Block I

Definitions

<div style="text-align: center;">

1.   Discrete dynamical systems

</div>

Dynamical system

Let:

> $\cdot M$ manifold
>
> $\cdot T$ monoid
>
> $\cdot \phi \; : \; M \times T \to M$

Then, (M, T, ϕ) is a dynamical system if:

> $\cdot \forall \; x \in X \; :$
>
> > $\phi(x, 0) = 0$
> >
> > $\forall \; t_1, t_2 \in T \; :$
> >
> > > $\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$

Dimension

Let:

> $\cdot (M, T, \phi)$ dynamical system

We name dimension of (M, T, ϕ) to:

> $\cdot \dim(M)$

We denote:

> $\cdot dim(M) = n \; : \; (M, T, \phi)$ n-D

Discrete & Continuous

Let:

$\cdot \, (M, T, \phi)$ dynamical system

Then, (M, T, ϕ) is discrete if:

$\cdot \, T \overset{c}{\sim} \mathbb{N}$

Then, (M, T, ϕ) is continuous if:

$\cdot \, T \subset \mathbb{R}$

$\cdot \, T$ open

Defined by a function

Let:

$\cdot \, (M, T, \phi)$ dynamical system

$\cdot \, f \, : \, M \to M$

Then, (M, T, ϕ) is a dynamical system defined by f if:

$\cdot \, T = \mathbb{N}$

$\cdot \quad \begin{array}{rcl} \phi \, : M \times \mathbb{N} & \longrightarrow & M \\ (x, n) & \longmapsto & f^n(x) \end{array}$

We denote:

$\cdot \, (M, T, \phi)$ dynamical system defined by $f \, : \, (M, \mathbb{N}, f)$

$\cdot \, f \in \mathcal{C}^n(M) \, : \, (M, \mathbb{N}, f) \ \mathcal{C}^n$

Orbit

Let:

> $\cdot$ $(M, \mathbb{N}, f)$ functional dynamical system

> $\cdot$ $x \in M$

We name orbit of x to:

> $\cdot$ $\{f^n(x)\}_{n \in \mathbb{N}}$

We denote:

> $\cdot$ $o(x)$

Periodicity

Let:

> $\cdot$ $(M, \mathbb{N}, f)$ dynamical system

> $\cdot$ $x \in M$

> $\cdot$ $m \in \mathbb{N}$

Then, x is a m-periodic point if:

> $\cdot$ $f^m(x) = x$

We denote:

> $\cdot$ $\{x \in M \mid f(x) = x\}$: $\text{Fix}(f)$

Stability

Let:

      $\cdot \, f \; : \; \mathbb{R}^n \to \mathbb{R}^n$

      $\cdot \, p \in \mathbb{R}^n$ m-periodic point

Then, p is stable if:

      $\cdot \, \forall \, \varepsilon \in \mathbb{R}^+$:

            $\exists \, \delta \in \mathbb{R}^+$:

                  $\forall \, x \in B(p, \delta)$:

                        $\forall \, n \in \mathbb{N}$:

                              $f^{nm}(x) \in B(p, \varepsilon)$

Then, p is unstable if:

      $\cdot \, p$ not stable

Attractive & Repulsive

Let:

    $\cdot\, f\,:\,\mathbb{R}^n \to \mathbb{R}^n$

    $\cdot\, p \in \mathbb{R}^n$ m-periodic point

Then, p is attractive if:

    $\cdot\, p$ stable

    $\cdot\,\ \exists\, \varepsilon \in \mathbb{R}^+\,:$

        $\forall\, x \in B(p,\varepsilon)\,:$

            $f^{nm}(x) \xrightarrow{\ n\ } p$

Then, p is repulsive if:

    $\cdot\, p$ attractive by f^{-1}

    $\cdot\, \forall\, \mathcal{U} \subset M\ {}_{\shortparallel}\ \mathcal{U}$ open $\wedge\, x \in \mathcal{U}\,:$

        $\forall\, x' \in \mathcal{U}\ {}_{\shortparallel}\ x' \neq x\,:$

            $\exists\, N \in \mathbb{N}\,:$

                $\forall\, n \in \mathbb{N}\ {}_{\shortparallel}\ n \geq N\,:$

                    $f^{nm}(x') \notin \mathcal{U}$

Fixed point character

Let:

· $(M, \mathbb{N}, f)$ functional dynamical system

We name Fixed point character to:
$$f : \text{Fix}(f) \ \longrightarrow \ \{-1, 0, 1\}$$

.
$$x \ \longmapsto \ \begin{cases} +1 & x \text{ repulsive} \\ -1 & x \text{ attractive} \\ 0 & x \text{ no attractive no repulsive} \end{cases}$$

We denote:

· $f \ : \ \chi_f$

Attraction set

Let:

· $(M, \mathbb{N}, f)$ dynamical system

· $x \in M$ attractive m-periodic point

· $o(x)$ orbit of x

We name attraction set of x to:

· $\{y \in M \mid \exists\, x' \in o(x) \ : \ f^{nm}(y) \xrightarrow{n} x'\}$

We denote:

· $A(x)$

Multiplier

Let:

    $\cdot\,(M, \mathbb{N}, f)$ $\mathcal{C}^1$ dynamical system

    $\cdot\,x \in M$

We name multiplier of x to:

    $\cdot\,f'(x)$

We denote:

    $\cdot\,m(x)$

    $\cdot\,|m(x)| = 1\;:\;x$ neutral point

Feeble point

Let:

    $\cdot\,(M, \mathbb{N}, f)$ $\mathcal{C}^3$ dynamical system

    $\cdot\,x \in M$

Then, x is feeble point if:

    $\cdot\,x$ neutral point

    $\cdot\,f''(x) = 0$

Sarkovskii's order

We name Sarkovskii's order to:

$$\cdot \ a = 2^n a', b = 2^m b'$$

$$\cdot \ a <_s b \leftrightarrow \begin{cases} m < n & a' = b' = 1 \\ & a' = 1, b' \neq 1 \\ a' < b' & a' = b' \neq 1 \\ n < m & 1 \neq a' \neq b' \end{cases}$$

Chaos

Let:

$\cdot \ (\mathbb{R}, \mathbb{N}, f)$ dynamical system

Then, $(\mathbb{R}, \mathbb{N}, f)$ is chaotic if:

$\cdot \ \text{Fix}(f)$ dense in $\mathbb{R}$

$\cdot \ \exists \ x \in \mathbb{R} :$

$\qquad o(x)$ dense in $\mathbb{R}$

$\cdot \ \forall \ x \in \mathbb{R} :$

$\qquad \exists \ \varepsilon \in \mathbb{R}^+ :$

$\qquad \qquad \forall \ \delta \in \mathbb{R}^+ :$

$\qquad \qquad \qquad \exists \ \tilde{x} \in B(x, \delta) :$

$$\lim_n o(\tilde{x}) \notin B(x, \varepsilon)$$

Topological equivalence

Let:

> $\cdot$ $(M, \mathbb{N}, f_1), (M, \mathbb{N}, f_2')$ dynamical systems

Then, $(M, \mathbb{N}, f_1)$ is topologically equivalent to $(M, \mathbb{N}, f_2)$ if:

> $\cdot$ $\text{Fix}(f) = \text{Fix}(f')$
>
> $\cdot$ $\forall\, x \in \text{Fix}(f):$
>
> $$\chi_{f_1}(x) = \chi_{f_2}(x)$$

We denote:

> $\cdot$ $(M, \mathbb{N}, f) \sim (M, \mathbb{N}, f')$

Bifurcation

Let:

> $\cdot$ $\Lambda \subset M$
>
> $\cdot$ $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ
>
> $\cdot$ $\lambda_0 \in \Lambda$

Then, λ_0 is a bifurcation value if:

> $\cdot$ $\forall\, \varepsilon \in \mathbb{R}^+:$
>
> $\exists\, \lambda_1, \lambda_2 \in B(\lambda_0, \varepsilon):$
>
> $(M, \mathbb{N}, f_{\lambda_1}) \not\sim (M, \mathbb{N}, f_{\lambda_2})$

Saddle-node bifurcation

Let:

$\cdot \, \Lambda \subset M$

$\cdot \, (M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ

$\cdot \, \lambda_0 \in \Lambda$

$\cdot \, x_0 \in M$

Then, λ_0 is a saddle-node bifurcation value at x_0 if:

$\cdot \, x_0 \in \text{Fix}(f_{\lambda_0})$

$\cdot \, x_0$ neutral point of f_{λ_0}

$\cdot \, \partial_\lambda f_\lambda(x_0) \neq 0$

$\cdot \, \partial_{xx} f_\lambda(x_0) \neq 0$

Pitchfork bifurcation

Let:

$\cdot\, \Lambda \subset M$

$\cdot\, (M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ

$\cdot\, \lambda_0 \in \Lambda$

$\cdot\, x_0 \in M$

Then, λ_0 is Pitchfork bifurcation value at x_0 if:

$\cdot\, x_0 \in \text{Fix}(f_{\lambda_0})$

$\cdot\, x_0$ neutral point of f_{λ_0}

$\cdot\, \partial_\lambda f_\lambda(x_0, \lambda_0) = 0$

$\cdot\, \partial_{x^2} f_\lambda(x_0, \lambda_0) = 0$

$\cdot\, \partial_{\lambda x} f_\lambda(x_0, \lambda_0) \neq 0$

$\cdot\, \partial_{x^3} f_\lambda(x_0, \lambda_0) \neq 0$

Period doubling bifurcation

Let:

$\cdot \Lambda \subset M$

$\cdot (M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ dynamical systems parametrized by Λ

$\cdot \lambda_0 \in \Lambda$

$\cdot x_0 \in M$

Then, λ_0 is Period doubling bifurcation value at x_0 if:

$\cdot \lambda_0$ Pitchfork bifurcation value at x_0 of f^2

Invariant curve

Let:

$\cdot \gamma$ differentiable curve

$\cdot p \in \mathbb{R}^n$

Then, γ is invariant if:

$\cdot \forall\ x \in \gamma* :$

$$o(x) \subset \gamma*$$

Then, γ is converges to p if:

$\cdot \forall\ x \in \gamma* :$

$$o(x) \xrightarrow{n} p$$

2. 2-D linear dynamical systems

Linear system

Let:

· $(M, \mathbb{N}, f)$ functional dynamical system

Then, $(M, \mathbb{N}, f)$ is linear if:

· $\exists\, A \in \mathcal{M}_{n\times n}(\mathbb{R}) :$

$$
\begin{aligned}
f : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\
x &\longmapsto Ax
\end{aligned}
$$

Multiplier

Let:

· $(M, \mathbb{N}, f)$ functional dynamical system

· $x \in M$

We namemultiplier of x to:

· $Df(p)$