

# Complex Analysis

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# Contents



# Block I

# Definitions

<b>1. The field of complex numbers</b>
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*introduction*

## The field of complex numbers

Let:

$$\begin{aligned} \cdot \quad & + : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ & ((a, b), (c, d)) \longmapsto (a + c, b + d) \\ \cdot \quad & \cdot : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ & ((a, b), (c, d)) \longmapsto (ac - bd, ad + bd) \end{aligned}$$

We name the field of complex numbers to:

$$\cdot (\mathbb{R}^2, +, \cdot)$$

We denote:

$$\cdot (\mathbb{R}^2, +, \cdot) : \mathbb{C}$$

$$\cdot (0, 1) \in \mathbb{C} : i$$

$$\cdot (a, b) \in \mathbb{C} : a + bi$$

$$\cdot \pi_1(\mathbb{C}) : Re$$

$$\cdot \pi_2(\mathbb{C}) : Im$$

## Conjugation

We name complex conjugation to:

$$\begin{aligned} \cdot \quad & f : \mathbb{C} \longrightarrow \mathbb{C} \\ & (a, b) \longmapsto (a, -b) \end{aligned}$$

We denote:

$$\cdot f((a, b)) : \overline{(a, b)}$$

**Norm**

We name complex norm to:

$$\begin{array}{lcl} \cdot & f : \mathbb{C} & \longrightarrow \mathbb{R} \\ & (a, b) & \longmapsto \sqrt{a^2 + b^2} \end{array}$$

We denote:

$$\cdot f((a, b)) : |(a, b)|$$

**Polar transformation**

We name polar transformation to:

$$\begin{array}{lcl} \cdot & f : \mathbb{C} & \longrightarrow \mathbb{R}^+ \times [0, 2\pi) \\ & (a, b) & \longmapsto (\sqrt{|(a, b)|}, \arctan(\frac{b}{a})) \end{array}$$

We denote:

$$\cdot f((a, b)) : (r, \theta)$$

**Unit sphere projection**

We name unit sphere projection to:

$$\begin{array}{lcl} \cdot & f : \mathbb{C} & \longrightarrow S^1 \\ & z & \longmapsto \frac{z}{|z|} \end{array}$$

We denote:

$$\cdot f(z) : \pi(z)$$



## Roots of unity

Let:

$$\cdot z \in \mathbb{C}$$

Then,  $z$  is a root of unity if:

$$\cdot \exists n \in \mathbb{N} :$$

$$z^n = 1$$

We denote:

$$\cdot \{z \in \mathbb{C} \mid z \text{ root of unity} \} : S^1$$

## Disk

Let:

$$\cdot p \in \mathbb{C}$$

$$\cdot r \in \mathbb{R}^+ \setminus \{0\}$$

We name Disk centered in  $p$  and radius  $r$  to:

$$\cdot \{z \in \mathbb{C} \mid |z - p| < r\}$$

We denote:

$$\cdot \{z \in \mathbb{C} \mid |z - p| < r\} : D^1$$

### Component decomposition

Let:

$$f : \mathbb{C} \rightarrow \mathbb{C}$$

We name real component of  $f$  to:

$$\begin{aligned} f_{Re} : \mathbb{C} &\longrightarrow \mathbb{R} \\ z &\longmapsto Re(f(z)) \end{aligned}$$

We name imaginary component of  $f$  to:

$$\begin{aligned} f_{Im} : \mathbb{C} &\longrightarrow \mathbb{R} \\ z &\longmapsto Im(f(z)) \end{aligned}$$

We name component decomposition of  $f$  to:

$$\begin{aligned} f_{\mathbb{R}^2} : \mathbb{C} &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (f_{Re}(x + yi), f_{Im}(x + yi)) \end{aligned}$$



**2. Holomorphic functions****Incremental quotient**

Let:

$$\cdot \mathcal{U} \subset \mathbb{C} \text{ open}$$

$$\cdot f : \mathcal{U} \rightarrow \mathbb{C}$$

$$\cdot p \in \mathcal{U}$$

We name incremental quotient of  $f$  in  $p$  to:

$$\cdot \lim_{z \rightarrow p} \frac{f(z) - f(p)}{z - p}$$

We denote:

$$\cdot f'(p)$$

## Holomorphic function

Let:

$$\cdot \mathcal{U} \subset \mathbb{C} \text{ open}$$

$$\cdot f : \mathcal{U} \rightarrow \mathbb{C}$$

$$\cdot p \in \mathcal{U}$$

Then,  $f$  is holomorphic over  $p$  if:

$$\cdot \exists f'(p)$$

Then,  $f$  is holomorphic over  $U$  if:

$$\cdot \forall p \in \mathcal{U} :$$

$$\exists f'(p)$$

We denote:

$$\cdot \{f : \mathcal{U} \rightarrow \mathbb{C} \mid f \text{ holomorphic over } \mathcal{U}\} : \mathcal{H}(\mathcal{U})$$

$$\cdot f \in \mathcal{H}(\mathbb{C}) : f \text{ entire}$$

### Cauchy-Riemman equations

Let:

$$\cdot u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\cdot \begin{array}{ccc} f : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (u((x, y)), v((x, y))) \end{array}$$

Then,  $f$  is satisfies the Cauchy-Riemman equations if:

$$\cdot \exists u_x, u_y, v_x, v_y$$

$$\cdot u_x = v_y$$

$$\cdot u_y = -v_x$$

We denote:

$$\cdot u_x + iv_x : f_x$$

$$\cdot u_y + iv_y : f_y$$

**Conformal**

Let:

$$\cdot \mathcal{U} \subset \mathbb{C} \text{ open}$$

$$\cdot f : \mathcal{U} \rightarrow \mathbb{C}$$

$$\cdot z \in \mathcal{U}$$

Then,  $f$  is conformal in  $z$  if:

$$\cdot \exists c \in \mathbb{C} :$$

$$\forall I \subset \mathbb{R} \quad 0 \in I :$$

$$\forall \gamma : I \rightarrow \mathbb{R}^2 \quad \gamma \text{ differentiable} \wedge \gamma(0) = z \wedge \gamma'(0) \neq$$

$0 :$

$$\frac{(f \circ \gamma)'(0)}{\gamma'(0)} = c$$

Then,  $f$  is conformal if:

$$\cdot \forall z \in \mathcal{U} :$$

$$f \text{ conformal in } z$$

### Power series

Let:

$$\cdot \sum_{n \geq 0} a_n f_n \text{ complex valued sequence}$$

Then,  $\sum_{n \geq 0} a_n f_n$  is a power series if:

$$\cdot \forall n \in \mathbb{N} :$$

$$\begin{array}{ccc} f_n : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & (z - a)^n \end{array}$$

### Convergence radius

Let:

$$\cdot \text{statements}$$

.

