1. Holomorphic functions

Cauchy-Riemman

Let:

    $\cdot\; f \;:\; \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies CR

Then, holds:

    $\cdot\; f_x = f'(z)$

    $\cdot\; f_y = -i f_y$

    $\cdot\; \partial_{f(z)} z = 0$

Demonstration:

    $f_x = u_x + i v_x = a + bi = f'(z)$

    $f_y = i(a + ib) = i f'(z)$

    $f'(z) = -i f_y$

    notation

tangent venctor

Let:

· γ differentiable plane arc „ $\forall\, t \in I$:

$$\gamma'(t) \neq 0$$

Then, holds:

· $\gamma'(t)$ tangent to γ

Demonstration:

demonstration

arc images

Let:

> $\cdot\, f \in \mathcal{H}(\mathcal{U})$
>
> $\cdot\, \gamma$ differentiable plane arc $\quad_{"}\; \gamma \subset \mathcal{U}$
>
> $\cdot\, \sigma = f(\gamma)$
>
> $\cdot\, z_0 = \gamma(0)$

Then, holds:

> $\cdot\, \sigma' = f'(\gamma)\gamma'$
>
> $\cdot\, \gamma'(0) \neq 0 \rightarrow f'(z_0) \neq 0$
>
> $\cdot\, \sigma'(0) = f'(z_0)\gamma'(0)$
>
> $\cdot\, |\sigma'(0)| = |f'(z_0)||\gamma'(0)|$
>
> $\cdot\, arg\sigma'(0) = arg\gamma'(0) + argf(z_0)$
>
> $\cdot$ f aplica una homotecia mas una rotacion constante a todos los

vectores tangentes que salen de z0

Demonstration:

> obvio

Holomorphic functions are conform

Let:

$\cdot\ f\ :\ \mathcal{U} \to \mathbb{C}$

.

Then, holds:

$\cdot\ f$ holomorph in z " $f'(z) \neq 0 \leftrightarrow f$ conform

Demonstration:

$\to$):

already seen

$\leftarrow$):

too hard

Convergence of complex series

Let:

$\cdot \sum\limits_{n \geq 0} c_n$ complex series

Then, holds:

$\cdot \sum\limits_{n \geq 0} c_n$ convergent $\leftrightarrow \sum\limits_{n \geq 0} Rec_n$ convergent $\land \sum Imc_n n[0]$ con-

vergent

Demonstration:

$demonstration$

Absolutely convergent are convergent

Let:

$\cdot \sum c_n n[0]$ absolutely convergent

Then, holds:

$\cdot \sum c_n n[0]$ convergent

Demonstration:

$S_k := \sum c_n n[0][k]$

$\forall \, m \in \mathbb{N} \,_{\shortparallel} \, m < k :$

$$|S_k - S_m| = |\sum c_n n[m+1][k]| \leq \sum |c_n| n[m+1][k]$$

$$\leq \sum |c_n| n[m+1] \xrightarrow{n} 0$$

$$|S_k - S_m| \xrightarrow{n} 0 \rightarrow (S_k)_k \text{ convergent} \rightarrow \sum c_n n[0] \text{ conver-}$$

gent

Series and norm

Let:

$\cdot \sum c_n n[0]$ convergent

Then, holds:

$\cdot |c_n| \xrightarrow{n} 0$

Demonstration:

$\sum c_n n[0]$ convergent $\leftrightarrow (S_n)_n$ convergent

$\rightarrow$ Cauchy $|S_n - S_m| \xrightarrow{n} 0$ por n y m $\rightarrow |S_n - S_{n-1}| \xrightarrow{n} 0$

$\rightarrow |c_n| \xrightarrow{n} 0$

Root test

Let:

$$\cdot \sum_{n \geq 0} c_n \text{ real series}$$

$$\cdot \, l \in \mathbb{R} \quad \text{\guillemotleft} \quad \overline{\lim_{k}} \, |c_k|^{\frac{1}{k}} = l$$

Then, holds:

$$\cdot \, l > 1 \rightarrow \sum_{n \geq 0} c_n \notin \mathbb{R}$$

$$\cdot \, l < 1 \rightarrow \sum_{n \geq 0} c_n \in \mathbb{R}$$

Demonstration:

demonstration

Quotient test

Let:

· $\sum_{n\geq 0} c_n$ real series

· $l \in \mathbb{R}$ "

Then, holds:

· $\exists\, l \in \mathbb{R}:$

$$\lim_{k} \frac{c_{k+1}}{c_k} = l$$

· $\overline{\lim_{c_k}} |c_k|^{\frac{1}{k}} k = l$

Power series theorem

Let:

$$\cdot \sum_{n\geq 0} a_n c^n \text{ power series}$$

Then, holds:

$\cdot\, |z - a| < R \rightarrow$ absolutely convergent

$\cdot\, |z - a| > R \rightarrow$ divergent

$\cdot\,$ convergent in $D(a, R)$

$$\cdot\quad \begin{array}{rcl} f : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \sum_{n\geq 0} c_n(z-a)^n \end{array} \in \mathcal{H}(D(a,R))$$

$\cdot\, \forall\, z \in \mathbb{C} :$

$$f'(z) = \sum_{n\geq 0} nc_n(z-a)^{n-1} \text{ convergent}$$

$\cdot\,$ convergence radius of f' = convergence radius of f

Demonstration:

$\forall\, z \in \mathbb{C} \;_{\shortparallel}\; |z - a| < R :$

Root test over $\sum_{n\geq 0} |c_n||z-a|^n \, rlimit(|c_n||z-a|^n)^{\frac{1}{n}} n = |z -$

$a| rlimit|c_n|^{\frac{1}{n}} n = \dfrac{|z-a|}{R} < 1$ Root test $\rightarrow$ absolutely convergent

$\forall\, z \in \mathbb{C} \;_{\shortparallel}\; |z - a| < R :$

$\forall\, \rho \in \mathbb{R} \;_{\shortparallel}\; |z - a| < \rho < R :$

$\frac{1}{\rho} < \frac{1}{R}$

$rlimit|c_n|^{1/n} n = \frac{1}{R} 1$ exists partial of $|c_n|^{1/n}$

$|c_n||z-a|^n > \dfrac{|z-a|^n}{\rho^n}$ no $\xrightarrow{n} 0$

General term test $\rightarrow$ divergent

Power series theorem

Let:

    .

