FV5 scheme for Poisson equation

The Poisson problem on the square

We consider the following Poisson problem with Dirichlet boundary conditions
$$\left\{\begin{aligned} -\Delta u &= f \text{ on } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}\right.$$
on the square domain $\Omega = [0,1] \times [0,1]$ with
$$f = 2\pi^2 sin(\pi x) sin(\pi y).$$

The unique solution of the problem is
$$u = -sin(\pi x) sin(\pi y).$$

The Poisson equation is a particular case of the diffusion problem
$$-\nabla \cdot (K\vec{\nabla} u) = f$$
and the associated diffusion flux is
$$F(u) = K\nabla u.$$

The FV5 scheme for the Laplace equation

The domain Ω is decomposed into cells C_i.

$|C_i|$ is the measure of the cell C_i.

f_{ij} is the interface between two cells C_i and C_j.

s_{ij} is the measure of the interface f_{ij}.

d_{ij} is the distance between the centers of mass of the two cells C_i and C_j.

The discrete Poisson problem is
$$-\frac{1}{|C_i|}\sum s_{ij} F_{ij} = f_i,$$
where u_i is the approximation of u in the cell C_i,

f_i is the approximation of f in the cell C_i,

F_{ij} is a numerical approximation of the outward normal diffusion flux from cell i to cell j.

In the case of the scheme FV5, we use the formula
$$F_{ij} = \frac{u_j - u_i}{d_{ij}}.$$

The script

```
#Discrétisation du second membre et extraction du nb max de voisins d'une cellu
le
#===============================================================================
==
my_RHSfield = cdmath.Field("RHS_field", cdmath.CELLS, my_mesh, 1)
maxNbNeighbours=0#This is to determine the number of non zero coefficients in t
he sparse finite element rigidity matrix

for i in range(nbCells):
    Ci = my_mesh.getCell(i)
    x = Ci.x()
    y = Ci.y()

    my_RHSfield[i]=2*pi*pi*sin(pi*x)*sin(pi*y)#mettre la fonction definie au se
cond membre de l edp
    # compute maximum number of neighbours
    maxNbNeighbours= max(1+Ci.getNumberOfFaces(),maxNbNeighbours)

# Construction de la matrice et du vecteur second membre du système linéaire
#==============================================================================
Rigidite=cdmath.SparseMatrixPetsc(nbCells,nbCells,maxNbNeighbours)# warning : t
hird argument is max number of non zero coefficients per line of the matrix
RHS=cdmath.Vector(nbCells)
#Parcours des cellules du domaine
for i in range(nbCells):
    RHS[i]=my_RHSfield[i] #la valeur moyenne du second membre f dans la cellule
i
    Ci=my_mesh.getCell(i)
    for j in range(Ci.getNumberOfFaces()):# parcours des faces voisinnes
        Fj=my_mesh.getFace(Ci.getFaceId(j))
        if not Fj.isBorder():
            k=Fj.getCellId(0)
            if k==i :
                k=Fj.getCellId(1)
            Ck=my_mesh.getCell(k)
            distance=Ci.getBarryCenter().distance(Ck.getBarryCenter())
            coeff=Fj.getMeasure()/Ci.getMeasure()/distance
            Rigidite.setValue(i,k,-coeff) # terme extradiagonal
        else:
            coeff=Fj.getMeasure()/Ci.getMeasure()/Ci.getBarryCenter().distance(
Fj.getBarryCenter())
        Rigidite.addValue(i,i,coeff) # terme diagonal


# Résolution du système linéaire
#=================================
LS=cdmath.LinearSolver(Rigidite,RHS,500,1.E-6,"GMRES","ILU")
SolSyst=LS.solve()

# Automatic postprocessing :  save 2D picture and plot diagonal data
#===========================
diag_data=VTK_routines.Extract_field_data_over_line_to_numpyArray(my_ResultFiel
d,[0,1,0],[1,0,0], resolution)
plt.legend()
plt.xlabel('Position on diagonal line')
plt.ylabel('Value on diagonal line')
if len(sys.argv) >1 :
```

Regular grid

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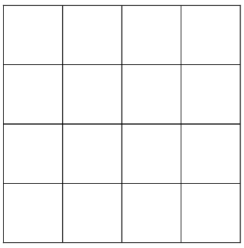
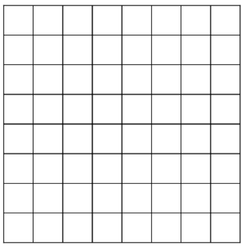
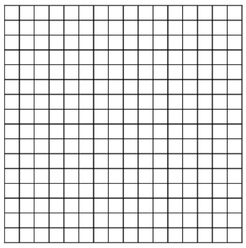
<tr><td></td><td></td><td></td></tr>
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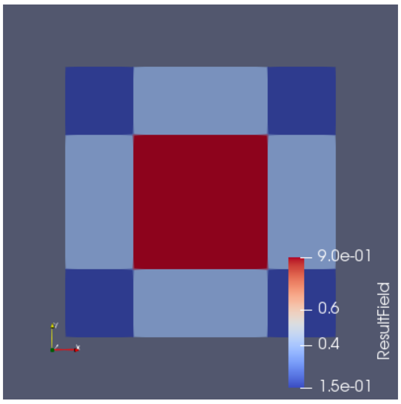
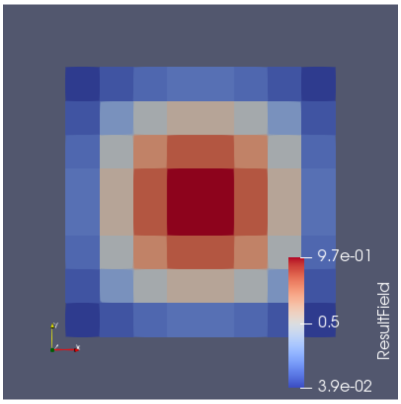
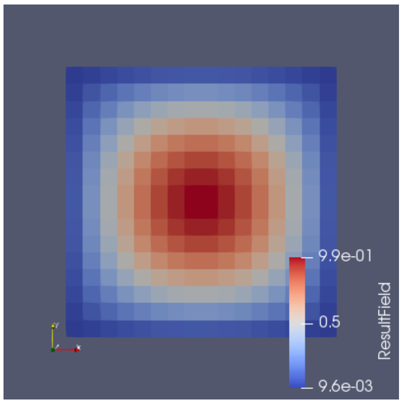
<tr><td>
</td><td>
</td><td>
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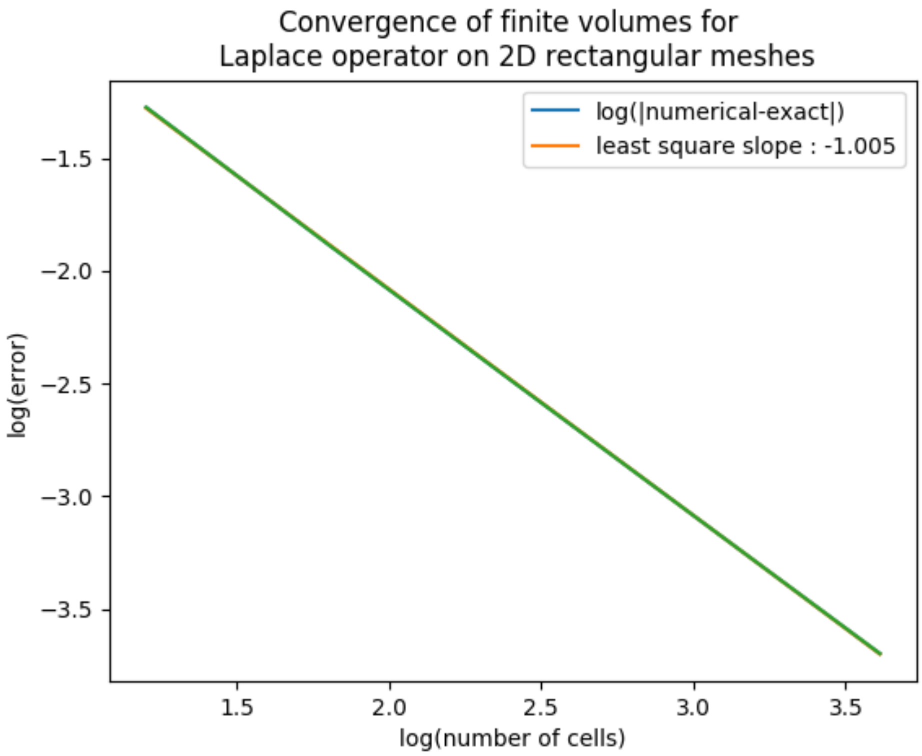

