

Dynamical systems

Martin Azpillaga

March 3, 2014

Block I

Definitions

1. One-dimensional discrete dynamical systems
Dynamical system

Let:

- M manifold
- T monoid
- $\phi : M \times T \rightarrow M$

Then, (M, T, ϕ) is a dynamical system if:

- $\forall x \in X :$

$$\phi(x, 0) = x$$

$$\forall t_1, t_2 \in T :$$

$$\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$$

Dimension

Let:

- (M, T, ϕ) dynamical system

We name dimension of (M, T, ϕ) to:

- $\dim(M)$

We denote:

- $\dim(M) = n : (M, T, \phi)$ n-D dynamical system

Discrete & Continuous

Let:

· (M, T, ϕ) dynamical system

Then, (M, T, ϕ) is discrete if:

· $T \simeq \mathbb{N}$

Then, (M, T, ϕ) is continuous if:

· $T \subset \mathbb{R}$

· T open

Defined by a function

Let:

· (M, T, ϕ) dynamical system

· $f : M \rightarrow M$

Then, (M, T, ϕ) is a dynamical system defined by f if:

· $T = \mathbb{N}$

·
$$\begin{array}{ccc} \phi : M \times \mathbb{N} & \longrightarrow & M \\ (x, n) & \longmapsto & f^n(x) \end{array}$$

We denote:

· (M, T, ϕ) dynamical system defined by $f : (M, \mathbb{N}, f)$

· $f \in \mathcal{C}^n(M) : (M, \mathbb{N}, f)$ \mathcal{C}^n dynamical system

Orbit

Let:

· (M, \mathbb{N}, f) functional dynamical system

· $x \in M$

We name orbit of x to:

· $\{f^n(x)\}_{n \in \mathbb{N}}$

We denote:

· $o(x)$

n-periodic point

Let:

· (M, \mathbb{N}, f) functional dynamical system

· $x \in M$

· $n \in \mathbb{N}$

Then, x is a n-periodic point if:

· $f^n(x) = x$

· $\forall n' \in \mathbb{N} \quad n' < n :$

$$f^{n'}(x) \neq x$$

We denote:

· $\{x \in M \mid f(x) = x\} : Fix(f)$

Attractive & Repulsive

Let:

- (M, \mathbb{N}, f) metrical dynamical system
- $x \in M$ m-periodic point

Then, x is attractive if:

- $\exists \mathcal{U} \in M :$

\mathcal{U} open

$\forall x' \in \mathcal{U} :$

$\exists N \in \mathbb{N} :$

$\forall n \in \mathbb{N} \quad \text{,,} \quad n \geq N :$

$f^{nm}(x') \in \mathcal{U}$

Then, x is repulsive if:

- $\forall \mathcal{U} \subset M \quad \text{,,} \quad \mathcal{U} \text{ open} \quad \wedge \quad x \in \mathcal{U} :$

$\forall x' \in \mathcal{U} \quad \text{,,} \quad x' \neq x :$

$\exists N \in \mathbb{N} :$

$\forall n \in \mathbb{N} \quad \text{,,} \quad n \geq N :$

$f^{nm}(x') \notin \mathcal{U}$

Fixed point character

Let:

$\cdot (M, \mathbb{N}, f)$ functional dynamical system

We name Fixed point character to:

$$\begin{aligned} f : Fix(f) &\longrightarrow \{+, -\} \\ \cdot \quad \quad \quad x &\longmapsto \begin{cases} + & x \text{ repulsive} \\ - & x \text{ attractive} \end{cases} \end{aligned}$$

We denote:

$$\cdot f : \chi$$

Attraction set

Let:

$\cdot (M, \mathbb{N}, f)$ dynamical system

$\cdot x \in M$ attractive m-periodic point

$\cdot o(x)$ orbit of x

We name attraction set of x to:

$$\cdot \{y \in M \mid \exists x' \in o(x) : f^{nm}(y) \xrightarrow{n} x'\}$$

We denote:

$$\cdot A(x)$$

Neutral point

Let:

· (M, \mathbb{N}, f) differentiable dynamical system

· $x \in M$

Then, x is a neutral point if:

· $f'(x) \in \{-1, 1\}$

Feeble point

Let:

· (M, \mathbb{N}, f) \mathcal{C}^3 dynamical system

· $x \in M$ neutral point

Then, x is feeble point if:

· $f''(x) = 0$

Saddle point

Let:

$$\cdot \mathcal{U} \subset \mathbb{R}^n$$

$$\cdot f \in \mathcal{C}^1(\mathcal{U})$$

$$\cdot x \in \mathcal{U}$$

Then, x is a saddle point if:

$$\cdot f'(x) = 0$$

Homeomorphism

Let:

$$\cdot (X_1, \tau_1), (X_2, \tau_2) \text{ topological spaces}$$

$$\cdot f : X_1 \rightarrow X_2$$

Then, f is a homeomorphism if:

$$\cdot f \text{ bijective}$$

$$\cdot f \in \mathcal{C}(X_1)$$

$$\cdot f^{-1} \in \mathcal{C}(X_2)$$

We denote:

$$\cdot \{f : X_1 \rightarrow X_2 \mid f \text{ homeomorphism} \} : \text{Homeo}(X_1)$$

Multiplier

Let:

· $(M, \mathbb{N}, f) \mathcal{C}^1$ dynamical system

· $x \in M$

We name multiplier of x to:

· $f'(x)$

Logistic

Let:

· $f : \mathbb{R} \rightarrow \mathbb{R}$

· (M, T, ϕ) dynamical system defined by f

Then, (M, T, ϕ) is a logistic dynamical system if:

· $\exists a \in \mathbb{R} :$

$$\begin{array}{lll} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & ax(1-x) \end{array}$$

Chaos

Let:

$$\cdot f : \mathbb{R} \rightarrow \mathbb{R}$$

$$\cdot (M, T, \phi) \text{ dynamical system defined by } f$$

Then, (M, T, ϕ) is chaotic if:

$$\cdot \text{Fix}(f) \text{ dense in } \mathbb{R}$$

$$\cdot \exists x \in \mathbb{R} :$$

$$o(x) \text{ dense in } \mathbb{R}$$

$$\cdot f \text{ sensibility of } x_0$$

Sarkovskii's order

We name Sarkovskii's order to:

$$\cdot a = 2^n a', b = 2^m b'$$

$$\cdot a < b \leftrightarrow \begin{cases} m < n & a' = b' = 1 \\ * & a' = 1, b' \neq 1 \\ a' < b' & a' = b' \neq 1 \\ n < m & 1 \neq a' \neq b' \end{cases}$$

We denote:

$$\cdot a < b : a < b$$

Topologically equivalent

Let:

· (M, \mathbb{N}, f) functional dynamical system

· (M, \mathbb{N}, f') functional dynamical system

Then, (M, \mathbb{N}, f) is topologically equivalent to (M, \mathbb{N}, f') if:

· $Fix(f) = Fix(f')$

· $\forall x \in Fix(f) :$

$$character_f(x) = character_{f'}(x)$$

We denote:

· $(M, \mathbb{N}, f) \sim (M, \mathbb{N}, f')$

Bifurcation point

Let:

· $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ functional dynamical system family

· $\lambda' \in \Lambda$

Then, λ' is a bifurcation value if:

· $\forall \varepsilon \in \mathbb{R}^+ :$

$\exists \lambda'' \in (\lambda' - \varepsilon, \lambda' + \varepsilon) :$

$(M, \mathbb{N}, f_{\lambda''})$ not topologically equivalent to $(M, \mathbb{N}, f_{\lambda'})$

Saddle-node bifurcation

Let:

· *statements*

.

Then, *item* is a/an entity if:

· *conditions*

.

We denote:

· *property : notation*

.

Pitchfork bifurcation

Let:

- *statements*
-

Then, *item* is a/an entity if:

- *conditions*
-

We denote:

- *property : notation*
-

Period doubling bifurcation

Let:

· *statements*

·

Then, *item* is a/an entity if:

· *conditions*

·

We denote:

· *property : notation*

·

Block II

Propositions

1. One-dimensional discrete dynamical systems
--

introduction

Fixed points theorem

Let:

- $I \subset \mathbb{R}$ open
- $f : I \rightarrow I$ differentiable
- $x \in I$

Then, holds:

- $|f'(x)| < 1 \rightarrow x$ attractive
- $|f'(x)| > 1 \rightarrow x$ repulsive

Demonstration:

demonstration

Attractiveness of periodic points does not involve the chosen point

Let:

· (M, \mathbb{N}, f) functional dynamical system

· $x \in M$ n-periodic point

· $\{x_i\}_{i=1}^r$ orbit of x

Then, holds:

· x attractive $\leftrightarrow \forall x' \in o(x) :$

x' attractive

Demonstration:

$\forall x' \in o(x) :$

$$f^{n'}(x') = \prod_{i=1}^r f'(x_i) = f^{n'}(x)$$

Partition of attraction set

Let:

- (M, \mathbb{N}, f) functional dynamical system
- x n-periodic point
- $o(x)$ orbit of x

Then, holds:

- $\forall x' \in o(x) :$

$$\exists \mathcal{U} \subset M \text{ open} :$$

$$\forall y \in \mathcal{U} :$$

$$f^n(y) \xrightarrow{n} x'$$

Demonstration:

demonstration

Homeomorphisms are monotonous

Let:

$\cdot f : \mathbb{R} \rightarrow \mathbb{R}$ homeomorphism

Then, holds:

$\cdot f$ monotonous

Demonstration:

no demonstration

Homeomorphisms and n-periodic points

Let:

$\cdot f : \mathbb{R} \rightarrow \mathbb{R}$ homeomorphism (M, T, ϕ) dynamical system defined by f

Then, holds:

$$\cdot \forall n \in \mathbb{N} :$$

$$\exists x \in M \text{ „ } x \text{ n-periodic point}$$

Demonstration:

graphically

Sarkovskii's theorem

Let:

$$\cdot f : I \rightarrow I$$

$$\cdot (M, T, \phi) \text{ dynamical system}$$

Then, holds:

$$\cdot \exists x \in M :$$

$$o(x) \text{ k-period}$$

$$\cdot \rightarrow \forall l \in \mathbb{N} \quad \text{, } l > k :$$

$$\exists x' \in M :$$

$$x' \text{ l-period}$$

Characterization of sella-node bifurcation points

Let:

$$\cdot I \subset \mathbb{R}\{f_\lambda : I \rightarrow I\}_{\lambda \in \Lambda}$$

$$\cdot x \in I$$

Then, holds:

$$\cdot x \text{ SN bifurcation point} \leftrightarrow f(x) = x, f'(x) = 1$$

$$\cdot \partial_\lambda f \neq 0$$

$$\cdot f''(x) \neq 0$$

Demonstration:

demonstration

Pitchfork bifurcation

Let:

- (M, T, f_λ) dynamical system family
- $x \in M$

Then, x is pitchfork bifurcation if:

- x fixed point
- born of 2 fixed points
-

Characterization of Pitchfork bifurcations

Let:

- (M, T, f_λ) dynamical system family
- $x \in M$

Then, holds:

- x Pitchfork \leftrightarrow
- $f(x) = x$
- $f'(x) = 1$
- $\partial_{x^2} f = 0$
- $\partial_\lambda f = 0$

Demonstration:

no demonstration

Block III

Examples

1. One-dimensional discrete dynamical systems
--

Examples of what are and what are not one-dimensional dynamical systems

Analysis of logistic dynamical systems

Let:

· (M, T, ϕ) logistical dynamical system defined by f

Then, holds:

$$\cdot \operatorname{Fix}(f) = \{0, \frac{a-1}{a}\}$$

$$\cdot \operatorname{Per}_2(f) =$$

Demonstration:

demonstration

Quadratic function bifurcations

Let:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto a - x^2 \end{aligned}$$

(M, T, f_c) dynamical system family

Then, f bifurcates in $-1/4$:

$$f_{-1/4}(x) = x \leftrightarrow x = -\frac{1}{2}$$

$$f'_{-1/4}(x) = -2x$$

$$f'_{-1/4}\left(-\frac{1}{2}\right) = 1$$

$$\partial_a f = 1 \neq 0$$

$$\partial_{x^2} f = -2 \neq 0$$

$$\text{sgn}(1 \neq -2) = - \rightarrow -\frac{1}{2} \text{ SN}$$

Block IV

Problems

MODELS I SISTEMES DINÀMICS

Llista 1: Aplicacions unidimensionals

B.1. Trobeu els punts fixos i les òrbites de període 2 de les següents funcions. En el cas que apareixin paràmetres, feu-ho en funció d'aquests.

