Convergence_WaveSystem_Centered_SQUARE

November 26, 2018

1 Centered scheme for the Wave System

1.1 The Wave System on the square

We consider the following Wave system with periodic boundary conditions

$$\left\{ \begin{array}{ll} \partial_t p + c^2 \nabla \cdot \vec{q} = 0 \\ \partial_t \vec{q} + \quad \vec{\nabla} p = 0 \end{array} \right. .$$

The wave system can be written in matrix form

$$\partial_t \left(\begin{array}{c} p \\ \vec{q} \end{array} \right) + \left(\begin{array}{cc} 0 & c^2 \nabla \cdot \\ \vec{\nabla} & 0 \end{array} \right) \left(\begin{array}{c} p \\ \vec{q} \end{array} \right) = 0$$

In d space dimensions the wave system is an hyperbolic system of $d+1$ equations

$$\partial_t U + \sum_{i=1}^{d} A_i \partial_{x_i} U = 0, \quad U = {}^t(p, \vec{q})$$

where the jacobian matrix is

$$A(\vec{n}) = \sum_{i=1}^{d} n_i A_i = \left(\begin{array}{cc} 0 & c^{2t}\vec{n} \\ \vec{n} & 0 \end{array} \right), \quad \vec{n} \in \mathbb{R}^d.$$

has $d+1$ eigenvalues $-c, 0, \ldots, 0, c$.

The wave system also takes the conservative form

$$\partial_t U_i + \nabla \cdot F(U) = 0,$$

where the flux matrix F is defined by

$$F(U)\vec{n} = A(\vec{n})U, \quad \vec{n} \in \mathbb{R}.$$

On the square domain $\Omega = [0,1] \times [0,1]$ we consider the initial data

$$\left\{ \begin{array}{l} p_0(x,y) = constant \\ q_{x0}(x,y) = \sin(\pi x)\cos(\pi y) \\ q_{y0}(x,y) = -\sin(\pi y)\cos(\pi x) \end{array} \right. .$$

The initial data (p_0, q_x, q_y) is a stationary solution of the wave system.

1.2 The Centered scheme for the Wave System

The domain Ω is decomposed into cells C_i.

- $|C_i|$ is the measure of the cell C_i.
- f_{ij} is the interface between two cells C_i and C_j.
- s_{ij} is the measure of the interface f_{ij}.
- d_{ij} is the distance between the centers of mass of the two cells C_i and C_j.

The semi-discrete colocated finite volume equation is

$$\partial_t U + \frac{1}{|C_i|} \sum s_{ij} F_{ij} = 0,$$

where U_i is the approximation of U in the cell C_i.

F_{ij} is a numerical approximation of the outward normal interfacial flux from cell i to cell j usually in the upwind form

$$F_{ij} = \frac{F(U_i) + F(U_j)}{2}\vec{n} - D(\vec{n})\frac{U_i - U_j}{2}.$$

In the case of the centered scheme the upwind matrix is

$$D_{centered}(\vec{n}) = 0.$$

1.3 Some Bibliography

Most of the literature on finite volume methods for hyperbolic equations focus on scalar equation whilst we are interested in systems of equation. Moreover the rare references that apply to the wave system are generally restricted to the upwind scheme. However the following two references apply to a general upwinding function

- Stability of finite volumes with a general upwinding matrix
  Ndjinga, Michaël. "L2 stability of nonlinear finite-volume schemes for linear hyperbolic systems." Comptes Rendus Mathematique 351.17 (2013): 707-711.
- Convergence of finite volumes with a general upwinding matrix
  Ndjinga, Michaël. "Weak Convergence of Nonlinear Finite Volume Schemes for Linear Hyperbolic Systems." Finite Volumes for Complex Applications VII-Methods and Theoretical Aspects. Springer, Cham, 2014. 411-419.

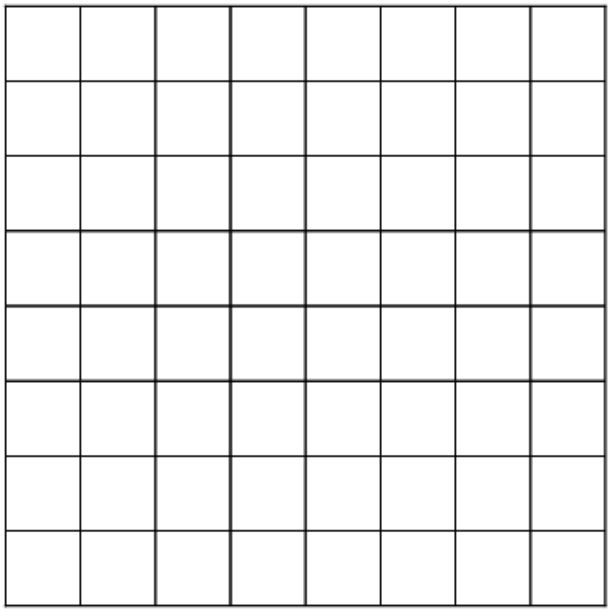
1.4 The script

```
#Condition initiale
pressure_field, velocity_field = initial_conditions_wave_system(my_mesh)

#Pas de temps
dt = cfl * dx_min / c0
```

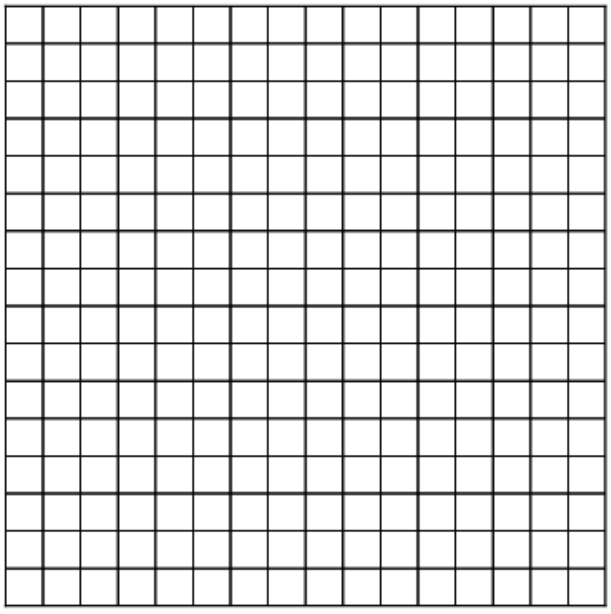
```
#Matrice des systèmes linéaires
divMat=computeDivergenceMatrix(my_mesh,nbVoisinsMax,dt,test_bc)

# Construction du vecteur inconnu
Un=cdmath.Vector(nbCells*(dim+1))
for k in range(nbCells):
    Un[k*(dim+1)+0] =     pressure_field[k]
    Un[k*(dim+1)+1] =rho0*velocity_field[k,0]
    Un[k*(dim+1)+2] =rho0*velocity_field[k,1]


# Création du solveur linéaire
LS=cdmath.LinearSolver(divMat,Un,iterGMRESMax, precision, "GMRES","ILU")

# Time loop
while (it<ntmax and time <= tmax and not isStationary):
    LS.setSndMember(Un)
    Un=LS.solve();
    Un.writeVTK

# Automatic postprocessing :  save 2D picture and plot diagonal data
#============================
diag_data=VTK_routines.Extract_field_data_over_line_to_numpyArray(my_ResultField,[0
plt.legend()
plt.xlabel('Position on diagonal line')
plt.ylabel('Value on diagonal line')
if len(sys.argv) >1 :
    plt.title('Plot over diagonal line for finite volumes \n for Wave system on a 2
    plt.plot(curv_abs, diag_data, label= str(nbCells)+ ' cells mesh')
    plt.savefig("FiniteVolumes2D_square_ResultField_"+str(nbCells)+ '_cells'+"_Plot
```

