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Spread polynomials, rotations and the butterfly effect

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Abstract

The spread between two lines in rational trigonometry replaces the concept of angle, allowing the complete specification of many geometrical and dynamical situations which have traditionally been viewed approximately. This paper investigates the case of powers of a rational spread rotation, and in particular, a curious periodicity in the prime power decomposition of the associated values of the spread polynomials, which are the analogs in rational trigonometry of the Chebyshev polynomials of the first kind. Rational trigonometry over finite fields plays a role, together with non-Euclidean geometries.

1 Introduction

This paper investigates the role of the *spread polynomials* $S_n(s)$ of *rational trigonometry* in understanding a simple dynamical system arising in elementary geometry. Spread polynomials were introduced in [6], and arise naturally when Euclidean geometry is studied *algebraically,* with angles replaced by spreads, and they play also a role in geometry over finite fields, and in fact over general fields not of characteristic two. Furthermore their importance extends to *elliptic* and *hyperbolic* geometries, see [9]. Spread polynomials also have interesting number theoretical properties that set them apart from Chebyshev and other orthogonal polynomials, see [6] and [2].

In this paper we study a phenomenon which occurs when we analyze iterates, or powers, of a particular rotation ρ in the Euclidean plane. If ρ is the rotation by an angle θ which is a rational multiple of π, then the subsequent powers ρ^n are easily understood. However when θ is an irrational multiple of π, the situation exhibits chaotic features, and illustrates the 'butterfly effect': the future state of the system is determined by the accuracy by which we know the real numbers which specify initial conditions, and even a small error will eventually result in complete uncertainty—this is the standard view.

With rational trigonometry, *increasing uncertainty* is replaced with *increasing complexity.* By measuring rotations ρ with a *spread* s, not an *angle* θ, the spread polynomials arise naturally from iterates, and the chaos disappears, replaced instead by an increasing escalation in the size and complexity of the numbers $S_n(s)$ that describe the evolution of the system. These numbers are far from random, and they tend to incorporate interesting, and indeed mysterious, number theoretic information.

For a rational spread s, we will prove a theorem about the spreads $S_n(s)$ associated to the powers ρ^n of the rotation ρ with initial spread s. Such rational spreads are common in geometry, for example any configuration of lines with rational equations has only rational spreads. Of particular interest is that we will need to examine *finite geometries* in order to understand the situation over the rational numbers, and that forms of *non-Euclidean geometry* also arise. Our work can also be viewed in the context of investigations of Chebyshev polynomials over a finite field, initiated already in [5]; see also [1] and [3].

1.1 Powers of a rotation

To provide some motivation for the spread polynomials, recall that with the ISO size standards, an $A4$ sheet of paper has proportions in the ratio $\sqrt{2}:1$. So if you fold a piece in two lengthwise, the result has the same proportions but only one half of the area, and is called $A5$ size. The angle in degrees between the long side l_0 and the diagonal l_1 of an $A4$ sheet may be described by the formula

$$\theta_{ISO} \equiv \left(180 \arcsin \sqrt{1/3}\right)/\pi.$$

This has a numerical value in degrees of approximately $\theta_{ISO} \approx 35.26$.

Suppose we rotate l_1 by θ_{ISO} to get l_2, then rotate l_2 by θ_{ISO} to get l_3, and so on. We could also say that l_2 is the reflection of l_0 in l_1, that l_3 is the reflection of l_1 in l_2, and so on. This gives a sequence $l_1, l_2, \cdots$ of concurrent lines whose respective angles with l_0 form the sequence $\theta_{ISO}, 2\theta_{ISO}, 3\theta_{ISO}, \cdots$. Since θ_{ISO} is not a rational multiple of π, these sequences of lines and angles will never repeat, and so the lines l_n are all distinct, as are the angles $n\theta_{ISO}$.

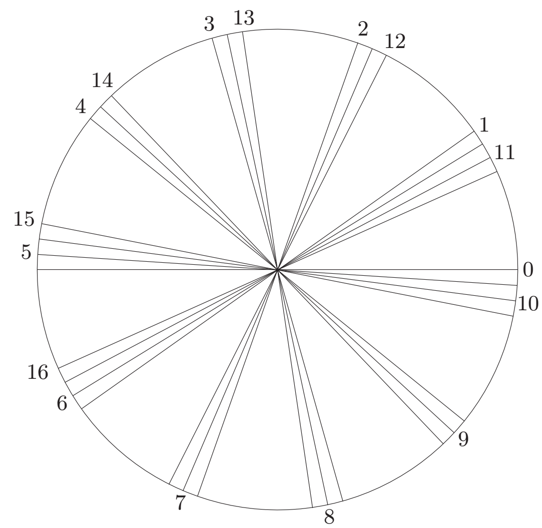

Figure 1: Multiples of θ_{ISO}, or equivalently $s = 1/3$

Figure 1 shows the first sixteen such lines, exhibiting the famous *three gaps phenomenon,* which was originally a conjecture of Steinhaus, (see for example [4]): for any n, there are at most three different angles found between any two adjacent lines in the set $\{l_0, l_1, l_2, \cdots l_n\}$.

The angle between l_0 and l_{16} is approximately

$$16\theta_{ISO} \approx 16 \times 35.26 \bmod 360 = 564.16 - 360 = 204.16.$$

We have lost information, however, and with the better initial approximation $\theta_{ISO} \approx 35.264\,389\,682$ we get instead

$$16\theta_{ISO} \approx 204.230\,235.$$

But no matter what accuracy we have for θ_{ISO}, computing larger and larger multiples of it, mod 360, inevitably results in increasing error—until after a finite number of multiples *all knowledge of the position is lost*. This is a simple example of the well-known *butterfly effect*, a feature of a wide range of dynamical systems, and an inevitable consequence of working in the framework of classical trigonometry, which generally deals only with *approximations* to real values. The usual idea is that our understanding of the future evolution of many completely specified systems is limited by the precision with which we know the real numbers that appear in the initial conditions.

With *rational trigonometry,* introduced in [6], we set our sights higher—we aim to describe such a system *completely precisely,* until we run out of computing power, memory space, or patience. The price we pay for *more accuracy* is—*more complexity*. For dynamical systems which can be expressed using rational numbers and polynomial transformations, the future evolution is completely known, but will generally be increasingly difficult and expensive to write down as time goes on. We will show that rotation by the angle θ_{ISO} exhibits this phenomenon, and brings out interesting number theoretical questions.

