1. 1st laboratory

Existence of holomorphic functions

Let:

- $f \in \mathcal{H}(\mathbb{D})$

Study:

- $\exists\ f \in \mathcal{H}(\mathbb{D})$:

  $\forall\ n \in \mathbb{N}$,, $n \geq 2$:

  $a)\ f(\pm\frac{1}{n}) = \frac{1}{2n+1}$

  $b)\ f(\pm\frac{1}{n}) = \frac{1}{n^2}$

  $c)\ |f(\frac{1}{n})| = \frac{1}{\log(n+1)}$

  $d)\ |f(\frac{1}{n})| = \frac{n}{n+1}$

Demonstration:

$a)$:

$$E_1 := \{+\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N}\}$$

$$E_2 := \{-\frac{1}{n} + 0i \in \mathbb{C} \mid n \in \mathbb{N}\}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} = \lim_{n} \frac{f(\frac{1}{n})}{\frac{1}{n}} - \frac{f(0)}{\frac{1}{n}} = \frac{1}{2} - \lim_{n} \frac{f(0)}{\frac{1}{n}}$$

$$\lim_{E_1} \frac{f(z) - f(0)}{z - 0} \begin{cases} = \frac{1}{2} & f(0) = 0 \\ \notin \mathbb{C} & f(0) \neq 0 \end{cases}$$

Case $f(0) = 0$:

$$\lim_{E_2} \frac{f(z) - f(0)}{z - 0} = \lim_{n} \frac{f(\frac{1}{n})}{\frac{1}{n}} = -\frac{1}{2} \neq \lim_{E_1} \frac{f(z) - f(0)}{z - 0}$$

$\nexists\, f \in \mathcal{H}(0) \;\text{„}\; f$ satisfies $a)$

In particular:

$$\nexists\, f \in \mathcal{H}(\mathbb{D}) \;\text{„}\; f \text{ satisfies } a)$$

$b)$:

$$\begin{array}{rcl} f\ :\mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & z^2 \end{array}$$

$\forall\, n \in \mathbb{N} \;\text{„}\; n \geq 2$:

$$f(\pm\tfrac{1}{n}) = \tfrac{1}{n^2}$$

f satisfies $b)$

$$\begin{array}{rcl} \bar{f}\ :\mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (u(x,y), v(x,y)) = (x^2 - y^2, 2xy) \end{array}$$

$\bar{f} \in Pol(\mathbb{R}^2) \to \bar{f}$ differentiable in $\mathbb{R}^2$

$\forall\ (x,y) \in \mathbb{R}^2$:

$$\partial_x u(x,y) = 2x = \partial_y v(x,y)$$

$$\partial_y u(x,y) = -2y = -\partial_x v(x,y)$$

f satisfies CR

$\therefore)\ f \in \mathcal{H}(\mathbb{R}^2)$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

$c)$:

Suppose $\exists\, f \in \mathcal{H}(\mathbb{D}) \;\text{„}\; f$ satisfies $c)$

$f \in \mathcal{C}^0(\mathbb{D}) \to f(0) = f(\lim\limits_{n} \dfrac{1}{n}) = \lim\limits_{n}\ f(\dfrac{1}{n}) = 0$

$$\left|\lim_{E_1} \frac{f(z)-f(0)}{z-0}\right| = \lim_{n} \frac{\left|f(\frac{1}{n})\right|}{\frac{1}{n}} \notin \mathbb{C}$$

$f \notin \mathcal{H}(0)$ absurd

$d)$:

$$\begin{array}{rcl} f\ :\mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \frac{1}{z+1} \end{array}$$

$\forall\ n \in \mathbb{N}\ \text{,,}\ n \geq 2$:

$$\left|f(\tfrac{1}{n})\right| = \frac{1}{\frac{1}{n}+1} = \frac{n}{n+1}$$

f satisfies $d)$

$$\begin{array}{rcl} \bar{f}\ :\mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (u(x,y),v(x,y)) = (\frac{x+1}{(x+1)^2+y^2}, \frac{-y}{(x+1)^2-y^2}) \end{array}$$

$\bar{f} \in Rat(\mathbb{R}^2) \wedge \forall\ (x,y) \in \mathbb{R}^2$:

$$(x+1)^2 + y^2 \neq 0$$

$\bar{f}$ differentiable in $\mathbb{R}^2$

$\forall\ (x,y) \in \mathbb{R}^2$:

$$\partial_x u(x,y) = \frac{y^2-(x+1)^2}{((x+1)^2+y^2)^2} = \partial_y v(x,y)$$

$$\partial_y u(x,y) = \frac{-2y(x+1)}{((x+1)^2+y^2)^2} = -\partial_x v(x,y)$$

f satisfies CR

$\therefore)\ f \in \mathcal{H}(\mathbb{R}^2)$

In particular:

$$f \in \mathcal{H}(\mathbb{D})$$

Constant tests

Let:

- $\Omega \subset \mathbb{C}$ region
- $f \in \mathcal{H}(\Omega)$

Then, holds:

- $Re(f) \equiv 0 \ \vee \ Im(f) \equiv 0 \rightarrow f \in \mathrm{Cst}(\mathbb{C})$
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Demonstration:

demonstration