# Dynamical systems

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March 5, 2014

# Contents

# Block I

# Definitions

# 1. Discrete dynamical systems

## Dynamical system

Let:

- $M$ manifold
- $T$ monoid
- $\phi : M \times T \to M$

Then, $(M, T, \phi)$ is a dynamical system if:

- $\forall\ x \in X$ :

  $\phi(x, 0) = 0$

  $\forall\ t_1, t_2 \in T$ :

  $$\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$$

## Dimension

Let:

- $(M, T, \phi)$ dynamical system

We name dimension of $(M, T, \phi)$ to:

- $\dim(M)$

We denote:

- $dim(M) = n : (M, T, \phi)$ n-D

## Discrete & Continuous

Let:

$\cdot$ $(M, T, \phi)$ dynamical system

Then, $(M, T, \phi)$ is discrete if:

$\cdot$ $T \underset{\sim}{\subset} \mathbb{N}$

Then, $(M, T, \phi)$ is continuous if:

$\cdot$ $T \subset \mathbb{R}$

$\cdot$ $T$ open

## Defined by a function

Let:

$\cdot$ $(M, T, \phi)$ dynamical system

$\cdot$ $f : M \to M$

Then, $(M, T, \phi)$ is a dynamical system defined by $f$ if:

$\cdot$ $T = \mathbb{N}$

$\cdot$ $\begin{array}{rcl} \phi : M \times \mathbb{N} & \longrightarrow & M \\ (x, n) & \longmapsto & f^n(x) \end{array}$

We denote:

$\cdot$ $(M, T, \phi)$ dynamical system defined by $f$ : $(M, \mathbb{N}, f)$

$\cdot$ $f \in \mathcal{C}^n(M)$ : $(M, \mathbb{N}, f)$ $\mathcal{C}^n$

**Orbit**

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$

We name orbit of $x$ to:

- $\{f^n(x)\}_{n \in \mathbb{N}}$

We denote:

- $o(x)$

**Periodicity**

Let:

- $(M, \mathbb{N}, f)$ dynamical system
- $x \in M$
- $m \in \mathbb{N}$

Then, $x$ is a m-periodic point if:

- $f^m(x) = x$

We denote:

- $\{x \in M \mid f(x) = x\} \ : \ \text{Fix}(f)$

**Stability**

Let:

- $f : \mathbb{R}^n \to \mathbb{R}^n$
- $p \in \mathbb{R}^n$ m-periodic point

Then, $p$ is stable if:

- $\forall\ \varepsilon \in \mathbb{R}^+ :$

  $\exists\ \delta \in \mathbb{R}^+ :$

  $\forall\ x \in B(p, \delta) :$

  $\forall\ n \in \mathbb{N} :$

  $f^{nm}(x) \in B(p, \varepsilon)$

Then, $p$ is unstable if:

- $p$ not stable

**Attractive & Repulsive**

Let:

- $f : \mathbb{R}^n \to \mathbb{R}^n$
- $p \in \mathbb{R}^n$ m-periodic point

Then, $p$ is attractive if:

- $p$ stable
- $\exists\ \varepsilon \in \mathbb{R}^+ :$

  $\forall\ x \in B(p, \varepsilon) :$

  $f^{nm}(x) \xrightarrow{n} p$

Then, $p$ is repulsive if:

- $p$ attractive by $f^{-1}$
- $\forall\ \mathcal{U} \subset M \ \text{„}\ \mathcal{U}$ open $\wedge\ x \in \mathcal{U} :$

  $\forall\ x' \in \mathcal{U} \ \text{„}\ x' \neq x :$

  $\exists\ N \in \mathbb{N} :$

  $\forall\ n \in \mathbb{N} \ \text{„}\ n \geq N :$

  $f^{nm}(x') \notin \mathcal{U}$

### Fixed point character

Let:

$\cdot\ (M, \mathbb{N}, f)$ functional dynamical system

We name Fixed point character to:

$$\cdot\ \begin{array}{rcl} f\ :\mathrm{Fix}(f) & \longrightarrow & \{-1,0,1\} \\ x & \longmapsto & \begin{cases} +1 & x \text{ repulsive} \\ -1 & x \text{ attractive} \\ 0 & x \text{ no attractive no repulsive} \end{cases} \end{array}$$

We denote:

$\cdot\ f\ :\ \chi_f$

### Attraction set

Let:

