# Mapping MAX-2-SAT to Ising model

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- Results.

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- Ising formulations of many NP-Hard problems [1]

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- Ising formulations of many NP-Hard problems [1]
- The coefficients of Ising model will define our calculation problem
- SAT – Automated theorem proving, CSP(Constraint satisfiability problem)
- 3-SAT – A very important problem in the world of in computational complexity theory

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- $P \subset NP$[2]. We can check the correctness of the solution in polynomial time, simply by solving it ourselves.
- A problem $H$ is **NP-Hard** when each $L \in NP$ can be reduced in polynomial time to $H$;

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# 2-SAT Problem

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- If an assignment does not satisfies $\Omega_i$, then exactly 6 of the 10 clause in $F$ are satisfiable.

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**Theorem**

The 2-SAT is solvable in polynomial time.

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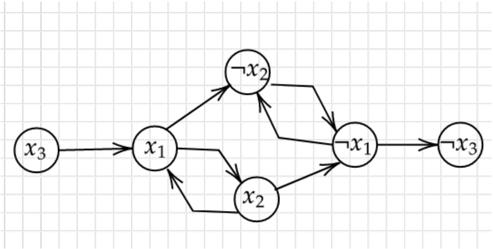


Figure: Reduction graph for $F$

# Strongly connected component

## Definition

Strongly connected component of the directed graph $G = (V, E)$ we call a subset $A \subset V$ such that $\forall_{i \neq j} \quad _{x_i, x_j \in A}$ there is a path between $x_i$ and $x_j$.

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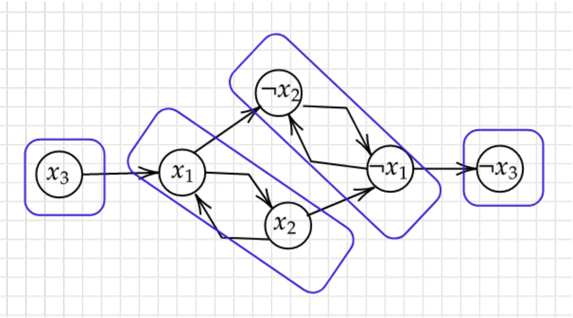


Figure: Strongly connected component

# Draft of proof for 2-SAT- lemma

## Lemma

If both vertices $x_i$ and $\neg x_i$ are in a strong connected component then the formula is unsatisfiable.

## Proof.

Recall that if $x_i$ and $\neg x_i$ are in a strong connected component then there is a pathway between $x_i \rightsquigarrow \neg x_i$ and $\neg x_i \rightsquigarrow x_i$.

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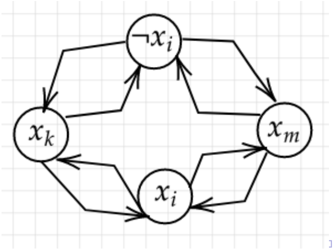
$$[(x_i \Rightarrow x_m) \wedge (x_m \Rightarrow \neg x_i)] \Rightarrow (x_i \Rightarrow \neg x_i)$$



# Phase transition

### Definition

The clause density for the CNF formula: 2-CNF is defined as ratio:

$$\alpha = \frac{M}{n}, \tag{2}$$

where $M$ number of clauses and $n$ number of variables in $F$

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- For $\alpha = \frac{5}{3}$ only one of the interpretation satisfiable $F$.

# Phase transition for 2-SAT and 3-SAT

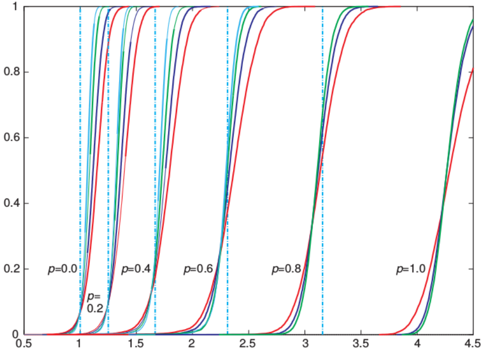


Figure: Phase transition for 2-SAT$(p = 0)$ from work [3] on the Y axis percentage of unfulfilled formula (UNSAT), on the X axis the clause density

[3] Determining computational complexity from characteristic 'phase transitions' Monasson, Rémi and Zecchina, Riccardo and Kirkpatrick, Scott and Selman, Bart and Troyansky, Lidror

# Tensor product of two vector space

## Definition

The tensor product $V \otimes W$ of two vector spaces $V$ and $W$ over the same field is a vector spaces, endowed with a bilinear mapping

$$(v, w) : V \times W :\mapsto v \otimes w \in V \otimes W$$

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If $\{v_s : s \in S\}$ and $\{w_t, t \in T\}$ are bases of $V$ and $W$, and $dim(V) = m$ and $dim(W) = n$ then $mn$ elements form a basic of $V \otimes W$. If $v \in V$ and $w \in W$ then coordinate vector of $v \otimes w$ over this basis is the outer product[4] of the coordinate vectors of v and w over the corresponding bases.

