Convergence_WaveSystem_Staggered_SQUARE_squares

November 26, 2018

1 Staggered scheme for Wave System on square meshes

1.1 The Wave System on the square

We consider the following Wave system with periodic boundary conditions

$$\left\{\begin{array}{l}\partial_t p + c^2\nabla\cdot\vec{q} = 0\\ \partial_t\vec{q} + \quad \vec{\nabla}p = 0\end{array}\right. .$$

The wave system can be written in matrix form

$$\partial_t\left(\begin{array}{c}p\\ \vec{q}\end{array}\right) + \left(\begin{array}{cc}0 & c^2\nabla\cdot\\ \vec{\nabla} & 0\end{array}\right)\left(\begin{array}{c}p\\ \vec{q}\end{array}\right) = \left(\begin{array}{c}0\\ \vec{0}\end{array}\right)$$

In d space dimensions the wave system is an hyperbolic system of $d+1$ equations

$$\partial_t U + \sum_{i=1}^{d} A_i \partial_{x_i} U = 0, \quad U = {}^t(p,\vec{q})$$

where the jacobian matrix is

$$A(\vec{n}) = \sum_{i=1}^{d} n_i A_i = \left(\begin{array}{cc}0 & c^{2t}\vec{n}\\ \vec{n} & 0\end{array}\right), \quad \vec{n}\in\mathbb{R}^d.$$

has $d+1$ eigenvalues $-c, 0, \dots, 0, c$.

On the square domain $\Omega = [0,1]\times[0,1]$ we consider the initial data

$$\left\{\begin{array}{l}p_0(x,y) = constant\\ q_{x0}(x,y) = \sin(\pi x)\cos(\pi y)\\ q_{y0}(x,y) = -\sin(\pi y)\cos(\pi x)\end{array}\right. .$$

The initial data (p_0, q_x, q_y) is a stationary solution of the wave system.

1.2 A 2D Staggered scheme for the Wave System

In $2D$, the linear wave system can be written using the cartesian coordinate system $\vec{q} = (q_x, q_y)$ as

$$\left\{\begin{array}{llc}\partial_t p & + & c^2(\partial_x q_x + \partial_y q_y) = 0\\ \partial_t q_x & + & \partial_x p = 0\\ \partial_t q_y & + & \partial_y p = 0.\end{array}\right.$$

We consider a $2D$ rectangular grid made of $N = n_x \times n_y$ cells.
The cells are indexed by two integers $i = 1, \dots, n_x$ (x-direction), and $j = 1, \dots, n_y$ (y-direction).

The pressure p is discretised at the cell centers and is indexed with integer values $p_{i,j}, i = 1, \dots, n_x, j = 1, \dots, n_y$.

The horizontal component q_x of the momentum is discretised at the vertical cell interfaces and is indexed with a half-integer followed by an integer $q_{i-\frac{1}{2},j}, i = 1, \dots, n_x, j = 1, \dots, n_y$.

The vertical component q_y of the momentum is discretised at the horizontal cell interfaces and is indexed with an integer followed by a half-integer $q_{i,j-\frac{1}{2}}, i = 1, \dots, n_x, j = 1, \dots, n_y$.

The discrete equations read

$$\left\{\begin{array}{lll} \partial_t p_{i,j} & + & c^2\dfrac{q_{i+\frac{1}{2},j} - q_{i-\frac{1}{2},j}}{\triangle x} + c^2\dfrac{q_{i,j+\frac{1}{2}} - q_{i,j-\frac{1}{2}}}{\triangle y} = 0 \\ \partial_t q_{i-\frac{1}{2},j} & + & \dfrac{p_{i,j} - p_{i-1,j}}{\triangle x} = 0 \\ \partial_t q_{i,j-\frac{1}{2}} & + & \dfrac{p_{i,j} - p_{i,j-1}}{\triangle y} = 0, \end{array}\right.$$

for $i = 1, \dots, n_x$, $j = 1, \dots, n_y$
with the notations $p_0 = p_{n_x}$, $q_{n_x+\frac{1}{2},j} = q_{\frac{1}{2},j}$ and $q_{i,n_y+\frac{1}{2}} = q_{i,\frac{1}{2}}$ at the periodic boundaries.

