Pythagorean Fifths on the Unit Circle

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Musical pitch, consonance

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- Velocity of hammer?

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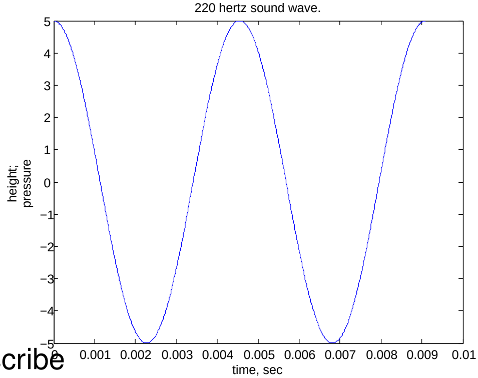
- Recall that period $= 2\pi/440\pi = 1/220$ *seconds/cycle*.
  So frequency $= 1/\textit{period} = 220$ *cycles/sec*.

The vibrating string

$$y(t) = A\cos(440\pi t)$$



$y(t)$ can describe

- Displacement of a point on the string in time.
- Pressure at a point in space over time (like at your eardrum)

Pitch, consonance, frequency ratios, intervals

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Pythagoras found certain ratios “consonant", others dissonant. When two musical notes are sounded, the musical *interval* corresponds to the frequency ratio.

Three consonant intervals

The Unison

The frequency ratio is 1, "A" 220 hz and "A" 220 hz.

The Octave

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Frequency ratio is 2, eg. "*A*" 220 *hz* and "*A*" 440 *hz*.

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Frequency ratio is $\frac{3}{2}$, eg. "*A*" 220 *hz* and "*E*" 330 *hz*. Sort of...

A musical scale

Build a scale by “stacking fifths".

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Start with C and C one octave higher (to avoid sharps, flats-black keys). Multiply successively by $\dfrac{3}{2}$, when this takes us outside the octave, mult by $\dfrac{1}{2}$ to get note one octave lower. All notes will then be in the same octave.

note	C		D		E		F		G		A		B		C
frq	1		$\frac{9}{8}$		$\frac{81}{64}$		$\frac{4}{3}$		$\frac{3}{2}$		$\frac{27}{16}$		$\frac{243}{128}$		2
intvl		$\frac{9}{8}$		$\frac{9}{8}$		$\frac{256}{243}$		$\frac{9}{8}$		$\frac{9}{8}$		$\frac{9}{8}$		$\frac{256}{243}$	

All good so far...

The black keys-sharps, flats

Keep stacking fifths, obtain

- F#-G $\sim \frac{256}{243}$ *Pythagorean diatonic semitone*.
- F-F# $\sim \frac{2187}{2048}$ *Pythagorean chromatic semitone*.
- B#-C $\sim \frac{531441}{524288}$ *Pythagorean comma*.

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... But B# and C are *enharmonically equivalent.*

Problems-playing in different keys, ...

Even-tempered tuning

We want 12 tones; all notes the same “distance" apart; that is the semitones, or half-step intervals have same ratio.

$2^{1/12} \approx 1.059463094$ diatonic semitone.

Compare

- 1.053497942
- 1.067871094
- 1.059463094

Close, but...

12 tones on the unit circle

We identify points on the unit circle with notes. Given a point, (x, y), write

$$(x, y) = (\cos\theta, \sin\theta).$$

So θ is the angle. So, eg.,

$(0, 1) \leftrightarrow \theta = 0$

$(1, 0) \leftrightarrow \theta = \pi/2.$

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We can identify musical intervals with the angle θ for a given interval. We would like our definition to have the following properties:

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Properties:

- Octave to corresponds to $\theta = 2\pi$,
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So Define, for an interval with frequency ratio r,

$$\theta(r) = 2\pi \log_2 r. \tag{1}$$

12 tones on the unit circle

$$\theta(r) = 2\pi \log_2 r$$

Works since the stacked interval with ratios r_1 and r_2 is the product $r_1 r_2$, and

$$\begin{aligned}
\theta(r_1 r_2) &= 2\pi \log_2 r_1 r_2 \\
&= 2\pi \log_2 r_1 + 2\pi \log_2 r_2 \\
&= \theta(r_1) + \theta(r_2). \qquad (2)
\end{aligned}$$

So we simply add the angles.

12 tones on the unit circle

It follows that the points on the unit circle for the twelve tones of the even-tempered scale are all equally spaced on the unit circle, and the ratio for one half-step must satisfy $r^{12} = 2$, which gives

$$r = 2^{1/12} \approx 1.059463094.$$

We find

$$\theta(2^{1/12}) = \frac{2\pi}{12} = \frac{\pi}{6}.$$

Also note that notes outside the octave simply wrap around the circle.

