Probability

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January 16, 2014

Contents

Block I

Definitions

1. Random Variables

The study of probabilities was started by Blaise Pascal.

Probability Space

Let:

- Ω set
- $\mathcal{A}\sigma$-algebra
- $\mathcal{P} : \mathcal{A} \to [0,1]$

Then, $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space if:

- $\mathcal{P}(\Omega) = 1$
- $\forall \{A_i\}_{i\geq 0} \subset \mathcal{A}$ " $\{A_i\}_{i\geq 0}$ mutually disjoint :

$$\mathcal{P}(\bigcup_{i\geq 0} A_i) = \sum_{i\geq 0} \mathcal{P}(A_i)$$

Conditioned Probability

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space $B \in \mathcal{A}$

We call probability conditioned by B over $(\Omega, \mathcal{A}, \mathcal{P})$ to:

- $$\begin{array}{rcl} \mathcal{P}|_B : \mathcal{A} & \longrightarrow & [0,1] \\ A & \longmapsto & \frac{\mathcal{P}(A\cap B)}{\mathcal{P}(B)} \end{array}$$

Independency

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space $\{A_i\}_{i=1}^n \subset \mathcal{A}$

Then, $\{A_i\}_{i=1}^n$ is an independent family if:

- $\forall \{k_i\}_{i=1}^r \subset [0,n]_{\mathbb{N}}$:

$$\mathcal{P}(\bigcap_{i=1}^r A_{k_i}) = \prod_{i=1}^n A_{k_i}$$

Random Variable

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \to \mathbb{R}$

Then, X is a random variable if:

- $\forall A \in \mathbb{B}(\mathbb{R})$:

  $X^{-1}(A) \in \mathcal{A}$

Law

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \to \mathbb{R}$ random variable

We call law of X to:

- $$\begin{array}{rcl} \mathcal{P}_X : \mathbb{B}(\mathbb{R}) & \longrightarrow & [0,1] \\ A & \longmapsto & \mathcal{P}(X^{-1}(A)) \end{array}$$

Distribution Function

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \to \mathbb{R}$ random variable

We call cumulative distribution function of X to:

- $$\begin{array}{rcl} F_x : \mathbb{R} & \longrightarrow & [0,1] \\ x & \longmapsto & \mathcal{P}(X \leq x) \end{array}$$

Discrete Random Variable

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \to \mathbb{R}$ random variable

Then, X is a discrete random variable if:

- $X(\Omega) \underset{\sim}{\subset} \mathbb{N}$

Probability Mass function

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \to \mathbb{R}$ discrete random variable

We call probability mass function of X to:

- $\begin{array}{rcl} p : X(\Omega) & \longrightarrow & [0,1] \\ x & \longmapsto & P(X = x) \end{array}$

Density function

Let:

- $f : \mathbb{R} \to \mathbb{R}$

Then, f is a density function if:

- $f \geq 0$
- $f\mathcal{R}(\mathbb{R})$
- $\int_{\mathbb{R}} f(x)dx = 1$

Absolutely Continuous random variable

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \to \mathbb{R}$ random variable
- F_X distribution function of X

Then, X is an absolutely continuous random variable if:

- $\exists\ f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ " $\forall\ x \in \mathbb{R}$:

  $F_x(x) = \int_{-\infty}^{x} f(x)dx$

We denote:

- Xan absolutely continuous random variable : X abs cont

Expected Value

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \to \mathbb{R}$ discrete random variable
- p probability mass function of X
- $X' : \Omega \to \mathbb{R}$ abs cont
- f_X density function of X'

Then, X is expectable if:

- $\sum_{x \in X(\Omega)} |x| p(x) \in \mathbb{R}$

We call expected value of X to:

- $\sum_{x \in X(\Omega)} x p(x)$

Then, X' is expectable if:

- $\int_{\mathbb{R}} |x| f(x) dx \in \mathbb{R}$

We call expected value of X' to:

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We denote:

- expected value of $X : E(X)$

Varaible change

Let:

- $\cdot$ $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $\cdot$ $X : \Omega \to \mathbb{R}$ abs cont
- $\cdot$ $I = \mathring{I} \subset \mathbb{R}$ „ $\mathcal{P}(X \in I) = 1$
- $\cdot$ $J = \mathring{J} \subset \mathbb{R}$
- $\cdot$ $g : I \to J$

Then, g is a variable change over X if:

- $\cdot$ g biyective
- $\cdot$ $g, g^{-1} \in \mathcal{C}^1$

2. Random Vectors

Generalization of random variables to n dimension

Aleatory vector

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ probability space
- $X : \Omega \to \mathbb{R}^m$

Then, X is a random vector if:

- $\forall\ i \in [1, m]_{\mathbb{N}}$:

  X_i random variable

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ith Marginal distribution function

Let:

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- $X : \Omega \to \mathbb{R}^m$ random vector
- $i \in [1, m]_{\mathbb{N}}$

We call ith marginal distribution function of X to:

- $\begin{array}{rcl} f : \mathbb{R} & \longrightarrow & [0,1] \\ x & \longmapsto & \mathcal{P}(X_i \leq x) \end{array}$

We denote:

- $f : F_i$

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Block II

Propositions

Block III

Examples

Block IV

Exercises