We are therefore led to a linear system of $3N = 3n_x \times n_y$ ODEs to solve.

1.3 The 2D Staggered scheme in matrix form

Define the unknown vector of the semi-discrete system as

$$\mathcal{U} = \begin{pmatrix} \mathcal{P} \\ \mathcal{Q}_x \\ \mathcal{Q}_y \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix}, \quad \mathcal{Q}_x = \begin{pmatrix} q^x_1 \\ \vdots \\ q^x_N \end{pmatrix}, \quad \mathcal{Q}_y = \begin{pmatrix} q^y_1 \\ \vdots \\ q^y_N \end{pmatrix},$$

where the global index for the pressure unknown for any cell (i, j) is $p_k = p_{jn_x+i}$
the global index for the x-momentum unknown for any vertical cell interface $(i-\frac{1}{2}, j)$ is $q^x_k = q_{jn_x+i}$
and global index for the the y-momentum unknown for any horizontal cell interface $(i, j - \frac{1}{2})$ $q^y_k = q_{jn_x+i}$.

With these notations, the discrete equations read for $k = 0, \dots, N$

$$\left\{\begin{array}{lll} \partial_t p_{jn_x+i} & + & c^2\dfrac{q^x_{jn_x+(i+1)\%n_x} - q^x_{jn_x+i}}{\triangle x} + c^2\dfrac{q^y_{((j+1)\%n_y)n_x+i} - q^y_{jn_x+i}}{\triangle y} = 0 \\ \partial_t q^x_{jn_x+i} & + & \dfrac{p_{jn_x+i} - p_{jn_x+(i-1)\%n_x}}{\triangle x} = 0 \\ \partial_t q^y_{jn_x+i} & + & \dfrac{p_{jn_x+i} - p_{((j-1)\%n_y)n_x+i}}{\triangle y} = 0 \end{array}\right. .$$

The discrete staggered scheme takes the matrix form

$$\partial_t \mathcal{U} + \mathcal{M}\mathcal{U} = 0,$$

with

$$
\begin{aligned}
\mathcal{M} &= \begin{pmatrix} 0 & c^2\mathcal{C}_x^{2d} & c^2\mathcal{C}_y^{2d} \\ -{}^t\mathcal{C}_x^{2d} & 0 & 0 \\ -{}^t\mathcal{C}_y^{2d} & 0 & 0 \end{pmatrix} \in \mathcal{M}_{3n_xn_y}(\mathbb{R}), \qquad \mathcal{C}_x^{2d}, \mathcal{C}_y^{2d} \in \mathcal{M}_{n_xn_y}(\mathbb{R}) \\
\mathcal{C}_x^{2d} &= \frac{1}{\Delta x} \begin{pmatrix} \mathcal{C}^{1d} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathcal{C}^{1d} \end{pmatrix} \in \mathcal{M}_{n_xn_y}(\mathbb{R}), \qquad \mathcal{C}^{1d} \in \mathcal{M}_{n_x}(\mathbb{R}) \\
\mathcal{C}_y^{2d} &= \frac{1}{\Delta y} \begin{pmatrix} -\mathbb{I}_{n_x} & \mathbb{I}_{n_x} & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \mathbb{I}_{n_x} \\ \mathbb{I}_{n_x} & 0 & 0 & -\mathbb{I}_{n_x} \end{pmatrix} \in \mathcal{M}_{n_xn_y}(\mathbb{R}).
\end{aligned}
$$