---

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# The Pauli matrixes

## Definition

The Pauli matrixes are set of three $2 \times 2$ complex matrices which are Hermitian and ununitary, with eigenvalues $+1, -1$. They are

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Identity matrix is given as:

$$\mathbb{1}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# The Kronecker product

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$A \in M_{n\times m}(\mathbb{C}), B \in M_{p\times q}(\mathbb{C})$ is the block matrix
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$$\mathbb{1}_2 \otimes \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

# The Ising Model – unitary matrices

### Definition

The Ising Model is a weighted graph, whose vertex are qubits. The weights of the vertices are indicated as $h_i$ for $i = 1, \ldots, |V|$, and weights of the edges are determined by $J_{i,j}$.

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$$H_p = \sum_{\{ij\} \in E} J_{i,j} \sigma_i^{(z)} \sigma_j^{(z)} + \sum_{j \in V} h_j \sigma_j^{(z)} \tag{3}$$

# With simple agreement:

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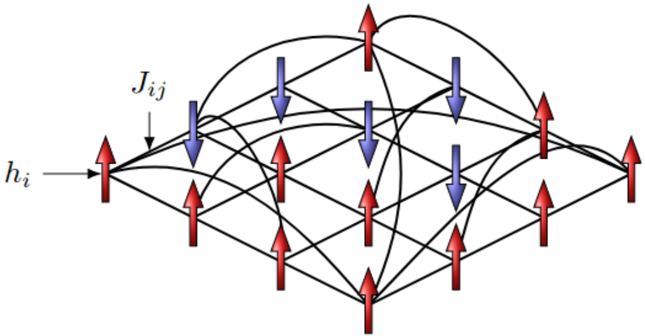
# Ising Model – Intuition



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### Remark

Ground state of $H_p$ is the vector of the form $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$[5] for some $\{x_i\}_{i=1}^n$, where $x_i \in \{0, 1\}$.

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# The Ising model solution:

**Remark**

$H_p$ is hermitian operator, so it has real values $(E_0 \leqslant E_1 \ldots \leqslant E_{2^n})$.
Ground state is its lowest-energy state $E_0$.

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### Definition

Ising's computational problem is to find the ground state $|\psi\rangle$ of Eigenfunction $H_p$ (such as in 10).

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It also define variable:

# Reduction - Step 1

The first step is transforming from Boolean $\mathbb{B} = \{T, F\}$ to binary variables $x_i = \{0, 1\}$, letting *TRUE* $\mapsto 0$ and *FALSE* $\mapsto 1$. The truth table of *OR* becomes as multiplication of the binary variables:

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

$\longrightarrow$

| $x_i \cdot x_j$ | $x_j$ | $x_i$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |
| 1 | 1 | 1 |

(6)

It also define variable:

$$v_j^k = \begin{cases} -1 & \text{if } x_j \text{ appears negated in } k\text{th clause} \\ 1 & \text{if } x_j \text{ appears unnegated in } k\text{th clause} \\ 0 & \text{if } x_j \text{ does not appear in } k\text{th clause} \end{cases} \tag{7}$$

# Reduction- Step 2

## Definition

The local Hamiltonian for the $\Omega_k$ clause is defined as

$$H_{\Omega_k} = \frac{\mathbb{1} - v^k_{j_1}\sigma^z_{j_1}}{2}\frac{\mathbb{1} - v^k_{j_2}\sigma^z_{j_2}}{2} \tag{8}$$

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### Lemma

If exist an assignment $\{x_{j_1}, x_{j_2}\} \in \{0,1\}^2$, which violate the clause $\Omega_k$ then minimal energy of $H_{\Omega_k}$ is 1. If not exist an assignment $\{x_{j_1}, x_{j_2}\} \in \{0,1\}^2$, which satisfied clause $\Omega_k$, then energy of $H_{\Omega_k}$ is 0.

# Example

Example of local Hamiltonian for

$$F = \Omega_1 = (\neg x_1 \vee x_2)$$

then for such a formula reading from $v_j$ we obtain form:

$$H_{\Omega_1} = \frac{1}{4}[\mathbb{1} - (-1 \cdot \sigma_1^z \otimes \mathbb{1}_2)] \cdot [\mathbb{1} - (\mathbb{1}_2 \otimes \sigma_2^z)] =$$

$$\frac{1}{4} \cdot \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Example c.d

Let us note that $H_{\Omega_1}$ has energies $E_i$ and the corresponding states $|\psi_i\rangle$ :

$$E_1 = 1 \mapsto |\psi_1\rangle = |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad E_2 = 0 \mapsto |\psi_2\rangle = |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$E_3 = 0 \mapsto |\psi_3\rangle = |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad E_4 = 0 \mapsto |\psi_4\rangle = |00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Logical assigments which, which violates the formula is: $x_1 = 0$, $x_2 = 1$, remark(we write this as $TRUE \mapsto 0$ and $FALSE \mapsto 1$

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that is the sum of the energy after all $M$ clause contained in the 2-SAT.

