

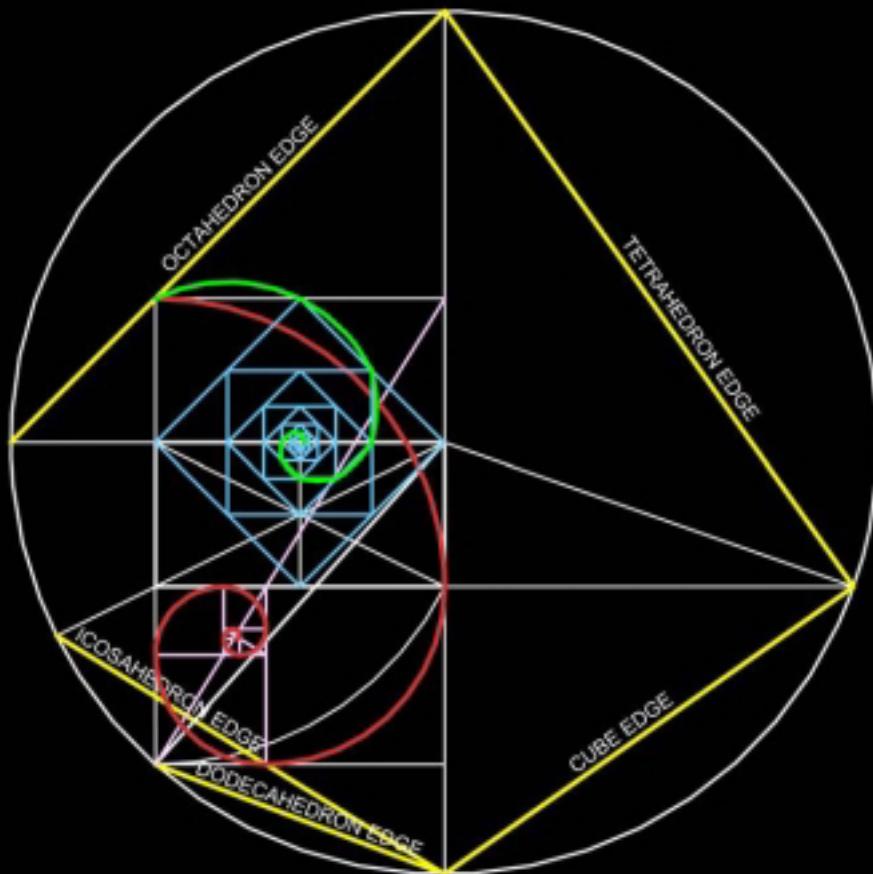
# **Ad Quadratum Construction**

**and**

## **Study of the Regular Polyhedra**

**by**

**Jean Le Mée**



The Louis and Jeanette Brooks  
Engineering Design Center

Department of Mechanical Engineering  
Albert Nerken School of Engineering  
The Cooper Union for the Advancement of Science & Art  
New York, NY 10003

Studies in the Design Process  
and Design Education  
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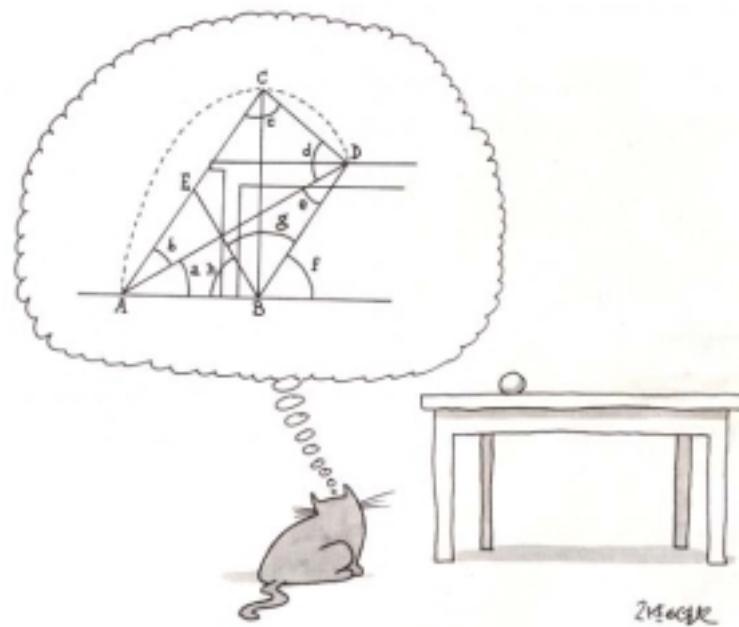
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Particular mention should also be made of my late cat friend Gaya, who always expertly landed on my papers with a most Euclidean precision. Her intuitive understanding of geometry was a great inspiration.



## Preface

The present *ad quadratum* method of generating the regular polyhedra and their study was developed in the context of the design of the Millennium Sphere project.

The Millennium Sphere was Cooper Union's submission for an international time-capsule competition sponsored by the New York Times Magazine. The project was designed in the spring 1999 semester by Professors Jean Le Mée and Manuel Báez, along with a team of Cooper Union students. The proposal was chosen as a finalist (honorable mention) and was included in an exhibition (Dec. 1, 1999-May 31, 2000) at the American Museum of Natural History in New York City. It was featured in the New York Times Magazine (Dec. 5, 1999), and shown at the exhibition "Art and Mathematics 2000" at the Cooper Union (Nov. 7 - Dec. 15, 2000). It is now on permanent exhibition at the Cathedral of St. John the Divine in Manhattan.

The capsule itself was to have a capacity of between 8 and 27 cubic feet, be watertight, airtight and have as few seams as possible. It had to be resistant to changes in temperature, corrosion and pressure. The capsule was to contain a range of printed materials, photographs and artifacts chosen by the editors of the New York Times Magazine. Each object would be individually wrapped and sealed in archival boxes and containers. The capsule should be emptied of oxygen and pumped full of an inert gas such as argon before being sealed.

The capsule was to protect its content for the next millennium. It was suggested that the capsule be beautiful enough to be deemed worthy of preservation without at the same time attracting looters. In any case it should be able to withstand rough treatment.

The capsule had to be located somewhere in Manhattan either in a museum or on "sacred ground."

The technical details of the capsule design are available in the project proposal. The design called for a sealed titanium sphere housed in the center of the Millennium Sphere. In fact, as explained in the booklet presenting the project and available on the web (<http://www.cooper.edu/sphere.html>), three such capsules were proposed, to be located one in the air, one in the earth, and the other in the crypt of a cathedral under an igneous rock, all next to the presence, symbolic or real, of water. To fulfill its function of preservation for a thousand years, the capsule was included into a system that we called **The Millennium Sphere or Ariadne's Clew**.

**The Millennium Sphere**, also called **Ariadne's Clew**, is a memory system to be situated in the Cathedral of St. John the Divine in Manhattan. It is designed not only to hold selected material in a capsule to be opened in one thousand years as required, but to perpetuate the knowledge of the existence of the time capsule itself for all that period, since one of the misfortunes that befalls time capsules is that their location often gets to be forgotten.

The realization of the project results from the interplay of three fundamental elements:

- The millennium sphere proper, holding in its core the sealed material to be preserved for 1,000 years;
- A marker in the form of a labyrinth, to help the remembrance;
- A ritual to keep the memory alive.

The capsule holds the past, the ritual the present, the marker the future.

As holder of the past, **the capsule** design wants to be, through its structure and proportions, an embodiment of traditional knowledge such as found in sacred geometry. We thereby endow it (through this geometric design) with a high esthetic appeal, and capture the essence of some fundamental ideas, which have given birth to – or sustained throughout the ages – some of the most profound speculations in philosophy, science, and religion. The container then becomes content, notwithstanding its payload. It is at the same time symbolically evocative of the heavens and of the atom but also of Ariadne's Clew, which she gave Theseus to help him out of the labyrinth.

The design of the capsule is based on the inner geometry of the five platonic forms, exhibiting their evolution and growth. What our *ad quadratum ex-planation* illustrates is the little-known fact that the entire geometry of the platonic forms results, like the Pythagorean musical scale, from the simple ratio of the first three integers governing the values of the angle of any two consecutive diagonals in each of the respective figures – The internal or Maraldi angle.

From the generating diagram we see a whole compendium of ancient science springing to life:

- The construction of platonic forms and their evolution;
- The development of the labyrinth;
- The spiral of growth and the golden spiral;
- The Heliconic square and Pythagorean musical tuning;
- The determination of Pythagorean triples and many other wonders.

**The Labyrinth**, marker of the site and image of the human condition, reminds us that our own Minotaur – personal and societal – lies at the center and needs to be dealt with.

**The ritual**, link between sphere and labyrinth, between heaven and earth, keeps the memory alive. Every fifty years, the sphere will be brought down to touch the labyrinth center – heaven on earth but also atom on earth – reminder of the possibilities and risks, of the promises and atonement marked by the Jubilee year.

In the few weeks available during the course of the project there was no time to elaborate on the principles and the methods used in the design. A more thorough presentation is now possible. This possibility and the desire of many viewers to get acquainted with the

principles of design and the meaning of the Millennium Sphere has now prompted the present work which specifically addresses the geometry of the sphere. It is therefore not a work of mathematical speculation on the platonic forms aiming at abstract generalized formulation but the reflections of a designer attempting to clarify in his own mind the intricate and beautiful connections existing in some apparently simple figures.

What it brings out however is a “view from the center” that considerably simplifies and unifies the understanding of the structure of all regular polyhedra – convex and stellated – and integrates it quite naturally into a rich tradition of speculative thought be it geometrical, musical, astronomical or philosophical that has been part of our intellectual heritage for centuries.

Similar efforts will be dedicated to the labyrinth and the ritual.

Interested readers may view the project at <http://www.cooper.edu/sphere.html>

Other studies and applications based on this work and pursued in collaboration with the “Form Studies Unit” at the Carleton University Center for Applied Architectural Research (CCAAR) in Ottawa will be subsequently issued.

Voltaire has Candide say that each man should cultivate his own garden. Though he may have a point, it is however in digging into my friend Manuel Báez’s *Phenomenological Garden* in search of our Millennium Sphere that stellated regular polyhedra started to grow on me as mistletoe on an apple tree. Though those around me may at times have felt as if polyhedra were sucking away some of my substance or turning into a semi-obsession, these wondrous figures have also provided a most interesting visual and intellectual experience. Of course when it’s over the whole thing appears rather evident at a glance and one wonders why all this slugging into the underbrush of the garden! It is somewhat like a labyrinth – to be explored anon – whose pattern is quite clear from some distance above but seems nothing but endless turns, u-turns and re-turns in the treading of it. Ariadne’s *clew* is then the only *clue* for our reassurance.

The hope here is that the interested reader may take our clues and gather the fruits without too much digging.

# Ad- Quadratum Construction and Study of Regular Polyhedra

*As a geometer who fully concentrates  
In squaring the circle and succeeds not  
Pondering principles that he would need  
Such was I at that new sight:  
Wishing to see how image to sphere conformed  
And how one within the other found its place.*

- Dante, Paradiso XXXII 1 (133-138)

## Introduction:

The Ad Quadratum method is a geometric construction for the platonic polyhedra based on the simple trigonometry of the internal (or Maraldi) angle, i.e., the angle between two consecutive diagonals in a regular polyhedron<sup>1</sup>. This trigonometry is based on the ratio of the first three integers 1, 2, and 3. For example, in a cube, the internal angle  $\hat{C}_i$  is such that  $\cos \hat{C}_i = \frac{1}{3}$  (fig. 1A.) For the tetrahedron,  $\cos \hat{T}_i = -\frac{1}{3}$  (fig. 1B); For the octahedron,  $\sin \hat{O}_i = 1$ ; For the dodecahedron,  $\sin \hat{D}_i = \frac{2}{3}$ ; and for the icosohedron,  $\tan \hat{I}_i = 2$ .

Through properties of duality, it is easy to show that the internal angle of one solid are related to the dihedral angle<sup>2</sup> (fig. 1C) of its dual through simple relationships.

Thus the dihedral angle of the cube  $\hat{C}_D = \hat{O}_i$ , the octahedron internal angle; conversely, the octahedron dihedral angle  $\hat{O}_D = \pi - \hat{C}_i$  where  $\hat{C}_i$  is the cube internal angle. Analogous relationships obtain for all pairs of duals as we shall see later.

The Ad-Quadratum method allows for easy compass and ruler construction of all regular polyhedra. It yields much more as we shall establish: from the construction of the Pythagorean triples, the gnomic golden rectangle series, the golden spiral, the exponential spiral, the tuning of the monochord, and much more. All this based on “the little matter of distinguishing one, two and three” to use the words of Socrates in The Republic<sup>3</sup>.

---

<sup>1</sup> i.e., those pairs of symmetry axes passing through the center of the circumsphere and two adjacent vertices belonging to the polyhedron under consideration. Mr. James Armstrong of London first drew my attention to the interest of the internal angles viewpoint.

<sup>2</sup> The dihedral angle is the angle between two intersecting planes (here the adjoining faces) measured between two lines contained in the planes and mutually perpendicular at a common point to the line of intersection. On the other hand, the dual or reciprocal of a polyhedron is another polyhedron, the vertices of which are the centers of the faces of its dual and conversely. Thus cube and octahedron, dodecahedron and icosahedron are duals of one another while the tetrahedron is its own dual, being self-reciprocating.