1.4 Staggered scheme stability

With the new unknown variable

$$
\mathcal{V} = \begin{pmatrix} \frac{1}{c}\mathcal{P} \\ \mathcal{Q}_x \\ \mathcal{Q}_y \end{pmatrix},
$$

which yields the discrete system

$$
\partial_t \mathcal{V} + \mathcal{M}'\mathcal{V} = 0,
$$

with the antisymmetric matrix :

$$
\mathcal{M}' = \begin{pmatrix} 0 & c\mathcal{C}_x^{2d} & c\mathcal{C}_y^{2d} \\ -c{}^t\mathcal{C}_x^{2d} & 0 & 0 \\ -c{}^t\mathcal{C}_y^{2d} & 0 & 0 \end{pmatrix}.
$$

Hence the norm of $\mathcal{V}$ is constant :

$$
\frac{1}{2}\partial_t ||\mathcal{V}||^2 = {}^t\mathcal{V}\partial_t\mathcal{V} = -\frac{c}{\Delta x}{}^t\mathcal{V} \begin{pmatrix} 0 & c\mathcal{C}_x^{2d} & c\mathcal{C}_y^{2d} \\ -c{}^t\mathcal{C}_x^{2d} & 0 & 0 \\ -c{}^t\mathcal{C}_y^{2d} & 0 & 0 \end{pmatrix} \mathcal{V} = 0.
$$

Since

$$
||\mathcal{U}(t)|| \leq \max\left\{1, \frac{1}{c}\right\} ||\mathcal{V}(t)|| = \max\left\{1, \frac{1}{c}\right\} ||\mathcal{V}(0)||,
$$

we deduce that $||\mathcal{U}||$ is bounded and the scheme is therefore stable.

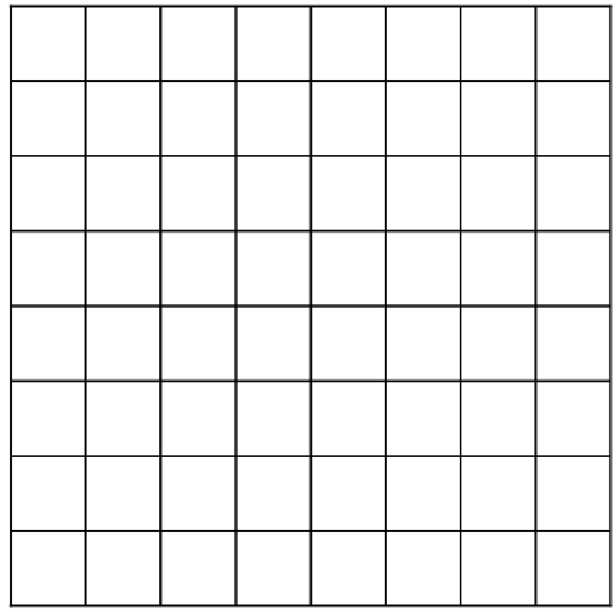
1.5 The script

```
#Condition initiale :
#Warning : the velocity is based on cells with the principle that the x component i
pressure_field, velocity_field = initial_conditions_wave_system(my_mesh)

#Pas de temps
dt = cfl * dx_min / c0

#Matrice des systèmes linéaires
```

