Block I

Tasks

1. 1st laboratory

Existence of holomorphic functions

Let:

$\cdot\ f \in \mathcal{H}(\mathbb{D})$

Study:

$\cdot\ \exists\, f \in \mathcal{H}(\mathbb{D})$:

$\forall\, n \in \mathbb{N}$ „ $n \geq 2$:

$a)\ f(\pm\frac{1}{n}) = \frac{1}{2n+1}$

$b)\ f(\pm\frac{1}{n}) = \frac{1}{n^2}$

$c)\ |f(\frac{1}{n})| = \frac{1}{\log(n+1)}$

$d)\ |f(\frac{1}{n})| = \frac{n}{n+1}$

Demonstration:

$a)$:

$E_1 := \{+\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N}\}$

$E_2 := \{-\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N}\}$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} = \lim_{n} \frac{f(\frac{1}{n})}{\frac{1}{n}} - \frac{f(0)}{\frac{1}{n}} = \frac{1}{2} - \lim_{n} \frac{f(0)}{\frac{1}{n}}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} \begin{cases} = \frac{1}{2} & f(0) = 0 \\ \notin \mathbb{C} & f(0) \neq 0 \end{cases}$$

Case $f(0) = 0$:

$$\lim_{E_2} \frac{f(z) - f(0)}{z - 0} = \lim_{n} \frac{f(-\frac{1}{n})}{-\frac{1}{n}} = -\frac{1}{2} \neq \lim_{E_1} \frac{f(z) - f(0)}{z - 0}$$

$$\nexists \ f \in \mathcal{H}(0) \ \ " \ \ f \text{ satisfies } a)$$

In particular:

$$\nexists \ f \in \mathcal{H}(\mathbb{D}) \ \ " \ \ f \text{ satisfies } a)$$

$b)$:

$$\begin{aligned} f \ : \mathbb{C} \ &\longrightarrow \ \mathbb{C} \\ z \ &\longmapsto \ z^2 \end{aligned}$$

$\forall \ n \in \mathbb{N} \ \ " \ \ n \geq 2$:

$$f(\pm\tfrac{1}{n}) = \tfrac{1}{n^2}$$

f satisfies $b)$

$$\begin{aligned} \bar{f} \ : \mathbb{R}^2 \ &\longrightarrow \ \mathbb{R}^2 \\ (x,y) \ &\longmapsto \ (u(x,y), v(x,y)) = (x^2 - y^2, 2xy) \end{aligned}$$

$\bar{f} \in \text{Pol}(\mathbb{R}^2) \rightarrow \bar{f}$ differentiable in $\mathbb{R}^2$

$\forall \ (x,y) \in \mathbb{R}^2$:

$$\partial_x u(x,y) = 2x = \partial_y v(x,y)$$

$$\partial_y u(x,y) = -2y = -\partial_x v(x,y)$$

f satisfies CR

$\therefore) \ \ f \in \mathcal{H}(\mathbb{R}^2)$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

$c)$:

Suppose $\exists \ f \in \mathcal{H}(\mathbb{D}) \ \ " \ \ f$ satisfies $c)$

$$f \in \mathcal{C}^0(\mathbb{D}) \rightarrow f(0) = f(\lim_n \frac{1}{n}) = \lim_n \ f(\frac{1}{n}) = 0$$

$$\left|\lim_{E_1} \frac{f(z)-f(0)}{z-0}\right| = \lim_{n} \frac{\left|f(\frac{1}{n})\right|}{\frac{1}{n}} \notin \mathbb{C}$$

$f \notin \mathcal{H}(0)$ absurd

$d)$:

$$\begin{aligned} f :\mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \tfrac{1}{z+1} \end{aligned}$$

$\forall\ n \in \mathbb{N}$ „ $n \geq 2$:

$$\left|f(\tfrac{1}{n})\right| = \frac{1}{\frac{1}{n}+1} = \frac{n}{n+1}$$

f satisfies $d)$

$$\begin{aligned} \bar{f} :\mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (u(x,y), v(x,y)) = \left(\tfrac{x+1}{(x+1)^2+y^2}, \tfrac{-y}{(x+1)^2-y^2}\right) \end{aligned}$$