$\cdot\ (M, \mathbb{N}, f)$ dynamical system

$\cdot\ x \in M$ attractive m-periodic point

$\cdot\ o(x)$ orbit of $x$

We name attraction set of $x$ to:

$\cdot\ \{y \in M \mid \exists\, x' \in o(x)\ :\ f^{nm}(y) \xrightarrow{n} x'\}$

We denote:

$\cdot\ A(x)$

**Multiplier**

Let:

$\cdot\ (M, \mathbb{N}, f)\ \ \mathcal{C}^1$ dynamical system

$\cdot\ x \in M$

We name multiplier of $x$ to:

$\cdot\ f'(x)$

We denote:

$\cdot\ m(x)$

$\cdot\ |m(x)| = 1\ :\ x$ neutral point

**Feeble point**

Let:

$\cdot\ (M, \mathbb{N}, f)\ \ \mathcal{C}^3$ dynamical system

$\cdot\ x \in M$

Then, $x$ is feeble point if:

$\cdot\ x$ neutral point

$\cdot\ f''(x) = 0$

### Sarkovskii's order

We name Sarkovskii's order to:

$\cdot\ a = 2^n a', b = 2^m b'$

$$\cdot\ a <_s b \leftrightarrow \begin{cases} m < n & a' = b' = 1 \\ & a' = 1, b' \neq 1 \\ a' < b' & a' = b' \neq 1 \\ n < m & 1 \neq a' \neq b' \end{cases}$$

### Chaos

Let:

$\cdot\ (\mathbb{R}, \mathbb{N}, f)$ dynamical system

Then, $(\mathbb{R}, \mathbb{N}, f)$ is chaotic if:

$\cdot\ \text{Fix}(f)$ dense in $\mathbb{R}$

$\cdot\ \exists\ x \in \mathbb{R}$ :

$o(x)$ dense in $\mathbb{R}$

$\cdot\ \forall\ x \in \mathbb{R}$ :

$\exists\ \varepsilon \in \mathbb{R}^+$ :

$\forall\ \delta \in \mathbb{R}^+$ :

$\exists\ \tilde{x} \in B(x, \delta)$ :

$$\lim_n\ o(\tilde{x}) \notin B(x, \varepsilon)$$

### Topological equivalence

Let:

- $(M, \mathbb{N}, f_1), (M, \mathbb{N}, f_2')$ dynamical systems

Then, $(M, \mathbb{N}, f_1)$ is topologically equivalent to $(M, \mathbb{N}, f_2)$ if:

- $\text{Fix}(f) = \text{Fix}(f')$
- $\forall\ x \in \text{Fix}(f)$ :

  $\chi_{f_1}(x) = \chi_{f_2}(x)$

We denote:

- $(M, \mathbb{N}, f) \sim (M, \mathbb{N}, f')$

### Bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by $\Lambda$
- $\lambda_0 \in \Lambda$

Then, $\lambda_0$ is a bifurcation value if:

- $\forall\ \varepsilon \in \mathbb{R}^+$ :

  $\exists\ \lambda_1, \lambda_2 \in B(\lambda_0, \varepsilon)$ :

  $(M, \mathbb{N}, f_{\lambda_1}) \nsim (M, \mathbb{N}, f_{\lambda_2})$

### Saddle-node bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by $\Lambda$
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, $\lambda_0$ is a saddle-node bifurcation value at $x_0$ if:

- $x_0 \in \mathrm{Fix}(f_{\lambda_0})$
- $x_0$ neutral point of $f_{\lambda_0}$
- $\partial_\lambda f_\lambda(x_0) \neq 0$
- $\partial_{xx} f_\lambda(x_0) \neq 0$

### Pitchfork bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by $\Lambda$
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, $\lambda_0$ is Pitchfork bifurcation value at $x_0$ if:

- $x_0 \in \mathrm{Fix}(f_{\lambda_0})$
- $x_0$ neutral point of $f_{\lambda_0}$
- $\partial_\lambda f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{x^2} f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{\lambda x} f_\lambda(x_0, \lambda_0) \neq 0$
- $\partial_{x^3} f_\lambda(x_0, \lambda_0) \neq 0$

### Period doubling bifurcation

Let:

· $\Lambda \subset M$

· $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by $\Lambda$

· $\lambda_0 \in \Lambda$

· $x_0 \in M$

Then, $\lambda_0$ is Period doubling bifurcation value at $x_0$ if:

· $\lambda_0$ Pitchfork bifurcation value at $x_0$ of $f^2$

### Invariant curve

Let:

· $\gamma$ differentiable curve

· $p \in \mathbb{R}^n$

Then, $\gamma$ is invariant if:

· $\forall\ x \in \gamma *$ :

$$o(x) \subset \gamma *$$

Then, $\gamma$ is converges to $p$if:

· $\forall\ x \in \gamma *$ :

$$o(x) \xrightarrow{n} p$$

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# Block II

# Propositions

# 1. Discrete dynamical systems

*introduction*

### Fixed points theorem

Let:

- $I \subset \mathbb{R}$ open
- $f : I \to I$ differentiable
- $x \in I$

Then, holds:

- $|f'(x)| < 1 \to x$ attractive
- $|f'(x)| > 1 \to x$ repulsive

Demonstration:

*demonstration*

**Attractiveness of periodic points does not involve the chosen point**

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$ n-periodic point
- $\{x_i\}_{i=1}^{r}$ orbit of $x$

Then, holds:

- $x$ attractive $\leftrightarrow \forall\ x' \in o(x)$ :

  $x'$ attractive

Demonstration:

$\forall\ x' \in o(x)$ :

$$f^{n'}(x') = \prod_{i=1}^{r} f'(x_i) = f^{n'}(x)$$

**Partition of attraction set**

Let:

$\cdot$ $(M, \mathbb{N}, f)$ functional dynamical system

$\cdot$ $x$ n-periodic point

$\cdot$ $o(x)$ orbit of $x$

Then, holds:

$\cdot \ \forall \ x' \in o(x) \ :$

$\exists \, \mathcal{U} \subset M$ open :

$\forall \ y \in \mathcal{U} \ :$

$$f^n(y) \xrightarrow{n} x'$$

Demonstration:

*demonstration*

### Homeomorphisms are monotonous

Let:

- $f : \mathbb{R} \to \mathbb{R}$ homeomorphism

Then, holds:

- $f$ monotonous

Demonstration:

no demonstration

### Homeomorphisms and n-periodic points

Let:

$\cdot\ f\ :\ \mathbb{R} \to \mathbb{R}$ homeomorphism $(M, T, \phi)$ dynamical system defined by $f$

Then, holds:

$\cdot\ \forall\ n \in \mathbb{N}\ :$

$\nexists\ x \in M \ \ ״\ \ x$ n-periodic point

Demonstration:

graphically

**Sarkovskii's theorem**

Let:

$\cdot\, f \,:\, I \to I$

$\cdot\, (M, \mathbb{N}, f)$ dynamical system

Then, holds:

$\cdot\ \ \exists\, x \in M\,:$

$o(x)$ k-period

$\cdot\ \ \to \forall\, l \in \mathbb{N}$ " $l > k\,:$

$\exists\, x' \in M\,:$

$x'$ l-period

### Invariance of stability over periods

Let:

- $(\mathbb{R}^n, \mathbb{N}, f)$ n-D dynamical system
- $p \in \mathbb{R}^n$ k-periodic point
- $\chi$ character of periodic points

Then, holds:

- $\exists\ \sigma \in \mathrm{Im}(\chi)$ :

  $\forall\ x \in o(p)$ :

  $$\chi(x) = \sigma$$

Demonstration:

i will

# Block III

# Examples

# 1. One-dimensional discrete dynamical systems

$Examples of what are and what are not one-dimensional dynamical systems$

**logistic function**

Let:

$\cdot$ $(M, T, \phi)$ logistical dynamical system defined by $f$

Then, holds:

$\cdot$ $Fix(f) = \{0, \frac{a-1}{a}\}$

$\cdot$ $Per_2(f) =$

Demonstration:

*demonstration*

**Quadratic function**

Let:

$$\cdot \quad \begin{array}{rccc} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & a - x^2 \end{array}$$

$\cdot$ $(M, T, f_c)$ dynamical system family

Then, $f$ is bifurcates in -1/4 :

$f_{-\frac{1}{4}}(x) = x \leftrightarrow x = -\frac{1}{2}$

$f'_{-\frac{1}{4}}(x) = -2x$

$f'_{-\frac{1}{4}}(-\frac{1}{2}) = 1$

$\partial_a f = 1 \neq 0$

$\partial_{x^2} f = -2 \neq 0$

$sgn(1 * -2) = - \rightarrow -\frac{1}{2}$ SN

**Henon's application**

Let:

$$\cdot \quad \begin{array}{rcl} f\ :\mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (-x^2+0.4y,x) \end{array}$$

Study:

$\cdot$ Fixed points of $f$

Demonstration:

$(0,0),(-0$

$6,-0$

$6)$ fixed points

# Block IV

# Problems

## MODELS I SISTEMES DINÀMICS

### Llista 1: Aplicacions unidimensionals

**B.1.** Trobeu els punts fixos i les òrbites de període 2 de les següents funcions. En el cas que apareixin paràmetres, feu-ho en funció d'aquests.

