Complex Analysis

Martin Azpillaga

February 23, 2014

Contents

Block I

Definitions

1. The field of complex numbers

introduction

The field of complex numbers

Let:

- $\begin{array}{rcl} + : \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ ((a,b),(c,d)) & \longmapsto & (a+c, b+d) \end{array}$
- $\begin{array}{rcl} \cdot : \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ ((a,b),(c,d)) & \longmapsto & (ac-bd, ad+bd) \end{array}$

We name the field of complex numbers to:

- $(\mathbb{R}^2, +, \cdot)$

We denote:

- $(\mathbb{R}^2, +, \cdot) \, : \, \mathbb{C}$
- $(0,1) \in \mathbb{C} \, : \, i$
- $(a,b) \in \mathbb{C} \, : \, a + bi$
- $\pi_1(\mathbb{C}) \, : \, Re$
- $\pi_2(\mathbb{C}) \, : \, Im$

Conjugation

We name complex conjugation to:

- $\begin{array}{rcl} f : \mathbb{C} & \longrightarrow & \mathbb{C} \\ (a,b) & \longmapsto & (a,-b) \end{array}$

We denote:

- $f((a,b)) \, : \, \overline{(a,b)}$

Norm

We name complex norm to:

$$\cdot \begin{array}{rcl} f : \mathbb{C} & \longrightarrow & \mathbb{R} \\ (a,b) & \longmapsto & \sqrt{a^2+b^2} \end{array}$$

We denote:

$$\cdot\, f((a,b)) \,:\, |(a,b)|$$

Polar transformation

We name polar transformation to:

$$\cdot \begin{array}{rcl} f : \mathbb{C} & \longrightarrow & \mathbb{R}^+ \times [0,2\pi) \\ (a,b) & \longmapsto & (\sqrt{|(a,b)|}, \arctan(\frac{b}{a})) \end{array}$$

We denote:

$$\cdot\, f((a,b)) \,:\, (r,\theta)$$

Unit sphere projection

We name unit sphere projection to:

$$\cdot \begin{array}{rcl} f : \mathbb{C} & \longrightarrow & S^1 \\ z & \longmapsto & \frac{z}{|z|} \end{array}$$

We denote:

$$\cdot\, f(z) \,:\, \pi(z)$$

Roots of unity

Let:

- $z \in \mathbb{C}$

Then, z is a root of unity if:

- $\exists\, n \in \mathbb{N}$:

  $z^n = 1$

We denote:

- $\{z \in \mathbb{C} \mid z \text{ root of unity } \} \;:\; S^1$

Disk

Let:

- $p \in \mathbb{C}$
- $r \in \mathbb{R}^+ \setminus \{0\}$

We name Disk centered in p and radius r to:

- $\{z \in \mathbb{C} \mid |z - p| < r\}$

We denote:

- $\{z \in \mathbb{C} \mid |z - p| < r\} \;:\; D^1$

Component decomposition

Let:

- $f : \mathbb{C} \to \mathbb{C}$

We name real component of f to:

- $$\begin{aligned} f_{Re} : \mathbb{C} &\longrightarrow \mathbb{R} \\ z &\longmapsto Re(f(z)) \end{aligned}$$

We name imaginary component of f to:

- $$\begin{aligned} f_{Im} : \mathbb{C} &\longrightarrow \mathbb{R} \\ z &\longmapsto Im(f(z)) \end{aligned}$$

We name component decomposition of f to:

- $$\begin{aligned} f_{\mathbb{R}^2} : \mathbb{C} &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (f_{Re}(x + yi), f_{Im}(x + yi) \end{aligned}$$

2. Holomorphic functions

Incremental quotient

Let:

- $\mathcal{U} \subset \mathbb{C}$ open
- $f : \mathcal{U} \to \mathbb{C}$
- $p \in \mathcal{U}$

We name incremental quotient of f in p to:

- $\lim_{z \to p} \frac{f(z) - f(p)}{z - p}$

We denote:

- $f'(p)$

Holomorphic function

Let:

- $\mathcal{U} \subset \mathbb{C}$ open
- $f : \mathcal{U} \to \mathbb{C}$
- $p \in \mathcal{U}$

Then, f is holomorphic over p if:

- $\exists f'(p)$

Then, f is holomorphic over U if:

- $\forall\ p \in \mathcal{U}$:

  $\exists f'(p)$

We denote:

- $\{f : \mathcal{U} \to \mathbb{C} \mid f \text{ holomorphic over } \mathcal{U}\} : \mathcal{H}(\mathcal{U})$
- $f \in \mathcal{H}(\mathbb{C}) : f$ entire

Cauchy-Riemman equations

Let:

- $u, v \,:\, \mathbb{R}^2 \to \mathbb{R}$
- $\begin{array}{rcl} f \,:\mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (u((x,y)), v((x,y))) \end{array}$

Then, f is satisfies the Cauchy-Riemman equations if:

- $\exists\, u_x, u_y, v_x, v_y$
- $u_x = v_y$
- $u_y = -v_x$

We denote:

- $u_x + iv_x \,:\, f_x$
- $u_y + iv_y \,:\, f_y$

Conformal

Let:

- $\mathcal{U} \subset \mathbb{C}$ open
- $f : \mathcal{U} \to \mathbb{C}$
- $z \in \mathcal{U}$

Then, f is conformal in z if:

- $\exists\, c \in \mathbb{C}$:

  $\forall\, I \subset \mathbb{R}$ " $0 \in I$:

  $\forall\, \gamma : I \to \mathbb{R}^2$ " γ differentiable $\land\ \gamma(0) = z \land \gamma'(0) \neq 0$:

$$\frac{(f \circ \gamma)'(0)}{\gamma'(0)} = c$$

Then, f is conformal if:

- $\forall\, z \in \mathcal{U}$:

  f conformal in z

Block II

Propositions

1. The field of complex numbers

introduction

go

2. Holomorphic functions

Cauchy-Riemman

Let:

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies CR

Then, holds:

- $f_x = f'(z)$
- $f_y = -if_y$
- $\frac{\partial}{\partial f(z)} z = 0$

Demonstration:

$f_x = u_x + iv_x = a + bi = f'(z)$

$f_y = i(a + ib) = if'(z)$

$f'(z) = -if_y$

notation

tangent venctor

Let:

· γ differentiable plane arc " $\forall\ t \in I$:

$\gamma'(t) \neq 0$

Then, holds:

· $\gamma'(t)$ tangent to γ

Demonstration:

demonstration

arc images

Let:

· $f \in \mathcal{H}(\mathcal{U})$

· γ differentiable plane arc „ $\gamma \subset \mathcal{U}$

· $\sigma = f(\gamma)$

· $z_0 = \gamma(0)$

Then, holds:

· $\sigma' = f'(\gamma)\gamma'$

· $\gamma'(0) \neq 0 \to f'(z_0) \neq 0$

· $\sigma'(0) = f'(z_0)\gamma'(0)$

· $|\sigma'(0)| = |f'(z_0)||\gamma'(0)|$

· $arg\sigma'(0) = arg\gamma'(0) + argf(z_0)$

· f aplica una homotecia mas una rotacion constante a todos los vectores tangentes que salen de z0

Demonstration:

obvio

Holomorphic functions are conform

Let:

- $f : \mathcal{U} \to \mathbb{C}$
-

Then, holds:

- $f$holomorph in z „ $f'(z) \neq 0 \leftrightarrow f$ conform

Demonstration:

$\rightarrow$):

already seen

$\leftarrow$):

too hard

Block III

Examples

1. Holomorphic functions

introduction

go

2. Holomorphic functions

Conjugation

Let:

$$\bar{a} : \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longmapsto \bar{z}$$

.

Then, $\bar{a}$ is not holomorphic :

$u_x = 1$

$u_y = 0$

$v_x = 0$

$v_y = -1$

$\forall\ z \in \mathbb{C}$:

$-1 \neq 1 \rightarrow f$ not holomorphic in z

Quadratic norm

Let:

- $$\begin{aligned} f\ :\mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto |z|^2 \end{aligned}$$
- $f_{\mathbb{R}^2}$ component decomposition of f

Then, f is holomorphic in 0:

f differentiable in $\mathbb{R}^2$ polinomial

$\forall\ z \in \mathbb{C}$:

$u_x(x,y) = 2x$

$u_y(x,y) = 2y$

$v_x(x,y) = 0$

$v_y(x,y) = 0$

$u_x = v_y \leftrightarrow x = 0$

$u_y = -v_x \leftrightarrow y = 0$

f holomorphic function in $z \leftrightarrow z = 0$

Non preserving angles function

Let:

- $f(z) = z^2$

Then, f is conform in $\mathbb{R} \setminus \{0\}$:

$$f(\{(x,0) \in \mathbb{C} \mid x > 0\}) = \{(x,0) \in \mathbb{C} \mid x > 0\}$$

$$f(\{(x,0) \in \mathbb{C} \mid x < 0\}) = \{(x,0) \in \mathbb{C} \mid x > 0\}$$

$$ang(A,B) = \pi \neq 0 = ang(f(A), f(B))$$

Block IV

Problems

PROBLEMES D'ANÀLISI COMPLEXA
2n quadrimestre del curs 2013-2014.

