1. Holomorphic functions

Cauchy-Riemman

Let:

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies CR

Then, holds:

- $f_x = f'(z)$
- $f_y = -if_y$
- $\partial_{f(z)} z = 0$

Demonstration:

$$f_x = u_x + iv_x = a + bi = f'(z)$$

$$f_y = i(a + ib) = if'(z)$$

$$f'(z) = -if_y$$

notation

tangent venctor

Let:

$\cdot\ \gamma$ differentiable plane arc „ $\forall\ t \in I$:

$\gamma'(t) \neq 0$

Then, holds:

$\cdot\ \gamma'(t)$ tangent to γ

Demonstration:

demonstration

arc images

Let:

· $f \in \mathcal{H}(\mathcal{U})$

· γ differentiable plane arc " $\gamma \subset \mathcal{U}$

· $\sigma = f(\gamma)$

· $z_0 = \gamma(0)$

Then, holds:

· $\sigma' = f'(\gamma)\gamma'$

· $\gamma'(0) \neq 0 \rightarrow f'(z_0) \neq 0$

· $\sigma'(0) = f'(z_0)\gamma'(0)$

· $|\sigma'(0)| = |f'(z_0)||\gamma'(0)|$

· $arg\sigma'(0) = arg\gamma'(0) + argf(z_0)$

· f aplica una homotecia mas una rotacion constante a todos los vectores tangentes que salen de z0

Demonstration:

obvio

Holomorphic functions are conform

Let:

- $f : \mathcal{U} \to \mathbb{C}$
-

Then, holds:

- $f$holomorph in z „ $f'(z) \neq 0 \leftrightarrow f$ conform

Demonstration:

$\rightarrow$):

already seen

$\leftarrow$):

too hard

Convergence of complex series

Let:

- $\sum_{n\geq 0} c_n$ complex series

Then, holds:

- $\sum_{n\geq 0} c_n$ convergent $\leftrightarrow \sum_{n\geq 0} Rec_n$ convergent $\wedge \sum Imc_n n[0]$ convergent

Demonstration:

demonstration

Absolutely convergent are convergent

Let:

- $\sum c_n n[0]$ absolutely convergent

Then, holds:

- $\sum c_n n[0]$ convergent

Demonstration:

$S_k := \sum c_n n[0][k]$

$\forall\ m \in \mathbb{N}$ " $m < k$:

$|S_k - S_m| = |\sum c_n n[m+1][k]| \leq \sum |c_n| n[m+1][k]$

$\leq \sum |c_n| n[m+1] \xrightarrow{n} 0$

$|S_k - S_m| \xrightarrow{n} 0 \rightarrow (S_k)_k$ convergent $\rightarrow \sum c_n n[0]$ convergent

Series and norm

Let:

- $\sum c_n n[0]$ convergent

Then, holds:

- $|c_n| \xrightarrow{n} 0$

Demonstration:

$\sum c_n n[0]$ convergent $\leftrightarrow (S_n)_n$ convergent

$\to$ Cauchy $|S_n - S_m| \xrightarrow{n} 0$ por n y m $\to |S_n - S_{n-1}| \xrightarrow{n} 0$

$\to |c_n| \xrightarrow{n} 0$

Root test

Let:

- $\sum_{n\geq 0} c_n$ real series
- $l \in \mathbb{R}$ " $\overline{\lim_k} |c_k|^{\frac{1}{k}} = l$

Then, holds:

- $l > 1 \rightarrow \sum_{n\geq 0} c_n \notin \mathbb{R}$
- $l < 1 \rightarrow \sum_{n\geq 0} c_n \in \mathbb{R}$

Demonstration:

demonstration

Quotient test

Let:

- $\sum_{n \geq 0} c_n$ real series
- $l \in \mathbb{R}$ "

Then, holds:

- $\exists\, l \in \mathbb{R}$:

$$\lim_{k} \frac{c_{k+1}}{c_k} = l$$

- $\overline{\lim_{c_k}} \, |c_k|^{\frac{1}{k}} k = l$

Power series theorem

Let:

- $\sum_{n\geq 0} a_n c^n$ power series

Then, holds:

- $|z-a| < R \rightarrow$ absolutely convergent
- $|z-a| > R \rightarrow$ divergent
- convergent in $D(a, R)$
- $\begin{array}{rcl} f : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \sum_{n\geq 0} c_n (z-a)^n \end{array} \in \mathcal{H}(D(a,R))$
- $\forall\ z \in \mathbb{C}$:

  $$f'(z) = \sum_{n\geq 0} n c_n (z-a)^{n-1} \text{ convergent}$$

- convergence radius of f' = convergence radius of f

Demonstration:

$\forall\ z \in \mathbb{C}\ \ _{"}\ |z-a| < R$:

Root test over $\sum_{n\geq 0} |c_n||z-a|^n \lim_n \left(|c_n||z-a|^n\right)^{\frac{1}{n}} = |z-a| \lim_n |c_n|^{\frac{1}{n}} = \dfrac{|z-a|}{R} < 1$ Root test $\rightarrow$ absolutely convergent

$\forall\ z \in \mathbb{C}\ \ _{"}\ |z-a| < R$:

$\forall\ \rho \in \mathbb{R}\ \ _{"}\ |z-a| < \rho < R$:

$\frac{1}{\rho} < \frac{1}{R}$

$\lim_n |c_n|^{1/n} = \dfrac{1}{R} \rightarrow$ exists partial of $|c_n|^{1/n}$

$|c_n||z-a|^n > \frac{|z-a|^n}{\rho^n}$ no $\xrightarrow{n} 0$

General term test → divergent