1.5 Numerical results for centered scheme on cartesian meshes

1.5.1 Cartesian meshes



mesh 1 | mesh 2 | mesh 3 - | - - | -



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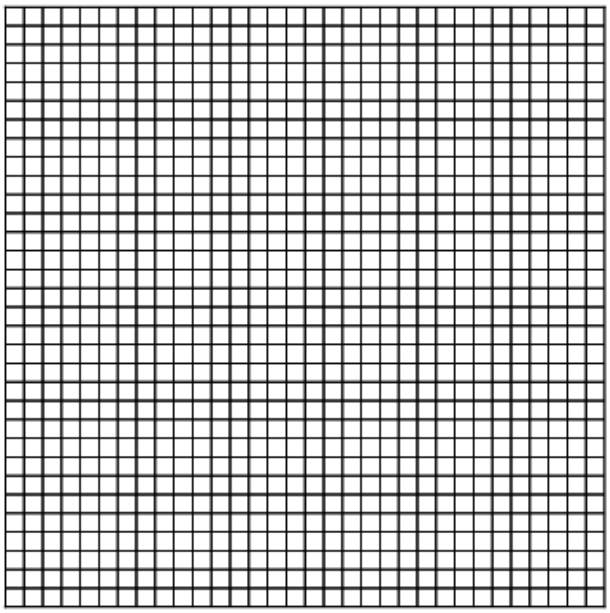
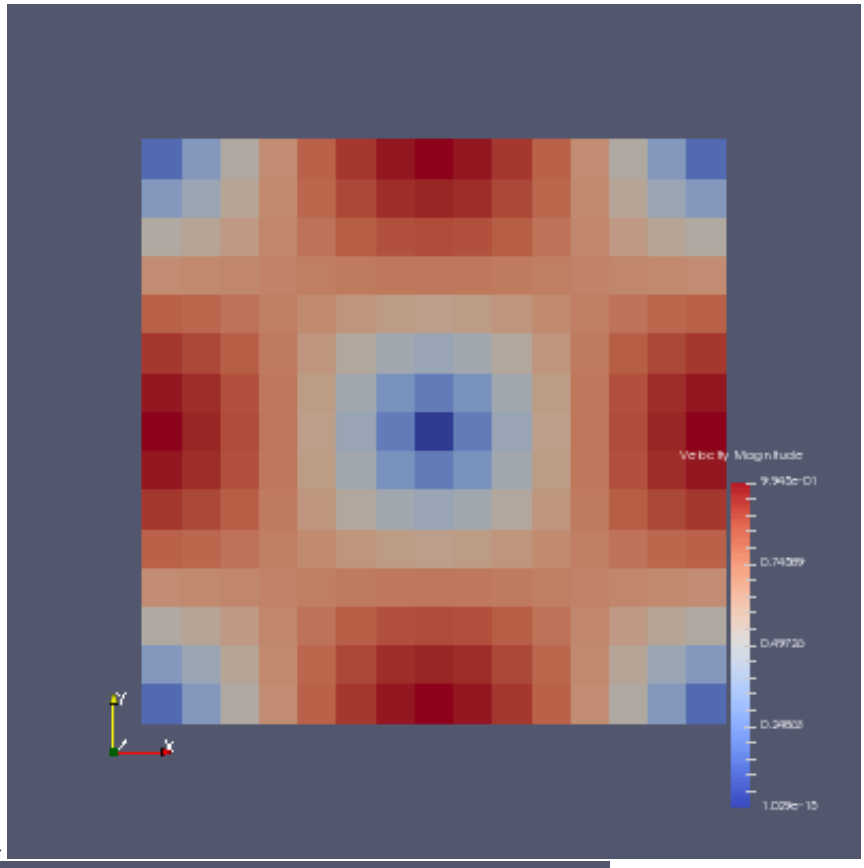


1.5.2 Velocity initial data (magnitude)

result 1 | result 2 | result 3 - | - - | -



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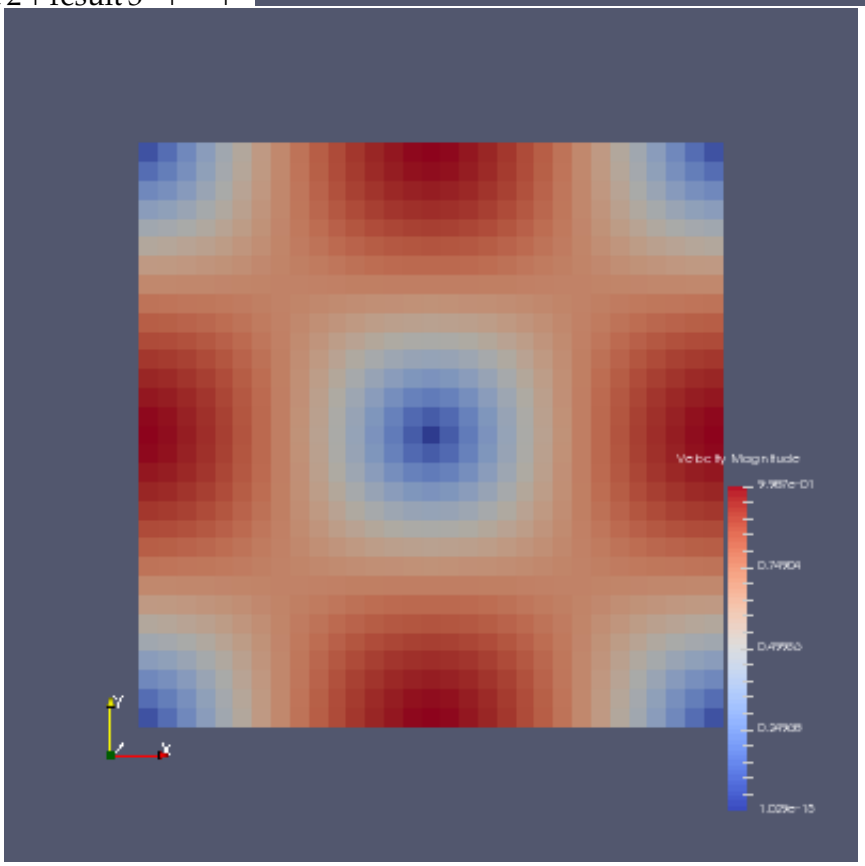