Llista 1: Els nombres complexos

B.1. Expresseu en la forma $a+ib$ els següents nombres:

(a) $(2+3i)(4+i)$ (c) $\frac{1}{4+i}$ (e) $\sqrt{i}$ (g) $\sqrt{9i}$
(b) $(4+2i)^2$ (d) $\frac{i}{4+i}$ (f) $\sqrt{-i}$ (h) $\sqrt{1+i}$

B.2. Si $z=x+iy$ trobeu les parts real i imaginària de les expressions següents:

(a) z^2 (b) $z(z+1)$ (c) $\frac{1}{z}$ (d) $\frac{1}{z-3}$.

B.3. És cert que

a) $\operatorname{Re}(z+w)=\operatorname{Re} z+\operatorname{Re} w$?
b) $\operatorname{Re}(zw)=(\operatorname{Re} z)(\operatorname{Re} w)$?
c) $\operatorname{Re}(\frac{z}{w})=\frac{\operatorname{Re} z}{\operatorname{Re} w}$?

B.4. Trobeu la forma polar dels nombres següents i dibuixeu-los.

(a) $3(1+\sqrt{3}\,i)$ (b) $2\sqrt{3}-2i$ (c) $-2+2i$ (d) $-1-i$

B.5. Sigui $(x+iy)/(x-iy)=a+ib$. Proveu que $a^2+b^2=1$.

B.6. Proveu que si $p(z)$ és un polinomi amb coeficients reals i z és un zero de p llavors $\bar{z}$ també ho és.

B.7. Descriviu els conjunts del pla que satisfan (recordeu que $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$.)

(a) $\operatorname{Im}\frac{z-a}{z}=0, a\in\mathbb{C}^*$ (b) $|z|=\operatorname{Re} z+1$ (c) $|z-2|>|z-3|$

SOL. B.1. a) $5+14i$; b)$12+16i$; c) $4/17-i/17$; d)$1/17+4i/17$; e) $\pm\sqrt{2}/2(1+i)$; f)$\pm\sqrt{2}/2(1-i)$; g) $\pm3\sqrt{2}/2(1+i)$; h) $\pm2^{1/4}(\cos(\pi/8)+i\sin(\pi/8))$.

B.2 a) x^2-y^2+2ixy; b) $x^2-y^2+x+i(y+2xy)$; c) $(x-iy)/(x^2+y^2)$; d) $(x-3-iy)/((x-3)^2+y^2)$.

B.3 a) si. b) no. c) no.

B.4 a) $6(\cos(\pi/3)+i\sin(\pi/3))$; b) $4(cos(\pi/6)-i\sin(\pi/6))$; c) $2\sqrt{2}(\cos(\pi/4)+i\sin(\pi/4))$; d) $\sqrt{2}(cos(3\pi/4)-i\sin(3\pi/4))$.

B.6 Conjugueu tot el polinomi.

B.7 a) Recta que passa per 0 i a; b) Paràbola horitzontal $x=(1/2)(y^2-1)$; c) $\{\operatorname{Re} z>3/2\}$.

1. Expresseu en la forma $a+ib$ els següents nombres:

(a) $\frac{1}{i}$ (c) $\frac{1}{2+i}+\frac{1}{2-i}$ (e) $\left(\frac{2+i}{3-2i}\right)^2$ (g) $\sqrt[4]{-i}$
(b) $\frac{1+i}{1-i}$ (d) $\frac{1}{2+i}+\frac{4-2i}{3+i}$ (f) $(1+i)^{100}+(1-i)^{100}$ (h) $(3+4i)^{\frac{1}{2}}$

2. Si $z=x+iy$ on $x,y\in\mathbb{R}$, trobeu les parts real i imaginària de:

PROBLEMES D'ANLISI COMPLEXA
2n quadrimestre del curs 2013-2014

Llista 2: Funcions de variable complexa i equacions de Cauchy-Riemann

B.1. Trobeu els punts on la funció f és derivable (en el sentit complex), en els següents casos, i calcula'n la derivada.

(a) $\cos|z|^2$ (c) e^{iz} (e) $\frac{1}{(z-1)^2(z^2+2)}$

(b) $|z|^4$ (d) $z+\frac{1}{z}$ (f) $\frac{1}{(z+\frac{1}{z})^2}$

Solució: (a) $\emptyset$; (b) $\emptyset$; (c) $\mathbb{C}$; $f'(z) = ie^{iz}$;(d) $\mathbb{C}\setminus\{0\}$; $f'(z) = 1 - |\frac{1}{z^2}$; (e) $\mathbb{C}\setminus\{1, \pm\sqrt{2}i\}$; (f) $\mathbb{C}$.

B.2. Determineu si aquestes funcions poden ser la part real d'una funció holomorfa, i en cas que ho siguin calculeu la part imaginària.

