

I. Linear systems
Real sequences of order 2

Let:

$$\cdot p, q \in \mathbb{R}$$

$$\cdot \forall a, b \in \mathbb{R} :$$

$$x_0 := a$$

$$x_1 := b$$

$$\forall n \in \mathbb{N} :$$

$$x_{n+2} := px_{n+1} + qx_n$$

Study:

$$\cdot \lim_n \frac{x_n}{x_{n+1}}$$

Start:

Consider:

$$A = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}$$

$$A \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}$$

Eigenvalue analysis:

$$\chi_A(t) = t^2 - \text{tr}(A)t + \det(A) = t^2 - pt - q$$

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \chi_A(\lambda) = 0\} = \left\{\frac{p}{2}\right\}$$

$$\lambda := \frac{p}{2}$$

$$E_\lambda(A) = \text{Ker}(A - \lambda \mathbb{1}) = \{(x, y) \in \mathbb{R}^2 \mid \text{-----}\}$$

$\dim(A) = 2 \wedge \sigma(A) = \{\lambda\} \wedge \gamma_A(\lambda) = 1 \rightarrow A$ no diagonalizable

Jordan form:

$$J := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

$$v_2 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_1 := (A - \lambda \mathbb{1})v_2 = \begin{pmatrix} p - \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}$$

$$C := (v_1 | v_2) = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}$$

$$C^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A = CJC^{-1} \rightarrow A^n = CJ^nC^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}$$

Sequence analysis:

$\forall a, b \in \mathbb{R} :$

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = A^n \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} 99 \\ 99 \end{pmatrix}$$

$$\lim_n \frac{x_n}{x_{n+1}} = \lambda = \frac{p}{2}$$

Graphical interpretation:

asdfghjkl