Block I

Definitions

1. The field of complex numbers

introduction

The field of complex numbers

Let:

- $\begin{array}{rcl} + : \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ ((a,b),(c,d)) & \longmapsto & (a+c, b+d) \end{array}$
- $\begin{array}{rcl} \cdot : \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ ((a,b),(c,d)) & \longmapsto & (ac-bd, ad+bd) \end{array}$

We name the field of complex numbers to:

- $(\mathbb{R}^2, +, \cdot)$

We denote:

- $(\mathbb{R}^2, +, \cdot) \ : \ \mathbb{C}$
- $(0,1) \in \mathbb{C} \ : \ i$
- $(a,b) \in \mathbb{C} \ : \ a + bi$
- $\pi_1(\mathbb{C}) \ : \ Re$
- $\pi_2(\mathbb{C}) \ : \ Im$

Conjugation

We name complex conjugation to:

- $\begin{array}{rcl} f : \mathbb{C} & \longrightarrow & \mathbb{C} \\ (a,b) & \longmapsto & (a,-b) \end{array}$

We denote:

- $f((a,b)) \ : \ \overline{(a,b)}$

Norm

We name complex norm to:

$$\cdot \begin{array}{rcl} f : \mathbb{C} & \longrightarrow & \mathbb{R} \\ (a,b) & \longmapsto & \sqrt{a^2+b^2} \end{array}$$

We denote:

- $f((a,b)) : |(a,b)|$

Polar transformation

We name polar transformation to:

$$\cdot \begin{array}{rcl} f : \mathbb{C} & \longrightarrow & \mathbb{R}^+ \times [0, 2\pi) \\ (a,b) & \longmapsto & (\sqrt{|(a,b)|}, \arctan(\frac{b}{a})) \end{array}$$

We denote:

- $f((a,b)) : (r, \theta)$

Unit sphere projection

We name unit sphere projection to:

$$\cdot \begin{array}{rcl} f : \mathbb{C} & \longrightarrow & S^1 \\ z & \longmapsto & \frac{z}{|z|} \end{array}$$

We denote:

- $f(z) : \pi(z)$

Roots of unity

Let:

- $z \in \mathbb{C}$

Then, z is a root of unity if:

- $\exists\, n \in \mathbb{N} \;\; " \; z^n = 1$

We denote:

- $\{z \in \mathbb{C} \mid z \text{ root of unity } \} \; : \; S^1$

Disk

Let:

- $p \in \mathbb{C}$
- $r \in \mathbb{R}^+ \setminus \{0\}$

We name Disk centered in p and radius r to:

- $\{z \in \mathbb{C} \mid |z-p| < r\}$

We denote:

- $\{z \in \mathbb{C} \mid |z-p| < r\} \; : \; D^1$

Components

Let:

- $f : \mathbb{C} \to \mathbb{C}$

We name real component of f to:

- $$\begin{array}{rcl} f_{Re} : \mathbb{C} & \longrightarrow & \mathbb{R} \\ z & \longmapsto & Re(f(z)) \end{array}$$

We name imaginary component of f to:

- $$\begin{array}{rcl} f_{Im} : \mathbb{C} & \longrightarrow & \mathbb{R} \\ z & \longmapsto & Im(f(z)) \end{array}$$

We name component decomposition of f to:

- $$\begin{array}{rcl} f_{\mathbb{R}^2} : \mathbb{C} & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (f_{Re}(x + yi), f_{Im}(x + yi) \end{array}$$

Component decomposition

Let:

- $f : \mathbb{C} \to \mathbb{C}$

- f_{Re}, f_{Im} real components of f

We name real dual of f to:

- $$\begin{array}{rcl} f_{\mathbb{R}^2} : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (f_{Re}(x + yi), f_{Im}(x + yi)) \end{array}$$

2. Holomorphic functions

Incremental quotient

Let:

- $\mathcal{U} \subset \mathbb{C}$ open
- $f : \mathcal{U} \to \mathbb{C}$
- $p \in \mathcal{U}$

We name incremental quotient of f in p to:

- $\lim_{z \to p} \frac{f(z) - f(p)}{z - p}$

We denote:

- $f'(p)$

Holomorphic function

Let:

- $\mathcal{U} \subset \mathbb{C}$ open
- $f : \mathcal{U} \to \mathbb{C}$
- $p \in \mathcal{U}$

Then, f is holomorphic over p if:

- $\exists f'(p)$

Then, f is holomorphic over U if:

- $\forall\, p \in \mathcal{U}$:

  $\exists f'(p)$

We denote:

- $\{f : \mathcal{U} \to \mathbb{C} \mid f \text{ holomorphic over } \mathcal{U}\} : \mathcal{H}(\mathcal{U})$
- $f \in \mathcal{H}(\mathbb{C}) : f$ entire