1. New

Let:

· $|\lambda| < 1 < |\mu|$

Then, holds:

· $A^n = \lambda^n, \mu^n$

· $\forall\ (x,y) \in \mathbb{R}^2$ 〃 $y \neq 0$:

$$o((x,y)) \xrightarrow{n} (0,\infty)$$

$\forall\ (x,y) \in \mathbb{R}^2$ 〃 $y = 0$:

$$o((x,y)) \xrightarrow{n} (0,0)$$

Demonstration:

$$x_n = \lambda^n x \xrightarrow{n} 0$$

$$y_n = \mu^n y \xrightarrow{n} \begin{cases} 0 & y = 0 \\ \infty & y \neq 0 \end{cases}$$

Stable & Unstable subspaces

Let:

$\cdot\, A \in \mathcal{M}_{2\times 2}(\mathbb{R})$

We name stable subspace of A to:

We name unstable subspace of A to:

We denote:

$\cdot\, E^s, E^u$

abstract

Let:

$\cdot\, A = \lambda, 1\lambda, 0$

Then, holds:

$\cdot\, \mid \lambda \mid < 1 \to o(x,y) \xrightarrow{n} (0,0)$

$\cdot\, \mid \lambda \mid > 1 \to o(x,y) \xrightarrow{n} (\infty,\infty)$

Demonstration:

$$A^n = \lambda^{n-1}(\lambda, n, 0, \lambda)$$

Origin stability theorem

Let:

- $A \in \mathcal{M}_{n\times n}(\mathbb{R})$
- $\begin{array}{rccc} f : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \longmapsto & Ax \end{array}$
- $\sigma(A)$ eigenvalues of A

Then, holds:

- $\forall\ i \in [1,n]_{\mathbb{N}}$:

  $\mid \lambda_i \mid < 1 \quad \to$ origin is attractive

  $\forall\ i \in [1,n]_{\mathbb{N}}$:

  $\mid \lambda_i \mid > 1 \quad \to$ origin is repulsive

  $\exists\ \lambda_1, \lambda_2 \in \sigma(A)$:

  $\mid \lambda_1 \mid > 1 \ \wedge \ \mid \lambda_2 \mid < 1 \quad \to \mathbb{R}^n = E^s + E^u$

Demonstration:

demonstration

Local stability of fixed points of non-linear applications

Let:

$\cdot\, f \,:\, \mathbb{R}^n \to \mathbb{R}^n$

$\cdot\, x \in \mathrm{Fix}(f)$

$\cdot\, \mathcal{U} \subset \mathbb{R}^n$ open " $x \in \mathcal{U} f \,:\, \mathcal{U} \to \mathbb{R}^n$ differentiable in x

Then, holds:

$\cdot\, \rho(Df(p)) < 1 \to x$ attractive

$\cdot\quad \forall\, \lambda \in \sigma(A)$:

$\quad |\,\lambda\,| > 1 \quad \to x$ repulsive

Demonstration:

Suppose $p = 0$

$A \,:=\, Df(0)$

Consider:

$||-|| \,:\, \mathbb{R}^n \to \mathbb{R}^n$ norm " $||A|| < 1$

$\exists\, \mathcal{U} \subset \mathbb{R}^n$ open :

$f \in \mathcal{C}^1(\mathcal{U}) \quad \forall\, v \in \mathbb{R}^n$:

$$f(v) = Av + \mu(v) \;\;\text{"}\;\; \lim_{v \to 0} \frac{\mu(v)}{||v||} = 0$$

$\forall\, \eta \in \mathbb{R}^+$:

$\exists\, r \in \mathbb{R}^+$:

$$||v|| < r \to ||\mu(v)|| \leq \eta ||v|| \qquad \eta \,:=\, a + \eta < 1$$

$\forall\ n \in \mathbb{N}$:

$\forall\ v \in \mathbb{D}(0, r)$:

$f^n(v) \in \mathbb{D}(0, r)$

$\|f^n(v)\| \le (a + \eta)^n \|v\|$

Induction over n:

$n = 0$ ok

Suppose true for n

$\|f^{n+1}(v)\| = \|f(f^n(v))\| \le \|Af^n(v)\| + \|\mu(f^n(v))\|$

$\le a\|f^n(v)\| + \eta\|f^n(v)\| = (a + \eta)\|f^n(v)\| \le (a + \eta)^{n+1}\|v\|$

$\forall\ \varepsilon \in \mathbb{R}^+$:

$\delta \ := \ \min(\varepsilon, r)$

$\forall\ v \in \mathbb{R}^n$ " $\|v\| < \delta$:

$\|f^n(v)\| \le (a + \eta)^n \|v\| \le \|v\| < \varepsilon$

$\|f^n(v)\| \xrightarrow{n} 0$

repulsive:

$D(f^{-1})(x) = (Df(x))^{-1}$

$A \ := \ D(f^{-1})(x)$

$A^{-1} = (Df(x))^{-1}$

$\sigma(A^{-1}) = \sigma(A)^{-1}$

x attractive by $f^{-1} \to x$ repulsive by f

Local stability of periodic points

Let:

- $f \;:\; \mathbb{R}^n \to \mathbb{R}^n$
- $x \in \mathbb{R}^n$ period k point
- $o(x)$ orbit of x

Then, holds:

- $\forall\; p_i \in o(x)\;:$

  $$A_i \;:=\; Df^k(p_i) = Df(p_{i-1}) - - - Df(p_k) - - - Df(p_i)$$

  $$\sigma(A_i) = \sigma(A_{i'})$$

Demonstration:

regla de la cadena da producto de matrices que no es conmutativo

Invariant stable & unstable manifold

Let:

- $f : \mathbb{R}^n \to \mathbb{R}^n$

We name invariant stable manifold of x to:

$$\{v \in \mathbb{R}^n \mid f^n \xrightarrow{n} 0\}$$

We name invariant unstable manifold of x to:

$$\{v \in \mathbb{R}^n \mid f^{-n} \xrightarrow{n} 0\}$$

We denote:

- stable invariant manifold : W^s
- unstable invariant manifold : W^u

Stable & unstable invariant manifolds

Let:

$$\cdot \quad \begin{array}{cccc} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & (\frac{x}{2}, 2y-7x^2) \end{array}$$

Study:

$\cdot$ stability of $\mathrm{Fix}(f)$

Start:

$$Df(x,y) = \begin{pmatrix} 1/2 & 0 \\ -4x & 2 \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

$\sigma(Df(0,0)) = \{1/2, 2\} \to (0,0)$ sella point

$\forall\ y \in \mathbb{R}$:

$$f(0,y) = (0,2y) \to f^n(0,y) = (0,2^n) \xrightarrow{n} \infty$$

$$\{(x,y) \in \mathbb{R}^2 \mid x = 0\} \subset W^u(0,0)$$

$C \ := \ \{(x,y) \in \mathbb{R}^2 \mid y = 4x^2\}$

$\forall\ (x,y) \in C$:

$$f(x,y) = f(x,4x) = (x/2, x^2) \in C$$

$$f^{-1}(x,y) = (2x, 16x^2) \in C$$

$$f^n(x,4x) = (x/2^n, x^2/2^{2n-2}) \xrightarrow{n} (0,0)$$

$W^s(0,0) = C$

Stable manifold theorem

Let:

$\cdot\, \mathcal{U} \subset \mathbb{R}^n$ open

$\cdot\, f : \mathcal{U} \to \mathbb{R}^n \in \mathcal{C}^r(\mathcal{U})$

$\cdot\, p \in \mathbb{R}^2$ sella fixed point

Then, holds:

$\cdot \quad \exists\, \gamma : [a,b] \to \mathbb{R}^2 \in \mathcal{C}^r([a,b])$:

$\gamma(0) = p$

$\gamma'(0) \neq 0$

$\forall\, i \in [1,r]_{\mathbb{N}}$:

$Df^r(p) = \gamma^r(0) \quad W^s(p)$ tangent $W^s(p)$

$W^u(p)$ tangent $E^u(p)$

Demonstration:

no demonstration

45. Stability of periodic points

Let:

$$\cdot \quad f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (2x+y^2-2, x^2-x-y^3) \end{array}$$

Study:

$\cdot$ Stability of $\operatorname{Fix}(f)$

Start:

$$f(1,1) = (2+1-2, 1-1-1) = (1,-1)$$

$$f(1,-1) = (2+1-2, 1-1+1) = (1,1)$$

$(1,1)$ period 2 point

$$Df(x,y) = \begin{pmatrix} 2 & 2y \\ 2x-1 & -3y^2 \end{pmatrix}$$

$$Df^2(1,1) = \begin{pmatrix} 2 & 10 & -1 & 11 \end{pmatrix} \neq Df^2(1,-1)$$

$$\chi_A(t) = t^2 - 13t + 32$$

$$\sigma(A) = \{3, 29 - --, 9, 70 - --\}$$

$\forall\ \lambda \in \sigma(A)$:

$|\ \lambda\ | > 1$

Stability theorem: $(1,1)$ repulsive

46. Explicit solution of non-linear applications

Let:

- $$\begin{array}{rccc} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & (2x-y^3, 1/2y) \end{array}$$

Then, holds:

- $\begin{pmatrix} 2x-y=t \\ 1/2y=s \end{pmatrix}$ determinated system $\rightarrow f$ inyective
- Suppose$(x_{n+1}, y_{n+1}) = (2^n(2x_0 - y_0^3 -$
-
-
- $-\frac{1}{4^{2n}}y_0^3), \frac{1}{2^{n+1}}y_0)$
-

Induction over n:

$n = 1$

$(x_1, y_1) = (2^0(2x_0 - y_0^3), 1/2y^0)$ok

$\forall\ n \in \mathbb{N}$:

Suppose true for n

$(x_{n+1}, y_{n+1}) = f(x_n, y_n)$ok

$C\ :=\ \{(x,y) \in \mathbb{R}^2 \mid x = \frac{8}{15}y^3\}$

c stable manifold of the origin

$\forall\ (x,y) \in C$:

$\forall\ n \in \mathbb{N}$:

Contractive application

Let:

· *statements*

·

Then, *item* is a/an entity if:

· *conditions*

·

We denote:

· *property* : *notation*

·

Fixed point theorem

Let:

· $\mathcal{U} \subset \mathbb{R}^n$ closed

· $f : \mathcal{U} \to \mathcal{U} \lambda$−contractive

Then, holds:

· $\mathrm{Fix}(f) = \{p\}$

· p global attractive fixed point

· a

· b

Demonstration:

$\forall\ x \in \mathcal{U}$:

$x_n \ := \ f^n(x)$

f contractive $\rightarrow \|x_n - y_n\| \leq \lambda^n \|x_0 - y_0\|$

In particular:

$y = f(x) = x_1 \rightarrow \|x_n - x_{n+1}\| \leq \lambda^n \|x_0 - x_1\|$

$(x_n)_{n \in \mathbb{N}}$ Cauchy sequence

$(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^n, \mathbb{R}^n$ complete $\rightarrow (x_n)_{n \in \mathbb{N}}$ convergent

f contractive $\rightarrow f$ continuous

$f(p) = f(\lim_n\ x_n) = \lim_n\ f(x_n) = \lim_n\ x_{n+1} = p$

p fixed point

$\forall\ p, q \in \mathcal{U}\ \ "\ \ p, q$ fixed points :

$\|f^n(p) - f^n(q)\| \leq \lambda^n \|p - q\| \xrightarrow{n} 0$

$\|p - q\| \xrightarrow{n} 0$

absurd

$\forall\ x \in \mathbb{R}^n$:

$\|f^n(x) - p\| = \|f^n(x) - f^n(p)\| \leq \lambda^n \|x - p\| \xrightarrow{n} 0$