1. 2nd laboratory

Power series

Study:

- $\sum_{n\geq 1} n(n+1)z^n$

Start:

Convergence domain:

$$\lim_n \frac{n(n+1)}{(n+1)(n+1)} = 1$$

Quotient test:

$$\overline{\lim_n} |c_n|^{\frac{1}{n}} = 1 \to \left(\overline{\lim_n} |c_n|^{\frac{1}{n}}\right)^{-1} = 1$$

Cauchy-Hadamard theorem:

$\sum_{n\geq 1} n(n+1)z^n$ absolutely convergent over $\mathbb{D}$

$\sum_{n\geq 1} n(n+1)z^n$ divergent over $\mathbb{C} \setminus \overline{\mathbb{D}}$

$\forall\ K \subset \mathbb{D}$ " K compact :

$$\sum_{n\geq 1} n(n+1)z^n \text{ uniformly convergent over } K$$

Sum:

Consider:

$$\begin{aligned} f : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n\geq 1} n(n+1)z^n \end{aligned}$$

$$\begin{aligned} g : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n\geq 1} n(n+1)z^{n-1} \end{aligned}$$

$$\begin{aligned} h : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n\geq 1} (n+1)z^n \end{aligned}$$

UCI theorem:

$\forall\ z \in \mathbb{D}$:

$$\int_0^z h(t)dt = \sum_{n\geq 1} z^{n+1} = \sum_{n\geq 0} z^n = \frac{1}{1-z}$$

$$h(z) = \partial_z \tfrac{1}{1-z} = \tfrac{1}{(1-z)^2}$$

$$\int_0^z g(t)dt = \sum_{n\geq 1} (n+1)z^n = h(z)$$

$$g(z) = \partial_z h(z) = \tfrac{2}{(1-z)^3}$$

$$f(z) = zg(z) = \tfrac{2z}{(1-z)^3} = x^{\frac{2z}{(1-z)^3}}$$

Application:

In particular:

$$\sum_{n\geq 1} (-1)^n \frac{n(n+1)}{2^n} = f(-\frac{1}{2}) = \frac{-2^3}{3^3}$$