Dynamical systems

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Contents

Block I

Definitions

1. Discrete dynamical systems

Dynamical system

Let:

- M manifold
- T monoid
- $\phi : M \times T \to M$

Then, (M, T, ϕ) is a dynamical system if:

- $\forall\ x \in X$:

  $\phi(x, 0) = 0$

  $\forall\ t_1, t_2 \in T$:

  $$\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$$

Dimension

Let:

- (M, T, ϕ) dynamical system

We name dimension of (M, T, ϕ) to:

- $\dim(M)$

We denote:

- $dim(M) = n\ :\ (M, T, \phi)$ n-D

Discrete & Continuous

Let:

· (M, T, ϕ) dynamical system

Then, (M, T, ϕ) is discrete if:

· $T \underset{\sim}{\subset} \mathbb{N}$

Then, (M, T, ϕ) is continuous if:

· $T \subset \mathbb{R}$

· T open

Defined by a function

Let:

· (M, T, ϕ) dynamical system

· $f : M \to M$

Then, (M, T, ϕ) is a dynamical system defined by f if:

· $T = \mathbb{N}$

· $\begin{array}{rcl} \phi : M \times \mathbb{N} & \longrightarrow & M \\ (x, n) & \longmapsto & f^n(x) \end{array}$

We denote:

· (M, T, ϕ) dynamical system defined by f : $(M, \mathbb{N}, f)$

· $f \in \mathcal{C}^n(M)$: $(M, \mathbb{N}, f)$ $\mathcal{C}^n$

Orbit

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$

We name orbit of x to:

- $\{f^n(x)\}_{n \in \mathbb{N}}$

We denote:

- $o(x)$

Periodicity

Let:

- $(M, \mathbb{N}, f)$ dynamical system
- $x \in M$
- $m \in \mathbb{N}$

Then, x is a m-periodic point if:

- $f^m(x) = x$

We denote:

- $\{x \in M \mid f(x) = x\}$: $\mathrm{Fix}(f)$

Stability

Let:

- $f : \mathbb{R}^n \to \mathbb{R}^n$
- $p \in \mathbb{R}^n$ m-periodic point

Then, p is stable if:

- $\forall\ \varepsilon \in \mathbb{R}^+ :$

  $\exists\ \delta \in \mathbb{R}^+ :$

  $\forall\ x \in B(p,\delta) :$

  $\forall\ n \in \mathbb{N} :$

  $f^{nm}(x) \in B(p,\varepsilon)$

Then, p is unstable if:

- p not stable

Attractive & Repulsive

Let:

$\cdot\, f \,:\, \mathbb{R}^n \to \mathbb{R}^n$

$\cdot\, p \in \mathbb{R}^n$ m-periodic point

Then, p is attractive if:

$\cdot\, p$ stable

$\cdot\;\; \exists\, \varepsilon \in \mathbb{R}^+ \,:$

$\forall\, x \in B(p, \varepsilon) \,:$

$f^{nm}(x) \xrightarrow{n} p$

Then, p is repulsive if:

$\cdot\, p$ attractive by f^{-1}

$\cdot\, \forall\, \mathcal{U} \subset M \;\;„\;\; \mathcal{U}$ open $\;\wedge\, x \in \mathcal{U} \,:$

$\forall\, x' \in \mathcal{U} \;\;„\;\; x' \neq x \,:$

$\exists\, N \in \mathbb{N} \,:$

$\forall\, n \in \mathbb{N} \;\;„\;\; n \geq N \,:$

$f^{nm}(x') \notin \mathcal{U}$

Fixed point character

Let:

· $(M, \mathbb{N}, f)$ functional dynamical system

We name Fixed point character to:

$$\cdot \; f : \begin{array}{rcl} \mathrm{Fix}(f) & \longrightarrow & \{-1, 0, 1\} \\ x & \longmapsto & \begin{cases} +1 & x \text{ repulsive} \\ -1 & x \text{ attractive} \\ 0 & x \text{ no attractive no repulsive} \end{cases} \end{array}$$

We denote:

· $f : \chi_f$

Attraction set

Let:

· $(M, \mathbb{N}, f)$ dynamical system

· $x \in M$ attractive m-periodic point

· $o(x)$ orbit of x

We name attraction set of x to:

· $\{y \in M \mid \exists\, x' \in o(x) : f^{nm}(y) \xrightarrow{n} x'\}$

We denote:

· $A(x)$

Multiplier

Let:

- $(M, \mathbb{N}, f)$ $\mathcal{C}^1$ dynamical system
- $x \in M$

We name multiplier of x to:

- $f'(x)$

We denote:

- $m(x)$
- $|m(x)| = 1$: x neutral point

Feeble point

Let:

- $(M, \mathbb{N}, f)$ $\mathcal{C}^3$ dynamical system
- $x \in M$

Then, x is feeble point if:

- x neutral point
- $f''(x) = 0$

Sarkovskii's order

We name Sarkovskii's order to:

$\cdot\ a = 2^n a', b = 2^m b'$

$$\cdot\ a <_s b \leftrightarrow \begin{cases} m < n & a' = b' = 1 \\ & a' = 1, b' \neq 1 \\ a' < b' & a' = b' \neq 1 \\ n < m & 1 \neq a' \neq b' \end{cases}$$

Chaos

Let:

$\cdot\ (\mathbb{R}, \mathbb{N}, f)$ dynamical system

Then, $(\mathbb{R}, \mathbb{N}, f)$ is chaotic if:

$\cdot\ \operatorname{Fix}(f)$ dense in $\mathbb{R}$

$\cdot\ \exists\, x \in \mathbb{R}$:

$o(x)$ dense in $\mathbb{R}$

$\cdot\ \forall\, x \in \mathbb{R}$:

$\exists\, \varepsilon \in \mathbb{R}^+$:

$\forall\, \delta \in \mathbb{R}^+$:

$\exists\, \tilde{x} \in B(x, \delta)$:

$$\lim_n o(\tilde{x}) \notin B(x, \varepsilon)$$

Topological equivalence

Let:

- $(M, \mathbb{N}, f_1), (M, \mathbb{N}, f_2')$ dynamical systems

Then, $(M, \mathbb{N}, f_1)$ is topologically equivalent to $(M, \mathbb{N}, f_2)$ if:

- $\text{Fix}(f) = \text{Fix}(f')$
- $\forall\ x \in \text{Fix}(f)$:

  $\chi_{f_1}(x) = \chi_{f_2}(x)$

We denote:

- $(M, \mathbb{N}, f) \sim (M, \mathbb{N}, f')$

Bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$

Then, λ_0 is a bifurcation value if:

- $\forall\ \varepsilon \in \mathbb{R}^+$:

  $\exists\ \lambda_1, \lambda_2 \in B(\lambda_0, \varepsilon)$:

  $(M, \mathbb{N}, f_{\lambda_1}) \nsim (M, \mathbb{N}, f_{\lambda_2})$

Saddle-node bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is a saddle-node bifurcation value at x_0 if:

- $x_0 \in \mathrm{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0) \neq 0$
- $\partial_{xx} f_\lambda(x_0) \neq 0$

Pitchfork bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is Pitchfork bifurcation value at x_0 if:

- $x_0 \in \mathrm{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{x^2} f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{\lambda x} f_\lambda(x_0, \lambda_0) \neq 0$
- $\partial_{x^3} f_\lambda(x_0, \lambda_0) \neq 0$

Period doubling bifurcation

Let:

- $\cdot\ \Lambda \subset M$
- $\cdot\ (M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\cdot\ \lambda_0 \in \Lambda$
- $\cdot\ x_0 \in M$

Then, λ_0 is Period doubling bifurcation value at x_0 if:

- $\cdot\ \lambda_0$ Pitchfork bifurcation value at x_0 of f^2

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Block II

Propositions

1. One-dimensional discrete dynamical systems

introduction

Fixed points theorem

Let:

- $I \subset \mathbb{R}$ open
- $f : I \to I$ differentiable
- $x \in I$

Then, holds:

- $|f'(x)| < 1 \to x$ attractive
- $|f'(x)| > 1 \to x$ repulsive

Demonstration:

demonstration

Attractiveness of periodic points does not involve the chosen point

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$ n-periodic point
- $\{x_i\}_{i=1}^{r}$ orbit of x