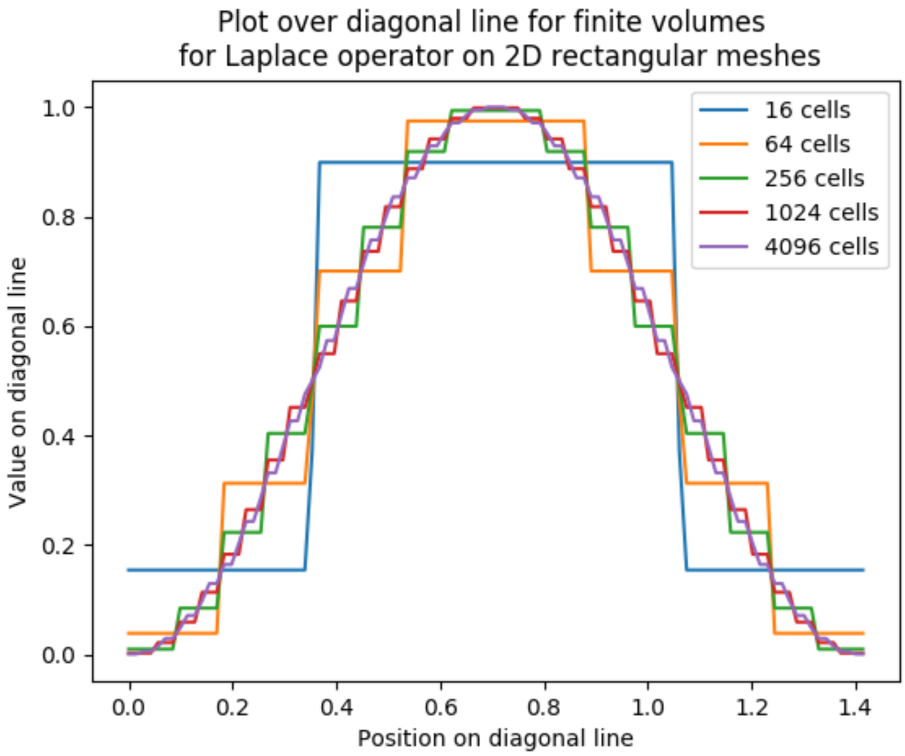

Deformed quadrangles

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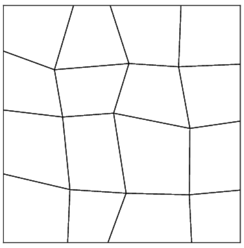	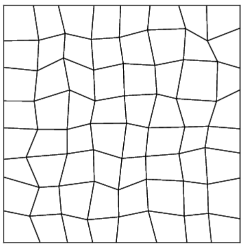	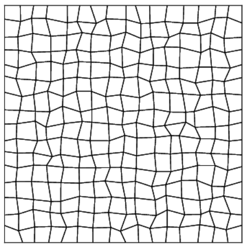

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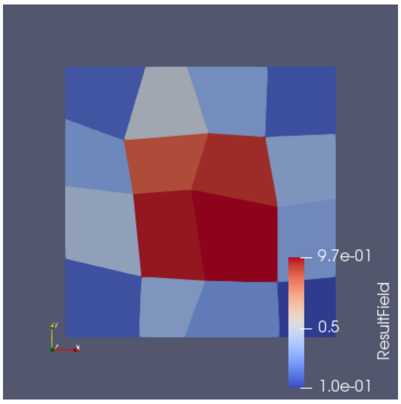	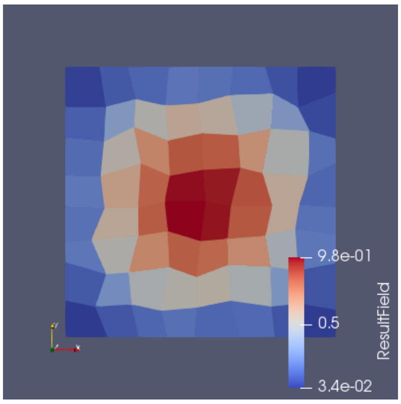	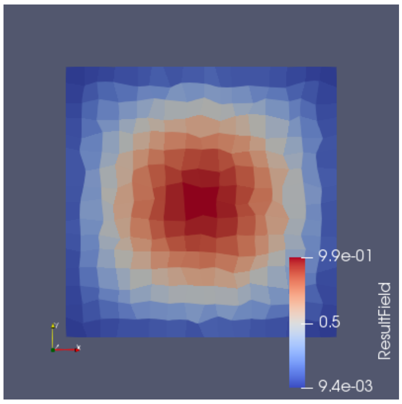