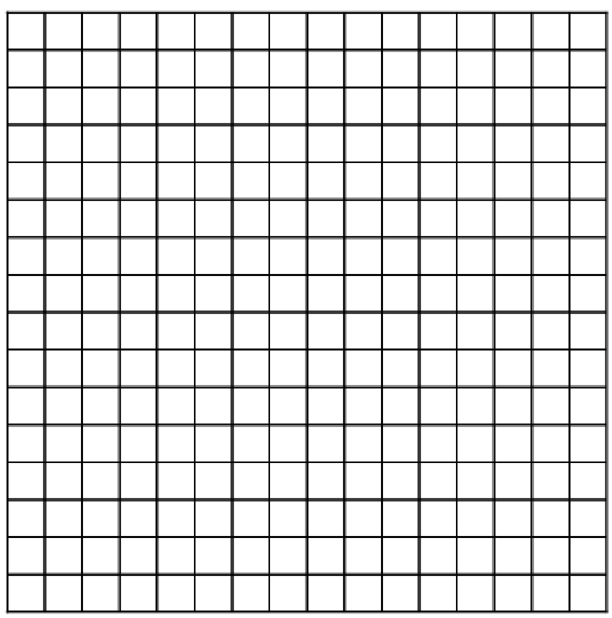
```
divMat=computeDivergenceMatrix(my_mesh,nbVoisinsMax,dt,test_bc)

# Construction du vecteur inconnu
Un=cdmath.Vector(nbCells*(dim+1))
for k in range(nbCells):
    Un[k*(dim+1)+0] =     pressure_field[k]
    Un[k*(dim+1)+1] =rho0*velocity_field[k,0]
    Un[k*(dim+1)+2] =rho0*velocity_field[k,1]


# Création du solveur linéaire
LS=cdmath.LinearSolver(divMat,Un,iterGMRESMax, precision, "GMRES","ILU")

# Time loop
while (it<ntmax and time <= tmax and not isStationary):
    LS.setSndMember(Un)
    Un=LS.solve();
    Un.writeVTK

# Automatic postprocessing :  save 2D picture and plot diagonal data
#===========================
diag_data=VTK_routines.Extract_field_data_over_line_to_numpyArray(my_ResultField,[(
plt.legend()
plt.xlabel('Position on diagonal line')
plt.ylabel('Value on diagonal line')
if len(sys.argv) >1 :
    plt.title('Plot over diagonal line for Staggered Finite Volumes \n for Wave sys
    plt.plot(curv_abs, diag_data, label= str(nbCells)+ ' cells mesh')
    plt.savefig("FiniteVolumes2D_square_ResultField_"+str(nbCells)+ '_cells'+"_Plot
```

