Block I

Laboratory

1. Fixed points cardinality

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Let:

$\cdot\, f\ :\ [0,1] \to [0,1] \in \mathcal{C}^2([0,1])$

$\cdot\, f(1) < 1$

$\cdot\, f'' > 0 \in [0,1]$

Show that:

$\cdot\, \#\,\{x \in [0,1] \mid f(x) = x\} = 1$

Demonstration:

$\#\,\{x \in [0,1] \mid f(x) = x\} \geq 1$:

Case $f(0) = 0$:

0 fixed point

Case $f(0) > 0$:

$$g\ : \begin{array}{ccc} [0,1] & \longrightarrow & [-1,1] \\ x & \longmapsto & f(x) - x \end{array} \in \mathcal{C}^2([0,1])$$

$g(0) = f(0) - 0 > 0$

$g(1) = f(1) - 1 < 0$

Bolzano's theorem:

$\exists\ x \in (0,1)$:

$g(x) = 0$

$f(x) = x$

$\#\,\{x \in [0,1] \mid f(x) = x\} \leq 1$:

$g'' > 0$ over $[0,1]$

Rolle's theorem:

$\#\,\{x \in (0,1) \mid g'(x) = 0\} \leq 1$

$\#\,\{x \in (0,1) \mid g(x) = 0\} \leq 2$

$\#\,\{x \in (0,1) \mid f(x) = x\} \leq 2$

$f'' > 0$ over $[0,1]$

Monotonicity test:

f' increasing in $[0,1]$

$\forall\ a < b \in [0,1)\ \ _{''}\ \ f(a) = a, f(b) = b$:

Mean Value Theorem:

$\exists\ c \in (a,b)$:

$f'(c) = \frac{f(b)-f(a)}{b-a} = 1$

$\exists\ d \in (b,1)$:

$$f'(d) = \frac{f(1)-f(b)}{1-b} < 1$$
$$f' \text{ increasing} \to f'(c) < f'(b) < f'(d)$$
$$1 < f'(b) < 1 \text{ absurd}$$
$$\therefore) \ \#\{x \in [0,1] \mid f(x) = x\} = 1$$

III. Bifurcation Theory

Bifurcation diagram

Let:

$$\cdot \begin{array}{rcl} f \ : \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (a,x) & \longmapsto & x^3 - 3x^2 + (5-a)x - 2 + a \end{array}$$

$\cdot \ \forall \ a \in \mathbb{R}$:

$$\begin{array}{rcl} f_a \ : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^3 - 3x^2 + (5-a)x - 2 + a \end{array}$$

Study:

$\cdot$ Bifurcations of $(\mathbb{R}, \mathbb{N}, \{f_a\}_{a\in\mathbb{R}})$

Start:

Fixed points:

$$f_a(x) = x \leftrightarrow x^3 - 3x^2 + (4-a)x - 2 + a = 0 \leftrightarrow (x-1)(x^2 - 2x + 2 - a) = 0$$

$$x^2 - 2x + 2 - a = 0 \leftrightarrow x = \pm\sqrt{a-1}$$

$\forall \ a \in \mathbb{R}$ " $a \leq 1$:

$$\text{Fix}(f_a) = \{1\}$$

$\forall \ a \in \mathbb{R}$ " $a > 1$:

$$\text{Fix}(f_a) = \{1, \pm\sqrt{a-1}\}$$

Stability:

$$\partial_x f(a,x) = 3x^2 - 6x + 5 - a$$

$\partial_{x^2} f(a,x) = 6x - 6$

$\partial_{x^3} f(a,x) = 6$

$|\partial_x f(a,1)| < 1 \leftrightarrow |2-a| < 1 \leftrightarrow a \in (1,3)$

$\partial_{x^2} f(1,1) = 0, \;\; \partial_{x^3} f(1,1) > 0$

$\partial_{x^2} f(3,1) = 0, \;\; \partial_{x^3} f(3,1) > 0$

$\forall\ a \in \mathbb{R}\ \text{„}\ a \leq 1 \ \vee\ a \geq 3:$

   1 repulsive

$\forall\ a \in \mathbb{R}\ \text{„}\ a \in (1,3):$

   1 attractive

$\forall\ a \in \mathbb{R}\ \text{„}\ a > 1:$

   $\left|\partial_x f(a, \pm\sqrt{a-1})\right| = |2a-1| > 1$

   $\pm\sqrt{a-1}$ repulsive

Pitchfork bifurcation at 1 :

$\partial_a f(1,1) = 1 - 1 = 0$

$\partial_{x^2} f(1,1) = 6 - 6 = 0$

$\partial_{ax} f(1,1) = -1 \neq 0$

$\partial_{x^3} f(1,1) = 6 \neq 0$

Period-doubling bifurcation at 3 :

$\partial_a f^2(3,1) = \partial_a f(3, f(3,1))\partial_a f(3,1) = (1-1)(1-1) = 0$

$\partial_{x^2} f^2(3,1) = \partial_{x^2} f(3, f(3,1))\partial_{x^2} f(3,1) = 0$

$$\partial_{ax} f^2(3,1) = \partial_{ax} f(3, f(3,1)) \partial_{ax} f(3,1) \neq 0$$