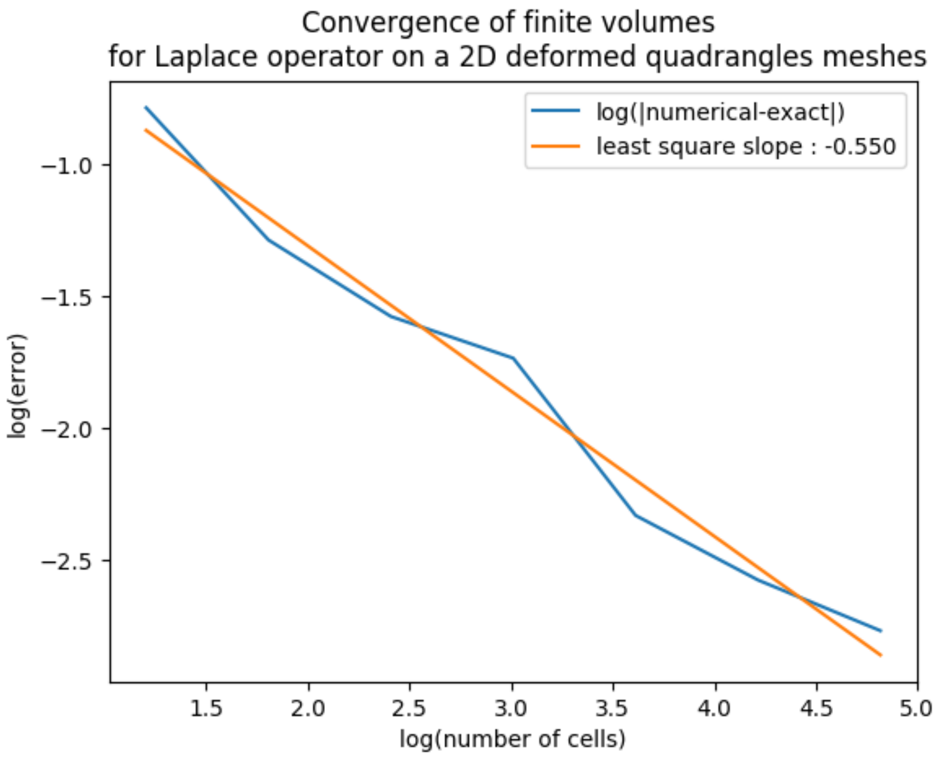

Triangular meshes

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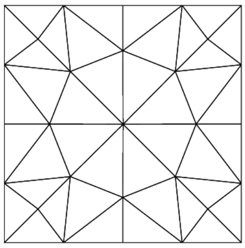
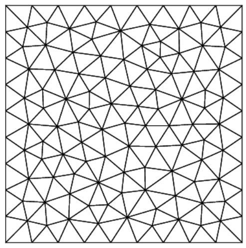
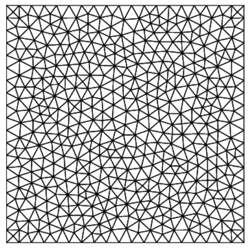
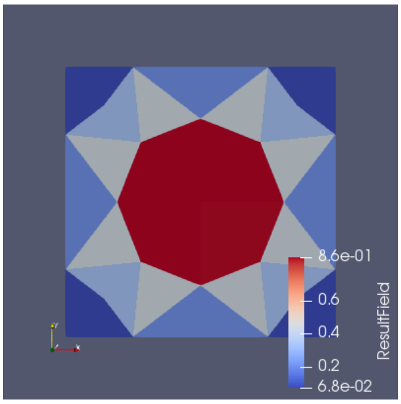
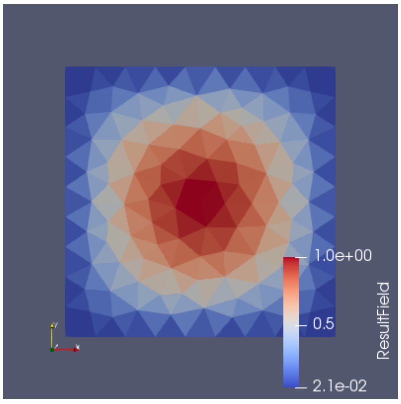
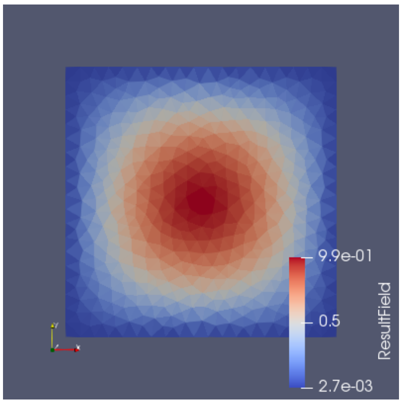
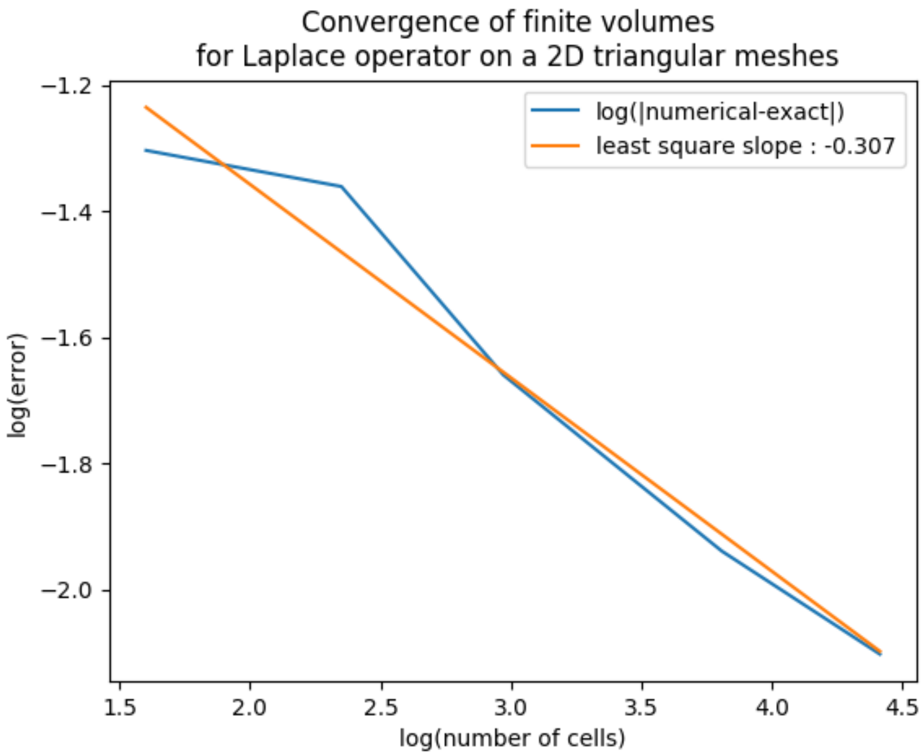
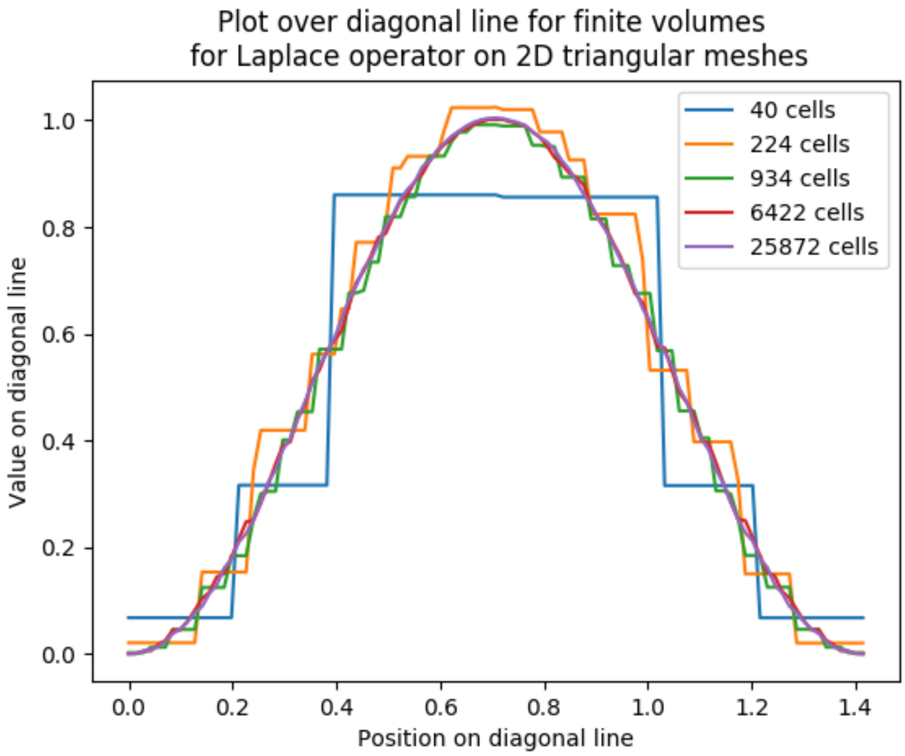
		

