Complex Analysis

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March 5, 2014

Contents

Block I

Definitions

1. The field of complex numbers

introduction

The field of complex numbers

Let:

- $\begin{array}{rcl} + \; : \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ ((a,b),(c,d)) & \longmapsto & (a+c, b+d) \end{array}$
- $\begin{array}{rcl} \cdot \; : \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ ((a,b),(c,d)) & \longmapsto & (ac-bd, ad+bd) \end{array}$

We name the field of complex numbers to:

- $(\mathbb{R}^2, +, \cdot)$

We denote:

- $(\mathbb{R}^2, +, \cdot) \; : \; \mathbb{C}$
- $(0,1) \in \mathbb{C} \; : \; i$
- $(a,b) \in \mathbb{C} \; : \; a+bi$
- $\pi_1(\mathbb{C}) \; : \; Re$
- $\pi_2(\mathbb{C}) \; : \; Im$

Conjugation

We name complex conjugation to:

- $\begin{array}{rcl} f \; : \mathbb{C} & \longrightarrow & \mathbb{C} \\ (a,b) & \longmapsto & (a,-b) \end{array}$

We denote:

- $f((a,b)) \; : \; \overline{(a,b)}$

Norm

We name complex norm to:

$$\cdot \begin{array}{rcl} f\ :\mathbb{C} & \longrightarrow & \mathbb{R} \\ (a,b) & \longmapsto & \sqrt{a^2+b^2} \end{array}$$

We denote:

$$\cdot\, f((a,b))\ :\ |(a,b)|$$

Polar transformation

We name polar transformation to:

$$\cdot \begin{array}{rcl} f\ :\mathbb{C} & \longrightarrow & \mathbb{R}^+ \times [0,2\pi) \\ (a,b) & \longmapsto & (\sqrt{|(a,b)|}, \arctan(\frac{b}{a})) \end{array}$$

We denote:

$$\cdot\, f((a,b))\ :\ (r,\theta)$$

Unit sphere projection

We name unit sphere projection to:

$$\cdot \begin{array}{rcl} f\ :\mathbb{C} & \longrightarrow & S^1 \\ z & \longmapsto & \frac{z}{|z|} \end{array}$$

We denote:

$$\cdot\, f(z)\ :\ \pi(z)$$

Roots of unity

Let:

- $z \in \mathbb{C}$

Then, z is a root of unity if:

- $\exists\ n \in \mathbb{N}$:

$$z^n = 1$$

We denote:

- $\{z \in \mathbb{C} \mid z \text{ root of unity } \}\ :\ S^1$

Disk

Let:

- $p \in \mathbb{C}$
- $r \in \mathbb{R}^+ \setminus \{0\}$

We name Disk centered in p and radius r to:

- $\{z \in \mathbb{C} \mid |z - p| < r\}$

We denote:

- $\{z \in \mathbb{C} \mid |z - p| < r\}\ :\ D^1$

Component decomposition

Let:

- $f : \mathbb{C} \to \mathbb{C}$

We name real component of f to:

- $$\begin{array}{rcl} f_{Re} : \mathbb{C} & \longrightarrow & \mathbb{R} \\ z & \longmapsto & Re(f(z)) \end{array}$$

We name imaginary component of f to:

- $$\begin{array}{rcl} f_{Im} : \mathbb{C} & \longrightarrow & \mathbb{R} \\ z & \longmapsto & Im(f(z)) \end{array}$$

We name component decomposition of f to:

- $$\begin{array}{rcl} f_{\mathbb{R}^2} : \mathbb{C} & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (f_{Re}(x + yi), f_{Im}(x + yi)) \end{array}$$

2. Holomorphic functions

Incremental quotient

Let:

- $\mathcal{U} \subset \mathbb{C}$ open
- $f : \mathcal{U} \to \mathbb{C}$
- $p \in \mathcal{U}$

We name incremental quotient of f in p to:

- $\lim_{z \to p} \dfrac{f(z) - f(p)}{z - p}$

We denote:

- $f'(p)$

Holomorphic function

Let:

- $\cdot\, \mathcal{U} \subset \mathbb{C}$ open
- $\cdot\, f : \mathcal{U} \to \mathbb{C}$
- $\cdot\, p \in \mathcal{U}$

Then, f is holomorphic over p if:

- $\cdot\, \exists f'(p)$

Then, f is holomorphic over U if:

- $\cdot\, \forall\, p \in \mathcal{U}$:

  $\exists f'(p)$

We denote:

- $\cdot\, \{f : \mathcal{U} \to \mathbb{C} \mid f$ holomorphic over $\mathcal{U}\} : \mathcal{H}(\mathcal{U})$
- $\cdot\, f \in \mathcal{H}(\mathbb{C}) : f$ entire

Cauchy-Riemman equations

Let:

- $u, v \,:\, \mathbb{R}^2 \to \mathbb{R}$
- $\begin{aligned} f \,:\mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (u((x,y)), v((x,y))) \end{aligned}$

Then, f is satisfies the Cauchy-Riemman equations if:

- $\exists\, u_x, u_y, v_x, v_y$
- $u_x = v_y$
- $u_y = -v_x$

We denote:

- $u_x + iv_x \,:\, f_x$
- $u_y + iv_y \,:\, f_y$

Conformal

Let:

$\cdot\, \mathcal{U} \subset \mathbb{C}$ open

$\cdot\, f : \mathcal{U} \to \mathbb{C}$

$\cdot\, z \in \mathcal{U}$

Then, f is conformal in z if:

$\cdot\; \exists\, c \in \mathbb{C}$:

$\forall\, I \subset \mathbb{R}$ „ $0 \in I$:

$\forall\, \gamma : I \to \mathbb{R}^2$ „ γ differentiable $\wedge\, \gamma(0) = z \,\wedge\, \gamma'(0) \neq$

0 :

$$\frac{(f \circ \gamma)'(0)}{\gamma'(0)} = c$$

Then, f is conformal if:

$\cdot\, \forall\, z \in \mathcal{U}$:

f conformal in z

Power series

Let:

- $\sum_{n\geq 0} a_n f_n$ complex valued sequence

Then, $\sum_{n\geq 0} a_n f_n$ is a power series if:

- $\forall\ n \in \mathbb{N}$:

$$\begin{array}{rcl} f_n : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & (z-a)^n \end{array}$$

Convergence radius

Let:

- *statements*
-

Then, *item* is a/an entity if:

- *conditions*
-

We denote:

- *property* : *notation*
-

Absolutely convergent

Let:

- $\sum c_n n[0]$ series

Then, $\sum c_n n[0]$ is absolutely convergent if:

- $\sum |c_n| n[0]$ convergent

Numeric series

Let:

- $(c_k)_{k\in\mathbb{N}}$

Then, *item* is a/an entity if:

- *conditions*
-

We denote:

- *property* : *notation*
-

Block II

Propositions

1. The field of complex numbers

introduction

go

2. Holomorphic functions

Cauchy-Riemman

Let:

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies CR

Then, holds:

- $f_x = f'(z)$
- $f_y = -if_y$
- $\partial_{f(z)} z = 0$

Demonstration:

$f_x = u_x + iv_x = a + bi = f'(z)$

$f_y = i(a + ib) = if'(z)$

$f'(z) = -if_y$

notation

tangent venctor

Let:

$\cdot\ \gamma$ differentiable plane arc " $\forall\ t \in I$:

$\gamma'(t) \neq 0$

Then, holds:

$\cdot\ \gamma'(t)$ tangent to γ

Demonstration:

demonstration

arc images

Let:

· $f \in \mathcal{H}(\mathcal{U})$

· γ differentiable plane arc " $\gamma \subset \mathcal{U}$

· $\sigma = f(\gamma)$

· $z_0 = \gamma(0)$

Then, holds:

· $\sigma' = f'(\gamma)\gamma'$

· $\gamma'(0) \neq 0 \rightarrow f'(z_0) \neq 0$

· $\sigma'(0) = f'(z_0)\gamma'(0)$

· $|\sigma'(0)| = |f'(z_0)||\gamma'(0)|$

· $arg\sigma'(0) = arg\gamma'(0) + argf(z_0)$

· f aplica una homotecia mas una rotacion constante a todos los vectores tangentes que salen de z0

Demonstration:

obvio

Holomorphic functions are conform

Let:

- $f : \mathcal{U} \to \mathbb{C}$
-

Then, holds:

- $f$holomorph in z „ $f'(z) \neq 0 \leftrightarrow f$ conform

Demonstration:

$\rightarrow$):

already seen

$\leftarrow$):

too hard

Convergence of complex series

Let:

$\cdot \sum_{n \geq 0} c_n$ complex series

Then, holds:

$\cdot \sum_{n \geq 0} c_n$ convergent $\leftrightarrow \sum_{n \geq 0} Rec_n$ convergent $\wedge \sum Imc_n n[0]$ convergent

Demonstration:

demonstration

Absolutely convergent are convergent

Let:

- $\sum c_n n[0]$ absolutely convergent

Then, holds:

- $\sum c_n n[0]$ convergent

Demonstration:

$S_k := \sum c_n n[0][k]$

$\forall\ m \in \mathbb{N}$ „ $m < k$:

$|S_k - S_m| = |\sum c_n n[m+1][k]| \leq \sum |c_n| n[m+1][k]$

$\leq \sum |c_n| n[m+1] \xrightarrow{n} 0$

$|S_k - S_m| \xrightarrow{n} 0 \to (S_k)_k$ convergent $\to \sum c_n n[0]$ convergent

Series and norm

Let:

- $\sum c_n n[0]$ convergent

Then, holds:

- $|c_n| \xrightarrow{n} 0$

Demonstration:

$\sum c_n n[0]$ convergent $\leftrightarrow (S_n)_n$ convergent

$\rightarrow$ Cauchy $|S_n - S_m| \xrightarrow{n} 0$ por n y m $\rightarrow |S_n - S_{n-1}| \xrightarrow{n} 0$

$\rightarrow |c_n| \xrightarrow{n} 0$

Root test

Let:

- $\sum\limits_{n \geq 0} c_n$ real series
- $l \in \mathbb{R}$ ″ $\overline{\lim\limits_{k}} |c_k|^{\frac{1}{k}} = l$

Then, holds:

- $l > 1 \to \sum\limits_{n \geq 0} c_n \notin \mathbb{R}$
- $l < 1 \to \sum\limits_{n \geq 0} c_n \in \mathbb{R}$

Demonstration:

$demonstration$

Quotient test

Let:

- $\sum_{n \geq 0} c_n$ real series
- $l \in \mathbb{R}$ "

Then, holds:

- $\exists\, l \in \mathbb{R}$:

$$\lim_{k} \frac{c_{k+1}}{c_k} = l$$

- $\overline{\lim_{c_k}} \, |c_k|^{\frac{1}{k}} k = l$

Power series theorem

Let:

- $\sum_{n\geq 0} a_n c^n$ power series

Then, holds:

- $|z-a| < R \to$ absolutely convergent
- $|z-a| > R \to$ divergent
- convergent in $D(a,R)$
- $\begin{array}{rccl} f: \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \sum_{n\geq 0} c_n (z-a)^n \end{array} \in \mathcal{H}(D(a,R))$
- $\forall\ z \in \mathbb{C}$:

$$f'(z) = \sum_{n\geq 0} nc_n(z-a)^{n-1} \text{ convergent}$$

- convergence radius of f' = convergence radius of f

Demonstration:

$\forall\ z \in \mathbb{C}$ " $|z-a| < R$:

Root test over $\sum_{n\geq 0} |c_n||z-a|^n rlimit(|c_n||z-a|^n)^{\frac{1}{n}} n = |z - a| rlimit |c_n|^{\frac{1}{n}} n = \frac{|z-a|}{R} < 1$ Root test $\to$ absolutely convergent

$\forall\ z \in \mathbb{C}$ " $|z-a| < R$:

$\forall\ \rho \in \mathbb{R}$ " $|z-a| < \rho < R$:

$\frac{1}{\rho} < \frac{1}{R}$

$rlimit|c_n|^{1/n} n = \frac{1}{R}\imath$ exists partial of $|c_n|^{1/n}$

$|c_n||z-a|^n > \frac{|z-a|^n}{\rho^n}$ no $\xrightarrow{n} 0$

General term test → divergent

Power series theorem

Let:

·

Then, holds:

$\cdot PartII, III, IV$

Demonstration:

Follow 3 steps

Step 1 : Uniform convergence in compacts of $D(a,R)$:

$$\begin{array}{rcl} g_n : \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & c_n(z-a)^n \end{array}$$

$\forall\ \rho \in \mathbb{R}$ " $\rho < R$:

$\forall\ z \in \overline{D(a,\rho)}$:

$$|g_n(z)| = |c_n||z-a|^n \leq |c_n|\rho^n$$

$M_n := |c_n|\rho^n$

$rlimit(|c_n|\rho^n)^{1/n}n = \rho rlimit|c_n|^{1/n}n = \frac{\rho}{R} < 1$

Root test→ $\sum_{n\geq 0} M_n$ convergent

M-Weierstrass→ $\sum_{n\geq 0} c_n(z-a)^n$ uniformly convergent over compacts of $D(a,\rho)$

$\sum_{n\geq 0} c_n(z-a)^n$ uniformly convergent over compacts of $D(a,r)$

$$f(z) := \sum_{n\geq 0} g_n(z)$$

g uniformly convergent $\to f$ continuous

Step 2 : IV. R' :

$$\tilde{f}(z) := \sum_{n\geq 1} nc_n(z-a)^{n-1}$$

$$\tilde{f}(z) = \sum_{n\geq 0} (n+1)c_{n+1}(z-a)^n$$

$\frac{1}{R'} = rlimit(n+1)|c_{n+1}|^{1/n}n = rlimit(n+1)^{1/n}|c_{n+1}|^{1/n}n = rlimit(|c_{n+1}|^{1/n+1})^{\frac{n+1}{n}}n = \frac{1}{R}$

$R' = R$

Step 3 : III :

$\tilde{f}$well defined in $D(a,R)$

$\forall\ z_0 \in D(a,R)$:

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - \tilde{f}(z_0)\right| \xrightarrow{n} 0?$$

$\forall\ n \in \mathbb{N}$:

$$S_n(z) := \sum_{k=0}^{n} f_k(z)$$

$$R_n(z) := \sum_{k\geq n+1} f_k(z)$$

$$f(z) = S_n(z) + R_n(z)$$

$$\tilde{f}(z) = \tilde{S_n}(z) + \tilde{R_n}(z)$$

$$R_n(z), \tilde{R_n}(z) \xrightarrow{n} 0$$

$\forall\ \varepsilon \in \mathbb{R}^+$:

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \left|\frac{S_n(z)-S_n(z_0)}{z-z_0} + \frac{R_n(z)-R_n(z_0)}{z-z_0} - \tilde{S_n}(z_0) - \tilde{R_n}(z_0)\right|$$

$\forall\ \rho \in \mathbb{R}$ „ $|z_0 - a| < \rho < R$:

$$\left|\tilde{R_n}(z_0)\right| \le \sum_{k \ge n+1} k|c_k|\rho^{k-1} < \frac{\varepsilon}{3}(n \ge n_1)$$

$$\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| \le \sum_{k \ge n+1} |c_k|\left|\frac{(z-a)^k - (z_0-a)^k}{z-z_0}\right|$$

$$\frac{a^k-b^k}{a-b} = a^{k-1} - a^{k-2}b + - - - + b^{k-1}$$

$$\le \sum_{k \ge n+1} |c_k|(|z-a|^{k-1} + |z-a|^{k-2}|z_0-a| + - - - +$$

$|z_0 - a|^{k-1}$

$$|z-a|, |z_0| < \rho$$

$$\le \sum_{k \ge n+1} |c_k|k\rho^{k-1} < \frac{\varepsilon}{3}(n \ge n_1)$$

$$\left|\frac{S_n(z)-S_n(z_0)}{z-z_0} - \tilde{S_n}(z_0)\right| < \frac{\varepsilon}{3}(S_n' = \tilde{S_n})(n \ge n_2)$$

