I. Bifurcation Theory

Bifurcation diagram

Let:

$$\cdot \quad \begin{array}{rcl} f : \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (a,x) & \longmapsto & x^3 - 3x^2 + (5-a)x - 2 + a \end{array}$$

$\cdot\ \forall\ a \in \mathbb{R}$:

$$\begin{array}{rcl} f_a : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^3 - 3x^2 + (5-a)x - 2 + a \end{array}$$

Study:

$\cdot$ Bifurcations of $(\mathbb{R}, \mathbb{N}, \{f_a\}_{a\in\mathbb{R}})$

Start:

Fixed points:

$$f_a(x) = x \leftrightarrow x^3 - 3x^2 + (4-a)x - 2 + a = 0 \leftrightarrow (x-1)(x^2 - 2x + 2 - a) = 0$$

$$x^2 - 2x + 2 - a = 0 \leftrightarrow x = \pm\sqrt{a-1}$$

$\forall\ a \in \mathbb{R}$ " $a \le 1$:

$$\text{Fix}(f_a) = \{1\}$$

$\forall\ a \in \mathbb{R}$ " $a > 1$:

$$\text{Fix}(f_a) = \{1, \pm\sqrt{a-1}\}$$

Stability:

$$\partial_x f(a,x) = 3x^2 - 6x + 5 - a$$

$$\partial_{x^2} f(a,x) = 6x - 6$$

$$\partial_{x^3} f(a,x) = 6$$

$|\partial_x f(a,1)| < 1 \leftrightarrow |2-a| < 1 \leftrightarrow a \in (1,3)$

$\partial_{x^2} f(1,1) = 0, \;\; \partial_{x^3} f(1,1) > 0$

$\partial_{x^2} f(3,1) = 0, \;\; \partial_{x^3} f(3,1) > 0$

$\forall\ a \in \mathbb{R}\ \text{„}\ a \leq 1\ \vee\ a \geq 3\ :$

1 repulsive

$\forall\ a \in \mathbb{R}\ \text{„}\ a \in (1,3)\ :$

1 attractive

$\forall\ a \in \mathbb{R}\ \text{„}\ a > 1\ :$

$\left|\partial_x f(a, \pm\sqrt{a-1})\right| = |2a-1| > 1$

$\pm\sqrt{a-1}$ repulsive

Pitchfork bifurcation at 1 :

$\partial_a f(1,1) = 1-1 = 0$

$\partial_{x^2} f(1,1) = 6-6 = 0$

$\partial_{ax} f(1,1) = -1 \neq 0$

$\partial_{x^3} f(1,1) = 6 \neq 0$

Period-doubling bifurcation at 3 :

$\partial_a f^2(3,1) = \partial_a f(f(3,1))\partial_a f(3,1) = 0$

$\partial_{x^2} f^2(3,1) = \partial_{x^2} f(f(3,1))\partial_{x^2} f(3,1) = 0$

$\partial_{ax} f^2(3,1) = \partial_{ax} f(f(3,1))\partial_{ax} f(3,1) \neq 0$

$\partial_{x^3} f^2(3,1) = \partial_{x^3} f(f(3,1))\partial_{x^3} f(3,1) \neq 0$

Source Code

```
#include "stdio.h"
#include "stdlib.h"
#include "math.h"
#include "string.h"

void plot( char *input_file, char *output_file )
{
  FILE *gnuplot;
  gnuplot = popen("gnuplot", "w");
  if( output_file )
  {
    fprintf(gnuplot, "set term svg\n");
    fprintf(gnuplot, "set out \"%s\"\n", output_file );
  }
  fprintf(gnuplot, "plot \"%s\" with dots\n", input_file);
  fflush(gnuplot);
  fclose(gnuplot);
}

double example_function( double param, double point )
{
  return pow(point,3) - 3*pow(point,2) + (5-param)*point - 2 + param;
}

void bifurcation_diagram( int param_min, int param_max, double param_step,
             int point_min, int point_max, int num_points,
             double (*f)(double,double), int num_iter, int tolerancy)
{
  FILE* file;
  double param, point;
  int i,j;

  srand(time(NULL));
  file = fopen("data.dat","w");

  for ( param = param_min; param < param_max; param += param_step )
  {
    for ( i = 0; i < num_points; i++ )
    {
      point = point_min + ((double) rand() / (double) RAND_MAX) * (point_max-
          point_min);

      for ( j = 0; j < num_iter && abs(point) < tolerancy; j++ )
      {
        point = (*f)(param,point);
      }

      if(abs(point) < tolerancy)
      {
        fprintf(file,"%lf %lf\n",param, point);
      }
    }
  }
  plot( "data.dat", "graph.svg" );
}
```

Example

```
int main(int argc, char const *argv[])
{
  bifurcation_diagram( 0, 5, 10e-3, 0, 5, 100, &example_function, 100, 10e1);

  return 0;
}
```

Output