Complex Analysis

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Block I

Definitions

1.   The field of complex numbers

introduction

The field of complex numbers

Let:

$$
\begin{aligned}
\cdot\quad + : \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\
((a,b),(c,d)) &\longmapsto (a+c, b+d) \\
\cdot\quad \cdot : \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\
((a,b),(c,d)) &\longmapsto (ac - bd, ad + bd)
\end{aligned}
$$

We name the field of complex numbers to:

$$\cdot\ (\mathbb{R}^2, +, \cdot)$$

We denote:

$$\cdot\ (\mathbb{R}^2, +, \cdot)\ :\ \mathbb{C}$$

$$\cdot\ (0,1) \in \mathbb{C}\ :\ i$$

$$\cdot\ (a,b) \in \mathbb{C}\ :\ a + bi$$

$$\cdot\ \pi_1(\mathbb{C})\ :\ Re$$

$$\cdot\ \pi_2(\mathbb{C})\ :\ Im$$

Conjugation

We name complex conjugation to:

$$
\begin{aligned}
\cdot\quad f : \mathbb{C} &\longrightarrow \mathbb{C} \\
(a,b) &\longmapsto (a, -b)
\end{aligned}
$$

We denote:

$$\cdot\ f((a,b))\ :\ \overline{(a,b)}$$

Norm

We name complex norm to:

$$\cdot \quad \begin{array}{rcl} f : \mathbb{C} & \longrightarrow & \mathbb{R} \\ (a,b) & \longmapsto & \sqrt{a^2 + b^2} \end{array}$$

We denote:

$$\cdot \, f((a,b)) \, : \, |(a,b)|$$

Polar transformation

We name polar transformation to:

$$\cdot \quad \begin{array}{rcl} f : \mathbb{C} & \longrightarrow & \mathbb{R}^+ \times [0, 2\pi) \\ (a,b) & \longmapsto & (\sqrt{|(a,b)|}, \arctan(\frac{b}{a})) \end{array}$$

We denote:

$$\cdot \, f((a,b)) \, : \, (r, \theta)$$

Unit sphere projection

We name unit sphere projection to:

$$\cdot \quad \begin{array}{rcl} f : \mathbb{C} & \longrightarrow & S^1 \\ z & \longmapsto & \frac{z}{|z|} \end{array}$$

We denote:

$$\cdot \, f(z) \, : \, \pi(z)$$

Roots of unity

Let:

- $z \in \mathbb{C}$

Then, z is a root of unity if:

- $\exists \, n \in \mathbb{N} :$

$$z^n = 1$$

We denote:

- $\{ z \in \mathbb{C} \mid z \text{ root of unity } \} \, : \, S^1$

Disk

Let:

- $p \in \mathbb{C}$

- $r \in \mathbb{R}^+ \smallsetminus \{0\}$

We name Disk centered in p and radius r to:

- $\{ z \in \mathbb{C} \mid |z - p| < r \}$

We denote:

- $\{ z \in \mathbb{C} \mid |z - p| < r \} \, : \, D^1$

Component decomposition

Let:

$$\cdot\ f\ :\ \mathbb{C} \to \mathbb{C}$$

We name real component of f to:

$$\cdot\quad \begin{array}{rcl} f_{Re}\ :\mathbb{C} & \longrightarrow & \mathbb{R} \\ z & \longmapsto & Re(f(z)) \end{array}$$

We name imaginary component of f to:

$$\cdot\quad \begin{array}{rcl} f_{Im}\ :\mathbb{C} & \longrightarrow & \mathbb{R} \\ z & \longmapsto & Im(f(z)) \end{array}$$

We name component decomposition of f to:

$$\cdot\quad \begin{array}{rcl} f_{\mathbb{R}^2}\ :\mathbb{C} & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (f_{Re}(x+yi), f_{Im}(x+yi) \end{array}$$

$$\boxed{\textbf{2.} \quad \textbf{Holomorphic functions}}$$

Incremental quotient

Let:

      $\cdot \, \mathcal{U} \subset \mathbb{C}$ open

      $\cdot \, f \, : \mathcal{U} \to \mathbb{C}$

      $\cdot \, p \in \mathcal{U}$

We name incremental quotient of f in p to:

      $\cdot \, \lim\limits_{z \to p} \dfrac{f(z) - f(p)}{z - p}$

We denote:

      $\cdot \, f'(p)$

Holomorphic function

Let:

$\cdot\ \mathcal{U} \subset \mathbb{C}$ open

$\cdot\ f : \mathcal{U} \to \mathbb{C}$

$\cdot\ p \in \mathcal{U}$

Then, f is holomorphic over p if:

$\cdot\ \exists f'(p)$

Then, f is holomorphic over U if:

$\cdot\ \forall\ p \in \mathcal{U} :$

$\quad\quad \exists f'(p)$

We denote:

$\cdot\ \{f : \mathcal{U} \to \mathbb{C} \mid f \text{ holomorphic over } \mathcal{U}\}\ :\ \mathcal{H}(\mathcal{U})$

$\cdot\ f \in \mathcal{H}(\mathbb{C})\ :\ f$ entire

Cauchy-Riemman equations

Let:

$\quad \cdot\, u, v\,:\, \mathbb{R}^2 \to \mathbb{R}$

$\quad . \quad \begin{array}{ll} f\,:\mathbb{R}^2 & \longrightarrow \quad \mathbb{R}^2 \\ (x,y) & \longmapsto \quad (u((x,y)), v((x,y))) \end{array}$

Then, f is satisfies the Cauchy-Riemman equations if:

$\quad \cdot\, \exists\, u_x, u_y, v_x, v_y$

$\quad \cdot\, u_x = v_y$

$\quad \cdot\, u_y = -v_x$

We denote:

$\quad \cdot\, u_x + i v_x\,:\, f_x$

$\quad \cdot\, u_y + i v_y\,:\, f_y$

Conformal

Let:

· $\mathcal{U} \subset \mathbb{C}$ open

· $f : \mathcal{U} \to \mathbb{C}$

· $z \in \mathcal{U}$

Then, f is conformal in z if:

· $\exists\, c \in \mathbb{C} :$

$\forall\, I \subset \mathbb{R}$ „ $0 \in I :$

$\forall\, \gamma : I \to \mathbb{R}^2$ „ γ differentiable $\wedge\ \gamma(0) = z\ \wedge\ \gamma'(0) \neq$

$0 :$

$$\frac{(f \circ \gamma)'(0)}{\gamma'(0)} = c$$

Then, f is conformal if:

· $\forall\, z \in \mathcal{U} :$

f conformal in z

Power series

Let:

· $\sum_{n \geq 0} a_n f_n$ complex valued sequence

Then, $\sum_{n \geq 0} a_n f_n$ is a power series if:

· $\forall\, n \in \mathbb{N}$:

$$f_n : \begin{array}{rcl} \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & (z-a)^n \end{array}$$

Convergence radius

Let:

· $statements$

·

Then, $item$ is a/an entity if:

· $conditions$

·

We denote:

· $property$: $notation$

·

Absolutely convergent

Let:

- $\sum c_n n[0]$ series

Then, $\sum c_n n[0]$ is absolutely convergent if:

- $\sum |c_n| n[0]$ convergent

Numeric series

Let:

- $(c_k)_{k \in \mathbb{N}}$

Then, *item* is a/an entity if:

- *conditions*

-

We denote:

- *property* : *notation*

-

Block II

Propositions

1. The field of complex numbers

introduction

go

$$\boxed{\textbf{2. \quad Holomorphic functions}}$$

Cauchy-Riemman

Let:

- $f \ : \ \mathbb{R}^2 \to \mathbb{R}^2$ satisfies CR

Then, holds:

- $f_x = f'(z)$

- $f_y = -i f_y$

- $\partial_{f(z)} z = 0$

Demonstration:

$$f_x = u_x + i v_x = a + bi = f'(z)$$

$$f_y = i(a + ib) = i f'(z)$$

$$f'(z) = -i f_y$$

notation

 tangent venctor

Let:

$\cdot\, \gamma$ differentiable plane arc   " $\forall\, t \in I :$

$\gamma'(t) \neq 0$

Then, holds:

$\cdot\, \gamma'(t)$ tangent to γ

Demonstration:

$demonstration$

arc images

Let:

$\cdot\, f \in \mathcal{H}(\mathcal{U})$

$\cdot\, \gamma$ differentiable plane arc $\quad_{\shortparallel}\; \gamma \subset \mathcal{U}$

$\cdot\, \sigma = f(\gamma)$

$\cdot\, z_0 = \gamma(0)$

Then, holds:

$\cdot\, \sigma' = f'(\gamma)\gamma'$

$\cdot\, \gamma'(0) \neq 0 \rightarrow f'(z_0) \neq 0$

$\cdot\, \sigma'(0) = f'(z_0)\gamma'(0)$

$\cdot\, |\sigma'(0)| = |f'(z_0)||\gamma'(0)|$

$\cdot\, arg\sigma'(0) = arg\gamma'(0) + argf(z_0)$

$\cdot$ f aplica una homotecia mas una rotacion constante a todos los

vectores tangentes que salen de z0

Demonstration:

obvio

Holomorphic functions are conform

Let:

$\cdot\, f\ :\ \mathcal{U} \to \mathbb{C}$

.

Then, holds:

$\cdot\, f$ holomorph in z $_{\shortparallel}$ $f'(z) \neq 0 \leftrightarrow f$ conform

Demonstration:

$\to$):

already seen

$\leftarrow$):

too hard

Convergence of complex series

Let:

$\cdot \sum\limits_{n\geq 0} c_n$ complex series

Then, holds:

$\cdot \sum\limits_{n\geq 0} c_n$ convergent $\leftrightarrow \sum\limits_{n\geq 0} Rec_n$ convergent $\wedge \sum Imc_n n[0]$ con-

vergent

Demonstration:

$demonstration$

Absolutely convergent are convergent

Let:

$\cdot \sum c_n n[0]$ absolutely convergent

Then, holds:

$\cdot \sum c_n n[0]$ convergent

Demonstration:

$S_k := \sum c_n n[0][k]$

$\forall\, m \in \mathbb{N} \text{ , } m < k :$

$$|S_k - S_m| = |\sum c_n n[m+1][k]| \le \sum |c_n| n[m+1][k]$$

$$\le \sum |c_n| n[m+1] \xrightarrow{n} 0$$

$$|S_k - S_m| \xrightarrow{n} 0 \to (S_k)_k \text{ convergent} \to \sum c_n n[0] \text{ conver-}$$

gent

Series and norm

Let:

$\cdot \sum c_n n[0]$ convergent

Then, holds:

$\cdot |c_n| \xrightarrow{n} 0$

Demonstration:

$\sum c_n n[0]$ convergent $\leftrightarrow (S_n)_n$ convergent

$\rightarrow$ Cauchy $|S_n - S_m| \xrightarrow{n} 0$ por n y m $\rightarrow |S_n - S_{n-1}| \xrightarrow{n} 0$

$\rightarrow |c_n| \xrightarrow{n} 0$

Root test

Let:

$\cdot \sum\limits_{n \geq 0} c_n$ real series

$\cdot l \in \mathbb{R} \quad {}_{\shortparallel} \quad \overline{\lim\limits_{k}} \, |c_k|^{\frac{1}{k}} = l$