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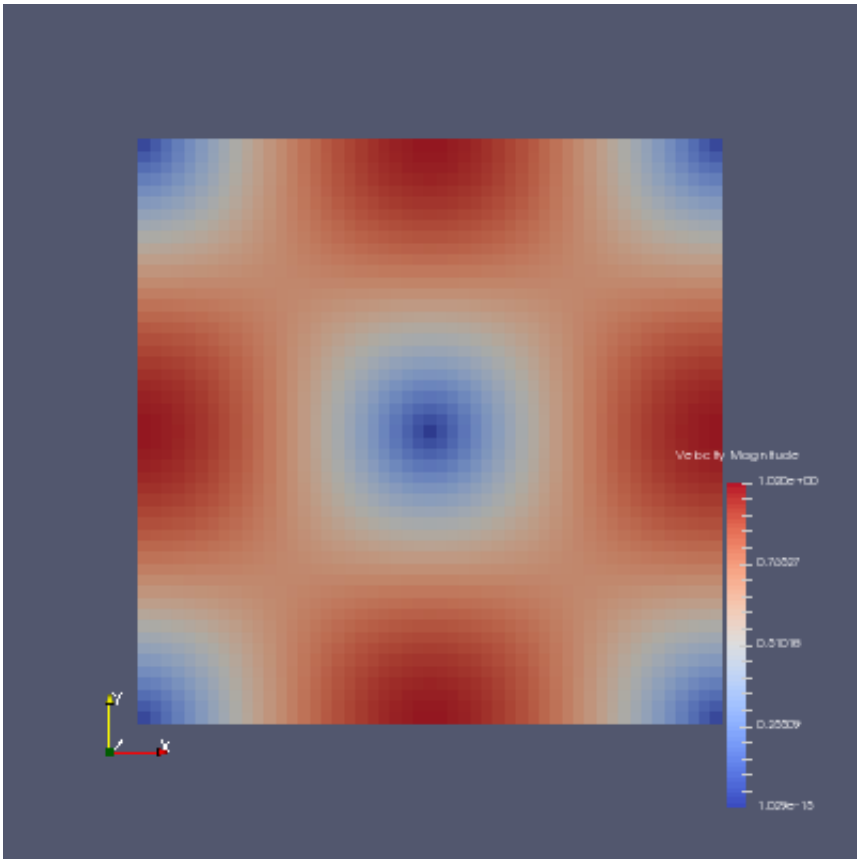
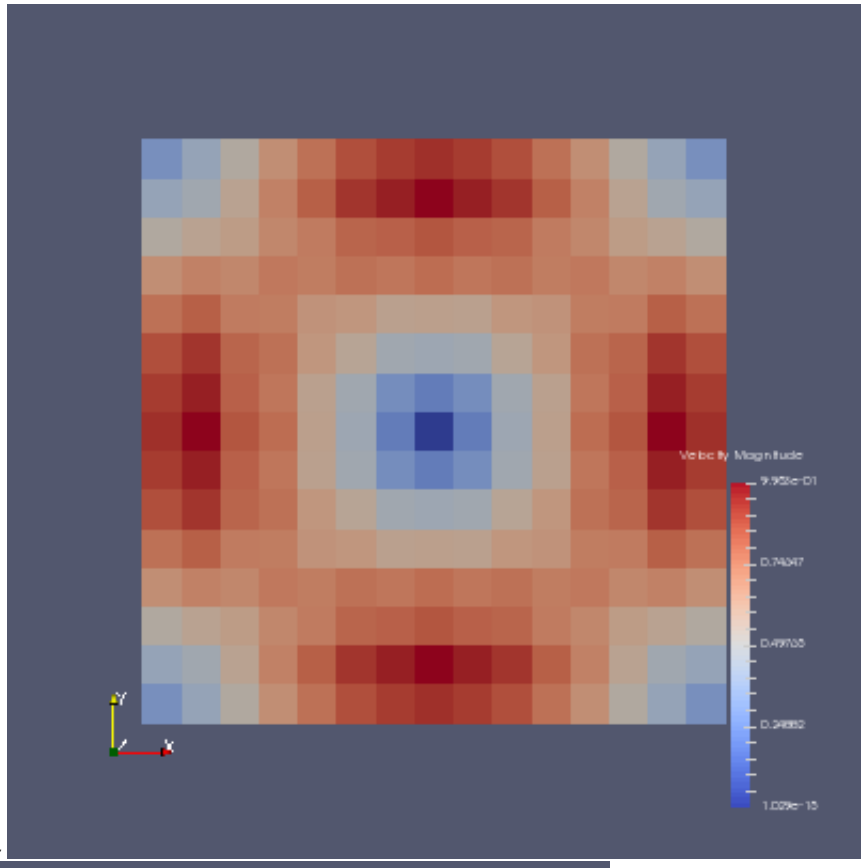
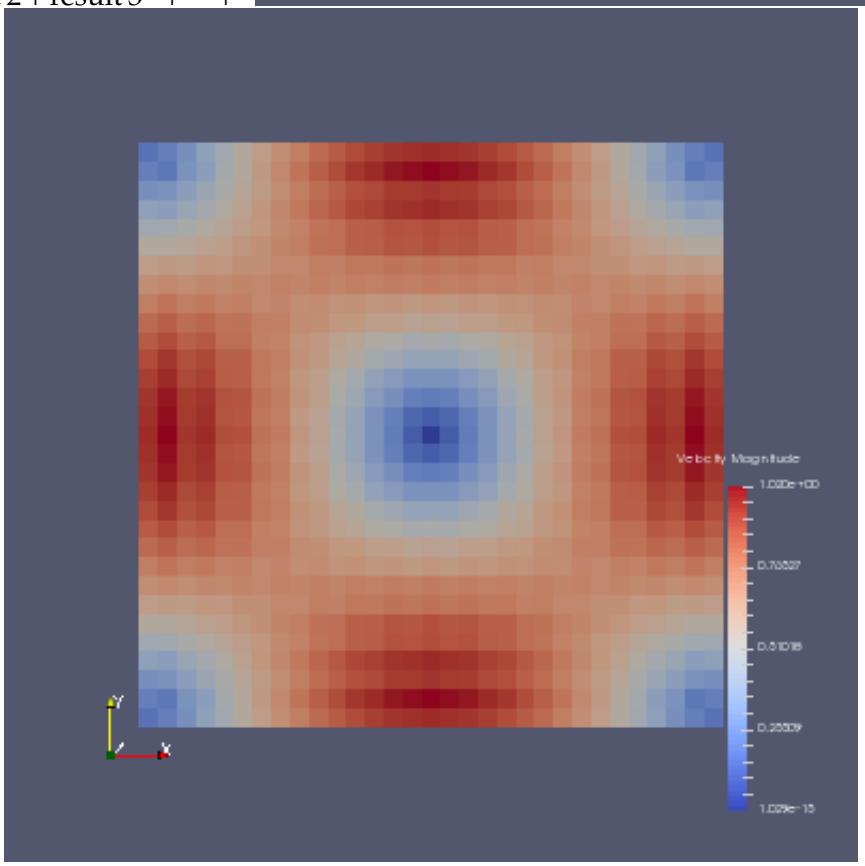

Velocity Magnitude
1.000e+00
0.75037
0.50019
0.25009
1.039e-10
Y
Z
X

1.5.3 Stationary velocity (magnitude)

result 1 | result 2 | result 3 - | - - | -





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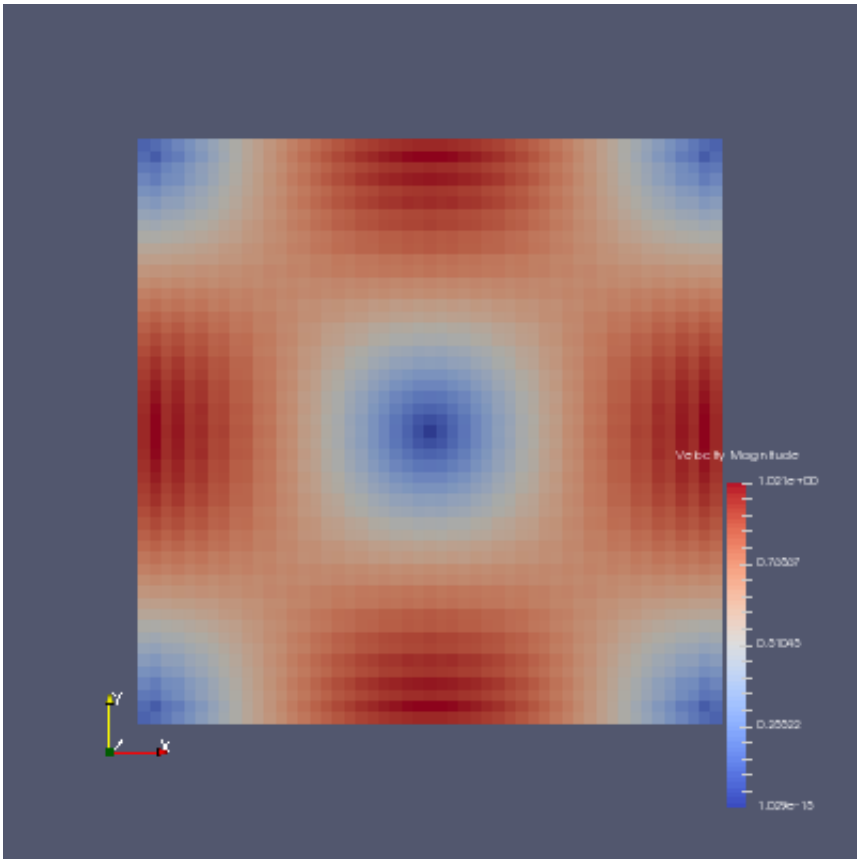
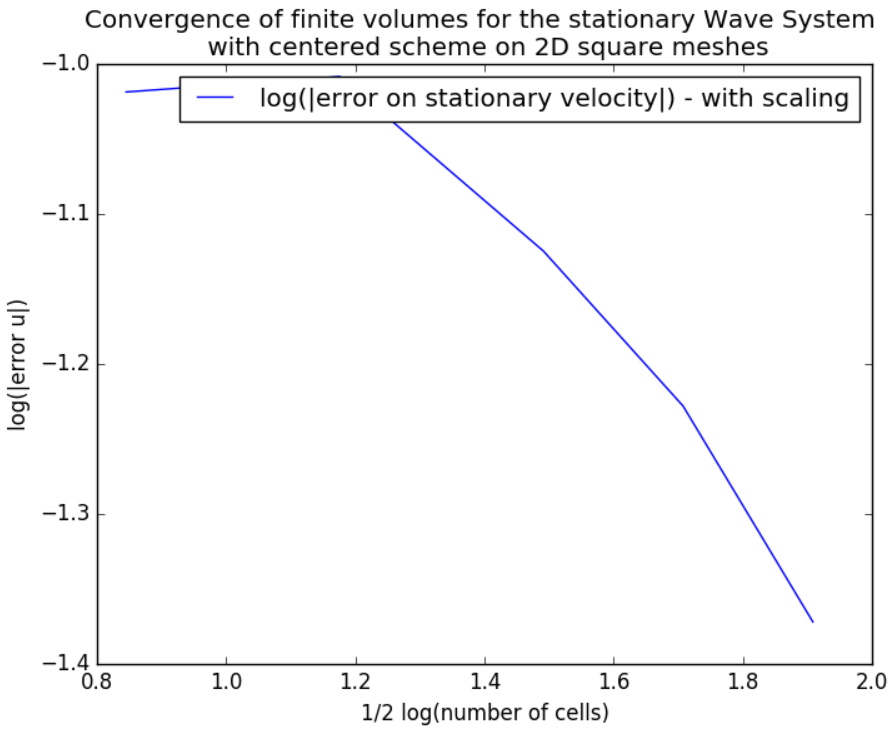
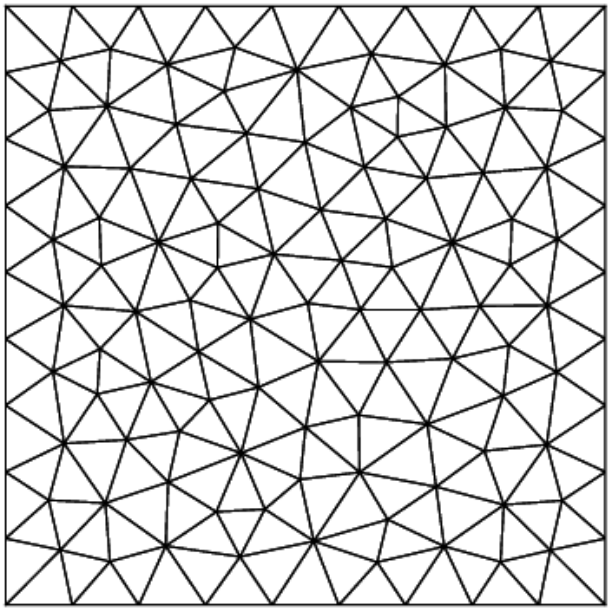