<sup>3</sup> Plato: Republic vii 522

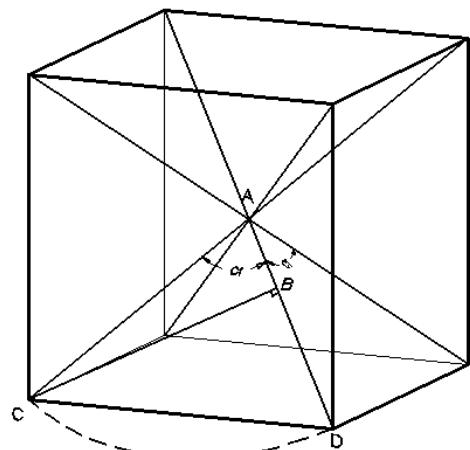
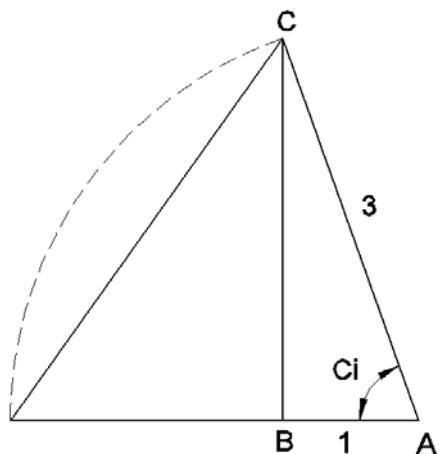


fig. 1A

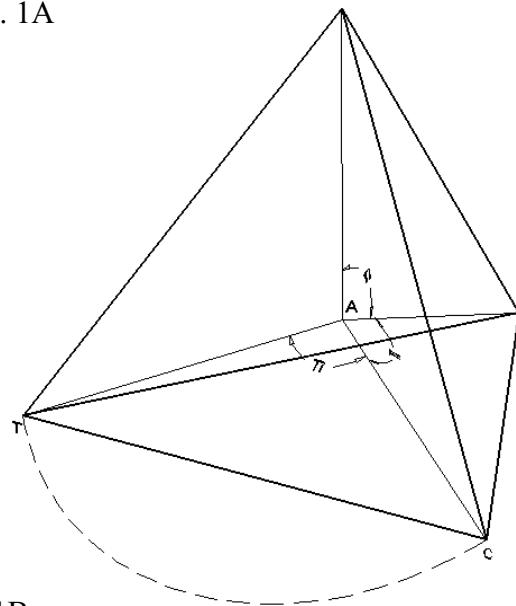
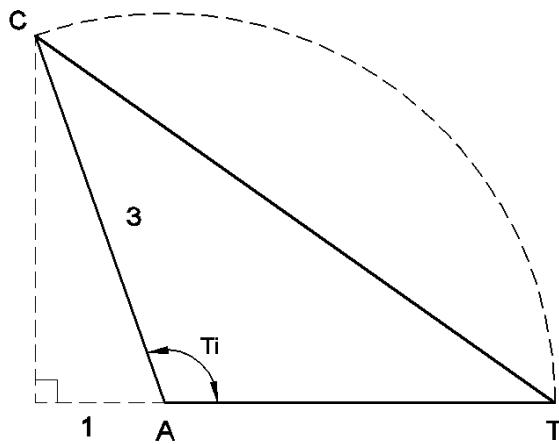


fig. 1B

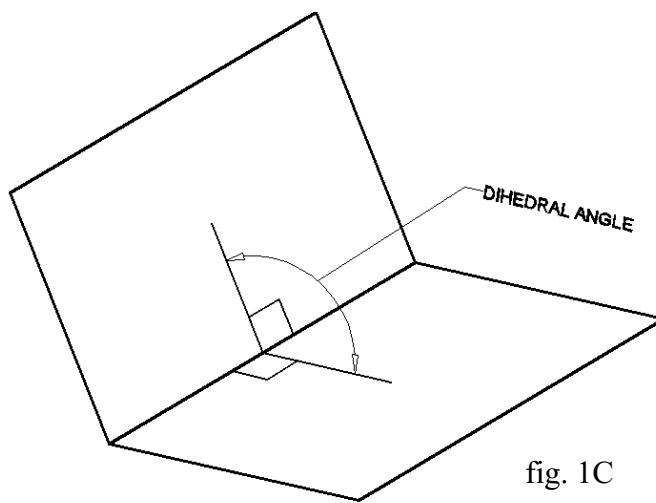


fig. 1C

## Generation of the Platonic Forms:

### 1. A view from the center

To appreciate the richness of this method, we begin by considering the generation of the five platonic forms or regular convex polyhedra (fig. 2). The symmetry groups or patterns of rotations and reflections inherent in these geometric forms are well known. Nevertheless, the viewpoint tends to remain external: Solids bounded by regular surfaces<sup>4</sup> - a view inherited no doubt from Euclid himself<sup>5</sup>.

There are, however, other possible viewpoints useful to the designer: either from a vertex, a face, or in our case, the center of the circumsphere common to the five regular convex polyhedra.

The move is not unlike Copernicus' bold step who sat himself on the sun and saw the complex ptolemaic universe dissolves into a well-ordered ballet of heliocentric orbits.

Taking therefore a central, internal viewpoint, seating ourselves, so to speak, at the center of our polyhedral universe, we consider the symmetries generated for each platonic form by the radii issuing from the circumsphere center to two adjacent vertices respectively (fig. 3).

These lines bursting from the center create by impact on the circumsphere the regular polyhedra. They delimit within the volume of the sphere regular pyramids with triangular, square or pentagonal basis as the case may be (fig. 1A, 1B, 3). Each of the platonic forms is constituted by a bunch of these pyramids (4 for the tetrahedron, with triangular basis; 6 for the cube, with square basis; 8 for the octahedron with triangular basis; 12 for the dodecahedron with pentagonal basis; and finally, 20 for the icosahedron with triangular basis), all with a common apex at the center of the circumsphere. Together they represent the only five possibilities of dividing Euclidian space equally.

This view of the fivefold division of space allows for a new definition of regularity, which applies to both convex and stellated polyhedra, as we shall see later.<sup>6</sup>

A polyhedron, convex or stellated, is regular if:

1. all its vertices lie on a common sphere;
2. the angles between any pair of consecutive radii joining the center of the sphere to the vertices (the Maraldi angle) are all equal.

<sup>4</sup> That is why we prefer to use the word *form* to that of *solid* traditionally used because it is more suggestive of the dependence of the shape of the polyhedra on their internal structure rather than on their external appearances.

<sup>5</sup> See, for instance, Euclid's account of the construction of the five platonic solids in T.L. Heath: A Manual of Greek Mathematics. Dover Pub. New York, 1963, pp. 251-254.

<sup>6</sup> See Polyhedra and Regularity – *Infra*, p. 43.

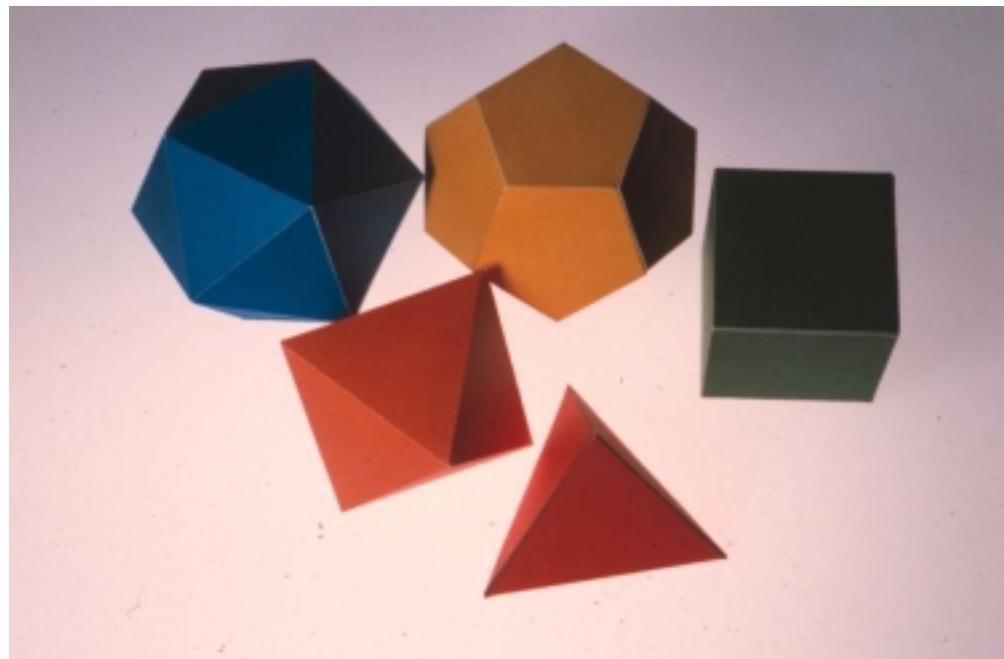


fig. 2

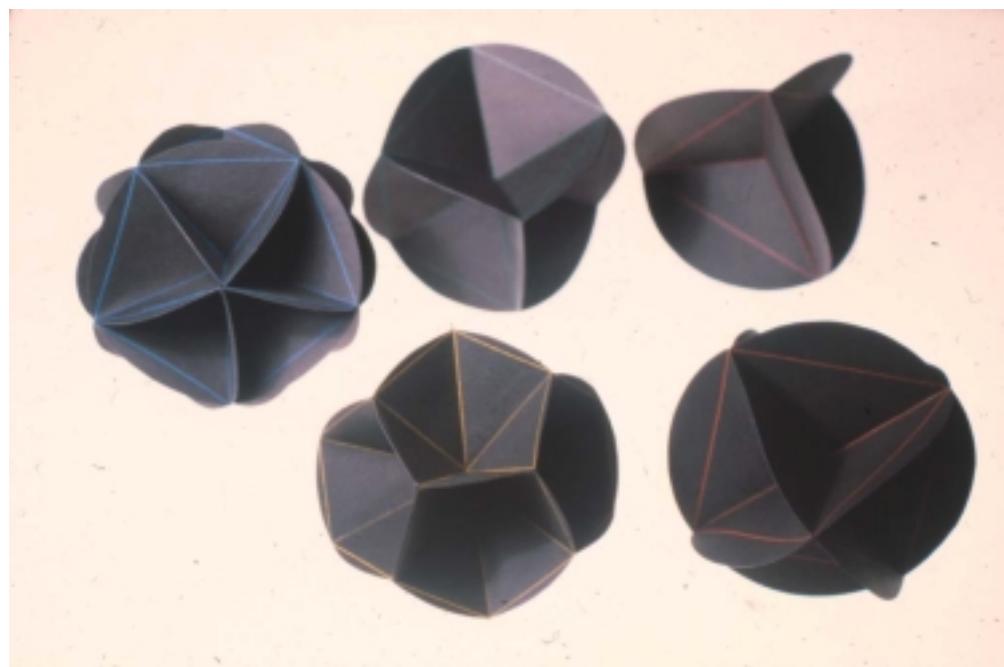


fig. 3

...which can be more briefly put as:

**To be regular, a polyhedron must be spherical and equimarialdian.**

*It will be convex if the network of lines (edges) projected on the sphere joining the vertices is locally planar (i.e., the segments between vertices do not cross). It will be stellated otherwise.*

The well-known duality properties ensure that the vertex radius of one figure is the face center radius of its dual.

**2. Complementary views:**

Other viewpoints are, of course, of conceptual interest to the designer. These methods of generation will be examined subsequently (See *Alternative Methods of Generating the Platonic Forms*). We can mention here in passing, the generation of platonic forms as the result of the interference of the circumsphere with spheres of equal diameter distributed initially tangentially to the circumsphere along axes corresponding to the radii perpendicular to the face of the respective polyhedra and being pressed together from all directions.

Another possibility yet is to start from the six directions of space at a point of origin (i.e., a Cartesian system of coordinates) and their afferent octahedron obtained by measuring equal distances along each axis. Then, along each of these directions, consider another octahedron as in a crystal like structure, compressing them all towards the center at the origin, the interferences eventually generate a cube out of which the tetrahedron and the icosahedron can be generated by very simple construction yielding finally the dodecahedron.

These methods of generation as well as others will be subsequently examined. At the present moment, however, we want to concentrate on the internal view and its relation with the Ad Quadratum method of construction. To this end, we begin by considering the convex polyhedra in relation to their common circumsphere.

**Platonic Forms and Circumsphere:**

The relationships between the radius of the circumsphere and the edge of the regular convex polyhedra have been known since Euclid at least. Theatetus, in the 4<sup>th</sup> Century B.C. , is generally credited with the discovery<sup>7</sup>. Designating by  $R$  the radius of the circumsphere common to all and by  $a$  the edge of the respective polyhedra we have:

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<sup>7</sup> See for instance T.H. Heath, Greek Mathematics, p. 106 and 134. Also, *infra*, Appendices: Circumradius and Edge Relationships.

