1. One-dimensional discrete dynamical systems

Dynamical system

Let:

$\cdot$ M manifold

$\cdot$ T monoid

$\cdot$ $\phi \;:\; M \times T \to M$

Then, (M, T, ϕ) is a dynamical system if:

$\cdot$ $\forall\; x \in X \;:$

$\phi(x, 0) = 0$

$\forall\; t_1, t_2 \in T \;:$

$$\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$$

Dimension

Let:

$\cdot$ (M, T, ϕ) dynamical system

We name dimension of (M, T, ϕ) to:

$\cdot$ $\dim(M)$

We denote:

$\cdot$ $dim(M) = n \;:\; (M, T, \phi)$ n-D dynamical system

Discrete & Continuous

Let:

- (M, T, ϕ) dynamical system

Then, (M, T, ϕ) is discrete if:

- $T \underset{\sim}{\subset} \mathbb{N}$

Then, (M, T, ϕ) is continuous if:

- $T \subset \mathbb{R}$
- T open

Defined by a function

Let:

- (M, T, ϕ) dynamical system
- $f : M \to M$

Then, (M, T, ϕ) is a dynamical system defined by f if:

- $T = \mathbb{N}$
- $\begin{array}{rcl} \phi : M \times \mathbb{N} & \longrightarrow & M \\ (x, n) & \longmapsto & f^n(x) \end{array}$

We denote:

- (M, T, ϕ) dynamical system defined by f : $(M, \mathbb{N}, f)$
- $f \in \mathcal{C}^n(M)$: $(M, \mathbb{N}, f)$ $\mathcal{C}^n$ dynamical system

Orbit

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$

We name orbit of x to:

- $\{f^n(x)\}_{n \in \mathbb{N}}$

We denote:

- $o(x)$

n-periodic point

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system
- $x \in M$
- $n \in \mathbb{N}$

Then, x is a n-periodic point if:

- $f^n(x) = x$
- $\forall\ n' \in \mathbb{N}\ \ „\ \ n' < n$:

  $f^{n'}(x) \neq x$

We denote:

- $\{x \in M \mid f(x) = x\}$: $Fix(f)$

Attractive & Repulsive

Let:

$\cdot\ (M, \mathbb{N}, f)$ metrical dynamical system

$\cdot\ x \in M$ m-periodic point

Then, x is attractive if:

$\cdot\ \exists\, \mathcal{U} \in M$:

$\mathcal{U}$ open

$\forall\ x' \in \mathcal{U}$:

$\exists\ N \in \mathbb{N}$:

$\forall\ n \in \mathbb{N}\ „\ n \geq N$:

$f^{nm}(x') \in \mathcal{U}$

Then, x is repulsive if:

$\cdot\ \forall\, \mathcal{U} \subset M\ „\ \mathcal{U}$ open $\wedge\ x \in \mathcal{U}$:

$\forall\ x' \in \mathcal{U}\ „\ x' \neq x$:

$\exists\ N \in \mathbb{N}$:

$\forall\ n \in \mathbb{N}\ „\ n \geq N$:

$f^{nm}(x') \notin \mathcal{U}$

Fixed point character

Let:

- $(M, \mathbb{N}, f)$ functional dynamical system

We name Fixed point character to:

- $$\begin{aligned} f\ : Fix(f) &\longrightarrow \{+,-\} \\ x &\longmapsto \begin{cases} + & x \text{ repulsive} \\ - & x \text{ attractive} \end{cases} \end{aligned}$$

We denote:

- $f\ :\ \chi$

Attraction set

Let:

- $(M, \mathbb{N}, f)$ dynamical system
- $x \in M$ attractive m-periodic point
- $o(x)$ orbit of x

We name attraction set of x to:

- $\{y \in M \mid \exists\, x' \in o(x)\ :\ f^{nm}(y) \xrightarrow{n} x'\}$

We denote:

- $A(x)$

Neutral point

Let:

- $(M, \mathbb{N}, f)$ differentiable dynamical system
- $x \in M$

Then, x is a neutral point if:

- $f'(x) \in \{-1, 1\}$

Feeble point

Let:

- $(M, \mathbb{N}, f)$ $\mathcal{C}^3$ dynamical system
- $x \in M$ neutral point

Then, x is feeble point if:

- $f''(x) = 0$

Saddle point

Let:

- $\mathcal{U} \subset \mathbb{R}^n$
- $f \in \mathcal{C}^1(\mathcal{U})$
- $x \in \mathcal{U}$

Then, x is a saddle point if:

- $f'(x) = 0$

Homeomorphism

Let:

- $(X_1, \tau_1), (X_2, \tau_2)$ topological spaces
- $f : X_1 \to X_2$

Then, f is a homeomorphism if:

- f biyective
- $f \in \mathcal{C}(X_1)$
- $f^{-1} \in \mathcal{C}(X_2)$

We denote:

- $\{f : X_1 \to X_2 \mid f \text{ homeomorphism } \} : Homeo(X_1)$

Multiplier

Let:

· $(M, \mathbb{N}, f)\mathcal{C}^1$ dynamical system

· $x \in M$

We name multiplier of x to:

· $f'(x)$

Logistic

Let:

· $f : \mathbb{R} \to \mathbb{R}$

· (M, T, ϕ) dynamical system defined by f

Then, (M, T, ϕ) is a logistic dynamical system if:

· $\exists\, a \in \mathbb{R}$:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto ax(1-x) \end{aligned}$$

Chaos

Let:

- $f : \mathbb{R} \to \mathbb{R}$
- (M, T, ϕ) dynamical system defined by f

Then, (M, T, ϕ) is chaotic if:

- $Fix(f)$ dense in $\mathbb{R}$
- $\exists\, x \in \mathbb{R}$:

  $o(x)$ dense in $\mathbb{R}$
- f sensibility of x_0

Sarkovskii's order

We name Sarkovskii's order to:

- $a = 2^n a', b = 2^m b'$
- $a < b \leftrightarrow \begin{cases} m < n & a' = b' = 1 \\ * & a' = 1, b' \neq 1 \\ a' < b' & a' = b' \neq 1 \\ n < m & 1 \neq a' \neq b' \end{cases}$

We denote:

- $a < b \,:\, a \prec b$

Topologically equivalent

Let:

· $(M, \mathbb{N}, f)$ functional dynamical system

· $(M, \mathbb{N}, f')$ functional dynamical system

Then, $(M, \mathbb{N}, f)$ is topologically equivalent to $(M, \mathbb{N}, f')$if:

· $Fix(f) = Fix(f')$

· $\forall\ x \in Fix(f)$:

$$character_f(x) = character_{f'}(x)$$

We denote:

· $(M, \mathbb{N}, f)\ (M, \mathbb{N}, f')$

Bifurcation point

Let:

$\cdot\ (M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ functional dynamical system family

$\cdot\ \lambda' \in \Lambda$

Then, λ' is a bifurcation value if:

$\cdot\ \forall\ \varepsilon \in \mathbb{R}^+ :$

$\exists\ \lambda'' \in (\lambda' - \varepsilon, \lambda' + \varepsilon) :$

$(M, \mathbb{N}, f_{\lambda''})$ not topologically equivalent to $(M, \mathbb{N}, f_{\lambda'})$