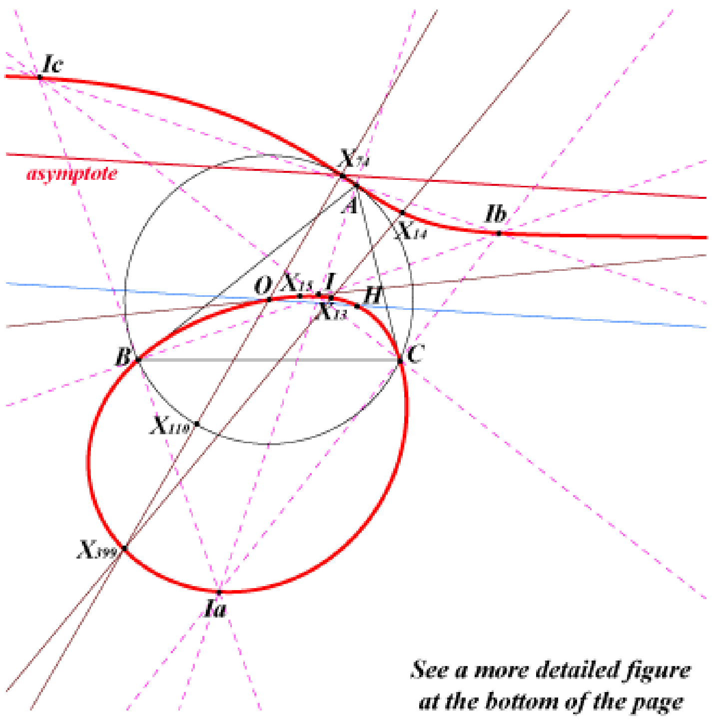
K001 Neuberg cubic = pK(X6, X30)



See a more detailed figure at the bottom of the page

Barycentric equation of the curve :

$$\sum_{\text{cyclic}} \left[a^2(b^2+c^2)+(b^2-c^2)^2-2a^4\right] x\,(c^2y^2-b^2z^2)=0$$

Points on the curve :

X(1), X(3), X(4), X(13), X(14), X(15), X(16), X(30), X(74), X(370), X(399), X(484), X(616), X(617), X(1138), X(1157), X(1263), X(1276), X(1277), X(1337), X(1338), X(2132), X(2133), X(3065), X(3440), X(3441), X(3464), X(3465), X(3466), X(3479), X(3480), X(3481), X(3482), X(3483), X(3484), X(5623), X(5624), X(5667) up to X(5685)

excenters; reflections of A, B, C in the sidelines of ABC; cevians of X(30)

vertices of 6 equilateral triangles erected on the sides of ABC

See table 19 for a description of these centers and more points on the Neuberg cubic.

other points Ua, Ub, Uc, etc below. See also table 16 and table 18.

five mates of X(370). See also K143, K144, Q033 and table 10 : X(370) and related curves.

Geometric properties :

The Neuberg cubic K001 is introduced in Neuberg's paper "Mémoire sur le tétraèdre" in Mémoires de l'Académie de Belgique, pp.1–70, 1884. The characterization given is related with properties of so-called "quadrangles involutifs" :

$$M \in \text{K001} \iff \begin{vmatrix} 1 & BC^2+AM^2 & BC^2\times AM^2 \\ 1 & CA^2+BM^2 & CA^2\times BM^2 \\ 1 & AB^2+CM^2 & AB^2\times CM^2 \end{vmatrix} = 0$$

K001 is sometimes called 21-point cubic or 37-point cubic in older literature.

K001 is the isogonal pK with pivot X(30) = infinite point of the Euler line : it is the locus of point P such that the line PP* is parallel to the Euler line (P* isogonal conjugate of P). Hence it is a member of the Euler pencil, see Table 27.

It follows that it is also the locus of P such that X(74), P, X(30)/P (Ceva conjugate) and X(2132), P, X(74)©P (crossconjugate) are collinear. X(74) is the isopivot (or secondary pivot) and X(2132) could be considered as a tertiary pivot.

K001 is a circular cubic with singular focus X(110), focus of the Kiepert parabola. It is also the orthopivotal cubic O(X3) and C(0) = C(infty) in "On two Remarkable Pencils of Cubics of the Triangle Plane" (see Downloads page).

The isotomic transform of K001 is K276. The inversive image of K001 in the circumcircle is Ki = K073. See also Inverses of Isocubics.

See Table 20 for cubics anharmonically equivalent to K001.

Related papers :

Neuberg Cubics

[Pairs and Triads of points on the Neuberg Cubic connected with Euler Lines and Brocard Axes]

[Isometric Parallel Chords on the Neuberg Cubic]

Locus properties : (see also Z. Cerin's papers in bibliography)

