1. New

Intergal of power series

Then, holds:

$$\cdot \int_{|z-z_0|=r} (z-z_0)^n dz = \begin{pmatrix} 0 & n \neq 1 \\ 2\pi i & n = -1 \end{pmatrix}$$

Demonstration:

n≥ 0 :

$\frac{(z-z_0)^{n+1}}{n} \in \int (z-z_0)^n$

$\frac{(z-z_0)^{n+1}}{n} \in \mathcal{H}(\mathcal{C})$

CFT over closed curve:

$$\int_{|z-z_0|=r} (z-z_0)^n dz = 0$$

n≤ -1 :

$\frac{(z-z_0)^{n+1}}{n} \in \int (z-z_0)^n$

$\frac{(z-z_0)^{n+1}}{n} \in \mathcal{H}(\mathcal{C} \setminus z_0)$

CFT over closed curve:

$$\int_{|z-z_0|=r} (z-z_0)^n dz = 0$$

$n = -1$

Integral formula of Cauchy over convex open sets

Let:

$\cdot\ \gamma$ closed curve

- $\Omega \subset \mathbb{C}$ convex open " $\gamma^* \subset \Omega$
- $f \in \mathcal{H}(\Omega)$
- $z \notin \gamma^*$

Then, holds:

- $\frac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(\omega)}{\omega - z} d\omega = f(z) Ind(\gamma, z)$

Demonstration:

$\forall\ z \notin \gamma^*$:

$$\begin{array}{cccc} \tilde{f} & : & \Omega & \longrightarrow & \mathbb{C} \\ & & \omega & \longmapsto & \begin{cases} \frac{f(\omega)-f(z)}{\omega - z} & \omega \neq z \\ f'(z) & \omega = z \end{cases} \end{array}$$

$\tilde{f} \in \mathcal{C}(\Omega)$

$\tilde{f} \in \mathcal{H}(\Omega \setminus \{z\})$

Cauchy's theorem:

$$\int_\gamma \tilde{f}(w) dw = 0$$

$z \notin \gamma^* \rightarrow \omega \neq z$

$$\int_\gamma \tilde{f}(\omega) d\omega = \int_\gamma \frac{f(\omega) - f(z)}{\omega - z} d\omega$$
$$= \int_\gamma \frac{f(\omega)}{\omega - z} d\omega - f(z) \int_\gamma \frac{1}{\omega - z} d\omega =$$
$$= \int_\gamma \frac{f(\omega)}{\omega - z} d\omega - f(z) 2\pi i Ind(\gamma, z) = 0$$
$$\frac{1}{2\pi i} \int_\gamma \frac{f(\omega)}{\omega - z} d\omega = f(z) Ind(\gamma, z)$$

Mean property

Let:

$\cdot\ \Omega \subset \mathbb{C}$ open

$\cdot\ f \in \mathcal{H}(\Omega)$

$\cdot\ D(a,r) \subset \Omega$

Then, holds:

$\cdot\ f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta$

Demonstration:

Integral formula:

$$f(a) = \tfrac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f(z)}{z-a} dz$$

$$\gamma \ := \ \partial D(a,r)$$

$$\gamma(\theta) = a + re^{i\theta}$$

$$\gamma'(\theta) = rie^{i\theta}$$

$$f(a) = \tfrac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} re^{i\theta} d\theta$$

Independence of γ

Let:

$\cdot\ \Omega \subset \mathbb{C}$ open

$\cdot\ f \in \mathcal{H}(\Omega)$

$\cdot\ \gamma, \tilde{\gamma}$ closed curve " $\gamma^* \subset \Omega$

$\cdot\ z \in \Omega$ " $Ind(\gamma, z) = Ind(\tilde{\gamma}, z)$

Then, holds:

$\cdot \displaystyle\int_\gamma \frac{f(\omega)}{\omega - z} d\omega = \int_{\tilde{\gamma}} \frac{f(\omega)}{\omega - z} d\omega$

Demonstration:

no demonstration

Integral formula application

Let:

$\cdot\ \gamma$ pasa por en medio de i,-i y rodea 1/2

Then, holds:

$\cdot \displaystyle\int_\gamma \frac{\cos(\frac{pi}{2}\omega)}{\omega} d\omega$

$\cdot \displaystyle\int_\gamma \frac{\cos(\frac{pi}{2}\omega)}{\omega - \frac{1}{2}} d\omega$

Demonstration:

$f(\omega) = \cos(\frac{\pi}{2}\omega) \in \mathcal{H}(\mathbb{C})$ convex

$$\int_\gamma \frac{\cos(\frac{pi}{2}\omega)}{\omega} d\omega = 2\pi i 1 1 = 2\pi i$$

$$\int_\gamma \frac{\cos(\frac{pi}{2}\omega)}{\omega - \frac{1}{2}} d\omega = 2\pi i \frac{\sqrt{2}}{2}(-1)$$

$$\int_\gamma \frac{\cos(\frac{pi}{2}\omega)}{\omega - i} d\omega = 2\pi i f(i) 0 = 0$$

$$\int_\gamma \frac{\cos(\frac{pi}{2}\omega)}{\omega(\omega^2 + 4)} d\omega = \int_\gamma \frac{\cos(\frac{pi}{2}\omega)}{\omega(\omega + 2i)(\omega - 2i)} d\omega$$

$2i, -2i \notin \gamma^*$

$$\int_\gamma \frac{\cos(\frac{\pi}{2}\omega)/(\omega^2 + 4)}{\omega} d\omega = 2\pi i \frac{1}{4} 1$$