TransportEquation1D_RegularGrid

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1 Finite volume approximation of the transport equation on 1D grids

1.1 The transport equation with periodic boundary conditions

We are interested in the finite volume approximation of the following partial differential equation

$$\partial_t u + c\partial_x u = 0,$$

on the 1D domain $[0,1]$ with periodic boundary condition at $x = 0$ and $x = 1$ and initial data

$$u_0(x) = \frac{1}{2}(1 + sin(\pi(4x - 0.5))1_{[0,0.5]} + 1_{[0.6,0.85]}.$$

The initial data consists in a smooth part ($x \in [0, 0.5]$) and a stiff part ($x \in [0.5, 1]$).

The exact solution is given by

$$u(x,t) = u_0(x - ct).$$

Since we sused periodic boundary condition, the exact solution is periodic with period $T = \frac{1}{c}$ and therefore

$$u(x, T) = u_0(x).$$

1.2 Finite volume approximations

In 1D finite volume approximations, the domain $\Omega = [0,1]$ is decomposed into N intervals $C_i = [x_i, x_{i+1}], i = 1, \ldots N$, and we seek the average values

$$u_i(t) = \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} u(x,t)dx$$

of the exact solution $u(x,t)$ in each cell C_i.
Similarly we decompose the time domain $\mathbb{R}_+$ into finite length intervals $[t_n, t_{n+1}]$. Denotig $\Delta t_n = t_{n+1} - t_n$ the time step and $\Delta x_i = x_{i+1} - x_i$ the space step, the double integration

$$\begin{aligned}
\frac{1}{\Delta x_i}\int_{x_i}^{x_{i+1}} \frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}} (\partial_t u + c\partial_x u)\, dt\, dx &= \frac{1}{\Delta x_i}\int_{x_i}^{x_{i+1}} \frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}} \partial_t u(x,t)\, dt\, dx + c\frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}} \frac{1}{\Delta x_i}\int_{x_i}^{x_{i+1}} \partial_x u(x,t)\, dx \\
&= \frac{1}{\Delta x_i}\int_{x_i}^{x_{i+1}} \frac{u(x,t_{n+1}) - u(x,t_n)}{\Delta t_n}\, dx + c\frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}} \frac{u(x_{i+1},t) - u(x_i,t)}{\Delta x_i}\, d
\end{aligned}$$