1. Denote by Pa, Pb, Pc the reflections of point P in BC, CA, AB respectively. Triangles ABC and PaPbPc are perspective if and only if P lies on the Neuberg cubic. The perspector Q lies on the Kn cubic and P, P*, Q are collinear. For this reason, the Neuberg cubic is called 2-pedal cubic in Pinkernell's paper. See also K216 and a generalization at Q067.
2. Denote by Oa, Ob, Oc the circumcenters of triangles PBC, PCA, PAB respectively. Triangles ABC and OaObOc are perspective if and only if P lies on the Neuberg cubic (together with C(O,R) and line at infinity). The perspector Q lies on the Napoleon cubic. Indeed, Q is the common point of the lines OP and HP*. Hence, Q* = OP* /\ HP and, since PP* and OH are parallel, QQ* passes through the midpoint of OH thus Q lies on the Napoleon cubic. Note that, if P is the perspector of an equilateral cevian triangle, Q is the center of the triangle. More generally, the mapping phi : M -> OM /\ HM* is an involution which commutes with isogonality (Jean-Pierre Ehrmann). If we replace the circumcenter by another center on the Euler line, we generally obtain another cubic such as K032 = Soddy cubic (with L = X(20)), K116 (with the nine-point center). See "On two remarkable pencils of cubics" in the Downloads page. On the other hand, with the Lemoine point K, we obtain the Lemoine quintic Q016.
3. The circumcenter of OaObOc lies on the Euler line if and only if P lies on the Neuberg cubic (together with C(O,R). (Jean-Pierre Ehrmann)
4. The Euler lines of triangles PBC, PCA, PAB concur (on the Euler line) if and only if P lies on the Neuberg cubic (together with C(O,R) and line at infinity). Further details below.
5. Denote by Ap, Bp, Cp the pedal triangle of P. The Euler lines of triangles PBpCp, PCpAp, PApBp concur (at Q) if and only if P lies on the Neuberg cubic (together with the line at infinity). The locus of Q is the circular sextic Q093.
6. The Brocard lines of triangles PBC, PCA, PAB concur (on the Brocard axis) if and only if P lies on the Neuberg cubic (together with C(O,R) and line at infinity). Further details below.
7. (P, X, Y) denotes the power of P with respect to the circle centered at X passing through Y. The locus of point P such that (P, A, B)*(P, B, C)*(P, C, A) = (P, B, A)*(P, C, B)*(P, A, C) is the Neuberg cubic. This property is equivalent to that given by Neuberg.
8. A tangent to the Kiepert parabola meets ABC sidelines at U, V, W. The circles C(U,A), C(V,B), C(W,C) form a pencil and intersect at two points of the Neuberg cubic collinear with O. (given in Neuberg's paper)
9. The isogonal conjugate of the tangent at X(74) to the circumcircle is the circum-parabola passing through X(476) with axis parallel to the Euler line. The locus of point P such that the polar lines (in this parabola) of P and its isogonal conjugate P* are parallel is the Neuberg cubic.
10. The in-conic with perspector X(1494) (isotomic conjugate of X(30)) is the parabola with focus X(74), directrix the perpendicular at H to the Euler line, axis the parallel at X(74) to the Euler line. The locus of point P such that the polar lines (in this parabola) of P and its isogonal conjugate P* are parallel is the Neuberg cubic.
11. For any point P, the locus of the perspectors of the equilateral triangles centered at P which are perspective with ABC is the orthopivotal cubic O(P). When P = O, we find the Neuberg cubic.
12. Let ABC be a triangle and P a point. The line AP intersects the circumcircle of the triangle PBC at P and A'. Similarly define B', C'. The orthocenters of BCA', CAB', ABC' form a triangle perspective to ABC if and only if P lies on the Neuberg cubic (together with the circumcircle, the line at infinity, the 3 circles with diameters BC, CA, AB). See also Q003 and Q030 for other loci related to the same configuration.
13. Locus of pivots of circular pKs which pass through the isodynamic points X(15), X(16). The locus of the poles is pK(X6 x X50, X6).
14. Locus of pivots of pK60+. The locus of the poles is K095.

Other properties :

See details in Special Isocubics, §6.5 and also table 18.

Ga, Gb, Gc are the vertices of the antimedial triangle.
Denote by (Ha) the hyperbola passing through B, C, Ga, the reflection A' of A in the line BC, the reflection Ah of H with respect to the second intersection of the altitude AH with the circumcircle. The asymptotes of (Ha) make 60° angles with

the sideline BC. (Hb) and (Hc) are defined similarly. See figure 1.

(Ha), (Hb), (Hc) have three points Ua, Ub, Uc in common which lie :

1. on the Neuberg cubic,
2. on the Lemoine cubic K009,
3. on K092, the only isotomic pK60,
4. on the circle C(L,2R) with L = X(20),
5. on the rectangular hyperbola passing through O, L, X(399), the reflection of H in X(110) and whose asymptotes are parallel to those of the Jerabek hyperbola,
6. on the rectangular hyperbola passing through O, H, the reflection of H in X(107) and whose asymptotes are parallel to those of the rectangular circum-hyperbola with center X(122),
7. on K080 = KO++. See figure 2.
8. on K142, another K60.
9. on several other curves as seen in table 18.

The orthocenter of triangle UaUbUc is O and its circumcenter is L. Thus ABC and UaUbUc share the same Euler line.

All rectangular hyperbolas passing through Ua, Ub, Uc, O are centered on the circle centered at X(550) with radius R (X(550) is the reflection of the nine point center X(5) in the circumcenter O).

