Dynamical systems

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Block I

Definitions

1. One-dimensional discrete dynamical systems

Dynamical system

Let:

- M manifold
- T monoid
- $\phi : M \times T \to M$

Then, (M, T, ϕ) is a dynamical system if:

- $\forall\ x \in X$:

  $\phi(x, 0) = 0$

  $\forall\ t_1, t_2 \in T$:

  $\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$

Dimension

Let:

- (M, T, ϕ) dynamical system

We name dimension of (M, T, ϕ) to:

- $\dim(M)$

We denote:

- $dim(M) = n\ :\ (M, T, \phi)$ n-D dynamical system

Discrete & Continuous

Let:

- (M, T, ϕ) dynamical system

Then, (M, T, ϕ) is discrete if:

- $T \overset{\subset}{\sim} \mathbb{N}$

Then, (M, T, ϕ) is continuous if:

- $T \subset \mathbb{R}$ „ T open

Defined by a function

Let:

- (M, T, ϕ) dynamical system
- $f : M \to M$

Then, (M, T, ϕ) is a dynamical system defined by f if:

- $T = \mathbb{N}$
- $$\begin{array}{rcl} \phi : M \times \mathbb{N} & \longrightarrow & M \\ (x, n) & \longmapsto & f^n(x) \end{array}$$

We denote:

- (M, T, ϕ) dynamical system defined by f : $(M, \mathbb{N}, f)$

Orbit

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system

- $x \in M$

We name orbit of x to:

- $\{f^n(x)\}_{n\in\mathbb{N}}$

We denote:

- $o(x)$

n-periodic point

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$
- $n \in \mathbb{N}$

Then, x is a n-periodic point if:

- $f^n(x) = x$
- $\forall\, n' \in \mathbb{N}$ „ $n' < n$:

  $f^{n'}(x) \neq x$

We denote:

- $n = 1$: x fixed point

Attractive & Repulsive

Let:

· $(M, \mathbb{N}, f)$ metrical dynamical system

· $x \in M$ m-periodic point

Then, x is attractive if:

· $\exists\, \mathcal{U} \subset M$ open :

$\forall\, y \in \mathcal{U}$:

$\exists\, N \in \mathbb{N}$:

$\forall\, n \in \mathbb{N}$ „ $n \geq N$:

$f^{nm}(y) \in \mathcal{U}$

Then, x is repulsive if:

· $\forall\, \mathcal{U} \subset M$ „ $x \in \mathcal{U}$:

$\forall\, y \in \mathcal{U}$:

$\exists\, N \in \mathbb{N}$:

$\forall\, n \in \mathbb{N}$ „ $n \geq N$:

$f^{nm}(x) \notin \mathcal{U}$

Attraction set

Let:

· $(M, \mathbb{N}, f)$ dynamical system

· $x \in M$ attractive m-periodic point

· $o(x)$ orbit of x

We name attraction set of x to:

· $\{y \in M \mid \exists\, x' \in o(x) \;:\; f^{nm}(y) \xrightarrow{n} x'\}$

We denote:

· $A(x)$

Neutral point

Let:

· $(M, \mathbb{N}, f)$ differentiable dynamical system

· $x \in M$

Then, x is a neutral point if:

· $f'(x) \in \{-1, 1\}$

Feeble attractive & repulsive points

Let:

- $(M, \mathbb{N}, f)$ $\mathcal{C}^3$ dynamical system
- $x \in M$

Then, x is feeble attractive point if:

- $f'(x) = 1$
- $f''(x) = 0$
- $f'''(x) > 0$

Then, x is feeble repulsive point if:

- $f'(x) = 1$
- $f''(x) = 0$
- $f'''(x) < 0$

Multiplier

Let:

- $(M, \mathbb{N}, f)\mathcal{C}^1$ dynamical system
- $x \in M$

We name multiplier of x to:

- $f'(x)$

Logistic

Let:

- $f : \mathbb{R} \to \mathbb{R}$
- (M, T, ϕ) dynamical system defined by f

Then, (M, T, ϕ) is a logistic dynamical system if:

- $\exists\, a \in \mathbb{R}$:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto ax(1-x) \end{aligned}$$

Chaos

Let:

- $f : \mathbb{R} \to \mathbb{R}$
- (M, T, ϕ) dynamical system defined by f

Then, (M, T, ϕ) is chaotic if:

- $Fix(f)$ dense in $\mathbb{R}$
- $\exists\, x \in \mathbb{R}$:

  $o(x)$ dense in $\mathbb{R}$

- f sensibility of x_0

Sarkovskii's order

We name Sarkovskii's order to:

$\cdot$ *naming*

$\cdot$

We denote:

$\cdot$ *property* : *notation*

$\cdot$

Saddle point

Let:

$\cdot\, \mathcal{U} \subset \mathbb{R}^n$

$\cdot\, f \in \mathcal{C}^1(\mathcal{U})$

$\cdot\, x \in \mathcal{U}$

Then, x is a saddle point if:

$\cdot\, f'(x) = 0$

Topologically equivalent

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $(M, \mathbb{N}, f')$ functional dynamical system

