2DPoisson_StiffBC_DISK

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In [5]: from IPython.display import display, Markdown
        with open('2DPoisson_StiffBC_DISK.md', 'r') as file1:
            PoissonStiffBCDISK = file1.read()

        display(Markdown(PoissonStiffBCDISK))
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1 Poisson problem with discontinuous boundary conditions

Let $\Omega \subset \mathbb{R}^d$ an open bounded set with Lipschitz boundary $\partial\Omega$, $f \in L^2(\partial\Omega)$ and $g \in L^2(\partial\Omega)$. We consider the following Dirichlet boundary value problem

$$\left\{ \begin{array}{c} -\Delta u = f \text{ on } \Omega \\ u = 0 \text{ on } \partial\Omega \end{array} \right.$$

The classical H^1-existence theory for elliptic equations using variational methods (section on the variational approach) requires that the boundary value g be in $H^{\frac{1}{2}}(\partial\Omega)$ which excludes for instance discontinuous boundary values in 2D. In physical and engineering sciences, boundary values are often discontinuous with $g \notin H^{\frac{1}{2}}(\partial\Omega)$.
We give the example of a function that is harmonic (hence smooth) on the open disk $D(0,1)$, bounded on the closed disk $\bar{D}(0,1)$ and admits a continuous limit on the unit circle $C(0,1)$ except at two points of discontinuity (see section on an Example of solution $u \notin H^1(\Omega)$ with stiff BC on the disk). Numerical results obtained with finite elements and finite volume method show convergence of both methods with order 1 for the L^2 norm (see section on Numerical results). This is to be compared with the order 2 convergence observed for both the finite elements and finite volume methods for solutions in $H^1(\Omega)$.

1.1 Classical variational approach

Provided $g \in H^{\frac{1}{2}}(\partial\Omega)$, we seek $u \in H^1_g(\Omega)$ such that

$$\forall v \in H^1_g(\Omega), \quad \int_\Omega \vec{\nabla} u \cdot \vec{\nabla} v - \int_{\partial\Omega} g \vec{\nabla} u \cdot \mathrm{d}\vec{s} = \int_\Omega v f,$$

where $H^1_g(\Omega)$ is the subspace of function having trace g on Ω :

$$H^1_g(\Omega) = \left\{ w \in H^1(\Omega), \quad w_{|\Omega} = g \right\}.$$

This functional setting is justified by the fact that the trace operator is continuous and bijective from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\Omega)$:

$$\begin{array}{ccc} H^1(\Omega) & \to & H^{\frac{1}{2}}(\Omega) \\ w & \to & w_{|\Omega} \end{array}.$$

Since the unknown u of the variational problem (**??**) is in $H^1(\Omega)$, its boundary value g is necessarily in $H^{\frac{1}{2}}(\Omega)$. Therefore we cannot consider a general function $g \in L^2(\Omega)$.\ The reason the variational approach cannot admit low regularity boundary functions $g \in L^2(\Omega)$ is that the notion of boundary value is quite strong in the Sobolev context. The trace operator on the boundary must be a continuous operator defined for function in $H^{\frac{1}{2}}$. It is not possible to build the trace operator for general functions on $L^2(\Omega)$ (see remark 4.3.13 and exercice 4.3.4 in Allaire Numerical Analysis book).

1.2 Example of solution $u \notin H^1(\Omega)$ with stiff BC on the disk

In this section we give an example of solution $u \notin H^1(\Omega)$ such that $u_{|\partial\Omega} = \lim_{x\to\partial\Omega} u \notin H^{\frac{1}{2}}(\Omega)$.\
We consider the particular case of the following function defined on $D(0,1)$:

$$u : \begin{array}{ccc} D(0,1) & \to & \mathbb{R} \\ (x,y) & \to & arctan\left(\frac{2x}{x^2+y^2-1}\right) = arctan\left(\frac{2r\cos(\theta)}{r^2-1}\right) \end{array}.$$

- u is bounded on the closed disk $\bar{D}(0,1)$
- $u \in C^\infty(D(0,1))$ and u is harmonic in $D(0,1)$ since:

$$\begin{array}{rcl} \partial_{xx}u + \partial_{yy}u & = & \partial_x\left(\frac{2(x^2+y^2-1)-4x^2}{(x^2+y^2-1)^2}\frac{1}{1+\left(\frac{2x}{x^2+y^2-1}\right)^2}\right) \\ & & +\partial_y\left(\frac{-4xy}{(x^2+y^2-1)^2}\frac{1}{1+\left(\frac{2x}{x^2+y^2-1}\right)^2}\right) \\ & = & \partial_x\left(\frac{2(x^2+y^2-1)-4x^2}{(x^2+y^2-1)^2+4x^2}\right) \\ & & +\partial_y\left(\frac{-4xy}{(x^2+y^2-1)^2+4x^2}\right) \\ & = & \frac{-4x((x^2+y^2-1)^2+4x^2)-(4x(x^2+y^2-1)+8x)(2(x^2+y^2-1)-4x^2)}{((x^2+y^2-1)^2+4x^2)^2} \\ & & +\frac{-4x((x^2+y^2-1)^2+4x^2)+4xy4y(x^2+y^2-1)}{((x^2+y^2-1)^2+4x^2)^2} \\ & = & \frac{-16x(x^2+y^2-1)^2-32x^3+32x^3+(-16x+16x^3+16xy^2)(x^2+y^2-1)}{((x^2+y^2-1)^2+4x^2)^2} \\ & = & \frac{16x(x^2+y^2-1)(-(x^2+y^2-1)-1+x^2+y^2)}{((x^2+y^2-1)^2+4x^2)^2} \\ & = & 0. \end{array}$$

- The limit values of u on the circle $C(0,1)$ exist except for $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$ and u admit two discontinuites on the circle :

$$u_{|\partial B(0,1)} : \begin{array}{ccc} C(0,1) & \to & \mathbb{R} \\ \theta & \to & \left\{\begin{array}{ll} \frac{\pi}{2} & \text{if } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ -\frac{\pi}{2} & \text{if } \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{array}\right. \end{array},$$

- $u \notin H^1(\Omega)$, $u_{|\partial\Omega} = \lim_{x\to\partial\Omega} u \notin H^{\frac{1}{2}}(\Omega)$