- | | |
|--|---|
| (a) $* f(x) = 2x(1-x)$, on $x \in \mathbb{R}$. | (c) $f(x) = x^2 + 1$, on $x \in \mathbb{R}$. |
| (b) $* f_c(x) = x^2 + c$, on $x, c \in \mathbb{R}$ (només punts fixos). | (d) $f_{a,b}(x) = ax + b$, on $a, b, x \in \mathbb{R}$. |
| | (e) $f(x) = 2x^2 - 5x$, on $x \in \mathbb{R}$. |

B.2. Fent servir anàlisi gràfic, dibuixeu el retrat de fases de

- | | |
|--|--|
| (a) $f(x) = x^2$, $x \in \mathbb{R}$. | (c) $f_a(x) = ax$, $x \in \mathbb{R}$, pels diferents valors de $a \in \mathbb{R}$. |
| (b) $f(x) = x(1-x)$, $x \in \mathbb{R}$. | |

B.3. * Trobeu els punts fixos atractors i les seves conques d'atracció per a la funció $f(x) = \frac{3x-x^3}{2}$, per $|x| \leq \sqrt{3}$.

B.4. Per a la funció logística $f_a(x) = ax(1-x)$, calculeu els punts fixos i els cicles de període 2 en funció del paràmetre, i determineu-ne l'estabilitat.

1. Estudieu el comportament asimptòtic de la successió $\{x_n\}_{n \in \mathbb{N}}$, pels diferents valors de x_0 indicats.

- | | |
|---|--|
| (a) $* x_{n+1} = \frac{\sqrt{x_n}}{2}$, $x_0 \geq 0$. | (b) $x_{n+1} = \frac{x_n}{2} + \frac{2}{x_n}$, $x_0 \geq 2$. |
|---|--|

2. Donada la successió $x_{n+1} = \frac{x_n+2}{x_n+1}$,

- (a) Trobeu el límit $L = \lim_{n \rightarrow \infty} x_n$ per a $x_0 \geq 0$.
- (b) Descriu el conjunt dels $x_0 < 0$ pels quals el límit $\lim_{n \rightarrow \infty} x_n$ existeix i no és igual a L , o bé no existeix. (Per exemple $x_0 = -1$).

3. (**Examen 2011**) Considereu el sistema dinàmic real definit per $x_{n+1} = \frac{x_n}{4} + x_n^3$. Trobeu el comportament asimptòtic de les òrbites per a tota condició inicial $x_0 \in \mathbb{R}$. Justifiqueu rigorosament les vostres afirmacions.

4. Demostreu rigurosament que $f(x) = \sin(x)$ té $x = 0$ com atractor global.

5. Demostreu que si $f: \mathbb{R} \rightarrow \mathbb{R}$ és derivable, x_0 és un punt fix i $|f'(x_0)| > 1$ llavors x_0 és un punt fix repulsor.

6. Signi $f: \mathbb{R} \rightarrow \mathbb{R}$ de classe \mathcal{C}^∞ i sigui x_0 un punt fix tal que $f'(x_0) = 1$. Doneu criteris sobre les derivades d'ordre superior, per determinar el retrat de fase local al voltant de x_0 . Apliqueu-ho a determinar l'estabilitat dels punts fixos de $x^3 - x$.

1. One-dimensional discrete dynamical system

introduction

Decreasing function orbits

Let:

· *declarations*

·

Show that:

· *statements*

·

Demonstration:

f corta en un punto

f decreasing $\rightarrow f^2$ increasing

$f^{2n} \xrightarrow{n}$ fixed point of f

9. Periodic points

Let:

$$\begin{aligned} f : \mathbb{R} \times \mathbb{R}^+ &\longrightarrow \mathbb{R} \\ (x, r) &\longmapsto r \frac{x}{1+x^2} \end{aligned}$$

Study:

- Periodic points of f

Demonstration:

Graphical analysis :

f odd

f has 2 extrema in ± 1

$f \xrightarrow{n} 0$

Fixed points :

$$f(x) = x \leftrightarrow x = \pm\sqrt{r-1}$$

$$f'(\pm\sqrt{r-1}) = \frac{2-r}{r}$$

n-periodic points :

$$f^n(x) = x$$

10. Global orbit analysis

Let:

$$\cdot f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \in \mathbb{C}^\infty$$

$$\cdot f(0) = 0$$

$$\cdot p \in \mathbb{R}^+ \setminus \{0\} \quad \text{,,} \quad f'(p) \geq 0$$

$$\cdot f' \text{ decreasing}$$

Show that:

$$\cdot \forall x \in \mathbb{R}^+ \setminus \{0\} :$$

$$f^n(x) \xrightarrow{n} p$$

Demonstration:

