New_Notebook1010

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1 P_1 finite element method for the Laplace equation with Dirichlet boundary conditions (by Sédrick Kameni Ngwamou, PhD student).

The goal of this work is to find the optimal position where we can install a radiator in order to optimize the temperature in a room.

1.1 1 - Variational formulation

Let $d \in \mathbb{N}^*$ and Ω a Lipschitz open subset of $\mathbb{R}^d$. Let $g \in H^{\frac{1}{2}}(\partial\Omega)$ a function definedon the boundary $\partial\Omega$. We are interested in the weak solutions of the following problem:

$$\begin{cases} -\triangle u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega, \end{cases} \qquad (1) \tag{1}$$

which means we are seeking for $u \in H^1_g(\Omega)$ such that

$$\forall v \in H^1_g(\Omega), \quad \int_\Omega \vec{\nabla} u \cdot \vec{\nabla} v - \int_{\partial\Omega} \vec{\nabla} u \cdot \vec{n}_x v \, ds_x = 0, \qquad (2) \tag{2}$$

where $H^1_g = \tilde{g} + H^1_0$ is the affine space

$$H^1_g = \{u \in H^1(\Omega), u_{|\partial\Omega} = g\} \tag{3}$$

with $u_{|\partial\Omega}$ denoting the trace of u on $\partial\Omega$, and $\tilde{g} \in H^1(\Omega)$ such that g is the trace of $\tilde{g}$ on $\partial\Omega$.

1.2 2 - Existence of the solution

Here we follow the method proposed by the remark 5.2.10 of [1] page 116. Using a change of variables, the boundary condition is set to zeros. The problem comes down to solving the Poisson problem with a source term in $H^{-1}(\Omega)$.

1.2.1 2.1 - Nonhomogeneous problem

Given that Ω is Lipschitz, the trace operator is surjective from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$ (see [1] remark 4.3.17, (or [2] remark 7-i) chapter 9 page 315). Then there exists a function $\tilde{g} \in H^1(\Omega)$ such that $\tilde{g}_{|\partial\Omega} = g$.

We want to prove the existence of the weak solution $\tilde{u} = u - \tilde{g} \in H^1_0(\Omega)$ of the following problem :