result 1	result 2	result 3
		





Hexagonal meshes

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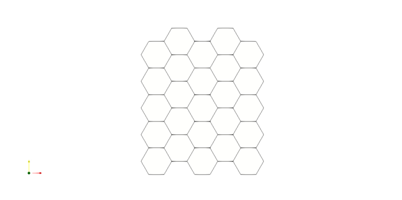	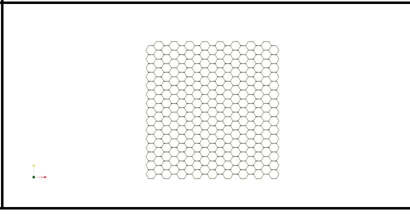	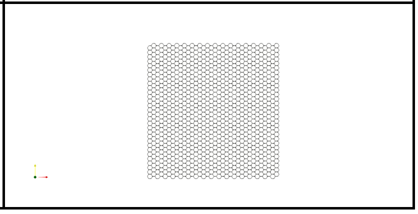

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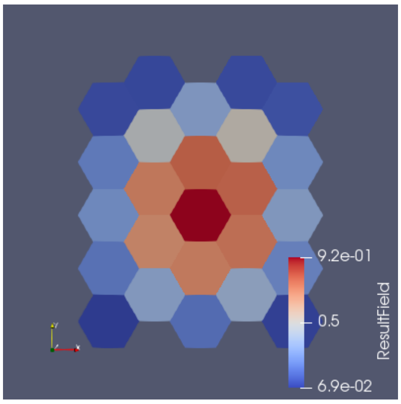	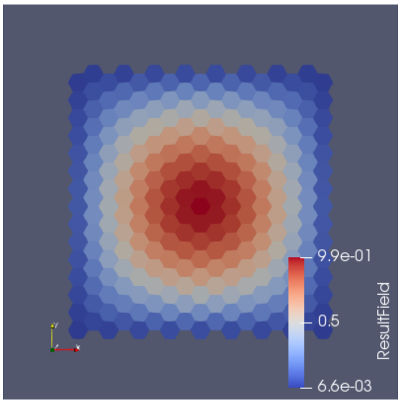	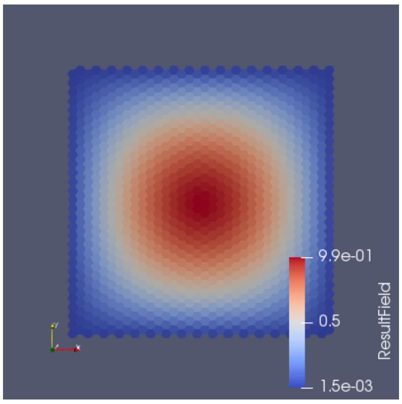

