

Dynamical systems

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Block I

Definitions

1. Discrete dynamical systems
Dynamical system

Let:

· M manifold

· $(T, +)$ monoid

· $\phi : T \times M \rightarrow M$

Then, (M, T, ϕ) is a dynamical system if:

· $\forall x \in X$:

· $\phi(x, 0) = x$

· $\forall t_1, t_2 \in T$:

· $\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$

Discrete

Let:

· (M, T, ϕ) dynamical system

Then, (M, T, ϕ) is discrete if:

· $T \simeq \mathbb{N}$

Dimension

Let:

· (M, T, ϕ) dynamical system

We name dimension of (M, T, ϕ) to:

$\dim(M)$

We denote:

· $\dim(M) = n : (M, T, \phi)$ n-D

Functional

Let:

· (M, T, ϕ) discrete dynamical system

Then, (M, T, ϕ) is functional if:

· $T \subset \mathbb{N}$

· $\exists f : M \rightarrow M:$

· $\forall (t, x) \in T \times M:$

· $\phi(t, x) = f^t(x)$

We denote:

· $T = \mathbb{N} : (M, f)$ functional dynamical system

· $f \in \mathcal{C}^n(M) : (M, f) \in \mathcal{C}^n$

Orbit

Let:

$\cdot (M, f)$ functional dynamical system

$\cdot x \in M$

We name orbit of x to:

$$\{f^n(x)\}_{n \in \mathbb{N}}$$

We denote:

$\cdot o(x)$

Periodicity

Let:

$\cdot (M, f)$ functional dynamical system

$\cdot x \in M$

$\cdot m \in \mathbb{N}$

Then, x is a period m point if:

$$\cdot f^m(x) = x$$

We denote:

$\cdot \{x \in M \mid x \text{ period } 1 \text{ point} \} : \text{Fix}(f)$

Stability

Let:

· (\mathbb{R}^n, f) functional dynamical system

· $p \in \mathbb{R}^n$ period m point

Then, p is stable for f if:

· $\forall \varepsilon \in \mathbb{R}^+$:

· $\exists \delta \in \mathbb{R}^+$:

· $\forall x \in B(p, \delta)$:

· $\forall n \in \mathbb{N}$:

· $f^{nm}(x) \in B(p, \varepsilon)$

Then, p is unstable if:

· p not stable

Then, p is attractive for f if:

· p stable

· $\exists \varepsilon \in \mathbb{R}^+$:

· $\forall x \in B(p, \varepsilon)$:

· $f^{nm}(x) \xrightarrow{n} p$

Then, p is repulsive if:

· p attractive by f^{-1}

Attraction set

Let:

- (M, f) functional dynamical system
- $x \in M$ attractive m-periodic point
- $o(x)$ orbit of x

We name attraction set of $o(x)$ to:

$$\{y \in M \mid \exists x' \in o(x) : f^{nm}(y) \xrightarrow{n} x'\}$$

We denote:

- $A(o(x))$

Multiplier

Let:

- (M, f) \mathcal{C}^1 functional dynamical system
- $x \in M$

We name multiplier of x to:

$$m(x) = f'(x)$$

We denote:

- $|m(x)| = 1$: x neutral point
- $|m(x)| = 1 \wedge f''(x) = 0$: x feeble point

Character

Let:

$\cdot (M, f)$ functional dynamical system

We name fixed point character to:

$$\begin{array}{rcl} f : \text{Fix}(f) & \longrightarrow & \{-2, -1, 0, 1, 2\} \\ x & \longmapsto & \begin{cases} +2 & x \text{ repulsive} \\ +1 & x \text{ unstable} \\ 0 & * \\ -1 & x \text{ stable} \\ -2 & x \text{ attractive} \end{cases} \end{array}$$

We denote:

$$\cdot f : \chi_f$$

Topological equivalence

Let:

$\cdot (M, f_1), (M, f_2)$ functional dynamical systems

Then, (M, \mathbb{N}, f_1) is topologically equivalent to (M, \mathbb{N}, f_2) if:

$$\cdot \text{Fix}(f_1) = \text{Fix}(f_2)$$

$$\cdot \forall x \in \text{Fix}(f_1):$$

$$\cdot \chi_{f_1}(x) = \chi_{f_2}(x)$$

We denote:

$$\cdot (M, f_1) \sim (M, f_2)$$

Bifurcation

Let:

- $\{(M, f_\lambda)\}_{\lambda \in \Lambda}$ functional dynamical systems
- $\lambda_0 \in \Lambda$

Then, λ_0 is a bifurcation parameter if:

- $\forall \varepsilon \in \mathbb{R}^+$:
- $\exists \lambda' \in B(\lambda_0, \varepsilon)$:
- $(M, f_{\lambda'}) \not\sim (M, f_{\lambda_0})$

Saddle-node bifurcation

Let:

- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ functional dynamical systems
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is a saddle-node bifurcation value at x_0 if:

- $x_0 \in \text{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0) \neq 0$
- $\partial_{xx} f_\lambda(x_0) \neq 0$

Pitchfork bifurcation

Let:

· $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ functional dynamical systems

· $\lambda_0 \in \Lambda$

· $x_0 \in M$

Then, λ_0 is Pitchfork bifurcation value at x_0 if:

· $x_0 \in \text{Fix}(f_{\lambda_0})$

· x_0 neutral point of f_{λ_0}

· $\partial_\lambda f_\lambda(x_0, \lambda_0) = 0$

· $\partial_{x^2} f_\lambda(x_0, \lambda_0) = 0$

· $\partial_{\lambda x} f_\lambda(x_0, \lambda_0) \neq 0$

· $\partial_{x^3} f_\lambda(x_0, \lambda_0) \neq 0$

Period doubling bifurcation

Let:

$$\cdot \Lambda \subset M$$

$$\cdot (M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda} \text{ functional dynamical systems}$$

$$\cdot \lambda_0 \in \Lambda$$

$$\cdot x_0 \in M$$

Then, λ_0 is Period doubling bifurcation value at x_0 if:

$$\cdot \lambda_0 \text{ Pitchfork bifurcation value at } x_0 \text{ of } f^2$$

Sarkovskii's order

We name Sarkovskii's order to:

$$\forall a, b \in \mathbb{N} \quad \begin{cases} a = 2^n a', b = 2^m b', 2^n \parallel a, 2^m \parallel b: \\ a <_s b \leftrightarrow \begin{cases} m < n & a' = b' = 1 \\ & a' = 1, b' \neq 1 \\ a' < b' & a' = b' \neq 1 \\ n < m & 1 \neq a' \neq b' \end{cases} \end{cases}$$

Chaos

Let:

· $(\mathbb{R}, \mathbb{N}, f)$ dynamical system

Then, $(\mathbb{R}, \mathbb{N}, f)$ is chaotic if:

· $\text{Fix}(f)$ dense in \mathbb{R}

· $\exists x \in \mathbb{R}$:

· $o(x)$ dense in \mathbb{R}

· $\forall x \in \mathbb{R}$:

· $\exists \varepsilon \in \mathbb{R}^+$:

· $\forall \delta \in \mathbb{R}^+$:

· $\exists \tilde{x} \in B(x, \delta)$:

· $\lim_n o(\tilde{x}) \notin B(\lim_n o(x), \varepsilon)$

2. Linear dynamical systems

Linear

Let:

· (\mathbb{R}^n, f) euclidean functional dynamical system

Then, (\mathbb{R}^n, f) is linear if:

· $\exists A \in \mathcal{M}_{n \times n}(\mathbb{R})$:

· $\forall x \in \mathbb{R}^n$:

· $f(x) = Ax$

Invariant curve

Let:

· γ differentiable curve

· $p \in \mathbb{R}^n$

Then, γ is invariant if:

· $\forall x \in \gamma^*:$

· $o(x) \subset \gamma^*$

Then, γ is converges to p if:

· $\forall x \in \gamma^*:$

· $o(x) \xrightarrow{n} p$