1.6 Numerical results

1.6.1 Cartesian meshes

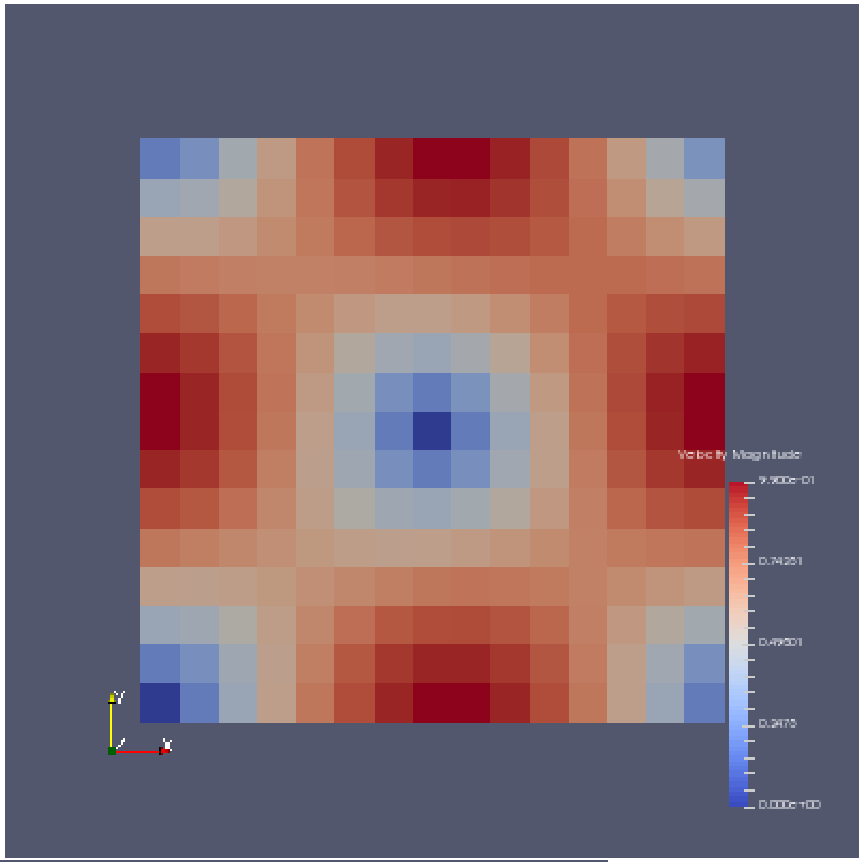


mesh 1 | mesh 2 | mesh 3 - | - - | -

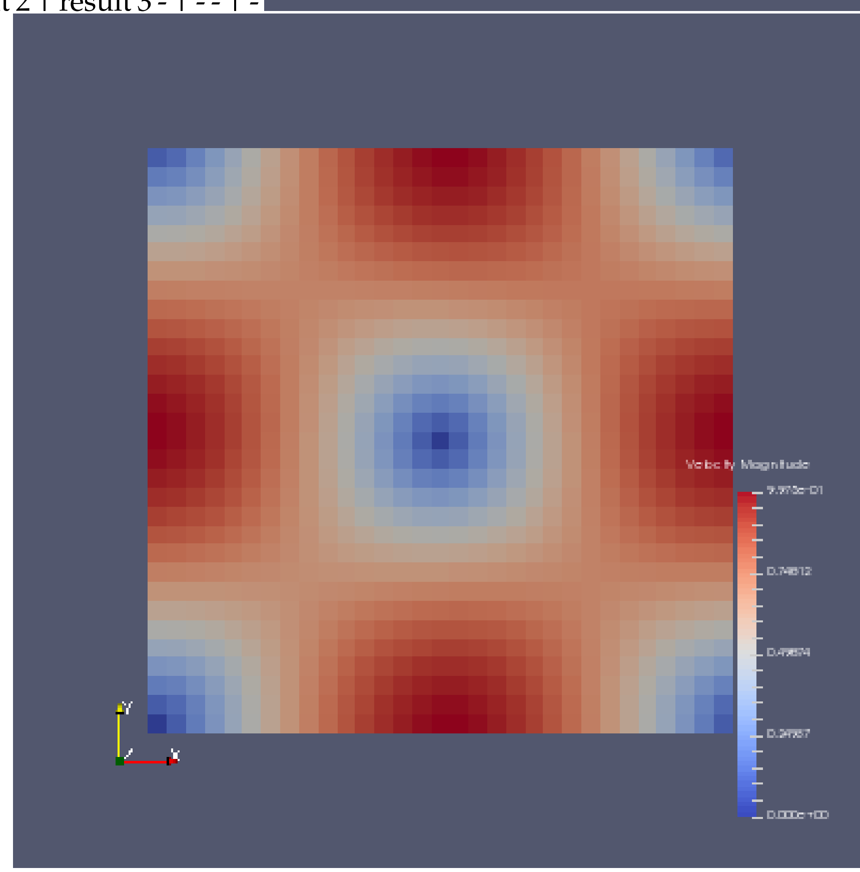


