

1. New

Power series theorem

Let:

.

Then, holds:

· *Part II, III, IV*

Demonstration:

Follow 3 steps

Step 1 : Uniform convergence in compacts of $D(a, R)$:

$$\begin{aligned} g_n : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto c_n(z-a)^n \end{aligned}$$

$$\forall \rho \in \mathbb{R} \quad \text{,} \quad \rho < R :$$

$$\forall z \in \overline{D(a, \rho)} :$$

$$|g_n(z)| = |c_n||z-a|^n \leq |c_n|\rho^n$$

$$M_n := |c_n|\rho^n$$

$$\lim_n (|c_n|\rho^n)^{1/n} = \rho \lim_n |c_n|^{1/n} = \frac{\rho}{R} < 1$$

$$\text{Root test} \rightarrow \sum_{n \geq 0} M_n \text{ convergent}$$

$$\text{M-Weierstrass} \rightarrow \sum_{n \geq 0} c_n(z-a)^n \text{ uniformly convergent}$$

over compacts of $D(a, \rho)$

$$\sum_{n \geq 0} c_n(z-a)^n \text{ uniformly convergent over compacts of } D(a, r)$$

$$f(z) := \sum_{n \geq 0} g_n(z)$$

g uniformly convergent $\rightarrow f$ continuous

Step 2 : IV. R' :

$$\tilde{f}(z) := \sum_{n \geq 1} n c_n (z - a)^{n-1}$$

$$\tilde{f}(z) = \sum_{n \geq 0} (n+1) c_{n+1} (z - a)^n$$

$$\frac{1}{R'} = \lim_n (n+1) |c_{n+1}|^{1/n} = \lim_n (n+1)^{1/n} |c_{n+1}|^{1/n} = \lim_n (|c_{n+1}|^{1/n+1})^{\frac{n+1}{n}} =$$

$$\frac{1}{R}$$

$$R' = R$$

Step 3 : III :

\tilde{f} well defined in $D(a, R)$

$\forall z_0 \in D(a, R) :$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - \tilde{f}(z_0) \right| \xrightarrow{n} 0?$$

$\forall n \in \mathbb{N} :$

$$S_n(z) := \sum_{k=0}^n f_k(z)$$

$$R_n(z) := \sum_{k \geq n+1} f_k(z)$$

$$f(z) = S_n(z) + R_n(z)$$

$$\tilde{f}(z) = \tilde{S}_n(z) + \tilde{R}_n(z)$$

$$R_n(z), \tilde{R}_n(z) \xrightarrow{n} 0$$

$$\forall \varepsilon \in \mathbb{R}^+ :$$

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| = \left| \frac{S_n(z)-S_n(z_0)}{z-z_0} + \frac{R_n(z)-R_n(z_0)}{z-z_0} - \tilde{S}_n(z_0) - \tilde{R}_n(z_0) \right|$$

$$\forall \rho \in \mathbb{R} \quad |z_0 - a| < \rho < R :$$

$$|\tilde{R}_n(z_0)| \leq \sum_{k \geq n+1} k |c_k| \rho^{k-1} < \frac{\varepsilon}{3} (n \geq n_1)$$

$$\begin{aligned} \left| \frac{R_n(z)-R_n(z_0)}{z-z_0} \right| &\leq \sum_{k \geq n+1} |c_k| \left| \frac{(z-a)^k - (z_0-a)^k}{z-z_0} \right| \\ \frac{a^k - b^k}{a-b} &= a^{k-1} - a^{k-2}b + \dots + b^{k-1} \\ &\leq \sum_{k \geq n+1} |c_k| (|z-a|^{k-1} + |z-a|^{k-2}|z_0-a| + \dots + \\ &\quad |z_0-a|^{k-1} \end{aligned}$$

$$|z-a|, |z_0| < \rho$$

$$\leq \sum_{k \geq n+1} |c_k| k \rho^{k-1} < \frac{\varepsilon}{3} (n \geq n_1)$$

$$\left| \frac{S_n(z)-S_n(z_0)}{z-z_0} - \tilde{S}_n(z_0) \right| < \frac{\varepsilon}{3} (S'_n = \tilde{S}_n) (n \geq n_2)$$

$$\forall n \in \mathbb{N} \quad n \geq \max(n_1, n_2) :$$

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| < \varepsilon$$

functions associated to series are \mathcal{C}^∞

Let:

$$\cdot f(z) = \sum c_n (z - a)^n n[0] \text{ series}$$

$$\cdot R \text{ radius of convergence of } f$$

Then, holds:

$$\cdot f \in \mathcal{C}^\infty \text{ over } D(a, R)$$

$$\cdot \forall n \in \mathbb{N} :$$

$$f^{(n)} \in \mathcal{H}(D(a, R))$$

$$\cdot c_k = \frac{f^{(k)}}{k!}$$

$$\cdot \text{ series associated to } f \text{ is unique}$$

Demonstration:

$$f^{(k)}(z) = \sum n(n-1) \dots (n-k+1) c_n (z-a)^{n-k} n[k]$$

$$f^{(k)}(a) = k! c_k$$

$$c_k = \frac{f^{(k)}(a)}{k!}$$

Geometric series

Let:

$$\cdot a : 0$$

$$\cdot c_n : 0$$

Then, $\sum z^n n[0]$ is convergent in \mathbb{D} :

$$R = \frac{c_n}{c_{n+1}} = 1 \quad \text{Then, holds:}$$

$$\cdot \sum z^n n[0] = \frac{1}{1-z}$$

$$\cdot \sum n z^{n-1} n[0] = \frac{1}{(1-z)^2}$$

$$\cdot \sum \frac{z^{n+1}}{n+1} n[0] = -\log(1-z)$$

Demonstration:

$$\forall z \in \mathbb{D} :$$

$$\sum z^n n[0] \text{ geometric series}$$

$$\sum z^n n[0] = \frac{1}{1-z}$$

II differentiating

III integrating

Series not centered in 0

Let:

$$\cdot a : i$$

$$\cdot c_n : \frac{n+1}{5^{n+1}}$$

Then, *item* is a/an entity :

$$\begin{aligned} & \sum \frac{n(z-i)^{n-1}}{5^n} n[1] \\ &= \frac{1}{5} \sum n \frac{z-i}{5} n^{n-1} n[1] = \frac{1}{5} \sum n u^{n-1} n[1] \end{aligned}$$

$$S(u) = \tilde{S}'(u)$$

$$\tilde{S}(u) = \frac{1}{5} \sum u^n n[1] = \frac{u}{5(1-u)}$$

$$S(u) = \frac{1}{5(1-u)^2}$$

$$S(z) = \frac{5}{(5+i-z)^2} \text{ over } D(i, 5)$$

Radius of convergence without quotient test

Let:

$$\cdot \sum \frac{(-1)^n}{n(n+1)} (z-2)^{n(n+1)} n[1]$$

Then, *R* is a/an entity :

$$\begin{aligned} & \lim_{c_{n+1}} c_n \nexists \\ & \lim_n \frac{1}{n(n+1)} \frac{1}{\frac{1}{n(n+1)}} = 1 \\ & \text{ignore zeros} \end{aligned}$$