Then, holds:

- x attractive $\leftrightarrow \forall\ x' \in o(x)$:

  x' attractive

Demonstration:

$\forall\ x' \in o(x)$:

$$f^{n'}(x') = \prod_{i=1}^{r} f'(x_i) = f^{n'}(x)$$

Partition of attraction set

Let:

$\cdot\ (M, \mathbb{N}, f)$ functional dynamical system

$\cdot\ x$ n-periodic point

$\cdot\ o(x)$ orbit of x

Then, holds:

$\cdot\ \forall\ x' \in o(x)\ :$

$\exists\, \mathcal{U} \subset M$ open $:$

$\forall\ y \in \mathcal{U}\ :$

$$f^n(y) \xrightarrow{n} x'$$

Demonstration:

demonstration

Homeomorphisms are monotonous

Let:

· $f : \mathbb{R} \rightarrow \mathbb{R}$ homeomorphism

Then, holds:

· f monotonous

Demonstration:

no demonstration

Homeomorphisms and n-periodic points

Let:

$\cdot\ f\ :\ \mathbb{R} \to \mathbb{R}$ homeomorphism (M, T, ϕ) dynamical system defined by f

Then, holds:

$\cdot\ \forall\ n \in \mathbb{N}$:

$\nexists\ x \in M$ „ x n-periodic point

Demonstration:

graphically

Sarkovskii's theorem

Let:

- $f : I \to I$
- (M, T, ϕ) dynamical system

Then, holds:

- $\exists\, x \in M$:

  $o(x)$ k-period

- $\to \forall\, l \in \mathbb{N}$ " $l > k$:

  $\exists\, x' \in M$:

  x' l-period

Characterization of sella-node bifurcation points

Let:

- $I \subset \mathbb{R}\{f_\lambda \,:\, I \to I\}_{\lambda \in \Lambda}$
- $x \in I$

Then, holds:

- x SN bifurcation point $\leftrightarrow f(x) = x, f'(x) = 1$
- $\partial_\lambda f \neq 0$
- $f''(x) \neq 0$

Demonstration:

demonstration

Pitchfork bifurcation

Let:

· (M, T, f_λ) dynamical system family

· $x \in M$

Then, x is pitchfork bifurcation if:

· x fixed point

· born of 2 fixed points

·

Characterization of Pitchfork bifurcations

Let:

· (M, T, f_λ) dynamical system family

· $x \in M$

Then, holds:

· x Pitchfork $\leftrightarrow$

· $f(x) = x$

· $f'(x) = 1$

· $\partial_{x^2} f = 0$

· $\partial_\lambda f = 0$

Demonstration:

no demonstration

Block III

Examples

1. One-dimensional discrete dynamical systems

$Examples of what are and what are not one-dimensional dynamical systems$

Analysis of logistic dynamical systems

Let:

· (M, T, ϕ) logistical dynamical system defined by f

Then, holds:

· $Fix(f) = \{0, \frac{a-1}{a}\}$

· $Per_2(f) =$

Demonstration:

demonstration

Quadratic function bifurcations

Let:

- $\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto a - x^2 \end{aligned}$
- (M, T, f_c) dynamical system family

Then, f is bifurcates in -1/4 :

$f_{-\frac{1}{4}}(x) = x \leftrightarrow x = -\frac{1}{2}$

$f'_{-\frac{1}{4}}(x) = -2x$

$f'_{-\frac{1}{4}}(-\frac{1}{2}) = 1$

$\partial_a f = 1 \neq 0$

$\partial_{x^2} f = -2 \neq 0$

$sgn(1 * -2) = - \rightarrow -\frac{1}{2}$ SN

Block IV

Problems

MODELS I SISTEMES DINÀMICS

Llista 1: Aplicacions unidimensionals

B.1. Trobeu els punts fixos i les òrbites de període 2 de les següents funcions. En el cas que apareixin paràmetres, feu-ho en funció d'aquests.

(a) * $f(x) = 2x(1-x)$, on $x \in \mathbb{R}$.

(b) * $f_c(x) = x^2 + c$, on $x, c \in \mathbb{R}$ (només punts fixos).