$\bar{f} \in \mathrm{Rat}(\mathbb{R}^2) \wedge \forall\ (x,y) \in \mathbb{R}^2$:

$$(x+1)^2 + y^2 \neq 0$$

$\bar{f}$ differentiable in $\mathbb{R}^2$

$\forall\ (x,y) \in \mathbb{R}^2$:

$$\partial_x u(x,y) = \frac{y^2-(x+1)^2}{((x+1)^2+y^2)^2} = \partial_y v(x,y)$$

$$\partial_y u(x,y) = \frac{-2y(x+1)}{((x+1)^2+y^2)^2} = -\partial_x v(x,y)$$

f satisfies CR

$\therefore)$ $f \in \mathcal{H}(\mathbb{R}^2)$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

Constant tests

Let:

$\cdot\ \Omega \subset \mathbb{C}$ region

$\cdot\ f \in \mathcal{H}(\Omega)$

Then, holds:

$\cdot\ f_{Re} = 0 \ \vee \ f_{Im} = 0 \to f \in \text{Cst}(\Omega)$

$\cdot\ |f| \in \text{Cst}(\Omega) \to f \in \text{Cst}(\Omega)$

$\cdot$ Imf circumference $\to f \in$ Cst

Demonstration:

$f_{Re} = 0 \ \vee \ f_{Im} = 0$:

$u \ := \ f_{Re}$

$v \ := \ f_{Im}$

$f \in \mathcal{H}(\Omega) \to f$ satisfies CR in Ω

$\partial_x u = \partial_y v = 0$

$\partial_y u = -\partial_x v = 0$

Null diferential test:

Ω connex $\to u, v \in$ Cst

$u, v \in$ Cst$\to f \in$ Cst

$|f| \in \text{Cst}(\Omega)$:

$$\begin{array}{rcl} |f| : \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & \sqrt{u(x,y)^2 + v(x,y)^2} \end{array}$$

$|f| \in \text{Cst} \rightarrow \exists\ a \in \mathbb{R}$:

$$\sqrt{u(x,y)^2 + v(x,y)^2} = a$$

$$u(x,y)^2 + v(x,y)^2 = a^2$$

$$2\partial_x u(x,y) + 2\partial_x v(x,y) = 0$$

$$2\partial_y u(x,y) + 2\partial_y v(x,y) = 0$$

$f \in \mathcal{H}(\Omega) \rightarrow f$ satisfies CR in Ω

$$2\partial_y v(x,y) + 2\partial_x v(x,y) = 0$$

$$-2\partial_x v(x,y) + 2\partial_y v(x,y) = 0$$

$$+:\ 4\partial_y v(x,y) = 0 \rightarrow \partial_y v(x,y) = 0$$

$$-:\ 4\partial_x v(x,y) = 0 \rightarrow \partial_x v(x,y) = 0$$

Null differential test:

Ω connex $\rightarrow u, v \in$ Cst

$u, v \in \text{Cst} \rightarrow f \in \text{Cst}$

$\text{Im}(f)$ circumference :

$\exists\ (x_0, y_0) \in \mathbb{R}^2, r \in \mathbb{R}^+$:

$$\text{Im}(f) = C_r(x_0, y_0)$$

$$\begin{array}{rcl} \bar{f}\ : \mathbb{R}^2 & \longrightarrow & C_r(x_0,y_0) \\ (x,y) & \longmapsto & (r\cos(x-x_0), r\sin(y-y_0)) \end{array}$$

$\forall\ (x,y) \in \Omega$:

$$\left|\bar{f}\right|(x,y) = \sqrt{r^2(\cos^2(x-x_0) + \sin^2(y-y_0)} = r$$

$|f| \in \text{Cst} \rightarrow f$ Cst

Real part of holomorphic functions

Let:

- $\Omega \subset \mathbb{R}^2$ region
- $u \in \mathcal{C}^2(\Omega)$ 〃 $\exists\ f \in \mathcal{H}(\Omega)$:

  $f_{Re} = u$

Show that:

- $\partial_{xx} u + \partial_{yy} u = 0$

Study:

- $\exists\ f \in \mathcal{H}(\Omega)$:

  $a)\ f_{Re}(x, y) = x^2 + y^2$

  $b)\ f_{Re}(x, y) = x(x+1) - y^2$

  $c)\ \forall\ \alpha \in \mathbb{R}$:

  $f_{Re} = y^3 + \alpha x^2 y\ \wedge\ \Omega = \mathbb{C}$

Demonstration:

$\partial_{xx} u + \partial_{yy} u = 0$:

$u\ :=\ f_{Re}, v\ :=\ f_{Im}$

$f \in \mathcal{H}(\Omega) \to f$ satisfies CR in Ω

$\partial_x u = \partial_y v \to \partial_{xx} u = \partial_{xy} v$

$\partial_y u = -\partial_x v \to \partial_{yy} u = -\partial_{xy} v$

$\therefore)\ \partial_{xx} u + \partial_{yy} u = 0$

$f_{Re}(x,y) = x^2 + y^2$:

$$\partial_{xx}u + \partial_{yy}u = 4 \neq 0$$

$$\nexists f \in \mathcal{H}(\Omega) \;\;„\;\; f_{Re}(x,y) = x^2 + y^2$$

$f_{Re}(x,y) = x(x+1) - y^2$:

$$\begin{array}{rcl} \bar{f}\ :\Omega & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (u(x,y), v(x,y)) = (x(x+1) - y^2, 2xy + y) \end{array}$$

$\bar{f} \in$ Pol$\rightarrow$ $\bar{f}$ differentiable in Ω

$\forall\ (x,y) \in \Omega$:

$$\partial_x u(x,y) = 2x + 1 = \partial_y v(x,y)$$

$$\partial_y u(x,y) = -2y = -\partial_x v(x,y)$$

f satisfies CR in Ω

$$f \in \mathcal{H}(\Omega)\ \wedge\ f_{Re} = u$$

$f_{Re}(x,y) = y^3 + \alpha x^2 y$:

f has to satisfy CR in $\mathbb{C}$:

$\forall\ (x,y) \in \mathbb{R}^2$:

$$\partial_y v(x,y) = \partial_x u(x,y) = 2\alpha xy$$

$$\partial_x v(x,y) = -\partial_y u(x,y) = -3y^2 - \alpha x^2$$

$$v(x,y) = \alpha x y^2 + c(x)$$

$$v(x,y) = -3xy^2 - \tfrac{\alpha}{3}x^3 + c(y)$$

$$\alpha = -3,\ c(x) = x^3,\ c(y) = 0$$

$$v(x,y) = -3xy^2 + x^3$$

2. 2nd laboratory

Power series

Study:

- $\sum_{n\geq 1} n(n+1)z^n$

Demonstration:

Naming:

R radius of convergence of the series

$\forall\ n \in \mathbb{N}$:

$$c_n \ := \ n(n+1)$$

Convergence domain:

$$\lim_n \frac{c_n}{c_{n+1}} = \lim_n \frac{n(n+1)}{(n+1)(n+1)} = 1$$

Quotient test:

$$R^{-1} = \overline{\lim_n} \, |c_n|^{\frac{1}{n}} = 1 \to R = 1$$

Cauchy-Hadamard theorem:

$\sum_{n\geq 1} n(n+1)z^n$ convergent over $\mathbb{D}$

$\sum_{n\geq 1} n(n+1)z^n$ divergent over $\mathbb{C} \setminus \overline{\mathbb{D}}$

$\forall\ K \subset \mathbb{D}$ „ K compact :

$$\sum_{n\geq 1} n(n+1)z^n \text{ uniformly convergent over } K$$

Define

$$\begin{aligned} f \ : \mathbb{D} \ &\longrightarrow \ \mathbb{C} \\ z \ &\longmapsto \ z \sum_{n\geq 1} n(n+1)z^{n-1} \end{aligned}$$

Sum:

$$\begin{aligned} g \ : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n\geq 1} n(n+1)z^{n-1} \end{aligned}$$

UCI theorem:

$$\int_0^z g(t)dt = \sum_{n\geq 1} (n+1)z^n$$

$$\begin{aligned} h \ : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n\geq 1} (n+1)z^n \end{aligned}$$

$$\int_0^z h(t)dt = \sum_{n\geq 1} z^{n+1} = \sum_{n\geq 0} z^n = \frac{1}{1-z}$$

$$h(z) = \partial_z \tfrac{1}{1-z} = \tfrac{1}{(1-z)^2}$$

$$g(z) = \partial_z h(z) = \tfrac{2}{(1-z)^3}$$

$$f(z) = \tfrac{2z}{(1-z)^3}$$

Application:

In particular:

$$\sum_{n\geq 1} (-1)^n \frac{n(n+1)}{2^n} = f(-\frac{1}{2}) = \frac{-2^3}{3^3}$$