I. Complex logarithm

holomorphisms with different definitions

Let:

$$\cdot \quad \begin{array}{rccc} \bar{log} : & \mathbb{C} \setminus \{z \in \mathbb{C} \mid Re(z) < 0, Im(z) = 0\} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & log(z) + 4\pi i \end{array}$$

Show that:

$$\cdot\, \bar{log} \in \mathcal{H}(\mathbb{C} \setminus \{z \in \mathbb{C} \mid Re(z) < 0, Im(z) = 0\})$$

Start:

Convergence domain:

$$\lim_{n} \frac{n(n+1)}{(n+1)(n+1)} = 1$$

Quotient test:

$$\overline{\lim_{n}}\, |c_n|^{\frac{1}{n}} = 1 \rightarrow \left(\overline{\lim_{n}}\, |c_n|^{\frac{1}{n}}\right)^{-1} = 1$$

Cauchy-Hadamard theorem:

$$\sum_{n \geq 1} n(n+1)z^n \text{ absolutely convergent over } \mathbb{D}$$

$$\sum_{n \geq 1} n(n+1)z^n \text{ divergent over } \mathbb{C} \setminus \overline{\mathbb{D}}$$

$$\forall\ K \subset \mathbb{D} \text{ ,, } K \text{ compact } :$$

$$\sum_{n \geq 1} n(n+1)z^n \text{ uniformly convergent over } K$$

Sum:

Consider:

$$\begin{array}{rccc} f : & \mathbb{D} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \sum_{n \geq 1} n(n+1)z^n \end{array}$$

$$\begin{array}{rccc} g: & \mathbb{D} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \sum_{n\geq 1} n(n+1)z^{n-1} \\ h: & \mathbb{D} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \sum_{n\geq 1} (n+1)z^{n} \end{array}$$

$\forall\ z \in \mathbb{D}$:

$$f(z) = zg(z)$$

UCI theorem:

$$\int_0^z g(t)dt = \sum_{n\geq 1} (n+1)z^n = h(z)$$

g, h well defined

$\forall\ z \in \mathbb{D}$:

UCI theorem:

$$\int_0^z h(t)dt = \sum_{n\geq 1} z^{n+1} = \sum_{n\geq 0} z^n = \frac{1}{1-z}$$

$h(z) = \partial_z \frac{1}{1-z} = \frac{1}{(1-z)^2}$

$g(z) = \partial_z h(z) = \frac{2}{(1-z)^3}$

$f(z) = zg(z) = \frac{2z}{(1-z)^3}$

Application:

$$\sum_{n\geq 1} (-1)^n \frac{n(n+1)}{2^n} = f(-\frac{1}{2}) = \frac{-2^3}{3^3}$$