1. Discrete dynamical systems

Dynamical system

Let:

- M manifold
- $(T, +)$ monoid
- $\phi : T \times M \to M$

Then, (M, T, ϕ) is a dynamical system if:

- $\forall\ x \in X$:
  - $\phi(x, 0) = 0$
  - $\forall\ t_1, t_2 \in T$:
    - $\phi(\phi(x, t_1), t_2) = \phi(x, t_1 + t_2)$

Discrete

Let:

- (M, T, ϕ) dynamical system

Then, (M, T, ϕ) is discrete if:

- $T \overset{\subseteq}{\sim} \mathbb{N}$

Dimension

Let:

$\cdot\ (M, T, \phi)$ dynamical system

We name dimension of (M, T, ϕ) to:

$\dim(M)$

We denote:

$\cdot\ dim(M) = n\ :\ (M, T, \phi)$ n-D

Functional

Let:

$\cdot\ (M, T, \phi)$ discrete dynamical system

Then, (M, T, ϕ) is functional if:

$\cdot\ T \subset \mathbb{N}$

$\cdot\ \ \exists\ f\ :\ M \to M$:

$\cdot\ \ \forall\ (t, x) \in T \times M$:

$\cdot\ \ \phi(t, x) = f^t(x)$

We denote:

$\cdot\ T = \mathbb{N}\ :\ (M, f)$ functional dynamical system

$\cdot\ f \in \mathcal{C}^n(M)\ :\ (M, f)\ \ \mathcal{C}^n$

Orbit

Let:

- (M, f) functional dynamical system
- $x \in M$

We name orbit of x to:

$$\{f^n(x)\}_{n\in\mathbb{N}}$$

We denote:

- $o(x)$

Periodicity

Let:

- (M, f) functional dynamical system
- $x \in M$
- $m \in \mathbb{N}$

Then, x is a period m point if:

- $f^m(x) = x$

We denote:

- $\{x \in M \mid x \text{ period 1 point }\} : \text{Fix}(f)$

Stability

Let:

- $(\mathbb{R}^n, f)$ functional dynamical system
- $p \in \mathbb{R}^n$ period m point

Then, p is stable for f if:

- $\forall\ \varepsilon \in \mathbb{R}^+$:
    - $\exists\ \delta \in \mathbb{R}^+$:
        - $\forall\ x \in B(p, \delta)$:
            - $\forall\ n \in \mathbb{N}$:
                - $f^{nm}(x) \in B(p, \varepsilon)$

Then, p is unstable if:

- p not stable

Then, p is attractive for f if:

- p stable
- $\exists\ \varepsilon \in \mathbb{R}^+$:
    - $\forall\ x \in B(p, \varepsilon)$:
        - $f^{nm}(x) \xrightarrow{n} p$

Then, p is repulsive if:

- p attractive by f^{-1}

Attraction set

Let:

$\cdot$ (M, f) functional dynamical system

$\cdot$ $x \in M$ attractive m-periodic point

$\cdot$ $o(x)$ orbit of x

We name attraction set of $o(x)$ to:

$$\{y \in M \mid \exists\, x' \in o(x) \;:\; f^{nm}(y) \xrightarrow{n} x'\}$$

We denote:

$\cdot$ $A(o(x))$

Multiplier

Let:

$\cdot$ (M, f) $\mathcal{C}^1$ functional dynamical system

$\cdot$ $x \in M$

We name multiplier of x to:

$$m(x) = f'(x)$$

We denote:

$\cdot$ $|m(x)| = 1 \;:\; x$ neutral point

$\cdot$ $|m(x)| = 1 \;\wedge\; f''(x) = 0 \;:\; x$ feeble point

Character

Let:

- (M, f) functional dynamical system

We name fixed point character to:

$$\begin{array}{rcl} f \ : \quad \mathrm{Fix}(f) & \longrightarrow & \{-2,-1,0,1,2\} \\ x & \longmapsto & \begin{cases} +2 & x \text{ repulsive} \\ +1 & x \text{ unstable} \\ 0 & * \\ -1 & x \text{ stable} \\ -2 & x \text{ attractive} \end{cases} \end{array}$$