The isogonal conjugates Ua*, Ub*, Uc* of Ua, Ub, Uc , the midpoints Va, Vb, Vc of triangle Ua*Ub*Uc*, their isogonal conjugates Va*, Vb*, Vc* are nine other points on the Neuberg cubic. Notice that Va, Vb, Vc are the complements of Ua*, Ub*, Uc* respectively, these latter points lying on the circle C(O,2R). Thus the translation with vector OH maps the triangle UaUbUc to the triangle Ua*Ub*Uc*. Note that the lines Ua*Va*, Ub*Vb*, Uc*Vc* concur at X(399), the Parry reflection point. See figure 3.

Hence we know the nine common points of the Neuberg cubic and the cubic which is its anticomplement : these are O, the Fermat points X(13) and X(14), the point at infinity X(30) of the Euler line, the three points Va, Vb, Vc and the circular points at infinity.

At last, remark that all the cubics passing through A, B, C, Ga, Gb, Gc, Ua, Ub, Uc are equilateral cubics and the equilateral triangle formed by the asymptotes has always center O. The only K60+ is K0++ = K080 which is at the same time a central cubic.

See also the related cubic K405 which contains Ua*, Ub*, Uc*.

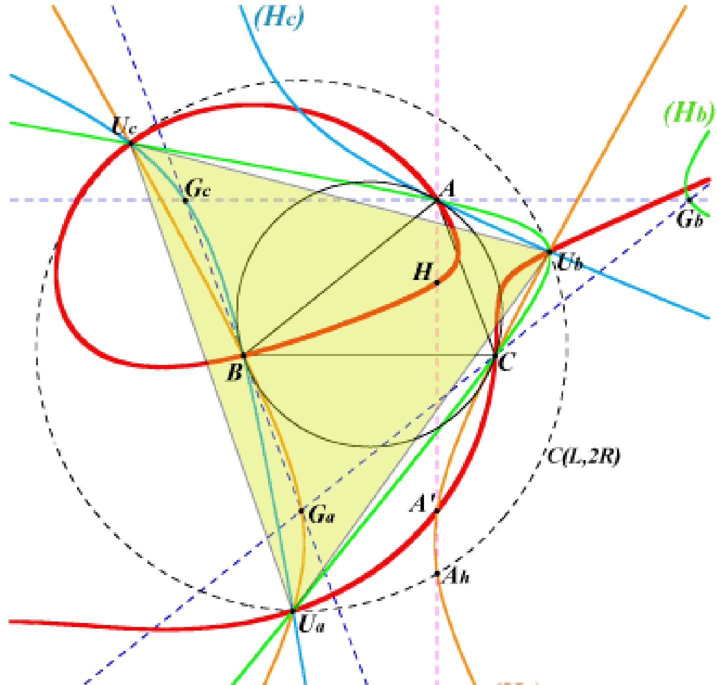


figure 1

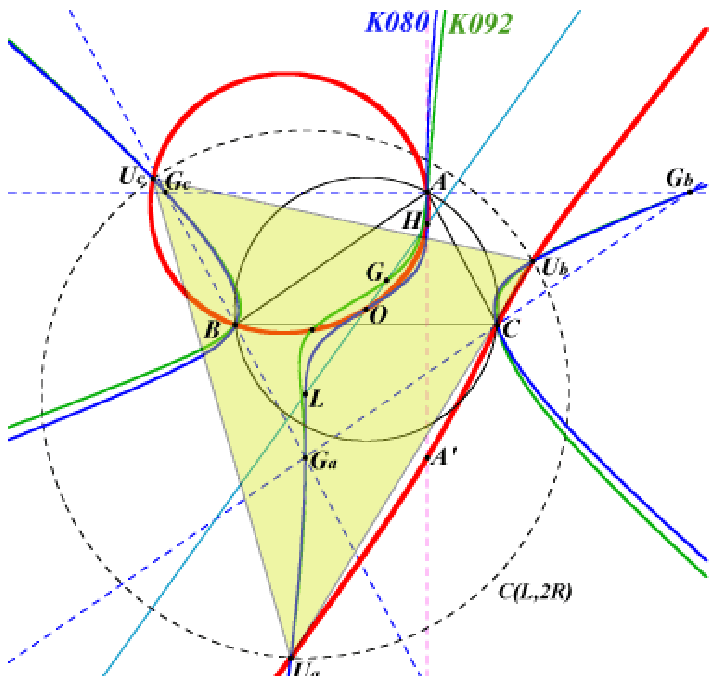


figure 2

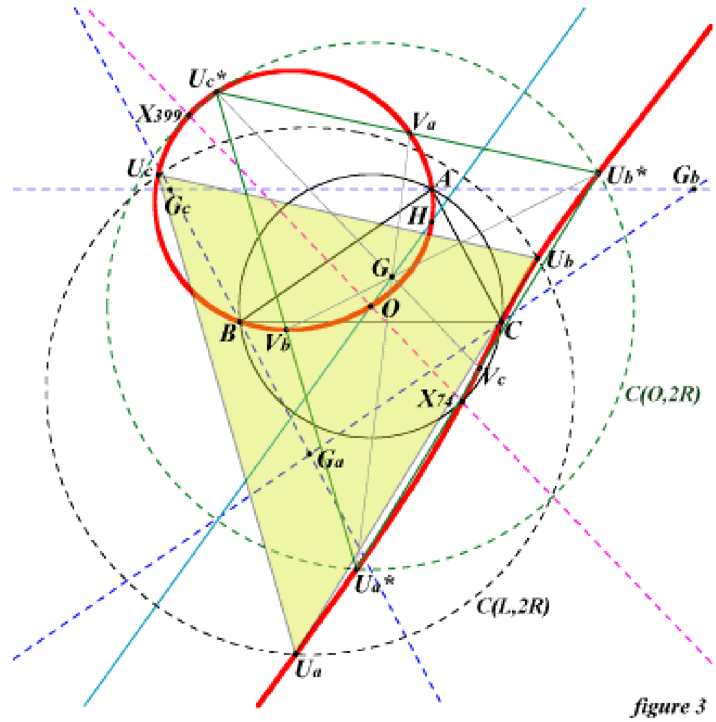
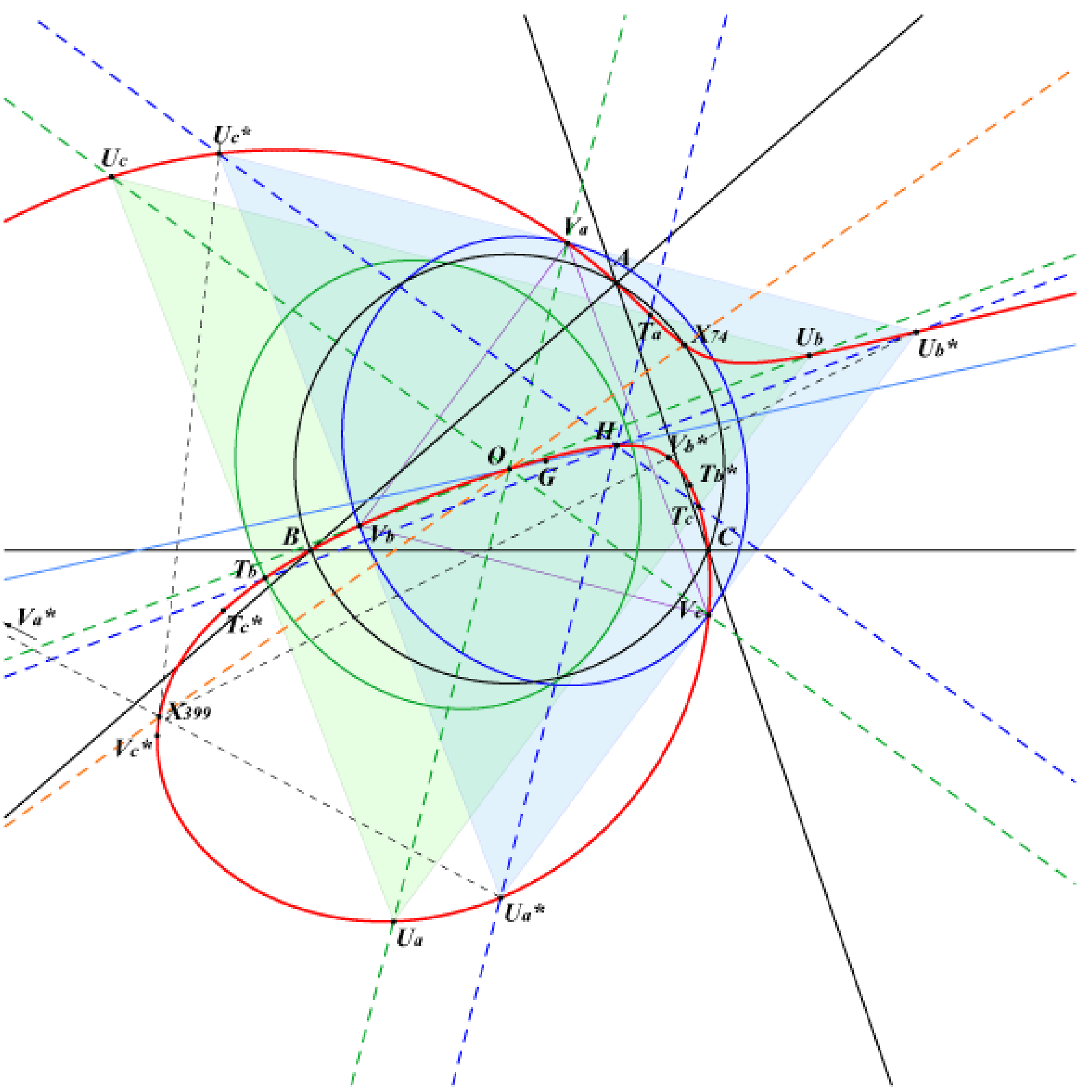
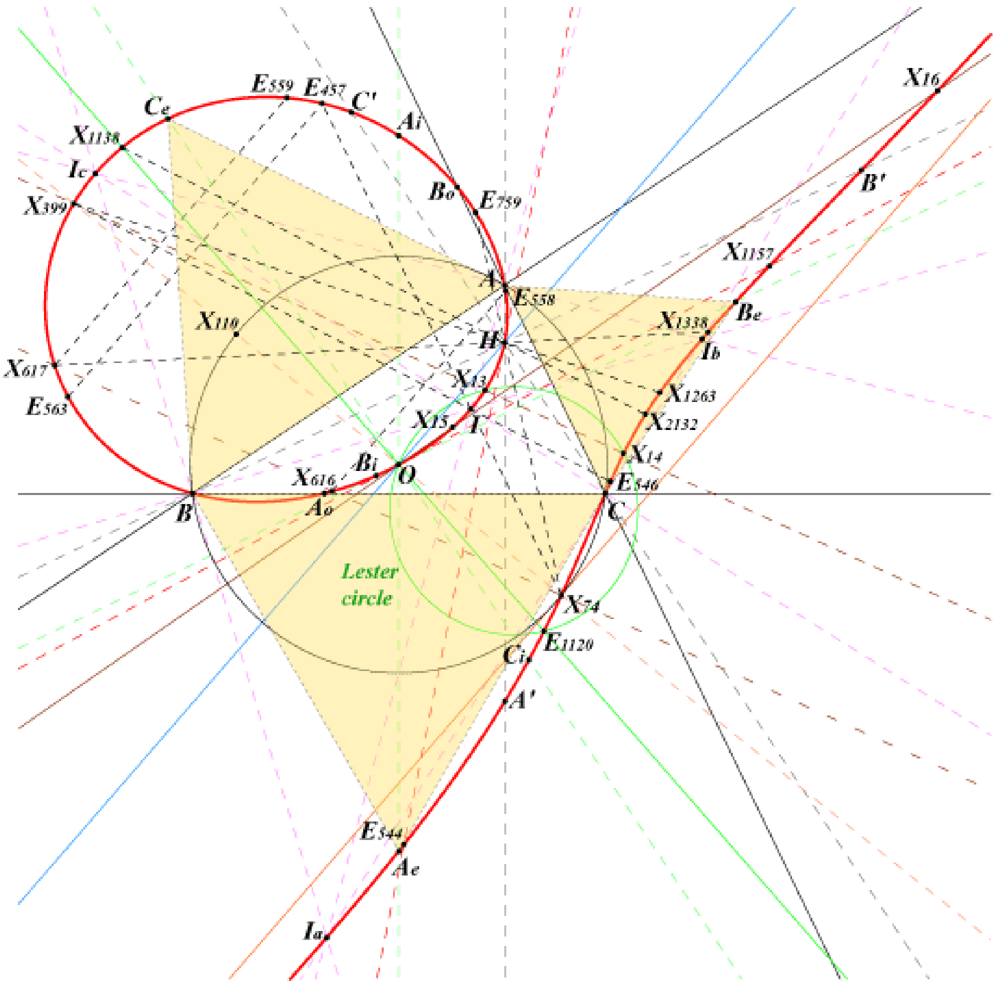