It will be important for us to consider not just rotations in the Euclidean plane, but also over finite prime fields. For this it is often more convenient to work not with a unit circle, but rather with the *associated projective line*. The metrical geometry of the projective line is described by a *projective version of rational trigonometry*, which also extends to higher projective spaces, and it is in terms of this that we formulate rotations and reflections.

1.2 Basic rational trigonometry

Here is a quick review of some main ideas from affine rational trigonometry in the plane; the main reference is [6], see also [10]. We work over a general field, not of characteristic two. The primary measurement is the *quadrance* between two points $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$, defined by

$$Q(A_1, A_2) \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Over the real numbers, quadrance is the square of distance—or better yet *distance is the square root of quadrance.* However in rational trigonometry we wish to avoid square roots, and so distance plays no role. This allows the theory to extend to more general fields.

The short side, long side and diagonal of an $A4$ sheet of paper form a right triangle with quadrances Q_1, Q_2 and Q_3 in the ratio $1:2:3$, and Pythagoras' theorem takes the form

$$Q_1 + Q_2 = Q_3.$$

In rational trigonometry the separation of two lines is measured by a *spread*, not an *angle*. A line l with equation $ax + by + c = 0$ is a **null line** precisely when $a^2 + b^2 = 0$. The **spread** between two non-null lines with equations

$$a_1 x + b_1 y + c_1 = 0 \qquad \text{and} \qquad a_2 x + b_2 y + c_2 = 0$$

is defined to be the number

$$s \equiv \frac{(a_1 b_2 - a_2 b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}.$$

This formula also gives the spread between lines with direction vectors (a_1, b_1) and (a_2, b_2). Over the real numbers the spread s between two lines lies between 0 and 1, being 0 when lines are parallel, 1 when lines are perpendicular, and in general the square of the sine of an angle θ between them. There are many such possible angles (e.g. θ, $\pi - \theta, 2\pi + \theta, 3\pi - \theta, \cdots$), but the square of the sine—the *spread*—is the same for all.

The spread between two non-null lines can be expressed as a ratio of two quadrances—an opposite quadrance to a hypotenuse quadrance in a right triangle formed from those lines, so for example the spread between the long side l_0 of a piece of $A4$ paper and its diagonal l_1 is $s(l_0, l_1) = s = 1/3$. Unlike the notion of angle, spread really is defined between *lines*, not *rays*. Using spreads instead of angles makes much of the study of triangles—that is, *trigonometry*—dramatically simpler. This is explained at length in [6].

Define a number s to be a **spread number** precisely when $s(1-s)$ is a square in the field. The Spread number theorem ([6, Chapter 6]) asserts that a number s is the spread between two lines precisely when s is a spread number. In the finite prime field $\mathbb{F}_p$ of p elements, there are $(p+3)/2$ or $(p+1)/2$ spread numbers, depending respectively on whether p is congruent to $3 \bmod 4$ or $1 \bmod 4$.

1.3 Definition of the spread polynomials

The **spread polynomials** $S_n(s)$ arise when we consider the rational analog of *multiples of an angle*. They may be defined recursively, over a general field, without any reference to geometry as follows:

$$\begin{aligned} S_0(s) &\equiv 0 \\ S_1(s) &\equiv s \\ S_n(s) &\equiv 2(1-2s) S_{n-1}(s) - S_{n-2}(s) + 2s. \end{aligned} \tag{1}$$

The coefficient of s^n in $S_n(s)$ is a power of four, so over any field not of characteristic two, $S_n(s)$ is a polynomial of degree n. The generating function for the spread polynomials was computed by M. Hirschhorn, it is

$$S(s,t) \equiv \sum_{n=0}^{\infty} S_n(s) t^n = \frac{ts(1+t)}{(1-t)(1-2t+t^2+4ts)}. \tag{2}$$

The spread polynomials are intimately linked to geometry, in the following sense. If two intersecting lines l_0 and l_1 in the plane make a spread of s, then we may reflect l_0 in l_1 to obtain l_2, reflect l_1 in l_2 to obtain l_3, and so on. We will see that the spread $s(l_0, l_n)$ is then $S_n(s)$. For this description we only need the lines through the origin, so it is really a statement about the associated *projective line*. In fact our statement of this fact, the Spread of a power theorem, will be stated in sufficient generality to include also some non-Euclidean geometries, and will involve a multiplicative structure on the *non-null points of the projective line*.

Another key fact about the spread polynomials, which connects with their geometric interpretation, is the following.

Theorem (Spread composition) *For any natural numbers n and m,*

$$S_n \circ S_m = S_{nm}.$$

We will give a proof later using the relations between spreads and rotations; for another see [6, page 110].

1.4 Table and graphs of spread polynomials

Here are the first few spread polynomials. Note that $S_2(s)$ is the *logistic map*.

$$\begin{aligned}
S_0(s) &= 0 \\
S_1(s) &= s \\
S_2(s) &= 4s - 4s^2 = 4s(1-s) \\
S_3(s) &= 9s - 24s^2 + 16s^3 = s(3-4s)^2 \\
S_4(s) &= 16s - 80s^2 + 128s^3 - 64s^4 = 16s(1-s)(1-2s)^2 \\
S_5(s) &= 25s - 200s^2 + 560s^3 - 640s^4 + 256s^5 = s\left(5 - 20s + 16s^2\right)^2 \\
S_6(s) &= 4s(1-s)(3-4s)^2(1-4s)^2 \\
S_7(s) &= s\left(7 - 56s + 112s^2 - 64s^3\right)^2 \\
S_8(s) &= 64s(1-s)(1-2s)^2\left(1 - 8s + 8s^2\right)^2 \\
S_9(s) &= s(3-4s)^2\left(3 - 36s + 96s^2 - 64s^3\right)^2 \\
S_{10}(s) &= 4s(1-s)\left(1 - 12s + 16s^2\right)^2\left(5 - 20s + 16s^2\right)^2
\end{aligned}$$

A remarkable fact, already suggested by this list, is that the spread polynomials factor in an interesting way, indeed in a more pleasant fashion than the Chebyshev polynomials. We will establish this in a future paper.

The next Figures show the first eight and twenty five spread polynomials over the real numbers in the range $0 \leq s \leq 1$. Note the interesting ghost patterns that begin to appear as we increase the number of polynomials shown; these are related to Lissajous curves, and such a phenomenon occurs also for Chebyshev polynomials.

Observe also that the spread polynomials are positive in this range, so do not form an orthogonal family of polynomials in the usual sense unless they are translated vertically.

