$$\boxed{\textbf{1. Fixed points cardinality}}$$

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Let:

$\cdot\, f \,:\, [0,1] \to [0,1] \in \mathcal{C}^2([0,1])$

$\cdot\, f(1) < 1$

$\cdot\, f'' > 0 \in [0,1]$

Show that:

$\cdot\, \# \{x \in [0,1] \mid f(x) = x\} = 1$

Demonstration:

$\# \{x \in [0,1] \mid f(x) = x\} \geq 1$:

Case $f(0) = 0$:

0 fixed point

Case $f(0) > 0$:

$$g \,:\, \begin{array}{ccc} [0,1] & \longrightarrow & [-1,1] \\ x & \longmapsto & f(x) - x \end{array} \;\in \mathcal{C}^2([0,1])$$

$g(0) = f(0) - 0 > 0$

$g(1) = f(1) - 1 < 0$

Bolzano's theorem:

$\exists\, x \in (0,1)$:

$g(x) = 0$

$f(x) = x$

$\# \{x \in [0,1] \mid f(x) = x\} \leq 1$:

$f'' > 0 \in [0,1]$

Rolle's theorem:

$\# \{x \in (0,1) \mid f'(x) = 0\} \leq 1$

$\# \{x \in (0,1) \mid f(x) = 0\} \leq 2$

$f'' > 0 \in [0,1]$

Monotonicity test:

f' increasing in $[0,1]$

$\forall \, a < b \in [0,1)$ „ $f(a) = a, f(b) = b$:

Medium Value Theorem:

$\exists \, c \in (a,b)$:

$f'(c) = \frac{f(b)-f(a)}{b-a} = 1$

$\exists \, d \in (b,1)$:

$f'(d) = \frac{f(1)-f(b)}{1-b} < 1$

f' increasing $\rightarrow f'(c) < f'(b) < f'(d)$

$1 < f'(b) < 1$ absurd

$\therefore) \; \# \{x \in [0,1] \mid f(x) = x\} = 1$