The circle of fifths; even-tempered tuning

$$\left(2^{7/12}\right)^{n} \ ; \quad n = 1, 2, 3, ...$$

Eventually we get a power of 2. (When??)
So

$$\theta(2^{7/12}) = 7n\frac{\pi}{6} \ ; \quad n = 1, 2, 3, ...$$

Eventually we get a multiple of 2π.

The circle of Pythagorean fifths

$$\left(\frac{3}{2}\right)^n \; ; \quad n = 1, 2, 3, ...$$

We never get a power of 2.

$$\theta\left(\frac{3}{2}n\right) \approx 3.67542779n; \quad n = 1, 2, 3, ...$$

We never get a multiple of 2π.

Asymptotic distribution of Pythagorean fifths

How do the points on the unit circle distribute themselves as n gets large?

It turns out that as $n \to \infty$, the set of points is “dense" on the unit circle.

Point mass measures; probability

If X continuous r.v. with probability density $\rho(t)$ the probability

$$P(a < X < b) = \int_a^b \rho(t)\, dt.$$

That is, the area under the curve.

Measure theory allows us to place the discrete probabilities in this framework using *point-mass measures*. Suppose X is a discrete r.v. and μ is a measure with mass m_i at $t_i,\ i = 1, 2, ..., n$, that give probabilities for X.

$$P(a < X < b) = \int_a^b d\mu = \sum m_i$$

where the sum is taken over all the i such that t_i is in the interval $[a, b]$.

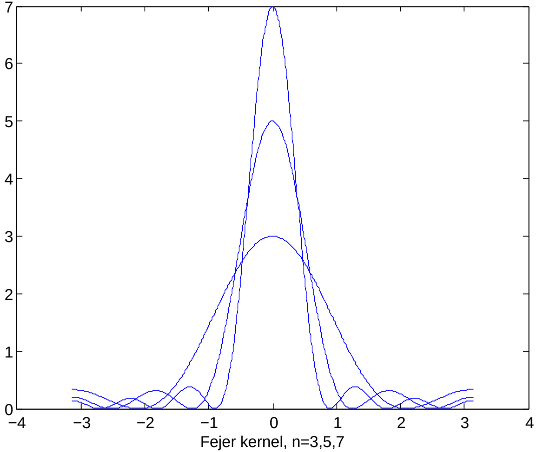
Point mass measures; probability

So integration with respect to a point-mass, or discrete, measure is just a sum.

For any f continuous on $[0, 1]$,

$$\int_0^1 f(t)\, d\mu(t) = \sum_{i=1}^{n} m_i f(t_i)$$

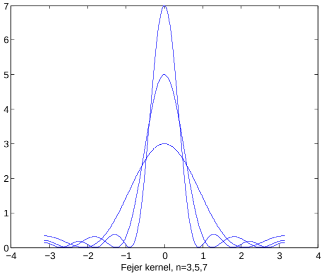
Approximate identities; densities that converge to a point mass.

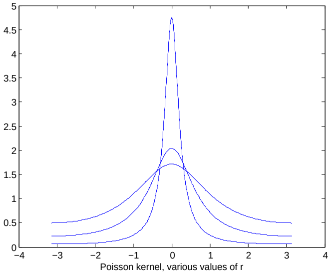


The Fejér kernel,

$$\phi_n(\theta) = \frac{1}{n}\left(\frac{\sin(n\theta/2)}{\sin(\theta/2)}\right)^2$$

The Fejér and Poisson kernels





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Fejér: $n \to \infty$

$$\psi_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$$

Poisson: $r \to 1$

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- On the unit circle, μ_n assigns weight $1/n$ at each of the points with angle $\theta\left(\frac{3}{2}n\right)$.

Weak-star convergence of point masses

We say that a sequence of meaures μ_n converges to μ weak-*, if

$$\int f(t)d\mu_n \to \int f(t)d\mu \ \ as \ \ n \to \infty$$

for all continuous functions f.

So if μ_n are discrete measures, we have

$$\sum_{i=1}^{n} m_i f(t_i) \to \int f(t)d\mu \ as \ \ n \to \infty$$

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$$\sum_{i=1}^{n} m_i f(t_i) \to \frac{1}{2\pi} \int_0^{2\pi} f(t)dt$$

  Again, the uniform density on $[0, 2\pi)$.