Then, $(M, \mathbb{N}, f)$ is topologically equivalent to $(M, \mathbb{N}, f')$if:

- $Fix(f) = Fix(f')$
- $\forall\ x \in Fix(f)$:

$$character_f(x) = character_{f'}(x)$$

We denote:

- $(M, \mathbb{N}, f)\ (M, \mathbb{N}, f')$

Bifurcation point

Let:

- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ functional dynamical system family
- $\lambda' \in \Lambda$

Then, λ' is a bifurcation value if:

- $\forall\ \varepsilon \in \mathbb{R}^+$:

  $\exists\ \lambda'' \in (\lambda' - \varepsilon, \lambda' + \varepsilon)$:

  $(M, \mathbb{N}, f_{\lambda''})$ not topologically equivalent to $(M, \mathbb{N}, f_{\lambda'})$

Block II

Propositions

1. One-dimensional discrete dynamical systems

introduction

Fixed points theorem

Let:

- $I \subset \mathbb{R}$ open
- $f : I \to I$ differentiable
- $x \in I$

Then, holds:

- $|f'(x)| < 1 \to x$ attractive
- $|f'(x)| > 1 \to x$ repulsive

Demonstration:

demonstration

Attractiveness of periodic points does not involve the chosen point

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$ n-periodic point
- $\{x_i\}_{i=1}^{r}$ orbit of x

Then, holds:

- x attractive $\leftrightarrow$ $\forall\ x' \in o(x)$:

  x' attractive

Demonstration:

$\forall\ x' \in o(x)$:

$$f^{n'}(x') = \prod_{i=1}^{r} f'(x_i) = f^{n'}(x)$$

Partition of attraction set

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system

- x n-periodic point

- $o(x)$ orbit of x

Then, holds:

- $\forall\, x' \in o(x)$:

  $\exists\, \mathcal{U} \subset M$ open :

  $\forall\, y \in \mathcal{U}$:

  $$f^n(y) \xrightarrow{n} x'$$

Demonstration:

demonstration

Homeomorphisms are monotonous

Let:

- $f : \mathbb{R} \to \mathbb{R}$ homeomorphism

Then, holds:

- f monotonous

Demonstration:

no demonstration

Homeomorphisms and n-periodic points

Let:

$\cdot\, f\ :\ \mathbb{R} \to \mathbb{R}$ homeomorphism (M, T, ϕ) dynamical system defined by f

Then, holds:

$\cdot\quad \forall\ n \in \mathbb{N}$:

$\nexists\ x \in M\ \ _{''}\ x$ n-periodic point

Demonstration:

graphically

Sarkovskii's theorem

Let:

· $f : I \to I$

· (M, T, ϕ) dynamical system

Then, holds:

· $\exists\, x \in M$:

$o(x)$ k-period

· $\to$ $\forall\, l \in \mathbb{N}$ „ $l > k$:

$\exists\, x' \in M$:

x' l-period

Block III

Examples

1. One-dimensional discrete dynamical systems

$Examplesofwhatareandwhatarenotone-dimensionaldynamicalsystems$

Analysis of logistic dynamical systems

Let:

- (M, T, ϕ) logistical dynamical system defined by f

Then, holds:

- $Fix(f) = \{0, \frac{a-1}{a}\}$
- $Per_2(f) =$

Demonstration:

demonstration

Block IV

Problems

MODELS I SISTEMES DINÀMICS

Llista 1: Aplicacions unidimensionals

B.1. Trobeu els punts fixos i les òrbites de període 2 de les següents funcions. En el cas que apareixin paràmetres, feu-ho en funció d'aquests.

(a) * $f(x) = 2x(1-x)$, on $x \in \mathbb{R}$.

(b) * $f_c(x) = x^2 + c$, on $x, c \in \mathbb{R}$ (només punts fixos).

(c) $f(x) = x^2 + 1$, on $x \in \mathbb{R}$.

(d) $f_{a,b}(x) = ax + b$, on $a, b, x \in \mathbb{R}$.

(e) $f(x) = 2x^2 - 5x$, on $x \in \mathbb{R}$.

B.2. Fent servir anàlisi gràfic, dibuixeu el retrat de fases de

(a) $f(x) = x^2, \ \ x \in \mathbb{R}$.

(b) $f(x) = x(1-x), \ \ x \in \mathbb{R}$.

(c) $f_a(x) = ax, \ \ x \in \mathbb{R}$, pels diferents valors de $a \in \mathbb{R}$.

B.3. * Trobeu els punts fixos atractors i les seves conques d'atracció per a la funció $f(x) = \frac{3x-x^3}{2}$, per $|x| \leq \sqrt{3}$.

B.4. Per a la funció logística $f_a(x) = ax(1-x)$, calculeu els punts fixos i els cicles de període 2 en funció del paràmetre, i determineu-ne l'estabilitat.

1. Estudieu el comportament asimptòtic de la successió $\{x_n\}_{n \in \mathbb{N}}$, pels diferents valors de x_0 indicats.

(a) * $x_{n+1} = \frac{\sqrt{x_n}}{2}$, $x_0 \geq 0$.

