1. Holomorphic functions

Cauchy-Riemman

Let:

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies CR

Then, holds:

- $f_x = f'(z)$
- $f_y = -if_y$
- $\partial_{f(z)} z = 0$

Demonstration:

$f_x = u_x + iv_x = a + bi = f'(z)$

$f_y = i(a + ib) = if'(z)$

$f'(z) = -if_y$

notation

tangent venctor

Let:

- γ differentiable plane arc $\;_{\shortparallel}\; \forall\ t \in I$:

  $\gamma'(t) \neq 0$

Then, holds:

- $\gamma'(t)$ tangent to γ

Demonstration:

demonstration

arc images

Let:

· $f \in \mathcal{H}(\mathcal{U})$

· γ differentiable plane arc " $\gamma \subset \mathcal{U}$

· $\sigma = f(\gamma)$

· $z_0 = \gamma(0)$

Then, holds:

· $\sigma' = f'(\gamma)\gamma'$

· $\gamma'(0) \neq 0 \to f'(z_0) \neq 0$

· $\sigma'(0) = f'(z_0)\gamma'(0)$

· $|\sigma'(0)| = |f'(z_0)||\gamma'(0)|$

· $arg\sigma'(0) = arg\gamma'(0) + argf(z_0)$

· f aplica una homotecia mas una rotacion constante a todos los vectores tangentes que salen de z0

Demonstration:

obvio

Holomorphic functions are conform

Let:

- $f : \mathcal{U} \to \mathbb{C}$
-

Then, holds:

- $f$holomorph in z „ $f'(z) \neq 0 \leftrightarrow f$ conform

Demonstration:

→):

already seen

←):

too hard

Convergence of complex series

Let:

- $\sum_{n\geq 0} c_n$ complex series

Then, holds:

- $\sum_{n\geq 0} c_n$ convergent $\leftrightarrow \sum_{n\geq 0} Rec_n$ convergent $\wedge \sum Imc_nn[0]$ convergent

Demonstration:

demonstration

Absolutely convergent are convergent

Let:

- $\sum c_n n[0]$ absolutely convergent

Then, holds:

- $\sum c_n n[0]$ convergent

Demonstration:

$S_k := \sum c_n n[0][k]$

$\forall\, m \in \mathbb{N}$ „ $m < k$:

$|S_k - S_m| = |\sum c_n n[m+1][k]| \leq \sum |c_n| n[m+1][k]$

$\leq \sum |c_n| n[m+1] \xrightarrow{n} 0$

$|S_k - S_m| \xrightarrow{n} 0 \rightarrow (S_k)_k$ convergent $\rightarrow \sum c_n n[0]$ convergent

Series and norm

Let:

· $\sum c_n n[0]$ convergent

Then, holds:

· $|c_n| \xrightarrow{n} 0$

Demonstration:

$\sum c_n n[0]$ convergent $\leftrightarrow (S_n)_n$ convergent

$\rightarrow$ Cauchy $|S_n - S_m| \xrightarrow{n} 0$ por n y m $\rightarrow |S_n - S_{n-1}| \xrightarrow{n} 0$

$\rightarrow |c_n| \xrightarrow{n} 0$

Root test

Let:

- $\sum_{n\geq 0} c_n$ real series
- $l \in \mathbb{R}$ „ $\overline{\lim_{k}} |c_k|^{\frac{1}{k}} = l$

Then, holds:

- $l > 1 \rightarrow \sum_{n\geq 0} c_n \notin \mathbb{R}$
- $l < 1 \rightarrow \sum_{n\geq 0} c_n \in \mathbb{R}$

Demonstration:

demonstration

Quotient test

Let:

- $\sum_{n\geq 0} c_n$ real series
- $l \in \mathbb{R}$ "

Then, holds:

- $\exists\, l \in \mathbb{R}$:

$$\lim_k \frac{c_{k+1}}{c_k} = l$$

- $\overline{\lim_{c_k}} |c_k|^{\frac{1}{k}} k = l$

Power series theorem

Let:

- $\sum_{n\geq 0} a_n c^n$ power series

Then, holds:

- $|z-a| < R \rightarrow$ absolutely convergent

- $|z-a| > R \rightarrow$ divergent

- convergent in $D(a,R)$
- $\begin{array}{rccc} f : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \sum_{n\geq 0} c_n(z-a)^n \end{array} \in \mathcal{H}(D(a,R))$
- $\forall\ z \in \mathbb{C}$:

  $$f'(z) = \sum_{n\geq 0} nc_n(z-a)^{n-1} \text{ convergent}$$

- convergence radius of f' = convergence radius of f

Demonstration:

$\forall\ z \in \mathbb{C}$,, $|z-a| < R$:

Root test over $\sum_{n\geq 0} |c_n||z-a|^n rlimit(|c_n||z-a|^n)^{\frac{1}{n}} n = |z-a| rlimit|c_n|^{\frac{1}{n}} n = \frac{|z-a|}{R} < 1$ Root test $\rightarrow$ absolutely convergent

$\forall\ z \in \mathbb{C}$,, $|z-a| < R$:

$\forall\ \rho \in \mathbb{R}$,, $|z-a| < \rho < R$:

$\frac{1}{\rho} < \frac{1}{R}$

$rlimit|c_n|^{1/n} n = \frac{1}{R}\imath$ exists partial of $|c_n|^{1/n}$

$|c_n||z-a|^n > \frac{|z-a|^n}{\rho^n}$ no $\xrightarrow{n} 0$

General term test → divergent

Power series theorem

Let:

·

Then, holds:

· $PartII, III, IV$

Demonstration:

Follow 3 steps

Step 1 : Uniform convergence in compacts of $D(a,R)$:

$$\begin{aligned} g_n : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto c_n(z-a)^n \end{aligned}$$

$\forall\ \rho \in \mathbb{R}$ " $\rho < R$:

$\forall\ z \in \overline{D(a,\rho)}$:

$$|g_n(z)| = |c_n||z-a|^n \leq |c_n|\rho^n$$

$M_n := |c_n|\rho^n$

$rlimit(|c_n|\rho^n)^{1/n}n = \rho rlimit|c_n|^{1/n}n = \frac{\rho}{R} < 1$

Root test$\rightarrow \sum_{n\geq 0} M_n$ convergent

M-Weierstrass$\rightarrow \sum_{n\geq 0} c_n(z-a)^n$ uniformly convergent over compacts of $D(a,\rho)$

$\sum_{n\geq 0} c_n(z-a)^n$ uniformly convergent over compacts of $D(a,r)$

$$f(z) \;:=\; \sum_{n\geq 0} g_n(z)$$

g uniformly convergent $\rightarrow f$ continuous

Step 2 : IV. R' :

$$\tilde{f}(z) \;:=\; \sum_{n\geq 1} nc_n(z-a)^{n-1}$$

$$\tilde{f}(z) = \sum_{n\geq 0}(n+1)c_{n+1}(z-a)^n$$

$\frac{1}{R'} = rlimit(n+1)|c_{n+1}|^{1/n}n = rlimit(n+1)^{1/n}|c_{n+1}|^{1/n}n =$ $rlimit(|c_{n+1}|^{1/n+1})^{\frac{n+1}{n}}n = \frac{1}{R}$

$$R' = R$$

Step 3 : III :

$\tilde{f}$well defined in $D(a,R)$

$\forall\ z_0 \in D(a,R)$:

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - \tilde{f}(z_0)\right| \xrightarrow{n} 0?$$

$\forall\ n \in \mathbb{N}$:

$$S_n(z) \;:=\; \sum_{k=0}^{n} f_k(z)$$

$$R_n(z) \;:=\; \sum_{k\geq n+1} f_k(z)$$

$$f(z) = S_n(z) + R_n(z)$$

$$\tilde{f}(z) = \tilde{S}_n(z) + \tilde{R}_n(z)$$

$$R_n(z), \tilde{R_n}(z) \xrightarrow{n} 0$$

$\forall\ \varepsilon \in \mathbb{R}^+$:

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \left|\frac{S_n(z)-S_n(z_0)}{z-z_0} + \frac{R_n(z)-R_n(z_0)}{z-z_0} - \tilde{S_n}(z_0) - \tilde{R_n}(z_0)\right|$$

$\forall\ \rho \in \mathbb{R}$ " $|z_0 - a| < \rho < R$:

$$\left|\tilde{R_n}(z_0)\right| \le \sum_{k \ge n+1} k|c_k|\rho^{k-1} < \frac{\varepsilon}{3}(n \ge n_1)$$

$$\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| \le \sum_{k \ge n+1} |c_k|\left|\frac{(z-a)^k - (z_0-a)^k}{z - z_0}\right|$$

$$\frac{a^k-b^k}{a-b} = a^{k-1} - a^{k-2}b + - - - + b^{k-1}$$

$$\le \sum_{k \ge n+1} |c_k|(|z-a|^{k-1} + |z-a|^{k-2}|z_0 - a| + - - - +$$

$|z_0 - a|^{k-1}$

$|z - a|, |z_0| < \rho$

$$\le \sum_{k \ge n+1} |c_k| k\rho^{k-1} < \frac{\varepsilon}{3}(n \ge n_1)$$

$$\left|\frac{S_n(z)-S_n(z_0)}{z-z_0} - \tilde{S_n}(z_0)\right| < \frac{\varepsilon}{3}(S'_n = \tilde{S_n})(n \ge n_2)$$

$\forall\ n \in \mathbb{N}$ " $n \ge \max(n_1, n_2)$:

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \varepsilon$$

functions associated to series are $\mathcal{C}^\infty$

Let:

- $f(z) = \sum c_n (z-a)^n n[0]$ series

- R radius of convergence of f

Then, holds:

- $f \in \mathcal{C}^\infty$ over $D(a,R)$

- $\forall\ n \in \mathbb{N}$:

  $f^{n)} \in \mathcal{H}(D(a,R))$

- $c_k = \frac{f^{k)}}{k!}$

- series associated to f is unique

Demonstration:

$f^{k)}(z) = \sum n(n-1) - - - (n-k+1)c_n(z-a)^{n-k} n[k]$

$f^{k)}(a) = k! c_k$

$c_k = \frac{f^{k)}(a)}{k!}$