Uc*
X399
Va
Uc
A
Ub*
Gb
Gc
H
Ub
G
O
B
Vb
C
Vc
X74
C(O,2R)
Ga
Ua*
C(L,2R)
Ua

figure 3

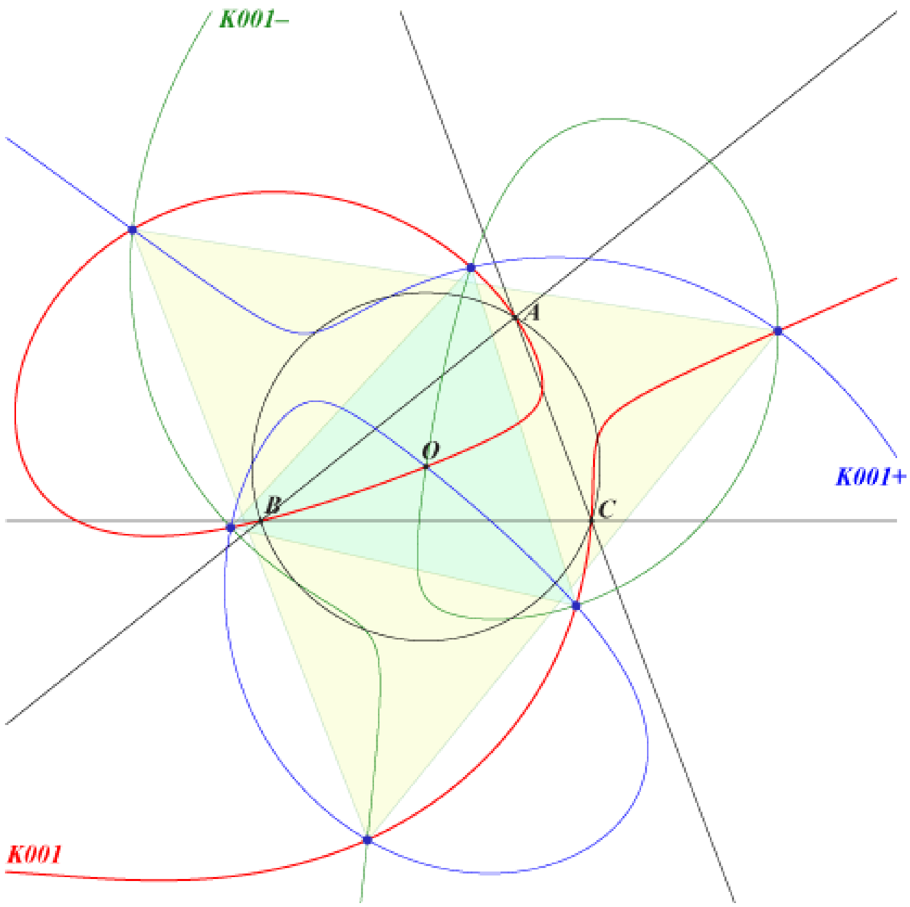

Uc*
Uc
Va
A
Ta
X74
Ub
Ub*
H
Vb*
O
G
Tb*
Tc
B
Vb
C
Tb
Vc
Va*
Tc*
X399
Vc*
Ua*
Ua

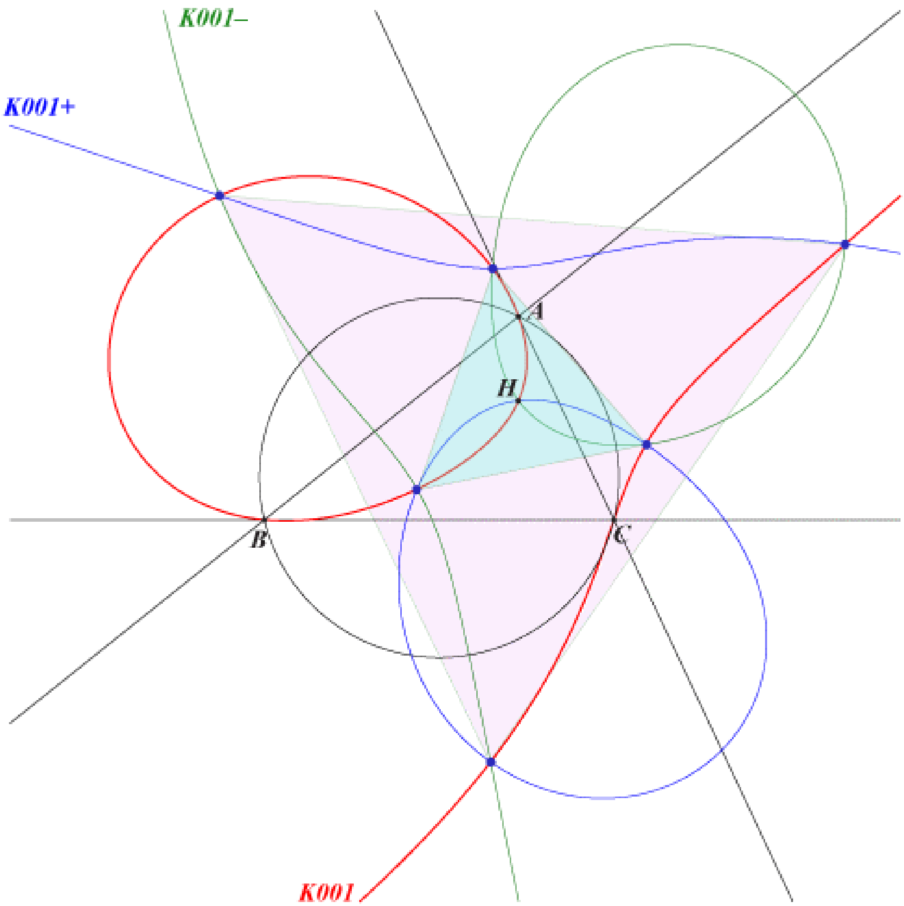
A closer view to the Neuberg cubic...



Four other equilateral triangles inscribed in the Neuberg cubic

Under the rotations with center O (resp. H) and angles +/-2π/3, the Neuberg cubic is transformed into two other circular cubics K001+ and K001–. These two cubics generate a pencil of circular cubics passing