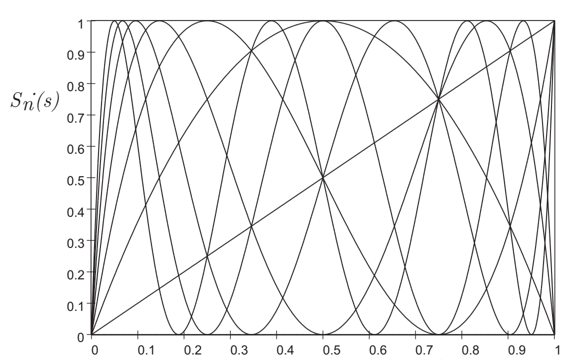

Figure 2: The first eight spread polynomials $S_0(s)$ to $S_7(s)$

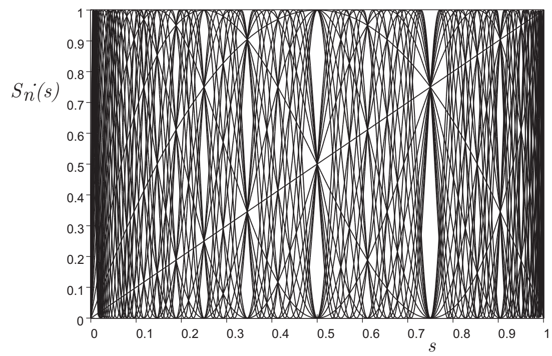

Figure 3: The first twenty five spread polynomials

Example 1 *Here are some values of spread polynomials in the field $\mathbb{F}_5$. The spread numbers in this field are* $0, 1$ *and* 3.

s	0	1	2	3	4
$S_0(s)$	0	0	0	0	0
$S_1(s)$	0	1	2	3	4
$S_2(s)$	0	0	2	1	2
$S_3(s)$	0	1	0	3	1
$S_4(s)$	0	0	2	0	2
$S_5(s)$	0	1	2	3	4

s	0	1	2	3	4
$S_6(s)$	0	0	0	1	0
$S_7(s)$	0	1	2	3	4
$S_8(s)$	0	0	2	0	2
$S_9(s)$	0	1	0	3	1
$S_{10}(s)$	0	0	2	1	2
$S_{11}(s)$	0	1	2	3	4

The pattern repeats with period 12, *that is* $S_n(s) = S_{n+12}(s)$ *for all* n *and for all* s. *However for values of* s *which are spread numbers,* $S_n(s) = S_{n+4}(s)$ *for all* n, *while for values of* s *which are non-spread numbers* $S_n(s) = S_{n+6}(s)$ *for all* n. $\diamond$

Example 2 *Here are the values of the spread polynomials in the field $\mathbb{F}_7$. The spread numbers in this field are* $0, 1, 3, 4$ *and* 5.

s	0	1	2	3	4	5	6
$S_0(s)$	0	0	0	0	0	0	0
$S_1(s)$	0	1	2	3	4	5	6
$S_2(s)$	0	0	6	4	1	4	6
$S_3(s)$	0	1	1	5	4	3	0
$S_4(s)$	0	0	6	1	0	1	6
$S_5(s)$	0	1	2	5	4	3	6
$S_6(s)$	0	0	0	4	1	4	0
$S_7(s)$	0	1	2	3	4	5	6
$S_8(s)$	0	0	6	0	0	0	6
$S_9(s)$	0	1	1	3	4	5	0
$S_{10}(s)$	0	0	6	4	1	4	6
$S_{11}(s)$	0	1	2	5	4	3	6
$S_{12}(s)$	0	0	0	1	0	1	0

s	0	1	2	3	4	5	6
$S_{13}(s)$	0	1	2	5	4	3	6
$S_{14}(s)$	0	0	6	4	1	4	6
$S_{15}(s)$	0	1	1	3	4	5	0
$S_{16}(s)$	0	0	6	0	0	0	6
$S_{17}(s)$	0	1	2	3	4	5	6
$S_{18}(s)$	0	0	0	4	1	4	0
$S_{19}(s)$	0	1	2	5	4	3	6
$S_{20}(s)$	0	0	6	1	0	1	6
$S_{21}(s)$	0	1	1	5	4	3	0
$S_{22}(s)$	0	0	6	4	1	4	6
$S_{23}(s)$	0	1	2	3	4	5	6
$S_{24}(s)$	0	0	0	0	0	0	0
$S_{25}(s)$	0	1	2	3	4	5	6

The pattern repeats with period 24, *that is* $S_n(s) = S_{n+24}(s)$ *for all* n *and for all* s. *For values of* s *which are spread numbers, however,* $S_n(s) = S_{n+8}(s)$ *for all* n, *while for the non-spread numbers* $S_n(s) = S_{n+6}(s)$. $\diamond$

1.5 Spread polynomials evaluated at $s = 1/3$

Consider the spread $s = 1/3$ formed by the long side l_0 and the diagonal l_1 of an $A4$ sheet of paper, as discussed previously. The numbers $S_n(1/3) = s(l_0, l_n)$ are the spreads formed by successive reflections, or rotations, of these two lines.

The recursive formula (1) allows us to calculate the following list of prime power factorizations:

$$
\begin{array}{ll}
S_1(1/3) = 3^{-1} & S_2(1/3) = 2^3 3^{-2} \\
S_3(1/3) = 5^2 3^{-3} & S_4(1/3) = 2^5 3^{-4} \\
S_5(1/3) = 3^{-5} & S_6(1/3) = 2^3 5^2 3^{-6} \\
S_7(1/3) = 43^2 3^{-7} & S_8(1/3) = 2^7 7^2 3^{-8} \\
S_9(1/3) = 5^2 19^2 3^{-9} & S_{10}(1/3) = 2^3 11^2 3^{-10} \\
S_{11}(1/3) = 197^2 3^{-11} & S_{12}(1/3) = 2^5 5^2 23^2 3^{-12} \\
S_{13}(1/3) = 1249^2 3^{-13} & S_{14}(1/3) = 2^3 13^2 43^2 3^{-14} \\
S_{15}(1/3) = 5^4 29^2 3^{-15} & S_{16}(1/3) = 2^9 7^2 17^2 3^{-16} \\
S_{17}(1/3) = 9791^2 3^{-17} & S_{18}(1/3) = 2^3 5^2 19^2 73^2 3^{-18} \\
S_{19}(1/3) = 26107^2 3^{-19} & S_{20}(1/3) = 2^5 11^2 241^2 3^{-20} \\
S_{21}(1/3) = 5^2 43^2 167^2 3^{-21} & S_{22}(1/3) = 2^3 197^2 263^2 3^{-22} \\
S_{23}(1/3) = 139^2 2207^2 3^{-23} & S_{24}(1/3) = 2^7 5^2 7^2 23^2 47^2 3^{-24} \\
S_{25}(1/3) = 149^2 1949^2 3^{-25} & S_{26}(1/3) = 2^3 131^2 1249^2 3^{-26} \\
S_{27}(1/3) = 5^2 19^2 53^2 433^2 3^{-27} & S_{28}(1/3) = 2^5 13^2 43^2 1511^2 3^{-28} \\
S_{29}(1/3) = 6973919^2 3^{-29} & S_{30}(1/3) = 2^3 5^4 11^2 29^2 239^2 3^{-30}
\end{array}
$$

