1. New

Cauchy-Riemman equations

Let:

$$\cdot \quad \begin{array}{rcl} f : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ x & \longmapsto & (u(x), v(x)) \end{array}$$

Then, f is satisfies the Cauchy-Riemman equations if:

- $u_x = v_y$
- $u_y = -v_x$

We denote:

- $u_x + iv_x \ : \ f_x$
- $u_y + iv_y \ : \ f_y$

Cauchy-Riemman

Let:

- $f \ : \ \mathbb{R}^2 \to \mathbb{R}^2$ satisfies CR

Then, holds:

- $f_x = f'(z)$
- $f_y = -if_y$
- $\frac{\partial}{\partial z} f(z) = 0$

Demonstration:

$$f_x = u_x + iv_x = a + bi = f'(z)$$

$$f_y = i(a + ib) = if'(z)$$

$f'(z) = -if_y$

notation

Conjugation

Let:

$$\begin{aligned} \bar{a} :\ & \mathbb{C} && \longrightarrow && \mathbb{C} \\ & z && \longmapsto && \bar{z} \end{aligned}.$$

Then, $\bar{a}$ is not holomorphic :

$$u_x = 1$$

$$u_y = 0$$

$$v_x = 0$$

$$v_y = -1$$

$\forall\ z \in \mathbb{C}$:

$$-1 \neq 1 \to f \text{ not holomorphic in } z$$

Quadratic norm

Let:

- $$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto |z|^2 \end{aligned}$$
- $f_{\mathbb{R}^2}$ component decomposition of f

Then, f is holomorphic in 0:

f differentiable in $\mathbb{R}^2$ polinomial

$\forall\ z \in \mathbb{C}$:

$u_x(x,y) = 2x$

$u_y(x,y) = 2y$

$v_x(x,y) = 0$

$v_y(x,y) = 0$

$u_x = v_y \leftrightarrow x = 0$

$u_y = -v_x \leftrightarrow y = 0$

f holomorphic function in $z \leftrightarrow z = 0$

3. Cauchy-Riemman

Let:

$\cdot\ f \in \mathcal{H}(\mathbb{C}) \ \shortparallel\ Ref + Imf = c_a$

Show that:

$\cdot\ \exists\ a' \in \mathbb{C} \ \shortparallel\ f = c_{a'}$

Demonstration:

u, v real components of f

$u(x,y) + v(x,y) = a$

differentiate respect x and y

$u_x + v_x = 0$

$u_y + v_y = 0$

f holomorphic $\rightarrow f$ CR

$u_x - u_y = 0$

$u_y + u_x = 0$

$u_x, u_y, v_x, v_y = 0$

$\exists\ a_1 \in \mathbb{R} \ \shortparallel\ u = c_{a_1}$

$\exists\ a_2 \in \mathbb{R} \ \shortparallel\ v = c_{a_2}$

$f = c_{(a_1,a_2)}$

B.2 a)

Let:

$$\cdot\ \begin{array}{rcl} u\ :\mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & \exp(y)\cos(x) \end{array}$$

Show that:

$$\cdot\ \exists\ f \in \mathcal{H}(\mathbb{C})\ {}_{\shortparallel}\ u \text{ real component of } f$$

Demonstration:

lab 1$\rightarrow u_{xx} + v_{yy} = 0$

$u_x = \exp(x)\cos(y)$

$u_{xx} = \exp(x)\cos(y)$

$u_y = -\exp(x)\sin(y)$

$u_{yy} = -\exp(x)\cos(y)$

ok

Calculate v using CR

$v_y = u_x = \exp(x)\cos(y)$

$v(x,y) = \int_{\mathbb{C}} \exp(x)\sin(y)dy = \exp(x)\sin(y) + \phi(x)$

$v_x = \frac{\partial}{\partial x}v = \exp(x)\sin(y) + \phi'(x)$

$-u_y = \exp(x)\sin(y) + \phi'(x)$

CR$\rightarrow \phi'(x) = 0$

$\forall\ c \in \mathbb{R}$:

$\phi(x) = c$ ok

$v(x,y) = \exp(x)\sin(y)$

$$\begin{array}{rcl} f\ :\mathbb{C} & \longrightarrow & \mathbb{C} \\ (x,y) & \longmapsto & (u(x,y), v(x,y)) \end{array}$$

Conformalidad

Plane arc

Let:

$$\cdot \quad \begin{array}{rcl} \gamma : I & \longrightarrow & \mathbb{R}^2 \\ t & \longmapsto & (x(t), y(t)) \end{array}$$

Then, γ is a plane arc if:

$\cdot\ \exists\ \varepsilon \in \mathbb{R}\ \ ''\ I = (-\varepsilon, \varepsilon)$

We denote:

$\cdot\ \exists \gamma'\ :\ \gamma$ differentiable

tangent venctor

Let:

$\cdot\ \gamma$ differentiable plane arc $\ ''\ \ \forall\ t \in I$:

$$\gamma'(t) \neq 0$$

Then, holds:

$\cdot\ \gamma'(t)$ tangent to γ

Demonstration:

demonstration

arc images

Let:

· $f \in \mathcal{H}(\mathcal{U})$

· γ differentiable plane arc " $\gamma \subset \mathcal{U}$

· $\sigma = f(\gamma)$

· $z_0 = \gamma(0)$

Then, holds:

· $\sigma' = f'(\gamma)\gamma'$

· $\gamma'(0) \neq 0 \to f'(z_0) \neq 0$

· $\sigma'(0) = f'(z_0)\gamma'(0)$

· $|\sigma'(0)| = |f'(z_0)||\gamma'(0)|$

· $arg\sigma'(0) = arg\gamma'(0) + argf(z_0)$

· f aplica una homotecia mas una rotacion constante a todos los vectores tangentes que salen de z0

Demonstration:

obvio

Conformalism

Let:

- f function $z_0 \in \mathcal{U}$

Then, f is conform in $z_0$if:

- $\forall\ \gamma\ :\ I \to \mathbb{R}^2\ \ \shortparallel \gamma$ differentiable :

  $\gamma(0) = z_0 \gamma'(0) \neq 0$

  $\sigma := f(\gamma)$

  $arg \sigma'(0) - arg \gamma'(0)$ constant

  $|\frac{\sigma'(0)}{\gamma'(0)}|$ constant

Holomorphic functions are conform

Let:

- $f\ :\ \mathcal{U} \to \mathbb{C}$
-

Then, holds:

- $f$holomorph in $z\ \ \shortparallel f'(z) \neq 0 \leftrightarrow f$ conform

Demonstration:

$\to$):

already seen

$\leftarrow$):

too hard

Preservation of angles

Let:

- γ_1, γ_2 plane arcs $\shortparallel$ $\gamma_1(0) = \gamma_2(0)$

Then, holds:

- angle of $\gamma_1'(0)$ and $\gamma_2'(0)$ = angle $\sigma_1'(0), \sigma_2'(0)$

Demonstration:

rotations and homotecies let angles invariant

Non preserving angles function

Let:

- $f(z) = z^2$

Then, f is conform in $\mathbb{R} \setminus \{0\}$:

$$f(\{(x,0) \in \mathbb{C} \mid x > 0\}) = \{(x,0) \in \mathbb{C} \mid x > 0\}$$

$$f(\{(x,0) \in \mathbb{C} \mid x < 0\}) = \{(x,0) \in \mathbb{C} \mid x > 0\}$$

$$ang(A,B) = \pi \neq 0 = ang(f(A), f(B))$$

Power Series. Marsden 183-190

Partial sum

Let:

- $\sum a_n z^n n[0]$ power series

We name partial sum of $\sum a_n z^n n[0]$ to:

- $\sum a_n z^n n[0][k]$

We denote:

- $S_k(z)$
- series is convergent : $\lim\limits_{k} S_k(z)$ convergent