$$\boxed{\textbf{I. \quad Power series}}$$

Power series

Study:

$$\cdot \sum_{n \geq 1} n(n+1)z^n$$

Start:

Convergence domain:

$$\lim_n \frac{n(n+1)}{(n+1)(n+1)} = 1$$

Quotient test:

$$\overline{\lim_n} |c_n|^{\frac{1}{n}} = 1 \rightarrow \left(\overline{\lim_n} |c_n|^{\frac{1}{n}} \right)^{-1} = 1$$

Cauchy-Hadamard theorem:

$\sum_{n \geq 1} n(n+1)z^n$ absolutely convergent over $\mathbb{D}$

$\sum_{n \geq 1} n(n+1)z^n$ divergent over $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$

$\forall \; K \subset \mathbb{D} \; _{_\shortparallel} \; K$ compact :

$$\sum_{n \geq 1} n(n+1)z^n \text{ uniformly convergent over } K$$

Sum:

Consider:

$$
\begin{array}{rcc}
f : & \mathbb{D} & \longrightarrow & \mathbb{C} \\
& z & \longmapsto & \displaystyle\sum_{n \geq 1} n(n+1)z^n
\end{array}
$$

$$
\begin{array}{rcc}
g : & \mathbb{D} & \longrightarrow & \mathbb{C} \\
& z & \longmapsto & \displaystyle\sum_{n \geq 1} n(n+1)z^{n-1}
\end{array}
$$

$$
\begin{array}{rcc}
h : & \mathbb{D} & \longrightarrow & \mathbb{C} \\
& z & \longmapsto & \displaystyle\sum_{n \geq 1} (n+1)z^n
\end{array}
$$

$\forall\ z \in \mathbb{D}:$

$$f(z) = zg(z)$$

UCI theorem:

$$\int_0^z g(t)dt = \sum_{n \geq 1} (n+1)z^n = h(z)$$

g, h well defined

$\forall\ z \in \mathbb{D}:$

UCI theorem:

$$\int_0^z h(t)dt = \sum_{n \geq 1} z^{n+1} = \sum_{n \geq 0} z^n = \frac{1}{1-z}$$

$$h(z) = \partial_z \frac{1}{1-z} = \frac{1}{(1-z)^2}$$

$$g(z) = \partial_z h(z) = \frac{2}{(1-z)^3}$$

$$f(z) = zg(z) = \frac{2z}{(1-z)^3}$$

Application:

$$\sum_{n \geq 1} (-1)^n \frac{n(n+1)}{2^n} = f(-\frac{1}{2}) = \frac{-2^3}{3^3}$$