Then, holds:

$\cdot l > 1 \to \sum\limits_{n \geq 0} c_n \notin \mathbb{R}$

$\cdot l < 1 \to \sum\limits_{n \geq 0} c_n \in \mathbb{R}$

Demonstration:

$demonstration$

Quotient test

Let:

> $\cdot \sum\limits_{n \geq 0} c_n$ real series
>
> $\cdot \, l \in \mathbb{R}$ "

Then, holds:

> $\cdot \; \exists \, l \in \mathbb{R} :$
>
> $$\lim_{k} \frac{c_{k+1}}{c_k} = l$$
>
> $\cdot \overline{\lim\limits_{c_k}} \, |c_k|^{\frac{1}{k}} \, k = l$

Power series theorem

Let:

$\cdot \sum\limits_{n \geq 0} a_n c^n$ power series

Then, holds:

$\cdot |z - a| < R \rightarrow$ absolutely convergent

$\cdot |z - a| > R \rightarrow$ divergent

$\cdot$ convergent in $D(a, R)$

$\cdot \begin{array}{rcl} f & : \mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \sum\limits_{n \geq 0} c_n (z - a)^n \end{array} \in \mathcal{H}(D(a, R))$

$\cdot \forall z \in \mathbb{C} :$

$$f'(z) = \sum\limits_{n \geq 0} n c_n (z - a)^{n-1} \text{ convergent}$$

$\cdot$ convergence radius of f' = convergence radius of f

Demonstration:

$\forall z \in \mathbb{C} \text{ "} |z - a| < R :$

Root test over $\sum\limits_{n \geq 0} |c_n| |z - a|^n rlimit (|c_n||z-a|^n)^{\frac{1}{n}} n = |z - a| rlimit |c_n|^{\frac{1}{n}} n = \dfrac{|z - a|}{R} < 1$ Root test $\rightarrow$ absolutely convergent

$\forall z \in \mathbb{C} \text{ "} |z - a| < R :$

$\forall \rho \in \mathbb{R} \text{ "} |z - a| < \rho < R :$

$\frac{1}{\rho} < \frac{1}{R}$

$rlimit |c_n|^{1/n} n = \frac{1}{R} 1$ exists partial of $|c_n|^{1/n}$

$|c_n| |z - a|^n > \dfrac{|z-a|^n}{\rho^n}$ no $\xrightarrow{n} 0$

General term test → divergent

Power series theorem

Let:

    .

Then, holds:

    $\cdot\, Part\, II, III, IV$

Demonstration:

    Follow 3 steps

Step 1 :   Uniform convergence in compacts of $D(a, R)$:

$$g_n : \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longmapsto c_n(z-a)^n$$

$$\forall\, \rho \in \mathbb{R} \;_{\shortparallel}\; \rho < R :$$

$$\forall\, z \in \overline{D(a, \rho)} :$$

$$|g_n(z)| = |c_n||z-a|^n \le |c_n|\rho^n$$

$$M_n := |c_n|\rho^n$$

$$rlimit(|c_n|\rho^n)^{1/n}n = \rho\, rlimit|c_n|^{1/n}n = \frac{\rho}{R} < 1$$

Root test$\to \sum_{n\ge 0} M_n$ convergent

M-Weierstrass$\to \sum_{n\ge 0} c_n(z-a)^n$ uniformly convergent

over compacts of $D(a, \rho)$

$$\sum_{n\ge 0} c_n(z-a)^n \text{ uniformly convergent over compacts of } D(a, r)$$

$$f(z) := \sum_{n \geq 0} g_n(z)$$

g uniformly convergent $\rightarrow f$ continuous

Step 2 : IV. R' :

$$\tilde{f}(z) := \sum_{n \geq 1} n c_n (z-a)^{n-1}$$

$$\tilde{f}(z) = \sum_{n \geq 0} (n+1) c_{n+1} (z-a)^n$$

$$\frac{1}{R'} = rlimit(n+1)|c_{n+1}|^{1/n} n = rlimit(n+1)^{1/n}|c_{n+1}|^{1/n} n =$$

$$rlimit(|c_{n+1}|^{1/n+1})^{\frac{n+1}{n}} n = \frac{1}{R}$$

$$R' = R$$

Step 3 : III :

$\tilde{f}$ well defined in $D(a, R)$

$\forall z_0 \in D(a, R)$:

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - \tilde{f}(z_0) \right| \xrightarrow{n} 0?$$

$\forall n \in \mathbb{N}$:

$$S_n(z) := \sum_{k=0}^{n} f_k(z)$$

$$R_n(z) := \sum_{k \geq n+1} f_k(z)$$

$$f(z) = S_n(z) + R_n(z)$$

$$\tilde{f}(z) = \tilde{S}_n(z) + \tilde{R}_n(z)$$

$$R_n(z), \tilde{R}_n(z) \xrightarrow{n} 0$$

$\forall\, \varepsilon \in \mathbb{R}^+ :$

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \left|\frac{S_n(z)-S_n(z_0)}{z-z_0} + \frac{R_n(z)-R_n(z_0)}{z-z_0} - \tilde{S}_n(z_0) - \tilde{R}_n(z_0)\right|$$