Then, *item* is a/an entity if:

$$\cdot \text{conditions}$$

.

We denote:

$$\cdot \text{property} : \text{notation}$$

.



### Absolutely convergent

Let:

·  $\sum c_n n[0]$  series

Then,  $\sum c_n n[0]$  is absolutely convergent if:

·  $\sum |c_n| n[0]$  convergent

### Numeric series

Let:

·  $(c_k)_{k \in \mathbb{N}}$

Then, *item* is a/an entity if:

· *conditions*

.

We denote:

· *property : notation*

.



# Block II

# Propositions

<b>1. The field of complex numbers</b>
--

*introduction*

go

**2. Holomorphic functions****Cauchy-Riemman**

Let:

$$\cdot f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ satisfies CR}$$

Then, holds:

$$\cdot f_x = f'(z)$$

$$\cdot f_y = -if_y$$

$$\cdot \partial_{f(z)} z = 0$$

Demonstration:

$$f_x = u_x + iv_x = a + bi = f'(z)$$

$$f_y = i(a + ib) = if'(z)$$

$$f'(z) = -if_y$$

notation

**tangent vector**

Let:

$\gamma$  differentiable plane arc  $\quad \forall t \in I :$

$$\gamma'(t) \neq 0$$

Then, holds:

$\gamma'(t)$  tangent to  $\gamma$

Demonstration:

*demonstration*

**arc images**

Let:

$$\cdot f \in \mathcal{H}(\mathcal{U})$$

$$\cdot \gamma \text{ differentiable plane arc} \quad \gamma \subset \mathcal{U}$$

$$\cdot \sigma = f(\gamma)$$

$$\cdot z_0 = \gamma(0)$$

Then, holds:

$$\cdot \sigma' = f'(\gamma)\gamma'$$

$$\cdot \gamma'(0) \neq 0 \rightarrow f'(z_0) \neq 0$$

$$\cdot \sigma'(0) = f'(z_0)\gamma'(0)$$

$$\cdot |\sigma'(0)| = |f'(z_0)| |\gamma'(0)|$$

$$\cdot \arg \sigma'(0) = \arg \gamma'(0) + \arg f'(z_0)$$

$$\cdot f \text{ aplica una homotecia mas una rotacion constante a todos los}$$

vectores tangentes que salen de  $z_0$

Demonstration:

obvio



**Holomorphic functions are conform**

Let:

$$f : \mathcal{U} \rightarrow \mathbb{C}$$

.

Then, holds:

$$f \text{ holomorph in } z \Leftrightarrow f'(z) \neq 0 \leftrightarrow f \text{ conform}$$

Demonstration:

$\rightarrow$ ):

already seen

$\leftarrow$ ):

too hard

**Convergence of complex series**

Let:

$$\cdot \sum_{n \geq 0} c_n \text{ complex series}$$

Then, holds:

$$\cdot \sum_{n \geq 0} c_n \text{ convergent} \leftrightarrow \sum_{n \geq 0} \text{Re} c_n \text{ convergent} \wedge \sum_{n \geq 0} \text{Im} c_n \text{ convergent}$$

vergent

Demonstration:

*demonstration*

**Absolutely convergent are convergent**

Let:

$$\cdot \sum c_n n[0] \text{ absolutely convergent}$$

Then, holds:

$$\cdot \sum c_n n[0] \text{ convergent}$$

Demonstration:

$$S_k := \sum c_n n[0][k]$$

$$\forall m \in \mathbb{N} \quad m < k :$$

$$|S_k - S_m| = |\sum c_n n[m+1][k]| \leq \sum |c_n| n[m+1][k]$$

$$\leq \sum |c_n| n[m+1] \xrightarrow{n} 0$$

$$|S_k - S_m| \xrightarrow{n} 0 \rightarrow (S_k)_k \text{ convergent} \rightarrow \sum c_n n[0] \text{ convergent}$$

gent

**Series and norm**

Let:

$$\cdot \sum c_n n[0] \text{ convergent}$$

Then, holds:

$$\cdot |c_n| \xrightarrow{n} 0$$

Demonstration:

$$\sum c_n n[0] \text{ convergent} \leftrightarrow (S_n)_n \text{ convergent}$$

$$\rightarrow \text{Cauchy } |S_n - S_m| \xrightarrow{n} 0 \text{ por n y m} \rightarrow |S_n - S_{n-1}| \xrightarrow{n} 0$$

$$\rightarrow |c_n| \xrightarrow{n} 0$$

**Root test**

Let:

$$\begin{aligned} & \cdot \sum_{n \geq 0} c_n \text{ real series} \\ & \cdot l \in \mathbb{R} \quad \text{, } \lim_{k \rightarrow \infty} |c_k|^{\frac{1}{k}} = l \end{aligned}$$

Then, holds:

$$\begin{aligned} & \cdot l > 1 \rightarrow \sum_{n \geq 0} c_n \notin \mathbb{R} \\ & \cdot l < 1 \rightarrow \sum_{n \geq 0} c_n \in \mathbb{R} \end{aligned}$$

Demonstration:

*demonstration*

**Quotient test**

Let:

- $\sum_{n \geq 0} c_n$  real series
- $l \in \mathbb{R}$  „

Then, holds:

- $\exists l \in \mathbb{R} :$

$$\lim_k \frac{c_{k+1}}{c_k} = l$$

- $\overline{\lim}_{c_k} |c_k|^{\frac{1}{k}} k = l$

**Power series theorem**

Let:

$$\cdot \sum_{n \geq 0} a_n c^n \text{ power series}$$

Then, holds:

$$\cdot |z - a| < R \rightarrow \text{absolutely convergent}$$

$$\cdot |z - a| > R \rightarrow \text{divergent}$$

$$\cdot \text{convergent in } D(a, R)$$

$$\cdot \begin{array}{ccc} f : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \sum_{n \geq 0} c_n (z - a)^n \in \mathcal{H}(D(a, R)) \end{array}$$

$$\cdot \forall z \in \mathbb{C} :$$

$$f'(z) = \sum_{n \geq 0} n c_n (z - a)^{n-1} \text{ convergent}$$

$$\cdot \text{convergence radius of } f' = \text{convergence radius of } f$$

Demonstration:

$$\forall z \in \mathbb{C} \quad \text{,, } |z - a| < R :$$

$$\begin{array}{l} \text{Root test over } \sum_{n \geq 0} |c_n| |z - a|^n \text{ rlimit}(|c_n| |z - a|^n)^{\frac{1}{n}} = |z - a| \text{ rlimit} |c_n|^{\frac{1}{n}} = \frac{|z - a|}{R} < 1 \text{ Root test} \rightarrow \text{absolutely convergent} \end{array}$$

$$\forall z \in \mathbb{C} \quad \text{,, } |z - a| < R :$$

$$\forall \rho \in \mathbb{R} \quad \text{,, } |z - a| < \rho < R :$$

$$\frac{1}{\rho} < \frac{1}{R}$$

$$\text{rlimit} |c_n|^{1/n} = \frac{1}{R} \text{ exists partial of } |c_n|^{1/n}$$

$$|c_n| |z - a|^n > \frac{|z - a|^n}{\rho^n} \text{ no } \xrightarrow{n} 0$$

General term test  $\rightarrow$  divergent



**Power series theorem**

Let:

.

Then, holds:

· *Part II, III, IV*

Demonstration:

Follow 3 steps

Step 1 : Uniform convergence in compacts of  $D(a, R)$  :

$$\begin{aligned} g_n : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto c_n(z-a)^n \end{aligned}$$

$$\forall \rho \in \mathbb{R} \quad \text{,, } \rho < R :$$

$$\forall z \in \overline{D(a, \rho)} :$$

$$|g_n(z)| = |c_n||z-a|^n \leq |c_n|\rho^n$$

$$M_n := |c_n|\rho^n$$

$$\text{rlimit}(|c_n|\rho^n)^{1/n} = \text{prlimit}|c_n|^{1/n} = \frac{\rho}{R} < 1$$

$$\text{Root test} \rightarrow \sum_{n \geq 0} M_n \text{ convergent}$$

$$\text{M-Weierstrass} \rightarrow \sum_{n \geq 0} c_n(z-a)^n \text{ uniformly convergent}$$

over compacts of  $D(a, \rho)$

$$\sum_{n \geq 0} c_n(z-a)^n \text{ uniformly convergent over compacts of } D(a, r)$$

$$f(z) := \sum_{n \geq 0} g_n(z)$$

$g$  uniformly convergent  $\rightarrow f$  continuous

Step 2 : IV. R' :

$$\tilde{f}(z) := \sum_{n \geq 1} n c_n (z - a)^{n-1}$$

$$\tilde{f}(z) = \sum_{n \geq 0} (n+1) c_{n+1} (z - a)^n$$

$$\frac{1}{R'} = rlimit(n+1) |c_{n+1}|^{1/n} n = rlimit(n+1)^{1/n} |c_{n+1}|^{1/n} n =$$

$$rlimit(|c_{n+1}|^{1/n+1})^{\frac{n+1}{n}} n = \frac{1}{R}$$

$$R' = R$$

Step 3 : III :

$\tilde{f}$  well defined in  $D(a, R)$

$\forall z_0 \in D(a, R) :$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - \tilde{f}(z_0) \right| \xrightarrow{n} 0?$$

$\forall n \in \mathbb{N} :$

$$S_n(z) := \sum_{k=0}^n f_k(z)$$

$$R_n(z) := \sum_{k \geq n+1} f_k(z)$$

$$f(z) = S_n(z) + R_n(z)$$

$$\tilde{f}(z) = \tilde{S}_n(z) + \tilde{R}_n(z)$$

$$R_n(z), \tilde{R}_n(z) \xrightarrow{n} 0$$

$$\forall \varepsilon \in \mathbb{R}^+ :$$

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| = \left| \frac{S_n(z)-S_n(z_0)}{z-z_0} + \frac{R_n(z)-R_n(z_0)}{z-z_0} - \tilde{S}_n(z_0) - \tilde{R}_n(z_0) \right|$$

$$\forall \rho \in \mathbb{R} \quad \text{,, } |z_0 - a| < \rho < R :$$

$$|\tilde{R}_n(z_0)| \leq \sum_{k \geq n+1} k |c_k| \rho^{k-1} < \frac{\varepsilon}{3} (n \geq n_1)$$

$$\begin{aligned} \left| \frac{R_n(z)-R_n(z_0)}{z-z_0} \right| &\leq \sum_{k \geq n+1} |c_k| \left| \frac{(z-a)^k - (z_0-a)^k}{z-z_0} \right| \\ \frac{a^k - b^k}{a-b} &= a^{k-1} - a^{k-2}b + \dots + b^{k-1} \\ &\leq \sum_{k \geq n+1} |c_k| (|z-a|^{k-1} + |z-a|^{k-2}|z_0-a| + \dots + \\ &\quad |z_0-a|^{k-1} \end{aligned}$$

$$|z-a|, |z_0| < \rho$$

$$\leq \sum_{k \geq n+1} |c_k| k \rho^{k-1} < \frac{\varepsilon}{3} (n \geq n_1)$$

