

Probability

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Block I

Definitions

1. Random Variables

The study of probabilities was started by Blaise Pascal.

Probability Space

Let:

- Ω set
- \mathcal{A} - σ -algebra
- $\mathcal{P} : \mathcal{A} \rightarrow [0, 1]$

Then, $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space if:

- $\mathcal{P}(\Omega) = 1$
 - $\forall \{A_i\}_{i \geq 0} \subset \mathcal{A}$ \parallel $\{A_i\}_{i \geq 0}$ mutually disjoint :
- $$\mathcal{P}\left(\bigcup_{i \geq 0} A_i\right) = \sum_{i \geq 0} \mathcal{P}(A_i)$$

Conditioned Probability

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space $B \in \mathcal{A}$

We call probability conditioned by B over $(\Omega, \mathcal{A}, \mathcal{P})$ to:

$$\begin{aligned} \cdot \quad \mathcal{P}|_B : \mathcal{A} &\longrightarrow [0, 1] \\ \cdot \quad A &\longmapsto \frac{\mathcal{P}(A \cap B)}{\mathcal{P}(B)} \end{aligned}$$

Independency

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space $\{A_i\}_{i=1}^n \subset \mathcal{A}$

Then, $\{A_i\}_{i=1}^n$ is an independent family if:

- $\forall \{k_i\}_{i=1}^r \subset [0, n]_{\mathbb{N}}$:
- $$\mathcal{P}\left(\bigcap_{i=1}^r A_{k_i}\right) = \prod_{i=1}^r \mathcal{P}(A_{k_i})$$

Random Variable

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \rightarrow \mathbb{R}$

Then, X is a random variable if:

- $\forall A \in \mathbb{B}(\mathbb{R})$:
- $$X^{-1}(A) \in \mathcal{A}$$

Law

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \rightarrow \mathbb{R}$ random variable

We call law of X to:

$$\begin{aligned} \cdot \quad \mathcal{P}_X : \mathbb{B}(\mathbb{R}) &\longrightarrow [0, 1] \\ &A \longmapsto \mathcal{P}(X^{-1}(A)) \end{aligned}$$

Distribution Function

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \rightarrow \mathbb{R}$ random variable

We call cumulative distribution function of X to:

$$\begin{aligned} \cdot \quad F_x : \mathbb{R} &\longrightarrow [0, 1] \\ x &\longmapsto \mathcal{P}(X \leq x) \end{aligned}$$

Discrete Random Variable

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \rightarrow \mathbb{R}$ random variable

Then, X is a discrete random variable if:

- $X(\Omega) \lesssim \mathbb{N}$

Probability Mass function

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \rightarrow \mathbb{R}$ discrete random variable

We call probability mass function of X to:

$$\begin{aligned} p : X(\Omega) &\longrightarrow [0, 1] \\ x &\longmapsto P(X = x) \end{aligned}$$

Density function

Let:

$$\cdot f : \mathbb{R} \rightarrow \mathbb{R}$$

Then, f is a density function if:

$$\cdot f \geq 0$$

$$\cdot f \mathcal{R}(\mathbb{R})$$

$$\cdot \int_{\mathbb{R}} f(x) dx = 1$$

Absolutely Continuous random variable

Let:

$$\cdot (\Omega, \mathcal{A}, \mathcal{P}) \text{ probability space}$$

$$\cdot X : \Omega \rightarrow \mathbb{R} \text{ random variable}$$

$$\cdot F_X \text{ distribution function of } X$$

Then, X is an absolutely continuous random variable if:

$$\cdot \exists f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \quad \text{,,} \quad \forall x \in \mathbb{R}:$$

$$F_x(x) = \int_{-\infty}^x f(x) dx$$

We denote:

$$\cdot X \text{ an absolutely continuous random variable : } X \text{ abs cont}$$

Expected Value

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \rightarrow \mathbb{R}$ discrete random variable
- p probability mass function of X
- $X' : \Omega \rightarrow \mathbb{R}$ abs cont
- f_X density function of X'

Then, X is expectable if:

$$\cdot \sum_{x \in X(\Omega)} |x|p(x) \in \mathbb{R}$$

We call expected value of X to:

$$\cdot \sum_{x \in X(\Omega)} xp(x)$$

Then, X' is expectable if:

$$\cdot \int_{\mathbb{R}} |x|f(x)dx \in \mathbb{R}$$

We call expected value of X' to:

$$\cdot \int_{\mathbb{R}} xf(x)dx$$

We denote:

$$\cdot \text{expected value of } X : E(X)$$

Variable change

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \rightarrow \mathbb{R}$ abs cont
- $I = \overset{\circ}{I} \subset \mathbb{R} \quad \text{,,} \quad \mathcal{P}(X \in I) = 1$
- $J = \overset{\circ}{J} \subset \mathbb{R}$
- $g : I \rightarrow J$

Then, g is a variable change over X if:

- g bijective
- $g, g^{-1} \in \mathcal{C}^1$

2. Random Vectors

Generalization of random variables to n dimension

Aleatory vector

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \rightarrow \mathbb{R}^m$

Then, X is a random vector if:

- $\forall i \in [1, m]_{\mathbb{N}}$:
 X_i random variable

Law

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- $X : \Omega \rightarrow \mathbb{R}$ random variable

We call law of X to:

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Let:

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- $X : \Omega \rightarrow \mathbb{R}^m$ random vector

We call cumulative distribution function of X to:

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i th Marginal distribution function

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \rightarrow \mathbb{R}^m$ random vector
- $i \in [1, m]_{\mathbb{N}}$

We call i th marginal distribution function of X to:

$$\begin{aligned} \cdot f &: \mathbb{R} \longrightarrow [0, 1] \\ \cdot x &\longmapsto \mathcal{P}(X_i \leq x) \end{aligned}$$

We denote:

$$\cdot f : F_i$$

Discrete Random Variable

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \rightarrow \mathbb{R}$ random variable

Then, X is a discrete random variable if:

$$\cdot X(\Omega) \stackrel{\circ}{\sim} \mathbb{N}$$

Probability Mass function

Let:

· $(\Omega, \mathcal{A}, \mathcal{P})$ probability space

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We call probability mass function of X to:

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Block II

Propositions

Block III

Examples

Block IV

Exercises