$\forall\, \rho \in \mathbb{R}\ _{\shortmid\shortmid}\ |z_0 - a| < \rho < R :$

$$\left|\tilde{R}_n(z_0)\right| \leq \sum_{k \geq n+1} k|c_k|\rho^{k-1} < \frac{\varepsilon}{3}(n \geq n_1)$$

$$\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| \leq \sum_{k \geq n+1} |c_k|\frac{(z-a)^k - (z_0-a)^k}{z-z_0}|$$

$$\frac{a^k - b^k}{a-b} = a^{k-1} - a^{k-2}b + - - - + b^{k-1}$$

$$\leq \sum_{k \geq n+1} |c_k|(|z-a|^{k-1} + |z-a|^{k-2}|z_0-a| + - - - +$$

$|z_0 - a|^{k-1}$

$$|z-a|, |z_0| < \rho$$

$$\leq \sum_{k \geq n+1} |c_k|k\rho^{k-1} < \frac{\varepsilon}{3}(n \geq n_1)$$

$$\left|\frac{S_n(z)-S_n(z_0)}{z-z_0} - \tilde{S}_n(z_0)\right| < \frac{\varepsilon}{3}(S_n' = \tilde{S}_n)(n \geq n_2)$$

$\forall\, n \in \mathbb{N}\ _{\shortmid\shortmid}\ n \geq \max(n_1, n_2) :$

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \varepsilon$$

functions associated to series are $\mathcal{C}^\infty$

Let:

- $f(z) = \sum c_n(z-a)^n n[0]$ series

- R radius of convergence of f

Then, holds:

- $f \in \mathcal{C}^\infty$ over $D(a, R)$

- $\forall\, n \in \mathbb{N}$:

$$f^{n)} \in \mathcal{H}(D(a, R))$$

- $c_k = \frac{f^{k)}}{k!}$

- series associated to f is unique

Demonstration:

$$f^{k)}(z) = \sum n(n-1) --- (n-k+1) c_n(z-a)^{n-k} n[k]$$

$$f^{k)}(a) = k! c_k$$

$$c_k = \frac{f^{k)}(a)}{k!}$$

Block III

Examples

1.   Holomorphic functions

introduction

go

$$\boxed{\textbf{2.   Holomorphic functions}}$$

Conjugation

Let:

$$. \quad \begin{array}{ccc} \bar{a} \; : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \bar{z} \end{array}$$

Then, $\bar{a}$ is not holomorphic :

$$u_x = 1$$

$$u_y = 0$$

$$v_x = 0$$

$$v_y = -1$$

$$\forall \; z \in \mathbb{C} :$$

$$-1 \neq 1 \to f \text{ not holomorphic in } z$$

Quadratic norm

Let:

. $$f : \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longmapsto |z|^2$$

· $f_{\mathbb{R}^2}$ component decomposition of f

Then, f is holomorphic in 0:

f differentiable in $\mathbb{R}^2$ polinomial

$\forall\, z \in \mathbb{C}$:

$$u_x(x,y) = 2x$$

$$u_y(x,y) = 2y$$

$$v_x(x,y) = 0$$

$$v_y(x,y) = 0$$

$$u_x = v_y \leftrightarrow x = 0$$

$$u_y = -v_x \leftrightarrow y = 0$$

f holomorphic function in $z \leftrightarrow z = 0$

Non preserving angles function

Let:

$\cdot\, f(z) = z^2$

Then, f is conform in $\mathbb{R} \smallsetminus \{0\}$:

$$f(\{(x,0) \in \mathbb{C} \mid x > 0\}) = \{(x,0) \in \mathbb{C} \mid x > 0\}$$

$$f(\{(x,0) \in \mathbb{C} \mid x < 0\}) = \{(x,0) \in \mathbb{C} \mid x > 0\}$$

$$ang(A,B) = \pi \neq 0 = ang(f(A), f(B))$$

Exponential

Let:

$\cdot\, a\,:\,0$

$\cdot\, c_n\,:\,\frac{1}{n}$

Then, $\sum_{n \geq 0} c_n (z-a)^n$ is convergent in D1 :

$$\lim_n \frac{|c_n|}{|c_{n+1}|} = \lim_n \frac{n+1}{n} = 1 \to R = 1$$

CH$\to D(0,1)$ convergent

$\mathbb{C} \smallsetminus D(0,1)$ divergent

$$f' = f$$

Geometric series

Let:

$\cdot\, a\,:\,0$

$\cdot\, c_n : 0$

Then, $\sum z^n n[0]$ is convergent in $\mathbb{D}$:

$R = \frac{c_n}{c_{n+1}} = 1$ Then, holds:

$\cdot\ \sum z^n n[0] = \frac{1}{1-z}$

$\cdot\ \sum n z^{n-1} n[0] = \frac{1}{(1-z)^2}$

$\cdot\ \sum \frac{z^{n+1}}{n+1} n[0] = -\log(1-z)$

Demonstration:

$\forall\ z \in \mathbb{D}\ :$

$\sum z^n n[0]$ geometric series

$\sum z^n n[0] = \frac{1}{1-z}$

II differentiating

III integrating

Series not centered in 0

Let:

$\cdot\ a\ :\ i$

$\cdot\ c_n\ :\ \frac{n+1}{5^{n+1}}$

Then, *item* is a/an entity :

$\sum \frac{n(z-i)^{n-1}}{5^n} n[1]$

$= \frac{1}{5} \sum n \frac{z-i}{5}^{n-1} n[1] = \frac{1}{5} \sum n u^{n-1} n[1]$

$S(u) = \tilde{S}'(u)$

$\tilde{S}(u) = \frac{1}{5} \sum u^n n[1] = \frac{u}{5(1-u)}$

$S(u) = \frac{1}{5(1-u)^2}$

$S(z) = \frac{5}{(5+i-z)^2}$ over $D(i,5)$

Radius of convergence without quotient test

Let:

$\cdot\ \sum \frac{(-1)^n}{n(n+1)} (z-2)^{n(n+1)} n[1]$

Then, R is a/an entity :

$\lim_{c_{n+1}} c_n \nexists$

$\lim_n \frac{1}{n(n+1)}^{\frac{1}{n(n+1)}} = 1$

  ignore zeros

Block IV

Problems

PROBLEMES D'ANÀLISI COMPLEXA
2n quadrimestre del curs 2013-2014.

