Complex Analysis

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Definitions

1. The field of complex numbers

introduction

The field of complex numbers

Let:

$$
\begin{aligned}
+ \, &: \mathbb{R}^2 \times \mathbb{R}^2 \;\longrightarrow\; \mathbb{R}^2 \\
&((a,b),(c,d)) \;\longmapsto\; (a+c, b+d) \\
\cdot \, &: \mathbb{R}^2 \times \mathbb{R}^2 \;\longrightarrow\; \mathbb{R}^2 \\
&((a,b),(c,d)) \;\longmapsto\; (ac - bd, ad + bd)
\end{aligned}
$$

We name the field of complex numbers to:

$\cdot\,(\mathbb{R}^2, +, \cdot)$

We denote:

$\cdot\,(\mathbb{R}^2, +, \cdot) \,:\, \mathbb{C}$

$\cdot\,(0,1) \in \mathbb{C} \,:\, i$

$\cdot\,(a,b) \in \mathbb{C} \,:\, a + bi$

$\cdot\,\pi_1(\mathbb{C}) \,:\, Re$

$\cdot\,\pi_2(\mathbb{C}) \,:\, Im$

Conjugation

We name complex conjugation to:

$$
\begin{aligned}
f \, &: \mathbb{C} \;\longrightarrow\; \mathbb{C} \\
&(a,b) \;\longmapsto\; (a, -b)
\end{aligned}
$$

We denote:

$\cdot\,f((a,b)) \,:\, \overline{(a,b)}$

Norm

We name   complex norm   to:

$$
\cdot \quad
\begin{array}{rcl}
f : \mathbb{C} & \longrightarrow & \mathbb{R} \\
(a,b) & \longmapsto & \sqrt{a^2 + b^2}
\end{array}
$$

We denote:

$$
\cdot \; f((a,b)) \; : \; |(a,b)|
$$

Polar transformation

We name   polar transformation   to:

$$
\cdot \quad
\begin{array}{rcl}
f : \mathbb{C} & \longrightarrow & \mathbb{R}^+ \times [0, 2\pi) \\
(a,b) & \longmapsto & (\sqrt{|(a,b)|}, \arctan(\tfrac{b}{a}))
\end{array}
$$

We denote:

$$
\cdot \; f((a,b)) \; : \; (r, \theta)
$$

Unit sphere projection

We name   unit sphere projection   to:

$$
\cdot \quad
\begin{array}{rcl}
f : \mathbb{C} & \longrightarrow & S^1 \\
z & \longmapsto & \frac{z}{|z|}
\end{array}
$$

We denote:

$$
\cdot \; f(z) \; : \; \pi(z)
$$

Roots of unity

Let:

$\cdot\, z \in \mathbb{C}$

Then, z is a root of unity if:

$\cdot\, \exists\, n \in \mathbb{N} \;\;_{_\parallel}\; z^n = 1$

We denote:

$\cdot\, \{z \in \mathbb{C} \mid z \text{ root of unity } \} \,:\, S^1$

Disk

Let:

$\cdot\, p \in \mathbb{C}$

$\cdot\, r \in \mathbb{R}^+ \smallsetminus \{0\}$

We name Disk centered in p and radius r to:

$\cdot\, \{z \in \mathbb{C} \mid |z - p| < r\}$

We denote:

$\cdot\, \{z \in \mathbb{C} \mid |z - p| < r\} \,:\, D^1$

Component decomposition

Let:

$\cdot\, f \,:\, \mathbb{C} \to \mathbb{C}$

We name real component of f to:

$$\cdot \quad \begin{array}{lll} f_{Re} \,:\, \mathbb{C} & \longrightarrow & \mathbb{R} \\ \quad\quad z & \longmapsto & Re(f(z)) \end{array}$$

We name imaginary component of f to:

$$\cdot \quad \begin{array}{lll} f_{Im} \,:\, \mathbb{C} & \longrightarrow & \mathbb{R} \\ \quad\quad z & \longmapsto & Im(f(z)) \end{array}$$

We name component decomposition of f to:

$$\cdot \quad \begin{array}{lll} f_{\mathbb{R}^2} \,:\, \mathbb{C} & \longrightarrow & \mathbb{R}^2 \\ \quad (x,y) & \longmapsto & (f_{Re}(x+yi), f_{Im}(x+yi) \end{array}$$

$$\boxed{\textbf{2.} \quad \textbf{Holomorphic functions}}$$

Incremental quotient

Let:

     $\cdot\, \mathcal{U} \subset \mathbb{C}$ open

     $\cdot\, f \,:\, \mathcal{U} \to \mathbb{C}$

     $\cdot\, p \in \mathcal{U}$

We name incremental quotient of f in p to:

     $\cdot\, \displaystyle\lim_{z \to p} \frac{f(z) - f(p)}{z - p}$

We denote:

     $\cdot\, f'(p)$

Holomorphic function

Let:

$\cdot\, \mathcal{U} \subset \mathbb{C}$ open

$\cdot\, f : \mathcal{U} \to \mathbb{C}$

$\cdot\, p \in \mathcal{U}$

Then, f is holomorphic over p if:

$\cdot\, \exists f'(p)$

Then, f is holomorphic over U if:

$\cdot\quad \forall\, p \in \mathcal{U}:$

$\exists f'(p)$

We denote:

$\cdot\, \{f : \mathcal{U} \to \mathbb{C} \,|\, f \text{ holomorphic over } \mathcal{U}\} \,:\, \mathcal{H}(\mathcal{U})$

$\cdot\, f \in \mathcal{H}(\mathbb{C}) \,:\, f \text{ entire}$

Cauchy-Riemman equations

Let:

$\cdot\ u, v\ :\ \mathbb{R}^2 \to \mathbb{R}$

$\cdot$
$$\begin{array}{rcl} f : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (u((x,y)), v((x,y))) \end{array}$$

Then, f is satisfies the Cauchy-Riemman equations if:

$\cdot\ \exists\ u_x, u_y, v_x, v_y$

$\cdot\ u_x = v_y$

$\cdot\ u_y = -v_x$

We denote:

$\cdot\ u_x + iv_x\ :\ f_x$

$\cdot\ u_y + iv_y\ :\ f_y$

Conformal

Let:

·$\mathcal{U} \subset \mathbb{C}$ open

·$f \,:\, \mathcal{U} \to \mathbb{C}$

·$z \in \mathcal{U}$

Then, f is conformal in z if:

·$\exists\, c \in \mathbb{C}$ „ $\forall\, I \subset \mathbb{R}$ „ $0 \in I$:

$\forall\, \gamma \,:\, I \to \mathbb{R}^2$ „ γ differentiable $\wedge\, \gamma(0) = z\, \wedge\, \gamma'(0) \neq 0$:

$\frac{(f \circ \gamma)'(0)}{\gamma'(0)} = c$

Then, f is conformal if:

· $\forall\, z \in \mathcal{U}$:

f conformal in z

Block II

Propositions

1.   The field of complex numbers

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Block III

Examples

1. Holomorphic functions

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Block IV

Problems

Llista 1: Els nombres complexos

B.1. Expresseu en la forma $a + ib$ els següents nombres:

    (a) $(2 + 3i)(4 + i)$ (c) $\frac{1}{4+i}$ (e) $\sqrt{i}$ (g) $\sqrt{9i}$

    (b) $(4 + 2i)^2$ (d) $\frac{i}{4+i}$ (f) $\sqrt{-i}$ (h) $\sqrt{1+i}$

B.2. Si $z = x + iy$ trobeu les parts real i imaginària de les expressions següents:

    (a) z^2 (b) $z(z + 1)$ (c) $\frac{1}{z}$ (d) $\frac{1}{z-3}$.

B.3. És cert que

    a) $\operatorname{Re}(z + w) = \operatorname{Re} z + \operatorname{Re} w$?

    b) $\operatorname{Re}(zw) = (\operatorname{Re} z)(\operatorname{Re} w)$?

    c) $\operatorname{Re}(\frac{z}{w}) = \frac{\operatorname{Re} z}{\operatorname{Re} w}$?

B.4. Trobeu la forma polar dels nombres següents i dibuixeu-los.

    (a) $3(1 + \sqrt{3}\,i)$ (b) $2\sqrt{3} - 2i$ (c) $-2 + 2i$ (d) $-1 - i$

B.5. Sigui $(x + iy)/(x - iy) = a + ib$. Proveu que $a^2 + b^2 = 1$.

B.6. Proveu que si $p(z)$ és un polinomi amb coeficients reals i z és un zero de p llavors $\bar{z}$ també ho és.

B.7. Descriviu els conjunts del pla que satisfan (recordeu que $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.)

    (a) $\operatorname{Im} \frac{z-a}{z} = 0, a \in \mathbb{C}^*$ (b) $|z| = \operatorname{Re} z + 1$ (c) $|z - 2| > |z - 3|$

SOL. B.1. a) $5 + 14i$; b)$12 + 16i$; c) $4/17 - i/17$; d)$1/17 + 4i/17$; e) $\pm\sqrt{2}/2(1 + i)$; f)$\pm\sqrt{2}/2(1 - i)$; g) $\pm 3\sqrt{2}/2(1 + i)$;
h) $\pm 2^{1/4}(\cos(\pi/8) + i\sin(\pi/8))$.