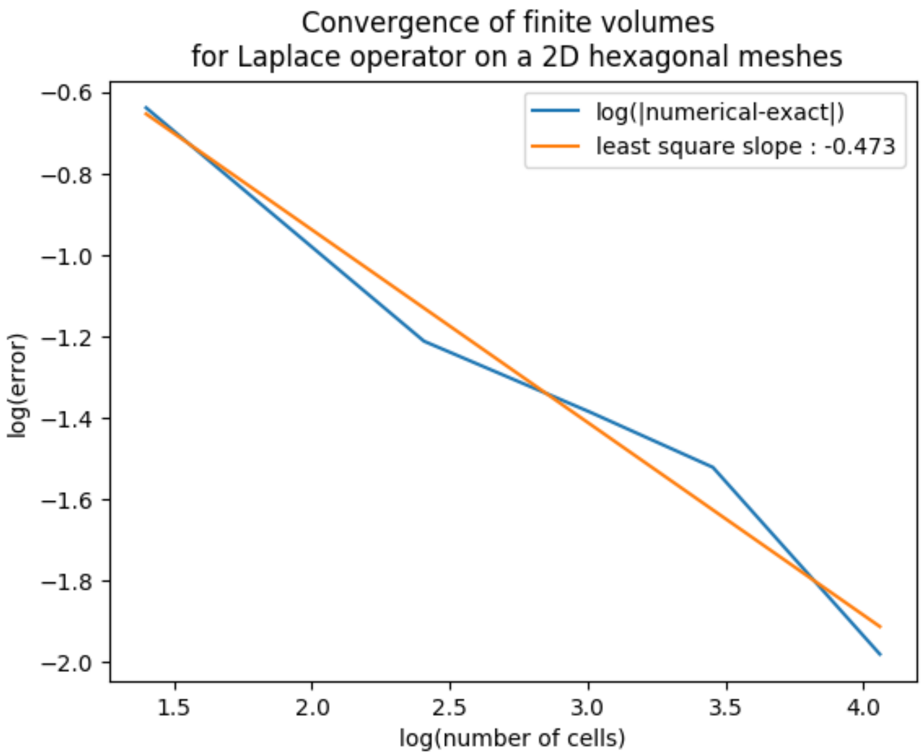

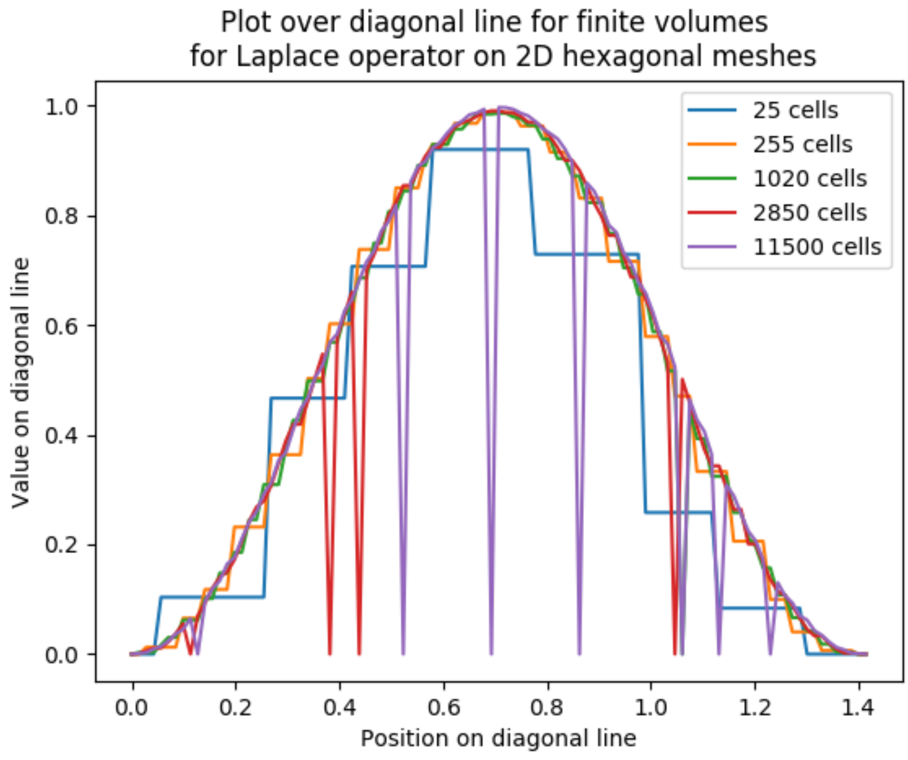

Brick wall meshes

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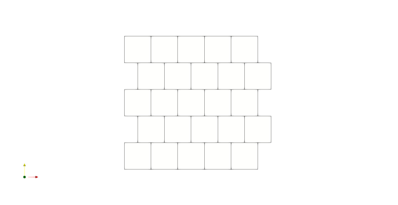	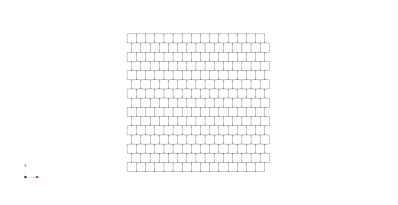	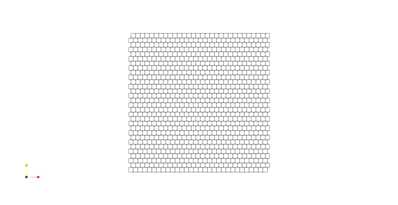

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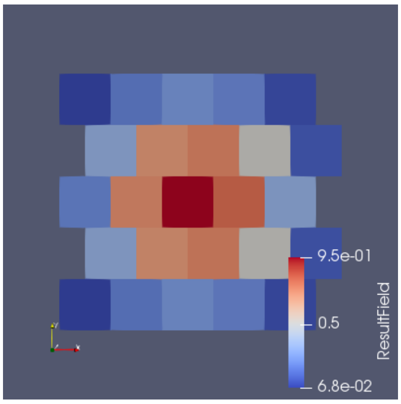	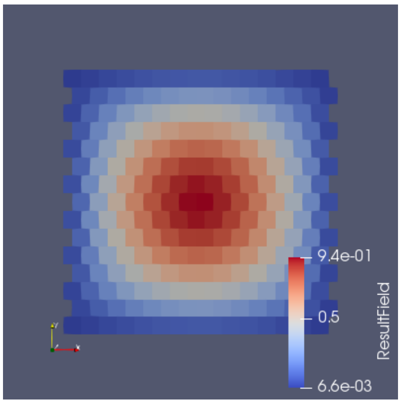	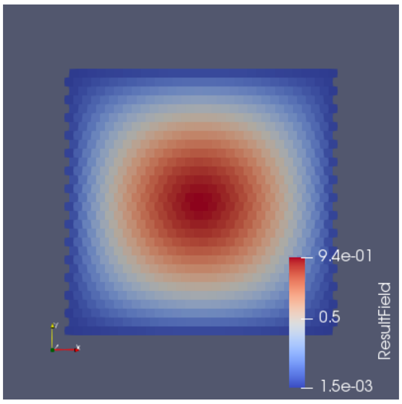