$$\left| \frac{S_n(z)-S_n(z_0)}{z-z_0} - \tilde{S}_n(z_0) \right| < \frac{\varepsilon}{3} (S'_n = \tilde{S}_n) (n \geq n_2)$$

$$\forall n \in \mathbb{N} \quad \text{,, } n \geq \max(n_1, n_2) :$$

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| < \varepsilon$$

functions associated to series are  $\mathcal{C}^\infty$

Let:

$$\cdot f(z) = \sum c_n (z - a)^n \quad n[0] \text{ series}$$

$$\cdot R \text{ radius of convergence of } f$$

Then, holds:

$$\cdot f \in \mathcal{C}^\infty \text{ over } D(a, R)$$

$$\cdot \forall n \in \mathbb{N} :$$

$$f^{(n)} \in \mathcal{H}(D(a, R))$$

$$\cdot c_k = \frac{f^{(k)}}{k!}$$

$$\cdot \text{series associated to } f \text{ is unique}$$

Demonstration:

$$f^{(k)}(z) = \sum n(n-1) \dots (n-k+1) c_n (z-a)^{n-k} \quad n[k]$$

$$f^{(k)}(a) = k! c_k$$

$$c_k = \frac{f^{(k)}(a)}{k!}$$

# Block III

# Examples

<b>1. Holomorphic functions</b>
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*introduction*

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go

**2. Holomorphic functions****Conjugation**

Let:

$$\begin{aligned} \bar{a} : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \bar{z} \end{aligned}$$

Then,  $\bar{a}$  is not holomorphic :

$$u_x = 1$$

$$u_y = 0$$

$$v_x = 0$$

$$v_y = -1$$

$$\forall z \in \mathbb{C} :$$

$$-1 \neq 1 \rightarrow f \text{ not holomorphic in } z$$



**Quadratic norm**

Let:

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto |z|^2 \end{aligned}$$

$\cdot f_{\mathbb{R}^2}$  component decomposition of  $f$

Then,  $f$  is holomorphic in 0:

$f$  differentiable in  $\mathbb{R}^2$  polynomial

$\forall z \in \mathbb{C} :$

$$u_x(x, y) = 2x$$

$$u_y(x, y) = 2y$$

$$v_x(x, y) = 0$$

$$v_y(x, y) = 0$$

$$u_x = v_y \leftrightarrow x = 0$$

$$u_y = -v_x \leftrightarrow y = 0$$

$f$  holomorphic function in  $z \leftrightarrow z = 0$

**Non preserving angles function**

Let:

$$\cdot f(z) = z^2$$

Then,  $f$  is conform in  $\mathbb{R} \setminus \{0\}$ :

$$f(\{(x, 0) \in \mathbb{C} \mid x > 0\}) = \{(x, 0) \in \mathbb{C} \mid x > 0\}$$

$$f(\{(x, 0) \in \mathbb{C} \mid x < 0\}) = \{(x, 0) \in \mathbb{C} \mid x > 0\}$$

$$\text{ang}(A, B) = \pi \neq 0 = \text{ang}(f(A), f(B))$$

**Exponential**

Let:

$$\cdot a : 0$$

$$\cdot c_n : \frac{1}{n}$$

Then,  $\sum_{n \geq 0} c_n (z - a)^n$  is convergent in  $D_1$  :

$$\lim_n \frac{|c_n|}{|c_{n+1}|} = \lim_n \frac{n+1}{n} = 1 \rightarrow R = 1$$

$\mathbb{CH} \rightarrow D(0, 1)$  convergent

$\mathbb{C} \setminus D(0, 1)$  divergent

$$f' = f$$

**Geometric series**

Let:

$$\cdot a : 0$$

$$\cdot c_n : 0$$

Then,  $\sum z^n n[0]$  is convergent in  $\mathbb{D}$ :

$$R = \frac{c_n}{c_{n+1}} = 1 \quad \text{Then, holds:}$$

$$\cdot \sum z^n n[0] = \frac{1}{1-z}$$

$$\cdot \sum n z^{n-1} n[0] = \frac{1}{(1-z)^2}$$

$$\cdot \sum \frac{z^{n+1}}{n+1} n[0] = -\log(1-z)$$

Demonstration:

$$\forall z \in \mathbb{D} :$$

$$\sum z^n n[0] \text{ geometric series}$$

$$\sum z^n n[0] = \frac{1}{1-z}$$

*II* differentiating

*III* integrating

### Series not centered in 0

Let:

$$\cdot a : i$$

$$\cdot c_n : \frac{n+1}{5^{n+1}}$$

Then, *item* is a/an entity :

$$\begin{aligned} & \sum \frac{n(z-i)^{n-1}}{5^n} n[1] \\ &= \frac{1}{5} \sum n \frac{z-i}{5} n^{n-1} n[1] = \frac{1}{5} \sum n u^{n-1} n[1] \end{aligned}$$

$$S(u) = \tilde{S}'(u)$$

$$\tilde{S}(u) = \frac{1}{5} \sum u^n n[1] = \frac{u}{5(1-u)}$$

$$S(u) = \frac{1}{5(1-u)^2}$$

$$S(z) = \frac{5}{(5+i-z)^2} \text{ over } D(i, 5)$$

### Radius of convergence without quotient test

Let:

$$\cdot \sum \frac{(-1)^n}{n(n+1)} (z-2)^{n(n+1)} n[1]$$

Then, *R* is a/an entity :

$$\begin{aligned} & \lim_{c_{n+1}} c_n \nexists \\ & \lim_n \frac{1}{n(n+1)} \frac{1}{\frac{1}{n(n+1)}} = 1 \\ & \text{ignore zeros} \end{aligned}$$