$\forall\ n \in \mathbb{N}$ „ $n \ge \max(n_1, n_2)$:

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \varepsilon$$

functions associated to series are $\mathcal{C}^\infty$

Let:

$\cdot$ $f(z) = \sum c_n (z-a)^n n[0]$ series

$\cdot$ R radius of convergence of f

Then, holds:

$\cdot$ $f \in \mathcal{C}^\infty$ over $D(a,R)$

$\cdot$ $\forall\ n \in \mathbb{N}$:

$$f^{n)} \in \mathcal{H}(D(a,R))$$

$\cdot$ $c_k = \frac{f^{k)}}{k!}$

$\cdot$ series associated to f is unique

Demonstration:

$$f^{k)}(z) = \sum n(n-1) --- (n-k+1) c_n (z-a)^{n-k} n[k]$$

$$f^{k)}(a) = k! c_k$$

$$c_k = \frac{f^{k)}(a)}{k!}$$

Block III

Examples

1. Holomorphic functions

introduction

go

2. Holomorphic functions

Conjugation

Let:

$$.\quad \begin{array}{rccc} \bar{a} & : \mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \bar{z} \end{array}$$

Then, $\bar{a}$ is not holomorphic :

$u_x = 1$

$u_y = 0$

$v_x = 0$

$v_y = -1$

$\forall\ z \in \mathbb{C}$:

$-1 \neq 1 \to f$ not holomorphic in z

Quadratic norm

Let:

- $$\begin{array}{rlcl} f & : \mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & |z|^2 \end{array}$$
- $f_{\mathbb{R}^2}$ component decomposition of f

Then, f is holomorphic in 0:

f differentiable in $\mathbb{R}^2$ polinomial

$\forall\ z \in \mathbb{C}$:

$u_x(x, y) = 2x$

$u_y(x, y) = 2y$

$v_x(x, y) = 0$

$v_y(x, y) = 0$

$u_x = v_y \leftrightarrow x = 0$

$u_y = -v_x \leftrightarrow y = 0$

f holomorphic function in $z \leftrightarrow z = 0$

Non preserving angles function

Let:

$\cdot\ f(z) = z^2$

Then, f is conform in $\mathbb{R} \setminus \{0\}$:

$f(\{(x,0) \in \mathbb{C} \mid x > 0\}) = \{(x,0) \in \mathbb{C} \mid x > 0\}$

$f(\{(x,0) \in \mathbb{C} \mid x < 0\}) = \{(x,0) \in \mathbb{C} \mid x > 0\}$

$ang(A,B) = \pi \neq 0 = ang(f(A), f(B))$

Exponential

Let:

$\cdot\ a\ :\ 0$

$\cdot\ c_n\ :\ \frac{1}{n}$

Then, $\sum_{n\geq 0} c_n(z-a)^n$ is convergent in D1 :

$\lim_n \frac{|c_n|}{|c_{n+1}|} = \lim_n \frac{n+1}{n} = 1 \to R = 1$

CH$\to$ $D(0,1)$ convergent

$\mathbb{C} \setminus D(0,1)$ divergent

$f' = f$

Geometric series

Let:

- $a \; : \; 0$
- $c_n : 0$

Then, $\sum z^n n[0]$ is convergent in $\mathbb{D}$:

$R = \frac{c_n}{c_{n+1}} = 1$ Then, holds:

- $\sum z^n n[0] = \frac{1}{1-z}$
- $\sum n z^{n-1} n[0] = \frac{1}{(1-z)^2}$
- $\sum \frac{z^{n+1}}{n+1} n[0] = -\log(1-z)$

Demonstration:

$\forall \; z \in \mathbb{D}$:

$\sum z^n n[0]$ geometric series

$\sum z^n n[0] = \frac{1}{1-z}$

II differentiating

III integrating

Series not centered in 0

Let:

- a : i
- c_n : $\frac{n+1}{5^{n+1}}$

Then, *item* is a/an entity :

$\sum \frac{n(z-i)^{n-1}}{5^n} n[1]$

$= \frac{1}{5} \sum n \frac{z-i}{5}^{n-1} n[1] = \frac{1}{5} \sum n u^{n-1} n[1]$

$S(u) = \tilde{S}'(u)$

$\tilde{S}(u) = \frac{1}{5} \sum u^n n[1] = \frac{u}{5(1-u)}$

$S(u) = \frac{1}{5(1-u)^2}$

$S(z) = \frac{5}{(5+i-z)^2}$ over $D(i,5)$

Radius of convergence without quotient test

Let:

- $\sum \frac{(-1)^n}{n(n+1)} (z-2)^{n(n+1)} n[1]$

Then, R is a/an entity :

$\lim_{c_{n+1}} c_n \nexists$

$\lim_n \frac{1}{n(n+1)}^{\frac{1}{n(n+1)}} = 1$

ignore zeros

Block IV

Problems

PROBLEMES D'ANÀLISI COMPLEXA
2n quadrimestre del curs 2013-2014.

Llista 1: Els nombres complexos

B.1. Expresseu en la forma $a+ib$ els següents nombres:

(a) $(2+3i)(4+i)$ (c) $\frac{1}{4+i}$ (e) $\sqrt{i}$ (g) $\sqrt{9i}$
(b) $(4+2i)^2$ (d) $\frac{i}{4+i}$ (f) $\sqrt{-i}$ (h) $\sqrt{1+i}$

B.2. Si $z=x+iy$ trobeu les parts real i imaginària de les expressions següents:

(a) z^2 (b) $z(z+1)$ (c) $\frac{1}{z}$ (d) $\frac{1}{z-3}$.