$$f' \text{ decreasing} \rightarrow f'' < 0 \rightarrow f \text{ concave}$$

$$f \text{ positive} \rightarrow f \text{ has no extrema} \rightarrow f' > 0 \rightarrow f \text{ increasing}$$

$$f \text{ has only one fixed point}$$

$$\text{Suppose 2 fixed points : } p, p'$$

$$IVT \rightarrow \exists c \in (0, p') :$$

$$f'(c) = 1$$

$$f'(p) < 1 \rightarrow p \text{ attractive } IVT \rightarrow \text{dont exist more fixed points}$$

$$\rightarrow f'(c') = 1 \not\leq 1$$

$$\forall x \in (0, p) :$$

$$f(x) > x$$

$$\forall x \in \mathbb{R} \quad \text{,,} \quad x > p :$$

$$f(x) < x$$

$$f \text{ increasing} \rightarrow f([0, p]) = [0, p]$$

Block V

Laboratory

1. Orbit analysis

Martin Azpillaga

Let:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^3 + \frac{1}{4}x \end{aligned}$$

Study:

- Orbit behavior of the real dynamical system defined by f

Demonstration:

Formalization :

Consider (M, T, ϕ) where:

$$M = \mathbb{R}$$

$$T = \mathbb{N}$$

$$\begin{aligned} \phi : \mathbb{R} \times \mathbb{N} &\longrightarrow \mathbb{R} \\ (x, n) &\longmapsto f^n(x) \end{aligned}$$

Study the orbits of $(\mathbb{R}, \mathbb{N}, \phi)$

We will denote $f^n(x)$ as x_n

Fixed points :

$$\forall x \in \mathbb{R} :$$

$$f(x) = x \leftrightarrow x^3 + \frac{1}{4}x - x = 0 \leftrightarrow x^3 - \frac{3}{4}x = 0$$

$$\leftrightarrow x = 0 \vee x^2 - \frac{3}{4} = 0$$

$$x \text{ fixed point} \leftrightarrow x \in \{0, \pm \frac{\sqrt{3}}{2}\}$$

Graphic analysis :

Parity:

$$\forall x \in \mathbb{R} :$$

$$f(-x) = (-x)^3 + \frac{(-x)}{4} = -(x^3 + \frac{x}{4}) = -f(x)$$

f is odd

Monotonicity:

$$\forall x \in \mathbb{R} :$$

$$f'(x) = 3x^2 + \frac{1}{4} > 0$$

f is increasing over \mathbb{R}

Convexity:

$$\forall x \in \mathbb{R}^- :$$

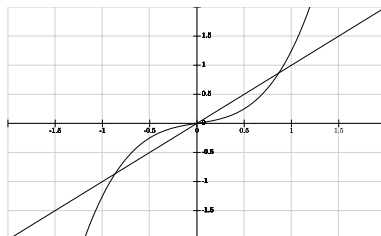
$$f''(x) = 6x \leq 0$$

$$\forall x \in \mathbb{R}^+ :$$

$$f''(x) = 6x \geq 0$$

f is concave over \mathbb{R}^- and convex over \mathbb{R}^+

Graphic representation :



$$\text{I} \forall x \in (-\infty, -\frac{\sqrt{3}}{2}) :$$

Induction over n :

$$f \text{ increasing} \rightarrow f(x_n) < f(-\frac{\sqrt{3}}{2})$$

$$x_{n+1} \in (-\infty, -\frac{\sqrt{3}}{2})$$

$$\therefore o(x) \text{ is enclosed in } (-\infty, -\frac{\sqrt{3}}{2})$$

Induction over n :

$$x_n^2 > \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) > 0$$

$$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) < 0$$

$$\therefore o(x) \text{ decreasing}$$

$$\nexists x < -\frac{\sqrt{3}}{2} \quad \text{" } x \text{ fixed point} \rightarrow o(x) \xrightarrow{n} -\infty$$

$$\text{II} \forall x \in (-\frac{\sqrt{3}}{2}, 0) :$$

Induction over n :

$$f \text{ increasing} \rightarrow f(-\frac{\sqrt{3}}{2}) < f(x_n) < f(0)$$

$$x_{n+1} \in (-\frac{\sqrt{3}}{2}, 0)$$

$$\therefore o(x) \text{ is enclosed in } (-\frac{\sqrt{3}}{2}, 0)$$

Induction over n :