# Block IV

# Problems

**PROBLEMES D'ANÀLISI COMPLEXA**  
**2n quadrimestre del curs 2013-2014.**

**Llista 1: Els nombres complexos**

**B.1.** Expressen en la forma  $a + ib$  els següents nombres:

- |                       |                     |                 |                  |
|-----------------------|---------------------|-----------------|------------------|
| (a) $(2 + 3i)(4 + i)$ | (c) $\frac{1}{4+i}$ | (e) $\sqrt{i}$  | (g) $\sqrt{9i}$  |
| (b) $(4 + 2i)^2$      | (d) $\frac{i}{4+i}$ | (f) $\sqrt{-i}$ | (h) $\sqrt{1+i}$ |

**B.2.** Si  $z = x + iy$  trobeu les parts real i imaginària de les expressions següents:

- |           |                |                   |                     |
|-----------|----------------|-------------------|---------------------|
| (a) $z^2$ | (b) $z(z + 1)$ | (c) $\frac{1}{z}$ | (d) $\frac{1}{z-3}$ |
|-----------|----------------|-------------------|---------------------|

**B.3.** És cert que

- a)  $\operatorname{Re}(z + w) = \operatorname{Re} z + \operatorname{Re} w$ ?
- b)  $\operatorname{Re}(zw) = (\operatorname{Re} z)(\operatorname{Re} w)$ ?
- c)  $\operatorname{Re}\left(\frac{z}{w}\right) = \frac{\operatorname{Re} z}{\operatorname{Re} w}$ ?

**B.4.** Trobeu la forma polar dels nombres següents i dibuixeu-los.

- |                        |                      |               |              |
|------------------------|----------------------|---------------|--------------|
| (a) $3(1 + \sqrt{3}i)$ | (b) $2\sqrt{3} - 2i$ | (c) $-2 + 2i$ | (d) $-1 - i$ |
|------------------------|----------------------|---------------|--------------|

**B.5.** Sigui  $(x + iy)/(x - iy) = a + ib$ . Proveu que  $a^2 + b^2 = 1$ .

**B.6.** Proveu que si  $p(z)$  és un polinomi amb coeficients reals i  $z$  és un zero de  $p$  llavors  $\bar{z}$  també ho és.

**B.7.** Descriu els conjunts del pla que satisfan (recordeu que  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ .)

- |   |                                     |                         |
|---|-------------------------------------|-------------------------|
| (a) $\operatorname{Im} \frac{z-a}{z} = 0, a \in \mathbb{C}^*$ | (b) $ z  = \operatorname{Re} z + 1$ | (c) $ z - 2  >  z - 3 $ |
|---|-------------------------------------|-------------------------|

**SOL.** B.1. a)  $5 + 14i$ ; b)  $12 + 16i$ ; c)  $4/17 - i/17$ ; d)  $1/17 + 4i/17$ ; e)  $\pm\sqrt{2}/2(1 + i)$ ; f)  $\pm\sqrt{2}/2(1 - i)$ ; g)  $\pm 3\sqrt{2}/2(1 + i)$ ; h)  $\pm 2^{1/4}(\cos(\pi/8) + i \sin(\pi/8))$ .

B.2 a)  $x^2 - y^2 + 2ixy$ ; b)  $x^2 - y^2 + x + i(y + 2xy)$ ; c)  $(x - iy)/(x^2 + y^2)$ ; d)  $(x - 3 - iy)/((x - 3)^2 + y^2)$ .

B.3 a) si. b) no. c) no.

B.4 a)  $6(\cos(\pi/3) + i \sin(\pi/3))$ ; b)  $4(\cos(\pi/6) - i \sin(\pi/6))$ ; c)  $2\sqrt{2}(\cos(\pi/4) + i \sin(\pi/4))$ ; d)  $\sqrt{2}(\cos(3\pi/4) - i \sin(3\pi/4))$ .

B.6 Conjugueu tot el polinomi.

B.7 a) Recta que passa per 0 i  $a$ ; b) Paràbola horitzontal  $x = (1/2)(y^2 - 1)$ ; c)  $\{\operatorname{Re} z > 3/2\}$ .

**1.** Expressen en la forma  $a + ib$  els següents nombres:

- |                       |  |                                       |                              |
|-----------------------|--|---------------------------------------|------------------------------|
| (a) $\frac{1}{i}$     | (c) $\frac{1}{2+i} + \frac{1}{2-i}$    | (e) $\left(\frac{2+i}{3-2i}\right)^2$ | (g) $\sqrt[4]{-i}$           |
| (b) $\frac{1+i}{1-i}$ | (d) $\frac{1}{2+i} + \frac{4-2i}{3+i}$ | (f) $(1+i)^{100} + (1-i)^{100}$       | (h) $(3 + 4i)^{\frac{1}{2}}$ |

**2.** Si  $z = x + iy$  on  $x, y \in \mathbb{R}$ , trobeu les parts real i imaginària de:

**PROBLEMES D'ANLISI COMPLEXA**  
**2n quadrimestre del curs 2013-2014**

**Llista 2: Funcions de variable complexa i equacions de Cauchy-Riemann**

**B.1.** Trobeu els punts on la funció  $f$  és derivable (en el sentit complex), en els següents casos, i calculeu la derivada.

(a)  $\cos |z|^2$

(c)  $e^{iz}$

(e)  $\frac{1}{(z-1)^2(z^2+2)}$

(b)  $|z|^4$

(d)  $z + \frac{1}{z}$

(f)  $\frac{1}{(z+\frac{1}{z})^2}$

**Solució:** (a)  $\emptyset$ ; (b)  $\emptyset$ ; (c)  $\mathbb{C}$ ;  $f'(z) = ie^{iz}$ ; (d)  $\mathbb{C} \setminus \{0\}$ ;  $f'(z) = 1 - \frac{1}{z^2}$ ; (e)  $\mathbb{C} \setminus \{1, \pm\sqrt{2}i\}$ ; (f)  $\mathbb{C}$ .

**B.2.** Determineu si aquestes funcions poden ser la part real d'una funció holomorfa, i en cas que ho siguin calculeu la part imaginària.

