FiniteElements3DPoisson_CUBE

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1 Simulation of the 3D Poisson equation on the cube

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2.1 1 - Finite element method for 3D Poisson equation

2.1.1 1.1 - The Poisson problem on a cube

We consider the following Poisson problem with Dirichlet boundary conditions

$$\left\{ \begin{array}{rcl} -\Delta u = f, & \text{in} & \Omega \\ u = 0, & \text{on} & \partial\Omega, \end{array} \right.$$

where $\Omega = [0,1] \times [0,1] \times [0,1]$ is the unit cube in $\mathbb{R}^3$, the right hand side $f \in \mathbf{L}^2(\Omega)$ and the unknown u is to be sought in $\mathbf{H}_0^1(\Omega)$.

With the following choice for f :

$$f = 3\pi^3 \sin(\pi x)\sin(\pi y)\sin(\pi z),$$

he unique solution of the problem is

$$u(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z).$$

2.1.2 1.2 - Existence and uniqueness of the solution

Weak formulation Using the **Green-Ostrograski** formula, the above problem is transformed into the following **weak formulation** by:

Find $u \in \mathbf{H}_0^1$ such that:

$$\forall v \in \mathbf{H}_0^1, \quad \int_\Omega \vec{\nabla} u.\vec{\nabla} v dx = \int_\Omega f v dx. \qquad (1) \qquad (1)$$

Existence of the weak solution The bilinear form $a(u,v) = \int_\Omega \vec{\nabla} u.\vec{\nabla} v dx$, $a(.,.)$ is continuous and coercive thanks to **Poincaré inequality**. The linear form $b(v) = \int_\Omega f v dx$, $b(.)$ is continuous. Thanks to these properties, the **Lax-Milgram theorem** ensures the existence and uniqueness of the **weak solution** of the variational formulation (1).

Regularity of the solution The weak solution $u \in \mathbf{H}_0^1$ of the variational problem (1) actually belongs to $\mathbf{H}_0^2$ that is it is twice weakly differentiable with $-\Delta u = f$ in the weak sense.

More generally, for $m \geq 0$, if $\Omega \in (\mathbb{R}^N)$ is an open bounded set with piecewise $\mathcal{C}^{m+2}$ boundary and $f \in \mathbf{H}^m(\Omega)$. Then the unique solution $u_f \in \mathbf{H}_0^1(\Omega)$ of the weak formulation (1) belongs to $\mathbf{H}^{m+2}(\Omega)$. Furthermore, u_f depends continuously of $f \in \mathbf{H}^m(\Omega)$:

$$\exists C > 0, \ \ \forall f \in \mathbf{L}^2, \quad ||u_f||_{\mathbf{H}_0^{m+2}(\Omega)} \leq C||f||_{\mathbf{H}^m(\Omega)}.$$

2.1.3 1.3 - The P1 finite element for Poisson problem

The P_1 finite element is the set $(\mathbf{K}, \mathbf{\Sigma}, \mathbf{P})$ in which

- $\mathbf{K}$ is a geometric element of $\mathbb{R}^N$.
- $\mathbf{\Sigma}$ is a finite set of linear forms: $\tau_i : \mathbf{P} \rightarrow \mathbb{R}$.
- $\mathbf{P}$ is a finite dimensional set of polynomials of degree one.

Let $\mathbf{V}_h \subset \mathbf{H}_0^1(\Omega)$, u_h the projection of the exact solution u on $\mathbf{V}_h$ and v_h the projection of the test function v on $\mathbf{V}_h$. Then the above variational formulation becomes

$$\text{find } \ u_h \in V_h \ \text{ such that } \ a(u_h, v_h) = b(v_h), \quad \forall v_h \in V_h, \tag{2}$$

where $a_h(u_h, v_h)$ is a symmetric positive-definite bilinear form.

Taking V_h generated by the nodal basis functions $(\phi_j)_{1 \leq j \leq N_h}$, we have $u_h = \sum_{j=1}^{N_h} u_j \phi_j$, and the above problem becomes

$$\text{find } \ u_h \in \mathbb{R}^{N_h} \ \text{ such that } \ a(\sum_{j=1}^{N_h} u_j \phi_j, \phi_i) = b(\phi_i), \quad \forall 1 \leq i \leq N_h.$$

Which can be written in the form of a linear system

$$\mathcal{A}_h U_h = b_h, \tag{3}$$

with matrix $(\mathcal{A}_h)_{ij} = (a_{ij})$, right hand side $(b_h)_i = (b_j)$, and unknown $U_h = (u_1, ..., u_{N_h})$.

$$a_{ij} = a(\phi_i, \phi_j) = \int_\Omega \vec{\nabla} \phi_i . \vec{\nabla} \phi_j dx. \tag{4}$$

$$b_j = b(\phi_i) = \int_\Omega f \phi_i dx. \tag{5}$$

$\mathcal{A}_h$ is a symmetric, positive definite M-matrix, hence the existence of a discrete solution.

The domain Ω is decomposed into thetrahedral cells $\mathcal{T}_k$. * $|\mathcal{T}_k|$ is the measure of the cell $\mathcal{T}_k$.

- $\phi_j(\vec{x}_{s_i^k}) = \delta_{ij}$ is the test function on the cell $\mathcal{T}_k$ at the node $\vec{x}_{s_i^k}$, where s_1^k, s_2^k, s_3^k and s_4^k are the vertices of the cell $\mathcal{T}_k$.

  $\mathcal{A}_h$ is an $n \times n$ symetric positive definite sparse matrix with entries given by:

  $$a_{ij} = \sum_k \int_{\mathcal{T}_k} \vec{\nabla} \phi_i . \vec{\nabla} \phi_j = \sum_k a_i^k,$$