I. Linear maps

Real sequence of order 2

Let:

- $p, q \in \mathbb{R}$
- $\forall\ a, b \in \mathbb{R}$:

  $x_0 \ := \ a$

  $x_1 \ := \ b$

  $\forall\ n \in \mathbb{N}$:

  $x_{n+2} \ := \ px_{n+1} + q_{x_n}$

Study:

- $\lim\limits_{n} \dfrac{x_n}{x_{n+1}}$

Start:

Consider:

$$A = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$$

$$A \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}$$

Eigenvalue analysis:

$$\chi_A(t) = t^2 - \mathrm{tr}(A)\, t + \det(A) = t^2 - pt - q$$

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \chi_A(\lambda) = 0\} = \{\tfrac{p}{2}\}$$

$$\lambda \ := \ \tfrac{p}{2}$$

$$E_\lambda(A) = \mathrm{Ker}(A - \lambda \mathbb{1}) = \{(x, y) \in \mathbb{R}^2 \mid \text{-----------}\}$$

$\dim(A) = 2 \ \wedge \ \sigma(A) = \{\lambda\} \ \wedge \ \gamma_A(\lambda) = 1 \rightarrow A$ no diagonalizable

Jordan form:

$$J \ := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

$$v_2 \ := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_1 \ := \ (A - \lambda \mathbb{1})v_2 = \begin{pmatrix} p - \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$$

$$C \ := \ (v_1|v_2) = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}$$

$$C^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A = CJC^{-1} \rightarrow A^n = CJ^nC^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}$$

Sequence analysis:

$$\forall \ a, b \in \mathbb{R} \ :$$

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = A^n \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} 99 \\ 99 \end{pmatrix}$$

$$\lim_n \ \frac{x_n}{x_{n+1}} = \lambda = \frac{p}{2}$$

Graphical interpretation:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} \in \mathbb{R}^2$$

$\frac{x_{n+1}}{x_n}$ slope of the line throgh $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}$

$$\lim_n \ \frac{x_{n+1}}{x_n} = \frac{p}{2}$$

the slope of the line approaches the stable subspace E^s as n becomes bigger