1. Holomorphic functions

Cauchy-Riemman

Let:

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies CR

Then, holds:

- $f_x = f'(z)$
- $f_y = -if_y$
- $\partial_{f(z)} z = 0$

Demonstration:

$$f_x = u_x + iv_x = a + bi = f'(z)$$

$$f_y = i(a + ib) = if'(z)$$

$$f'(z) = -if_y$$

notation

tangent venctor

Let:

· γ differentiable plane arc ,, $\forall\ t \in I$:

$\gamma'(t) \neq 0$

Then, holds:

· $\gamma'(t)$ tangent to γ

Demonstration:

demonstration

arc images

Let:

· $f \in \mathcal{H}(\mathcal{U})$

· γ differentiable plane arc " $\gamma \subset \mathcal{U}$

· $\sigma = f(\gamma)$

· $z_0 = \gamma(0)$

Then, holds:

· $\sigma' = f'(\gamma)\gamma'$

· $\gamma'(0) \neq 0 \to f'(z_0) \neq 0$

· $\sigma'(0) = f'(z_0)\gamma'(0)$

· $|\sigma'(0)| = |f'(z_0)||\gamma'(0)|$

· $arg\sigma'(0) = arg\gamma'(0) + argf(z_0)$

· f aplica una homotecia mas una rotacion constante a todos los vectores tangentes que salen de z0

Demonstration:

obvio

Holomorphic functions are conform

Let:

- $f : \mathcal{U} \to \mathbb{C}$
-

Then, holds:

- $f$holomorph in z „ $f'(z) \neq 0 \leftrightarrow f$ conform

Demonstration:

$\to$):

already seen

$\leftarrow$):

too hard

Convergence of complex series

Let:

· $\sum_{n \geq 0} c_n$ complex series

Then, holds:

· $\sum_{n \geq 0} c_n$ convergent $\leftrightarrow \sum_{n \geq 0} Rec_n$ convergent $\wedge \sum Imc_n n[0]$ convergent

Demonstration:

demonstration

Absolutely convergent are convergent

Let:

- $\sum c_n n[0]$ absolutely convergent

Then, holds:

- $\sum c_n n[0]$ convergent

Demonstration:

$S_k := \sum c_n n[0][k]$

$\forall\ m \in \mathbb{N}$ " $m < k$:

$|S_k - S_m| = |\sum c_n n[m+1][k]| \leq \sum |c_n| n[m+1][k]$

$\leq \sum |c_n| n[m+1] \xrightarrow{n} 0$

$|S_k - S_m| \xrightarrow{n} 0 \rightarrow (S_k)_k$ convergent $\rightarrow \sum c_n n[0]$ convergent

Series and norm

Let:

- $\sum c_n n[0]$ convergent

Then, holds:

- $|c_n| \xrightarrow{n} 0$

Demonstration:

$\sum c_n n[0]$ convergent $\leftrightarrow (S_n)_n$ convergent

$\rightarrow$ Cauchy $|S_n - S_m| \xrightarrow{n} 0$ por n y m $\rightarrow |S_n - S_{n-1}| \xrightarrow{n} 0$

$\rightarrow |c_n| \xrightarrow{n} 0$

Root test

Let:

$$\cdot \sum_{n \geq 0} c_n \text{ real series}$$

$$\cdot\ l \in \mathbb{R} \quad \text{″} \quad \overline{\lim_{k}}\, |c_k|^{\frac{1}{k}} = l$$

Then, holds:

$$\cdot\ l > 1 \to \sum_{n \geq 0} c_n \notin \mathbb{R}$$

$$\cdot\ l < 1 \to \sum_{n \geq 0} c_n \in \mathbb{R}$$

Demonstration:

$$demonstration$$

Quotient test

Let:

- $\sum_{n\geq 0} c_n$ real series
- $l \in \mathbb{R}$ "

Then, holds:

- $\exists\, l \in \mathbb{R}$:

$$\lim_{k} \frac{c_{k+1}}{c_k} = l$$

- $\overline{\lim_{c_k}} |c_k|^{\frac{1}{k}} k = l$

Power series theorem

Let:

- $\sum_{n\geq 0} a_n c^n$ power series

Then, holds:

- $|z-a| < R \to$ absolutely convergent
- $|z-a| > R \to$ divergent
- convergent in $D(a,R)$
- $\begin{array}{rcl} f : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \sum_{n\geq 0} c_n (z-a)^n \end{array} \in \mathcal{H}(D(a,R))$
- $\forall\ z \in \mathbb{C}$:

$$f'(z) = \sum_{n\geq 0} n c_n (z-a)^{n-1} \text{ convergent}$$

- convergence radius of f' = convergence radius of f

Demonstration:

$\forall\ z \in \mathbb{C}$ " $|z-a| < R$:

Root test over $\sum_{n\geq 0} |c_n||z-a|^n \lim_n (|c_n||z-a|^n)^{\frac{1}{n}} = |z-a| \lim_n |c_n|^{\frac{1}{n}} = \dfrac{|z-a|}{R} < 1$ Root test $\to$ absolutely convergent

$\forall\ z \in \mathbb{C}$ " $|z-a| < R$:

$\forall\ \rho \in \mathbb{R}$ " $|z-a| < \rho < R$:

$\frac{1}{\rho} < \frac{1}{R}$

$\lim_n |c_n|^{1/n} = \dfrac{1}{R} \to$ exists partial of $|c_n|^{1/n}$

$|c_n||z-a|^n > \frac{|z-a|^n}{\rho^n}$ no $\xrightarrow{n} 0$

General term test → divergent

Power series theorem

Let:

·

Then, holds:

$\cdot\, Part II, III, IV$

Demonstration:

Follow 3 steps

Step 1 : Uniform convergence in compacts of $D(a,R)$:

$$\begin{aligned} g_n : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto c_n(z-a)^n \end{aligned}$$

$\forall\ \rho \in \mathbb{R}$ „ $\rho < R$:

$\forall\ z \in \overline{D(a,\rho)}$:

$$|g_n(z)| = |c_n||z-a|^n \leq |c_n|\rho^n$$

$M_n := |c_n|\rho^n$

$\lim_n (|c_n|\rho^n)^{1/n} = \rho \lim_n |c_n|^{1/n} = \dfrac{\rho}{R} < 1$

Root test→ $\sum_{n\geq 0} M_n$ convergent

M-Weierstrass→ $\sum_{n\geq 0} c_n(z-a)^n$ uniformly convergent over compacts of $D(a,\rho)$

$\sum_{n\geq 0} c_n(z-a)^n$ uniformly convergent over compacts of $D(a,r)$

$$f(z) := \sum_{n\geq 0} g_n(z)$$

g uniformly convergent $\to f$ continuous

Step 2 : IV. R' :

$$\tilde{f}(z) := \sum_{n\geq 1} nc_n(z-a)^{n-1}$$

$$\tilde{f}(z) = \sum_{n\geq 0}(n+1)c_{n+1}(z-a)^n$$

$\frac{1}{R'} = \lim_n (n+1)|c_{n+1}|^{1/n} = \lim_n (n+1)^{1/n}|c_{n+1}|^{1/n} = \lim_n \left(|c_{n+1}|^{1/n+1}\right)^{\frac{n+1}{n}} =$

$$\frac{1}{R}$$

$$R' = R$$

Step 3 : III :

$\tilde{f}$well defined in $D(a,R)$

$\forall\ z_0 \in D(a,R)$:

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - \tilde{f}(z_0)\right| \xrightarrow{n} 0?$$

$\forall\ n \in \mathbb{N}$:

$$S_n(z) := \sum_{k=0}^{n} f_k(z)$$

$$R_n(z) := \sum_{k\geq n+1} f_k(z)$$

$$f(z) = S_n(z) + R_n(z)$$

$$\tilde{f}(z) = \tilde{S_n}(z) + \tilde{R_n}(z)$$

$$R_n(z), \tilde{R_n}(z) \xrightarrow{n} 0$$

$\forall\ \varepsilon \in \mathbb{R}^+$:

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \left|\frac{S_n(z)-S_n(z_0)}{z-z_0} + \frac{R_n(z)-R_n(z_0)}{z-z_0} - \tilde{S_n}(z_0) - \tilde{R_n}(z_0)\right|$$

$\forall\ \rho \in \mathbb{R}$ " $|z_0 - a| < \rho < R$:

$$\left|\tilde{R_n}(z_0)\right| \leq \sum_{k \geq n+1} k|c_k|\rho^{k-1} < \frac{\varepsilon}{3}(n \geq n_1)$$

$$\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| \leq \sum_{k \geq n+1} |c_k|\left|\frac{(z-a)^k - (z_0-a)^k}{z-z_0}\right|$$

$$\frac{a^k-b^k}{a-b} = a^{k-1} - a^{k-2}b + - - - + b^{k-1}$$

$$\leq \sum_{k \geq n+1} |c_k|(|z-a|^{k-1} + |z-a|^{k-2}|z_0-a| + - - - +$$

$$|z_0 - a|^{k-1}$$

$$|z-a|, |z_0| < \rho$$

$$\leq \sum_{k \geq n+1} |c_k|k\rho^{k-1} < \frac{\varepsilon}{3}(n \geq n_1)$$

$$\left|\frac{S_n(z)-S_n(z_0)}{z-z_0} - \tilde{S_n}(z_0)\right| < \frac{\varepsilon}{3}(S_n' = \tilde{S_n})(n \geq n_2)$$

$\forall\ n \in \mathbb{N}$ " $n \geq \max(n_1, n_2)$:

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \varepsilon$$

functions associated to series are $\mathcal{C}^{\infty}$

Let:

- $f(z) = \sum c_n (z-a)^n n[0]$ series
- R radius of convergence of f

Then, holds:

- $f \in \mathcal{C}^{\infty}$ over $D(a,R)$
- $\forall\ n \in \mathbb{N}$:

  $f^{n)} \in \mathcal{H}(D(a,R))$
- $c_k = \frac{f^{k)}}{k!}$
- series associated to f is unique

Demonstration:

$f^{k)}(z) = \sum n(n-1) - - - (n-k+1) c_n (z-a)^{n-k} n[k]$

$f^{k)}(a) = k! c_k$

$c_k = \frac{f^{k)}(a)}{k!}$