B.3. És cert que

a) $\operatorname{Re}(z+w)=\operatorname{Re} z+\operatorname{Re} w$?
b) $\operatorname{Re}(zw)=(\operatorname{Re} z)(\operatorname{Re} w)$?
c) $\operatorname{Re}(\frac{z}{w})=\frac{\operatorname{Re} z}{\operatorname{Re} w}$?

B.4. Trobeu la forma polar dels nombres següents i dibuixeu-los.

(a) $3(1+\sqrt{3}\,i)$ (b) $2\sqrt{3}-2i$ (c) $-2+2i$ (d) $-1-i$

B.5. Sigui $(x+iy)/(x-iy)=a+ib$. Proveu que $a^2+b^2=1$.

B.6. Proveu que si $p(z)$ és un polinomi amb coeficients reals i z és un zero de p llavors $\bar{z}$ també ho és.

B.7. Descriviu els conjunts del pla que satisfan (recordeu que $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$.)

(a) $\operatorname{Im}\frac{z-a}{z}=0, a\in\mathbb{C}^*$ (b) $|z|=\operatorname{Re} z+1$ (c) $|z-2|>|z-3|$

SOL. B.1. a) $5+14i$; b)$12+16i$; c) $4/17-i/17$; d)$1/17+4i/17$; e) $\pm\sqrt{2}/2(1+i)$; f)$\pm\sqrt{2}/2(1-i)$; g) $\pm3\sqrt{2}/2(1+i)$; h) $\pm 2^{1/4}(\cos(\pi/8)+i\sin(\pi/8))$.

B.2 a) x^2-y^2+2ixy; b) $x^2-y^2+x+i(y+2xy)$; c) $(x-iy)/(x^2+y^2)$; d) $(x-3-iy)/((x-3)^2+y^2)$.

B.3 a) si. b) no. c) no.

B.4 a) $6(\cos(\pi/3)+i\sin(\pi/3))$; b) $4(cos(\pi/6)-i\sin(\pi/6))$; c) $2\sqrt{2}(\cos(\pi/4)+i\sin(\pi/4))$; d) $\sqrt{2}(cos(3\pi/4)-i\sin(3\pi/4))$.

B.6 Conjugueu tot el polinomi.

B.7 a) Recta que passa per 0 i a; b) Paràbola horitzontal $x=(1/2)(y^2-1)$; c) $\{\operatorname{Re} z>3/2\}$.

1. Expresseu en la forma $a+ib$ els següents nombres:

(a) $\frac{1}{i}$ (c) $\frac{1}{2+i}+\frac{1}{2-i}$ (e) $\left(\frac{2+i}{3-2i}\right)^2$ (g) $\sqrt[4]{-i}$
(b) $\frac{1+i}{1-i}$ (d) $\frac{1}{2+i}+\frac{4-2i}{3+i}$ (f) $(1+i)^{100}+(1-i)^{100}$ (h) $(3+4i)^{\frac{1}{2}}$

2. Si $z=x+iy$ on $x,y\in\mathbb{R}$, trobeu les parts real i imaginària de:

PROBLEMES D'ANLISI COMPLEXA
2n quadrimestre del curs 2013-2014

Llista 2: Funcions de variable complexa i equacions de Cauchy-Riemann

B.1. Trobeu els punts on la funció f és derivable (en el sentit complex), en els següents casos, i calcula'n la derivada.

(a) $\cos|z|^2$ (c) e^{iz} (e) $\frac{1}{(z-1)^2(z^2+2)}$

(b) $|z|^4$ (d) $z+\frac{1}{z}$ (f) $\frac{1}{(z+\frac{1}{z})^2}$

Solució: (a) $\emptyset$; (b) $\emptyset$; (c) $\mathbb{C}$; $f'(z) = ie^{iz}$;(d) $\mathbb{C}\setminus\{0\}$; $f'(z) = 1 - |\frac{1}{z^2}$; (e) $\mathbb{C}\setminus\{1, \pm\sqrt{2}i\}$; (f) $\mathbb{C}$.

B.2. Determineu si aquestes funcions poden ser la part real d'una funció holomorfa, i en cas que ho siguin calculeu la part imaginària.

(a) $e^x \cos y$ (b) $x^3 + 6xy^2$ (c) $\log(x^2+y^2)$

Solució: (a) $e^x \sin y$; $f(z) = e^z$; (b) No ho és; (c) $2\arctan(y/x)$; $(f(z) = \log(z^2)$.

B.3. Sigui f una funció holomorfa en un obert $\Omega \subset \mathbb{C}$ i $z_0 \in \Omega$ tal que $f'(z_0) \neq 0$. Quin angle formen les corbes $\operatorname{Re} f(z) = \operatorname{Re} f(z_0)$ i $\operatorname{Im} f(z) = \operatorname{Im} f(z_0)$ en un punt z_0 ?
Solució: $\pi/2$.