through O (resp. H) and therefore having six other common points. This pencil contains the Neuberg cubic itself hence there are two equilateral triangles with center O (resp. H) inscribed in the Neuberg cubic. See the two figures below.





the Neuberg cubic and other related curves

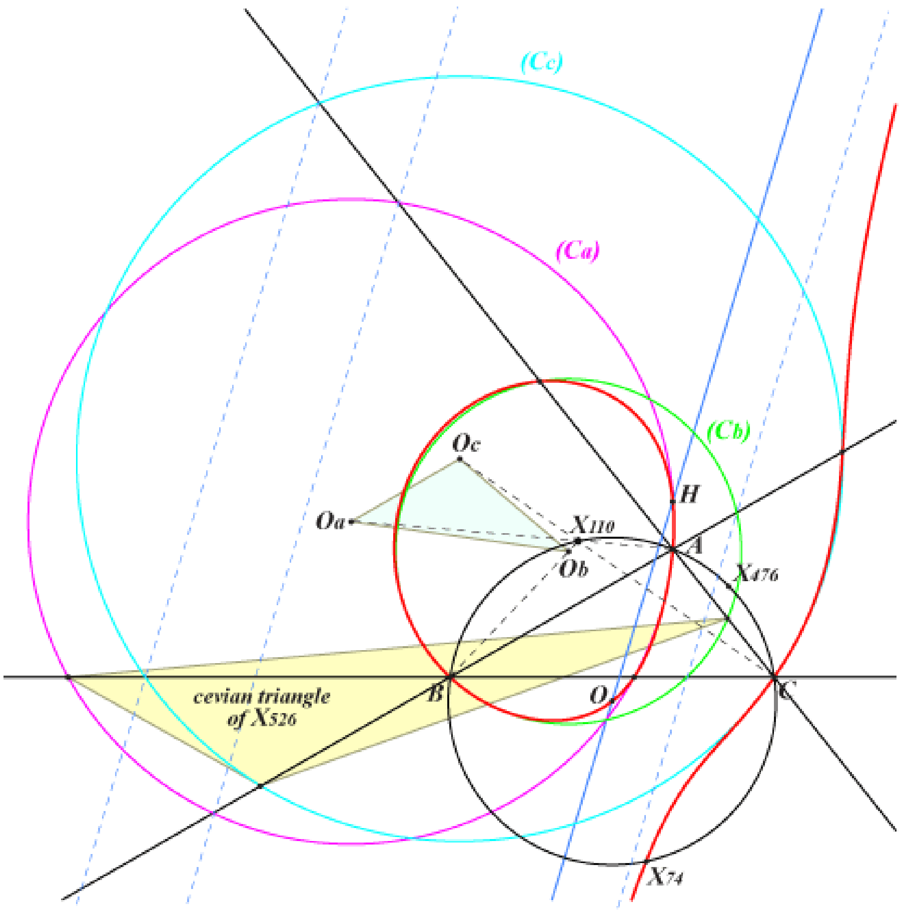
A line passing through O meets the Neuberg cubic again at two points M, N (see property 8 above). The midpoint P of MN lies on the Stammler strophoid K038. Note that these points M, N lie on a same circum-conic passing through X(1138).