Let's make some empirical observations about the above table. For each n, $S_n(1/3)$ is a fraction whose denominator is 3^n. The numerator is divisible by 2 precisely when n is even, and the power of 2 appearing is odd. The other factor of the numerator is a square. Prime factors of the numerator seem to occur periodically. For an odd prime $p \neq 3$, define $m(p)$ to be the smallest natural number m such that for all positive multiples n of m, $S_n(1/3)$ has a factor of p, if such an m exists. From the table, we may guess that this number for small primes p is:

$$
\begin{array}{llll}
m(5) = 3 & m(7) = 8 & m(11) = 10 & m(13) = 14 \\
m(17) = 16 & m(19) = 9 & m(23) = 12 & m(29) = 15.
\end{array}
$$

However larger primes also appear in the table, for example perhaps

$$m(6973919) = 29.$$

The aim of this paper is to try to begin to explain these numbers, and to show that the phenomenon is not dependent on the initial spread $s = 1/3$.

We adopt the convention that a rational number α is **divisible by a prime** p precisely when $\alpha = p\beta$ with β a rational number which can be expressed as $\beta = c/d$ with c and d integers, and d not divisible by p. In this case we say p is a **factor** of α. The following is the main result of this paper.

Theorem (Spread Periodicity) *For any rational number $s \equiv a/b$ and any prime p not dividing b, there is a natural number m such that $S_n(s)$ is divisible by p precisely when m divides n. This number m is a divisor of either $p-1$ or $p+1$.*

Corollary *For any rational number $s \equiv a/b$, any prime p not dividing b occurs infinitely often as a factor of the numbers $S_n(s)$* for $n = 1, 2, 3, \cdots$.

In addition, if we find a prime p appearing as a factor of $S_k(s)$, then we can be sure that p will appear as a factor of any spread $S_{nk}(s)$, for $n = 1, 2, 3, \cdots$. For example, from the above observations it follows that $S_{58}(1/3)$ is divisible by 6973919. Note however that we are not able to address the more difficult problem of determining $m(p)$ for a given p.

To prove the Spread periodicity theorem, we will explore the metrical geometry of the *one-dimensional projective line,* and show that finite geometries, both *Euclidean and non-Euclidean*, play a role.

2 Geometry of the projective line

The metrical structure of one-dimensional geometry over a general field $\mathbb{F}$, not of characteristic two, was investigated recently in [7]. There are two distinctly different contexts: *affine* and *projective*, and in this paper it is the projective setting that is of primary interest, because we are interested in rotations and reflections of one-dimensional subspaces of a two-dimensional space $\mathbb{F}^2$, which essentially takes place in the one-dimensional projective line.

A vector in $\mathbb{F}^2$ will typically be denoted U or V. If U is a non-zero vector, then $u = [U]$ represents the corresponding **projective point**, or **p-point** for short, namely the line OU through U and the origin $O \equiv [0,0]$. If $U \equiv [x,y]$ then we write $u = [U] \equiv [x:y]$, with the usual convention for proportions that $[x_1 : y_1] = [x_2 : y_2]$ precisely when $x_1 y_2 - x_2 y_1 = 0$. The projective points constitute the **projective line** $\mathbb{P}^1$.

Now fix a symmetric bilinear form on $\mathbb{F}^2$,

$$[x_1, y_1] \cdot [x_2, y_2] \equiv a x_1 x_2 + b\left(x_1 y_2 + x_2 y_1\right) + c y_1 y_2 \tag{3}$$

which is **non-degenerate**, that is

$$ac - b^2 \neq 0.$$

The projective point $u \equiv [U]$ is then **null** precisely when $U \cdot U = 0$. It is important to realize that such a bilinear form on $\mathbb{F}^2$ can naturally provide a *metrical structure on the associated projective line* $\mathbb{P}^1$.

The **projective quadrance**, or **p-quadrance** for short, between the non-null p-points $u_1 \equiv [U_1]$ and $u_2 \equiv [U_2]$ is the number

$$q\left(u_1, u_2\right) \equiv 1 - \frac{\left(U_1 \cdot U_2\right)^2}{\left(U_1 \cdot U_1\right)\left(U_2 \cdot U_2\right)}.$$

This is well-defined, and if $u_1 \equiv [x_1 : y_1]$ and $u_2 \equiv [x_2 : y_2]$, then a generalization of the well-known **Fibonacci's identity**

$$\left(x_1 x_2 + y_1 y_2\right)^2 + \left(x_1 y_2 - x_2 y_1\right)^2 = \left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right)$$

gives

$$\begin{aligned} q\left(u_1, u_2\right) &= 1 - \frac{\left(a x_1 x_2 + b\left(x_1 y_2 + x_2 y_1\right) + c y_1 y_2\right)^2}{\left(a x_1^2 + 2b x_1 y_1 + c y_1^2\right)\left(a x_2^2 + 2b x_2 y_2 + c y_2^2\right)} \\ &= \frac{\left(ac - b^2\right)\left(x_1 y_2 - x_2 y_1\right)^2}{\left(a x_1^2 + 2b x_1 y_1 + c y_1^2\right)\left(a x_2^2 + 2b x_2 y_2 + c y_2^2\right)}. \end{aligned}$$

This shows that $q\left(u_1, u_2\right) = 0$ precisely when $u_1 = u_2$. The pair $\left(\mathbb{P}^1, q\right)$ denotes the projective line together with the projective quadrance q.

The following is the fundamental formula for projective trigonometry in such a one-dimensional setting, with a proof similar to the one in [8]. In planar rational trigonometry this law is called the *Triple spread formula*, and is the analog of the fact that the sum of the angles in a triangle in the real Cartesian plane is π.