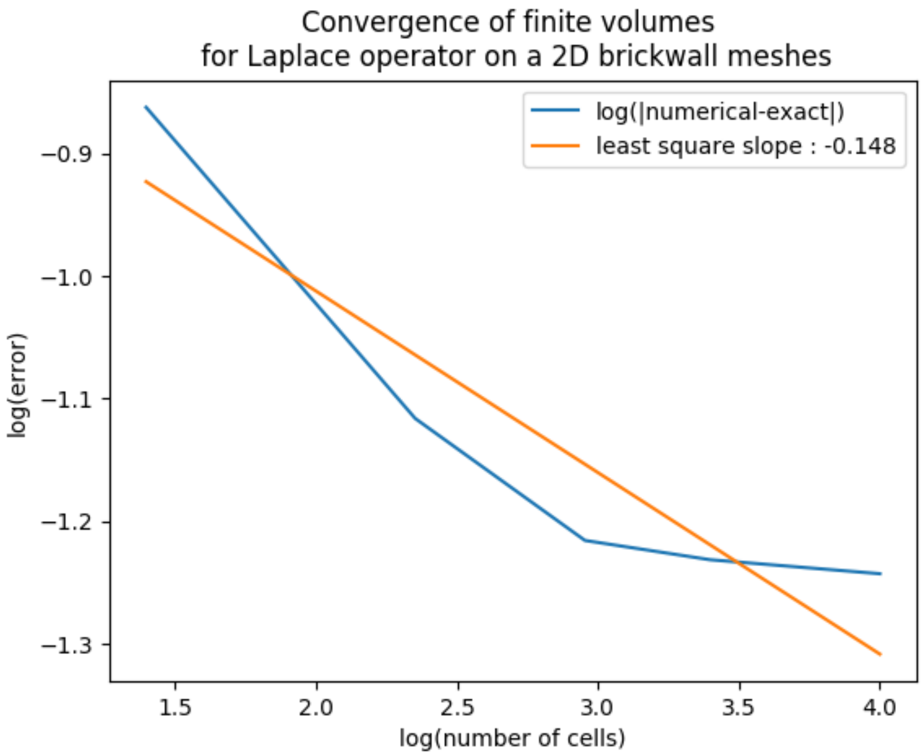

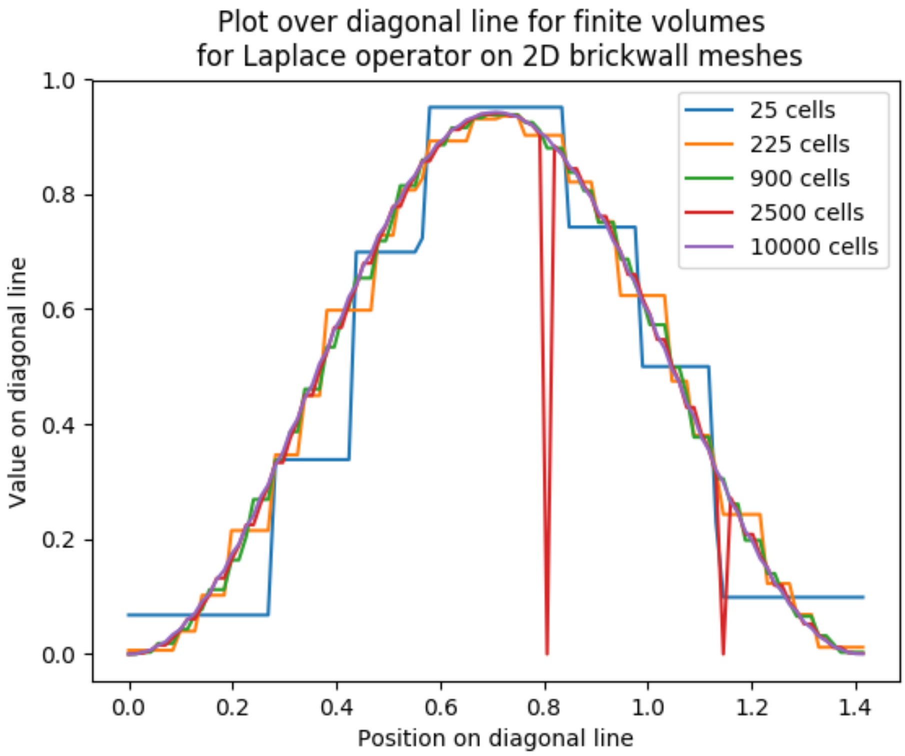

Locally refined meshes

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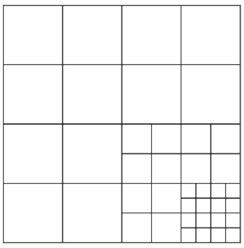	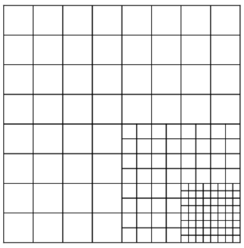	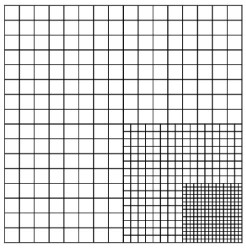

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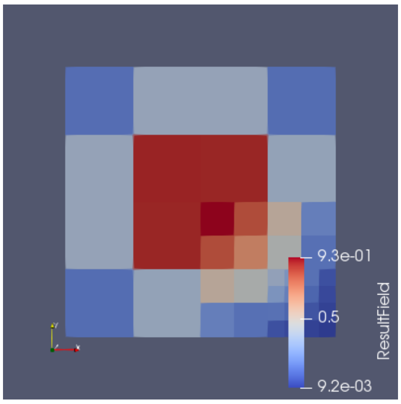	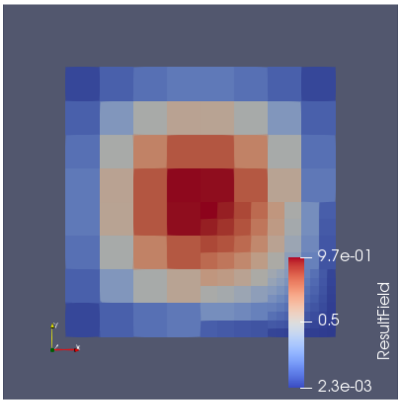	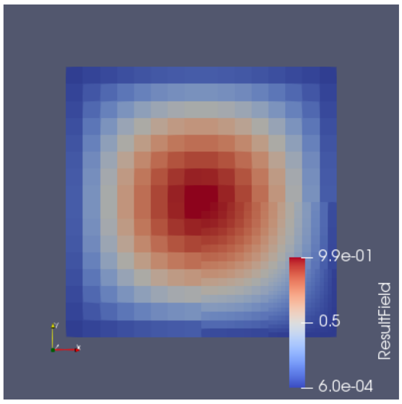