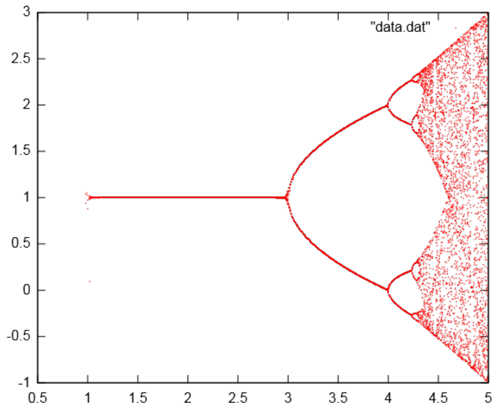
$$\partial_{x^3} f^2(3,1) = \partial_{x^3} f(3, f(3,1)) \partial_{x^3} f(3,1) \neq 0$$

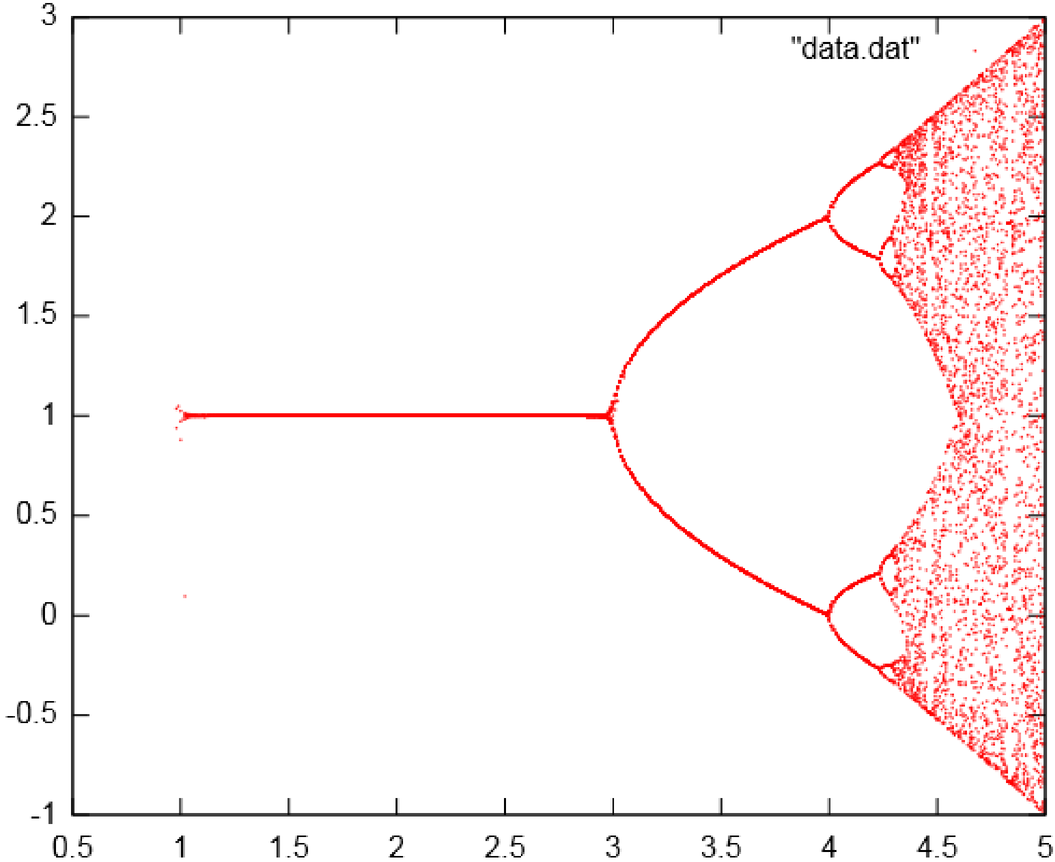
Source Code

```
#include "stdio.h"
#include "stdlib.h"
#include "math.h"
#include "string.h"

void plot( char *input_file, char *output_file )
{
  FILE *gnuplot;
  gnuplot = popen("gnuplot", "w");
  if( output_file )
  {
    fprintf(gnuplot, "set term svg\n");
    fprintf(gnuplot, "set out \"%s\"\n", output_file );
  }
  fprintf(gnuplot, "plot \"%s\" with dots\n", input_file);
  fflush(gnuplot);
  fclose(gnuplot);
}
double example_function( double param, double point )
{
  return pow(point,3) - 3*pow(point,2) + (5-param)*point - 2 + param;
}

void bifurcation_diagram( int param_min, int param_max, double param_step,
 int point_min, int point_max, int num_points,
 double (*f)(double,double), int num_iter, int tolerancy)
{
  FILE* file;
  double param, point;
  int i,j;

  srand(time(NULL));
  file = fopen("data.dat","w");

  for ( param = param_min; param < param_max; param += param_step )
  {
    for ( i = 0; i < num_points; i++ )
    {
      point = point_min + ((double) rand() / (double) RAND_MAX) * (point_max-
          point_min);

      for ( j = 0; j < num_iter && abs(point) < tolerancy; j++ )
      {
        point = (*f)(param,point);
      }

      if(abs(point) < tolerancy)
      {
        fprintf(file,"%lf %lf\n",param, point);
      }
    }
  }
  plot( "data.dat", "graph.svg" );
}
```

```
int main(int argc, char const *argv[])
{
  bifurcation_diagram( 0, 5, 10e-3, 0, 5, 100, &example_function, 100, 10e1);

  return 0;
}
```

Bifurcation Diagram





"data.dat"
3
2.5
2
1.5
1
0.5
0
-0.5
-1
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5