Theorem 3 (Projective triple quad formula) *If $u \equiv [U]$, $v \equiv [V]$ and $w \equiv [W]$ are non-null p-points, then the p-quadrances $q_w \equiv q(u,v)$, $q_u \equiv q(v,w)$ and $q_v \equiv q(u,w)$ satisfy*

$$\left(q_u + q_v + q_w\right)^2 = 2\left(q_u^2 + q_v^2 + q_w^2\right) + 4 q_u q_v q_w.$$

Proof. If we write $a_U \equiv U \cdot U$ and $b_{UV} \equiv U \cdot V$ then

$$q_u = \frac{a_V a_W - b_{VW}^2}{a_V a_W} \qquad q_v = \frac{a_U a_W - b_{UW}^2}{a_U a_W} \qquad \text{and} \qquad q_w = \frac{a_U a_V - b_{UV}^2}{a_U a_V}.$$

The following is an algebraic identity in the abstract variables a_U, a_V, a_W, b_{UV}, b_{UW} and b_{VW}:

$$\begin{aligned} &\left(q_u + q_v + q_w\right)^2 - 2\left(q_u^2 + q_v^2 + q_w^2\right) - 4 q_u q_v q_w \\ &= \left(a_U a_V a_W + 2 b_{UV} b_{UW} b_{VW} - a_U b_{VW}^2 - a_V b_{UW}^2 - a_W b_{UV}^2\right) \\ &\times \left(2 b_{UV} b_{UW} b_{VW} - a_U a_V a_W + a_U b_{VW}^2 + a_V b_{UW}^2 + a_W b_{UV}^2\right) a_W^{-2} a_V^{-2} a_U^{-2}. \end{aligned}$$

But the first factor on the right hand side is the determinant

$$\begin{vmatrix} a_U & b_{UV} & b_{UW} \\ b_{UV} & a_V & b_{VW} \\ b_{UW} & b_{VW} & a_W \end{vmatrix} = \begin{vmatrix} U \cdot U & U \cdot V & U \cdot W \\ U \cdot V & V \cdot V & V \cdot W \\ U \cdot W & V \cdot W & W \cdot W \end{vmatrix}$$

which is zero since U, V and W are coplanar. ■

Example 4 *For an integer k, the symmetric bilinear form*

$$(x_1, y_1) \cdot (x_2, y_2) \equiv x_1 x_2 + k y_1 y_2$$

is non-degenerate provided that the characteristic of $\mathbb{F}$ does not divide k, which we henceforth assume. In particular we assume that k is non-zero. If $u_1 \equiv [x_1 : y_1]$ and $u_2 \equiv [x_2 : y_2]$ then the associated p-quadrance q_k is

$$\begin{aligned} q_k (u_1, u_2) &\equiv 1 - \frac{(x_1 x_2 + k y_1 y_2)^2}{(x_1^2 + k y_1^2)(x_2^2 + k y_2^2)} \\ &= \frac{k (x_1 y_2 - x_2 y_1)^2}{(x_1^2 + k y_1^2)(x_2^2 + k y_2^2)}. \quad \diamond \end{aligned}$$

3 Isometries of the projective line

Let's now show how spread polynomials link to reflections and rotations in one-dimensional geometry. Suppose q is some fixed choice of p-quadrance on $\mathbb{P}^1$. An **isometry** of $(\mathbb{P}^1, q)$ is a map $\tau : u \to u\tau$ that inputs and outputs non-null p-points, and satisfies for any non-null p-points u and v

$$q(u, v) = q(u\tau, v\tau).$$

For a 2×2 matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

representing the following linear transformation on $\mathbb{F}^2$:

$$[x, y] \to [x, y] \begin{pmatrix} a & b \\ c & d \end{pmatrix} = [ax + cy, bx + dy]$$

define the corresponding **projective transformation** τ

$$[x : y] \tau = [ax + cy : bx + dy]$$

and denote it by

$$\tau \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The matrix for such a projective transformation is determined only up to a scalar, so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$$

for any non-zero number λ. We adapt the following from [8].

Theorem 5 (Isometries of the projective line) *An isometry of the projective space $(\mathbb{P}^1, q_k)$ must be a projective transformation of one of the following types:*

$$\rho_{[a:b]} \equiv \begin{bmatrix} a & b \\ -kb & a \end{bmatrix} \qquad \text{or} \qquad \sigma_{[a:b]} \equiv \begin{bmatrix} a & b \\ kb & -a \end{bmatrix}$$

for some non-null projective point $[a : b]$.

Proof. Suppose that τ is an isometry of $(\mathbb{P}^1, q_k)$ and that it sends $i_1 \equiv [1 : 0]$ to $[a : b]$ and $i_2 \equiv [0 : 1]$ to $[c : d]$. Since i_1 and i_2 are non-null p-points, $[a : b]$ and $[c : d]$ must also be non-null. Since $q_k (i_1, i_2) = 0$, we must have

$$ac + kbd = 0.$$

So $[c : d] = [kb : -a]$. Now given an arbitrary non-null projective point $u \equiv [x : y]$ with $u\tau = v \equiv [w : z]$,

$$q_k (u, i_1) = \frac{k y^2}{x^2 + k y^2} = q_k (v, [a : b]) = \frac{k (az - bw)^2}{(a^2 + kb^2)(w^2 + kz^2)}$$

and

$$q_k(u, i_2) = \frac{x^2}{x^2 + ky^2} = q_k(v, [kb : -a]) = \frac{(aw + kbz)^2}{(a^2 + kb^2)(w^2 + kz^2)}.$$

Comparing these two equations gives

$$x^2 : y^2 = (aw + kbz)^2 : (az - bw)^2$$

so that either

$$x : y = (aw + kbz) : (az - bw) \qquad \text{or} \qquad x : y = -(aw + kbz) : (az - bw).$$

Using matrix notation, either

$$[x : y] = [w : z] \begin{bmatrix} a & -b \\ kb & a \end{bmatrix} \qquad \text{or} \qquad [x : y] = [w : z] \begin{bmatrix} -a & -b \\ -kb & a \end{bmatrix}.$$

Inverting gives either

$$\tau = \begin{bmatrix} a & b \\ -kb & a \end{bmatrix} \qquad \text{or} \qquad \tau = \begin{bmatrix} a & b \\ kb & -a \end{bmatrix}.$$

If $[a : b]$ is non-null, then both of these are isometries, due to the equations

$$\begin{bmatrix} a & b \\ -kb & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} a & b \\ -kb & a \end{bmatrix}^T = \begin{bmatrix} a^2 + b^2k & 0 \\ 0 & a^2k + b^2k^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ kb & -a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} a & b \\ kb & -a \end{bmatrix}^T = \begin{bmatrix} a^2 + b^2k & 0 \\ 0 & a^2k + b^2k^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}. \quad \blacksquare$$

We call

$$\rho_{[a:b]} \equiv \begin{bmatrix} a & b \\ -kb & a \end{bmatrix} \qquad \text{and} \qquad \sigma_{[a:b]} \equiv \begin{bmatrix} a & b \\ kb & -a \end{bmatrix}$$

respectively a **projective rotation** and a **projective reflection**. These realizations as projective matrices extend isometries also to null points.