$$x_n^2 < \frac{3}{4} \rightarrow (x_n^2 - \frac{3}{4}) < 0$$

$$x_{n+1} - x_n = x_n^3 - \frac{3}{4}x_n = x_n(x_n^2 - \frac{3}{4}) > 0$$

$$\therefore o(x) \text{ increasing}$$

$$o(x) \text{ convergent} \wedge 0 \text{ fixed point} \rightarrow o(x) \xrightarrow{n} 0$$

$$\text{III} \forall x \in (0, \frac{\sqrt{3}}{2}) :$$

Induction over n :

$$-x_n \in (-\frac{\sqrt{3}}{2}, 0)$$

$$\text{II} \rightarrow f(-x_n) \in (-\frac{\sqrt{3}}{2}, 0) \wedge f(-x_n) > -x_n$$

$$f \text{ odd} \rightarrow f(x_n) = -f(-x_n) \in (0, \frac{\sqrt{3}}{2})$$

$$f \text{ odd} \rightarrow f(x_n) = -f(-x_n) < x_n$$

$$\therefore) o(x) \text{ is enclosed in } (0, \frac{\sqrt{3}}{2}) \wedge o(x) \text{ decreasing}$$

$$o(x) \text{ convergent} \wedge 0 \text{ fixed point} \rightarrow o(x) \xrightarrow{n} 0$$

$$\text{IV} \forall x \in \mathbb{R} \quad \text{II} \quad x > \frac{\sqrt{3}}{2} :$$

Induction over n :

$$-x_n \in (\frac{\sqrt{3}}{2}, \infty)$$

$$\text{I} \rightarrow f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty) \wedge f(-x_n) < -x_n$$

$$f \text{ odd} \rightarrow f(x_n) = -f(-x_n) \in (\frac{\sqrt{3}}{2}, \infty)$$

$$f \text{ odd} \rightarrow f(x_n) = -f(-x_n) > x_n$$

$$\therefore) o(x) \text{ is inf bounded by in } \frac{\sqrt{3}}{2} \wedge o(x) \text{ increasing}$$

$$o(x) \text{ convergent}$$

$$\nexists x > \frac{\sqrt{3}}{2} \quad \text{II} \quad x \text{ fixed point} \rightarrow o(x) \xrightarrow{n} +\infty$$

2. Fixed points cardinality

Martin Azpillaga

Let:

$$\cdot f : [0, 1] \rightarrow [0, 1] \in \mathcal{C}^2([0, 1])$$

$$\cdot f(1) < 1$$

$$\cdot f'' > 0 \in [0, 1]$$

Show that:

$$\cdot \# \{x \in [0, 1] \mid f(x) = x\} = 1$$

Demonstration:

$$\# \{x \in [0, 1] \mid f(x) = x\} \geq 1:$$

Case $f(0) = 0$:

0 fixed point

Case $f(0) > 0$:

$$\begin{array}{ccc} g : [0, 1] & \longrightarrow & [-1, 1] \\ x & \longmapsto & f(x) - x \end{array} \in \mathcal{C}^2([0, 1])$$

$$g(0) = f(0) - 0 > 0$$

$$g(1) = f(1) - 1 < 0$$

Bolzano's theorem:

$$\exists x \in (0, 1) :$$

$$g(x) = 0$$

$$f(x) = x$$

$$\# \{x \in [0, 1] \mid f(x) = x\} \leq 1:$$

$$g'' > 0 \text{ over } [0, 1]$$

Rolle's theorem:

$$\# \{x \in (0, 1) \mid g'(x) = 0\} \leq 1$$

$$\# \{x \in (0, 1) \mid g(x) = 0\} \leq 2$$

$$\# \{x \in (0, 1) \mid f(x) = x\} \leq 2$$

$$f'' > 0 \text{ over } [0, 1]$$

Monotonicity test:

$$f' \text{ increasing in } [0, 1]$$

$$\forall a < b \in [0, 1) \quad f(a) = a, f(b) = b :$$

Mean Value Theorem:

$$\exists c \in (a, b) :$$

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 1$$

$$\exists d \in (b, 1) :$$

$$f'(d) = \frac{f(1) - f(b)}{1 - b} < 1$$

$$f' \text{ increasing} \rightarrow f'(c) < f'(b) < f'(d)$$

$$1 < f'(b) < 1 \text{ absurd}$$

$$\therefore \# \{x \in [0, 1] \mid f(x) = x\} = 1$$