1. New

Intergal of power series

Then, holds:

$$\cdot \int_{|z-z_0|=r} (z-z_0)^n dz = \begin{pmatrix} 0 & n \neq 1 \\ 2\pi i & n = -1 \end{pmatrix}$$

Demonstration:

n≥ 0 :

$\frac{(z-z_0)^{n+1}}{n} \in \int (z-z_0)^n$

$\frac{(z-z_0)^{n+1}}{n} \in \mathcal{H}(\mathcal{C})$

CFT over closed curve:

$$\int_{|z-z_0|=r} (z-z_0)^n dz = 0$$

n≤ -1 :

$\frac{(z-z_0)^{n+1}}{n} \in \int (z-z_0)^n$

$\frac{(z-z_0)^{n+1}}{n} \in \mathcal{H}(\mathcal{C} \setminus z_0)$

CFT over closed curve:

$$\int_{|z-z_0|=r} (z-z_0)^n dz = 0$$

$n = -1$

Integral formula of Cauchy over convex open sets

Let:

$\cdot\ \gamma$ closed curve

- $\Omega \subset \mathbb{C}$ convex open " $\gamma^* \subset \Omega$
- $f \in \mathcal{H}(\Omega)$
- $z \notin \gamma^*$

Then, holds:

- $\frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{\omega - z}d\omega = f(z)Ind(\gamma, z)$

Demonstration:

$\forall\ z \notin \gamma^*$:

$$\begin{array}{rccc} \tilde{f} : & \Omega & \longrightarrow & \mathbb{C} \\ & \omega & \longmapsto & \begin{cases} \frac{f(\omega)-f(z)}{\omega - z} & \omega \neq z \\ f'(z) & \omega = z \end{cases} \end{array}$$

$\tilde{f} \in \mathcal{C}(\Omega)$

$\tilde{f} \in \mathcal{H}(\Omega \setminus \{z\})$

Cauchy's theorem:

$$\int_\gamma \tilde{f}(w)dw = 0$$

$z \notin \gamma^* \to \omega \neq z$

$$\int_\gamma \tilde{f}(\omega)d\omega = \int_\gamma \frac{f(\omega) - f(z)}{\omega - z}d\omega$$
$$= \int_\gamma \frac{f(\omega)}{\omega - z}d\omega - f(z)\int_\gamma \frac{1}{\omega - z}d\omega =$$
$$= \int_\gamma \frac{f(\omega)}{\omega - z}d\omega - f(z)2\pi i Ind(\gamma, z) = 0$$
$$\frac{1}{2\pi i}\int_\gamma \frac{f(\omega)}{\omega - z}d\omega = f(z)Ind(\gamma, z)$$

Mean property

Let:

$\cdot\ \Omega \subset \mathbb{C}$ open

$\cdot\ f \in \mathcal{H}(\Omega)$

$\cdot\ D(a, r) \subset \Omega$

Then, holds:

$\cdot\ f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta$

Demonstration:

Integral formula:

$$f(a) = \tfrac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f(z)}{z-a} dz$$

$$\gamma \ := \ \partial D(a, r)$$

$$\gamma(\theta) = a + re^{i\theta}$$

$$\gamma'(\theta) = rie^{i\theta}$$

$$f(a) = \tfrac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}} re^{i\theta} d\theta$$

Independence of γ

Let:

$\cdot\ \Omega \subset \mathbb{C}$ open

$\cdot\ f \in \mathcal{H}(\Omega)$

$\cdot\ \gamma, \tilde{\gamma}$ closed curve " $\gamma^* \subset \Omega$

$\cdot\ z \in \Omega$ " $Ind(\gamma, z) = Ind(\tilde{\gamma}, z)$

Then, holds:

$\cdot \displaystyle\int_{\gamma} \frac{f(\omega)}{\omega - z} d\omega = \int_{\tilde{\gamma}} \frac{f(\omega)}{\omega - z} d\omega$

Demonstration:

no demonstration

Integral formula application

Let:

$\cdot\ \gamma$ pasa por en medio de i,-i y rodea 1/2

Then, holds:

$\cdot \displaystyle\int_{\gamma} \frac{\cos(\frac{pi}{2}\omega)}{\omega} d\omega$

$\cdot \displaystyle\int_{\gamma} \frac{\cos(\frac{pi}{2}\omega)}{\omega - \frac{1}{2}} d\omega$

Demonstration:

$$f(\omega) = \cos(\tfrac{\pi}{2}\omega) \in \mathcal{H}(\mathbb{C}) \text{ convex}$$

$$\int_{\gamma} \frac{\cos(\frac{pi}{2}\omega)}{\omega} d\omega = 2\pi i 1 1 = 2\pi i$$

$$\int_{\gamma} \frac{\cos(\frac{pi}{2}\omega)}{\omega - \frac{1}{2}} d\omega = 2\pi i \frac{\sqrt{2}}{2}(-1)$$

$$\int_{\gamma} \frac{\cos(\frac{pi}{2}\omega)}{\omega - i} d\omega = 2\pi i f(i) 0 = 0$$

$$\int_{\gamma} \frac{\cos(\frac{pi}{2}\omega)}{\omega(\omega^2 + 4)} d\omega = \int_{\gamma} \frac{\cos(\frac{pi}{2}\omega)}{\omega(\omega + 2i)(\omega - 2i)} d\omega$$

$$2i, -2i \notin \gamma^*$$

$$\int_{\gamma} \frac{\cos(\frac{\pi}{2}\omega)/(\omega^2 + 4)}{\omega} d\omega = 2\pi i \frac{1}{4} 1$$