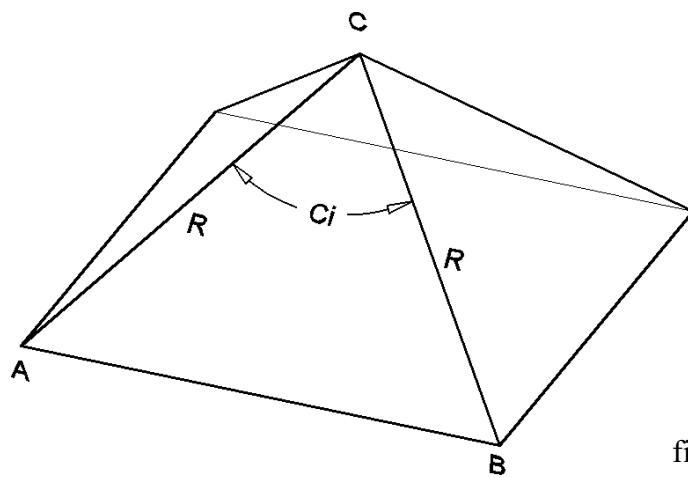


fig. 4A

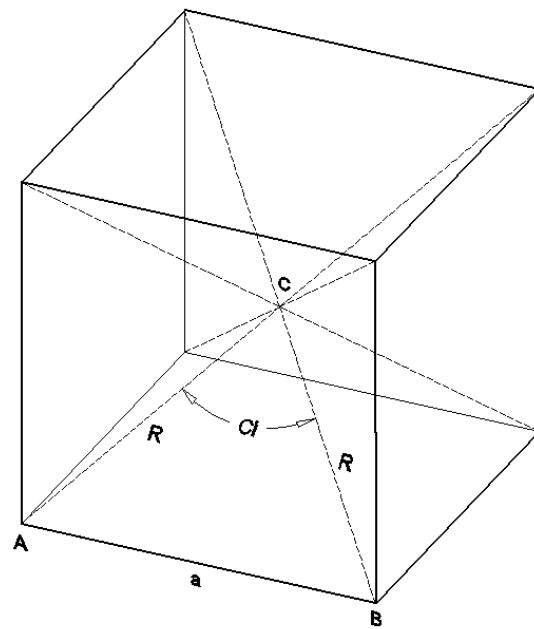


fig. 4B

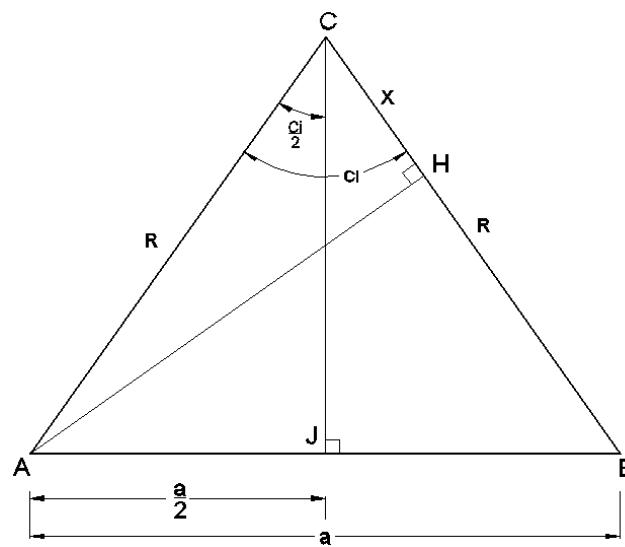


fig. 5

Tetrahedron:	Cube:	Octahedron:	Dodecahedron:	Icosahedron:
$R = \frac{a}{4}\sqrt{6}$	$R = \frac{a}{2}\sqrt{3}$	$R = \frac{a}{2}\sqrt{2}$	$R = \frac{a\sqrt{3}(1+\sqrt{5})}{4}$	$R = \frac{a}{4}\sqrt{2(5+\sqrt{5})}$
$a = \frac{2}{3}\sqrt{6}R$	$a = \frac{2\sqrt{3}}{3}R$	$a = \sqrt{2}R$	$a = \frac{\sqrt{3}}{3}(\sqrt{5}-1)R$	$a = \frac{\sqrt{10}}{5}\sqrt{5-\sqrt{5}}R$

We now proceed to show that the internal structure of the 5 regular convex polyhedra can be accounted for through the simple trigonometric ratios defined within the pyramids whose bases are the polyhedra faces and the apexes the center of the circumsphere.

### **Cube:**

We begin with the cube, more familiar and easy to visualize.

It will be seen that the cube can be considered as made up of 6 equal pyramids (fig. 4A) with square bases (the faces of the cube) and a common apex at C the meeting point of the diagonals and center of the circumsphere (fig. 4B). Let A and B be two contiguous vertices on the cube. Then  $AB = a$ , the cube edge and  $CA = CB = R$ , with  $R$ =radius of circumsphere. Let  $\hat{ACB} = \hat{C}_i$  be the cube internal angle. Projecting ABC in true size, we obtain figure (fig. 5):

Drop the perpendiculars  $CJ$  on  $AB$  and  $AH$  on of  $CB$ .

Now consider  $\sin \frac{\hat{C}_i}{2} = \frac{a/2}{R}$  or  $\frac{a}{2R}$ . But  $\cos 2A = 1 - 2 \sin^2 A$  (Dwight 403.22)<sup>8</sup>

$$\therefore \cos \hat{C}_i = 1 - 2 \sin^2 \frac{\hat{C}_i}{2} \quad (1)$$

$$\cos \hat{C}_i = 1 - 2 \frac{a^2}{4R^2} = 1 - \frac{a^2}{2R^2}$$

$$\text{Now } a = \frac{2\sqrt{3}R}{3} \text{ or } a^2 = \frac{4 \cdot 3}{9}R^2 = \frac{4}{3}R^2$$

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<sup>8</sup> Dwight, H.B.: Tables of Integrals and other mathematical data. The MacMillan Co. New York 1947

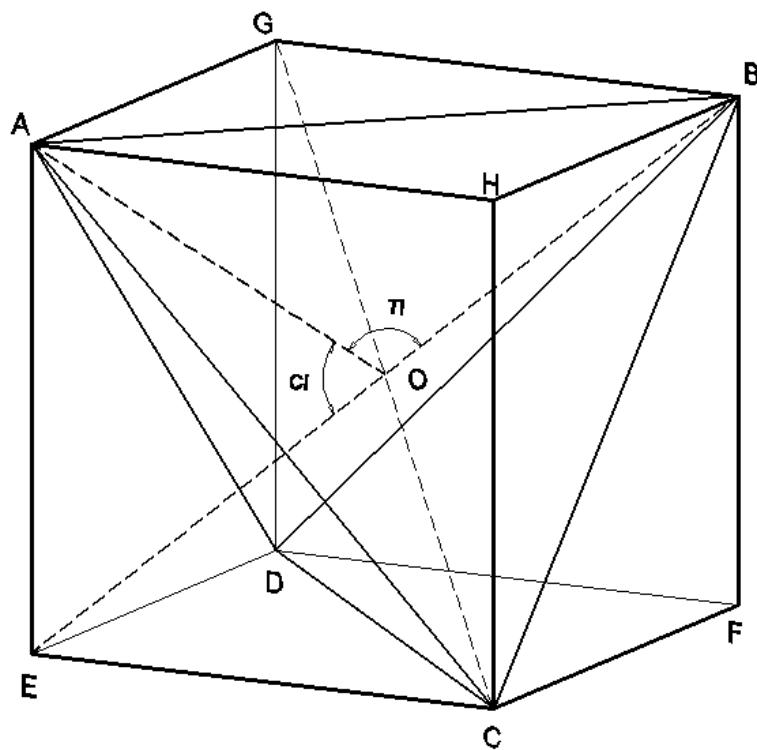


fig. 6

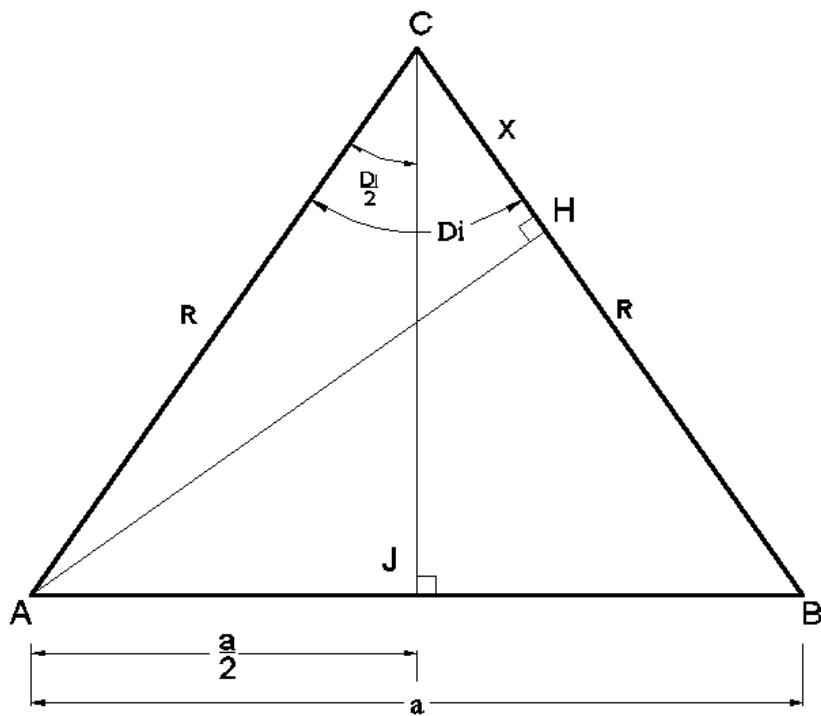


fig. 7

$$\therefore \text{from (1)} \cos \hat{C} = 1 - \frac{\frac{4}{3}R^2}{2R^2} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\boxed{\cos \hat{C}_i = \frac{1}{3}}$$

Designate  $CH$  by  $x$ , then  $\cos \hat{C} = \frac{x}{R}$  or  $x = R \cos \hat{C}$

So that if  $R=3$ , we have:  $x=1$ .

### **Tetrahedron:**

Passing to the tetrahedron, we can see by inspection that

$$\hat{T}_i = \pi - \hat{C}_i$$

This follows from looking at the cube containing the regular tetrahedron whose edges are the diagonals of the faces of the cube and is therefore circumscribed by the same sphere (fig.6).

The Tetrahedron is ABCD, the cube AGBHCEDF. O is the center of the circumsphere common to both. Consider cube diagonal EB and the half diagonal AO. They form a plane with  $A\hat{O}E = \hat{C}_i$  and  $A\hat{O}B = \hat{T}_i$ .

Since EB is a straight line through O it is seen that  $\hat{C}_i + \hat{T}_i = \pi$ , or  $\hat{T}_i = \pi - \hat{C}_i$ .

It follows that

$$\boxed{\cos \hat{T}_i = -\frac{1}{3}}$$

### **Octahedron:**

The case of the octahedron is trivial in the sense that its internal angle  $\hat{O}_i$  is a right angle,