Llista 1: Els nombres complexos

B.1. Expresseu en la forma $a + ib$ els següents nombres:

    (a) $(2 + 3i)(4 + i)$ (c) $\frac{1}{4+i}$ (e) $\sqrt{i}$ (g) $\sqrt{9i}$

    (b) $(4 + 2i)^2$ (d) $\frac{i}{4+i}$ (f) $\sqrt{-i}$ (h) $\sqrt{1+i}$

B.2. Si $z = x + iy$ trobeu les parts real i imaginària de les expressions següents:

    (a) z^2 (b) $z(z + 1)$ (c) $\frac{1}{z}$ (d) $\frac{1}{z-3}$.

B.3. És cert que

    a) $\operatorname{Re}(z + w) = \operatorname{Re} z + \operatorname{Re} w$?

    b) $\operatorname{Re}(zw) = (\operatorname{Re} z)(\operatorname{Re} w)$?

    c) $\operatorname{Re}(\frac{z}{w}) = \frac{\operatorname{Re} z}{\operatorname{Re} w}$?

B.4. Trobeu la forma polar dels nombres següents i dibuixeu-los.

    (a) $3(1 + \sqrt{3}\,i)$ (b) $2\sqrt{3} - 2i$ (c) $-2 + 2i$ (d) $-1 - i$

B.5. Sigui $(x + iy)/(x - iy) = a + ib$. Proveu que $a^2 + b^2 = 1$.

B.6. Proveu que si $p(z)$ és un polinomi amb coeficients reals i z és un zero de p llavors $\bar{z}$ també ho és.

B.7. Descriviu els conjunts del pla que satisfan (recordeu que $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.)

    (a) $\operatorname{Im} \frac{z-a}{z} = 0, a \in \mathbb{C}^*$ (b) $|z| = \operatorname{Re} z + 1$ (c) $|z - 2| > |z - 3|$

SOL. B.1. a) $5 + 14i$; b)$12 + 16i$; c) $4/17 - i/17$; d)$1/17 + 4i/17$; e) $\pm\sqrt{2}/2(1 + i)$; f)$\pm\sqrt{2}/2(1 - i)$; g) $\pm 3\sqrt{2}/2(1 + i)$;
h) $\pm 2^{1/4}(\cos(\pi/8) + i\sin(\pi/8))$.

    B.2 a) $x^2 - y^2 + 2ixy$; b) $x^2 - y^2 + x + i(y + 2xy)$; c) $(x - iy)/(x^2 + y^2)$; d) $(x - 3 - iy)/((x - 3)^2 + y^2)$.

    B.3 a) si. b) no. c) no.

    B.4 a) $6(\cos(\pi/3) + i\sin(\pi/3))$; b) $4(\cos(\pi/6) - i\sin(\pi/6))$; c) $2\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$; d) $\sqrt{2}(\cos(3\pi/4) -i\sin(3\pi/4))$.

    B.6 Conjugueu tot el polinomi.

    B.7 a) Recta que passa per 0 i a; b) Paràbola horitzontal $x = (1/2)(y^2 - 1)$; c) $\{\operatorname{Re} z > 3/2\}$.

1. Expresseu en la forma $a + ib$ els següents nombres:

    (a) $\frac{1}{i}$ (c) $\frac{1}{2+i} + \frac{1}{2-i}$ (e) $\left(\frac{2+i}{3-2i}\right)^2$ (g) $\sqrt[4]{-i}$

    (b) $\frac{1+i}{1-i}$ (d) $\frac{1}{2+i} + \frac{4-2i}{3+i}$ (f) $(1+i)^{100} + (1-i)^{100}$ (h) $(3 + 4i)^{\frac{1}{2}}$

2. Si $z = x + iy$ on $x, y \in \mathbb{R}$, trobeu les parts real i imaginària de:

<div align="center">1</div>

Llista 2: Funcions de variable complexa i equacions de Cauchy-Riemann

B.1. Trobeu els punts on la funció f és derivable (en el sentit complex), en els següents casos, i calcula'n la derivada.

(a) $\cos |z|^2$

(c) e^{iz}

(e) $\frac{1}{(z-1)^2(z^2+2)}$

(b) $|z|^4$

(d) $z + \frac{1}{z}$

(f) $\frac{1}{(z+\frac{1}{z})^2}$

Solució: (a) $\emptyset$; (b) $\emptyset$; (c) $\mathbb{C}$; $f'(z) = ie^{iz}$;(d) $\mathbb{C} \setminus \{0\}$; $f'(z) = 1 - |\frac{1}{z^2}|$; (e) $\mathbb{C} \setminus \{1, \pm\sqrt{2}i\}$; (f) $\mathbb{C}$.

B.2. Determineu si aquestes funcions poden ser la part real d'una funció holomorfa, i en cas que ho siguin calculeu la part imaginària.

(a) $e^x \cos y$

(b) $x^3 + 6xy^2$

(c) $\log(x^2 + y^2)$

Solució: (a) $e^x \sin y$; $f(z) = e^z$; (b) No ho és; (c) $2 \arctan(y/x)$; ($f(z) = \log(z^2)$).

B.3. Sigui f una funció holomorfa en un obert $\Omega \subset \mathbb{C}$ i $z_0 \in \Omega$ tal que $f'(z_0) \neq 0$. Quin angle formen les corbes $\operatorname{Re} f(z) = \operatorname{Re} f(z_0)$ i $\operatorname{Im} f(z) = \operatorname{Im} f(z_0)$ en un punt z_0 ?
Solució: $\pi/2$.