(a) * $f(x) = 2x(1-x)$, on $x \in \mathbb{R}$.

(b) * $f_c(x) = x^2 + c$, on $x, c \in \mathbb{R}$ (només punts fixos).

(c) $f(x) = x^2 + 1$, on $x \in \mathbb{R}$.

(d) $f_{a,b}(x) = ax + b$, on $a, b, x \in \mathbb{R}$.

(e) $f(x) = 2x^2 - 5x$, on $x \in \mathbb{R}$.

**B.2.** Fent servir anàlisi gràfic, dibuixeu el retrat de fases de

(a) $f(x) = x^2, \ \ x \in \mathbb{R}$.

(b) $f(x) = x(1-x), \ \ x \in \mathbb{R}$.

(c) $f_a(x) = ax, \ \ x \in \mathbb{R}$, pels diferents valors de $a \in \mathbb{R}$.

**B.3.** * Trobeu els punts fixos atractors i les seves conques d'atracció per a la funció $f(x) = \frac{3x - x^3}{2}$, per $|x| \leq \sqrt{3}$.

**B.4.** Per a la funció logística $f_a(x) = ax(1-x)$, calculeu els punts fixos i els cicles de període 2 en funció del paràmetre, i determineu-ne l'estabilitat.

**1.** Estudieu el comportament asimptòtic de la successió $\{x_n\}_{n \in \mathbb{N}}$, pels diferents valors de $x_0$ indicats.

(a) * $x_{n+1} = \frac{\sqrt{x_n}}{2}$, $x_0 \geq 0$.

(b) $x_{n+1} = \frac{x_n}{2} + \frac{2}{x_n}$, $x_0 \geq 2$.

**2.** Donada la successió $x_{n+1} = \frac{x_n + 2}{x_n + 1}$,

(a) Trobeu el límit $L = \lim_{n \to \infty} x_n$ per a $x_0 \geq 0$.

(b) Descriviu el conjunt dels $x_0 < 0$ pels quals el límit $\lim_{n \to \infty} x_n$ existeix i no és igual a $L$, o bé no existeix. (Per exemple $x_0 = -1$).

**3.** **(Examen 2011)** Considereu el sistema dinàmic real definit per $x_{n+1} = \frac{x_n}{4} + x_n^3$. Trobeu el comportament asimptòtic de les òrbites per a tota condició inicial $x_0 \in \mathbb{R}$. Justifiqueu rigorosament les vostres afirmacions.

**4.** Demostreu rigurosament que $f(x) = \sin(x)$ té $x = 0$ com atractor global.

**5.** Demostreu que si $f : \mathbb{R} \to \mathbb{R}$ és derivable, $x_0$ és un punt fix i $|f'(x_0)| > 1$ llavors $x_0$ és un punt fix repulsor.

**6.** Sigui $f : \mathbb{R} \to \mathbb{R}$ de classe $\mathcal{C}^\infty$ i sigui $x_0$ un punt fix tal que $f'(x_0) = 1$. Doneu criteris sobre les derivades d'ordre superior, per determinar el retrat de fase local al voltant de $x_0$. Apliqueu-ho a determinar l'estabilitat dels punts fixos de $x^3 - x$.

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# 1. One-dimensional discrete dynamical system

*introduction*

### Decreasing function orbits

Let:

$\cdot$ *declarations*

$\cdot$

Show that:

$\cdot$ *statements*

$\cdot$

Demonstration:

$f$ corta en un punto

$f$ decreasing $\rightarrow f^2$ increasing

$f^{2n} \xrightarrow{n}$ fixed point of $f$

## 9. Periodic points

Let:

$$\cdot \quad \begin{array}{rrcl} f & : \mathbb{R} \times \mathbb{R}^+ & \longrightarrow & \mathbb{R} \\ & (x, r) & \longmapsto & r\frac{x}{1+x^2} \end{array}$$

Study:

- Periodic points of $f$

Demonstration:

Graphical analysis :