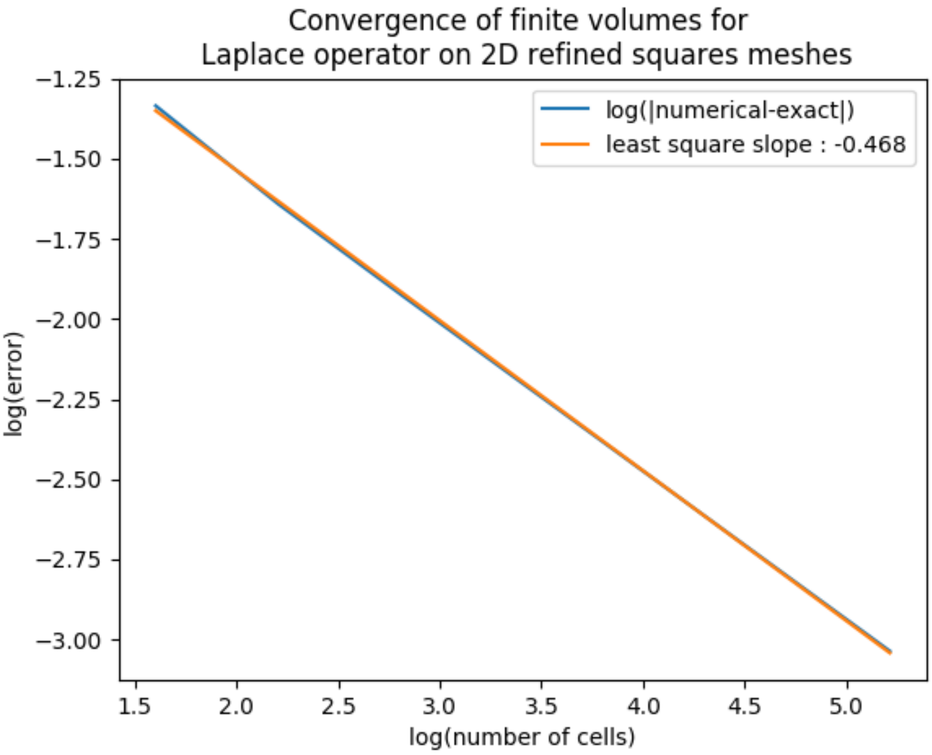

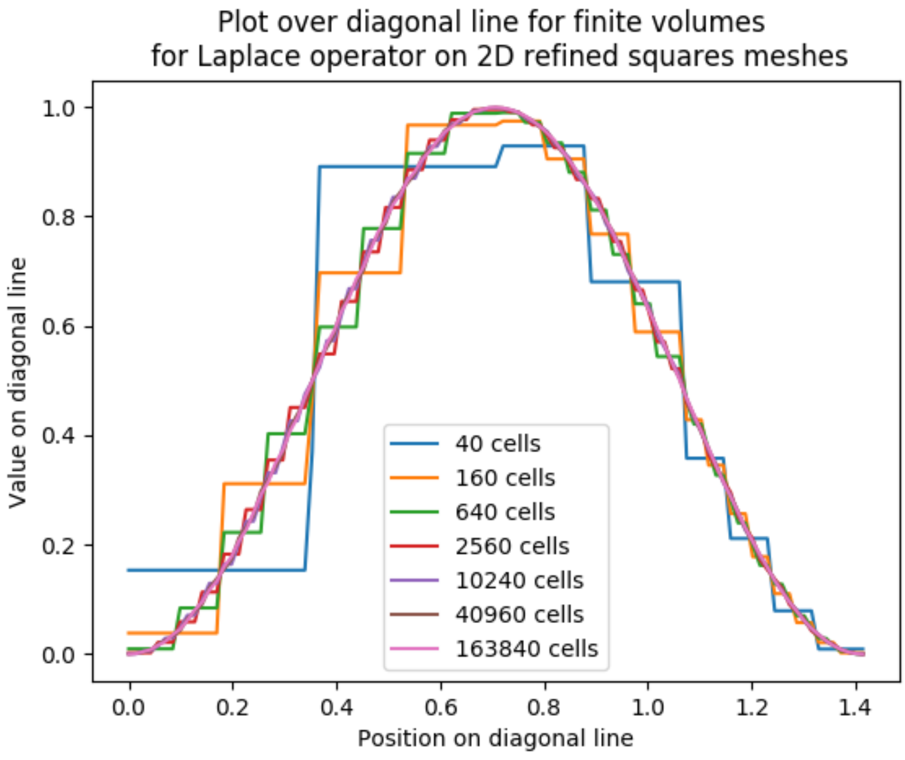

Checkerboard meshes

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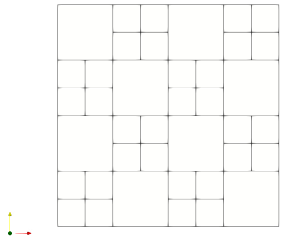	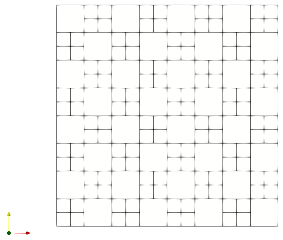	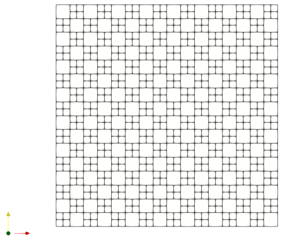