(a)  $e^x \cos y$

(b)  $x^3 + 6xy^2$

(c)  $\log(x^2 + y^2)$

**Solució:** (a)  $e^x \sin y$ ;  $f(z) = e^z$ ; (b) No ho és; (c)  $2 \arctan(y/x)$ ;  $f(z) = \log(z^2)$ .

**B.3.** Sigui  $f$  una funció holomorfa en un obert  $\Omega \subset \mathbb{C}$  i  $z_0 \in \Omega$  tal que  $f'(z_0) \neq 0$ . Quin angle formen les corbes  $\operatorname{Re} f(z) = \operatorname{Re} f(z_0)$  i  $\operatorname{Im} f(z) = \operatorname{Im} f(z_0)$  en un punt  $z_0$ ?

**Solució:**  $\pi/2$ .

**1.** Trobeu els punts on la funció  $f$  és derivable (en el sentit complex), en els següents casos:

(a)  $f(z) = |z|$

(d)  $f(z) = z + z\bar{z}$

(b)  $\cosh x \cos y + i \sinh x \sin y$

(c)  $f(z) = \operatorname{Re} z$

(e)  $f(z) = \operatorname{Im} e^{\bar{z}} + i \operatorname{Re} e^z$

**2.** Sigui  $\Omega \subset \mathbb{C}$  un obert,  $z_0 \in \Omega$  i  $f : \Omega \rightarrow \mathbb{C}$  una funció.

a) Identificant  $\mathbb{R}^2$  amb  $\mathbb{C}$  de la forma habitual, demostreu que si  $f$  és diferenciable en  $z_0$ , llavors

$$Df(z_0)(z) = \frac{\partial f}{\partial z}(z_0) \cdot z + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot \bar{z} \quad (z \in \mathbb{C}),$$

on

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

b) Proveu que  $f$  és holomorfa en  $\Omega$  si, i només si,  $f$  és diferenciable i  $\frac{\partial f}{\partial \bar{z}} = 0$  en  $\Omega$ . En tal cas,  $f' = \frac{\partial f}{\partial z}$ .

**3.** Demostreu que si  $f$  és diferenciable en un obert de  $\mathbb{C}$ , llavors

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}} \quad \text{i} \quad \overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}.$$

<b>1. The field of complex numbers</b>
--

*introduction*



**entity**

Let:

· *statements*

·

Then, *item* is a/an entity if:

· *conditions*

·

We denote:

· *property : notation*

·

<b>2. Holomorphic functions</b>
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### 3. Cauchy-Riemann

Let:

$$\cdot f \in \mathcal{H}(\mathbb{C}) \quad \text{,,} \quad \operatorname{Re} f + i \operatorname{Im} f = c_a$$

Show that:

$$\cdot \exists a' \in \mathbb{C} :$$

$$f = c_{a'}$$

Demonstration:

$u, v$  real components of  $f$

$$u(x, y) + v(x, y) = a$$

differentiate respect  $x$  and  $y$

$$u_x + v_x = 0$$

$$u_y + v_y = 0$$

$f$  holomorphic  $\rightarrow f$  CR

$$u_x - u_y = 0$$

$$u_y + u_x = 0$$

$$u_x, u_y, v_x, v_y = 0$$

$$\exists a_1 \in \mathbb{R} :$$

$$u = c_{a_1}$$

$$\exists a_2 \in \mathbb{R} :$$

$$v = c_{a_2}$$

$$f = c_{(a_1, a_2)}$$

**B.2 a)**

Let:

$$\begin{aligned} u : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto \exp(y) \cos(x) \end{aligned}$$

Show that:

$$\exists f \in \mathcal{H}(\mathbb{C}) :$$

$u$  real component of  $f$

Demonstration:

$$\text{lab } 1 \rightarrow u_{xx} + v_{yy} = 0$$

$$u_x = \exp(x) \cos(y)$$

$$u_{xx} = \exp(x) \cos(y)$$

$$u_y = -\exp(x) \sin(y)$$

$$u_{yy} = -\exp(x) \cos(y)$$

ok

Calculate  $v$  using CR

$$v_y = u_x = \exp(x) \cos(y)$$

$$v(x, y) = \int_{\mathbb{C}} \exp(x) \sin(y) dy = \exp(x) \sin(y) + \phi(x)$$

$$v_x = \partial_v x = \exp(x) \sin(y) + \phi'(x)$$

$$-u_y = \exp(x) \sin(y) + \phi'(x)$$

$$\text{CR} \rightarrow \phi'(x) = 0$$

$$\forall c \in \mathbb{R} :$$

$$\phi(x) = c \text{ ok}$$

$$v(x, y) = \exp(x) \sin(y)$$

### Preservation of angles

Let:

$$\gamma_1, \gamma_2 \text{ plane arcs} \quad \gamma_1(0) = \gamma_2(0)$$

Then, holds:

$$\text{angle of } \gamma_1'(0) \text{ and } \gamma_2'(0) = \text{angle } \sigma_1'(0), \sigma_2'(0)$$

Demonstration:

rotations and homotecies let angles invariant



# Block V

## Tasks

**1. 1st laboratory**
**Existence of holomorphic functions**

Let:

$$\cdot f \in \mathcal{H}(\mathbb{D})$$

Study:

$$\cdot \exists f \in \mathcal{H}(\mathbb{D}) :$$

$$\forall n \in \mathbb{N} \quad n \geq 2 :$$

$$a) f\left(\pm \frac{1}{n}\right) = \frac{1}{2n+1}$$

$$b) f\left(\pm \frac{1}{n}\right) = \frac{1}{n^2}$$

$$c) \left|f\left(\frac{1}{n}\right)\right| = \frac{1}{\log(n+1)}$$

$$d) \left|f\left(\frac{1}{n}\right)\right| = \frac{n}{n+1}$$

Demonstration:

a):

$$E_1 := \left\{ +\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N} \right\}$$

$$E_2 := \left\{ -\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N} \right\}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} = \lim_n \frac{f\left(\frac{1}{n}\right) - f(0)}{\frac{1}{n}} = \frac{f(0)}{\frac{1}{n}} = \frac{1}{2} - \lim_n \frac{f(0)}{\frac{1}{n}}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} \begin{cases} = \frac{1}{2} & f(0) = 0 \\ \notin \mathbb{C} & f(0) \neq 0 \end{cases}$$