Velocity Magnitude
1.021e+00
0.70507
0.51045
0.25522
1.039e-10


Convergence of finite volumes for the stationary Wave System
with centered scheme on 2D square meshes
log(|error on stationary velocity|) - with scaling
log(|error u|)
1/2 log(number of cells)
−1.0
−1.1
−1.2
−1.3
−1.4
0.8
1.0
1.2
1.4
1.6
1.8
2.0

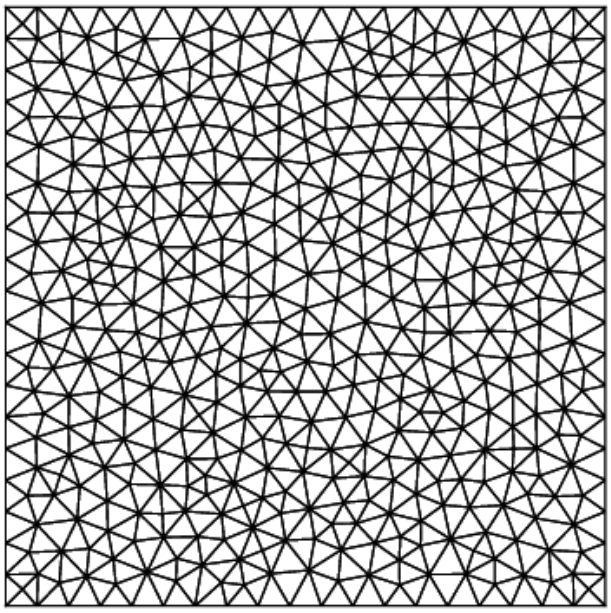
1.5.4 Convergence on stationary velocity

1.6 Numerical results for centered scheme on triangular meshes

1.6.1 Triangular meshes



mesh 1 | mesh 2 | mesh 3 - | - - | -



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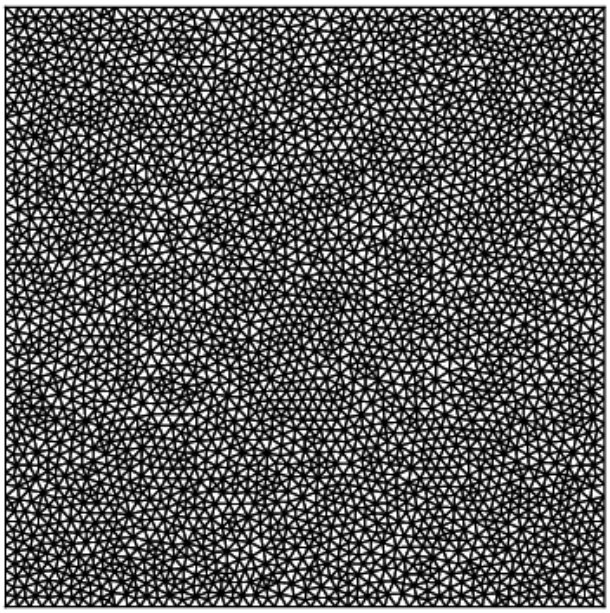
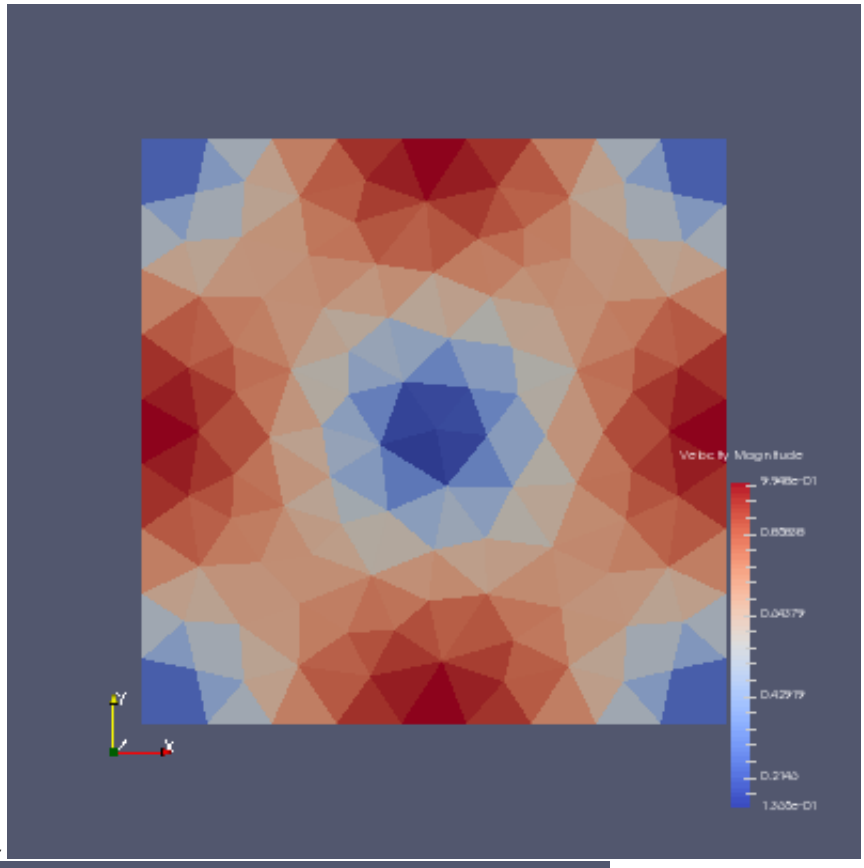