1. Trobeu els punts on la funció f és derivable (en el sentit complex), en els següents casos:

(a) $f(z) = |z|$

(d) $f(z) = z + z\bar{z}$

(b) $\cosh x \cos y + i \sinh x \sin y$

(c) $f(z) = \operatorname{Re} z$

(e) $f(z) = \operatorname{Im} e^{\bar{z}} + i \operatorname{Re} e^z$

2. Sigui $\Omega \subset \mathbb{C}$ un obert, $z_0 \in \Omega$ i $f : \Omega \to \mathbb{C}$ una funció.

a) Identificant $\mathbb{R}^2$ amb $\mathbb{C}$ de la forma habitual, demostreu que si f és diferenciable en z_0, llavors

$$Df(z_0)(z) = \frac{\partial f}{\partial z}(z_0) \cdot z + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot \bar{z} \qquad (z \in \mathbb{C}),$$

on

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \ \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

b) Proveu que f és holomorfa en Ω si, i només si, f és diferenciable i $\frac{\partial f}{\partial \bar{z}} = 0$ en Ω.
En tal cas, $f' = \frac{\partial f}{\partial z}$.

3. Demostreu que si f és diferenciable en un obert de $\mathbb{C}$, llavors

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}} \quad i \quad \overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}.$$

1. The field of complex numbers

introduction

entity

Let:

· *statements*

·

Then, *item* is a/an entity if:

· *conditions*

·

We denote:

· *property* : *notation*

·

2. Holomorphic functions

3. Cauchy-Riemman

Let:

$\cdot\, f \in \mathcal{H}(\mathbb{C})$ „ $Re f + Im f = c_a$

Show that:

$\cdot\ \exists\, a' \in \mathbb{C}:$

$$f = c_{a'}$$

Demonstration:

u, v real components of f

$u(x, y) + v(x, y) = a$

differentiate respect x and y

$u_x + v_x = 0$

$u_y + v_y = 0$

f holomorphic $\rightarrow f$ CR

$u_x - u_y = 0$

$u_y + u_x = 0$

$u_x, u_y, v_x, v_y = 0$

$\exists\, a_1 \in \mathbb{R}:$

$$u = c_{a_1}$$

$\exists\, a_2 \in \mathbb{R}:$

$$v = c_{a_2}$$

$$f = c_{(a_1, a_2)}$$

B.2 a)

Let:

$$. \quad \begin{array}{rcl} u : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & \exp(y)\cos(x) \end{array}$$

Show that:

· $\exists\, f \in \mathcal{H}(\mathbb{C}) :$

u real component of f

Demonstration:

lab 1 $\to u_{xx} + v_{yy} = 0$

$u_x = \exp(x)\cos(y)$

$u_{xx} = \exp(x)\cos(y)$

$u_y = -\exp(x)\sin(y)$

$u_{yy} = -\exp(x)\cos(y)$

ok

Calculate v using CR

$v_y = u_x = \exp(x)\cos(y)$

$v(x, y) = \int_{\mathbb{C}} \exp(x)\sin(y)dy = \exp(x)\sin(y) + \phi(x)$

$v_x = \partial_v x = \exp(x)\sin(y) + \phi'(x)$

$-u_y = \exp(x)\sin(y) + \phi'(x)$

CR $\to \phi'(x) = 0$

$\forall\, c \in \mathbb{R} :$

$$\phi(x) = c \text{ ok} \qquad 54$$

$$v(x, y) = \exp(x)\sin(y)$$

Preservation of angles

Let:

$\cdot \, \gamma_1, \gamma_2$ plane arcs   „ $\gamma_1(0) = \gamma_2(0)$

Then, holds:

$\cdot$ angle of $\gamma_1'(0)$ and $\gamma_2'(0)$ = angle $\sigma_1'(0), \sigma_2'(0)$

Demonstration:

rotations and homotecies let angles invariant

Block V

Tasks

$$\boxed{\textbf{1. \quad 1st laboratory}}$$

Existence of holomorphic functions

Let:

- $f \in \mathcal{H}(\mathbb{D})$

Study:

- $\exists\, f \in \mathcal{H}(\mathbb{D})$:

  $\forall\, n \in \mathbb{N}$ „ $n \geq 2$:

  a) $f(\pm\frac{1}{n}) = \frac{1}{2n+1}$

  b) $f(\pm\frac{1}{n}) = \frac{1}{n^2}$

  c) $|f(\frac{1}{n})| = \frac{1}{\log(n+1)}$

  d) $|f(\frac{1}{n})| = \frac{n}{n+1}$

Demonstration:

a):

$$E_1 := \{+\tfrac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N}\}$$

$$E_2 := \{-\tfrac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N}\}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} = \lim_{n} \frac{f(\frac{1}{n})}{\frac{1}{n}} - \frac{f(0)}{\frac{1}{n}} = \frac{1}{2} - \lim_{n} \frac{f(0)}{\frac{1}{n}}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} \begin{cases} = \frac{1}{2} & f(0) = 0 \\ \notin \mathbb{C} & f(0) \neq 0 \end{cases}$$

Case $f(0) = 0$:

$$\lim_{E_2} \frac{f(z) - f(0)}{z - 0} = \lim_{n} \frac{f(-\frac{1}{n})}{-\frac{1}{n}} = -\frac{1}{2} \neq \lim_{E_1} \frac{f(z) - f(0)}{z - 0}$$

$$\nexists \, f \in \mathcal{H}(0) \,\text{\tiny\rm ,\!,}\, f \text{ satisfies } a)$$

In particular:

$$\nexists \, f \in \mathcal{H}(\mathbb{D}) \,\text{\tiny\rm ,\!,}\, f \text{ satisfies } a)$$

$b)$:

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z^2 \end{aligned}$$

$\forall \, n \in \mathbb{N} \,\text{\tiny\rm ,\!,}\, n \geq 2 :$

$$f\left(\pm \tfrac{1}{n}\right) = \tfrac{1}{n^2}$$

f satisfies $b)$

$$\begin{aligned} \bar{f} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (u(x,y), v(x,y)) = (x^2 - y^2, 2xy) \end{aligned}$$

$\bar{f} \in \mathrm{Pol}(\mathbb{R}^2) \to \bar{f}$ differentiable in $\mathbb{R}^2$

$\forall \, (x,y) \in \mathbb{R}^2 :$

$$\partial_x u(x,y) = 2x = \partial_y v(x,y)$$

$$\partial_y u(x,y) = -2y = -\partial_x v(x,y)$$

f satisfies CR

$\therefore) \; f \in \mathcal{H}(\mathbb{R}^2)$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

$c)$:

Suppose $\exists \, f \in \mathcal{H}(\mathbb{D}) \,\text{\tiny\rm ,\!,}\, f$ satisfies $c)$

$$f \in \mathcal{C}^0(\mathbb{D}) \to f(0) = f\left(\lim_n \frac{1}{n}\right) = \lim_n \, f\left(\frac{1}{n}\right) = 0$$

$$\left| \lim_{E_1} \frac{f(z) - f(0)}{z - 0} \right| = \lim_n \frac{\left| f\left(\frac{1}{n}\right) \right|}{\frac{1}{n}} \notin \mathbb{C}$$

$f \notin \mathcal{H}(0)$ absurd

d):

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \frac{1}{z+1} \end{aligned}$$

$\forall\, n \in \mathbb{N}\ _{_\shortmid\shortmid}\ n \geq 2$:

$$\left| f\left(\tfrac{1}{n}\right) \right| = \frac{1}{\frac{1}{n}+1} = \frac{n}{n+1}$$

f satisfies d)

$$\begin{aligned} \bar{f} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (u(x,y), v(x,y)) = \left(\tfrac{x+1}{(x+1)^2+y^2}, \tfrac{-y}{(x+1)^2-y^2} \right) \end{aligned}$$

$\bar{f} \in \mathrm{Rat}(\mathbb{R}^2) \wedge \forall\, (x,y) \in \mathbb{R}^2$:

$$(x+1)^2 + y^2 \neq 0$$

$\bar{f}$ differentiable in $\mathbb{R}^2$

$\forall\, (x,y) \in \mathbb{R}^2$:

$$\partial_x u(x,y) = \tfrac{y^2-(x+1)^2}{((x+1)^2+y^2)^2} = \partial_y v(x,y)$$

$$\partial_y u(x,y) = \tfrac{-2y(x+1)}{((x+1)^2+y^2)^2} = -\partial_x v(x,y)$$

f satisfies CR

$\therefore)\ f \in \mathcal{H}(\mathbb{R}^2)$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

Constant tests

Let:

· $\Omega \subset \mathbb{C}$ region

· $f \in \mathcal{H}(\Omega)$

Then, holds:

· $f_{Re} = 0 \;\vee\; f_{Im} = 0 \to f \in \mathrm{Cst}(\Omega)$

· $|f| \in \mathrm{Cst}(\Omega) \to f \in \mathrm{Cst}(\Omega)$

· $\mathrm{Im} f$ circumference $\to f \in \mathrm{Cst}$

Demonstration:

$f_{Re} = 0 \;\vee\; f_{Im} = 0$:

$u := f_{Re}$

$v := f_{Im}$

$f \in \mathcal{H}(\Omega) \to f$ satisfies CR in Ω

$\partial_x u = \partial_y v = 0$

$\partial_y u = -\partial_x v = 0$

Null diferential test:

Ω connex $\to u, v \in \mathrm{Cst}$

$u, v \in \mathrm{Cst} \to f \in \mathrm{Cst}$

$|f| \in \mathrm{Cst}(\Omega)$:

$$\begin{array}{rcl} |f| : \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & \sqrt{u(x,y)^2 + v(x,y)^2} \end{array}$$

$|f| \in \text{Cst} \to \exists\, a \in \mathbb{R} :$

$$\sqrt{u(x,y)^2 + v(x,y)^2} = a$$

$$u(x,y)^2 + v(x,y)^2 = a^2$$

$$2\partial_x u(x,y) + 2\partial_x v(x,y) = 0$$

$$2\partial_y u(x,y) + 2\partial_y v(x,y) = 0$$

$$f \in \mathcal{H}(\Omega) \to f \text{ satisfies CR in } \Omega$$

$$2\partial_y v(x,y) + 2\partial_x v(x,y) = 0$$

$$-2\partial_x v(x,y) + 2\partial_y v(x,y) = 0$$

$$+ : \ 4\partial_y v(x,y) = 0 \to \partial_y v(x,y) = 0$$

$$- : \ 4\partial_x v(x,y) = 0 \to \partial_x v(x,y) = 0$$

Null differential test:

$$\Omega \text{ connex} \to u, v \in \text{Cst}$$

$$u, v \in \text{Cst} \to f \in \text{Cst}$$

$\text{Im}(f)$ circumference :

$\exists\, (x_0, y_0) \in \mathbb{R}^2, r \in \mathbb{R}^+ :$

$$\text{Im}(f) = C_r(x_0, y_0)$$

$$\begin{array}{rcl} \bar{f} : \mathbb{R}^2 & \longrightarrow & C_r(x_0, y_0) \\ (x,y) & \longmapsto & (r\cos(x - x_0), r\sin(y - y_0)) \end{array}$$

$\forall\, (x,y) \in \Omega :$

$$\left|\bar{f}\right|(x,y) = \sqrt{r^2(\cos^2(x - x_0) + \sin^2(y - y_0))} = r$$

$$|f| \in \text{Cst} \to f \text{ Cst}$$

Real part of holomorphic functions

Let:

$\quad \cdot\, \Omega \subset \mathbb{R}^2$ region

$\quad \cdot\, u \in \mathcal{C}^2(\Omega)$ „ $\quad \exists\, f \in \mathcal{H}(\Omega) :$

$\qquad f_{Re} = u$

Show that:

$\quad \cdot\, \partial_{xx}u + \partial_{yy}u = 0$

Study:

$\quad \cdot\, \exists\, f \in \mathcal{H}(\Omega) :$

$\qquad a)\ f_{Re}(x,y) = x^2 + y^2$

$\qquad b)\ f_{Re}(x,y) = x(x+1) - y^2$

$\qquad c)\ \forall\, \alpha \in \mathbb{R} :$

$\qquad\qquad f_{Re} = y^3 + \alpha x^2 y \ \wedge\ \Omega = \mathbb{C}$

Demonstration:

$\quad \partial_{xx}u + \partial_{yy}u = 0:$

$\qquad u := f_{Re}, v := f_{Im}$

$\qquad f \in \mathcal{H}(\Omega) \rightarrow f$ satisfies CR in Ω

$\qquad \partial_x u = \partial_y v \rightarrow \partial_{xx}u = \partial_{xy}v$

$\qquad \partial_y u = -\partial_x v \rightarrow \partial_{yy}u = -\partial_{xy}v$

$\qquad \therefore)\ \partial_{xx}u + \partial_{yy}u = 0$

$f_{Re}(x,y) = x^2 + y^2$:

$$\partial_{xx}u + \partial_{yy}u = 4 \neq 0$$

$$\nexists f \in \mathcal{H}(\Omega) \; _{\shortparallel} \; f_{Re}(x,y) = x^2 + y^2$$

$f_{Re}(x,y) = x(x+1) - y^2$:

$$\begin{aligned} \bar{f} : \Omega &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (u(x,y),v(x,y)) = (x(x+1) - y^2, 2xy + y) \end{aligned}$$

$\bar{f} \in \text{Pol} \rightarrow \bar{f}$ differentiable in Ω

$\forall \, (x,y) \in \Omega :$

$$\partial_x u(x,y) = 2x + 1 = \partial_y v(x,y)$$

$$\partial_y u(x,y) = -2y = -\partial_x v(x,y)$$

f satisfies CR in Ω

$f \in \mathcal{H}(\Omega) \; \wedge \; f_{Re} = u$

$f_{Re}(x,y) = y^3 + \alpha x^2 y$:

f has to satisfy CR in $\mathbb{C}$:

$\forall \, (x,y) \in \mathbb{R}^2 :$

$$\partial_y v(x,y) = \partial_x u(x,y) = 2\alpha xy$$

$$\partial_x v(x,y) = -\partial_y u(x,y) = -3y^2 - \alpha x^2$$

$$v(x,y) = \alpha xy^2 + c(x)$$

$$v(x,y) = -3xy^2 - \frac{\alpha}{3}x^3 + c(y)$$

$$\alpha = -3, \; c(x) = x^3, \; c(y) = 0$$

$$v(x,y) = -3xy^2 + x^3$$

$$\boxed{\textbf{2. \quad 2nd laboratory}}$$

Power series

Study:

$$\cdot \sum_{n \geq 1} n(n+1)z^n$$

Demonstration:

Naming:

R radius of convergence of the series

$\forall\, n \in \mathbb{N}:$

$$c_n := n(n+1)$$

Convergence domain:

$$\lim_n \frac{c_n}{c_{n+1}} = \lim_n \frac{n(n+1)}{(n+1)(n+1)} = 1$$

Quotient test:

$$R^{-1} = \overline{\lim_n} \, |c_n|^{\frac{1}{n}} = 1 \to R = 1$$

Cauchy-Hadamard theorem:

$\displaystyle\sum_{n \geq 1} n(n+1)z^n$ convergent over $\mathbb{D}$

$\displaystyle\sum_{n \geq 1} n(n+1)z^n$ divergent over $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$

$\forall\, K \subset \mathbb{D}$ $_{\shortparallel}$ K compact :

$$\sum_{n \geq 1} n(n+1)z^n \text{ uniformly convergent over } K$$

Define

$$
\begin{aligned}
f : \mathbb{D} &\longrightarrow \mathbb{C} \\
z &\longmapsto z \sum_{n \geq 1} n(n+1)z^{n-1}
\end{aligned}
$$

Sum:

$$g \; : \mathbb{D} \; \longrightarrow \; \mathbb{C}$$
$$z \; \longmapsto \; \sum_{n \geq 1} n(n+1)z^{n-1}$$

UCI theorem:

$$\int_0^z g(t)dt = \sum_{n \geq 1}(n+1)z^n$$

$$h \; : \mathbb{D} \; \longrightarrow \; \mathbb{C}$$
$$z \; \longmapsto \; \sum_{n \geq 1}(n+1)z^n$$

$$\int_0^z h(t)dt = \sum_{n \geq 1} z^{n+1} = \sum_{n \geq 0} z^n = \frac{1}{1-z}$$

$$h(z) = \partial_z \frac{1}{1-z} = \frac{1}{(1-z)^2}$$

$$g(z) = \partial_z h(z) = \frac{2}{(1-z)^3}$$

$$f(z) = \frac{2z}{(1-z)^3}$$

Application:

In particular:

$$\sum_{n \geq 1}(-1)^n \frac{n(n+1)}{2^n} = f\left(-\frac{1}{2}\right) = \frac{-2^3}{3^3}$$

Bibliography

[1] Joaquim Ortega-Cerda. Funcions analtiques, February 2005.