Statistics

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Block I

Definitions

1. Statistical models & Statistics

introduction

Statistical model

Let:

- Ω set
- $\mathcal{A}$ sigma-algebra over Ω
- $\mathcal{P} \subset \mathcal{F}(\Omega, [0,1])$

Then, $(\Omega, \mathcal{A}, \mathcal{P})$ is a statistical model if:

- $\forall\ P \in \mathcal{P}$:

  $(\Omega, \mathcal{A}, P)$ probability space

Parametrization

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ statistical model
- $\Theta \subset \mathbb{R}^d$

Then, $(\Omega, \mathcal{A}, \mathcal{P})$ is parametrized by Θ if:

- $\mathcal{P} = \{P_\theta\}_{\theta \in \Theta}$

We denote:

- $(\Omega, \mathcal{A}, \mathcal{P})\ :\ (\Omega, \mathcal{A}, \{P_\theta\}_{\theta \in \Theta})$ d-D statistical model

Likelihood function

Let:

$\cdot\ (\Omega, \mathcal{A}, \{P_\theta\}_{\theta\in\Theta})$ d-D statistical model

We name likelihood function of $(\Omega, \mathcal{A}, \mathcal{P})$ to:

$$\cdot \quad \begin{array}{rccc} L6: & \Omega\times\Theta & \longrightarrow & [0,1] \\ & (x,\theta) & \longmapsto & P_\theta(x) \end{array}$$

Exponential model

Let:

$\cdot\ (\Omega, \mathcal{A}, \{P_\theta\}_{\theta\in\Theta})$ d-D statistical model

$\cdot\ L$ likelihood function of $(\Omega, \mathcal{A}, \mathcal{P})$

Then, $(\Omega, \mathcal{A}, \mathcal{P})$ is exponential if:

$\cdot\ \exists\ \{f_i\}_{i=0}^r \subset \mathcal{F}(\Omega, \mathbb{R})$:

$\forall\ i \in [0, r]_{\mathbb{N}}$:

f_i measurable

$\cdot\ \exists\ \{\phi_i\}_{i=0}^r \subset \mathcal{F}(\Theta, \mathbb{R})$:

$\forall\ x \in \Omega$:

$\forall\ \theta \in \Theta$:

$$L(x,\theta) = \exp(\sum_{i=1}^r f_i(x)\phi_i(\theta) + f(x) + \phi(\theta))$$

Statistic

Let:

- $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta \in \Theta})$ d-D statistical model
- $f : (\Omega, \mathcal{A}) \to (\mathbb{R}^m, \mathbb{B}(\mathbb{R}^m))$

Then, f is a statistic if:

- f measurable

Sufficiency

Let:

- $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta \in \Theta})$ d-D statistical model
- (M, Σ) measurable space
- $X : M \to \Omega$ random variable
- $T : \Omega \to \mathbb{R}^m$ statistic

Then, T is sufficient if:

- $\forall\ \theta_1, \theta_2 \in \Theta :$

  $\forall\ x \in M :$

  $\forall\ t \in \mathbb{R}^m :$

$$P_{\theta_1}(X = x \mid T \circ X = t) = P_{\theta_2}(X = x \mid T \circ X = t)$$

2. Information & Decision

Regularity

Let:

- $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta\in\Theta})$ 1-D **real** statistical model
- L likelihood function of $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta\in\Theta})$

Then, $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta\in\Theta})$ is regular if:

- Θ open
- $\forall\ \theta_1, \theta_2 \in \Theta$:

  $$\{x \in \Omega \mid L(x, \theta_1) = 0\} = \{x \in \Omega \mid L(x, \theta_2) = 0\}$$

- $\forall\ \theta \in \Theta$:

  $\exists\ f\ :\ \Omega \to \mathbb{R}^+$:

  $\exists\ \mathcal{E}_\theta \subset \Theta$:

  $\mathcal{E}_\theta$ neighborhood of θ

  $\forall\ \theta' \in \mathcal{E}_\theta$:

  $$\mid \partial_\theta \log(L(x,\theta)) \mid \vee \mid \partial_{\theta^2} \log(L(x,\theta))\theta \mid\ \le$$

$f(x)$

- $\forall\ \theta \in \Theta$:

  $$E_x\left(\mid \partial_\theta \log(L(x,\theta)) \mid^2 \right) \text{ finite}$$

Fisher's information

Let:

$\cdot$ $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta \in \Theta})$ 1-D regular statistical model

$\cdot$ L likelihood function of $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta \in \Theta})$

We name Fisher's information of $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta \in \Theta})$ to:

$$\cdot \begin{array}{cccc} f : & \Theta & \longrightarrow & \mathbb{R} \\ & \theta & \longmapsto & E_x\Big(\mid \log(L(x,\theta)) \mid^2 \Big) \end{array}$$

We denote:

$\cdot$ $f(\theta) \, : \, I_F(\theta)$

Kullback's information

Let:

$\cdot$ $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta \in \Theta})$ d-D statistical model

We name Kullback's information of $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta \in \Theta})$ to:

$$\cdot \begin{array}{cccc} f : & \Theta^2 & \longrightarrow & \mathbb{R} \\ & (\theta_1, \theta_2) & \longmapsto & E_{\theta_2}\Big(\log(\frac{L(x,\theta_1)}{L(x,\theta_2)}) \Big) \end{array}$$

We denote:

$\cdot$ $f((\theta_1, \theta_2)) \, : \, I_K(\theta_1 \mid \theta_2)$

Decision

Let:

· $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta\in\Theta})$ d-D **measurable** statistical model

· $(D, \mathcal{D})$ measurable space

· $f : \Omega \to D$

Then, f is a decision if:

· f measurable

We denote:

· $\{f : \Omega \to D \mid f \text{ measurable }\} : \Xi$

Decision order

Let:

· $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta\in\Theta})$ d-D **measurable** statistical model

· $(D, \mathcal{D})$ measurable space

· $f_1, f_2 : \Omega \to D$ decisions

Then, f_1 is better than f_2 if:

· *conditions*

·

Loss function

Let:

· $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta \in \Theta})$ d-D statistical model

· $(D, \mathcal{D})$ measurable space

· $W : D \times \Theta \to \mathbb{R}^+$

Then, W is a loss function if:

· W measurable

· $\forall\, d \in D$ 〃 d correct :

$$W(d, \theta) = 0$$

· $\forall\, d_1, d_2 \in D$ 〃 d_1 better than d_2 :

$$W(d_1, \theta) \leq W(d_2, \theta)$$

Risk function

Let:

· $(\Omega, \mathcal{A}, \{P_\theta\}_{\theta \in \Theta})$ d-D statistical model

· $(D, \mathcal{D})$ measurable space

· $W : D \times \Theta \to \mathbb{R}^+$ loss function

We name risk function of W to:

$$\cdot \quad \begin{array}{rccc} R : & \Xi \times \Theta & \longrightarrow & \mathbb{R} \\ & (\chi, \theta) & \longmapsto & E_x\left(\, (W(\chi(x), \theta))\, \right) \end{array}$$

3. Punctual Estimations

introduction

UMV

Let:

· *statements*

·

Then, *item* is a/an entity if:

· *conditions*

·

We denote:

· *property* : *notation*

·

Same conditions

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ parametric statistical model
- $X : \Omega \to \mathbb{R}$ random variable
- $\Theta \subset \mathbb{R}$ interval
- χ_F real estimator with integrable quadratic
-

Then, *item* is a/an entity if:

- $\forall\ \theta \in \Theta$:

  $\exists\ h : \mathbb{R} \to \mathbb{R}$:

  $h \geq 0$

  h integrable

  $\exists\ \mathcal{U} \subset \mathbb{R}$:

  $\theta \in \mathcal{U}$

  $|T(x)\partial_\theta L(x, \theta)| \leq h$

Efficient

Let:

- mismas condiciones

Then, T is efficient if:

- $Var_\theta T = \frac{g'(\theta)^2}{I(\theta)}$

Block II

Propositions

1. Statistic models

introduction

Characterization of regular exponential models

Let:

$\cdot$ $(\Omega, \mathcal{A}, \mathcal{P})$ 1-D exponential model parametrized by Θ

$\cdot$ $\exp(f(x)\phi(\theta) + f'(x) + \phi'(\theta))$ likelihood function of $(\Omega, \mathcal{A}, \mathcal{P})$

statistical model

Then, holds:

$\cdot$ Θ interval , $\phi, \phi' \in \mathcal{C}^2$

$\cdot$ $\forall\ \theta \in \Theta$:

$\phi'(\theta) \neq 0$

$E_\theta f^2(x) \in \mathbb{R}$

$\cdot$ $\rightarrow (\Omega, \mathcal{A}, \mathcal{P})$ regular

Demonstration:

no demonstration

2. Statistic models

introduction

Characterization of regular exponential models

Let:

- $(\Omega, \mathcal{A}, \mathcal{P})$ 1-D exponential model parametrized by Θ
- $\exp(f(x)\phi(\theta) + f'(x) + \phi'(\theta))$ likelihood function of $(\Omega, \mathcal{A}, \mathcal{P})$

statistical model

Then, holds:

- Θ interval , $\phi, \phi' \in \mathcal{C}^2$
- $\forall\ \theta \in \Theta$:

  $\phi'(\theta) \neq 0$

  $E_\theta f^2(x) \in \mathbb{R}$
- $\rightarrow (\Omega, \mathcal{A}, \mathcal{P})$ regular

Demonstration:

no demonstration

Block III

Examples

1. Statistic models

introduction

Exponential model

Let:

- $\cdot\, \Omega : \mathbb{R}^n$
- $\cdot\, \mathcal{A} : \mathbb{B}(\mathbb{R}^n)$
- $\cdot\, \theta : (\mu, \sigma^2)$
- $\cdot\, \Theta : \mathbb{R} \times \mathbb{R}^+$
- $\cdot\, \mathcal{P} : \{N(\theta)\}_{\theta \in \Theta}$

Then, $(\Omega, \mathcal{A}, \mathcal{P})$ is an exponential model :

$\forall\ x \in \mathbb{R}^n$:

$$f_\theta(x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(x_i - \mu)^2)$$
$$= (\tfrac{1}{\sqrt{2\pi\sigma^2}})^n \exp(-\tfrac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2)$$
$$L(x,\theta) = \exp(\tfrac{n}{2}\log(\tfrac{1}{2\pi\sigma^2}) - \tfrac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2 + \frac{n\bar{x}\mu}{\sigma^2} - \frac{n\mu^2}{2\sigma^2})$$
$$\begin{array}{rcccc} \phi & : & \Theta & \longrightarrow & \mathbb{R}^2 \\ & & (\mu,\sigma^2) & \longmapsto & (-\tfrac{\mu}{2\sigma^2}, \tfrac{n\mu}{\sigma^2}) \end{array}$$
$$\begin{array}{rcccc} \phi' & : & \Theta & \longrightarrow & \mathbb{R} \\ & & (\mu,\sigma^2) & \longmapsto & \tfrac{n}{2}\log(\tfrac{1}{2\pi\sigma^2}) - \tfrac{n\mu^2}{2\sigma^2} \end{array}$$
$$\begin{array}{rcccc} f & : & \Omega & \longrightarrow & \mathbb{R}^2 \\ & & x & \longmapsto & (\sum_{i=1}^{n} x_i^2, \bar{x}) \end{array}$$
$$L(x,\theta) = \exp(\phi'(\theta) - \phi(\theta)f(x))$$

$\therefore)$ $(\Omega, \mathcal{A}, \mathcal{P})$ exponential model

Block IV

Problems

1. Statistic models

introduction

go

Block V

Tasks