# Block I

# Laboratory

# 1. Orbit analysis

**Martin Azpillaga**

Let:

$$\cdot \quad \begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^3 + \tfrac{1}{4}x \end{aligned}$$

Study:

· Orbit behavior of the real dynamical system defined by $f$

Demonstration:

Formalization :

Consider $(M, T, \phi)$ where:

$M = \mathbb{R}$

$T = \mathbb{N}$

$$\begin{aligned} \phi : \mathbb{R} \times \mathbb{N} &\longrightarrow \mathbb{R} \\ (x, n) &\longmapsto f^n(x) \end{aligned}$$

Study the orbits of $(\mathbb{R}, \mathbb{N}, \phi)$

We will denote $f^n(x)$ as $x_n$

Fixed points :

$\forall\ x \in \mathbb{R}$ :

$$f(x) = x \leftrightarrow x^3 + \tfrac{1}{4}x - x = 0 \leftrightarrow x^3 - \tfrac{3}{4}x = 0$$

$$\leftrightarrow x = 0 \ \vee \ x^2 - \tfrac{3}{4} = 0$$

$$x \text{ fixed point} \leftrightarrow x \in \{0, \pm\tfrac{\sqrt{3}}{2}\}$$

Graphic analysis :

Parity:

$\forall\ x \in \mathbb{R}$ :

$$f(-x) = (-x)^3 + \frac{(-x)}{4} = -(x^3 + \frac{x}{4}) = -f(x)$$

$f$ is odd

Monotonicity:

$\forall\ x \in \mathbb{R}$ :

$$f'(x) = 3x^2 + \frac{1}{4} > 0$$

$f$ is increasing over $\mathbb{R}$

Convexity:

$\forall\ x \in \mathbb{R}^-$ :

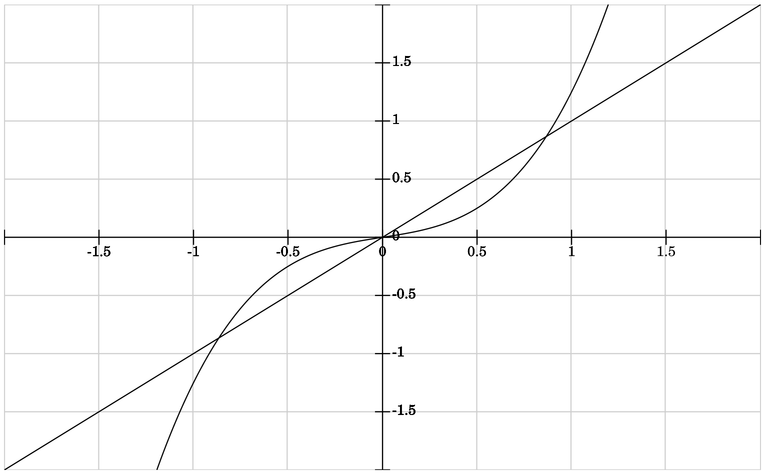
$$f''(x) = 6x \leq 0$$

$\forall\ x \in \mathbb{R}^+$ :

$$f''(x) = 6x \geq 0$$

$f$ is concave over $\mathbb{R}^-$ and convex over $\mathbb{R}^+$

Graphic representation :




$\underline{\text{I}}\forall\ x \in (-\infty, -\frac{\sqrt{3}}{2})$ :

Induction over $n$:

$$f \text{ incresing} \to f(x_n) < f(-\tfrac{\sqrt{3}}{2})$$

$$x_{n+1} \in (-\infty, -\tfrac{\sqrt{3}}{2})$$

$\therefore)\ o(x)$ is enclosed in $(-\infty, -\frac{\sqrt{3}}{2})$

Induction over $n$:

$$x_n^2 > \tfrac{3}{4} \to (x_n^2 - \tfrac{3}{4}) > 0$$

$$x_{n+1} - x_n = x_n^3 - \tfrac{3}{4}x_n = x_n(x_n^2 - \tfrac{3}{4}) < 0$$

$\therefore)\ o(x)$ decreasing

$\nexists\ x < -\frac{\sqrt{3}}{2}$ „ $x$ fixed point $\to o(x) \xrightarrow{n} -\infty$

II $\forall\ x \in (-\frac{\sqrt{3}}{2}, 0)$ :

Induction over $n$:

$f$ increasing $\to f(-\frac{\sqrt{3}}{2}) < f(x_n) < f(0)$

$x_{n+1} \in (-\frac{\sqrt{3}}{2}, 0)$

$\therefore)$ $o(x)$ is enclosed in $(-\frac{\sqrt{3}}{2}, 0)$

Induction over $n$:

$x_n^2 < \frac{3}{4} \to (x_n^2 - \frac{3}{4}) < 0$

$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) > 0$

$\therefore)$ $o(x)$ increasing

$o(x)$ convergent $\wedge\ 0$ fixed point $\to o(x) \xrightarrow{n} 0$

III $\forall\ x \in (0, \frac{\sqrt{3}}{2})$ :

Induction over $n$:

$-x_n \in (-\frac{\sqrt{3}}{2}, 0)$

II $\to f(-x_n) \in (-\frac{\sqrt{3}}{2}, 0)\ \wedge\ f(-x_n) > -x_n$

$f$ odd $\to f(x_n) = -f(-x_n) \in (0, \frac{\sqrt{3}}{2})$

$f$ odd $\to f(x_n) = -f(-x_n) < x_n$

$\therefore)$ $o(x)$ is enclosed in $(0, \frac{\sqrt{3}}{2})\ \wedge\ o(x)$ decreasing

$o(x)$ convergent $\wedge\ 0$ fixed point $\to o(x) \xrightarrow{n} 0$

IV $\forall\ x \in \mathbb{R}$ „ $x > \frac{\sqrt{3}}{2}$ :

Induction over $n$:

$-x_n \in (\frac{\sqrt{3}}{2}, \infty)$

$\underline{\text{I}} \to f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty) \ \wedge \ f(-x_n) < -x_n$

$f$ odd $\to f(x_n) = -f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty)$

$f$ odd $\to f(x_n) = -f(-x_n) > x_n$

$\therefore)$ $o(x)$ is inf bounded by in $\frac{\sqrt{3}}{2}$ $\wedge$ $o(x)$ increasing

$o(x)$ convergent

$\nexists\, x > \frac{\sqrt{3}}{2}$ „ $x$ fixed point $\to o(x) \xrightarrow{n} +\infty$

## 2. Fixed points cardinality

## III. Martin Azpillaga

Let:

$\cdot\ f\ :\ [0,1] \to [0,1] \in \mathcal{C}^2([0,1])$

$\cdot\ f(1) < 1$

$\cdot\ f'' > 0 \in [0,1]$

Show that:

$\cdot\ \#\ \{x \in [0,1] \mid f(x) = x\} = 1$

Demonstration:

$\#\ \{x \in [0,1] \mid f(x) = x\} \geq 1$:

Case $f(0) = 0$:

0 fixed point

Case $f(0) > 0$:

$$\begin{array}{rcl} g\ : [0,1] & \longrightarrow & [-1,1] \\ x & \longmapsto & f(x) - x \end{array} \in \mathcal{C}^2([0,1])$$

$g(0) = f(0) - 0 > 0$

$g(1) = f(1) - 1 < 0$

Bolzano's theorem:

$\exists\ x \in (0,1)$ :

$g(x) = 0$

$f(x) = x$

$\#\ \{x \in [0,1] \mid f(x) = x\} \leq 1$:

$g'' > 0$ over $[0,1]$

Rolle's theorem:

$\#\ \{x \in (0,1) \mid g'(x) = 0\} \leq 1$

$\#\ \{x \in (0,1) \mid g(x) = 0\} \leq 2$

$\#\ \{x \in (0,1) \mid f(x) = x\} \leq 2$

$f'' > 0$ over $[0,1]$

Monotonicity test:

$f'$ increasing in $[0,1]$

$\forall\ a < b \in [0,1)\ \shortparallel\ f(a) = a, f(b) = b$ :

Mean Value Theorem:

$\exists\ c \in (a,b)$ :

$f'(c) = \frac{f(b)-f(a)}{b-a} = 1$

$\exists\ d \in (b,1)$ :

$$\begin{aligned} &f'(d) = \tfrac{f(1)-f(b)}{1-b} < 1 \\ &f' \text{ increasing} \to f'(c) < f'(b) < f'(d) \\ &1 < f'(b) < 1 \text{ absurd} \end{aligned}$$

$\therefore)\ \#\{x \in [0,1] \mid f(x) = x\} = 1$

## IV. Bifurcation Theory

### Bifurcation diagram

Let:

$$\cdot \begin{array}{rcl} f\ :\mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (a,x) & \longmapsto & x^3 - 3x^2 + (5-a)x - 2 + a \end{array}$$

$\cdot\ \forall\ a \in \mathbb{R}$ :

$$\begin{array}{rcl} f_a\ :\mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^3 - 3x^2 + (5-a)x - 2 + a \end{array}$$

