

# TransportEquation1D\_RegularGrid

December 30, 2018

## 1 Finite volume approximation of the transport equation on 1D grids

### 1.1 The transport equation with periodic boundary conditions

We are interested in the finite volume approximation of the following partial differential equation

$$\partial_t u + c \partial_x u = 0,$$

on the 1D domain  $[0, 1]$  with periodic boundary condition at  $x = 0$  and  $x = 1$  and initial data

$$u_0(x) = \frac{1}{2}(1 + \sin(\pi(4x - 0.5)))1_{[0,0.5]} + 1_{[0.6,0.85]}.$$

The initial data consists in a smooth part ( $x \in [0, 0.5]$ ) and a stiff part ( $x \in [0.5, 1]$ ).

The exact solution is given by

$$u(x, t) = u_0(x - ct).$$

Since we used periodic boundary condition, the exact solution is periodic with period  $T = \frac{1}{c}$  and therefore

$$u(x, T) = u_0(x).$$

### 1.2 Finite volume approximations

In 1D finite volume approximations, the domain  $\Omega = [0, 1]$  is decomposed into  $N$  intervals  $C_i = [x_i, x_{i+1}]$ ,  $i = 1, \dots, N$ , and we seek the average values

$$u_i(t) = \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} u(x, t) dx$$

of the exact solution  $u(x, t)$  in each cell  $C_i$ .

Similarly we decompose the time domain  $\mathbb{R}_+$  into finite length intervals  $[t_n, t_{n+1}]$ . Denote  $\Delta t_n = t_{n+1} - t_n$  the time step and  $\Delta x_i = x_{i+1} - x_i$  the space step, the double integration

$$\begin{aligned} \frac{1}{\Delta x_i} \int_{x_i}^{x_{i+1}} \frac{1}{\Delta t_n} \int_{t_n}^{t_{n+1}} (\partial_t u + c \partial_x u) dt dx &= \frac{1}{\Delta x_i} \int_{x_i}^{x_{i+1}} \frac{1}{\Delta t_n} \int_{t_n}^{t_{n+1}} \partial_t u(x, t) dt dx + c \frac{1}{\Delta t_n} \int_{t_n}^{t_{n+1}} \frac{1}{\Delta x_i} \int_{x_i}^{x_{i+1}} \partial_x u(x, t) dx dt \\ &= \frac{1}{\Delta x_i} \int_{x_i}^{x_{i+1}} \frac{u(x, t_{n+1}) - u(x, t_n)}{\Delta t_n} dx + c \frac{1}{\Delta t_n} \int_{t_n}^{t_{n+1}} \frac{u(x_{i+1}, t) - u(x_i, t)}{\Delta x_i} dt \end{aligned}$$