(a) $e^x\cos y$ (b) x^3+6xy^2 (c) $\log(x^2+y^2)$

Solució: (a) $e^x\sin y$; $f(z) = e^z$; (b) No ho és; (c) $2\arctan(y/x)$; ($f(z) = \log(z^2)$).

B.3. Sigui f una funció holomorfa en un obert $\Omega\subset\mathbb{C}$ i $z_0\in\Omega$ tal que $f'(z_0)\neq 0$. Quin angle formen les corbes $\operatorname{Re} f(z) = \operatorname{Re} f(z_0)$ i $\operatorname{Im} f(z) = \operatorname{Im} f(z_0)$ en un punt z_0 ?
Solució: $\pi/2$.

1. Trobeu els punts on la funció f és derivable (en el sentit complex), en els següents casos:

(a) $f(z) = |z|$
(b) $\cosh x\cos y + i\sinh x\sin y$
(c) $f(z) = \operatorname{Re} z$
(d) $f(z) = z + z\overline{z}$
(e) $f(z) = \operatorname{Im} e^{\overline{z}} + i\operatorname{Re} e^z$

2. Sigui $\Omega\subset\mathbb{C}$ un obert, $z_0\in\Omega$ i $f:\Omega\to\mathbb{C}$ una funció.

a) Identificant $\mathbb{R}^2$ amb $\mathbb{C}$ de la forma habitual, demostreu que si f és diferenciable en z_0, llavors

$$Df(z_0)(z) = \frac{\partial f}{\partial z}(z_0)\cdot z + \frac{\partial f}{\partial\overline{z}}(z_0)\cdot\overline{z} \qquad (z\in\mathbb{C}),$$

on

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial\overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

b) Proveu que f és holomorfa en Ω si, i només si, f és diferenciable i $\frac{\partial f}{\partial\overline{z}} = 0$ en Ω. En tal cas, $f' = \frac{\partial f}{\partial z}$.

3. Demostreu que si f és diferenciable en un obert de $\mathbb{C}$, llavors

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial\overline{f}}{\partial\overline{z}} \quad \text{i} \quad \overline{\frac{\partial f}{\partial\overline{z}}} = \frac{\partial\overline{f}}{\partial z}.$$

1. The field of complex numbers

introduction

entity

Let:

· *statements*

·

Then, *item* is a/an entity if:

· *conditions*

·

We denote:

· *property* : *notation*

·

2. Holomorphic functions

3. Cauchy-Riemman

Let:

$\cdot\ f \in \mathcal{H}(\mathbb{C})$ „ $Ref + Imf = c_a$

Show that:

$\cdot\quad \exists\ a' \in \mathbb{C}$:

$f = c_{a'}$

Demonstration:

u, v real components of f

$u(x,y) + v(x,y) = a$

differentiate respect x and y

$u_x + v_x = 0$

$u_y + v_y = 0$

f holomorphic $\rightarrow f$ CR

$u_x - u_y = 0$

$u_y + u_x = 0$

$u_x, u_y, v_x, v_y = 0$

$\exists\ a_1 \in \mathbb{R}$:

$u = c_{a_1}$

$\exists\ a_2 \in \mathbb{R}$:

$v = c_{a_2}$

$f = c_{(a_1, a_2)}$

B.2 a)

Let:

$$\cdot \quad \begin{array}{rcl} u : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & \exp(y)\cos(x) \end{array}$$

Show that:

- $\exists\, f \in \mathcal{H}(\mathbb{C})$:

  u real component of f

Demonstration:

lab 1→ $u_{xx} + v_{yy} = 0$

$u_x = \exp(x)\cos(y)$

$u_{xx} = \exp(x)\cos(y)$

$u_y = -\exp(x)\sin(y)$

$u_{yy} = -\exp(x)\cos(y)$

ok

Calculate v using CR

$v_y = u_x = \exp(x)\cos(y)$

$v(x, y) = \int_{\mathbb{C}} \exp(x)\sin(y)dy = \exp(x)\sin(y) + \phi(x)$

$v_x = \frac{\partial}{\partial v}x = \exp(x)\sin(y) + \phi'(x)$

$-u_y = \exp(x)\sin(y) + \phi'(x)$

CR→ $\phi'(x) = 0$

$\forall\ c \in \mathbb{R}$:

$\phi(x) = c$ ok

$v(x, y) = \exp(x)\sin(y)$

Preservation of angles

Let:

- γ_1, γ_2 plane arcs „ $\gamma_1(0) = \gamma_2(0)$

Then, holds:

- angle of $\gamma_1'(0)$ and $\gamma_2'(0)$ = angle $\sigma_1'(0), \sigma_2'(0)$

Demonstration:

rotations and homotecies let angles invariant

Block V

Tasks