$$\hat{O}_i = \frac{\pi}{2}.$$

$\therefore$

$$\boxed{\sin \hat{O}_i = 1}$$

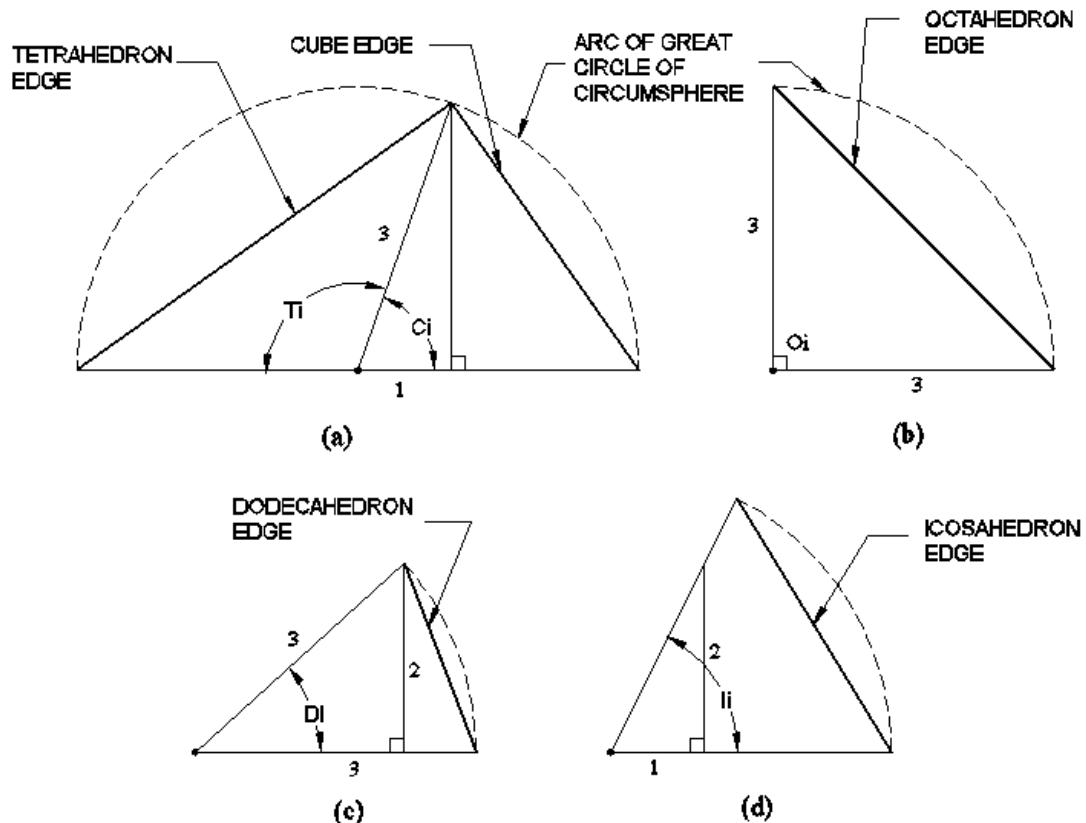


fig. 8

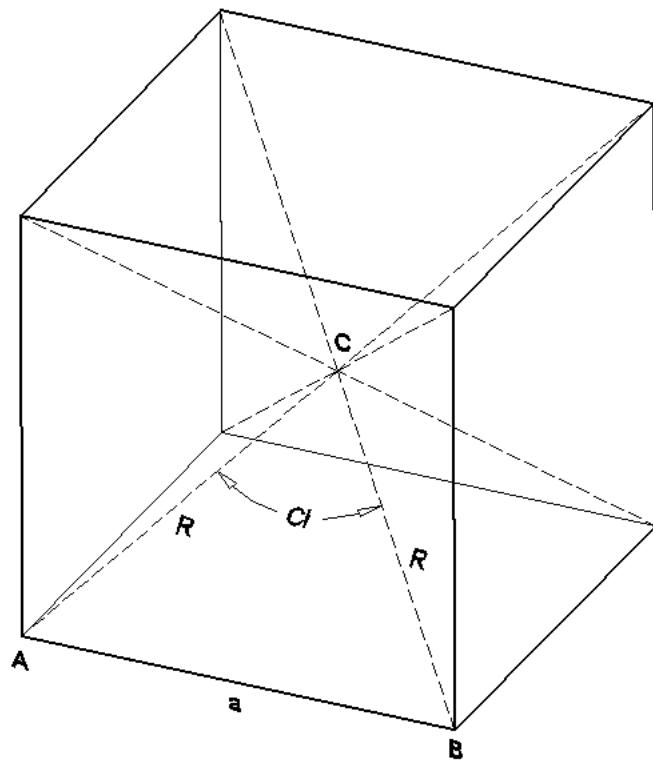


fig. 9

### Dodecahedron:

For the dodecahedron, we proceed as for the cube calling in this case its internal angle  $\hat{D}_i$ , the angle between two consecutive diagonals. These diagonals constitute the sides of a triangle, itself side of a pyramid with apex at the center of the circumsphere and having for base a regular pentagon, face of the dodecahedron.

Though neither equal nor similar, the isosceles triangle, face of the pyramid just described, has the same geometry as that for the cube previously examined (fig. 7). We can therefore write:

$$\cos \hat{D}_i = 1 - 2 \sin^2 \frac{\hat{D}_i}{2} = 1 - \frac{a^2}{2R^2}$$

Here, however:

$$a = \frac{\sqrt{3}}{3} (\sqrt{5} - 1) R \quad \therefore \quad a^2 = \frac{1}{3} (6 - 2\sqrt{5}) R^2$$

and therefore:

$$\frac{a^2}{2R^2} = 1 - \frac{\sqrt{5}}{3}$$

or:

$$\cos \hat{D}_i = 1 - \left(1 - \frac{\sqrt{5}}{3}\right) = \frac{\sqrt{5}}{3}$$

In triangle ACH we therefore have:

$$AC = R; \quad CH = \frac{\sqrt{5}}{3} R; \quad \text{and} \quad AH^2 = AC^2 - CH^2,$$

$$\text{or: } AH^2 = R^2 - \frac{5}{9} R^2 = R^2 \left(1 - \frac{5}{9}\right) = R^2 \left(\frac{4}{9}\right) \quad \therefore \quad AH = \frac{2}{3} R;$$

where from

$$\sin \hat{D}_i = \frac{AH}{AC} = \frac{2}{3}$$

$$\sin \hat{D}_i = \frac{2}{3}$$

### Icosahedron:

Proceeding similarly for the icosahedron, we have:

$$\cos \hat{I}_i = 1 - 2 \sin^2 \frac{\hat{I}_i}{2} = 1 - \frac{a^2}{2R^2},$$

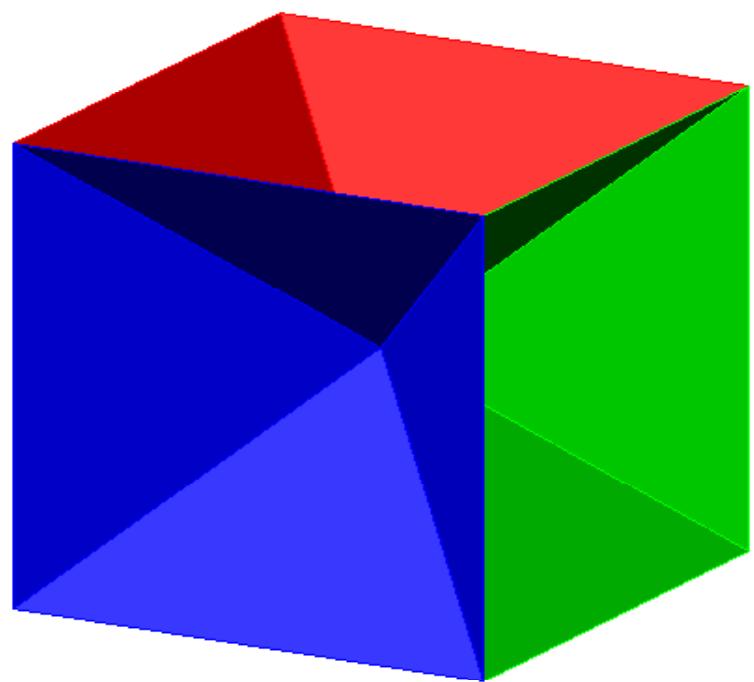


fig. 10

$$\text{with } a = \frac{\sqrt{10}}{5} \sqrt{5 - \sqrt{5}} R \quad \therefore \quad a^2 = \frac{10}{25} (5 - \sqrt{5}) R^2$$

$$\frac{a^2}{2R^2} = \frac{10}{50} (5 - \sqrt{5}) = \frac{5 - \sqrt{5}}{5} = 1 - \frac{\sqrt{5}}{5}$$

$$\therefore \cos \hat{I}_i = 1 - \left(1 - \frac{\sqrt{5}}{5}\right) = \frac{\sqrt{5}}{5}$$

$$\begin{aligned} \text{Again, in triangle ACH: } AC &= R; \quad CH = \frac{\sqrt{5}}{5} R \\ \therefore AH^2 &= R^2 - \frac{5}{25} R^2 = R^2 \left(1 - \frac{1}{5}\right) = \frac{4}{5} R^2 \end{aligned}$$

$$AH = \frac{2\sqrt{5}}{5} R,$$

$$\text{but } \frac{AH}{CH} = \tan \hat{I}_i = \frac{\frac{2\sqrt{5}}{5} R}{\frac{\sqrt{5}}{5} R} = 2$$

$$\therefore \boxed{\tan \hat{I}_i = 2}.$$

The internal angles of the five regular polyhedra can therefore be constructed very simply on the basis of right angle triangles. To simplify the construction and deal with integers, as ancient mathematicians would prefer (rather than with fractions), we obtain the results shown on fig. 8.

It now becomes possible to build the five regular polyhedra with compass and ruler and the simple measures 1, 2, and 3.

### **Construction of the Polyhedra – Example of the Cube**

Before proceeding to the Ad-Quadratum construction we build a cube made up of 6 square based pyramids with apexes at the center of the circumsphere and angle between two adjacent edges equal to  $\hat{C}_i$  (fig. 4B). Each such a pyramid (fig. 9, repeated from fig. 4A) is made up of 4 triangles as shown in fig. 8 (a).

Six of these joined together with a common apex constitute a cube (figs. 3 and 10).

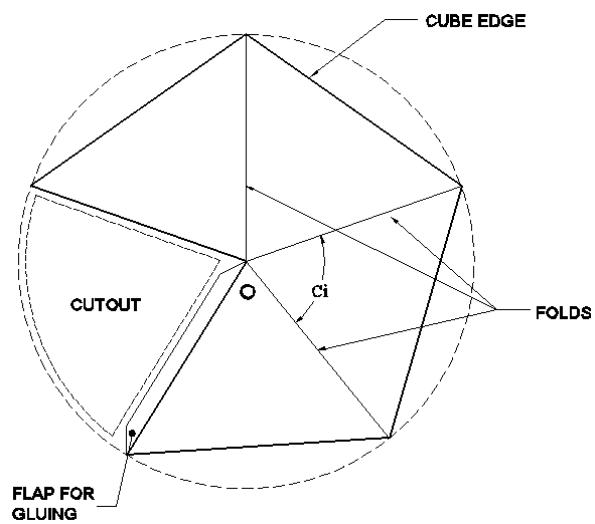


fig. 11

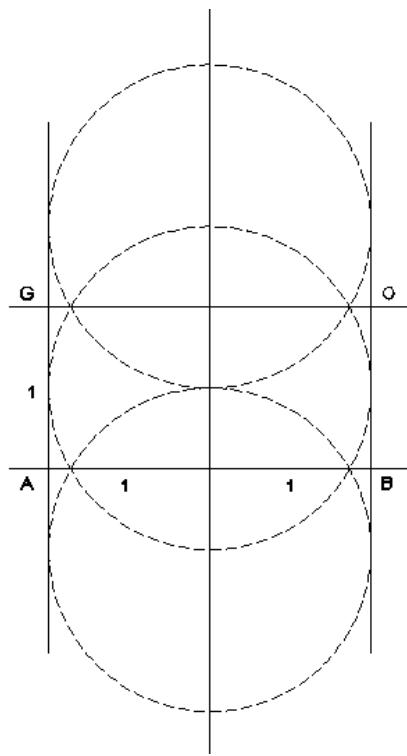


fig. 12

A simple way of constructing each individual pyramid is to plot four times  $\hat{C}_i$  along a great circle of the circumsphere (fig. 11). When cut and folded, we obtain the pyramid (including the arc of the great circle subtended by the cube edge). Six such constructions will yield the cube.

## The Ad Quadratum Construction

The Ad Quadratum construction begins with the construction of a double square each of side unity (fig. 12).

We therefore begin with a circle of unit radius to which we add two intersecting and equal circles, centered at the extremity of a diameter, respectively, so as to form a double vesica. The intersection of the center line and of the tangents to the circles with the secants through the 2 vesicas determine the double squares:  $A B O G$ .

It is on the basis of these two squares that the construction evolves (fig. 13). Together they form a rectangle  $ABOG$  with sides equal to 1 and 2 respectively.

$$\therefore \text{ Diagonal } GB = \sqrt{5} .$$

We can already note in passing that  $\tan A\hat{G}B = 2$   
and therefore that  $A\hat{G}B = A\hat{O}B = \hat{I}$ , the internal angle of the icosahedron.

Now swing  $GB$  down in the extension of  $GA$  to point  $D$ .

$$\text{In triangle } OGD, GD = \sqrt{5}, GO = 2.$$

$$\text{Therefore } DO = 3.$$

And therefore,

$$G\hat{D}O = D\hat{O}B = \hat{D} ,$$

the internal angle of the dodecahedron, since

$$\sin G\hat{D}O = \frac{GO}{DO} = \frac{2}{3} .$$

Now describe the circle centered at  $O$  with radius  $OD$  and extend  $AB$  to  $S$ ;  $OB$  to  $R$  and  $U$ ;  $OG$  to  $V$  and  $OA$  to  $W$ , where these extensions intercept the circle.

Then draw  $OS$ .

