

1. Fixed points cardinality

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Let:

$$\cdot f : [0, 1] \rightarrow [0, 1] \in \mathcal{C}^2([0, 1])$$

$$\cdot f(1) < 1$$

$$\cdot f'' > 0 \in [0, 1]$$

Show that:

$$\cdot \# \{x \in [0, 1] \mid f(x) = x\} = 1$$

Demonstration:

$$\# \{x \in [0, 1] \mid f(x) = x\} \geq 1:$$

Case $f(0) = 0$:

0 fixed point

Case $f(0) > 0$:

$$\begin{array}{ccc} g : [0, 1] & \longrightarrow & [-1, 1] \\ x & \longmapsto & f(x) - x \end{array} \in \mathcal{C}^2([0, 1])$$

$$g(0) = f(0) - 0 > 0$$

$$g(1) = f(1) - 1 < 0$$

Bolzano's theorem:

$$\exists x \in (0, 1):$$

$$g(x) = 0$$

$$f(x) = x$$

$$\# \{x \in [0, 1] \mid f(x) = x\} \leq 1:$$

$$g'' > 0 \text{ over } [0, 1]$$

Rolle's theorem:

$$\# \{x \in (0, 1) \mid g'(x) = 0\} \leq 1$$

$$\# \{x \in (0, 1) \mid g(x) = 0\} \leq 2$$

$$\# \{x \in (0, 1) \mid f(x) = x\} \leq 2$$

$$f'' > 0 \text{ over } [0, 1]$$

Monotonicity test:

$$f' \text{ increasing in } [0, 1]$$

$$\forall a < b \in [0, 1) \quad f(a) = a, f(b) = b:$$

Mean Value Theorem:

$$\exists c \in (a, b):$$

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 1$$

$$\exists d \in (b, 1):$$

$$f'(d) = \frac{f(1) - f(b)}{1 - b} < 1$$

$$f' \text{ increasing} \rightarrow f'(c) < f'(b) < f'(d)$$

$$1 < f'(b) < 1 \text{ absurd}$$

$$\therefore \# \{x \in [0, 1] \mid f(x) = x\} = 1$$