Study:

$\cdot$ Bifurcations of $(\mathbb{R}, \mathbb{N}, \{f_a\}_{a\in\mathbb{R}})$

Start:

Fixed points:

$f_a(x) = x \leftrightarrow x^3 - 3x^2 + (4-a)x - 2 + a = 0 \leftrightarrow (x-1)(x^2 - 2x + 2 - a) = 0$

$x^2 - 2x + 2 - a = 0 \leftrightarrow x = \pm\sqrt{a-1}$

$\forall\ a \in \mathbb{R}$ " $a \le 1$ :

$\mathrm{Fix}(f_a) = \{1\}$

$\forall\ a \in \mathbb{R}$ " $a > 1$ :

$\mathrm{Fix}(f_a) = \{1, \pm\sqrt{a-1}\}$

Stability:

$\partial_x f(a,x) = 3x^2 - 6x + 5 - a$

$\partial_{x^2} f(a,x) = 6x - 6$

$\partial_{x^3} f(a,x) = 6$

$|\partial_x f(a,1)| < 1 \leftrightarrow |2-a| < 1 \leftrightarrow a \in (1,3)$

$\partial_{x^2} f(1,1) = 0, \;\; \partial_{x^3} f(1,1) > 0$

$\partial_{x^2} f(3,1) = 0, \;\; \partial_{x^3} f(3,1) > 0$

$\forall\ a \in \mathbb{R}$ „ $a \leq 1 \ \vee \ a \geq 3$ :

  1 repulsive

$\forall\ a \in \mathbb{R}$ „ $a \in (1,3)$ :

  1 attractive

$\forall\ a \in \mathbb{R}$ „ $a > 1$ :

  $\left|\partial_x f(a, \pm\sqrt{a-1})\right| = |2a-1| > 1$

  $\pm\sqrt{a-1}$ repulsive

Pitchfork bifurcation at 1 :

$\partial_a f(1,1) = 1 - 1 = 0$

$\partial_{x^2} f(1,1) = 6 - 6 = 0$

$\partial_{ax} f(1,1) = -1 \neq 0$

$\partial_{x^3} f(1,1) = 6 \neq 0$

Period-doubling bifurcation at 3 :

$\partial_a f^2(3,1) = \partial_a f(f(3,1))\partial_a f(3,1) = 0$

$\partial_{x^2} f^2(3,1) = \partial_{x^2} f(f(3,1))\partial_{x^2} f(3,1) = 0$

$$\partial_{ax} f^2(3,1) = \partial_{ax} f(f(3,1))\partial_{ax} f(3,1) \neq 0$$

$$\partial_{x^3} f^2(3,1) = \partial_{x^3} f(f(3,1))\partial_{x^3} f(3,1) \neq 0$$

Source Code

```
#include "stdio.h"
#include "stdlib.h"
#include "math.h"
#include "string.h"

void plot( char *input_file, char *output_file )
{
  FILE *gnuplot;
  gnuplot = popen("gnuplot", "w");
  if( output_file )
  {
    fprintf(gnuplot, "set term svg\n");
    fprintf(gnuplot, "set out \"%s\"\n", output_file );
  }
  fprintf(gnuplot, "plot \"%s\" with dots\n", input_file);
  fflush(gnuplot);
  fclose(gnuplot);
}

double example_function( double param, double point )
{
  return pow(point,3) - 3*pow(point,2) + (5-param)*point - 2 + param;
}

void bifurcation_diagram( int param_min, int param_max, double param_step,
             int point_min, int point_max, int num_points,
             double (*f)(double,double), int num_iter, int tolerancy)
{
  FILE* file;
  double param, point;
  int i,j;

  srand(time(NULL));
  file = fopen("data.dat","w");

  for ( param = param_min; param < param_max; param += param_step )
  {
    for ( i = 0; i < num_points; i++ )
    {
      point = point_min + ((double) rand() / (double) RAND_MAX) * (point_max-
          point_min);

      for ( j = 0; j < num_iter && abs(point) < tolerancy; j++ )
      {
        point = (*f)(param,point);
      }

      if(abs(point) < tolerancy)
      {
        fprintf(file,"%lf %lf\n",param, point);
      }
    }
  }
  plot( "data.dat", "graph.svg" );
}
```

Example

```
int main(int argc, char const *argv[])
{
  bifurcation_diagram( 0, 5, 10e-3, 0, 5, 100, &example_function, 100, 10e1);

  return 0;
}
```

Output