

1. Discrete dynamical systems

Dynamical system

Let:

- M manifold
- T monoid
- $\phi : M \times T \rightarrow M$

Then, (M, T, ϕ) is a dynamical system if:

- $\forall x \in X :$

$$\phi(x, 0) = x$$

$$\forall t_1, t_2 \in T :$$

$$\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$$

Dimension

Let:

- (M, T, ϕ) dynamical system

We name dimension of (M, T, ϕ) to:

- $\dim(M)$

We denote:

- $\dim(M) = n : (M, T, \phi)$ n-D

Discrete & Continuous

Let:

· (M, T, ϕ) dynamical system

Then, (M, T, ϕ) is discrete if:

· $T \simeq \mathbb{N}$

Then, (M, T, ϕ) is continuous if:

· $T \subset \mathbb{R}$

· T open

Defined by a function

Let:

· (M, T, ϕ) dynamical system

· $f : M \rightarrow M$

Then, (M, T, ϕ) is a dynamical system defined by f if:

· $T = \mathbb{N}$

·
$$\begin{array}{ccc} \phi : M \times \mathbb{N} & \longrightarrow & M \\ (x, n) & \longmapsto & f^n(x) \end{array}$$

We denote:

· (M, T, ϕ) dynamical system defined by $f : (M, \mathbb{N}, f)$

· $f \in \mathcal{C}^n(M) : (M, \mathbb{N}, f) \in \mathcal{C}^n$

Orbit

Let:

- (M, \mathbb{N}, f) functional dynamical system
- $x \in M$

We name orbit of x to:

- $\{f^n(x)\}_{n \in \mathbb{N}}$

We denote:

- $o(x)$

Periodicity

Let:

- (M, \mathbb{N}, f) dynamical system
- $x \in M$
- $m \in \mathbb{N}$

Then, x is a m -periodic point if:

- $f^m(x) = x$

We denote:

- $\{x \in M \mid f(x) = x\} : \text{Fix}(f)$

Stability

Let:

$$\cdot f : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\cdot p \in \mathbb{R}^n \text{ m-periodic point}$$

Then, p is stable if:

$$\cdot \forall \varepsilon \in \mathbb{R}^+ :$$

$$\exists \delta \in \mathbb{R}^+ :$$

$$\forall x \in B(p, \delta) :$$

$$\forall n \in \mathbb{N} :$$

$$f^{nm}(x) \in B(p, \varepsilon)$$

Then, p is unstable if:

$$\cdot p \text{ not stable}$$

Attractive & Repulsive

Let:

$$\cdot f : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\cdot p \in \mathbb{R}^n \text{ m-periodic point}$$

Then, p is attractive if:

$$\cdot p \text{ stable}$$

$$\cdot \exists \varepsilon \in \mathbb{R}^+ :$$

$$\forall x \in B(p, \varepsilon) :$$

$$f^{nm}(x) \xrightarrow{n} p$$

Then, p is repulsive if:

$$\cdot p \text{ attractive by } f^{-1}$$

$$\cdot \forall \mathcal{U} \subset M \quad \mathcal{U} \text{ open} \wedge x \in \mathcal{U} :$$

$$\forall x' \in \mathcal{U} \quad x' \neq x :$$

$$\exists N \in \mathbb{N} :$$

$$\forall n \in \mathbb{N} \quad n \geq N :$$

$$f^{nm}(x') \notin \mathcal{U}$$

Fixed point character

Let:

$\cdot (M, \mathbb{N}, f)$ functional dynamical system

We name Fixed point character to:

$$\begin{aligned} f : \text{Fix}(f) &\longrightarrow \{-1, 0, 1\} \\ \cdot \quad \quad \quad x &\longmapsto \begin{cases} +1 & x \text{ repulsive} \\ -1 & x \text{ attractive} \\ 0 & x \text{ no attractive no repulsive} \end{cases} \end{aligned}$$

We denote:

$$\cdot f : \chi_f$$

Attraction set

Let:

$\cdot (M, \mathbb{N}, f)$ dynamical system

$\cdot x \in M$ attractive m-periodic point

$\cdot o(x)$ orbit of x

We name attraction set of x to:

$$\cdot \{y \in M \mid \exists x' \in o(x) : f^{nm}(y) \xrightarrow{n} x'\}$$

We denote:

$$\cdot A(x)$$

Multiplier

Let:

- (M, \mathbb{N}, f) \mathcal{C}^1 dynamical system
- $x \in M$

We name multiplier of x to:

- $f'(x)$

We denote:

- $m(x)$
- $|m(x)| = 1$: x neutral point

Feeble point

Let:

- (M, \mathbb{N}, f) \mathcal{C}^3 dynamical system
- $x \in M$

Then, x is feeble point if:

- x neutral point
- $f''(x) = 0$

Sarkovskii's order

We name Sarkovskii's order to:

$$\begin{aligned} & \cdot a = 2^n a', b = 2^m b' \\ & \cdot a <_s b \leftrightarrow \begin{cases} m < n & a' = b' = 1 \\ & a' = 1, b' \neq 1 \\ a' < b' & a' = b' \neq 1 \\ n < m & 1 \neq a' \neq b' \end{cases} \end{aligned}$$

Chaos

Let:

$$\cdot (\mathbb{R}, \mathbb{N}, f) \text{ dynamical system}$$

Then, $(\mathbb{R}, \mathbb{N}, f)$ is chaotic if:

$$\cdot \text{Fix}(f) \text{ dense in } \mathbb{R}$$

$$\cdot \exists x \in \mathbb{R} :$$

$$o(x) \text{ dense in } \mathbb{R}$$

$$\cdot \forall x \in \mathbb{R} :$$

$$\exists \varepsilon \in \mathbb{R}^+ :$$

$$\forall \delta \in \mathbb{R}^+ :$$

$$\exists \tilde{x} \in B(x, \delta) :$$

$$\lim_n o(\tilde{x}) \notin B(x, \varepsilon)$$

Topological equivalence

Let:

$$\cdot (M, \mathbb{N}, f_1), (M, \mathbb{N}, f'_2) \text{ dynamical systems}$$

Then, (M, \mathbb{N}, f_1) is topologically equivalent to (M, \mathbb{N}, f_2) if:

$$\cdot \text{Fix}(f) = \text{Fix}(f')$$

$$\cdot \forall x \in \text{Fix}(f) :$$

$$\chi_{f_1}(x) = \chi_{f_2}(x)$$

We denote:

$$\cdot (M, \mathbb{N}, f) \sim (M, \mathbb{N}, f')$$

Bifurcation

Let:

$$\cdot \Lambda \subset M$$

$$\cdot (M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda} \text{ dynamical systems parametrized by } \Lambda$$

$$\cdot \lambda_0 \in \Lambda$$

Then, λ_0 is a bifurcation value if:

$$\cdot \forall \varepsilon \in \mathbb{R}^+ :$$

$$\exists \lambda_1, \lambda_2 \in B(\lambda_0, \varepsilon) :$$

$$(M, \mathbb{N}, f_{\lambda_1}) \not\sim (M, \mathbb{N}, f_{\lambda_2})$$

Saddle-node bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is a saddle-node bifurcation value at x_0 if:

- $x_0 \in \text{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0) \neq 0$
- $\partial_{xx} f_\lambda(x_0) \neq 0$

Pitchfork bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is Pitchfork bifurcation value at x_0 if:

- $x_0 \in \text{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{x^2} f_\lambda(x_0, \lambda_0) = 0$
- $\partial_{\lambda x} f_\lambda(x_0, \lambda_0) \neq 0$
- $\partial_{x^3} f_\lambda(x_0, \lambda_0) \neq 0$

Period doubling bifurcation

Let:

$$\cdot \Lambda \subset M$$

$$\cdot (M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda} \text{ dynamical systems parametrized by } \Lambda$$

$$\cdot \lambda_0 \in \Lambda$$

$$\cdot x_0 \in M$$

Then, λ_0 is Period doubling bifurcation value at x_0 if:

$$\cdot \lambda_0 \text{ Pitchfork bifurcation value at } x_0 \text{ of } f^2$$