The **identity transformation**

$$\rho_{[1:0]} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a projective rotation.

Our convention for compositions is $u(\tau_1 \tau_2) \equiv (u\tau_1)\tau_2$.

Theorem 6 (Composition of isometries) *For any non-null p-points* $[a : b]$ *and* $[c : d]$

$$\begin{aligned} \sigma_{[a:b]}\sigma_{[c:d]} &= \rho_{[ac+kbd:ad-bc]} \qquad & \rho_{[a:b]}\rho_{[c:d]} &= \rho_{[ac-kbd:ad+bc]} \\ \rho_{[a:b]}\sigma_{[c:d]} &= \sigma_{[ac+kbd:ad-bc]} \qquad & \sigma_{[a:b]}\rho_{[c:d]} &= \sigma_{[ac-kbd:ad+bc]}. \end{aligned}$$

Proof. This is a straightforward verification. Another generalization of the Fibonacci identities,

$$(ac + kbd)^2 + k(ad - bc)^2 = \left(a^2 + kb^2\right)\left(c^2 + kd^2\right) = (ac - kbd)^2 + k(ad + bc)^2$$

shows that the resultant isometries are also associated to non-null points. $\blacksquare$

Define $G \equiv G(q_k)$ to be the group of isometries of q_k, and distinguish the subgroup $G_e \equiv G_e(q_k)$ of projective rotations $\rho_{[a:b]}$. These latter are naturally in bijection with the non-null p-points. From the Composition of isometries theorem, the subgroup G_e is commutative.

The coset $G_o \equiv G_o(q_k)$ consists of projective reflections $\sigma_{[a:b]}$, and these too are also naturally in bijection with the non-null p-points. The group G naturally acts on the space of non-null p-points. This action is transitive since $\sigma_{[a:b]}$ and $\rho_{[a:b]}$ both send $[1 : 0]$ to $[a : b]$.

Since projective rotations are in bijection with non-null p-points, we can *transfer the multiplicative structure*

$$\rho_{[a:b]}\rho_{[c:d]} = \rho_{[ac-kbd:ad+bc]}$$

of projective rotations to non-null p-points. So for $u \equiv [a:b]$ and $v \equiv [c:d]$ non-null p-points, define their k-**product** by the rule:

$$uv = [a:b]\,[c:d] \equiv [ac - kbd : ad + bc]\,.$$

When $k = 1$ this multiplication is familiar from the two-dimensional setting of complex numbers. Note however that we are here working in the one-dimensional situation, over a general field, and allowing different values of k. The resulting group is commutative, has **identity**

$$e \equiv [1:0]$$

and the **inverse** of $u \equiv [a:b]$ is $u^{-1} \equiv [a:-b]$. For a non-null projective point u, we let $u^n \equiv uu\cdots u$ (n times) denote the n**-th power** of u. Of course this depends on the prior choice $q = q_k$ of p-quadrance.

4 Spreads of p-points

Working in $\left(\mathbb{P}^1, q_k\right)$, define the k**-spread** $s_k(u)$ of the non-null p-point $u \equiv [a:b]$ to be the number

$$s_k(u) \equiv q_k(e,u) = \frac{kb^2}{a^2 + kb^2}.$$

Then $s_k(u) = 0$ precisely when $u = [1:0] = e$, and $s_k(u) = 1$ precisely when $u = [0:1]$. Since multiplication by v is the same as applying the projective rotation ρ_v, multiplication is an isometry, so that for any non-null p-points v and u,

$$q_k(v, vu) = s_k(u) = \frac{kb^2}{a^2 + kb^2}.$$

The next result connects spread polynomials and powers.

Theorem 7 (Spread of a power) *In $\left(\mathbb{P}^1, q_k\right)$, if u is a non-null projective point with $s_k(u) \equiv s$, then for any natural number n,*

$$s_k(u^n) = S_n(s)\,.$$

Proof. Suppose that $u \equiv [a:b]$ so that

$$s_k(u) = \frac{kb^2}{a^2 + b^2 k} \equiv s.$$

We know from the Isometries of the projective line theorem that the projective rotation ρ_u has the form

$$\rho_u = \begin{bmatrix} a & b \\ -kb & a \end{bmatrix}.$$

We diagonalize the matrix $\begin{pmatrix} a & b \\ -kb & a \end{pmatrix}$ using a number r satisfying $r^2 = -k$. If the field $\mathbb{F}$ does not contain such a number, the following equations take place in the quadratic extension $\mathbb{F}(r)$. First verify that

$$\begin{pmatrix} a & b \\ -kb & a \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ r & r \end{pmatrix} \begin{pmatrix} a+br & 0 \\ 0 & a-br \end{pmatrix} \begin{pmatrix} 1 & -1 \\ r & r \end{pmatrix}^{-1}.$$

Then for any natural number n,

$$\begin{aligned}\begin{pmatrix} a & b \\ -kb & a \end{pmatrix}^n &= \begin{pmatrix} 1 & -1 \\ r & r \end{pmatrix} \begin{pmatrix} (a+br)^n & 0 \\ 0 & (a-br)^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ r & r \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1}{2}(a+br)^n + \frac{1}{2}(a-br)^n & \frac{1}{2r}(a+br)^n - \frac{1}{2r}(a-br)^n \\ -\frac{r}{2}(a+br)^n - \frac{r}{2}(a-br)^n & \frac{1}{2}(a+br)^n + \frac{1}{2}(a-br)^n \end{pmatrix}.\end{aligned}$$

Taking the proportion determined by the first row gives

$$u^n = \left[r(a+br)^n + r(a-br)^n : (a+br)^n - (a-br)^n\right]$$

so

$$t_n \equiv s_k(u^n) = -\frac{((a+br)^n - (a-br)^n)^2}{4(a+br)^n(a-br)^n} = -\frac{(A^n - B^n)^2}{4A^nB^n}$$

where $A \equiv a + br$ and $B \equiv a - br$.

