Dynamical systems

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Block I

Definitions

1. One-dimensional discrete dynamical systems

Dynamical system

Let:

- M manifold
- T monoid
- $\phi : M \times T \to M$

Then, (M, T, ϕ) is a dynamical system if:

- $\forall\ x \in X$:

  $\phi(x, 0) = 0$

  $\forall\ t_1, t_2 \in T$:

  $\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$

Dimension

Let:

- (M, T, ϕ) dynamical system

We name dimension of (M, T, ϕ) to:

- $\dim(M)$

We denote:

- $dim(M) = n\ :\ (M, T, \phi)$ n-D dynamical system

Discrete & Continuous

Let:

- (M, T, ϕ) dynamical system

Then, (M, T, ϕ) is discrete if:

- $T \overset{\subset}{\sim} \mathbb{N}$

Then, (M, T, ϕ) is continuous if:

- $T \subset \mathbb{R}$ „ T open

Defined by a function

Let:

- (M, T, ϕ) dynamical system
- $f : M \to M$

Then, (M, T, ϕ) is a dynamical system defined by f if:

- $T = \mathbb{N}$
- $\begin{array}{rcl} \phi : M \times \mathbb{N} & \longrightarrow & M \\ (x, n) & \longmapsto & f^n(x) \end{array}$

We denote:

- (M, T, ϕ) dynamical system defined by f : $(M, \mathbb{N}, f)$

Orbit

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$

We name orbit of x to:

- $\{f^n(x)\}_{n\in\mathbb{N}}$

We denote:

- $o(x)$

n-periodic point

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$
- $n \in \mathbb{N}$

Then, x is a n-periodic point if:

- $f^n(x) = x$
- $\forall\ n' \in \mathbb{N}$ „ $n' < n$:

  $f^{n'}(x) \neq x$

We denote:

- $n = 1\ :\ x$ fixed point

Attractive & Repulsive

Let:

$\cdot\, (M, \mathbb{N}, f)$ metrical dynamical system

$\cdot\, x \in M$ m-periodic point

Then, x is attractive if:

$\cdot \quad \exists\, \mathcal{U} \subset M$ open :

$\forall\, y \in \mathcal{U}$:

$\exists\, N \in \mathbb{N}$:

$\forall\, n \in \mathbb{N}$ „ $n \geq N$:

$f^{nm}(y) \in \mathcal{U}$

Then, x is repulsive if:

$\cdot \quad \forall\, \mathcal{U} \subset M$ „ $x \in \mathcal{U}$:

$\forall\, y \in \mathcal{U}$:

$\exists\, N \in \mathbb{N}$:

$\forall\, n \in \mathbb{N}$ „ $n \geq N$:

$f^{nm}(x) \notin \mathcal{U}$

Attraction set

Let:

- $(M, \mathbb{N}, f)$ dynamical system
- $x \in M$ attractive m-periodic point
- $o(x)$ orbit of x

We name attraction set of x to:

- $\{y \in M \mid \exists\, x' \in o(x) \;:\; f^{nm}(y) \xrightarrow{n} x'\}$

We denote:

- $A(x)$

Neutral point

Let:

- $(M, \mathbb{N}, f)$ differentiable dynamical system
- $x \in M$

Then, x is a neutral point if:

- $f'(x) \in \{-1, 1\}$

Feeble attractive & repulsive points

Let:

· $(M, \mathbb{N}, f)$ $\mathcal{C}^3$ dynamical system

· $x \in M$

Then, x is feeble attractive point if:

· $f'(x) = 1$

· $f''(x) = 0$

· $f'''(x) > 0$

Then, x is feeble repulsive point if:

· $f'(x) = 1$

· $f''(x) = 0$

· $f'''(x) < 0$

Multiplier

Let:

· $(M, \mathbb{N}, f)\mathcal{C}^1$ dynamical system

· $x \in M$

We name multiplier of x to:

· $f'(x)$

Block II

Propositions

1. One-dimensional discrete dynamical systems

introduction

Fixed points theorem

Let:

$\cdot\ I \subset \mathbb{R}$ open

$\cdot\ f\ :\ I \to I$ differentiable

$\cdot\ x \in I$

Then, holds:

$\cdot\ |f'(x)| < 1 \to x$ attractive

$\cdot\ |f'(x)| > 1 \to x$ repulsive

Demonstration:

demonstration

Attractiveness of periodic points does not involve the chosen point

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$ n-periodic point
- $\{x_i\}_{i=1}^{r}$ orbit of x

Then, holds:

- x attractive $\leftrightarrow$ $\forall\ x' \in o(x)$:

  x' attractive

Demonstration:

$\forall\ x' \in o(x)$:

$$f^{n'}(x') = \prod_{i=1}^{r} f'(x_i) = f^{n'}(x)$$

Partition of attraction set

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- x n-periodic point
- $o(x)$ orbit of x

Then, holds:

- $\forall\, x' \in o(x)$:

  $\exists\, \mathcal{U} \subset M$ open :

  $\forall\, y \in \mathcal{U}$:

  $$f^n(y) \xrightarrow{n} x'$$

Demonstration:

demonstration

Block III

Examples

1. One-dimensional discrete dynamical systems

$Examplesofwhatareandwhatarenotone-dimensionaldynamicalsystems$

go

Block IV

Problems

MODELS I SISTEMES DINÀMICS

Llista 1: Aplicacions unidimensionals

B.1. Trobeu els punts fixos i les òrbites de període 2 de les següents funcions. En el cas que apareixin paràmetres, feu-ho en funció d'aquests.

(a) * $f(x) = 2x(1-x)$, on $x \in \mathbb{R}$.

(b) * $f_c(x) = x^2 + c$, on $x, c \in \mathbb{R}$ (només punts fixos).

(c) $f(x) = x^2 + 1$, on $x \in \mathbb{R}$.

(d) $f_{a,b}(x) = ax + b$, on $a, b, x \in \mathbb{R}$.

(e) $f(x) = 2x^2 - 5x$, on $x \in \mathbb{R}$.

B.2. Fent servir anàlisi gràfic, dibuixeu el retrat de fases de

(a) $f(x) = x^2, \quad x \in \mathbb{R}$.

(b) $f(x) = x(1-x), \quad x \in \mathbb{R}$.

(c) $f_a(x) = ax, \quad x \in \mathbb{R}$, pels diferents valors de $a \in \mathbb{R}$.

B.3. * Trobeu els punts fixos atractors i les seves conques d'atracció per a la funció $f(x) = \frac{3x - x^3}{2}$, per $|x| \leq \sqrt{3}$.

B.4. Per a la funció logística $f_a(x) = ax(1-x)$, calculeu els punts fixos i els cicles de període 2 en funció del paràmetre, i determineu-ne l'estabilitat.

1. Estudieu el comportament asimptòtic de la successió $\{x_n\}_{n \in \mathbb{N}}$, pels diferents valors de x_0 indicats.

(a) * $x_{n+1} = \frac{\sqrt{x_n}}{2}$, $x_0 \geq 0$.

(b) $x_{n+1} = \frac{x_n}{2} + \frac{2}{x_n}$, $x_0 \geq 2$.

2. Donada la successió $x_{n+1} = \frac{x_n + 2}{x_n + 1}$,

(a) Trobeu el límit $L = \lim_{n \to \infty} x_n$ per a $x_0 \geq 0$.

(b) Descriviu el conjunt dels $x_0 < 0$ pels quals el límit $\lim_{n \to \infty} x_n$ existeix i no és igual a L, o bé no existeix. (Per exemple $x_0 = -1$).

3. **(Examen 2011)** Considereu el sistema dinàmic real definit per $x_{n+1} = \frac{x_n}{4} + x_n^3$. Trobeu el comportament asimptòtic de les òrbites per a tota condició inicial $x_0 \in \mathbb{R}$. Justifiqueu rigorosament les vostres afirmacions.

4. Demostreu rigurosament que $f(x) = \sin(x)$ té $x = 0$ com atractor global.

5. Demostreu que si $f : \mathbb{R} \to \mathbb{R}$ és derivable, x_0 és un punt fix i $|f'(x_0)| > 1$ llavors x_0 és un punt fix repulsor.

6. Sigui $f : \mathbb{R} \to \mathbb{R}$ de classe $\mathcal{C}^\infty$ i sigui x_0 un punt fix tal que $f'(x_0) = 1$. Doneu criteris sobre les derivades d'ordre superior, per determinar el retrat de fase local al voltant de x_0. Apliqueu-ho a determinar l'estabilitat dels punts fixos de $x^3 - x$.

1. One-dimensional discrete dynamical system

introduction

go

Block V

Tasks

1. 1st laboratory

Orbit analysis

Let:

$$\cdot \quad \begin{array}{rcl} f: \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^3 + \frac{1}{4}x \end{array}$$

Study:

· Orbit behavior of the real dynamical system defined by f

Demonstration:

Formalization :

Consider (M, T, ϕ) where:

$M = \mathbb{R}$

$T = \mathbb{N}$

$$\begin{array}{rcl} \phi: \mathbb{R} \times \mathbb{N} & \longrightarrow & \mathbb{R} \\ (x, n) & \longmapsto & f^n(x) \end{array}$$

Study the orbits of $(\mathbb{R}, \mathbb{N}, \phi)$

We will denote $f^n(x)$ as x_n

Fixed points :