1.6.2 Velocity initial data (magnitude)

result 1 | result 2 | result 3 - | - - | -



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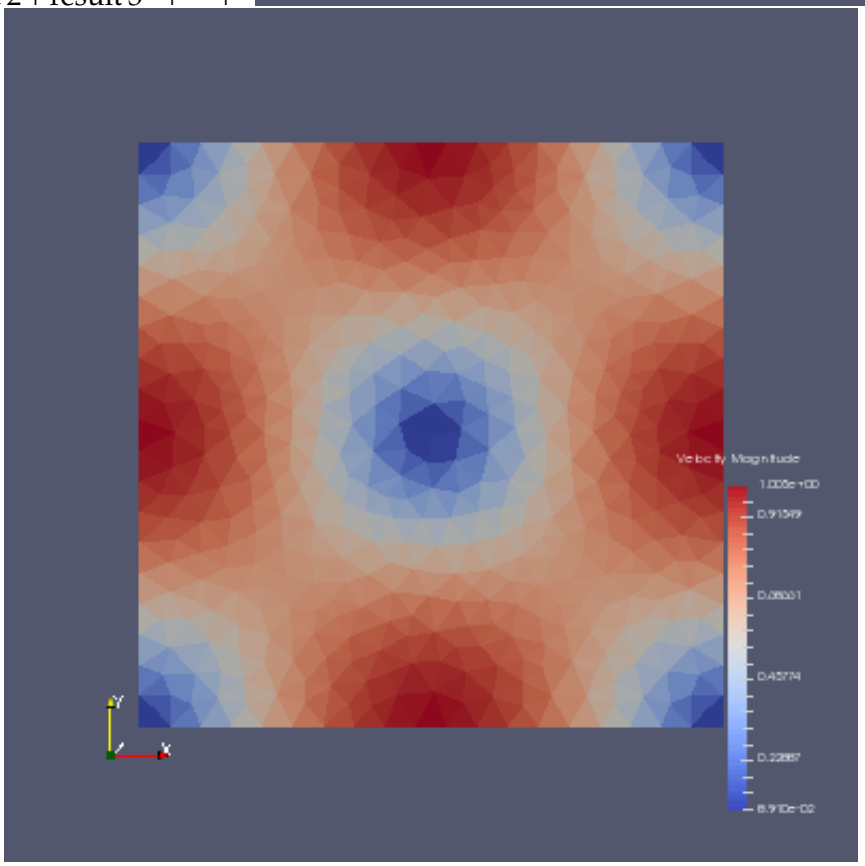

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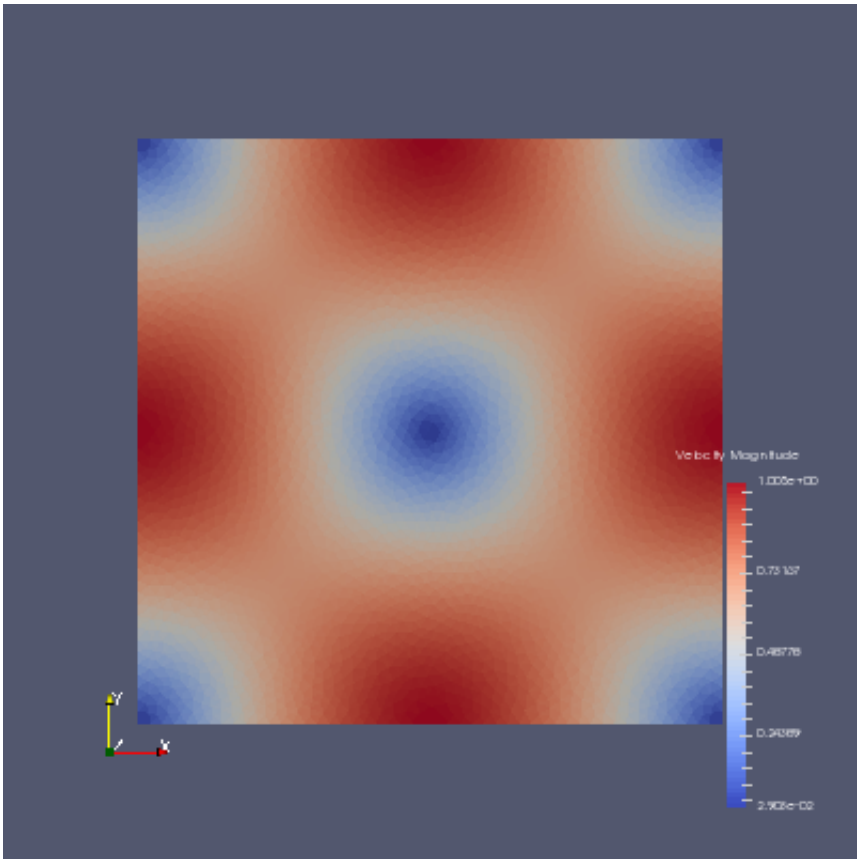
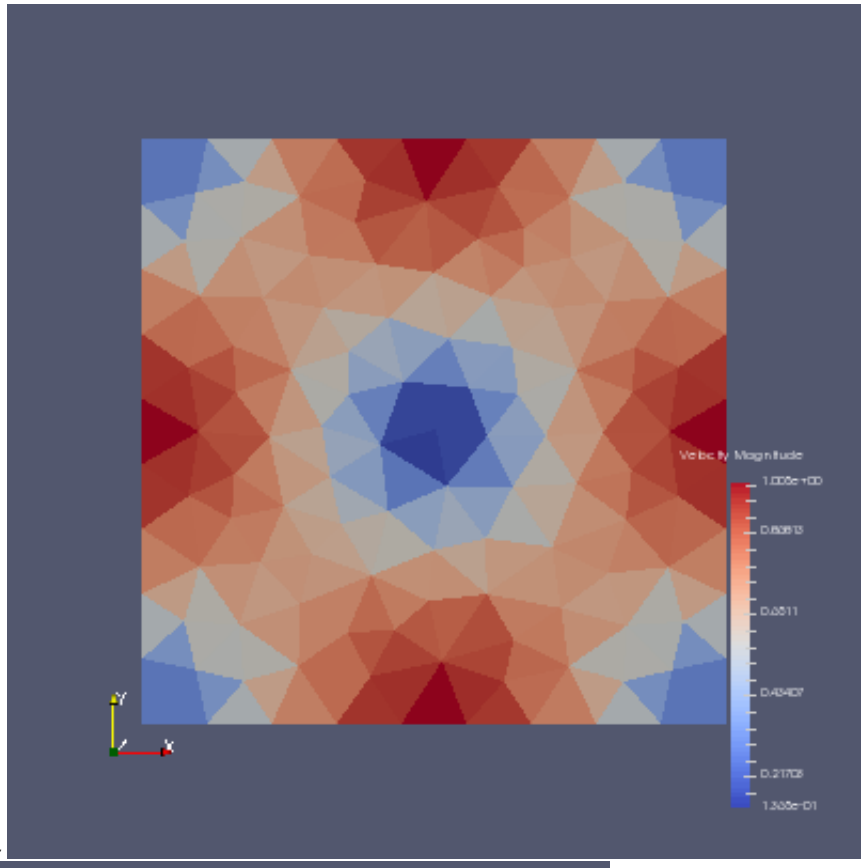

Velocity Magnitude
1.000e+00
0.75137
0.48778
0.24389
2.903e-02

1.6.3 Stationary velocity (magnitude)

result 1 | result 2 | result 3 - | - - | -



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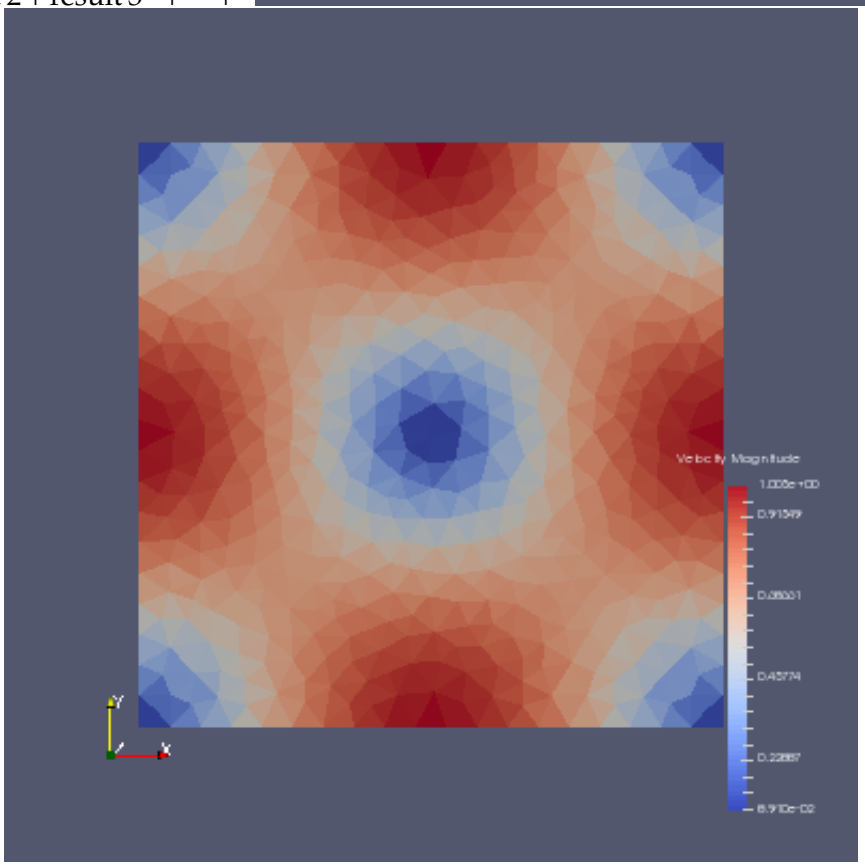