We denote:

- $f \ : \ \chi_f$

Topological equivalence

Let:

- $(M, f_1), (M, f_2)$ functional dynamical systems

Then, $(M, \mathbb{N}, f_1)$ is topologically equivalent to $(M, \mathbb{N}, f_2)$ if:

- $\mathrm{Fix}(f_1) = \mathrm{Fix}(f_2)$
- $\forall\ x \in \mathrm{Fix}(f_1)$:
    - $\chi_{f_1}(x) = \chi_{f_2}(x)$

We denote:

- $(M, f_1) \sim (M, f_2)$

Bifurcation

Let:

- $\{(M, f_\lambda)\}_{\lambda \in \Lambda}$ functional dynamical systems
- $\lambda_0 \in \Lambda$

Then, λ_0 is a bifurcation parameter if:

- $\forall\ \varepsilon \in \mathbb{R}^+$:
  - $\exists\ \lambda' \in B(\lambda_0, \varepsilon)$:
    - $(M, f_{\lambda'}) \not\sim (M, f_{\lambda_0})$

Saddle-node bifurcation

Let:

- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ functional dynamical systems
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is a saddle-node bifurcation value at x_0 if:

- $x_0 \in \mathrm{Fix}(f_{\lambda_0})$
- x_0 neutral point of f_{λ_0}
- $\partial_\lambda f_\lambda(x_0) \neq 0$
- $\partial_{xx} f_\lambda(x_0) \neq 0$

Pitchfork bifurcation

Let:

· $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ functional dynamical systems

· $\lambda_0 \in \Lambda$

· $x_0 \in M$

Then, λ_0 is Pitchfork bifurcation value at x_0 if:

· $x_0 \in \mathrm{Fix}(f_{\lambda_0})$

· x_0 neutral point of f_{λ_0}

· $\partial_\lambda f_\lambda(x_0, \lambda_0) = 0$

· $\partial_{x^2} f_\lambda(x_0, \lambda_0) = 0$

· $\partial_{\lambda x} f_\lambda(x_0, \lambda_0) \neq 0$

· $\partial_{x^3} f_\lambda(x_0, \lambda_0) \neq 0$

Period doubling bifurcation

Let:

- $\Lambda \subset M$
- $(M, \mathbb{N}, f_\lambda)_{\lambda \in \Lambda}$ functional dynamical systems
- $\lambda_0 \in \Lambda$
- $x_0 \in M$

Then, λ_0 is Period doubling bifurcation value at x_0 if:

- λ_0 Pitchfork bifurcation value at x_0 of f^2

Sarkovskii's order

We name Sarkovskii's order to:

$$\forall\ a, b \in \mathbb{N} \quad a = 2^n a', b = 2^m b', 2^n || a, 2^m || b:$$

$$a <_s b \leftrightarrow \begin{cases} m < n & a' = b' = 1 \\ & a' = 1, b' \neq 1 \\ a' < b' & a' = b' \neq 1 \\ n < m & 1 \neq a' \neq b' \end{cases}$$

Chaos

Let:

- $(\mathbb{R}, \mathbb{N}, f)$ dynamical system

Then, $(\mathbb{R}, \mathbb{N}, f)$ is chaotic if:

- $\text{Fix}(f)$ dense in $\mathbb{R}$
- $\exists\ x \in \mathbb{R}$:
  - $o(x)$ dense in $\mathbb{R}$
- $\forall\ x \in \mathbb{R}$:
  - $\exists\ \varepsilon \in \mathbb{R}^+$:
    - $\forall\ \delta \in \mathbb{R}^+$:
      - $\exists\ \tilde{x} \in B(x, \delta)$:
        - $\lim_n\ o(\tilde{x}) \notin B(\lim_n\ o(x), \varepsilon)$