Clearly $t_0 = 0$, and since $r^2 = -k$,

$$t_1 = \frac{kb^2}{a^2 + b^2k} = s.$$

To show that $t_n = S_n(s)$ for all natural numbers n, we establish the identity

$$t_n - 2(1-2s)t_{n-1} + t_{n-2} - 2s = 0 \tag{4}$$

for all $n \geq 2$. The left hand side of (4) is

$$-\frac{(A^n - B^n)^2}{4A^nB^n} + 2\left(1 + \frac{2(A-B)^2}{4AB}\right)\frac{(A^{n-1} - B^{n-1})^2}{4A^{n-1}B^{n-1}} - \frac{(A^{n-2} - B^{n-2})^2}{4A^{n-2}B^{n-2}} + \frac{2(A-B)^2}{4AB}$$

and if we remove a factor of $(4A^nB^n)^{-1}$ this becomes

$$\begin{aligned}&-A^{2n} + 2A^nB^n - B^{2n} + \left(A^2 + B^2\right)\left(A^{2(n-1)} - 2A^{n-1}B^{n-1} + B^{2(n-1)}\right)\\&-A^2B^2\left(A^{2(n-2)} - 2A^{n-2}B^{n-2} + B^{2(n-2)}\right) + 2A^{n-1}B^{n-1}\left(A^2 - 2AB + B^2\right),\end{aligned}$$

which after expansion is identically zero, so holds independent of the particular choices of A and B. ■

Theorem 8 (Spread composition) *For any natural numbers n and m,*

$$S_n \circ S_m = S_{nm}.$$

Proof. Working over the rational numbers with the Euclidean quadrance q, the Spread of a power theorem shows that if $s \equiv s(u)$ for some p-point u, then $s(u^m) = S_m(s)$. It follows that

$$S_n(S_m(s)) = s((u^m)^n) = s(u^{mn}) = S_{nm}(s).$$

Since this holds for more than nm different values of s, and both $S_n \circ S_m$ and S_{nm} are polynomials of degree nm, we conclude that

$$S_n \circ S_m = S_{nm}. \quad ■$$

5 Spread numbers in a prime field

Fix an integer k which is not zero in the field $\mathbb{F}_p$ of characteristic $p \neq 2$, in other words which is not divisible by p. Define a number a in $\mathbb{F}_p$ to be a **k-spread number** precisely when $a(1-a)$ is k times a square. If $k = 1$ this agrees with our earlier usage.

For p an odd prime, the finite prime field $\mathbb{F}_p$ contains an equal number of non-zero squares and non-squares. This follows for example from the standard fact that the multiplicative group of a finite field is cyclic. Note that for $p \equiv 3 \bmod 4$ every number is either a square or the negative of a square, since -1 is not a square, and so every number is either a 1-spread number or a (-1)-spread number. However for $p \equiv 1 \bmod 4$ the 1-square numbers and the (-1)-square numbers agree.

Every number a in a field is a k-spread number for at least one k, for example 0 and 1 are k-spread numbers for all k, and for other a we may choose $k \equiv a(1-a)$. The next result generalizes the Spread number theorem in [6, Theorem 34].

Theorem 9 (Spread number) *For any non-null p-points u and v in $\left(\mathbb{P}^1, q_k\right)$, the p-quadrance $q_k(u,v)$ is a k-spread number, and conversely for every k-spread number q there exist non-null p-points u and v with $q_k(u,v) = q$, and so there exists a non-null p-point w with $s_k(w) = q$.*

Proof. If $u \equiv [x_1 : y_1]$ and $v \equiv [x_2 : y_2]$ then

$$q_k(u,v) = \frac{k(x_1y_2 - x_2y_1)^2}{(x_1^2 + ky_1^2)(x_2^2 + ky_2^2)} \equiv q$$

in which case

$$q(1-q) = k\frac{(x_1x_2 + ky_1y_2)^2 (x_2y_1 - x_1y_2)^2}{(x_2^2 + ky_2^2)^2 (x_1^2 + ky_1^2)^2}$$

which is k times a square, so q is a k-spread number. Conversely suppose that q is a k-spread number, so that $q(1-q) = kr^2$ for some number r in the field. If $q = 1$ then it is the p-quadrance between $[1:0]$ and $[0:1]$. Otherwise the p-quadrance between the p-points $u \equiv [1:0]$ and $v \equiv [1-q:r]$ is

$$k\frac{r^2}{(1-q)^2 + kr^2} = \frac{q(1-q)}{(1-q)^2 + q(1-q)} = q.$$

Note that

$$(1-q)^2 + kr^2 = (1-q)^2 + q(1-q) = 1-q$$

is indeed non-zero so both u and v are non-null.

Once we have u and v we can multiply both by u^{-1} to obtain 1 and $w \equiv u^{-1}v$ so that

$$q_k(1,w) = s_k(w) = q. \quad \blacksquare$$

To determine k-spread numbers in a prime field $\mathbb{F}_p$ it suffices to know the squares in the field. In terms of the Legendre symbol

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a square in } \mathbb{F}_p \\ -1 & \text{otherwise} \end{cases}$$

q is a k-spread number precisely when

$$\left(\frac{kq(1-q)}{p}\right) = \left(\frac{-k}{p}\right)\left(\frac{q}{p}\right)\left(\frac{q-1}{p}\right) = 1. \tag{5}$$

So it suffices to know whether or not q, $q-1$ and $-k$ are squares. If $k = 1$ (Euclidean geometry) then the above gives a straightforward recipe for finding spread numbers from square numbers.

Example 10 *In $\mathbb{F}_{13}$, $\left(\frac{-1}{p}\right) = 1$ and since the square numbers are $0, 1, 3, 4, 9, 10$ and 12, the spread numbers are $q = 0, 1, 4, 6, 7, 8$ and 10, as these are those q (aside from 0 and 1) whose Legendre symbol agrees with that of $q-1$. Whereas in $\mathbb{F}_{11}$, $\left(\frac{-1}{p}\right) = -1$ and since the squares are $0, 1, 3, 4, 5$ and 9, the spread numbers are $0, 1, 2, 3, 6, 9$ and 10, those q (aside from 0 and 1) whose Legendre symbol disagree with that of $q-1$.* $\diamond$

6 Spread periodicity

Suppose $\mathbb{F} = \mathbb{F}_p$ for some odd prime p. In $(\mathbb{P}^1, q_k)$ there are exactly $p+1$ p-points, but the number of null p-points depends also on k; there are null p-points precisely when

$$x^2 + ky^2 = 0$$

has non-zero solutions. This occurs precisely when $-k$ is a square modulo p, and in this case there are exactly 2 null p-points, and so the order of the group of rotations G_e is $p-1$. Otherwise there are no null p-points and G_e has order $p+1$.

We regard an element of $\mathbb{F}_p$ as an expression of the form a/b, where a and b are integers with b not divisible by p, with the convention that $a/b = c/d$ precisely when

$$ad - bc \text{ is divisible by } p.$$

Thus every rational number a/b with b not divisible by p represents also an element of $\mathbb{F}_p$. So the values of the spread polynomial $S_n(s)$ over $\mathbb{F}_p$ are just obtained by reducing $\bmod\, p$.