It will be seen that in triangle  $BOS$ ,

$$\cos B\hat{O}S = \frac{1}{3} \text{ and therefore } B\hat{O}S = \hat{C}_i , \text{ the internal angle of the cube.}$$

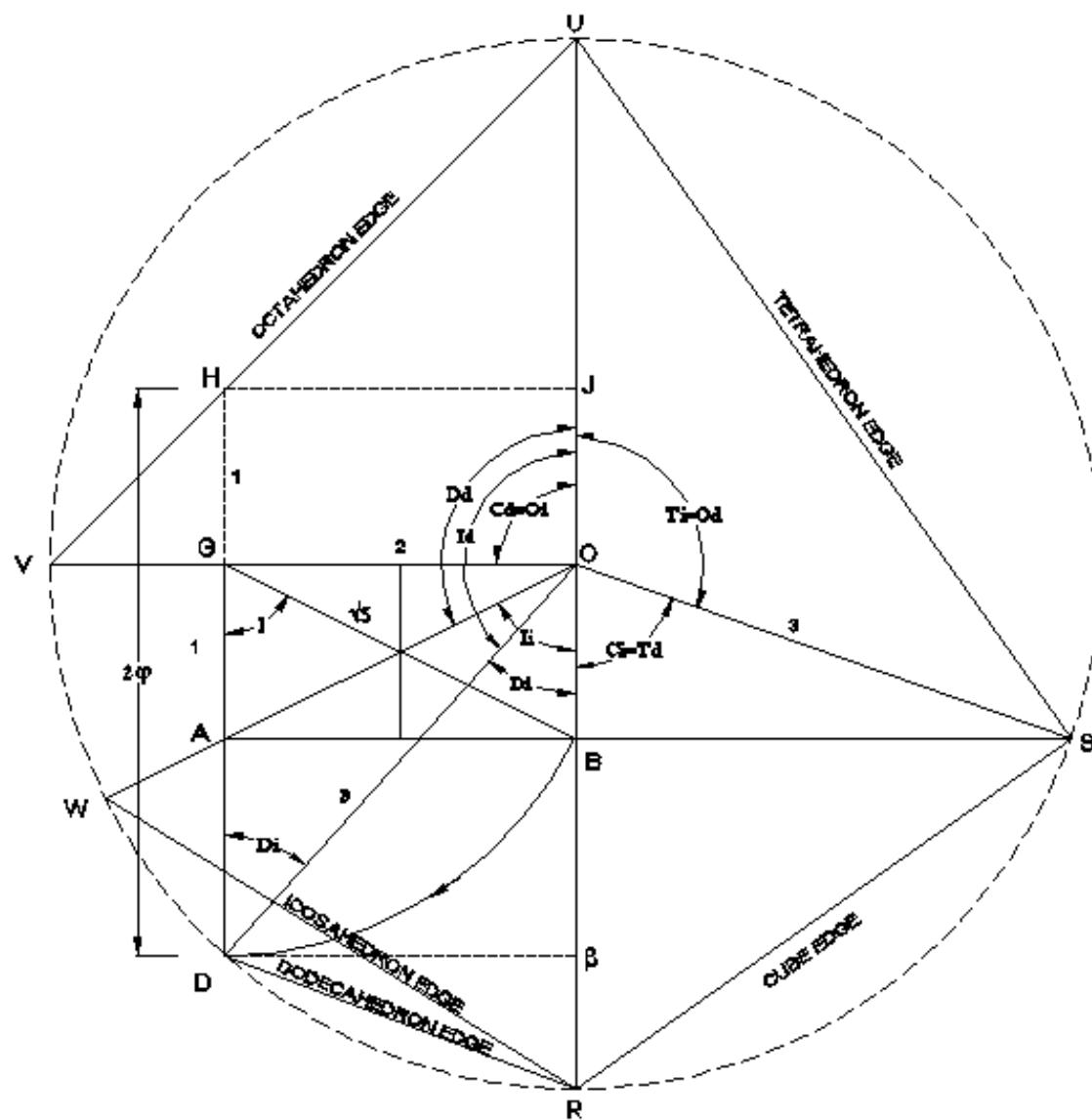


fig. 13

Similarly,  $\cos U\hat{O}S = -\frac{1}{3}$ , and therefore  $U\hat{O}S = \hat{T}_i$ , the internal angle of the tetrahedron.

Finally,

$$V\hat{O}U = \frac{\pi}{4} = \hat{O}_i,$$

the internal angle of the octahedron.

Noting that

$$G\hat{D}O = D\hat{O}R = \hat{D}_i,$$

and

$$D\hat{G}B = W\hat{O}R = \hat{I}_i,$$

we see the internal angles of the 5 regular polyhedra displayed on the diagram together with the chords subtending these angles representing the edges of the respective polyhedra, and the circumscribing circle, the great circle of the circumsphere.

We can also note as previously mentioned that the dihedral angles of the respective polyhedra can be read off this graph directly in virtue of the duality principle.

The dodecahedron and the icosahedron being dual of one another, it follows that the angle between the perpendiculars to two adjacent faces of the dodecahedron *is* the internal angle of the icosahedron. (Note that the circles in figures 14 through 17 are not on the circumsphere, though they are on spheres concentric with it. They lie in planes perpendicular to an edge of the respective polyhedra.)

$$\therefore \hat{I} + \hat{D}_D = \pi,$$

or:

$$\boxed{\hat{D}_D = \pi - \hat{I}_i}$$

The dodecahedron dihedral angle  $\hat{D}_D$  is the supplement of the icosahedron internal angle  $\hat{I}_i$ .

Conversely, the angle between the perpendiculars to two adjacent faces of the icosahedron *is* the internal angle of the dodecahedron.

$$\therefore \hat{I}_D + \hat{D}_i = \pi,$$

$$\text{or } \boxed{\hat{I}_D = \pi - \hat{D}_i}$$

The icosahedron dihedral angle  $\hat{D}_i$  is the supplement of the dodecahedron internal angle  $\hat{I}_D$ .

The same will obviously obtain for the cube and the octahedron.

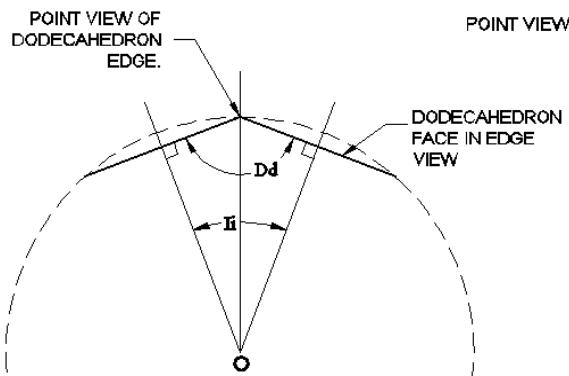


fig. 14

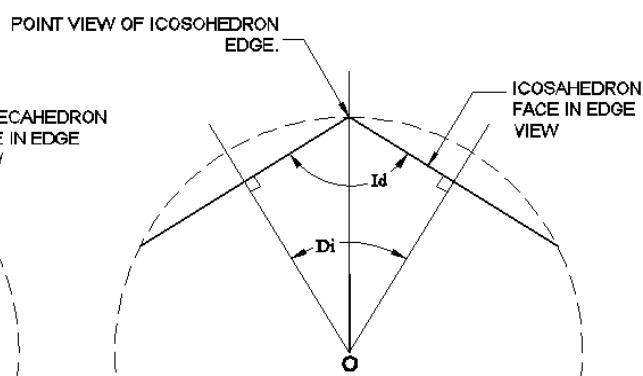


fig. 15

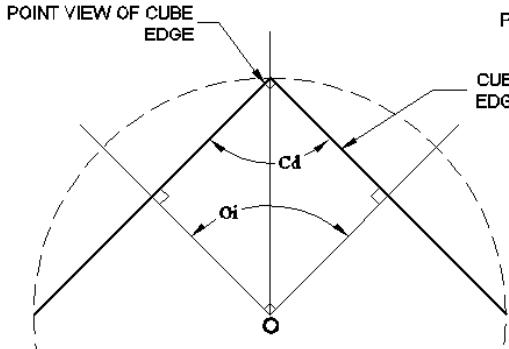


fig. 16

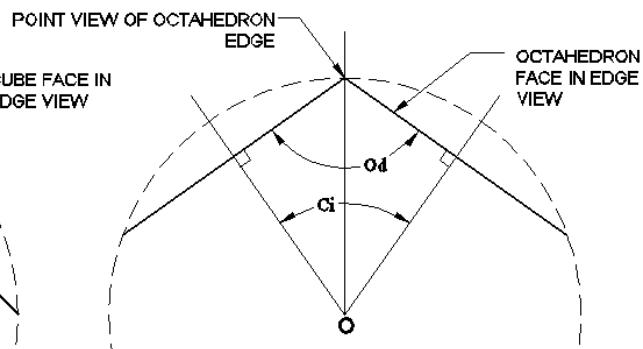


fig. 17

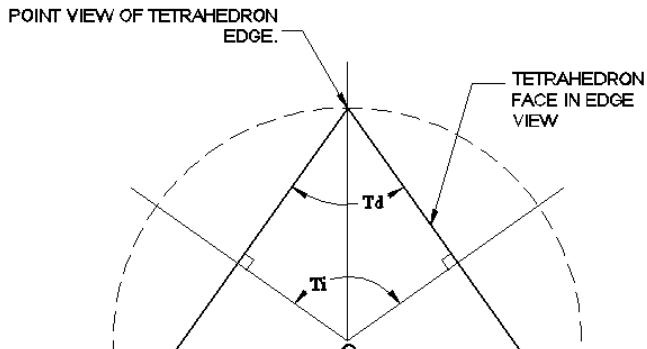


fig. 18

$$\boxed{\hat{C}_D = \hat{O}_i = \frac{\pi}{2}}$$

The cube dihedral angle  $\hat{C}_D$  is equal to the octahedron internal angle  $\hat{O}_i$

$$\therefore \hat{O}_D + \hat{C}_i = \pi$$

But, as we have seen previously in connection with fig. 6,

$$\hat{T}_i = \pi - \hat{C}_i$$

So that we can also conclude

$$\boxed{\hat{O}_D = \hat{T}_i}$$

The octahedron dihedral angle is therefore equal to the supplement of the internal angle of the cube  $\hat{C}_i$  and to the internal angle of the tetrahedron  $\hat{T}_i$ .

The tetrahedron being its own dual, we consider the relation between  $\hat{T}_D$  and  $\hat{T}_i$ . We see that (fig. 18):

$$\hat{T}_D + \hat{T}_i = \pi.$$

But we previously established that

$$\pi - \hat{T}_i = \hat{C}_i \quad \therefore \quad \hat{T}_D = \hat{C}_i$$

The tetrahedron dihedral angle  $\hat{T}_D$  is the supplement of its internal angle  $\hat{T}_i$ , and is also equal to the cube internal angle.

We can therefore establish the following table:

Form	Trigonometric Ratio	Internal Angle	Dihedral Angle
Tetrahedron	$\cos T_i = -\frac{1}{3}$	$T_i = 109.47^\circ$	$T_D = C_i = 70.53^\circ$
Cube	$\cos C_i = \frac{1}{3}$	$C_i = 70.53^\circ$	$C_D = O_i = 90^\circ$
Octahedron	$\sin O_i = 1$	$O_i = 90^\circ$	$O_D = T_i = \pi - C_i = 109.47^\circ$
Dodecahedron	$\sin D_i = \frac{2}{3}$	$D_i = 41.81^\circ$	$D_D = \pi - I_i = 116.56^\circ$
Icosahedron	$\tan I_i = 2$	$I_i = 63.44^\circ$	$I_D = \pi - D_i = 138.19^\circ$

Although the information regarding the dihedral angle is not required for the construction of the polyhedra, it is included here for the sake of completeness and to serve as a verification of their construction.

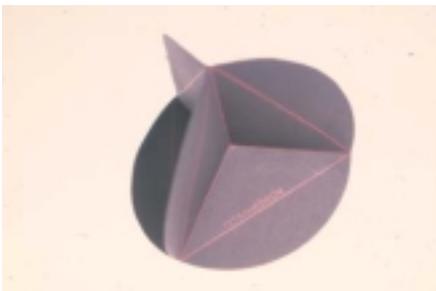


fig. 19A

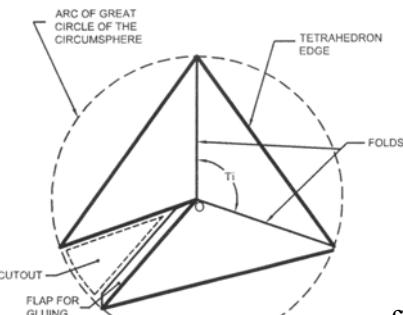


fig. 19B

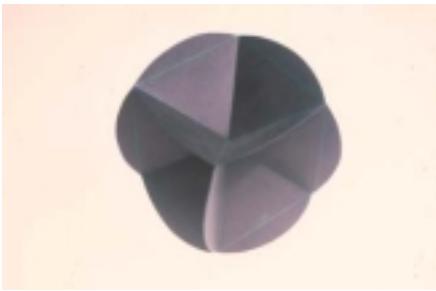


fig. 20A

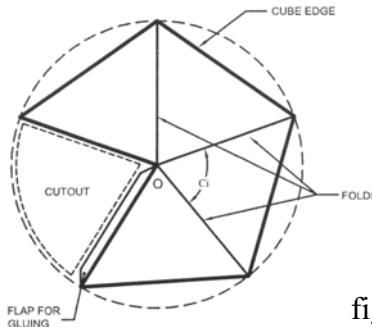


fig. 20B

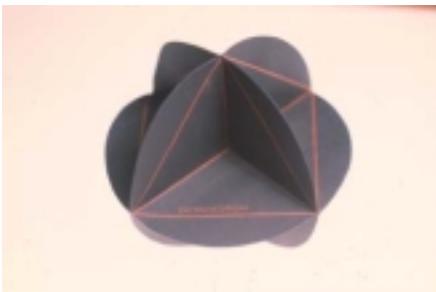


fig. 21A

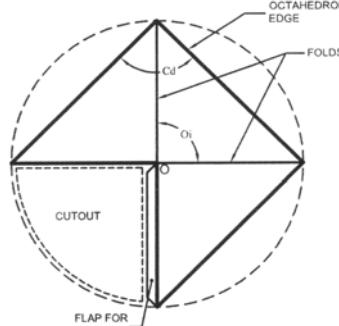


fig. 21B

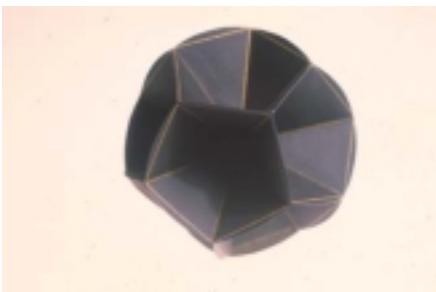


fig. 22A

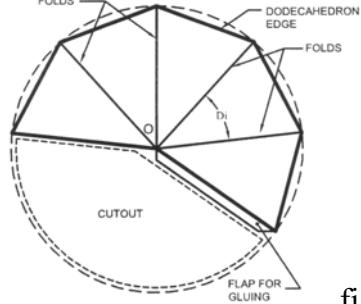


fig. 22B



fig. 23A

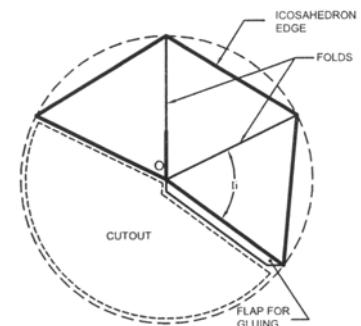


fig. 23B

As indicated for the cube (fig. 10 and fig. 11), we can now proceed with construction of each regular polyhedra.