(c) $f(x) = x^2 + 1$, on $x \in \mathbb{R}$.

(d) $f_{a,b}(x) = ax + b$, on $a, b, x \in \mathbb{R}$.

(e) $f(x) = 2x^2 - 5x$, on $x \in \mathbb{R}$.

B.2. Fent servir anàlisi gràfic, dibuixeu el retrat de fases de

(a) $f(x) = x^2$, $x \in \mathbb{R}$.

(b) $f(x) = x(1-x)$, $x \in \mathbb{R}$.

(c) $f_a(x) = ax$, $x \in \mathbb{R}$, pels diferents valors de $a \in \mathbb{R}$.

B.3. * Trobeu els punts fixos atractors i les seves conques d'atracció per a la funció $f(x) = \frac{3x-x^3}{2}$, per $|x| \leq \sqrt{3}$.

B.4. Per a la funció logística $f_a(x) = ax(1-x)$, calculeu els punts fixos i els cicles de període 2 en funció del paràmetre, i determineu-ne l'estabilitat.

1. Estudieu el comportament asimptòtic de la successió $\{x_n\}_{n\in\mathbb{N}}$, pels diferents valors de x_0 indicats.

(a) * $x_{n+1} = \frac{\sqrt{x_n}}{2}$, $x_0 \geq 0$.

(b) $x_{n+1} = \frac{x_n}{2} + \frac{2}{x_n}$, $x_0 \geq 2$.

2. Donada la successió $x_{n+1} = \frac{x_n+2}{x_n+1}$,

(a) Trobeu el límit $L = \lim_{n\to\infty} x_n$ per a $x_0 \geq 0$.

(b) Descriviu el conjunt dels $x_0 < 0$ pels quals el límit $\lim_{n\to\infty} x_n$ existeix i no és igual a L, o bé no existeix. (Per exemple $x_0 = -1$).

3. **(Examen 2011)** Considereu el sistema dinàmic real definit per $x_{n+1} = \frac{x_n}{4} + x_n^3$. Trobeu el comportament asimptòtic de les òrbites per a tota condició inicial $x_0 \in \mathbb{R}$. Justifiqueu rigorosament les vostres afirmacions.

4. Demostreu rigurosament que $f(x) = \sin(x)$ té $x = 0$ com atractor global.

5. Demostreu que si $f : \mathbb{R} \to \mathbb{R}$ és derivable, x_0 és un punt fix i $|f'(x_0)| > 1$ llavors x_0 és un punt fix repulsor.

6. Sigui $f : \mathbb{R} \to \mathbb{R}$ de classe $\mathcal{C}^\infty$ i sigui x_0 un punt fix tal que $f'(x_0) = 1$. Doneu criteris sobre les derivades d'ordre superior, per determinar el retrat de fase local al voltant de x_0. Apliqueu-ho a determinar l'estabilitat dels punts fixos de $x^3 - x$.

1. One-dimensional discrete dynamical system

introduction

Decreasing function orbits

Let:

$\cdot$ *declarations*

$\cdot$

Show that:

$\cdot$ *statements*

$\cdot$

Demonstration:

f corta en un punto

f decreasing $\rightarrow f^2$ increasing

$f^{2n} \xrightarrow{n}$ fixed point of f

9. Periodic points

Let:

$$\cdot \quad \begin{array}{rcl} f : \mathbb{R} \times \mathbb{R}^{+} & \longrightarrow & \mathbb{R} \\ (x, r) & \longmapsto & r \frac{x}{1+x^2} \end{array}$$

Study:

· Periodic points of f

Demonstration:

Graphical analysis :

f odd

f has 2 extrema in ± 1

$f \xrightarrow{n} 0$

Fixed points :

$f(x) = x \leftrightarrow x = \pm\sqrt{r-1}$

$f'(\pm\sqrt{r-1}) = \frac{2-r}{r}$

n-periodic points :

$f^n(x) = x$

10. Global orbit analysis

Let:

- $f : \mathbb{R}^+ \to \mathbb{R}^+ \in \mathbb{C}^\infty$
- $f(0) = 0$
- $p \in \mathbb{R}^+ \setminus \{0\}$ „ $f'(p) \geq 0$
- f' decreasing