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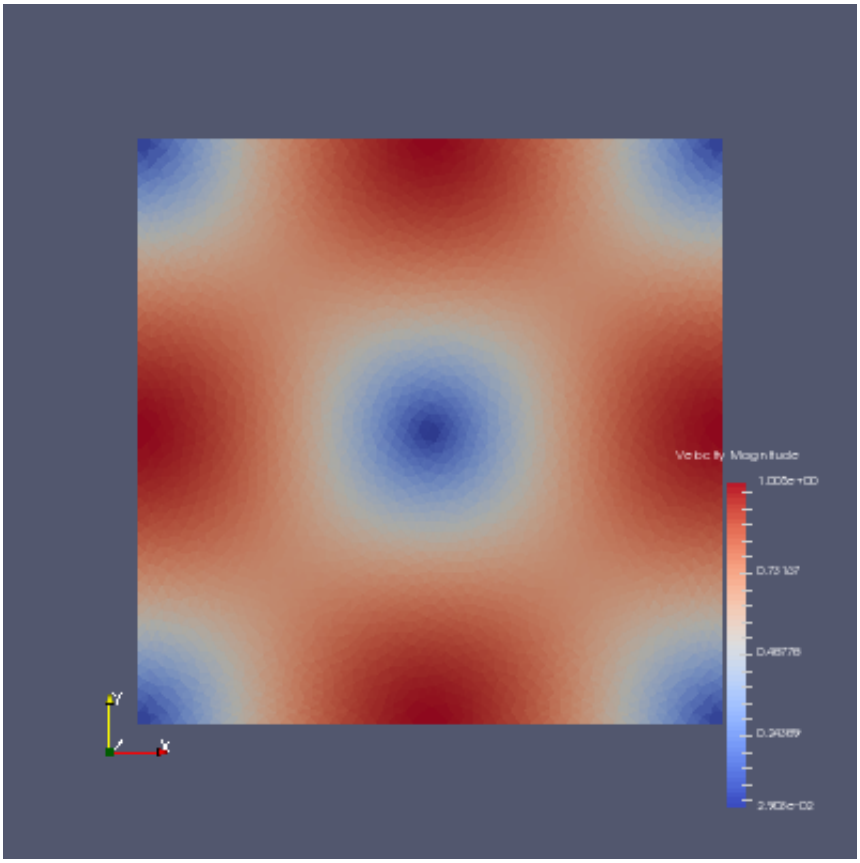
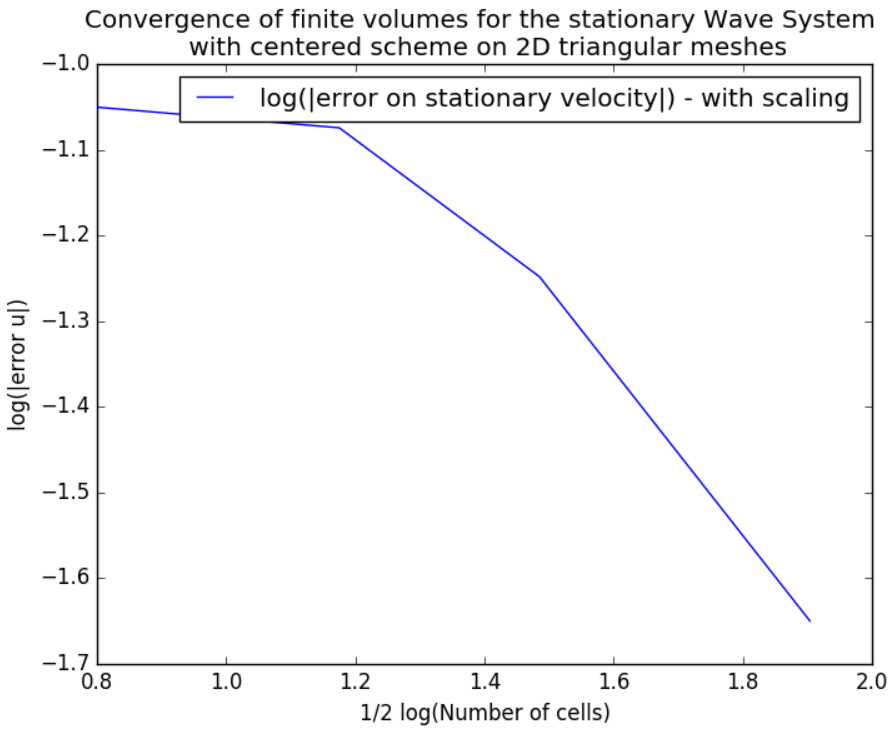
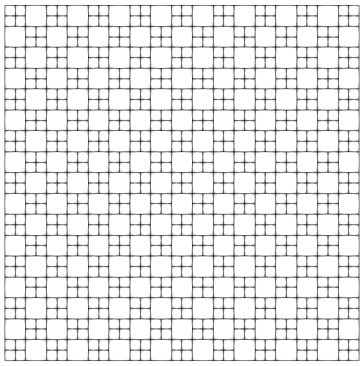

Velocity Magnitude
1.000e+00
0.75137
0.48778
0.24389
2.903e-02
Y
Z
X


Convergence of finite volumes for the stationary Wave System
with centered scheme on 2D triangular meshes
log(|error on stationary velocity|) - with scaling
log(|error u|)
−1.0
−1.1
−1.2
−1.3
−1.4
−1.5
−1.6
−1.7
0.8
1.0
1.2
1.4
1.6
1.8
2.0
1/2 log(Number of cells)

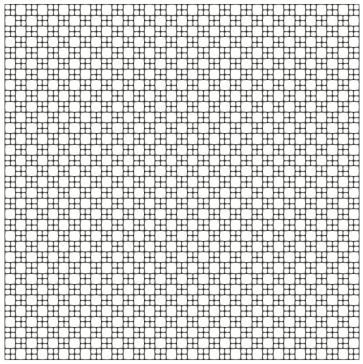
1.6.4 Convergence on stationary velocity