Theorem 11 (Spread periodicity) *Fix an odd prime p. For any element s in $\mathbb{F}_p$, there exists a positive integer m with the property that $S_n(s) = 0$ in $\mathbb{F}_p$ precisely when n is a multiple of m. Furthermore m divides either $p-1$ or $p+1$.*

Proof. Choose a non-zero integer k such that s is a k-spread number in $\mathbb{F}_p$. The Spread number theorem then asserts that there is a non-null projective point u in the projective line $(\mathbb{P}^1, q_k)$ over $\mathbb{F}_p$ with $s_k(u) = s$. Multiplication by u is a projective rotation ρ_u, and by the Spread of a power theorem $s_k(u^n) = S_n(s)$. So $S_n(s) = 0$ precisely when $u^n = e = [1, 0]$. Since ρ_u belongs to the finite commutative group G_e, $S_n(s) = 0$ precisely when n is a multiple of the order m of ρ_u in G_e. This m divides the order of the group G_e, which is either $p-1$ or $p+1$. ■

Corollary 12 *For any rational number $s = a/b$, any prime p not dividing b occurs infinitely often as a factor of the numbers $S_n(s)$ for $n = 1, 2, 3, \cdots$.*

Proof. This is an immediate consequence of the theorem. ■

To illustrate the theorem for $s = 1/3$, we will discuss the numbers $m(5)$, $m(7)$ and $m(19)$.

Example 13 *In $\mathbb{F}_5$ the squares are $0, 1$ and 4, while the 1-spread numbers are $0, 1$ and 3. In this field $s = 1/3 = 2$ and so $s(1-s) = 3$, which is not a square, but $s(1-s) = k \times r^2$ for $k = 2$ and $r = 2$. That means that s is a 2-spread number (and hence also a 3-spread number). There are no null p-points for either of the projective quadrances q_2 or q_3, since -2 and -3 are not squares in $\mathbb{F}_5$. Hence G_e has 6 elements in either case, and so the order $m(5)$ divides 6. In fact $m(5) = 3$. A projective point with 2-spread equal to 2 is $u \equiv [1-s : r] = [2 : 1]$. Then $u^2 = [1 : 2]$ and $u^3 = [1 : 0]$.* ⋄

Example 14 *In $\mathbb{F}_7$ the squares are $0, 1, 2$ and 4, while the 1-spread numbers are $0, 1, 3, 4$ and 5. In this field $s = 1/3 = 5$ is a spread number since $s(1-s) = k \times r^2$ with $k = 1$ and $r = 1$. Then $(\mathbb{P}^1, q_1)$ has no null p-points since -1 is not a square, so G_e has order 8. Thus $m(7)$ divides 8. In fact $m(7) = 8$. A projective point with 1-spread equal to 5 is $u \equiv [1-s : r] = [3 : 1]$. Then the powers of u are $u^2 = [6 : 1]$, $u^3 = [5 : 1]$, $u^4 = [0 : 1]$, $u^5 = [2 : 1]$, $u^6 = [1 : 1]$, $u^7 = [4 : 1]$ and $u^8 = [1 : 0]$.* ⋄

Example 15 *In $\mathbb{F}_{19}$ the squares are*

$$0, 1, 4, 5, 6, 7, 9, 11, 16, 17$$

while the 1-spread numbers are

$$0, 1, 2, 4, 8, 9, 10, 11, 12, 16, 18.$$

In this field $s = 13$ is not a 1-spread number but $s(1-s) = 15 = k \times r^2$ with $k = -1$ and $r = 2$, so s is a (-1)-spread number. Then $(\mathbb{P}^1, q_{-1})$ has null p-points since 1 is a square, so G_e has order 18. Thus $m(19)$ divides 18. In fact $m(19) = 9$. A projective point with (-1)-spread equal to 13 is $u \equiv [1-s : r] = [13 : 1]$. Then the powers of u are $u^2 = [8 : 1]$, $u^3 = [5 : 1]$, $u^4 = [10 : 1]$, $u^5 = [9 : 1]$, $u^6 = [14 : 1]$, $u^7 = [11 : 1]$, $u^8 = [6 : 1]$ and $u^9 = [1 : 0]$. ⋄

6.1 The special spreads

The spreads

$$s = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$$

play a special role in rational trigonometry. Over the real numbers, they correspond to angles of $0, 30°, 45°, 60°$ and $90°$ respectively, and also to the supplements of these angles. Over a fixed field, any spread polynomial evaluated at any one of these five values of s will also yield one of these five values.

For example if $s = 1/4$ then $S_2(s) = 3/4$, $S_3(s) = 1$, $S_4(s) = 3/4$, $S_5(s) = 1/4$ and $S_6(s) = 0$. It is not hard to check that

$$S_6(s) = 0$$

for any one of these values of s. Since this is true over the rationals, it is true over any other field not of characteristic two.

Example 16 *In $\mathbb{F}_7$, $s = 1/4 = 2$ which is a 3-spread number since $s(1-s) = 5 = k \times r^2$ with $k = 3$ and $r = 2$. Then q_3 has null p-points since -3 is a square, so G_e has order 6. Thus $m(7)$ divides 6. In fact $m(7) = 6$. A projective point with 3-spread equal to 2 is $u = [1-s : r] = [3, 1]$. Then the powers of u are $u^2 = [1 : 1]$, $u^3 = [0 : 1]$, $u^4 = [6 : 1]$, $u^5 = [4 : 1]$ and $u^6 = [1 : 0]$.* ⋄

Example 17 *In* $\mathbb{F}_{11}$, $s = 1/4 = 3$ *which is a* 1*-spread number and* $s(1-s) = 5 = 4^2$. *Then* $(\mathbb{P}^1, q_1)$ *has no null p-points since* -1 *is not a square, so* G_e *has order* 12. *Thus* $m(11)$ *divides* 12. *In fact* $m(11) = 6$. *A projective point with* 1*-spread equal to* 3 *is* $u = [1-s:r] = [-2:4] = [5:1]$. *Then the powers of* u *are* $u^2 = [9:1]$, $u^3 = [0:1]$, $u^4 = [2:1]$, $u^5 = [6:1]$ *and* $u^6 = [1:0]$. $\diamond$

7 Conclusion

The results of this paper indicate that there are further rich number theoretical aspects of the spread polynomials. Some of these will be studied in future work, particularly the parallels between Spread polynomials and Chebyshev polynomials, and the close affinity with *cyclotomic polynomials*.

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