$f$ odd

$f$ has 2 extrema in $\pm 1$

$f \xrightarrow{n} 0$

Fixed points :

$f(x) = x \leftrightarrow x = \pm\sqrt{r-1}$

$f'(\pm\sqrt{r-1}) = \frac{2-r}{r}$

n-periodic points :

$f^n(x) = x$

## 10. Global orbit analysis

Let:

- $f : \mathbb{R}^+ \to \mathbb{R}^+ \in \mathbb{C}^\infty$
- $f(0) = 0$
- $p \in \mathbb{R}^+ \setminus \{0\}$ 〃 $f'(p) \geq 0$
- $f'$ decreasing

Show that:

- $\forall\, x \in \mathbb{R}^+ \setminus \{0\}$ :

$$f^n(x) \xrightarrow{n} p$$

Demonstration:

$f'$ decreasing $\to f'' < 0 \to f$ concave

$f$ positive $\to f$ has no extrema $\to f' > 0 \to f$ increasing

$f$ has only one fixed point

Suppose 2 fixed points : $p, p'$

$IVT \to \exists\, c \in (0, p')$ :

$$f'(c) = 1$$

$f'(p) < 1 \to p$ attractive $IVT \to$ dont exist more fixed points $\to f'(c') = 1 \nleq 1$

$\forall\, x \in (0, p)$ :

$$f(x) > x$$

$\forall\, x \in \mathbb{R}$ 〃 $x > p$ :

$$f(x) < x$$



$f$ increasing $\to f([0,p]) = [0,p]$

# Block V

# Laboratory

# 1. Orbit analysis

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Let:

$$\cdot \quad \begin{array}{rcl} f: \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^3 + \frac{1}{4}x \end{array}$$

Study:

· Orbit behavior of the real dynamical system defined by $f$

Demonstration:

Formalization :

Consider $(M, T, \phi)$ where:

$M = \mathbb{R}$

$T = \mathbb{N}$

$$\begin{array}{rcl} \phi: \mathbb{R} \times \mathbb{N} & \longrightarrow & \mathbb{R} \\ (x, n) & \longmapsto & f^n(x) \end{array}$$

Study the orbits of $(\mathbb{R}, \mathbb{N}, \phi)$

We will denote $f^n(x)$ as $x_n$

Fixed points :

$\forall\ x \in \mathbb{R}$ :

$$f(x) = x \leftrightarrow x^3 + \frac{1}{4}x - x = 0 \leftrightarrow x^3 - \frac{3}{4}x = 0$$

$$\leftrightarrow x = 0 \ \vee \ x^2 - \frac{3}{4} = 0$$

$$x \text{ fixed point} \leftrightarrow x \in \{0, \pm\frac{\sqrt{3}}{2}\}$$

Graphic analysis :

Parity:

$\forall\ x \in \mathbb{R}$ :

$$f(-x) = (-x)^3 + \frac{(-x)}{4} = -(x^3 + \frac{x}{4}) = -f(x)$$

$f$ is odd

Monotonicity:

$\forall\ x \in \mathbb{R}$ :

$$f'(x) = 3x^2 + \frac{1}{4} > 0$$

$f$ is increasing over $\mathbb{R}$

Convexity:

$\forall\ x \in \mathbb{R}^-$ :

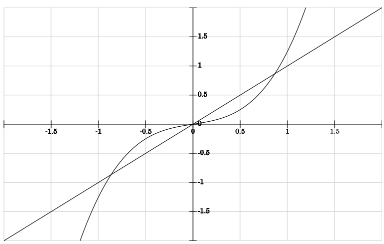
$$f''(x) = 6x \leq 0$$

$\forall\ x \in \mathbb{R}^+$ :

$$f''(x) = 6x \geq 0$$

$f$ is concave over $\mathbb{R}^-$ and convex over $\mathbb{R}^+$

Graphic representation :



I $\forall\ x \in (-\infty, -\frac{\sqrt{3}}{2})$ :

Induction over $n$:

$f$ incresing $\rightarrow f(x_n) < f(-\frac{\sqrt{3}}{2})$

$x_{n+1} \in (-\infty, -\frac{\sqrt{3}}{2})$

$\therefore)$ $o(x)$ is enclosed in $(-\infty, -\frac{\sqrt{3}}{2})$

Induction over $n$:

$x_n^2 > \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) > 0$

$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) < 0$

$\therefore)$ $o(x)$ decreasing

$\nexists\ x < -\frac{\sqrt{3}}{2}$ " $x$ fixed point $\rightarrow o(x) \xrightarrow{n} -\infty$

