1. 1st laboratory

Existence of holomorphic functions

Let:

$\cdot\ f \in \mathcal{H}(\mathbb{D})$

Study:

$\cdot\ \exists\ f \in \mathcal{H}(\mathbb{D})$:

$\forall\ n \in \mathbb{N}$ „ $n \geq 2$:

$a)\ f(\pm\frac{1}{n}) = \frac{1}{2n+1}$

$b)\ f(\pm\frac{1}{n}) = \frac{1}{n^2}$

$c)\ |f(\frac{1}{n})| = \frac{1}{\log(n+1)}$

$d)\ |f(\frac{1}{n})| = \frac{n}{n+1}$

Demonstration:

$a)$:

$$E_1 := \{+\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N}\}$$

$$E_2 := \{-\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N}\}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} = \lim_{n} \frac{f(\frac{1}{n})}{\frac{1}{n}} - \frac{f(0)}{\frac{1}{n}} = \frac{1}{2} - \lim_{n} \frac{f(0)}{\frac{1}{n}}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} \begin{cases} = \frac{1}{2} & f(0) = 0 \\ \notin \mathbb{C} & f(0) \neq 0 \end{cases}$$

Case $f(0) = 0$:

$$\lim_{E_2} \frac{f(z) - f(0)}{z - 0} = \lim_{n} \frac{f(\frac{1}{n})}{\frac{1}{n}} = -\frac{1}{2} \neq \lim_{E_1} \frac{f(z) - f(0)}{z - 0}$$

$\nexists\ f \in \mathcal{H}(0)$ 〃 f satisfies $a)$

In particular:

$$\nexists\ f \in \mathcal{H}(\mathbb{D}) \text{ 〃 } f \text{ satisfies } a)$$

$b)$:

$$\begin{aligned} f\ :\mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z^2 \end{aligned}$$

$\forall\ n \in \mathbb{N}$ 〃 $n \geq 2$:

$$f(\pm\tfrac{1}{n}) = \tfrac{1}{n^2}$$

f satisfies $b)$

$$\begin{aligned} \bar{f}\ :\mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (u(x,y),v(x,y)) = (x^2-y^2,2xy) \end{aligned}$$

$\bar{f} \in \mathrm{Pol}(\mathbb{R}^2) \to \bar{f}$ differentiable in $\mathbb{R}^2$

$\forall\ (x,y) \in \mathbb{R}^2$:

$$\partial_x u(x,y) = 2x = \partial_y v(x,y)$$

$$\partial_y u(x,y) = -2y = -\partial_x v(x,y)$$

f satisfies CR

$\therefore)\ f \in \mathcal{H}(\mathbb{R}^2)$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

$c)$:

Suppose $\exists\ f \in \mathcal{H}(\mathbb{D})$ 〃 f satisfies $c)$

$$f \in \mathcal{C}^0(\mathbb{D}) \to f(0) = f(\lim_n \frac{1}{n}) = \lim_n\ f(\frac{1}{n}) = 0$$

$$\left|\lim_{E_1} \frac{f(z)-f(0)}{z-0}\right| = \lim_{n} \frac{\left|f(\frac{1}{n})\right|}{\frac{1}{n}} \notin \mathbb{C}$$

$f \notin \mathcal{H}(0)$ absurd

$d)$:

$$\begin{aligned} f \ : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \tfrac{1}{z+1} \end{aligned}$$

$\forall \ n \in \mathbb{N}$ „ $n \geq 2$:

$$\left|f(\tfrac{1}{n})\right| = \frac{1}{\frac{1}{n}+1} = \frac{n}{n+1}$$

f satisfies $d)$

$$\begin{aligned} \bar{f} \ : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (u(x,y), v(x,y)) = \left(\tfrac{x+1}{(x+1)^2+y^2}, \tfrac{-y}{(x+1)^2-y^2}\right) \end{aligned}$$

$\bar{f} \in \mathrm{Rat}(\mathbb{R}^2) \wedge \forall \ (x,y) \in \mathbb{R}^2$:

$$(x+1)^2 + y^2 \neq 0$$

$\bar{f}$ differentiable in $\mathbb{R}^2$

$\forall \ (x,y) \in \mathbb{R}^2$:

$$\partial_x u(x,y) = \frac{y^2-(x+1)^2}{((x+1)^2+y^2)^2} = \partial_y v(x,y)$$

$$\partial_y u(x,y) = \frac{-2y(x+1)}{((x+1)^2+y^2)^2} = -\partial_x v(x,y)$$

f satisfies CR

$\therefore)$ $f \in \mathcal{H}(\mathbb{R}^2)$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

Constant tests

Let:

- $\Omega \subset \mathbb{C}$ region
- $f \in \mathcal{H}(\Omega)$

Then, holds:

- $f_{Re} = 0 \ \vee \ f_{Im} = 0 \to f \in \mathrm{Cst}(\Omega)$
- $|f| \in \mathrm{Cst}(\Omega) \to f \in \mathrm{Cst}(\Omega)$

Demonstration:

$f_{Re} = 0 \ \vee \ f_{Im} = 0$:

$u \ := \ f_{Re}$

$v \ := \ f_{Im}$

$f \in \mathcal{H}(\Omega) \to f$ satisfies CR in Ω

$\partial_x u = \partial_y v = 0$

$\partial_y u = -\partial_x v = 0$

Null diferential test:

Ω connex $\to u, v \in \mathrm{Cst}(\Omega)$

$u, v \in \mathrm{Cst}(\Omega) \to f \in \mathrm{Cst}(\Omega)$

$|f| \in \mathrm{Cst}(\Omega)$:

$u \ := \ f_{Re}$

$v \ := \ f_{Im}$

$$\begin{aligned} |f| : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto \sqrt{u(x,y)^2 + v(x,y)^2} \end{aligned}$$

$|f| \in \mathrm{Cst}(\mathbb{C}) \to \exists\, a \in \mathbb{R}$:

$$\sqrt{u(x,y)^2 + v(x,y)^2} = a$$

$$u(x,y)^2 + v(x,y)^2 = a^2$$

$$2\partial_x u(x,y) + 2\partial_x v(x,y) = 0$$

$$2\partial_y u(x,y) + 2\partial_y v(x,y) = 0$$

$f \in \mathcal{H}(\Omega) \to f$ satisfies CR in Ω

$$2\partial_y v(x,y) + 2\partial_x v(x,y) = 0$$

$$-2\partial_x v(x,y) + 2\partial_y v(x,y) = 0$$

$$+ :\ 4\partial_y v(x,y) = 0 \to \partial_y v(x,y) = 0$$

$$- :\ 4\partial_x v(x,y) = 0 \to \partial_x v(x,y) = 0$$

Null differential test:

Ω connex $\to u, v \in \mathrm{Cst}(\Omega)$

$u, v \in \mathrm{Cst}(\Omega) \to f \in \mathrm{Cst}(\Omega)$