Thus the tetrahedron will be made up of four pyramids (fig. 19A) built from the development shown (fig. 19B).

For the cube, as already seen, we have 6 pyramids (fig. 20A) with the following development shown (fig. 20B). (Repeats of fig. 10 and 11 respectively).

The octahedron will be made up of 8 pyramids (fig. 21A), developed as shown: (fig. 21B)

The dodecahedron will have 12 pyramids (fig. 22A) developed as shown (fig. 22B):

Finally, the icosahehedron will be made up of 20 pyramids (fig. 23A), with the development as shown: (fig. 23B)

## The Challenge of Abul Wefa

Abul Wefa was a tenth century Islamic philosopher<sup>9</sup> credited with “the feat of drawing all five Platonic solids using only a straightedge and a pair of compass at a fixed setting.” Such fixed compasses (known as “rusty” compasses), adds Hersey<sup>10</sup>, have been the tools of virtuoso geometrical draftsmanship in many periods.

We would like to show here, without laying claim to virtuosity eleven centuries after Abul Wefa, that our Adquadratum method can easily be modified to accomplish the deed.

We’ll consider having met the challenge if we can draw the adquadratum diagram with a “rusty” compass and a straightedge only.

We begin by tracing a line  $xy$  with the straightedge (fig. 24A). Setting our compass opening at  $R$ , radius of the circumsphere common to all the regular forms, we then proceed by drawing four intersecting circles whose centers  $O_1, O_2, O_3, O_4$  are on line  $xy$  and the circumferences of their neighbors as shown on fig. 24A. The three vesicas determine both a square  $ABCD$  of side equal to the circle diameter and the median  $EF$  of the square to which we add diagonals  $AC$  and  $BD$  and square  $EHFG$ .

Now, draw  $IJ$  (it passes through  $O_2$ ) and join  $AO_2$  and  $DO_2$  cutting  $GE$  and  $GF$  at  $K$  and  $L$  respectively. Join  $KL$  cutting  $XY$  at  $M$ . Call center point of square  $ABCD$ ,  $O_5$ .

Then

$$MO_5 = \frac{2}{3} GO_5$$

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<sup>9</sup> George L. Hersey: *Architecture and Geometry in the Age of the Baroque*, Chicago U.P. 2000, p. 88

<sup>10</sup> Hersey, op. cit. p. 88

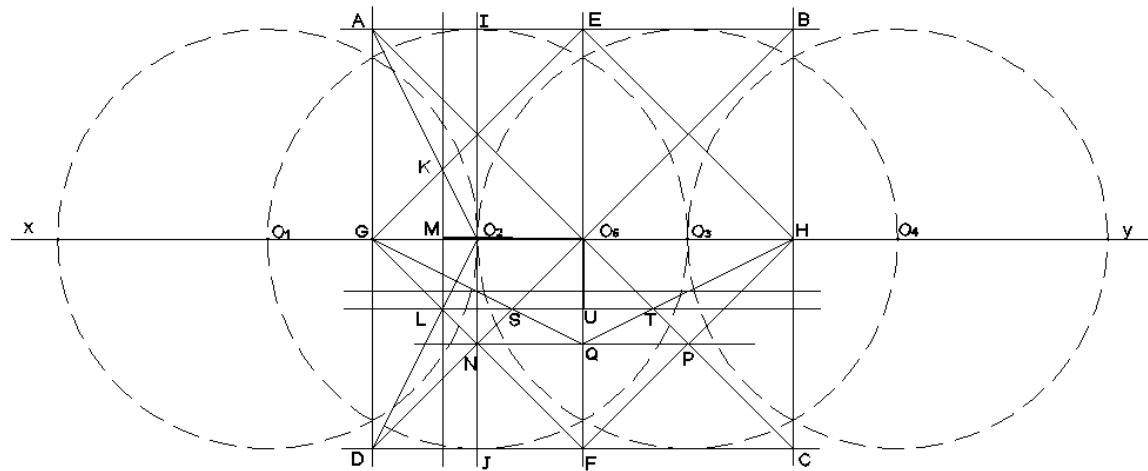


fig. 24A

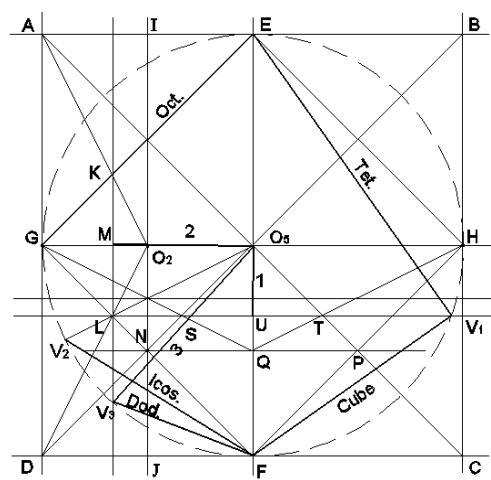


fig. 24B

or  $MO_5 = \frac{2}{3}R$

so that if we (arbitrarily) set  $R=3$ ,  $MO_5 = 2$ . This is easily established through similarity of triangles and is a standard construction for the harmonic series (see *Adquadratum & Music, Infra.*).

Now, repeat a similar construction along  $EF$ , i.e., draw  $NP$  cutting  $EF$  at  $Q$ . Then join  $G$  to  $Q$  and  $H$  to  $Q$ , cutting  $DB$  and  $AC$  at  $S$  and  $T$  respectively.

$ST$  cuts  $EF$  at  $U$ .

Then  $O_5U = \frac{1}{3}O_5F$

or  $O_5U = \frac{1}{3}R$

again with  $R=3$ ,  $O_5U = 1$ .

Now with the same “rusty” compass opening (fig. 24B), draw circle centered at  $O_5$ . It is inscribed within square  $ABCD$ . Extend  $ST$  on both sides so that it cuts circle  $O_5$  at  $V_1$ . On the left, it passes through  $L$ ;  $O_5L$  extended cuts the circumference at  $V_2$  and  $KL$  extensions will cut it at  $V_3$ .

Join  $FV_1, FV_2, FV_3, EV_1, EG$ .

These are the edges respectively of:

The cube, the icosahedron, the dodecahedron, the tetrahedron, and the octahedron. Each of them subtends the central angle that is the internal angle of the corresponding polyhedron.

Abul Wefa challenge has therefore been met.

## Ratio of Insphere to Circumsphere Radii

We take the circumsphere to be common to all five regular convex polyhedra.

The insphere is that sphere concentric with the circumsphere but tangent to the faces of a given polyhedron. Inspheres will therefore be generally different for the various polyhedra, though simple relations can be shown to exist between the radii of both spheres for each of the polyhedra:

Let  $R$  be the radius of the circumsphere.

$r$  be the radius of the inspheres

$a$  be the edge of the polyhedra



fig. 25A

**Tetrahedron:**

For the tetrahedron, as is well-known:

$$\begin{aligned} a &= \frac{2}{3}\sqrt{6}R \text{ and } r = \frac{a}{12}\sqrt{6}, \\ \text{so that } \frac{r}{R} &= \frac{2 \times 6}{3 \times 12} = \frac{1}{3}; \\ \text{and we note that } \frac{1}{3} &= \cos \hat{C}_i \quad \therefore \quad \frac{r}{R} = \cos \hat{C}_i. \end{aligned}$$

Note also that when  $R = 3$  (as in our Ad Quadratum construction),  $r = 1$ ;

While the height of the tetrahedron given by

$$h = \frac{a}{3}\sqrt{6} = \frac{2}{3}\sqrt{6}R \cdot \frac{\sqrt{6}}{3} = \frac{2 \times 6}{9}R = \frac{4}{3}R$$

So that for  $R = 3$ ,  $h = 4$ .

Having started with 1, 2, and 3, we find 4 and with it, the 3<sup>rd</sup> Dimension. Socrates would be pleased!

**Cube:**

$$a = \frac{2\sqrt{3}}{3}R \text{ and } r = \frac{a}{2} = \frac{\sqrt{3}}{3}R$$

$$\therefore \frac{r}{R} = \frac{\sqrt{3}}{3} = \sqrt{3} \cos \hat{C}_i$$

$$\text{noting that } \frac{\sqrt{3}}{2} = \cos \frac{\pi}{6},$$

$$\frac{r}{R} = 2 \cos \frac{\pi}{6} \cos \hat{C}_i$$

**Octahedron:**

$$a = R\sqrt{2}$$

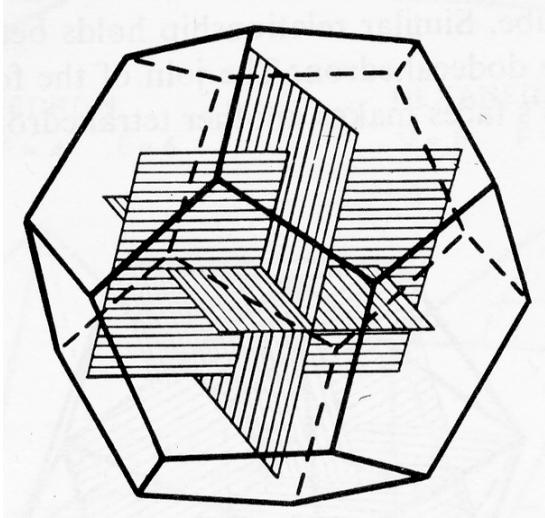


fig. 25B

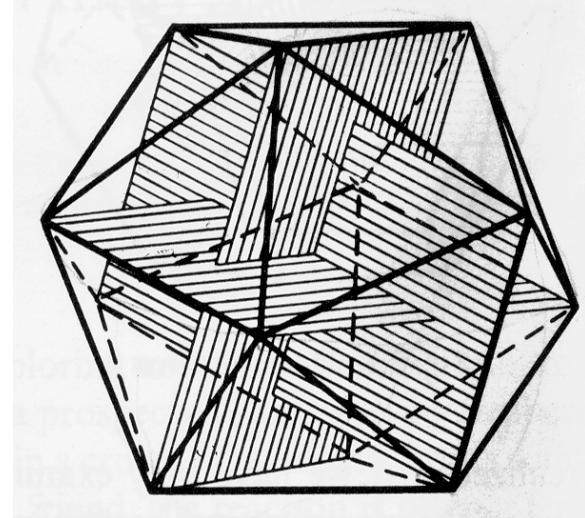


fig. 26

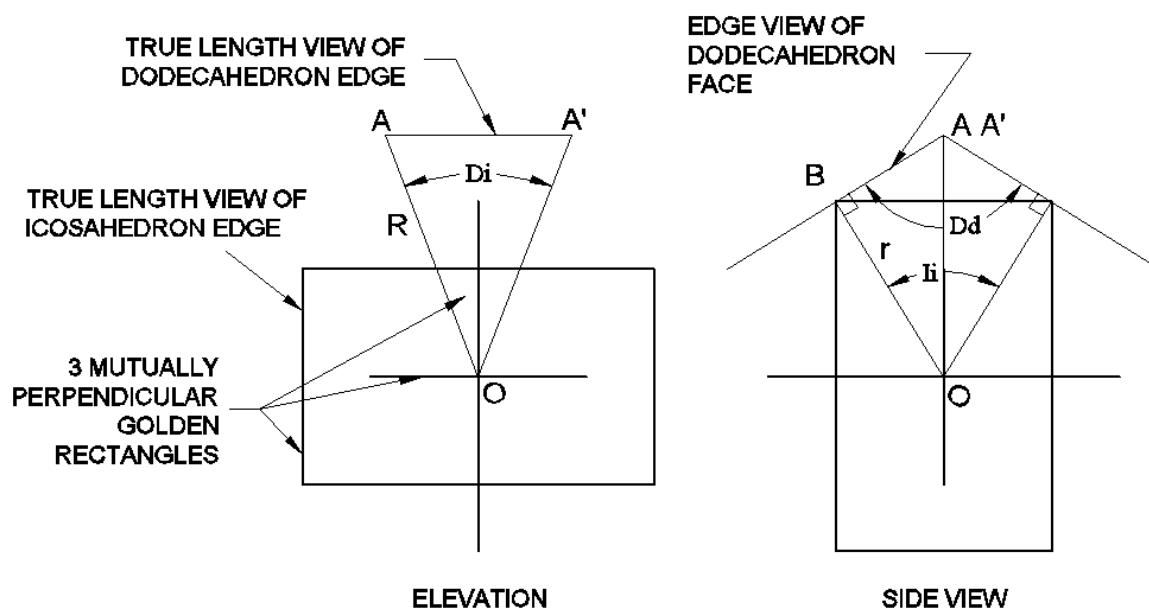


fig. 27

$$\begin{aligned}
 r &= \frac{a}{6} \sqrt{6} = \frac{R\sqrt{2}\sqrt{6}}{6} = \frac{R\sqrt{12}}{6} = \frac{2\sqrt{3}}{6} R \\
 \therefore \frac{r}{R} &= \frac{\sqrt{3}}{3} = \sqrt{3} \cos \hat{C}_i \\
 &= 2 \cos \frac{\pi}{6} \cos \hat{C}_i
 \end{aligned}$$

The cube and the octahedron having a common circumsphere have also a common insphere, and for  $R=3$ ,  $r=\sqrt{3}$ .