| |

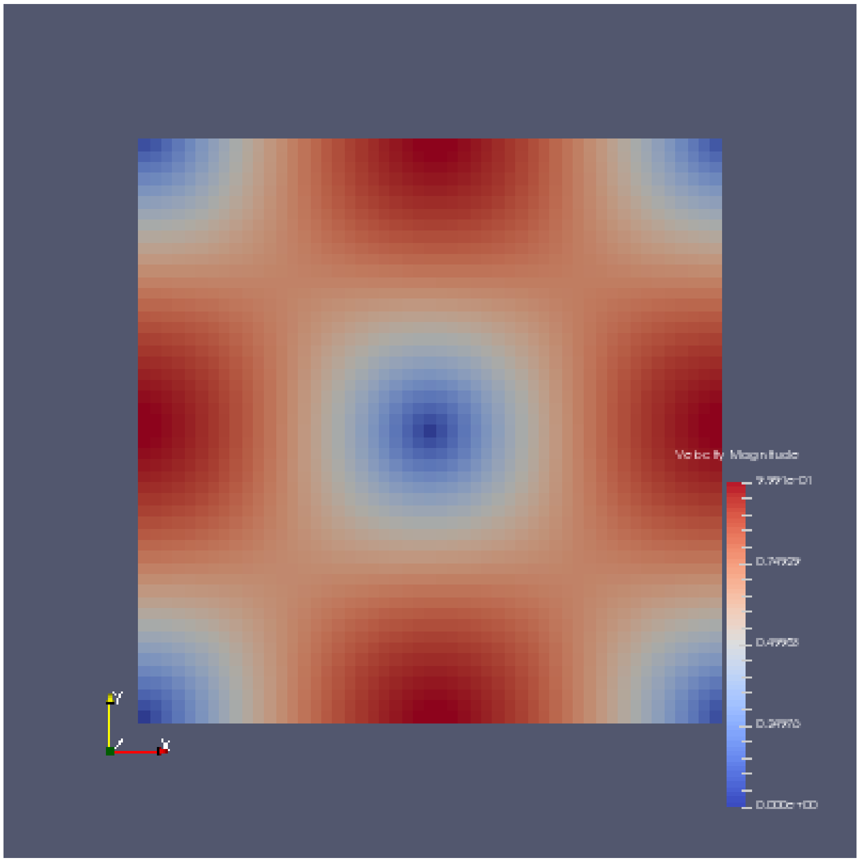
1.6.2 Velocity initial data (magnitude)



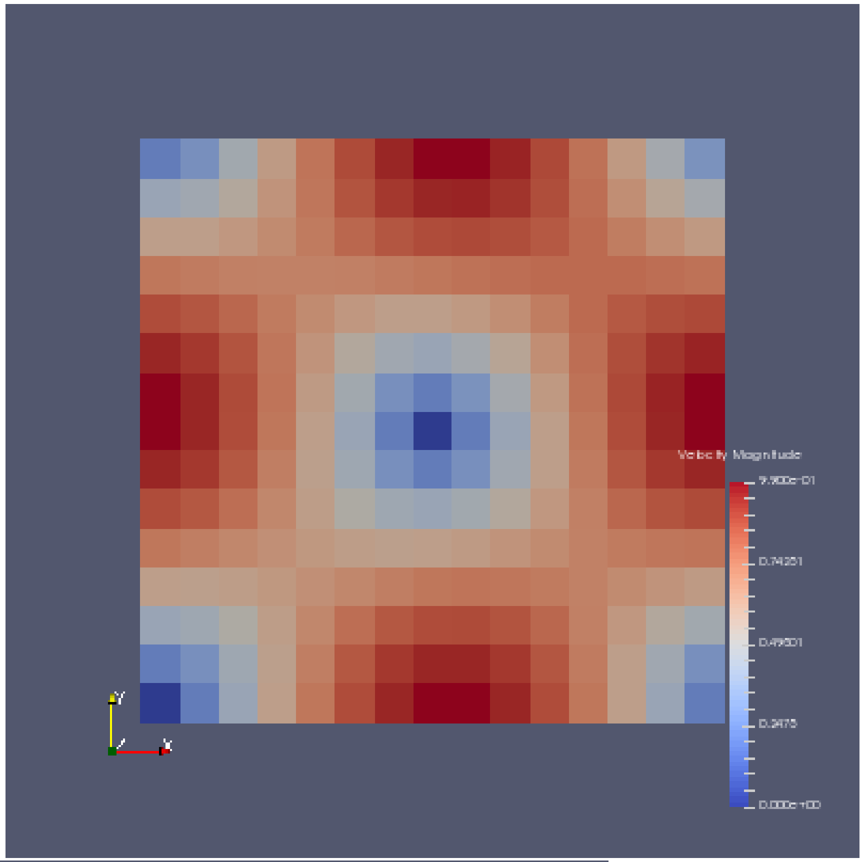
result 1 | result 2 | result 3 - | - - | -