II $\forall\ x \in (-\frac{\sqrt{3}}{2}, 0)$ :

Induction over $n$:

$f$ increasing $\rightarrow f(-\frac{\sqrt{3}}{2}) < f(x_n) < f(0)$

$x_{n+1} \in (-\frac{\sqrt{3}}{2}, 0)$

$\therefore)$ $o(x)$ is enclosed in $(-\frac{\sqrt{3}}{2}, 0)$

Induction over $n$:

$x_n^2 < \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) < 0$

$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) > 0$

$\therefore)$ $o(x)$ increasing

$o(x)$ convergent $\wedge$ 0 fixed point $\rightarrow o(x) \xrightarrow{n} 0$

$\mathrm{I\underline{I}I} \forall\ x \in (0, \frac{\sqrt{3}}{2})$ :

Induction over $n$:

$-x_n \in (-\frac{\sqrt{3}}{2}, 0)$

$\mathrm{I\underline{I}} \to f(-x_n) \in (-\frac{\sqrt{3}}{2}, 0)\ \wedge\ f(-x_n) > -x_n$

$f$ odd $\to f(x_n) = -f(-x_n) \in (0, \frac{\sqrt{3}}{2})$

$f$ odd $\to f(x_n) = -f(-x_n) < x_n$

$\therefore)$ $o(x)$ is enclosed in $(0, \frac{\sqrt{3}}{2})\ \wedge\ o(x)$ decreasing

$o(x)$ convergent $\wedge\, 0$ fixed point $\to o(x) \xrightarrow{n} 0$

$\mathrm{I\underline{V}} \forall\ x \in \mathbb{R}$ „ $x > \frac{\sqrt{3}}{2}$ :

Induction over $n$:

$-x_n \in (\frac{\sqrt{3}}{2}, \infty)$

$\underline{\mathrm{I}} \to f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty)\ \wedge\ f(-x_n) < -x_n$

$f$ odd $\to f(x_n) = -f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty)$

$f$ odd $\to f(x_n) = -f(-x_n) > x_n$

$\therefore)$ $o(x)$ is inf bounded by in $\frac{\sqrt{3}}{2}\ \wedge\ o(x)$ increasing

$o(x)$ convergent

$\nexists\ x > \frac{\sqrt{3}}{2}$ „ $x$ fixed point $\to o(x) \xrightarrow{n} +\infty$

## 2. Fixed points cardinality

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Let:

$\cdot\ f\ :\ [0,1] \to [0,1] \in \mathcal{C}^2([0,1])$

$\cdot\ f(1) < 1$

$\cdot\ f'' > 0 \in [0,1]$

Show that:

$\cdot\ \#\ \{x \in [0,1] \mid f(x) = x\} = 1$

Demonstration:

$\#\ \{x \in [0,1] \mid f(x) = x\} \geq 1$:

Case $f(0) = 0$:

0 fixed point

Case $f(0) > 0$:

$$\begin{array}{rcl} g\ :[0,1] & \longrightarrow & [-1,1] \\ x & \longmapsto & f(x) - x \end{array} \in \mathcal{C}^2([0,1])$$

$g(0) = f(0) - 0 > 0$

$g(1) = f(1) - 1 < 0$

Bolzano's theorem:

$\exists\ x \in (0,1)$ :

$g(x) = 0$

$f(x) = x$

$\# \{x \in [0,1] \mid f(x) = x\} \leq 1$:

$g'' > 0$ over $[0,1]$

Rolle's theorem:

$\# \{x \in (0,1) \mid g'(x) = 0\} \leq 1$

$\# \{x \in (0,1) \mid g(x) = 0\} \leq 2$

$\# \{x \in (0,1) \mid f(x) = x\} \leq 2$

$f'' > 0$ over $[0,1]$

Monotonicity test:

$f'$ increasing in $[0,1]$

$\forall\ a < b \in [0,1)$ " $f(a) = a, f(b) = b$ :

Mean Value Theorem:

$\exists\ c \in (a,b)$ :

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 1$$

$\exists\ d \in (b,1)$ :

$$f'(d) = \frac{f(1) - f(b)}{1 - b} < 1$$

$f'$ increasing $\rightarrow f'(c) < f'(b) < f'(d)$

$1 < f'(b) < 1$ absurd

$\therefore)\ \# \{x \in [0,1] \mid f(x) = x\} = 1$