

I. Power series

Power series

Study:

$$\cdot \sum_{n \geq 1} n(n+1)z^n$$

Start:

Convergence domain:

$$\lim_n \frac{n(n+1)}{(n+1)(n+1)} = 1$$

Quotient test:

$$\overline{\lim}_n |c_n|^{\frac{1}{n}} = 1 \rightarrow \left(\overline{\lim}_n |c_n|^{\frac{1}{n}} \right)^{-1} = 1$$

Cauchy-Hadamard theorem:

$$\sum_{n \geq 1} n(n+1)z^n \text{ absolutely convergent over } \mathbb{D}$$

$$\sum_{n \geq 1} n(n+1)z^n \text{ divergent over } \mathbb{C} \setminus \overline{\mathbb{D}}$$

$$\forall K \subset \mathbb{D} \quad K \text{ compact} :$$

$$\sum_{n \geq 1} n(n+1)z^n \text{ uniformly convergent over } K$$

Sum:

Consider:

$$\begin{aligned} f : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n \geq 1} n(n+1)z^n \\ g : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n \geq 1} n(n+1)z^{n-1} \\ h : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n \geq 1} (n+1)z^n \end{aligned}$$

UCI theorem:

$\forall z \in \mathbb{D}$:

$$\int_0^z h(t) dt = \sum_{n \geq 1} z^{n+1} = \sum_{n \geq 0} z^n = \frac{1}{1-z}$$

$$h(z) = \partial_z \frac{1}{1-z} = \frac{1}{(1-z)^2}$$

$$\int_0^z g(t) dt = \sum_{n \geq 1} (n+1) z^n = h(z)$$

$$g(z) = \partial_z h(z) = \frac{2}{(1-z)^3}$$

$$f(z) = zg(z) = \frac{2z}{(1-z)^3}$$

Application:

$$\sum_{n \geq 1} (-1)^n \frac{n(n+1)}{2^n} = f\left(-\frac{1}{2}\right) = \frac{-2^3}{3^3}$$