Similarly, a line passing through H meets the Neuberg cubic again at two points M, N and their midpoint lies on another strophoid K591 we shall call the Kiepert-Neuberg strophoid.

These strophoids are both pedal curves of the Kiepert parabola. More information in K591, K592, K593.

Osculating circles on the Neuberg cubic

The osculating circle (Ca) at A to the Neuberg



cubic passes through A, the traces of X(30) and X(526) on the sideline BC. Its center Oa lies on the line AX(110).

(Cb), (Cc) and their centers Ob, Oc are defined similarly.

The triangle OaObOc is perspective with

• ABC at X(110),

• the cevian triangle of X(476) at X(523),

• the anticevian triangle of any point on the cubic K130 = pK(X6, X476).

The locus of the perspector is the isogonal pK with pivot P = X(5)X(1117) /\ X(110)X(476) passing through the in/excenters, X(110), X(477), X(523), X(5663) and Oa, Ob, Oc.

P lies on the anticomplement of the Steiner deltoid.

Note that this cubic is tangent at X(523) to the line at infinity and tangent at X(110) to the circumcircle.

See the analogous cubic K695 and further details in the page Q073.

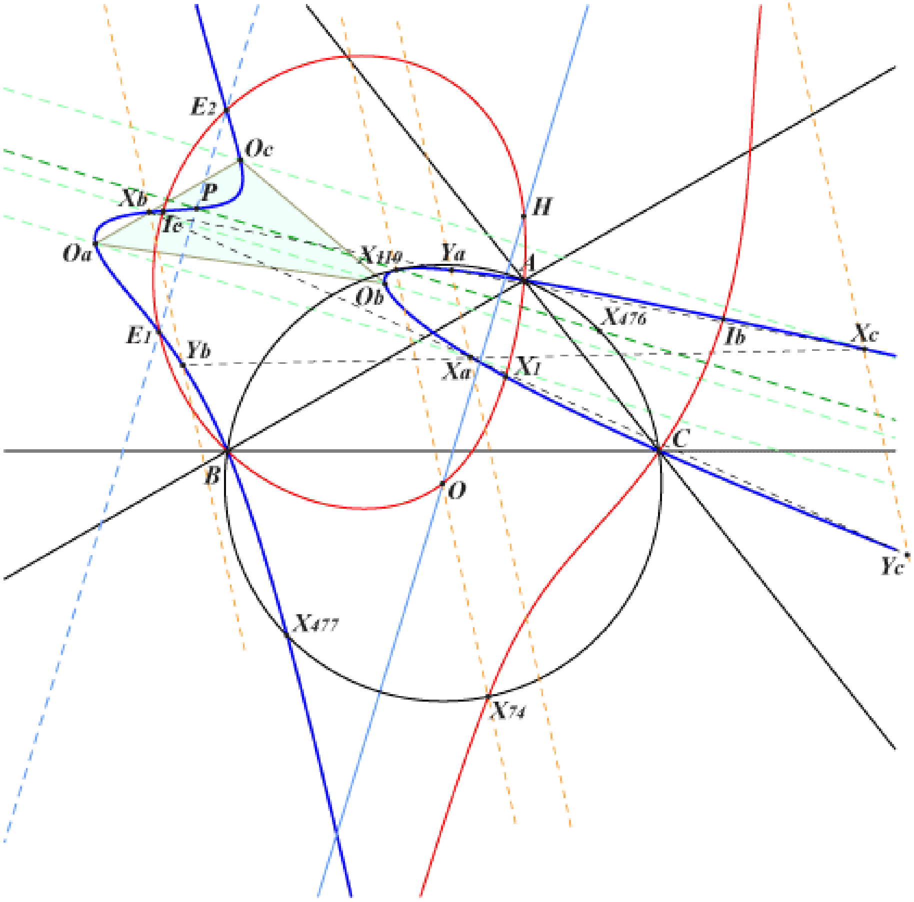
The third points Xa, Xb, Xc on the sidelines of OaObOc are the traces of X(523).