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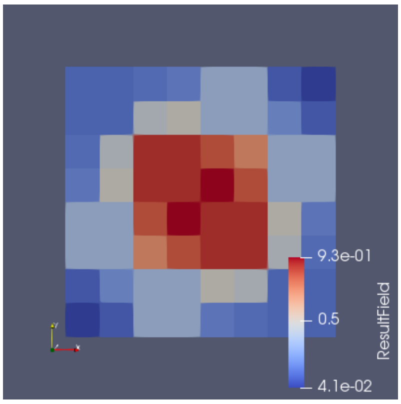
<td>
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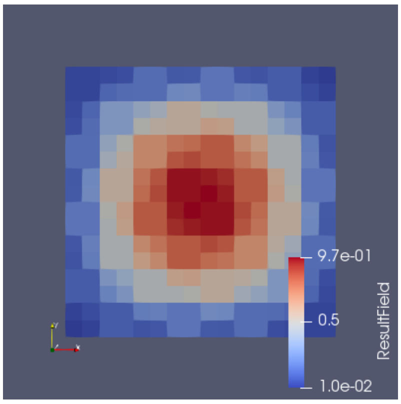
<td>
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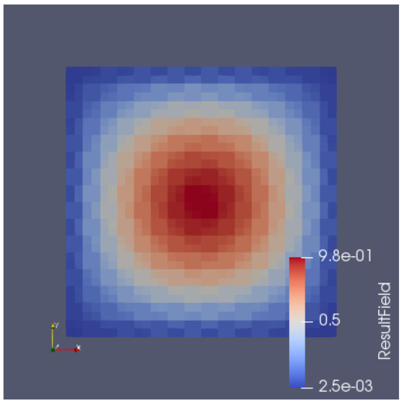
<td>
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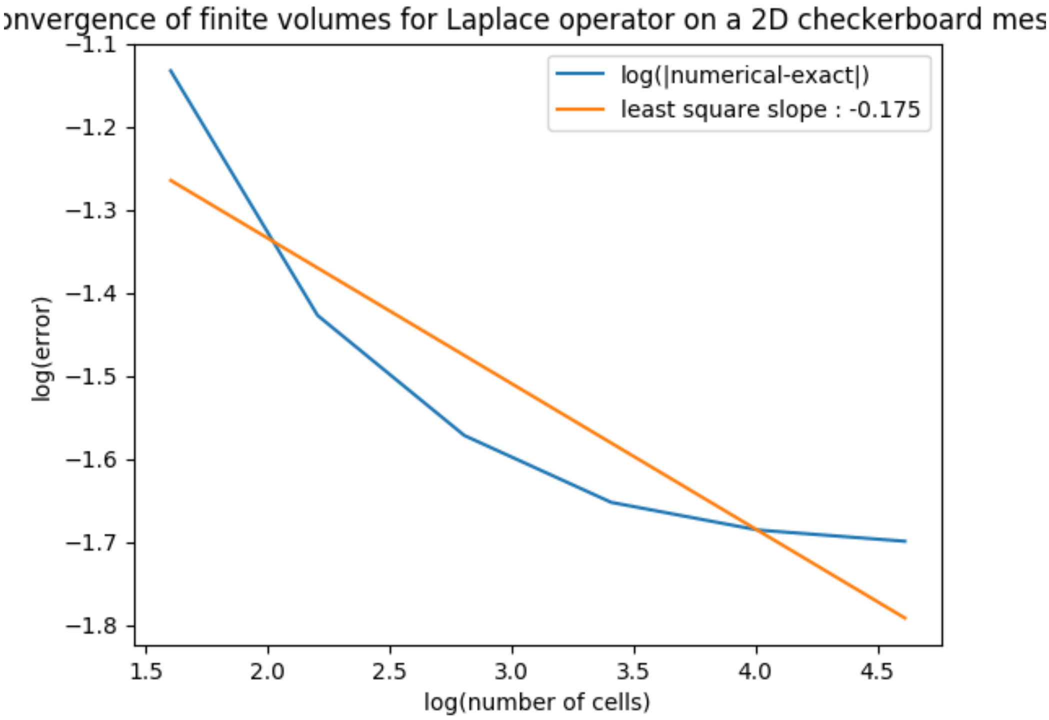

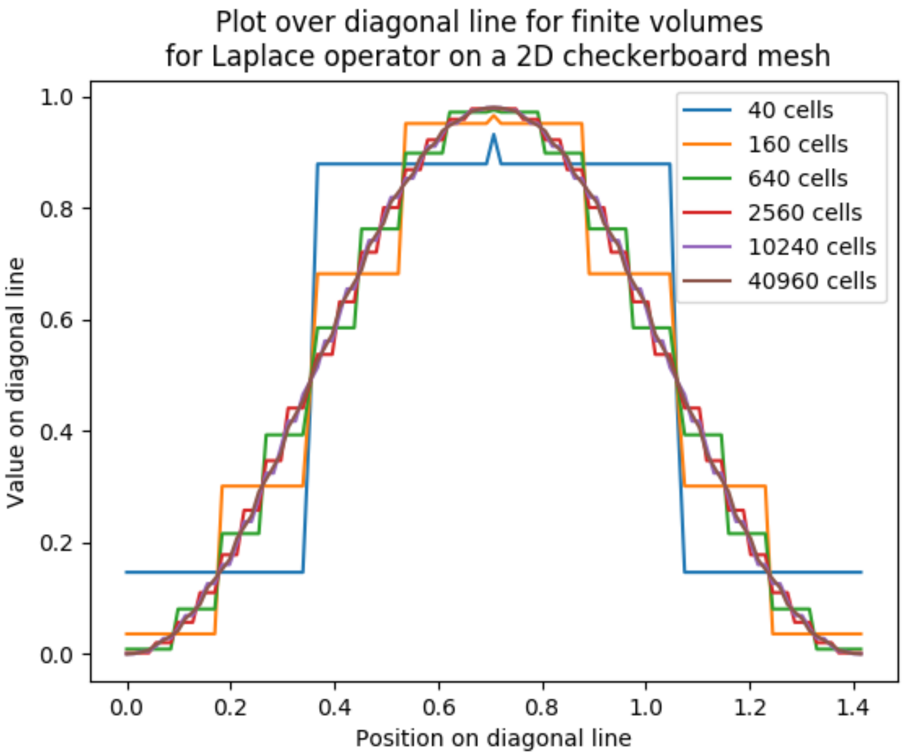
Plot over diagonal line for finite volumes
for Laplace operator on a 2D checkerboard mesh
40 cells
160 cells
640 cells
2560 cells
10240 cells
40960 cells
Value on diagonal line
Position on diagonal line
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4