Case  $f(0) = 0$ :

$$\lim_{E_2} \frac{f(z) - f(0)}{z - 0} = \lim_n \frac{f\left(-\frac{1}{n}\right) - f(0)}{-\frac{1}{n}} = -\frac{1}{2} \neq \lim_{E_1} \frac{f(z) - f(0)}{z - 0}$$



$$\nexists f \in \mathcal{H}(0) \quad \text{„} f \text{ satisfies } a)$$

In particular:

$$\nexists f \in \mathcal{H}(\mathbb{D}) \quad \text{„} f \text{ satisfies } a)$$

b):

$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z^2 \end{aligned}$$

$$\forall n \in \mathbb{N} \quad \text{„} n \geq 2 :$$

$$f\left(\pm \frac{1}{n}\right) = \frac{1}{n^2}$$

$f$  satisfies b)

$$\begin{aligned} \bar{f} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (u(x, y), v(x, y)) = (x^2 - y^2, 2xy) \\ \bar{f} \in \text{Pol}(\mathbb{R}^2) &\rightarrow \bar{f} \text{ differentiable in } \mathbb{R}^2 \end{aligned}$$

$$\forall (x, y) \in \mathbb{R}^2 :$$

$$\partial_x u(x, y) = 2x = \partial_y v(x, y)$$

$$\partial_y u(x, y) = -2y = -\partial_x v(x, y)$$

$f$  satisfies CR

$$\therefore f \in \mathcal{H}(\mathbb{R}^2)$$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

c):

Suppose  $\exists f \in \mathcal{H}(\mathbb{D}) \quad \text{„} f \text{ satisfies } c)$

$$f \in \mathcal{C}^0(\mathbb{D}) \rightarrow f(0) = f\left(\lim_n \frac{1}{n}\right) = \lim_n f\left(\frac{1}{n}\right) = 0$$

$$\left| \lim_{E_1} \frac{f(z) - f(0)}{z - 0} \right| = \lim_n \frac{\left| f\left(\frac{1}{n}\right) \right|}{\frac{1}{n}} \notin \mathbb{C}$$

$f \notin \mathcal{H}(0)$  absurd

d):

$$\begin{array}{ccc} f : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \frac{1}{z+1} \end{array}$$

$$\forall n \in \mathbb{N} \quad n \geq 2 :$$

$$\left| f\left(\frac{1}{n}\right) \right| = \frac{1}{\frac{1}{n}+1} = \frac{n}{n+1}$$

$f$  satisfies d)

$$\begin{array}{ccc} \bar{f} : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (u(x, y), v(x, y)) = \left( \frac{x+1}{(x+1)^2+y^2}, \frac{-y}{(x+1)^2+y^2} \right) \end{array}$$

$$\bar{f} \in \text{Rat}(\mathbb{R}^2) \wedge \forall (x, y) \in \mathbb{R}^2 :$$

$$(x+1)^2 + y^2 \neq 0$$

$\bar{f}$  differentiable in  $\mathbb{R}^2$

$$\forall (x, y) \in \mathbb{R}^2 :$$

$$\partial_x u(x, y) = \frac{y^2 - (x+1)^2}{((x+1)^2 + y^2)^2} = \partial_y v(x, y)$$

$$\partial_y u(x, y) = \frac{-2y(x+1)}{((x+1)^2 + y^2)^2} = -\partial_x v(x, y)$$

$f$  satisfies CR

$$\therefore f \in \mathcal{H}(\mathbb{R}^2)$$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

### Constant tests

Let:

$$\cdot \Omega \subset \mathbb{C} \text{ region}$$

$$\cdot f \in \mathcal{H}(\Omega)$$

Then, holds:

$$\cdot f_{Re} = 0 \vee f_{Im} = 0 \rightarrow f \in \text{Cst}(\Omega)$$

$$\cdot |f| \in \text{Cst}(\Omega) \rightarrow f \in \text{Cst}(\Omega)$$

$$\cdot \text{Im} f \text{ circumference} \rightarrow f \in \text{Cst}$$

Demonstration:

$$f_{Re} = 0 \vee f_{Im} = 0:$$

$$u := f_{Re}$$

$$v := f_{Im}$$

$$f \in \mathcal{H}(\Omega) \rightarrow f \text{ satisfies CR in } \Omega$$

$$\partial_x u = \partial_y v = 0$$

$$\partial_y u = -\partial_x v = 0$$

Null differential test:

$$\Omega \text{ connex} \rightarrow u, v \in \text{Cst}$$

$$u, v \in \text{Cst} \rightarrow f \in \text{Cst}$$

$$|f| \in \text{Cst}(\Omega):$$

$$\begin{aligned} |f| : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \sqrt{u(x, y)^2 + v(x, y)^2} \end{aligned}$$

$$|f| \in \text{Cst} \rightarrow \exists a \in \mathbb{R} :$$

$$\sqrt{u(x, y)^2 + v(x, y)^2} = a$$

$$u(x, y)^2 + v(x, y)^2 = a^2$$

$$2\partial_x u(x, y) + 2\partial_x v(x, y) = 0$$

$$2\partial_y u(x, y) + 2\partial_y v(x, y) = 0$$

$$f \in \mathcal{H}(\Omega) \rightarrow f \text{ satisfies CR in } \Omega$$

$$2\partial_y v(x, y) + 2\partial_x v(x, y) = 0$$

$$-2\partial_x v(x, y) + 2\partial_y v(x, y) = 0$$

$$+ : 4\partial_y v(x, y) = 0 \rightarrow \partial_y v(x, y) = 0$$

$$- : 4\partial_x v(x, y) = 0 \rightarrow \partial_x v(x, y) = 0$$

Null differential test:

$$\Omega \text{ connex} \rightarrow u, v \in \text{Cst}$$

$$u, v \in \text{Cst} \rightarrow f \in \text{Cst}$$

$\text{Im}(f)$  circumference :