Basic exercises

Let:

$$\cdot\ \begin{array}{rccc} \gamma : & [0,\pi] & \longrightarrow & \mathbb{C} \\ & t & \longmapsto & 1 - cos(t) + isin(t) \end{array}$$

$$\cdot\ \begin{array}{rccc} f : & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \frac{1}{2+z^2} \end{array}$$

$$\cdot\ \begin{array}{rccc} f_2 : & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \frac{sin(z)}{z^2} \end{array}$$

$$\cdot\ \begin{array}{rccc} f_3 : & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \frac{(1+z)^n}{z^{m+1}} \end{array}$$

Then, holds:

$$\cdot \left| \int_\gamma f(z)dz \right| \leq \pi$$

$$\cdot \left| \int_\gamma f_2(z)dz \right| \leq 2\pi(sin(1)^2 + sinh(1)^2)$$

Demonstration:

$$\left| \int_\gamma f(z)dz \right| \leq \sup_{z\in\gamma^*} \mid f(z) \mid long(\gamma)$$

$$\min_{z\in\gamma^*} |2+z^2| = 1 \rightarrow \sup_{z\in\gamma^*} = 1$$

$$\sup_{z\in\gamma^*} \mid f(z) \mid long(\gamma) \leq \pi$$

$$\mid sin(z) \mid^2 = \sin(x)^2 + \sin(hy) \leq (\sin(1))^2 + (sinh(1))^2$$

$$\left| \int_\gamma f_2(z)dz \right| \leq 2\pi(sin(1)^2 + sinh(1)^2)$$

$$\left| \int_\gamma f_3(z)dz \right| \text{Formula binomio}$$

Power Series II

Analytic

Let:

$\cdot\ f : \mathbb{C} \rightarrow \mathbb{C}$

Then, f is analytic if:

· exists power series development of f

Holomorphic functions are power series

Let:

· $\Omega \subset \mathbb{C}$ open

· $f \in \mathcal{H}(\mathbb{C})$

· $D(a, R) \subset \Omega$

Then, holds:

· $f(z) = \sum_{n \geq 0} c_n (z-a)^n$ where

· $c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(\omega)}{(\omega - a)^{n+1}} d\omega, \rho < R$

· $f \in \mathcal{C}^\infty(\Omega)$

Demonstration:

$\forall \ \rho \in (0, R)$:

$$\begin{array}{cccc} \gamma \ : & (0, 2\pi) & \longrightarrow & \mathbb{C} \\ & t & \longmapsto & a+^{it} \end{array}$$

$\forall \ z \in \Omega$ " $|z - a| < \rho$:

$Ind(\gamma, z) = 1$

Integral formula:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\omega)}{\omega - z} d\omega = *$$

$$\frac{1}{\omega - z} = \frac{1}{\omega - a} \frac{1}{1 - \frac{z-a}{\omega-a}}$$

$$\left| \frac{z-a}{\omega-a} \right| < 1$$
$$\frac{1}{1-\frac{z-a}{\omega-a}} = \sum_{n\geq 0} (\frac{z-a}{\omega-a})^n$$
$$\frac{1}{\omega-z} = \sum_{n\geq 0} (\frac{z-a}{\omega-a})^{n+1}$$

UCI theorem:

$$f(z) = * = \frac{1}{2\pi i} \int_\gamma f(\omega) \sum_{n\geq 0} \frac{(z-a)^n}{(\omega-a)^{n+1}} d\omega$$
$$= \frac{1}{2\pi i} \sum_{n\geq 0} \int_{|\omega-a|=\rho} \frac{f(\omega)}{(\omega-a)^{n+1}} d\omega (z-a)^n$$
$$= \sum_{n\geq 0} c_{n,\rho}(z-a)^n$$
$$c_{n,\rho} := \frac{1}{2\pi} \int_{|\omega-a|=\rho} \frac{f(\omega)}{(\omega-a)^{n+1}} d\omega$$

Taylor coefficients of power series:

$$c_{n,\rho} = \frac{f^{n)}(a)}{n!}$$

no dependency of rho $c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(\omega)}{(\omega-a)^{n+1}} d\omega, \rho <$

R

Integral formula for derivatives

Let:

- $\Omega \subset \mathbb{C}$ open
- $D(a,r) \subset \Omega$
- $f \in \mathcal{H}(\Omega)$

Then, holds:

- $f^{n)}(a) = \frac{n!}{2\pi i} \int_{|\omega-a|=\rho} \frac{f(\omega)}{(\omega-a)^{n+1}} d\omega$

Demonstration:

Taylor coefficients substitution

Integral formula for derivatives application

Let:

$\cdot\ statements$

Then, holds:

$$\cdot \int_{|\ z-1\ |=1} \frac{ze^z}{(z-1)^2} dz = \frac{2\pi i}{1!} f'(1) = 2\pi i 2e$$

Demonstration:

demonstration