Block I

Laboratory

1. Orbit analysis

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Let:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^3 + \tfrac{1}{4}x \end{aligned}.$$

Study:

· Orbit behavior of the real dynamical system defined by f

Demonstration:

Formalization :

Consider (M, T, ϕ) where:

$M = \mathbb{R}$

$T = \mathbb{N}$

$$\begin{aligned} \phi : \mathbb{R} \times \mathbb{N} &\longrightarrow \mathbb{R} \\ (x, n) &\longmapsto f^n(x) \end{aligned}$$

Study the orbits of $(\mathbb{R}, \mathbb{N}, \phi)$

We will denote $f^n(x)$ as x_n

Fixed points :

$\forall\ x \in \mathbb{R}$:

$$f(x) = x \leftrightarrow x^3 + \tfrac{1}{4}x - x = 0 \leftrightarrow x^3 - \tfrac{3}{4}x = 0$$

$$\leftrightarrow x = 0 \ \vee\ x^2 - \tfrac{3}{4} = 0$$

$$x \text{ fixed point} \leftrightarrow x \in \{0, \pm\tfrac{\sqrt{3}}{2}\}$$

Graphic analysis :

Parity:

$\forall\ x \in \mathbb{R}$:

$$f(-x) = (-x)^3 + \frac{(-x)}{4} = -(x^3 + \frac{x}{4}) = -f(x)$$

f is odd

Monotonicity:

$\forall\ x \in \mathbb{R}$:

$$f'(x) = 3x^2 + \frac{1}{4} > 0$$

f is increasing over $\mathbb{R}$

Convexity:

$\forall\ x \in \mathbb{R}^-$:

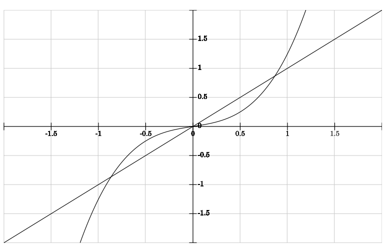
$$f''(x) = 6x \leq 0$$

$\forall\ x \in \mathbb{R}^+$:

$$f''(x) = 6x \geq 0$$

f is concave over $\mathbb{R}^-$ and convex over $\mathbb{R}^+$

Graphic representation :



$\underline{\text{I}} \forall\ x \in (-\infty, -\frac{\sqrt{3}}{2})$:

Induction over n:

f incresing $\rightarrow f(x_n) < f(-\frac{\sqrt{3}}{2})$

$x_{n+1} \in (-\infty, -\frac{\sqrt{3}}{2})$

$\therefore)$ $o(x)$ is enclosed in $(-\infty, -\frac{\sqrt{3}}{2})$

Induction over n:

$x_n^2 > \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) > 0$

$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) < 0$

$\therefore)$ $o(x)$ decreasing

$\nexists\ x < -\frac{\sqrt{3}}{2}$ „ x fixed point $\rightarrow o(x) \xrightarrow{n} -\infty$

$\underline{\text{II}} \forall\ x \in (-\frac{\sqrt{3}}{2}, 0)$:

Induction over n:

f increasing $\rightarrow f(-\frac{\sqrt{3}}{2}) < f(x_n) < f(0)$

$x_{n+1} \in (-\frac{\sqrt{3}}{2}, 0)$

$\therefore)$ $o(x)$ is enclosed in $(-\frac{\sqrt{3}}{2}, 0)$

Induction over n:

$x_n^2 < \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) < 0$

$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) > 0$

$\therefore)$ $o(x)$ increasing

$o(x)$ convergent $\wedge\ 0$ fixed point $\rightarrow o(x) \xrightarrow{n} 0$

$\mathrm{I}\underline{\mathrm{I}}\mathrm{I} \forall\, x \in (0, \frac{\sqrt{3}}{2})$:

Induction over n:

$-x_n \in (-\frac{\sqrt{3}}{2}, 0)$

$\underline{\mathrm{II}} \to f(-x_n) \in (-\frac{\sqrt{3}}{2}, 0) \;\wedge\; f(-x_n) > -x_n$

f odd $\to f(x_n) = -f(-x_n) \in (0, \frac{\sqrt{3}}{2})$

f odd $\to f(x_n) = -f(-x_n) < x_n$

$\therefore)$ $o(x)$ is enclosed in $(0, \frac{\sqrt{3}}{2}) \;\wedge\; o(x)$ decreasing

$o(x)$ convergent $\wedge$ 0 fixed point $\to o(x) \xrightarrow{n} 0$

$\mathrm{I}\underline{\mathrm{V}} \forall\, x \in \mathbb{R}$ „ $x > \frac{\sqrt{3}}{2}$:

Induction over n:

$-x_n \in (\frac{\sqrt{3}}{2}, \infty)$

$\underline{\mathrm{I}} \to f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty) \;\wedge\; f(-x_n) < -x_n$

f odd $\to f(x_n) = -f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty)$

f odd $\to f(x_n) = -f(-x_n) > x_n$

$\therefore)$ $o(x)$ is inf bounded by in $\frac{\sqrt{3}}{2} \;\wedge\; o(x)$ increasing

$o(x)$ convergent

$\nexists\, x > \frac{\sqrt{3}}{2}$ „ x fixed point $\to o(x) \xrightarrow{n} +\infty$

2. Fixed points cardinality

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Let:

$\cdot\, f \,:\, [0,1] \to [0,1] \in \mathcal{C}^2([0,1])$

$\cdot\, f(1) < 1$

$\cdot\, f'' > 0 \in [0,1]$

Show that:

$\cdot\, \#\, \{x \in [0,1] \mid f(x) = x\} = 1$

Demonstration:

$\#\, \{x \in [0,1] \mid f(x) = x\} \geq 1$:

Case $f(0) = 0$:

0 fixed point

Case $f(0) > 0$:

$$\begin{array}{rcl} g\ : [0,1] & \longrightarrow & [-1,1] \\ x & \longmapsto & f(x) - x \end{array} \in \mathcal{C}^2([0,1])$$

$g(0) = f(0) - 0 > 0$

$g(1) = f(1) - 1 < 0$

Bolzano's theorem:

$\exists\, x \in (0,1)$:

$g(x) = 0$

$f(x) = x$

$\# \{x \in [0,1] \mid f(x) = x\} \leq 1$:

$g'' > 0$ over $[0,1]$

Rolle's theorem:

$\# \{x \in (0,1) \mid g'(x) = 0\} \leq 1$

$\# \{x \in (0,1) \mid g(x) = 0\} \leq 2$

$\# \{x \in (0,1) \mid f(x) = x\} \leq 2$

$f'' > 0$ over $[0,1]$

Monotonicity test:

f' increasing in $[0,1]$

$\forall\ a < b \in [0,1)$ 〃 $f(a) = a, f(b) = b$:

Mean Value Theorem:

$\exists\ c \in (a,b)$:

$f'(c) = \frac{f(b)-f(a)}{b-a} = 1$

$\exists\ d \in (b,1)$:

$f'(d) = \frac{f(1)-f(b)}{1-b} < 1$

f' increasing $\rightarrow f'(c) < f'(b) < f'(d)$

$1 < f'(b) < 1$ absurd

$\therefore)\ \# \{x \in [0,1] \mid f(x) = x\} = 1$