### Dodecahedron:

Rather complex formulae exists for the dodecahedron. As can be found in handbooks:

$$a = \frac{\sqrt{3}}{3} (\sqrt{5} - 1) R \quad \text{and} \quad r = \frac{a}{4} \sqrt{\frac{50 + 22\sqrt{5}}{5}}.$$

It is, however, possible to derive a simpler formula based on the knowledge of the internal angles.

In a dodecahedron, the centers of the 12 pentagonal faces are at the corners of three mutually perpendicular golden rectangles (fig. 25A and 25B).

The same obtains for the vertices of the icosahedron<sup>11</sup> (fig. 26). As we previously saw, we have (fig. 27):

In the elevation view,

$$OA = R.$$

$\overline{OAA'}$  (projection of OA on vertical)

$$\therefore \overline{OAA'} = R \cos \frac{\hat{D}_i}{2}.$$

In side view,  $OB = r = \overline{OA} A' \cos \frac{\hat{I}_i}{2}$

$$\therefore \boxed{\frac{r}{R} = \cos \frac{\hat{D}_i}{2} \cos \frac{\hat{I}_i}{2}}$$

Calculation using this formula gives, to the third decimal:

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<sup>11</sup> Huntley, H.E.: The Divine Proportion, pp. 34-33, Dover, NY 1970.

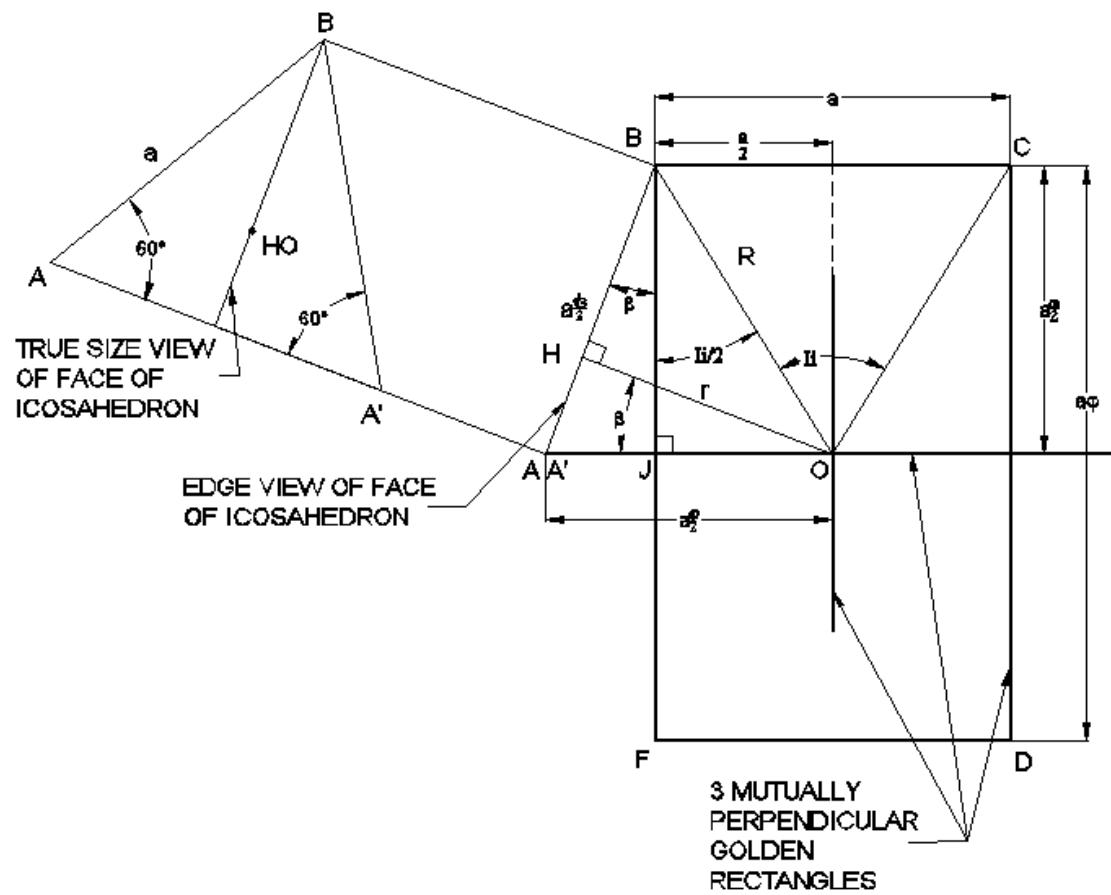


fig. 28

$$\frac{r}{R} = 0.794$$

The more complex formula stated previously yields the same result:

$$\frac{r}{R} = 0.794$$

The advantage of our formula, besides its simplicity of form and derivation, has the added advantages of relating the ratios of the radii to the inner structure of the polyhedron.

### **Icosahedron:**

Here also, a simple derivation based on the internal angle can be obtained by simple inspection.

The handbook formulae are:

$$a = \frac{\sqrt{10}}{5} \sqrt{5 - \sqrt{5}} R \quad \text{and} \quad r = \frac{a}{2} \sqrt{\frac{7 + 3\sqrt{5}}{6}}$$

Again, referring to figure 26, it is seen, as we previously explained, that the vertices of the icosahehedron are at the corners of 3 mutually perpendicular golden rectangles. This time, therefore,  $R$ , the radius of the circumsphere, is the half diagonal of one of the golden rectangles.

Considering the plane of such a golden rectangle, we have from the geometry of the figure: (fig. 28)

$$a = \text{edge of icosahehedron} = BC, \therefore CD = a\phi, \text{ where } \phi \text{ is the golden ratio } \frac{1+\sqrt{5}}{2}$$

$$R = \text{circumradius} = OB$$

$$r = \text{insphere radius} = OH \text{ perpendicular from } O, \text{ center of sphere to triangular face of icosahehedron.}$$

The other needed measurements appear on the figure.

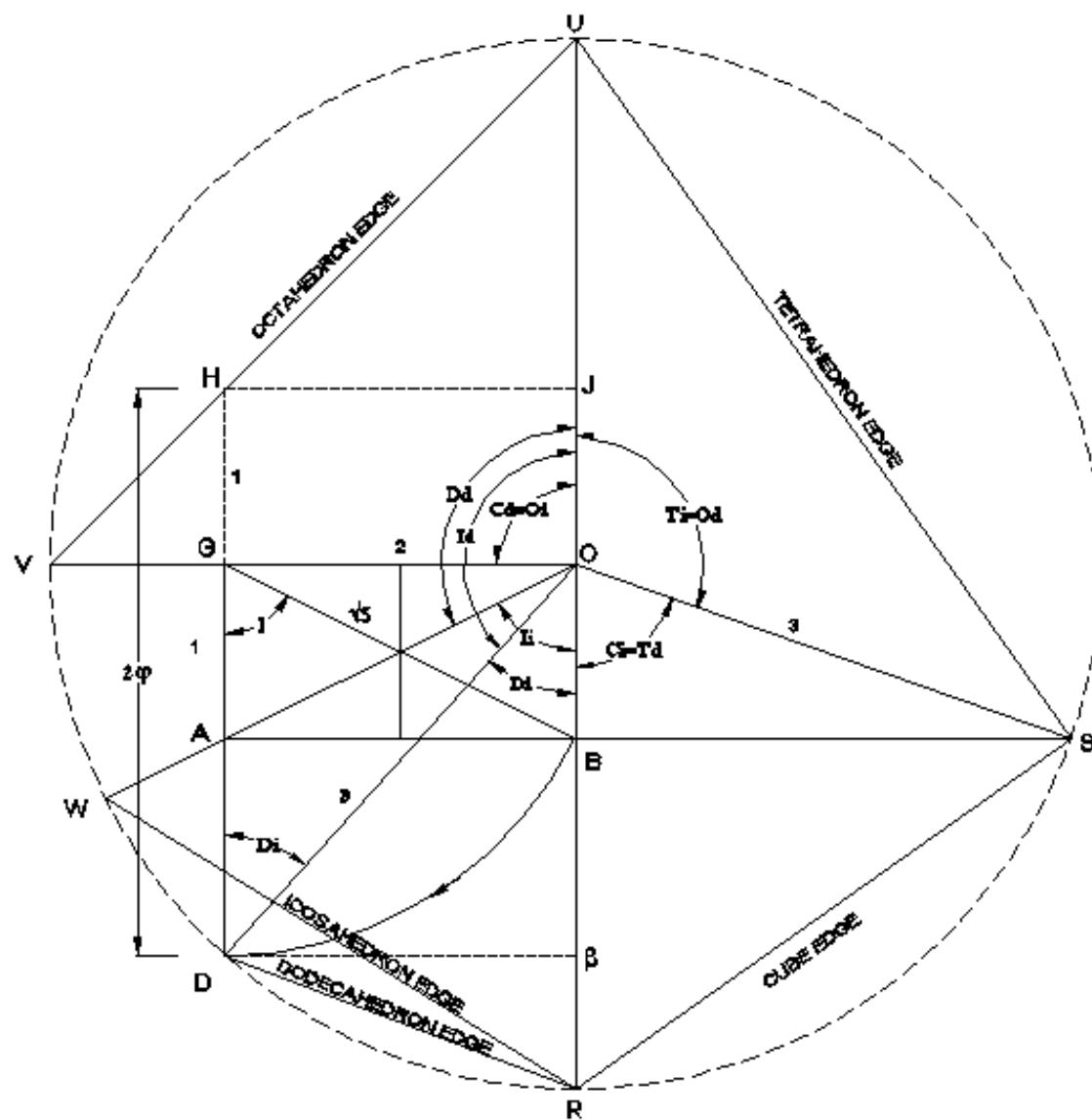


fig. 13

Now, consider triangles  $A\bar{A}'HO$  and  $A\bar{A}'BJ$ , where  $\bar{A}\bar{A}'$  is the point view of line  $AA'$ , simplified to  $A$  in the following:

$$\frac{AO}{AB} = \frac{HO}{BJ}$$

or, replacing by the values indicated on the figure

$$\frac{\frac{a\phi}{2}}{\frac{a\sqrt{3}}{2}} = \frac{r}{\frac{a\phi}{2}} \quad \therefore \quad \frac{\phi}{\sqrt{3}} = \frac{r}{\frac{a\phi}{2}}$$

But in  $BJO$ ,

$$R \cos \frac{\hat{I}_i}{2} = \frac{a\phi}{2}$$

$$\therefore \frac{\phi}{\sqrt{3}} = \frac{r}{R \cos \frac{\hat{I}_i}{2}},$$

$$\text{or: } \boxed{\frac{r}{R} = \frac{\phi}{\sqrt{3}} \cos \frac{\hat{I}_i}{2}}$$

Now referring to the initial adquadratum construction diagram (fig. 13), and considering triangle  $O\beta D$  we can write:

$$\cos \hat{D}_i = \frac{O\beta}{OD} = \frac{2\phi - 1}{3}.$$

And making use of the identity  $\cos \frac{A}{2} = \sqrt{\frac{1}{2}(1 + \cos A)}$ ,

$$\text{we have: } \cos \frac{\hat{D}_i}{2} = \sqrt{\frac{1}{2} \left( 1 + \frac{2\phi - 1}{3} \right)}$$

$$= \sqrt{\frac{1}{2} \left( \frac{2\phi + 2}{3} \right)} = \sqrt{\frac{\phi + 1}{3}}$$