# Reduction – Step 4

$$H_{Pr} = \sum_{\Omega_k \in F} H_{\Omega_k}, \tag{10}$$

Summing up after all clauses local Hamiltonians:

$$H_{Pr} = \frac{1}{4} \sum_{\Omega_k \in F} \mathbb{1} - v_{j_1}^k \sigma_{j_1}^z - v_{j_2}^k \sigma_{j_2}^z + v_{j_1}^k v_{j_2}^k \sigma_{j_1}^z \sigma_{j_2}^z \tag{11}$$

After rescaling by a factor of 4 and dropping the constant term we obtain:

$$h_{j_i} = -\sum_k v_{j_i}^k, \quad J_{j_1 j_2} = \sum_k v_{j_1}^k \cdot v_{j_2}^k, \tag{12}$$

# Reduction – algorithm

**procedure** InitIsing(F)

$M \leftarrow$ no. of clauses for formula$F$

$N \leftarrow$ no. of variables for formula $F$

$h \leftarrow$ create a zero vector size N

$J \leftarrow$ create zero marix size $N \times N$

$v \leftarrow$ create zero matrix size $M \times N$

**return** $N, m, J, h, v$

```
procedure 2CNFtoIsing(φ)
    for j = 1..M do
        get index of variables i_1, i_2
        var_1, var_2 ← +1, -1, 0 as in v_j^k 7
        v[j, i_1], v[j, i_2] ← var_1 ← var_2
    for j = 1..n do
        for i = 1..m do
            j_1 ← −1
            j_2 ← 0
            if v[j, i] <> 0 & j_1 == −1 then
                j_1 ← i + 1 continue
            if v[j, i] <> 0 & j_1 <> −1 then
                j_2 ← i + 1 break
        J[j_1, j_2] ← J[j_1, j_2] + v[j, j_1] · v[j, j_2]
        h[j_1] ← h[j_1] − v[j, j_1]
        h[j_2] ← h[j_2] − v[j, j_2]
    return J, h
```

# Reduction- step 5

## Remark

Note that we can equally transform the equation as:

$$H_{Pr} = \frac{1}{4} M \mathbb{1} + \frac{1}{4} H_p$$

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The result is $v, J$ and $h$, whereby $v$ is a size matrix $4 \times 3$

$$v = \begin{array}{c} \begin{array}{ccc} v_1^k & v_2^k & v_3^k \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \leftarrow \Omega_1 \\ \leftarrow \Omega_2 \\ \leftarrow \Omega_3 \\ \leftarrow \Omega_4 \end{array}$$

# Theorem

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For a given formula $\phi$ in the form of 2-CNF with $M$ clauses, the number of clauses to be satisfiable is $K$ if and only if when $E_{Ising} = 4(M - K) + M$, where $E_{Ising}$ is solution of the Ising model with coefficients $J, h$.

# Experiments:

- D-Wave:
  https://docs.dwavesys.com/docs/latest/c_gs_2.html

[6] Neural heuristics for SAT solving Sebastian Jaszczur and Michał Łuszczyk and Henryk Michalewski [1]

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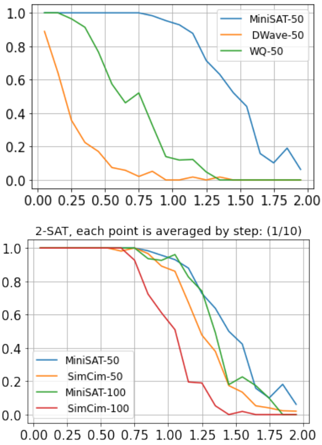
# Comparing 2-SAT with MiniSAT algorithm




(a) The percentage of formulas satisfiable averaged over what $\frac{1}{10}$ depends on picking $\alpha$.

# Comparing MAX-2-SAT with algorithm `akmaxsat`

For selected 100 formulas with $n \in \{300, 500, 700, 1000\}$

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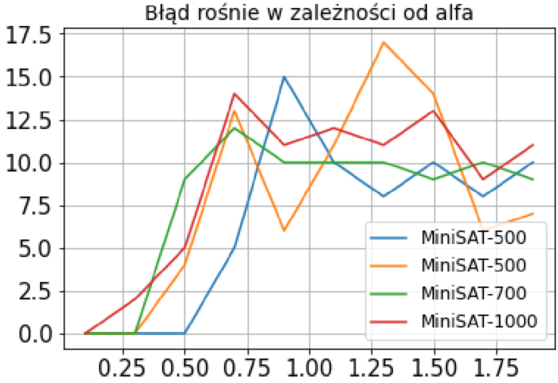


Error vs $\alpha$

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- New experiments with Pegasus Topology

Sebastian Jaszczur, Michał Łuszczyk, and Henryk Michalewski.

Neural heuristics for sat solving, 2020.

Egor S. Tiunov, Alexander E. Ulanov, and A. I. Lvovsky.
Annealing by simulating the coherent ising machine.
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