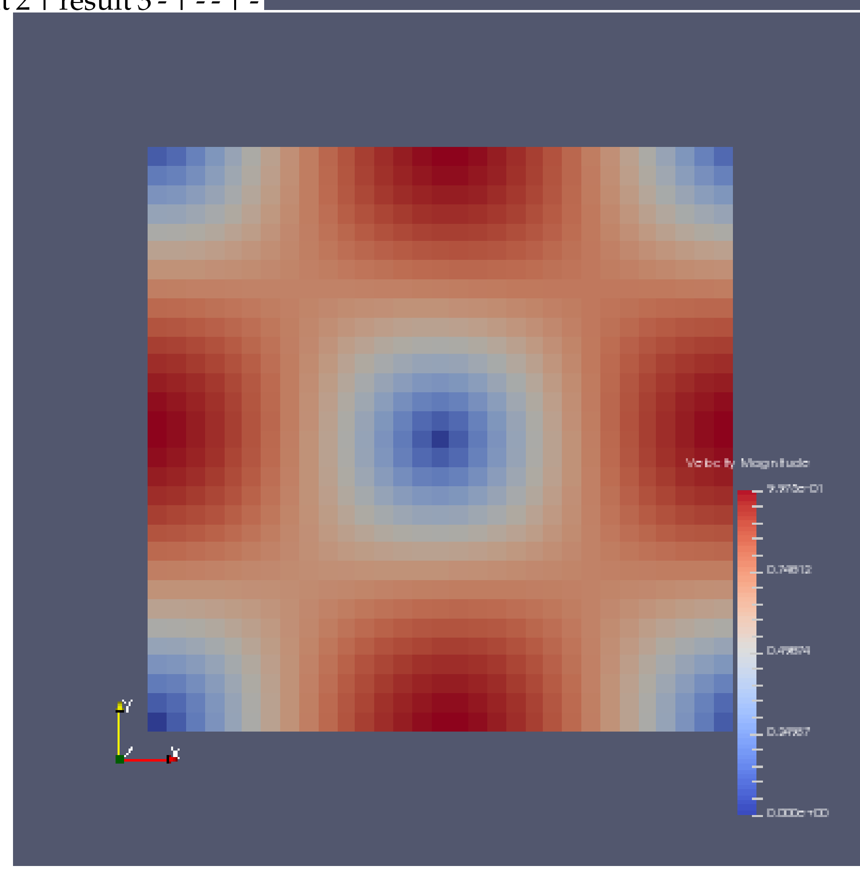
| |


Velocity Magnitude
0.000e+00

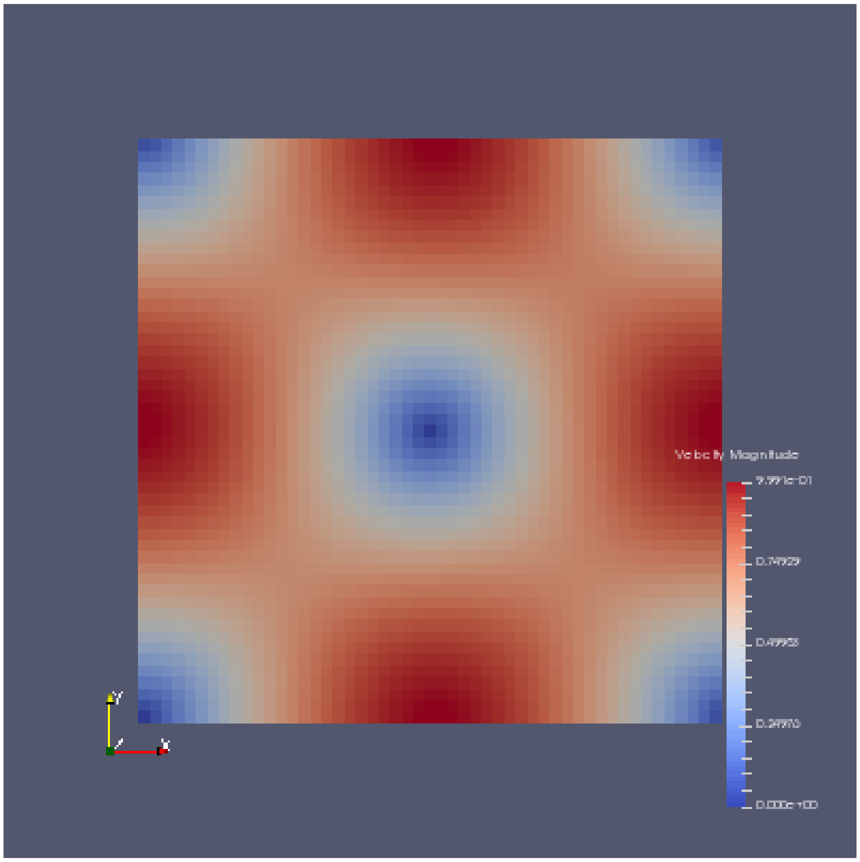
1.6.3 Stationary velocity (magnitude)