$$\exists (x_0, y_0) \in \mathbb{R}^2, r \in \mathbb{R}^+ :$$

$$\text{Im}(f) = C_r(x_0, y_0)$$

$$\begin{aligned} \bar{f} : \mathbb{R}^2 &\longrightarrow C_r(x_0, y_0) \\ (x, y) &\longmapsto (r \cos(x - x_0), r \sin(y - y_0)) \end{aligned}$$

$$\forall (x, y) \in \Omega :$$

$$|\bar{f}|(x, y) = \sqrt{r^2(\cos^2(x - x_0) + \sin^2(y - y_0))} = r$$

$$|f| \in \text{Cst} \rightarrow f \in \text{Cst}$$

### Real part of holomorphic functions

Let:

$$\cdot \Omega \subset \mathbb{R}^2 \text{ region}$$

$$\cdot u \in \mathcal{C}^2(\Omega) \quad \text{,,} \quad \exists f \in \mathcal{H}(\Omega) :$$

$$f_{Re} = u$$

Show that:

$$\cdot \partial_{xx}u + \partial_{yy}u = 0$$

Study:

$$\cdot \exists f \in \mathcal{H}(\Omega) :$$

$$a) f_{Re}(x, y) = x^2 + y^2$$

$$b) f_{Re}(x, y) = x(x+1) - y^2$$

$$c) \forall \alpha \in \mathbb{R} :$$

$$f_{Re} = y^3 + \alpha x^2 y \wedge \Omega = \mathbb{C}$$

Demonstration:

$$\partial_{xx}u + \partial_{yy}u = 0:$$

$$u := f_{Re}, v := f_{Im}$$

$$f \in \mathcal{H}(\Omega) \rightarrow f \text{ satisfies CR in } \Omega$$

$$\partial_x u = \partial_y v \rightarrow \partial_{xx}u = \partial_{xy}v$$

$$\partial_y u = -\partial_x v \rightarrow \partial_{yy}u = -\partial_{xy}v$$

$$\therefore \partial_{xx}u + \partial_{yy}u = 0$$

$$f_{Re}(x, y) = x^2 + y^2:$$

$$\partial_{xx}u + \partial_{yy}u = 4 \neq 0$$

$$\nexists f \in \mathcal{H}(\Omega) \quad \text{,} \quad f_{Re}(x, y) = x^2 + y^2$$

$$f_{Re}(x, y) = x(x+1) - y^2:$$

$$\begin{aligned} \bar{f} : \Omega &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (u(x, y), v(x, y)) = (x(x+1) - y^2, 2xy + y) \end{aligned}$$

$$\bar{f} \in \text{Pol} \rightarrow \bar{f} \text{ differentiable in } \Omega$$

$$\forall (x, y) \in \Omega :$$

$$\partial_x u(x, y) = 2x + 1 = \partial_y v(x, y)$$

$$\partial_y u(x, y) = -2y = -\partial_x v(x, y)$$

$$f \text{ satisfies CR in } \Omega$$

$$f \in \mathcal{H}(\Omega) \wedge f_{Re} = u$$

$$f_{Re}(x, y) = y^3 + \alpha x^2 y:$$

$$f \text{ has to satisfy CR in } \mathbb{C}:$$

$$\forall (x, y) \in \mathbb{R}^2 :$$

$$\partial_y v(x, y) = \partial_x u(x, y) = 2\alpha xy$$

$$\partial_x v(x, y) = -\partial_y u(x, y) = -3y^2 - \alpha x^2$$

$$v(x, y) = \alpha xy^2 + c(x)$$

$$v(x, y) = -3xy^2 - \frac{\alpha}{3}x^3 + c(y)$$

$$\alpha = -3, \quad c(x) = x^3, \quad c(y) = 0$$

$$v(x, y) = -3xy^2 + x^3$$



## 2. 2nd laboratory

### Power series

Study:

$$\cdot \sum_{n \geq 1} n(n+1)z^n$$

Demonstration:

Naming:

$R$  radius of convergence of the series

$\forall n \in \mathbb{N} :$

$$c_n := n(n+1)$$

Convergence domain:

$$\lim_n \frac{c_n}{c_{n+1}} = \lim_n \frac{n(n+1)}{(n+1)(n+1)} = 1$$

Quotient test:

$$R^{-1} = \overline{\lim}_n |c_n|^{\frac{1}{n}} = 1 \rightarrow R = 1$$

Cauchy-Hadamard theorem:

$\sum_{n \geq 1} n(n+1)z^n$  convergent over  $\mathbb{D}$

$\sum_{n \geq 1} n(n+1)z^n$  divergent over  $\mathbb{C} \setminus \overline{\mathbb{D}}$

$\forall K \subset \mathbb{D} \quad \parallel K \text{ compact} :$

$\sum_{n \geq 1} n(n+1)z^n$  uniformly convergent over  $K$

Define

$$\begin{aligned} f : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto z \sum_{n \geq 1} n(n+1)z^{n-1} \end{aligned}$$



Sum:

$$\begin{aligned} g : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n \geq 1} n(n+1)z^{n-1} \end{aligned}$$

UCI theorem:

$$\int_0^z g(t) dt = \sum_{n \geq 1} (n+1)z^n$$

$$\begin{aligned} h : \mathbb{D} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n \geq 1} (n+1)z^n \end{aligned}$$

$$\int_0^z h(t) dt = \sum_{n \geq 1} z^{n+1} = \sum_{n \geq 0} z^n = \frac{1}{1-z}$$

$$h(z) = \partial_z \frac{1}{1-z} = \frac{1}{(1-z)^2}$$

$$g(z) = \partial_z h(z) = \frac{2}{(1-z)^3}$$

$$f(z) = \frac{2z}{(1-z)^3}$$

Application:

In particular:

$$\sum_{n \geq 1} (-1)^n \frac{n(n+1)}{2^n} = f\left(-\frac{1}{2}\right) = \frac{-2^3}{3^3}$$