$ m-periodic point

Then, p is attractive if:

- p stable
- $\exists\ \varepsilon \in \mathbb{R}^+$:
  - $\forall\ x \in B(p, \varepsilon)$:
    - $f^{nm}(x) \xrightarrow{n} p$

Then, p is repulsive if:

- p attractive by f^{-1}
- $\forall\ \mathcal{U} \subset M \ {}_{''}\ \mathcal{U}$ open $\ \wedge\ x \in \mathcal{U}$:
  - $\forall\ x' \in \mathcal{U} \ {}_{''}\ x' \neq x$:
    - $\exists\ N \in \mathbb{N}$:

$\cdot\ \forall\, n \in \mathbb{N}$ " $n \geq N$:
$\cdot\ f^{nm}(x') \notin \mathcal{U}$

Fixedpointcharacter Let:
$\cdot\ (M, \mathbb{N}, f)$ functional dynamical system
We name Fixed point character to:

$$f\ :\quad \mathrm{Fix}(f) \longrightarrow \{-1,0,1\}$$
$$x \longmapsto \begin{cases} +1 & x \text{ repulsive} \\ -1 & x \text{ attractive} \\ 0 & x \text{ no attractive no repulsive} \end{cases}$$

We denote:
$\cdot\ f\ :\ \chi_f$
Attractionset Let:
$\cdot\ (M, \mathbb{N}, f)$ dynamical system
$\cdot\ x \in M$ attractive m-periodic point
$\cdot\ o(x)$ orbit of x
We name attraction set of x to:
$\{y \in M \mid \exists\, x' \in o(x)\ :\ f^{nm}(y) \xrightarrow{n} x'\}$
We denote:
$\cdot\ A(x)$
Multiplier Let:
$\cdot\ (M, \mathbb{N}, f)$ $\mathcal{C}^1$ dynamical system
$\cdot\ x \in M$
We name multiplier of x to:
$f'(x)$
We denote:
$\cdot\ m(x)$
$\cdot\ |m(x)| = 1\ :\ x$ neutral point
Feeblepoint Let:
$\cdot\ (M, \mathbb{N}, f)$ $\mathcal{C}^3$ dynamical system
$\cdot\ x \in M$
Then, x is feeble point if:
$\cdot\ x$ neutral point
$\cdot\ f''(x) = 0$
Sarkovskii'sorder We name Sarkovskii's order to:
$a = 2^n a', b = 2^m b'$

$$a <_s b \leftrightarrow \begin{cases} m < n & a' = b' = 1 \\ & a' = 1, b' \neq 1 \\ a' < b' & a' = b' \neq 1 \\ n < m & 1 \neq a' \neq b' \end{cases}$$

Chaos Let:

- $(\mathbb{R}, \mathbb{N}, f)$ dynamical system

Then, $(\mathbb{R}, \mathbb{N}, f)$ is chaotic if:

- $\text{Fix}(f)$ dense in $\mathbb{R}$
- $\exists\ x \in \mathbb{R}$:
  - $o(x)$ dense in $\mathbb{R}$

- $\forall\ x \in \mathbb{R}$:
  - $\exists\ \varepsilon \in \mathbb{R}^+$:
    - $\forall\ \delta \in \mathbb{R}^+$:
      - $\exists\ \tilde{x} \in B(x, \delta)$:
        - $\lim\limits_{n}\ o(\tilde{x}) \notin B(x, \varepsilon)$

Topologicalequivalence Let:

- $(M, \mathbb{N}, f_1), (M, \mathbb{N}, f_2')$ dynamical systems

Then, $(M, \mathbb{N}, f_1)$ is topologically equivalent to $(M, \mathbb{N}, f_2)$ if:

- $\text{Fix}(f) = \text{Fix}(f')$
- $\forall\ x \in \text{Fix}(f)$:
  - $\chi_{f_1}(x) = \chi_{f_2}(x)$

We denote:

- $(M, \mathbb{N}, f) \sim (M, \mathbb{N}, f')$

Bifurcation Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$

Then, λ_0 is a bifurcation value if:

- $\forall\ \varepsilon \in \mathbb{R}^+$:
  - $\exists\ \lambda_1, \lambda_2 \in B(\lambda_0, \varepsilon)$:
    - $(M, \mathbb{N}, f_{\lambda_1}) \not\sim (M, \mathbb{N}, f_{\lambda_2})$

$Saddle-nodebifurcation$ Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda\in\Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is a saddle-node bifurcation value at x_0 if:

- $x_0 \in \text{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0) \neq 0$
- $\partial_{xx} f_\lambda(x_0) \neq 0$

$Pitchforkbifurcation$ Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda\in\Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is Pitchfork bifurcation value at x_0 if:

- $x_0 \in \text{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{x^2} f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{\lambda x} f_\lambda(x_0, \lambda_0) \neq 0$
- $\partial_{x^3} f_\lambda(x_0, \lambda_0) \neq 0$

$Perioddoublingbifurcation$ Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda\in\Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is Period doubling bifurcation value at x_0 if:

- λ_0 Pitchfork bifurcation value at x_0 of f^2

$Invariantcurve$ Let:

- γ differentiable curve
- $p \in \mathbb{R}^n$

Then, γ is invariant if:

- $\forall\ x \in \gamma*$:
  - $o(x) \subset \gamma*$

Then, γ is converges to $p$if:

- $\forall\ x \in \gamma*$:
  - $o(x) \xrightarrow{n} p$

2. 2-D linear dynamical systems

Linearsystem Let:

· $(M, \mathbb{N}, f)$ functional dynamical system

Then, $(M, \mathbb{N}, f)$ is linear if:

· $\exists\ A \in \mathcal{M}_{n\times n}(\mathbb{R})$:

$$\cdot \begin{array}{rccc} f: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \longmapsto & Ax \end{array}$$

Multiplier Let:

· $(M, \mathbb{N}, f)$ functional dynamical system

· $x \in M$

We namemultiplier of x to:

$Df(p)$