1. Trobeu els punts on la funció f és derivable (en el sentit complex), en els següents casos:

(a) $f(z) = |z|$ (d) $f(z) = z + z\bar{z}$

(b) $\cosh x \cos y + i \sinh x \sin y$

(c) $f(z) = \operatorname{Re} z$ (e) $f(z) = \operatorname{Im} e^{\bar{z}} + i \operatorname{Re} e^z$

2. Sigui $\Omega \subset \mathbb{C}$ un obert, $z_0 \in \Omega$ i $f : \Omega \to \mathbb{C}$ una funció.

a) Identificant $\mathbb{R}^2$ amb $\mathbb{C}$ de la forma habitual, demostreu que si f és diferenciable en z_0, llavors

$$Df(z_0)(z) = \frac{\partial f}{\partial z}(z_0)\cdot z + \frac{\partial f}{\partial \bar{z}}(z_0)\cdot \bar{z} \qquad (z \in \mathbb{C}),$$

on

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

b) Proveu que f és holomorfa en Ω si, i només si, f és diferenciable i $\frac{\partial f}{\partial \bar{z}} = 0$ en Ω. En tal cas, $f' = \frac{\partial f}{\partial z}$.

3. Demostreu que si f és diferenciable en un obert de $\mathbb{C}$, llavors

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}} \quad \text{i} \quad \overline{\frac{\partial f}{\partial \bar{z}}} = \frac{\partial \bar{f}}{\partial z}.$$

1. The field of complex numbers

introduction

entity

Let:

- *statements*
-

Then, *item* is a/an entity if:

- *conditions*
-

We denote:

- *property* : *notation*
-

2. Holomorphic functions

3. Cauchy-Riemman

Let:

$\cdot\ f \in \mathcal{H}(\mathbb{C})$ „ $Ref + Imf = c_a$

Show that:

$\cdot\ \ \exists\ a' \in \mathbb{C}$:

$f = c_{a'}$

Demonstration:

u, v real components of f

$u(x,y) + v(x,y) = a$

differentiate respect x and y

$u_x + v_x = 0$

$u_y + v_y = 0$

f holomorphic $\to f$ CR

$u_x - u_y = 0$

$u_y + u_x = 0$

$u_x, u_y, v_x, v_y = 0$

$\exists\ a_1 \in \mathbb{R}$:

$u = c_{a_1}$

$\exists\ a_2 \in \mathbb{R}$:

$v = c_{a_2}$

$f = c_{(a_1,a_2)}$

B.2 a)

Let:

$$\cdot \begin{array}{rcl} u : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & \exp(y)\cos(x) \end{array}$$

Show that:

$\cdot\ \exists\ f \in \mathcal{H}(\mathbb{C})$:

u real component of f

Demonstration:

lab 1→ $u_{xx} + v_{yy} = 0$

$u_x = \exp(x)\cos(y)$

$u_{xx} = \exp(x)\cos(y)$

$u_y = -\exp(x)\sin(y)$

$u_{yy} = -\exp(x)\cos(y)$

ok

Calculate v using CR

$v_y = u_x = \exp(x)\cos(y)$

$v(x, y) = \int_{\mathbb{C}} \exp(x)\sin(y)dy = \exp(x)\sin(y) + \phi(x)$

$v_x = \partial_v x = \exp(x)\sin(y) + \phi'(x)$

$-u_y = \exp(x)\sin(y) + \phi'(x)$

CR→ $\phi'(x) = 0$

$\forall\ c \in \mathbb{R}$:

$\phi(x) = c$ ok

$v(x, y) = \exp(x)\sin(y)$

Preservation of angles

Let:

- γ_1, γ_2 plane arcs " $\gamma_1(0) = \gamma_2(0)$

Then, holds:

- angle of $\gamma_1'(0)$ and $\gamma_2'(0)$ = angle $\sigma_1'(0), \sigma_2'(0)$

Demonstration:

rotations and homotecies let angles invariant

Block V

Tasks

1. 1st laboratory

Existence of holomorphic functions

Let:

$\cdot\ f \in \mathcal{H}(\mathbb{D})$

Study:

$\cdot\ \exists\ f \in \mathcal{H}(\mathbb{D})$:

$\forall\ n \in \mathbb{N}$ „ $n \geq 2$:

$a)\ f(\pm\frac{1}{n}) = \frac{1}{2n+1}$

$b)\ f(\pm\frac{1}{n}) = \frac{1}{n^2}$

$c)\ |f(\frac{1}{n})| = \frac{1}{\log(n+1)}$

$d)\ |f(\frac{1}{n})| = \frac{n}{n+1}$

Demonstration:

$a)$:

$$E_1 := \{+\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N}\}$$

$$E_2 := \{-\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N}\}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} = \lim_{n} \frac{f(\frac{1}{n})}{\frac{1}{n}} - \frac{f(0)}{\frac{1}{n}} = \frac{1}{2} - \lim_{n} \frac{f(0)}{\frac{1}{n}}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} \begin{cases} = \frac{1}{2} & f(0) = 0 \\ \notin \mathbb{C} & f(0) \neq 0 \end{cases}$$

Case $f(0) = 0$:

$$\lim_{E_2} \frac{f(z) - f(0)}{z - 0} = \lim_{n} \frac{f(-\frac{1}{n})}{-\frac{1}{n}} = -\frac{1}{2} \neq \lim_{E_1} \frac{f(z) - f(0)}{z - 0}$$

$$\nexists\ f \in \mathcal{H}(0) \ \ \text{"} \ \ f \text{ satisfies } a)$$

In particular:

$$\nexists\ f \in \mathcal{H}(\mathbb{D}) \ \ \text{"} \ \ f \text{ satisfies } a)$$

