I. Logarithm determinations

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Let:

$$\cdot \quad \begin{array}{rccc} log : & \mathbb{C} \setminus (-\infty, 0] & \longrightarrow & \mathbb{C} \\ & 1 & \longmapsto & 4\pi i \end{array} \text{ logarithm determination}$$

$$\cdot \quad \begin{array}{rccc} f : & \mathbb{C} \setminus [1, \infty) & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & -log(2-2z) \end{array}$$

$\cdot \quad \forall\ z \in \mathbb{D}(\frac{1}{2}, \frac{1}{2})$:

$$S(z) = \sum_{n \geq 1} \frac{(2z-1)^n}{n}$$

Then, holds:

$\cdot\ f \in \mathcal{H}(\mathbb{C} \setminus [1, \infty))$

$\cdot\ f = S + 4\pi i$ over $\mathbb{D}(\frac{1}{2}, \frac{1}{2})$

Demonstration:

$$\begin{array}{rccc} g : & \mathbb{C} \setminus [1, \infty) & \longrightarrow & \mathbb{C} \setminus (-\infty, 0] \\ & z & \longmapsto & 2-2z \end{array}$$

g well defined :

$\forall\ z \in \mathbb{C} \setminus [1, \infty)$ „ $Im(g(z)) = 0$:

$$Im(2-2z) = 0 \rightarrow -2Im(z) = 0 \rightarrow Im(z) = 0 \rightarrow Re(z) < 1$$

$$Re(g(z)) = Re(2-2z) = 2 - 2Re(z) > 0$$

$g \in$ Pol$\mathbb{C} \setminus [1, \infty)) \rightarrow g \in \mathcal{H}(\mathbb{C} \setminus [1, \infty))$

$log \in \mathcal{H}(\mathbb{C} \setminus (-\infty, 0])$

$f = -log \circ g \in \mathcal{H}(\mathbb{C} \setminus [1, \infty))$

In particular:

$log(1) = 4\pi i \to ln(1) + iarg(1) = 4\pi i \to arg(1) = 4\pi i$

$f(0) = -log(2) = -ln(\mid 2 \mid) - i\arg(2) = -ln(2) - i\arg(1) = -ln(2) - 4\pi i$

$f(-i) = -log(2+2i) = -ln(\mid 2+2i \mid) - i\arg(2+2i) = -ln(\sqrt{8}) - i(\arg(1) + \frac{\pi}{2}) = -ln(\sqrt{8}) - i(\frac{17\pi}{4})$

S well defined:

$$\sum_{n\geq 1} \frac{(2z-1)^n}{n} = \sum_{n\geq 1} \frac{2^n(z-\frac{1}{2})^n}{n}$$

$$\lim_n \frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} = \lim_n \frac{2(n+1)}{n} = 2$$

Quotient test:

$$\lim_n \left(\frac{2^n}{n}\right)^{-\frac{1}{n}} = 2^{-1} = \frac{1}{2}$$

Cauchy-Hadamard theorem:

$S(z)$ convergent over $\mathbb{D}(\frac{1}{2}, \frac{1}{2})$

Sum of S:

$x \ := \ z - \frac{1}{2}$

$$\sum_{n\geq 1} \frac{(2z-1)^n}{n} = \sum_{n\geq 1} \frac{2^n x^n}{n} = \sum_{n\geq 1} \frac{(2x)^n}{n}$$

UCD theorem:

$$S'(z) = \sum_{n\geq 1} 2^n x^{n-1} = 2\sum_{n\geq 1} (2x)^{n-1} = \frac{2}{1-2x}$$

$S \in \int \frac{2}{1-2x} dx = \{-log(1-2x) + c\}_{c\in\mathbb{C}}$

$S(0) = 0 \to S = -log(1-2x) = -log(2-2z)$

$\mathbb{D}(\frac{1}{2}, \frac{1}{2}) \cap \mathbb{C} \setminus (-\infty, 0] = \varnothing \to S(z) = -Log(2-2z)$

log and Log relationship:

log, Log well defined over $\mathbb{C} \setminus (-\infty, 0]$

$\forall\ z \in \mathbb{C} \setminus (-\infty, 0]$:

$log(z) - Log(z) = ln(z) + iarg(z) - ln(z) - iArg(z) = i(arg(z) - Arg(z)) =$

$= i((Arg(z) + arg(1)) - Arg(z)) = 4\pi i$

$log(z) = Log(z) + 4\pi i$

$\forall\ z \in \mathbb{D}(\frac{1}{2}, \frac{1}{2})$:

$S(z) - f(z) = -Log(2 - 2z) + log(2 - 2z) = i(arg(2 - 2z) - Arg(z))$

$z \in \mathbb{C} \setminus (-\infty, 0] \rightarrow S(z) - f(z) = 4\pi i$

$S(z) = f(z) + 4\pi i$