Show that:

- $\forall\ x \in \mathbb{R}^+ \setminus \{0\}$:

  $f^n(x) \xrightarrow{n} p$

Demonstration:

f' decreasing $\to f'' < 0 \to f$ concave

f positive $\to f$ has no extrema $\to f' > 0 \to f$ increasing

f has only one fixed point

Suppose 2 fixed points : p, p'

$IVT \to \exists\ c \in (0, p')$:

$f'(c) = 1$

$f'(p) < 1 \to p$ attractive $IVT \to$ dont exist more fixed points $\to f'(c') = 1 \nleq 1$

$\forall\ x \in (0, p)$:

$f(x) > x$

$\forall\ x \in \mathbb{R}$ „ $x > p$:

$f(x) < x$

f increasing $\to f([0, p]) = [0, p]$

Block V

Laboratory

1. Orbit analysis

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Let:

$$. \begin{array}{rcl} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^3 + \frac{1}{4}x \end{array}$$

Study:

· Orbit behavior of the real dynamical system defined by f

Demonstration:

Formalization :

Consider (M, T, ϕ) where:

$M = \mathbb{R}$

$T = \mathbb{N}$

$$\begin{array}{rcl} \phi : \mathbb{R} \times \mathbb{N} & \longrightarrow & \mathbb{R} \\ (x, n) & \longmapsto & f^n(x) \end{array}$$

Study the orbits of $(\mathbb{R}, \mathbb{N}, \phi)$

We will denote $f^n(x)$ as x_n

Fixed points :

$\forall\ x \in \mathbb{R}$:

$$f(x) = x \leftrightarrow x^3 + \frac{1}{4}x - x = 0 \leftrightarrow x^3 - \frac{3}{4}x = 0$$

$$\leftrightarrow x = 0 \ \vee \ x^2 - \frac{3}{4} = 0$$

x fixed point $\leftrightarrow x \in \{0, \pm\frac{\sqrt{3}}{2}\}$

Graphic analysis :

Parity:

$\forall\ x \in \mathbb{R}$:

$$f(-x) = (-x)^3 + \frac{(-x)}{4} = -(x^3 + \frac{x}{4}) = -f(x)$$

f is odd

Monotonicity:

$\forall\ x \in \mathbb{R}$:

$$f'(x) = 3x^2 + \frac{1}{4} > 0$$

f is increasing over $\mathbb{R}$

Convexity:

$\forall\ x \in \mathbb{R}^-$:

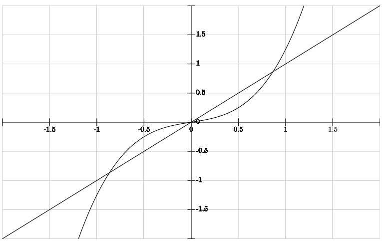
$$f''(x) = 6x \leq 0$$

$\forall\ x \in \mathbb{R}^+$:

$$f''(x) = 6x \geq 0$$

f is concave over $\mathbb{R}^-$ and convex over $\mathbb{R}^+$

Graphic representation :



$\underline{\mathrm{I}}\forall\ x \in (-\infty, -\frac{\sqrt{3}}{2})$:

Induction over n:

f incresing $\to f(x_n) < f(-\frac{\sqrt{3}}{2})$

$x_{n+1} \in (-\infty, -\frac{\sqrt{3}}{2})$

$\therefore)$ $o(x)$ is enclosed in $(-\infty, -\frac{\sqrt{3}}{2})$

Induction over n:

$x_n^2 > \frac{3}{4} \to (x_n^2 - \frac{3}{4}) > 0$

$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) < 0$

$\therefore)$ $o(x)$ decreasing

$\nexists\ x < -\frac{\sqrt{3}}{2}$ „ x fixed point $\to o(x) \xrightarrow{n} -\infty$

$\underline{\mathrm{II}}\forall\ x \in (-\frac{\sqrt{3}}{2}, 0)$:

Induction over n:

f increasing $\to f(-\frac{\sqrt{3}}{2}) < f(x_n) < f(0)$

$x_{n+1} \in (-\frac{\sqrt{3}}{2}, 0)$

$\therefore)$ $o(x)$ is enclosed in $(-\frac{\sqrt{3}}{2}, 0)$

Induction over n:

$x_n^2 < \frac{3}{4} \to (x_n^2 - \frac{3}{4}) < 0$

$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) > 0$

$\therefore)$ $o(x)$ increasing

$o(x)$ convergent $\wedge$ 0 fixed point $\to o(x) \xrightarrow{n} 0$

$\mathrm{I\underline{I}I} \forall\ x \in (0, \frac{\sqrt{3}}{2})$:

Induction over n:

$-x_n \in (-\frac{\sqrt{3}}{2}, 0)$

$\underline{\mathrm{I}}\mathrm{I} \rightarrow f(-x_n) \in (-\frac{\sqrt{3}}{2}, 0)\ \wedge\ f(-x_n) > -x_n$

f odd $\rightarrow f(x_n) = -f(-x_n) \in (0, \frac{\sqrt{3}}{2})$

f odd $\rightarrow f(x_n) = -f(-x_n) < x_n$

$\therefore)$ $o(x)$ is enclosed in $(0, \frac{\sqrt{3}}{2})\ \wedge\ o(x)$ decreasing

$o(x)$ convergent $\wedge$ 0 fixed point $\rightarrow o(x) \xrightarrow{n} 0$

$\mathrm{I\underline{V}} \forall\ x \in \mathbb{R}$ " $x > \frac{\sqrt{3}}{2}$:

Induction over n:

$-x_n \in (\frac{\sqrt{3}}{2}, \infty)$

$\underline{\mathrm{I}} \rightarrow f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty)\ \wedge\ f(-x_n) < -x_n$

f odd $\rightarrow f(x_n) = -f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty)$

f odd $\rightarrow f(x_n) = -f(-x_n) > x_n$

$\therefore)$ $o(x)$ is inf bounded by in $\frac{\sqrt{3}}{2}\ \wedge\ o(x)$ increasing

$o(x)$ convergent

$\nexists\ x > \frac{\sqrt{3}}{2}$ " x fixed point $\rightarrow o(x) \xrightarrow{n} +\infty$

2. Fixed points cardinality

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Let:

$\cdot\, f \,:\, [0,1] \to [0,1] \in \mathcal{C}^2([0,1])$

$\cdot\, f(1) < 1$

$\cdot\, f'' > 0 \in [0,1]$

Show that:

$\cdot\, \#\, \{x \in [0,1] \mid f(x) = x\} = 1$

Demonstration:

$\#\, \{x \in [0,1] \mid f(x) = x\} \geq 1$:

Case $f(0) = 0$:

0 fixed point

Case $f(0) > 0$:

$$\begin{array}{rcl} g\ :[0,1] & \longrightarrow & [-1,1] \\ x & \longmapsto & f(x) - x \end{array} \in \mathcal{C}^2([0,1])$$

$g(0) = f(0) - 0 > 0$

$g(1) = f(1) - 1 < 0$

Bolzano's theorem:

$\exists\, x \in (0,1)$:

$g(x) = 0$

$f(x) = x$

$\# \{x \in [0,1] \mid f(x) = x\} \le 1$:

$g'' > 0$ over $[0,1]$

Rolle's theorem:

$\# \{x \in (0,1) \mid g'(x) = 0\} \le 1$

$\# \{x \in (0,1) \mid g(x) = 0\} \le 2$

$\# \{x \in (0,1) \mid f(x) = x\} \le 2$

$f'' > 0$ over $[0,1]$

Monotonicity test:

f' increasing in $[0,1]$

$\forall\ a < b \in [0,1)$ „ $f(a) = a, f(b) = b$:

Mean Value Theorem:

$\exists\ c \in (a,b)$:

$$f'(c) = \frac{f(b)-f(a)}{b-a} = 1$$

$\exists\ d \in (b,1)$:

$$f'(d) = \frac{f(1)-f(b)}{1-b} < 1$$

f' increasing $\to f'(c) < f'(b) < f'(d)$

$1 < f'(b) < 1$ absurd

$\therefore)\ \# \{x \in [0,1] \mid f(x) = x\} = 1$