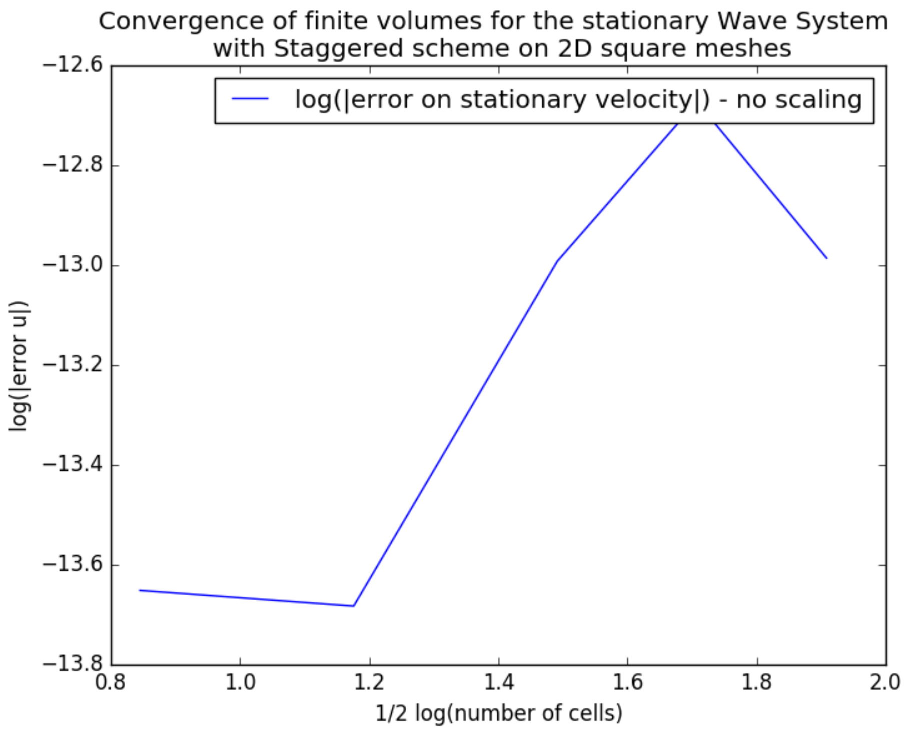
result 1 | result 2 | result 3 - | - - | -



| |



1.7 Convergence on stationary velocity


Convergence of finite volumes for the stationary Wave System
with Staggered scheme on 2D square meshes
log(|error on stationary velocity|) - no scaling
log(|error u|)
1/2 log(number of cells)
−12.6
−12.8
−13.0
−13.2
−13.4
−13.6
−13.8
0.8
1.0
1.2
1.4
1.6
1.8
2.0