

1. Orbit analysis

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Let:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^3 + \frac{1}{4}x \end{aligned}$$

Study:

- Orbit behavior of the real dynamical system defined by f

Demonstration:

Formalization :

Consider (M, T, ϕ) where:

$$M = \mathbb{R}$$

$$T = \mathbb{N}$$

$$\begin{aligned} \phi : \mathbb{R} \times \mathbb{N} &\longrightarrow \mathbb{R} \\ (x, n) &\longmapsto f^n(x) \end{aligned}$$

Study the orbits of $(\mathbb{R}, \mathbb{N}, \phi)$

We will denote $f^n(x)$ as x_n

Fixed points :

$$\forall x \in \mathbb{R}:$$

$$f(x) = x \leftrightarrow x^3 + \frac{1}{4}x - x = 0 \leftrightarrow x^3 - \frac{3}{4}x = 0$$

$$\leftrightarrow x = 0 \vee x^2 - \frac{3}{4} = 0$$

$$x \text{ fixed point} \leftrightarrow x \in \{0, \pm \frac{\sqrt{3}}{2}\}$$

Graphic analysis :

Parity:

$$\forall x \in \mathbb{R}:$$

$$f(-x) = (-x)^3 + \frac{(-x)}{4} = -(x^3 + \frac{x}{4}) = -f(x)$$

f is odd

Monotonicity:

$$\forall x \in \mathbb{R}:$$

$$f'(x) = 3x^2 + \frac{1}{4} > 0$$

f is increasing over \mathbb{R}

Convexity:

$$\forall x \in \mathbb{R}^-:$$

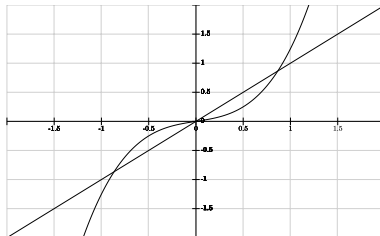
$$f''(x) = 6x \leq 0$$

$$\forall x \in \mathbb{R}^+:$$

$$f''(x) = 6x \geq 0$$

f is concave over \mathbb{R}^- and convex over \mathbb{R}^+

Graphic representation :



I $\forall x \in (-\infty, -\frac{\sqrt{3}}{2})$:

Induction over n :

$$f \text{ incresing} \rightarrow f(x_n) < f(-\frac{\sqrt{3}}{2})$$

$$x_{n+1} \in (-\infty, -\frac{\sqrt{3}}{2})$$

$$\therefore) o(x) \text{ is enclosed in } (-\infty, -\frac{\sqrt{3}}{2})$$

Induction over n :

$$x_n^2 > \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) > 0$$

$$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) < 0$$

$$\therefore) o(x) \text{ decreasing}$$

$$\nexists x < -\frac{\sqrt{3}}{2} \quad \text{„} x \text{ fixed point} \rightarrow o(x) \xrightarrow{n} -\infty$$

II $\forall x \in (-\frac{\sqrt{3}}{2}, 0)$:

Induction over n :

$$f \text{ increasing} \rightarrow f(-\frac{\sqrt{3}}{2}) < f(x_n) < f(0)$$

$$x_{n+1} \in (-\frac{\sqrt{3}}{2}, 0)$$

$$\therefore) o(x) \text{ is enclosed in } (-\frac{\sqrt{3}}{2}, 0)$$

Induction over n :

$$x_n^2 < \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) < 0$$

$$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) > 0$$

$$\therefore) o(x) \text{ increasing}$$

$$o(x) \text{ convergent} \wedge 0 \text{ fixed point} \rightarrow o(x) \xrightarrow{n} 0$$

III $\forall x \in (0, \frac{\sqrt{3}}{2})$:

Induction over n :

$$-x_n \in (-\frac{\sqrt{3}}{2}, 0)$$

$$\underline{\text{I}} \rightarrow f(-x_n) \in (-\frac{\sqrt{3}}{2}, 0) \wedge f(-x_n) > -x_n$$

$$f \text{ odd} \rightarrow f(x_n) = -f(-x_n) \in (0, \frac{\sqrt{3}}{2})$$

$$f \text{ odd} \rightarrow f(x_n) = -f(-x_n) < x_n$$

$$\therefore o(x) \text{ is enclosed in } (0, \frac{\sqrt{3}}{2}) \wedge o(x) \text{ decreasing}$$

$$o(x) \text{ convergent} \wedge 0 \text{ fixed point} \rightarrow o(x) \xrightarrow{n} 0$$

IV $\forall x \in \mathbb{R} \quad \text{,, } x > \frac{\sqrt{3}}{2}$:

Induction over n :

$$-x_n \in (\frac{\sqrt{3}}{2}, \infty)$$

$$\underline{\text{I}} \rightarrow f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty) \wedge f(-x_n) < -x_n$$

$$f \text{ odd} \rightarrow f(x_n) = -f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty)$$

$$f \text{ odd} \rightarrow f(x_n) = -f(-x_n) > x_n$$

$$\therefore o(x) \text{ is inf bounded by in } \frac{\sqrt{3}}{2} \wedge o(x) \text{ increasing}$$

$$o(x) \text{ convergent}$$

$$\nexists x > \frac{\sqrt{3}}{2} \quad \text{,, } x \text{ fixed point} \rightarrow o(x) \xrightarrow{n} +\infty$$