B.2 a) $x^2 - y^2 + 2ixy$; b) $x^2 - y^2 + x + i(y + 2xy)$; c) $(x - iy)/(x^2 + y^2)$; d) $(x - 3 - iy)/((x - 3)^2 + y^2)$.

B.3 a) si. b) no. c) no.

B.4 a) $6(\cos(\pi/3) + i\sin(\pi/3))$; b) $4(\cos(\pi/6) - i\sin(\pi/6))$; c) $2\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$; d) $\sqrt{2}(\cos(3\pi/4) - i\sin(3\pi/4))$.

B.6 Conjugueu tot el polinomi.

B.7 a) Recta que passa per 0 i a; b) Paràbola horitzontal $x = (1/2)(y^2 - 1)$; c) $\{\operatorname{Re} z > 3/2\}$.

1. Expresseu en la forma $a + ib$ els següents nombres:

    (a) $\frac{1}{i}$ (c) $\frac{1}{2+i} + \frac{1}{2-i}$ (e) $\left(\frac{2+i}{3-2i}\right)^2$ (g) $\sqrt[4]{-i}$

    (b) $\frac{1+i}{1-i}$ (d) $\frac{1}{2+i} + \frac{4-2i}{3+i}$ (f) $(1+i)^{100} + (1-i)^{100}$ (h) $(3 + 4i)^{\frac{1}{2}}$

2. Si $z = x + iy$ on $x, y \in \mathbb{R}$, trobeu les parts real i imaginària de:

Llista 2: Funcions de variable complexa i equacions de Cauchy-Riemann

B.1. Trobeu els punts on la funció f és derivable (en el sentit complex), en els següents casos, i calcula'n la derivada.

(a) $\cos |z|^2$

(c) e^{iz}

(e) $\frac{1}{(z-1)^2(z^2+2)}$

(b) $|z|^4$

(d) $z + \frac{1}{z}$

(f) $\frac{1}{(z+\frac{1}{z})^2}$

Solució: (a) $\emptyset$; (b) $\emptyset$; (c) $\mathbb{C}$; $f'(z) = ie^{iz}$;(d) $\mathbb{C} \setminus \{0\}$; $f'(z) = 1 - |\frac{1}{z^2}|$; (e) $\mathbb{C} \setminus \{1, \pm\sqrt{2}i\}$; (f) $\mathbb{C}$.

B.2. Determineu si aquestes funcions poden ser la part real d'una funció holomorfa, i en cas que ho siguin calculeu la part imaginària.

(a) $e^x \cos y$

(b) $x^3 + 6xy^2$

(c) $\log(x^2 + y^2)$

Solució: (a) $e^x \sin y$; $f(z) = e^z$; (b) No ho és; (c) $2 \arctan(y/x)$; $(f(z) = \log(z^2))$.

B.3. Sigui f una funció holomorfa en un obert $\Omega \subset \mathbb{C}$ i $z_0 \in \Omega$ tal que $f'(z_0) \neq 0$. Quin angle formen les corbes $\operatorname{Re} f(z) = \operatorname{Re} f(z_0)$ i $\operatorname{Im} f(z) = \operatorname{Im} f(z_0)$ en un punt z_0 ?
Solució: $\pi/2$.

1. Trobeu els punts on la funció f és derivable (en el sentit complex), en els següents casos:

(a) $f(z) = |z|$

(d) $f(z) = z + z\bar{z}$

(b) $\cosh x \cos y + i \sinh x \sin y$

(c) $f(z) = \operatorname{Re} z$

(e) $f(z) = \operatorname{Im} e^{\bar{z}} + i \operatorname{Re} e^z$

2. Sigui $\Omega \subset \mathbb{C}$ un obert, $z_0 \in \Omega$ i $f : \Omega \to \mathbb{C}$ una funció.

a) Identificant $\mathbb{R}^2$ amb $\mathbb{C}$ de la forma habitual, demostreu que si f és diferenciable en z_0, llavors

$$Df(z_0)(z) = \frac{\partial f}{\partial z}(z_0) \cdot z + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot \bar{z} \qquad (z \in \mathbb{C}),$$

on

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

b) Proveu que f és holomorfa en Ω si, i només si, f és diferenciable i $\frac{\partial f}{\partial \bar{z}} = 0$ en Ω. En tal cas, $f' = \frac{\partial f}{\partial z}$.

3. Demostreu que si f és diferenciable en un obert de $\mathbb{C}$, llavors

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}} \quad \text{i} \quad \overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}.$$

1.   The field of complex numbers

introduction

entity

Let:

· *statements*

·

Then, *item* is a/an entity if:

· *conditions*

·

We denote:

· *property* : *notation*

·

Block V

Tasks