(b) $x_{n+1} = \frac{x_n}{2} + \frac{2}{x_n}$, $x_0 \geq 2$.

2. Donada la successió $x_{n+1} = \frac{x_n + 2}{x_n + 1}$,

(a) Trobeu el límit $L = \lim_{n\to\infty} x_n$ per a $x_0 \geq 0$.

(b) Descriviu el conjunt dels $x_0 < 0$ pels quals el límit $\lim_{n\to\infty} x_n$ existeix i no és igual a L, o bé no existeix. (Per exemple $x_0 = -1$).

3. **(Examen 2011)** Considereu el sistema dinàmic real definit per $x_{n+1} = \frac{x_n}{4} + x_n^3$. Trobeu el comportament asimptòtic de les òrbites per a tota condició inicial $x_0 \in \mathbb{R}$. Justifiqueu rigorosament les vostres afirmacions.

4. Demostreu rigurosament que $f(x) = \sin(x)$ té $x = 0$ com atractor global.

5. Demostreu que si $f : \mathbb{R} \to \mathbb{R}$ és derivable, x_0 és un punt fix i $|f'(x_0)| > 1$ llavors x_0 és un punt fix repulsor.

6. Sigui $f : \mathbb{R} \to \mathbb{R}$ de classe $\mathcal{C}^\infty$ i sigui x_0 un punt fix tal que $f'(x_0) = 1$. Doneu criteris sobre les derivades d'ordre superior, per determinar el retrat de fase local al voltant de x_0. Apliqueu-ho a determinar l'estabilitat dels punts fixos de $x^3 - x$.

1. One-dimensional discrete dynamical system

introduction

Decreasing function orbits

Let:

$\cdot$ *declarations*

$\cdot$

Show that:

$\cdot$ *statements*

$\cdot$

Demonstration:

f corta en un punto

f decreasing $\rightarrow f^2$ increasing

$f^{2n} \xrightarrow{n}$ fixed point of f

10.

Let:

- $f\ :\ \mathbb{R}^+ \to \mathbb{R}^+ \in \mathbb{C}^\infty$
- $f(0) = 0$
- $p \in \mathbb{R}^+ \setminus \{0\}$ „ $f'(p) \geq 0$
- f' decreasing

Show that:

- $\forall\ p \in \mathbb{R}^+ \setminus \{0\}$:

$$f^n(x) \xrightarrow{n} p$$

Demonstration:

f' decreasing $\to f'' < 0 \to f$ concave

f positive $\to f$ has no extrema $\to f' > 0 \to f$ increasing

f has only one fixed point

Suppose 2 fixed points : p, p'

$IVT \to\ \exists\ c \in (0, p')$:

$$f'(c) = 1$$

$f'(p) < 1 \to p$ attractive $IVT \to$ dont exist more fixed points $\to f'(c') = 1 \nleq 1$

$\forall\ x \in (0, p)$:

$$f(x) > x$$

$\forall\ x \in \mathbb{R}$ „ $x > p$:

$$f(x) < x$$

f increasing $\to f([0, p]) = [0, p]$

Block V

Laboratory

1. Fixed points cardinality

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Let:

$\cdot\, f \,:\, [0,1] \to [0,1] \in \mathcal{C}^2([0,1])$

$\cdot\, f(1) < 1$

$\cdot\, f'' > 0 \in [0,1]$

Show that:

$\cdot\, \#\,\{x \in [0,1] \mid f(x) = x\} = 1$

Demonstration:

$\#\,\{x \in [0,1] \mid f(x) = x\} \geq 1$:

Case $f(0) = 0$:

0 fixed point

Case $f(0) > 0$:

$$\begin{array}{rcl} g\;:[0,1] & \longrightarrow & [-1,1] \\ x & \longmapsto & f(x) - x \end{array} \in \mathcal{C}^2([0,1])$$

$g(0) = f(0) - 0 > 0$

$g(1) = f(1) - 1 < 0$

Bolzano's theorem:

$\exists\, x \in (0,1)$:

$g(x) = 0$

$f(x) = x$

$\# \{x \in [0,1] \mid f(x) = x\} \leq 1$:

$g'' > 0$ over $[0,1]$

Rolle's theorem:

$\# \{x \in (0,1) \mid g'(x) = 0\} \leq 1$

$\# \{x \in (0,1) \mid g(x) = 0\} \leq 2$

$\# \{x \in (0,1) \mid f(x) = x\} \leq 2$

$f'' > 0$ over $[0,1]$

Monotonicity test:

f' increasing in $[0,1]$

$\forall\ a < b \in [0,1)$ " $f(a) = a, f(b) = b$:

Mean Value Theorem:

$\exists\ c \in (a,b)$:

$$f'(c) = \frac{f(b)-f(a)}{b-a} = 1$$

$\exists\ d \in (b,1)$:

$$f'(d) = \frac{f(1)-f(b)}{1-b} < 1$$

f' increasing $\to f'(c) < f'(b) < f'(d)$

$1 < f'(b) < 1$ absurd

$\therefore)\ \# \{x \in [0,1] \mid f(x) = x\} = 1$