The third points Ya, Yb, Yc on the sidelines of XaXbXc are the traces of the isogonal conjugate X(5663) of X(477) i.e. the lines XaYa, XbYb, XcYc are parallel to the line X(3)X(74)X(110).

The perspectors of any two triangles ABC, IaIbIc, OaObOc, XaXbXc, YaYbYc and the cevian triangle of P are centers on the cubic.

The last two common points with the Neuberg cubic are E1, E2 on the parallel at P to the Euler line.

This cubic is also a pK with respect to OaObOc with pivot X(523), isopivot the infinite point X(5663) of the line X(3)X(74)X(110). Hence the tangents at Oa, Ob, Oc are parallel to this latter line.



Polar conics and Poloconic of the line at infinity in the Neuberg cubic

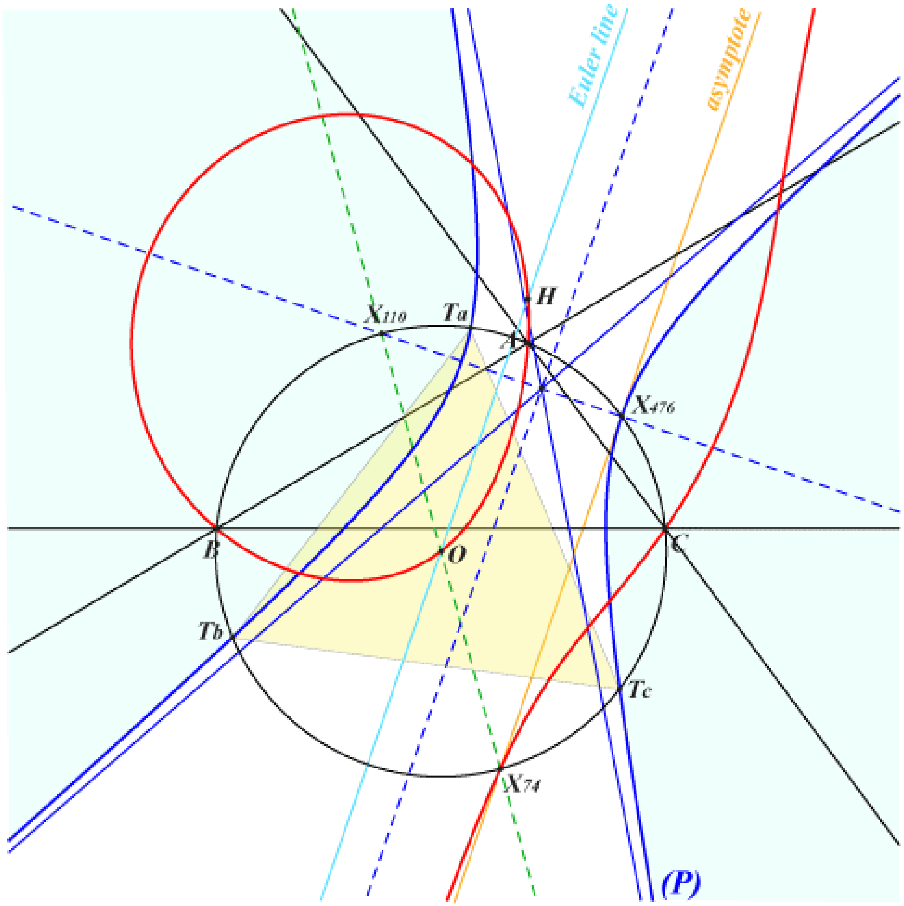
Let M be a point and C(M) its polar conic.

C(M) is

- a circle when M = X(110), the singular focus,
- a rectangular hyperbola when M lies on the Euler line, the orthic line of the cubic,
- a parabola when M lies on (P), the poloconic of the line at infinity,
- an ellipse when M lies inside the (light blue) region that contains X(110),
- a hyperbola when M lies outside this same region.

(P) is a very remarkable hyperbola passing through X(476) and the vertices of the circumtangential triangle. It has two asymptotes making an angle of 60° thus its eccentricity is 2.

X(110) is one of its foci and the related directrix is the Euler line.



The tangent at X(476) is the real asymptote of the Neuberg cubic.

Euler lines and Brocard axes of triangles PBC, PCA, PAB and the Neuberg cubic

For any point P on the Neuberg cubic, recall that the Euler lines (Ea), (Eb), (Ec) of triangles PBC, PCA, PAB concur at M on the Euler line (E) of ABC.

Conversely, for a given point M on (E), we seek points on the Neuberg cubic having this property.

There are two such points P, Q on K001 which are the common points of the rectangular circum-hyperbola (H) through M and the line (L) passing through X(399) and the reflection of O in M. Thus, the Euler lines of the six triangles PBC, PCA, PAB, QBC, QCA, QAB concur at M on (E).

It follows that K001 can be seen as the locus of the intersections of (H) and (L) when M traverses (E).

If X(399) is replaced by X(265), we obtain the cubic K638 whose isogonal transform is K639. Each of these two cubics have nine identified common points with the Neuberg cubic.

Similarly, for a given point M on the Brocard axis, there are three points P1, P2, P3 such that the Brocard lines of the nine analogous triangles concur at M.

More details here.