$b)$:

$$\begin{array}{rcl} f\ :\mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & z^2 \end{array}$$

$\forall\ n \in \mathbb{N}\ \ \text{"}\ \ n \geq 2$:

$$f(\pm\tfrac{1}{n}) = \tfrac{1}{n^2}$$

f satisfies $b)$

$$\begin{array}{rcl} \bar{f}\ :\mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (u(x,y), v(x,y)) = (x^2 - y^2, 2xy) \end{array}$$

$\bar{f} \in \mathrm{Pol}(\mathbb{R}^2) \to \bar{f}$ differentiable in $\mathbb{R}^2$

$\forall\ (x,y) \in \mathbb{R}^2$:

$$\partial_x u(x,y) = 2x = \partial_y v(x,y)$$

$$\partial_y u(x,y) = -2y = -\partial_x v(x,y)$$

f satisfies CR

$\therefore)\ \ f \in \mathcal{H}(\mathbb{R}^2)$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

$c)$:

Suppose $\exists\ f \in \mathcal{H}(\mathbb{D})\ \ \text{"}\ \ f$ satisfies $c)$

$$f \in \mathcal{C}^0(\mathbb{D}) \to f(0) = f(\lim_n \frac{1}{n}) = \lim_n\ f(\frac{1}{n}) = 0$$

$$\left|\lim_{E_1} \frac{f(z) - f(0)}{z - 0}\right| = \lim_n \frac{\left|f(\frac{1}{n})\right|}{\frac{1}{n}} \notin \mathbb{C}$$

$f \notin \mathcal{H}(0)$ absurd

$d)$:

$$\begin{aligned} f \ : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \tfrac{1}{z+1} \end{aligned}$$

$\forall\ n \in \mathbb{N}$ „ $n \geq 2$:

$$\left|f(\tfrac{1}{n})\right| = \frac{1}{\frac{1}{n}+1} = \frac{n}{n+1}$$

f satisfies $d)$

$$\begin{aligned} \bar{f} \ : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (u(x,y), v(x,y)) = \left(\tfrac{x+1}{(x+1)^2+y^2}, \tfrac{-y}{(x+1)^2-y^2}\right) \end{aligned}$$

$\bar{f} \in \mathrm{Rat}(\mathbb{R}^2) \wedge \forall\ (x,y) \in \mathbb{R}^2$:

$$(x+1)^2 + y^2 \neq 0$$

$\bar{f}$ differentiable in $\mathbb{R}^2$

$\forall\ (x,y) \in \mathbb{R}^2$:

$$\partial_x u(x,y) = \frac{y^2-(x+1)^2}{((x+1)^2+y^2)^2} = \partial_y v(x,y)$$

$$\partial_y u(x,y) = \frac{-2y(x+1)}{((x+1)^2+y^2)^2} = -\partial_x v(x,y)$$

f satisfies CR

$\therefore)$ $f \in \mathcal{H}(\mathbb{R}^2)$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

Constant tests

Let:

$\cdot\ \Omega \subset \mathbb{C}$ region

$\cdot\ f \in \mathcal{H}(\Omega)$

Then, holds:

$\cdot\ f_{Re} = 0 \ \vee \ f_{Im} = 0 \to f \in \text{Cst}(\Omega)$

$\cdot\ |f| \in \text{Cst}(\Omega) \to f \in \text{Cst}(\Omega)$

$\cdot$ Imf circumference $\to f \in$ Cst

Demonstration:

$f_{Re} = 0 \ \vee \ f_{Im} = 0$:

$u \ := \ f_{Re}$

$v \ := \ f_{Im}$

$f \in \mathcal{H}(\Omega) \to f$ satisfies CR in Ω

$\partial_x u = \partial_y v = 0$

$\partial_y u = -\partial_x v = 0$

Null diferential test:

Ω connex $\to u, v \in$ Cst

$u, v \in$ Cst$\to f \in$ Cst

$|f| \in \text{Cst}(\Omega)$:

$$\begin{array}{rcl} |f| : \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \sqrt{u(x,y)^2 + v(x,y)^2} \end{array}$$

$|f| \in \text{Cst} \rightarrow \exists\ a \in \mathbb{R}$:

$$\sqrt{u(x,y)^2 + v(x,y)^2} = a$$

$$u(x,y)^2 + v(x,y)^2 = a^2$$

$$2\partial_x u(x,y) + 2\partial_x v(x,y) = 0$$

$$2\partial_y u(x,y) + 2\partial_y v(x,y) = 0$$

$f \in \mathcal{H}(\Omega) \rightarrow f$ satisfies CR in Ω

$$2\partial_y v(x,y) + 2\partial_x v(x,y) = 0$$

$$-2\partial_x v(x,y) + 2\partial_y v(x,y) = 0$$

$$+:\ 4\partial_y v(x,y) = 0 \rightarrow \partial_y v(x,y) = 0$$

$$-:\ 4\partial_x v(x,y) = 0 \rightarrow \partial_x v(x,y) = 0$$

Null differential test:

Ω connex $\rightarrow u, v \in$ Cst

$u, v \in \text{Cst} \rightarrow f \in \text{Cst}$

$\text{Im}(f)$ circumference :

$\exists\ (x_0, y_0) \in \mathbb{R}^2, r \in \mathbb{R}^+$:

$$\text{Im}(f) = C_r(x_0, y_0)$$

$$\begin{array}{rcl} \bar{f}\ : \mathbb{R}^2 & \longrightarrow & C_r(x_0, y_0) \\ (x,y) & \longmapsto & (r\cos(x - x_0), r\sin(y - y_0)) \end{array}$$