Then, holds:

    $\cdot\, Part\, II, III, IV$

Demonstration:

    Follow 3 steps

Step 1 :   Uniform convergence in compacts of $D(a, R)$:

$$\begin{aligned} g_n \, : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto c_n(z-a)^n \end{aligned}$$

$\forall\, \rho \in \mathbb{R} \; _{"} \; \rho < R :$

$\forall\, z \in \overline{D(a, \rho)} :$

$$|g_n(z)| = |c_n||z - a|^n \le |c_n|\rho^n$$

$$M_n := |c_n|\rho^n$$

$$rlimit(|c_n|\rho^n)^{1/n}n = \rho\, rlimit|c_n|^{1/n}n = \frac{\rho}{R} < 1$$

Root test$\rightarrow \sum\limits_{n\ge 0} M_n$ convergent

M-Weierstrass$\rightarrow \sum\limits_{n\ge 0} c_n(z-a)^n$ uniformly convergent

over compacts of $D(a, \rho)$

$$\sum\limits_{n\ge 0} c_n(z-a)^n \text{ uniformly convergent over compacts of } D(a, r)$$

$$f(z) := \sum_{n \geq 0} g_n(z)$$

g uniformly convergent $\rightarrow f$ continuous

Step 2 : IV. R' :

$$\tilde{f}(z) := \sum_{n \geq 1} n c_n (z-a)^{n-1}$$

$$\tilde{f}(z) = \sum_{n \geq 0} (n+1) c_{n+1} (z-a)^n$$

$$\frac{1}{R'} = rlimit(n+1)|c_{n+1}|^{1/n} n = rlimit(n+1)^{1/n}|c_{n+1}|^{1/n} n =$$

$$rlimit(|c_{n+1}|^{1/n+1})^{\frac{n+1}{n}} n = \frac{1}{R}$$

$$R' = R$$

Step 3 : III :

$\tilde{f}$ well defined in $D(a, R)$

$\forall \, z_0 \in D(a, R) :$

$$\left| \frac{f(z)-f(z_0)}{z-z_0} - \tilde{f}(z_0) \right| \xrightarrow{n} 0 ?$$

$\forall \, n \in \mathbb{N} :$

$$S_n(z) := \sum_{k=0}^{n} f_k(z)$$

$$R_n(z) := \sum_{k \geq n+1} f_k(z)$$

$$f(z) = S_n(z) + R_n(z)$$

$$\tilde{f}(z) = \tilde{S}_n(z) + \tilde{R}_n(z)$$

$$R_n(z), \tilde{R}_n(z) \xrightarrow{n} 0$$

$\forall \, \varepsilon \in \mathbb{R}^+ :$

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| = \left| \frac{S_n(z)-S_n(z_0)}{z-z_0} + \frac{R_n(z)-R_n(z_0)}{z-z_0} - \tilde{S}_n(z_0) - \tilde{R}_n(z_0) \right|$$

$\forall \, \rho \in \mathbb{R} \;\;_{\shortparallel} \; |z_0 - a| < \rho < R :$

$$\left| \tilde{R}_n(z_0) \right| \le \sum_{k \ge n+1} k|c_k|\rho^{k-1} < \frac{\varepsilon}{3}(n \ge n_1)$$

$$\left| \frac{R_n(z)-R_n(z_0)}{z-z_0} \right| \le \sum_{k \ge n+1} |c_k| \frac{(z-a)^k - (z_0-a)^k}{z-z_0}|$$

$$\frac{a^k-b^k}{a-b} = a^{k-1} - a^{k-2}b + --- + b^{k-1}$$

$$\le \sum_{k \ge n+1} |c_k|(|z-a|^{k-1} + |z-a|^{k-2}|z_0-a| + ---+$$

$|z_0 - a|^{k-1}$

$|z-a|, |z_0| < \rho$

$$\le \sum_{k \ge n+1} |c_k|k\rho^{k-1} < \frac{\varepsilon}{3}(n \ge n_1)$$

$$\left| \frac{S_n(z)-S_n(z_0)}{z-z_0} - \tilde{S}_n(z_0) \right| < \frac{\varepsilon}{3}(S_n' = \tilde{S}_n)(n \ge n_2)$$

$\forall \, n \in \mathbb{N} \;\;_{\shortparallel} \; n \ge \max(n_1, n_2) :$

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| < \varepsilon$$

functions associated to series are $\mathcal{C}^\infty$

Let:

$\cdot\, f(z) = \sum c_n (z-a)^n n[0]$ series

$\cdot\, R$ radius of convergence of f

Then, holds:

$\cdot\, f \in \mathcal{C}^\infty$ over $D(a, R)$

$\cdot\, \forall\, n \in \mathbb{N} :$

$$f^{n)} \in \mathcal{H}(D(a, R))$$

$\cdot\, c_k = \frac{f^{k)}}{k!}$

$\cdot$ series associated to f is unique

Demonstration:

$f^{k)}(z) = \sum n(n-1) - - - (n-k+1)c_n (z-a)^{n-k} n[k]$

$f^{k)}(a) = k! c_k$

$c_k = \frac{f^{k)}(a)}{k!}$