

1. 2nd laboratory

Power series

Study:

$$\cdot \sum_{n \geq 1} n(n+1)z^n$$

Demonstration:

Naming:

R radius of convergence of the series

$\forall n \in \mathbb{N} :$

$$c_n := n(n+1)$$

Convergence domain:

$$\lim_n \frac{c_n}{c_{n+1}} = \lim_n \frac{n(n+1)}{(n+1)(n+1)} = 1$$

Quotient test:

$$R^{-1} = \overline{\lim}_n |c_n|^{\frac{1}{n}} = 1 \rightarrow R = 1$$

Cauchy-Hadamard theorem:

$\sum_{n \geq 1} n(n+1)z^n$ convergent over \mathbb{D}

$\sum_{n \geq 1} n(n+1)z^n$ divergent over $\mathbb{C} \setminus \overline{\mathbb{D}}$

$\forall K \subset \mathbb{D} \quad \parallel K \text{ compact} :$

$\sum_{n \geq 1} n(n+1)z^n$ uniformly convergent over K

Define

$$\begin{aligned} f : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto z \sum_{n \geq 1} n(n+1)z^{n-1} \end{aligned}$$

Sum:

$$\begin{aligned} g : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n \geq 1} n(n+1)z^{n-1} \end{aligned}$$

UCI theorem:

$$\int_0^z g(t) dt = \sum_{n \geq 1} (n+1)z^n$$

$$\begin{aligned} h : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n \geq 1} (n+1)z^n \end{aligned}$$

$$\int_0^z h(t) dt = \sum_{n \geq 1} z^{n+1} = \sum_{n \geq 0} z^n = \frac{1}{1-z}$$

$$h(z) = \partial_z \frac{1}{1-z} = \frac{1}{(1-z)^2}$$

$$g(z) = \partial_z h(z) = \frac{2}{(1-z)^3}$$

$$f(z) = \frac{2z}{(1-z)^3}$$

Application:

In particular:

$$\sum_{n \geq 1} (-1)^n \frac{n(n+1)}{2^n} = f\left(-\frac{1}{2}\right) = \frac{-2^3}{3^3}$$