$\forall\ (x,y) \in \Omega$:

$$\left|\bar{f}\right|(x,y) = \sqrt{r^2(\cos^2(x - x_0) + \sin^2(y - y_0)} = r$$

$|f| \in \text{Cst} \rightarrow f$ Cst

Real part of holomorphic functions

Let:

$\cdot\ \Omega \subset \mathbb{R}^2$ region

$\cdot\ u \in \mathcal{C}^2(\Omega)$ " $\exists\ f \in \mathcal{H}(\Omega)$:

$$f_{Re} = u$$

Show that:

$\cdot\ \partial_{xx}u + \partial_{yy}u = 0$

Study:

$\cdot\ \exists\ f \in \mathcal{H}(\Omega)$:

$a)\ f_{Re}(x,y) = x^2 + y^2$

$b)\ f_{Re}(x,y) = x(x+1) - y^2$

$c)\ \forall\ \alpha \in \mathbb{R}$:

$$f_{Re} = y^3 + \alpha x^2 y\ \wedge\ \Omega = \mathbb{C}$$

Demonstration:

$\partial_{xx}u + \partial_{yy}u = 0$:

$u\ :=\ f_{Re}, v\ :=\ f_{Im}$

$f \in \mathcal{H}(\Omega) \to f$ satisfies CR in Ω

$\partial_x u = \partial_y v \to \partial_{xx}u = \partial_{xy}v$

$\partial_y u = -\partial_x v \to \partial_{yy}u = -\partial_{xy}v$

$\therefore)\ \partial_{xx}u + \partial_{yy}u = 0$

$f_{Re}(x, y) = x^2 + y^2$:

$$\partial_{xx}u + \partial_{yy}u = 4 \neq 0$$

$$\nexists f \in \mathcal{H}(\Omega) \;\; " \;\; f_{Re}(x, y) = x^2 + y^2$$

$f_{Re}(x, y) = x(x + 1) - y^2$:

$$\begin{array}{rcl} \bar{f} \; : \Omega & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (u(x, y), v(x, y)) = (x(x + 1) - y^2, 2xy + y) \end{array}$$

$\bar{f} \in$ Pol$\rightarrow$ $\bar{f}$ differentiable in Ω

$\forall \; (x, y) \in \Omega$:

$$\partial_x u(x, y) = 2x + 1 = \partial_y v(x, y)$$

$$\partial_y u(x, y) = -2y = -\partial_x v(x, y)$$

f satisfies CR in Ω

$$f \in \mathcal{H}(\Omega) \;\wedge\; f_{Re} = u$$

$f_{Re}(x, y) = y^3 + \alpha x^2 y$:

f has to satisfy CR in $\mathbb{C}$:

$\forall \; (x, y) \in \mathbb{R}^2$:

$$\partial_y v(x, y) = \partial_x u(x, y) = 2\alpha xy$$

$$\partial_x v(x, y) = -\partial_y u(x, y) = -3y^2 - \alpha x^2$$

$$v(x, y) = \alpha x y^2 + c(x)$$

$$v(x, y) = -3xy^2 - \tfrac{\alpha}{3}x^3 + c(y)$$

$$\alpha = -3, \; c(x) = x^3, \; c(y) = 0$$

$$v(x, y) = -3xy^2 + x^3$$

2. 2nd laboratory

Power series

Study:

- $\sum_{n\geq 1} n(n+1)z^n$

Start:

Convergence domain:

$$\lim_n \frac{n(n+1)}{(n+1)(n+1)} = 1$$

Quotient test:

$$\overline{\lim_n} |c_n|^{\frac{1}{n}} = 1 \to \left(\overline{\lim_n} |c_n|^{\frac{1}{n}}\right)^{-1} = 1$$

Cauchy-Hadamard theorem:

$\sum_{n\geq 1} n(n+1)z^n$ absolutely convergent over $\mathbb{D}$

$\sum_{n\geq 1} n(n+1)z^n$ divergent over $\mathbb{C} \setminus \overline{\mathbb{D}}$

$\forall\ K \subset \mathbb{D}$ " K compact :

$\sum_{n\geq 1} n(n+1)z^n$ uniformly convergent over K

Sum:

Consider:

$$\begin{array}{rcl} f : \mathbb{D} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \sum_{n\geq 1} n(n+1)z^n \\ g : \mathbb{D} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \sum_{n\geq 1} n(n+1)z^{n-1} \\ h : \mathbb{D} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \sum_{n\geq 1} (n+1)z^n \end{array}$$

UCI theorem:

$\forall\ z \in \mathbb{D}$:

$$\int_0^z h(t)dt = \sum_{n\geq 1} z^{n+1} = \sum_{n\geq 0} z^n = \frac{1}{1-z}$$

$$h(z) = \partial_z \tfrac{1}{1-z} = \tfrac{1}{(1-z)^2}$$

$$\int_0^z g(t)dt = \sum_{n\geq 1} (n+1)z^n = h(z)$$

$$g(z) = \partial_z h(z) = \tfrac{2}{(1-z)^3}$$

$$f(z) = zg(z) = \tfrac{2z}{(1-z)^3}$$

Application:

In particular:

$$\sum_{n\geq 1} (-1)^n \frac{n(n+1)}{2^n} = f(-\frac{1}{2}) = \frac{-2^3}{3^3}$$