$\forall\ x \in \mathbb{R}$:

$f(x) = x \leftrightarrow x^3 + \frac{1}{4}x - x = 0 \leftrightarrow x^3 - \frac{3}{4}x = 0$

$\leftrightarrow x = 0 \ \vee\ x^2 - \frac{3}{4} = 0$

x fixed point $\leftrightarrow x \in \{0, \pm\frac{\sqrt{3}}{2}\}$

Graphic analysis :

Parity:

$\forall\ x \in \mathbb{R}$:

$$f(-x) = (-x)^3 + \frac{(-x)}{4} = -(x^3 + \frac{x}{4}) = -f(x)$$

f is odd

Monotonicity:

$\forall\ x \in \mathbb{R}$:

$$f'(x) = 3x^2 + \frac{1}{4} > 0$$

f is increasing over $\mathbb{R}$

Convexity:

$\forall\ x \in \mathbb{R}^-$:

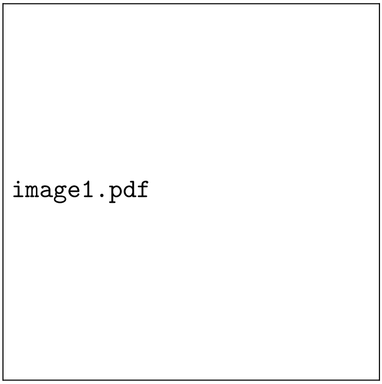
$$f''(x) = 6x \leq 0$$

$\forall\ x \in \mathbb{R}^+$:

$$f''(x) = 6x \geq 0$$

f is concave over $\mathbb{R}^-$ and convex over $\mathbb{R}^+$

Graphic representation :



$\underline{\text{I}}$ $\forall\, x \in (-\infty, -\frac{\sqrt{3}}{2})$:

Induction over n:

f incresing $\rightarrow f(x_n) < f(-\frac{\sqrt{3}}{2})$

$x_{n+1} \in (-\infty, -\frac{\sqrt{3}}{2})$

$\therefore)$ $o(x)$ is enclosed in $(-\infty, -\frac{\sqrt{3}}{2})$

Induction over n:

$x_n^2 > \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) > 0$

$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) < 0$

$\therefore)$ $o(x)$ decreasing

$\nexists\, x < -\frac{\sqrt{3}}{2}$ „ x fixed point $\rightarrow o(x) \xrightarrow{n} -\infty$

$\underline{\text{II}}$ $\forall\, x \in (-\frac{\sqrt{3}}{2}, 0)$:

Induction over n:

f increasing $\rightarrow f(-\frac{\sqrt{3}}{2}) < f(x_n) < f(0)$

$x_{n+1} \in (-\frac{\sqrt{3}}{2}, 0)$

$\therefore)$ $o(x)$ is enclosed in $(-\frac{\sqrt{3}}{2}, 0)$

Induction over n:

$x_n^2 < \frac{3}{4} \to (x_n^2 - \frac{3}{4}) < 0$

$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) > 0$

$\therefore)$ $o(x)$ increasing

$o(x)$ convergent $\wedge$ 0 fixed point $\to o(x) \xrightarrow{n} 0$

$\underline{\text{III}}$ $\forall\, x \in (0, \frac{\sqrt{3}}{2})$:

Induction over n:

$-x_n \in (-\frac{\sqrt{3}}{2}, 0)$

$\underline{\text{II}} \to f(-x_n) \in (-\frac{\sqrt{3}}{2}, 0) \;\wedge\; f(-x_n) > -x_n$

f odd $\to f(x_n) = -f(-x_n) \in (0, \frac{\sqrt{3}}{2})$

f odd $\to f(x_n) = -f(-x_n) < x_n$

$\therefore)$ $o(x)$ is enclosed in $(0, \frac{\sqrt{3}}{2})$ $\wedge$ $o(x)$ decreasing

$o(x)$ convergent $\wedge$ 0 fixed point $\to o(x) \xrightarrow{n} 0$

$\underline{\text{IV}}$ $\forall\, x \in \mathbb{R}$ " $x > \frac{\sqrt{3}}{2}$:

Induction over n:

$-x_n \in (\frac{\sqrt{3}}{2}, \infty)$

$\underline{\text{I}} \to f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty) \;\wedge\; f(-x_n) < -x_n$

f odd $\to f(x_n) = -f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty)$

f odd $\to f(x_n) = -f(-x_n) > x_n$

$\therefore)$ $o(x)$ is inf bounded by in $\frac{\sqrt{3}}{2}$ $\wedge$ $o(x)$ increasing

$o(x)$ convergent

$\nexists\, x > \frac{\sqrt{3}}{2}$ „ x fixed point $\rightarrow o(x) \xrightarrow{n} +\infty$