1.7 Numerical results for centered scheme on checkerboard meshes

1.7.1 Checkerboard meshes



mesh 1 | mesh 2 | mesh 3 - | - - | -



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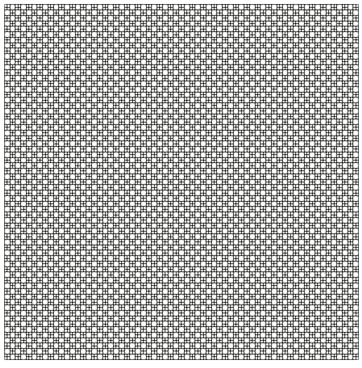
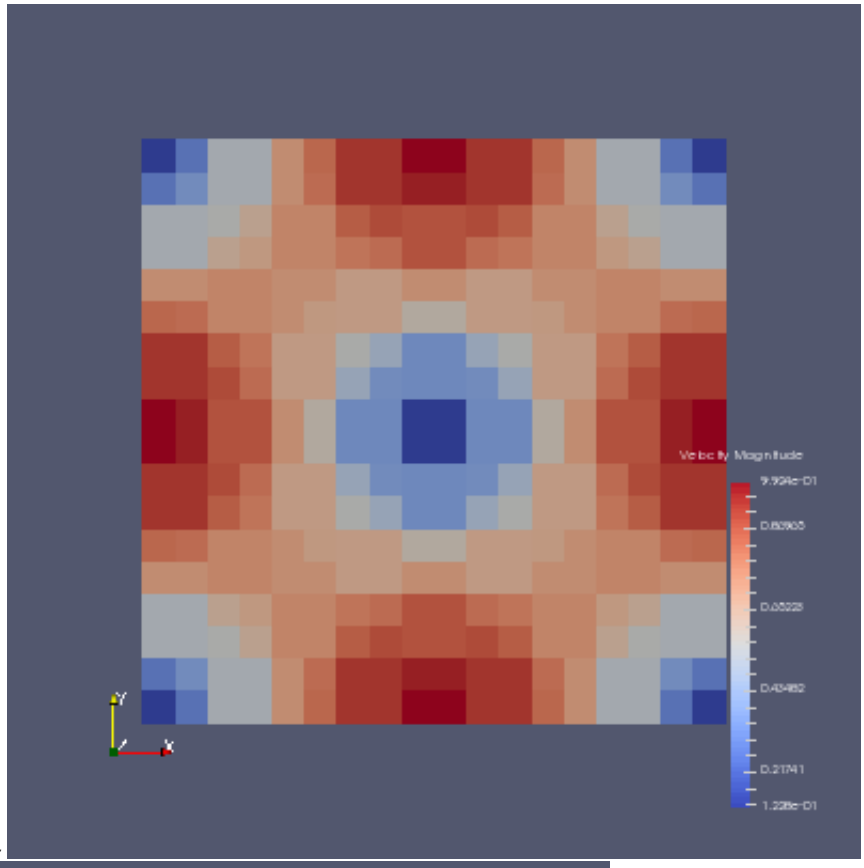


1.7.2 Velocity initial data (magnitude)

result 1 | result 2 | result 3 - | - - | -



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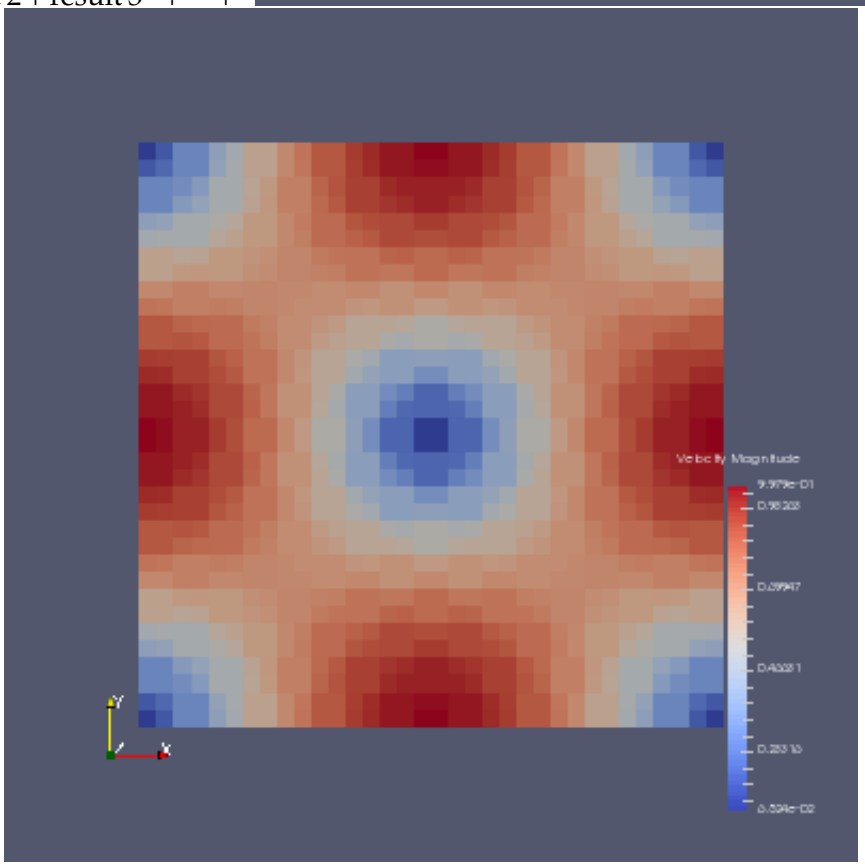

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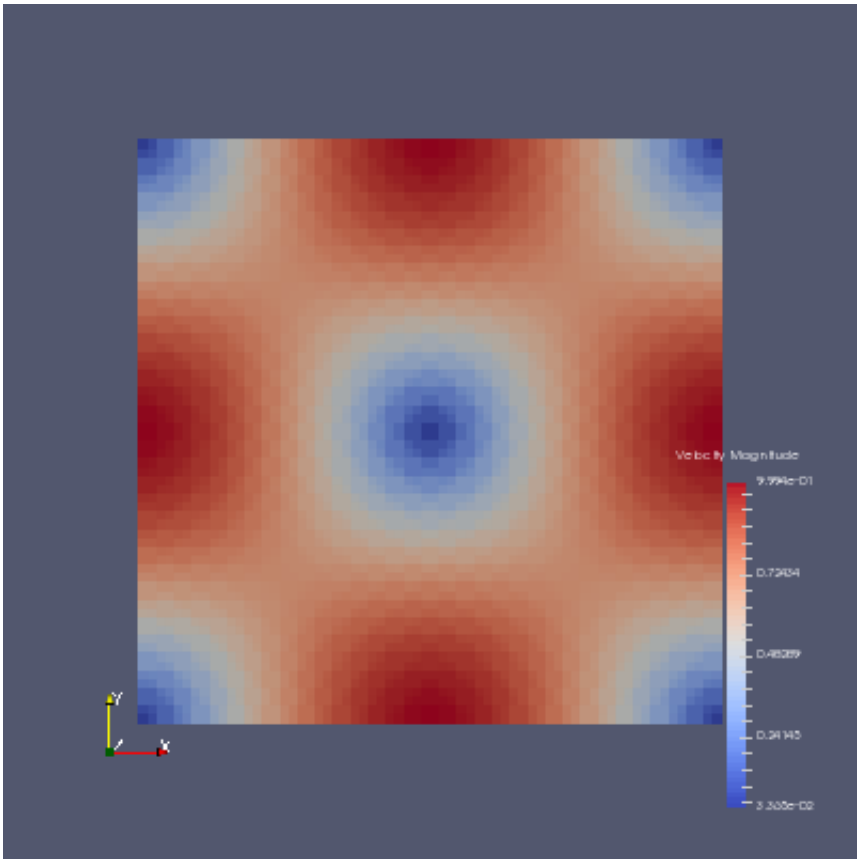
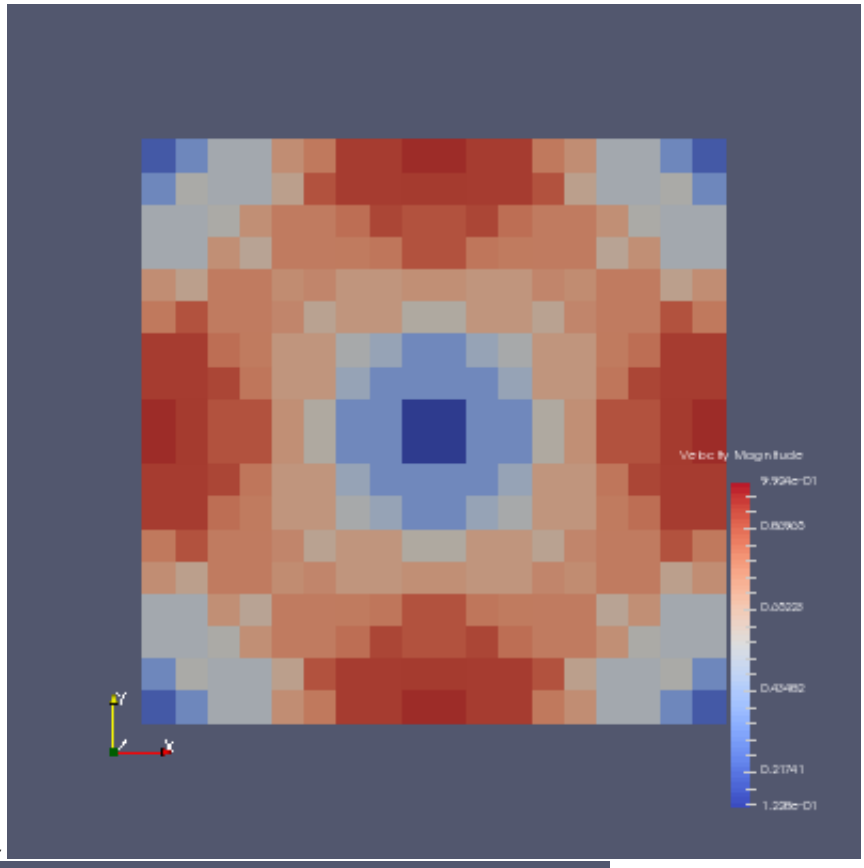
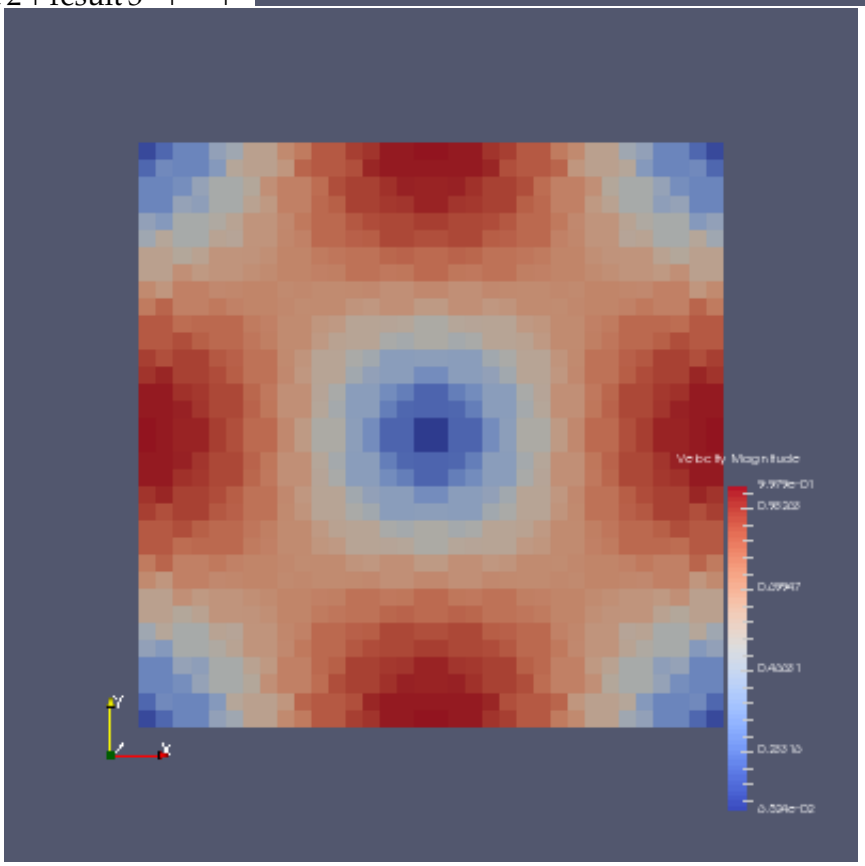

Velocity Magnitude
9.994e-01
0.73404
0.48089
0.24145
3.203e-02
Y
Z
X

1.7.3 Stationary velocity (magnitude)

result 1 | result 2 | result 3 - | - - | -





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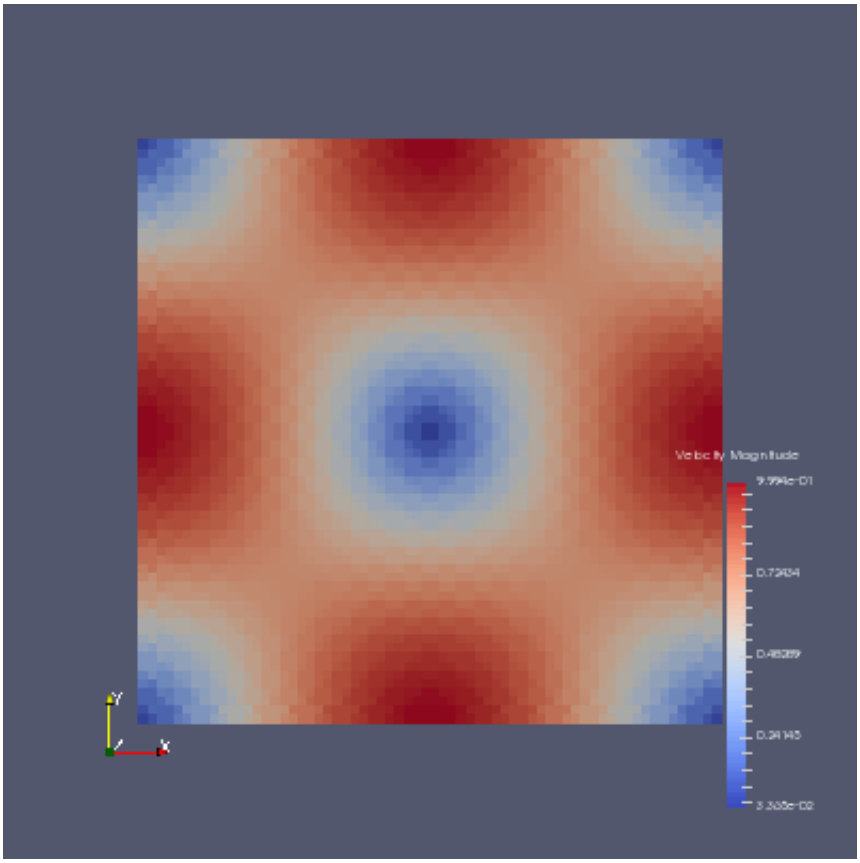
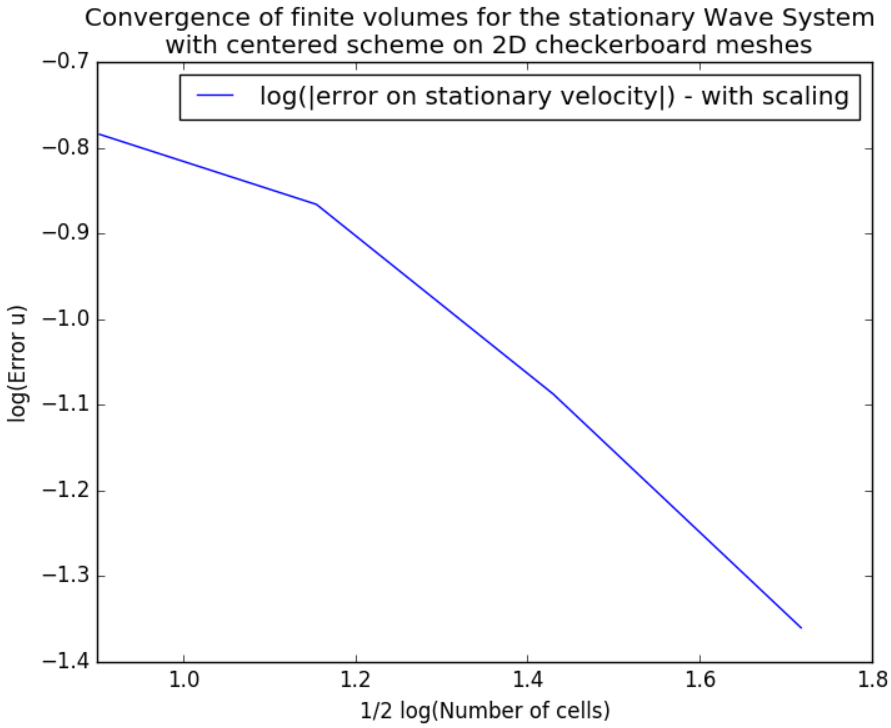


1.7.4 Convergence on stationary velocity


Convergence of finite volumes for the stationary Wave System
with centered scheme on 2D checkerboard meshes
log(|error on stationary velocity|) - with scaling
log(Error u)
1/2 log(Number of cells)
−0.7
−0.8
−0.9
−1.0
−1.1
−1.2
−1.3
−1.4
1.0
1.2
1.4
1.6
1.8