But, from the well-known identity  $\phi + 1 = \phi^2$

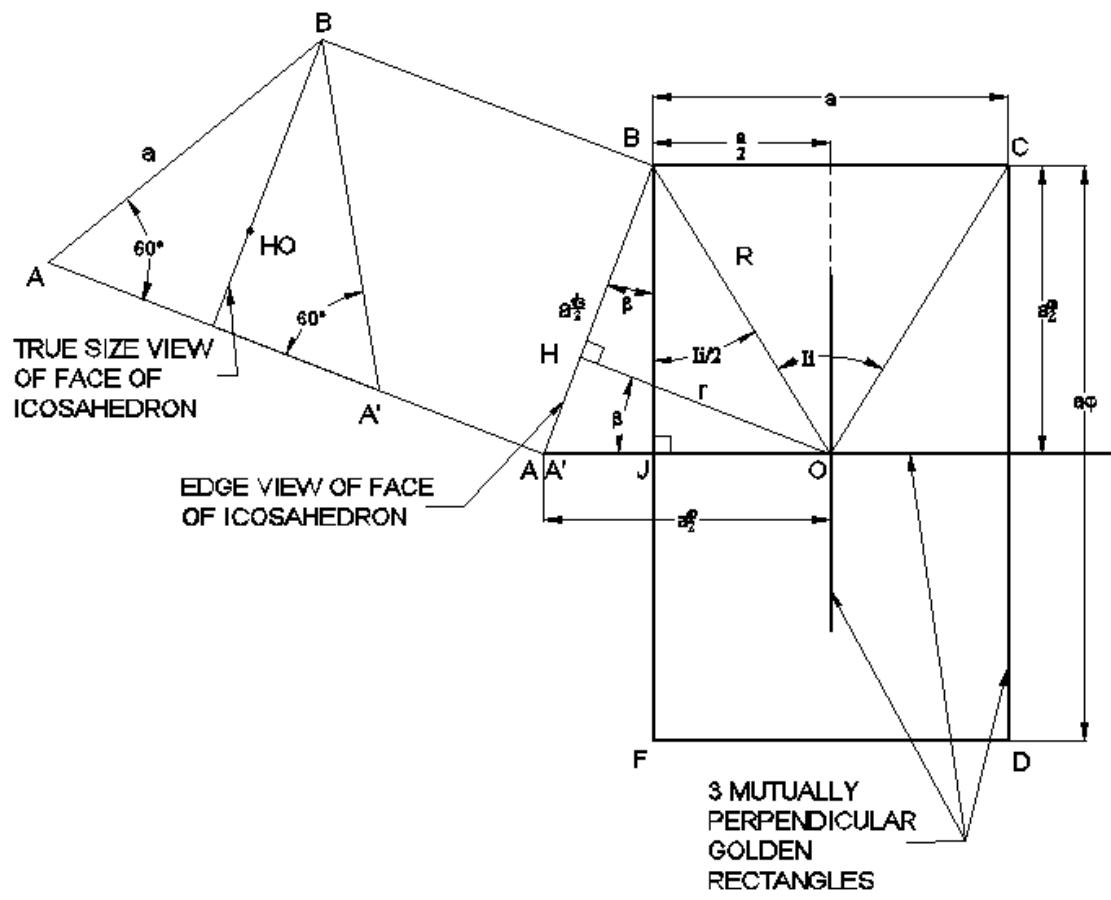


fig. 28

$$\therefore \cos \frac{\hat{D}_i}{2} = \sqrt{\frac{\phi^2}{3}} = \frac{\phi}{\sqrt{3}}$$

We can therefore rewrite the ratio of circum- to inradii as:

$$\boxed{\frac{r}{R} = \cos \frac{\hat{D}_i}{2} \cos \frac{\hat{I}_i}{2}}$$

Which is precisely the formula derived earlier for the dodecahedron, confirming Apollonius remark<sup>12</sup> that both dodecahedron and icosahedron having a common circumsphere have also a common insphere.

From fig. 28, it can also be seen that we can write in triangle BJO:

$$\begin{aligned} \cos \frac{\hat{I}_i}{2} &= \frac{a\phi}{R} ; \text{ with } R = \sqrt{\frac{a^2}{4} + \frac{a^2\phi^2}{4}} = \frac{a}{2}\sqrt{1+\phi^2} \\ \therefore \cos \frac{\hat{I}_i}{2} &= \frac{\phi}{\sqrt{1+\phi^2}} \end{aligned}$$

It follows that

$$\begin{aligned} \frac{r}{R} &= \cos \frac{\hat{D}_i}{2} \cos \frac{\hat{I}_i}{2} \\ &= \frac{\phi}{\sqrt{3}} \frac{\phi}{\sqrt{1+\phi^2}} = \frac{\phi^2}{\sqrt{3}\sqrt{\phi^2+1}} \end{aligned}$$

or, since  $\phi^2 = \phi + 1$

$$\boxed{\frac{r}{R} = \frac{1}{\sqrt{3}} \left( \frac{\phi+1}{\sqrt{\phi+2}} \right)}$$

(numerically 0.794 as found before.)

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<sup>12</sup> Heath, T.H., op. cit. p. 254

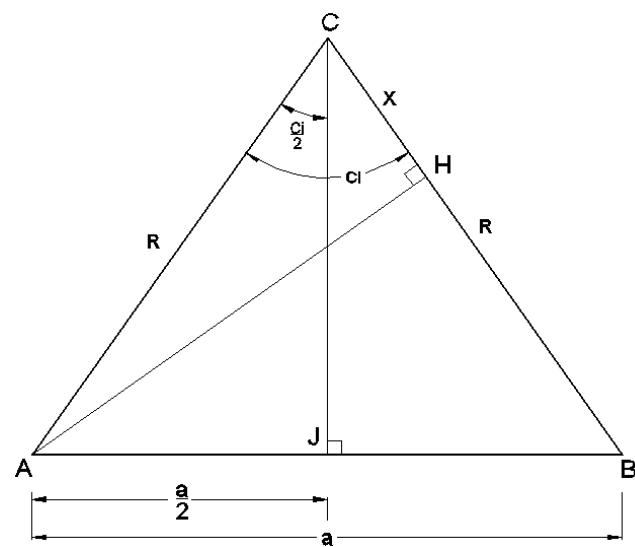


fig. 5

## Ratio of Intersphere to Circumsphere Radii

Besides the insphere just described, another sphere can be considered in relation to the platonic polyhedra --the intermediate sphere or intersphere<sup>13</sup>. This sphere is defined by the midpoints of each edge of a platonic polyhedron, marking the intercept of interpenetrating duals.

The radius of this intersphere can be readily calculated for each polyhedron.

Let  $r_i$  be the radius of the intersphere.

Referring to fig. 5, which applies to all the regular convex polyhedra, and represents one of the lateral faces of the pyramids making up each polyhedron, it is seen that the radius of the intersphere will be CJ and in all cases,  $CJ = r_i = R \cos \frac{\hat{C}_i}{2}$ ; where here  $\frac{\hat{C}_i}{2}$  is half the internal angle of a particular polyhedron.

From the identity  $\cos \frac{A}{2} = \sqrt{\frac{1}{2}(1 + \cos A)}$  we therefore obtain for each polyhedron, in turn:

### **Tetrahedron:**

$$\frac{r_i}{R} = \cos \frac{\hat{T}_i}{2} = \sqrt{\frac{1}{2}\left(1 - \frac{1}{3}\right)} = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}$$

### **Cube:**

$$\frac{r_i}{R} = \cos \frac{\hat{C}_i}{2} = \sqrt{\frac{1}{2}\left(1 + \frac{1}{3}\right)} = \sqrt{\frac{2}{3}}$$

### **Octahedron:**

$$\frac{r_i}{R} = \cos \frac{\hat{O}_i}{2}$$

$$\text{but } \sin \hat{O}_i = 1 \therefore \cos \hat{O}_i = 0$$

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<sup>13</sup> Kappraff, Jay: Connection, McGraw-Hill, Inc., New York, 1991, p. 266.

and  $\cos \frac{\hat{O}_i}{2} = \sqrt{\frac{1}{2}(1-0)} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

**Dodecahedron:**

$$\frac{r_i}{R} = \cos \frac{\hat{D}_i}{2} = \frac{\phi}{\sqrt{3}}$$

As previously established,

$$\text{with } \phi = \frac{1+\sqrt{5}}{2}$$

$$\frac{r_i}{R} = \frac{1+\sqrt{5}}{2\sqrt{3}} = \frac{(1+\sqrt{5})\sqrt{3}}{6}$$

**Icosahedron:**

Making use of previous results, we can write:

$$\frac{r_i}{R} = \cos \frac{\hat{I}_i}{2} = \frac{\phi}{\sqrt{1+\phi^2}} = \frac{\phi}{\sqrt{\phi+2}}$$

**Ratio of Insphere to Intersphere Radii:**

Finally we can also establish the ratio of the insphere radius  $r$  to the intersphere radius  $r_i$ ,

$\frac{r}{r_i}$  for each polyhedron,

$$\text{since } \frac{r}{r_i} = \frac{\frac{r}{R}}{\frac{r_i}{R}} = \frac{\frac{r}{R}}{\frac{\phi}{\sqrt{\phi+2}}} = \frac{r}{\phi}$$

**Tetrahedron:**

$$\frac{r}{R} = \cos \hat{C}_i = \frac{1}{3}; \quad \frac{r_i}{R} = \cos \frac{\hat{T}_i}{2} = \frac{\sqrt{3}}{3}$$

$$\therefore \frac{r}{r_i} = \frac{\cos \hat{C}_i}{\cos \frac{\hat{T}_i}{2}} = \frac{\frac{1}{3}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = 0.577$$

**Cube:**

$$\frac{r}{R} = \sqrt{3} \cos \hat{C}_i = \frac{\sqrt{3}}{3}; \quad \frac{r_i}{R} = \cos \frac{\hat{C}_i}{2} = \sqrt{\frac{2}{3}}$$

$$\frac{r}{r_i} = \frac{\sqrt{3} \cos \hat{C}_i}{\cos \frac{\hat{C}_i}{2}} = \frac{\frac{\sqrt{3}}{3}}{\frac{\sqrt{2}}{\sqrt{3}}} = \frac{3}{3\sqrt{2}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} = 0.707$$

**Octahedron:**

$$\frac{r}{R} = \sqrt{3} \cos \hat{C}_i = \frac{\sqrt{3}}{3}; \quad \frac{r_i}{R} = \cos \frac{\hat{O}_i}{2} = \frac{\sqrt{2}}{2} = \sqrt{\frac{1}{2}}$$

$$\frac{r}{r_i} = \frac{\frac{\sqrt{3}}{3}}{\frac{\sqrt{2}}{2}} = \sqrt{\frac{2}{3}} = 0.816$$

**Dodecahedron:**

$$\frac{r}{R} = \cos \frac{\hat{D}_i}{2} \cos \frac{\hat{I}_i}{2} = \frac{1}{\sqrt{3}} \left( \frac{\phi+1}{\sqrt{\phi+2}} \right); \quad \frac{r_i}{R} = \cos \frac{\hat{D}_i}{2} = \frac{\phi}{\sqrt{3}}$$

$$\frac{r}{r_i} = \frac{\frac{\phi^2}{\sqrt{3}\sqrt{\phi^2+1}}}{\frac{\phi}{\sqrt{3}}} = \frac{\phi}{\sqrt{\phi^2+1}} = \frac{\phi}{\sqrt{\phi+2}} = 0.850$$

**Icosahedron:**

$$\frac{r}{R} = \cos \frac{\hat{D}_i}{2} \cos \frac{\hat{I}_i}{2} \quad \text{but} \quad \frac{r_i}{R} = \cos \frac{\hat{I}_i}{2};$$

so that:  $\frac{r}{r_i} = \cos \frac{\hat{D}_i}{2};$

and, as previously established we have

$$\cos \frac{\hat{D}_i}{2} = \frac{\phi}{\sqrt{3}}$$

$$\therefore \frac{r}{r_i} = \frac{\phi}{\sqrt{3}} = 0.934$$

As an effect of the duality principle we can see that

- for the tetrahedron

$$\frac{r_i}{R} = \frac{r}{r_i} = \sqrt{\frac{1}{3}} = \sqrt{\cos C_i} = \cos \frac{T_i}{2}$$

- for the cube and the octahedron

$$\frac{r_i}{R} \Big|_{\text{cube}} = \frac{r}{r_i} \Big|_{\text{oct.}} = \sqrt{\frac{2}{3}} = \sqrt{\sin D_i} = \cos \frac{C_i}{2}$$

$$\text{and } \frac{r_i}{R} \Big|_{\text{oct.}} = \frac{r}{r_i} \Big|_{\text{cube}} = \sqrt{\frac{1}{2}} = \sqrt{\frac{1}{\tan I_i}} = \sqrt{\operatorname{ctn} I_i} = \cos \frac{O_i}{2}$$

- for the dodecahedron and the icosahedron

$$\frac{r_i}{R} \Big|_{\text{dod.}} = \frac{r}{r_i} \Big|_{\text{icos.}} = \frac{\phi}{\sqrt{3}} = \cos \frac{D_i}{2}$$

$$\text{and } \frac{r_i}{R} \Big|_{\text{icos.}} = \frac{r}{r_i} \Big|_{\text{dod.}} = \frac{\phi}{\sqrt{\phi+2}} = \cos \frac{I_i}{2}$$

### Comparison with Orthoscheme Approach:

Schlafli<sup>14</sup> established general formulae giving the ratio between the radii of these spheres based on the study of the decomposition of the polyhedra in *orthoschemes* which are oppositely congruent tetrahedra making up our pyramids. The decomposition is made along the axis of the pyramid giving as many orthoschemes as sides to the face of the polyhedron under study. Furthermore, the four faces of these tetrahedra are right triangles, and the lengths of the three edges meeting at the polyhedron center are radii of the circumsphere, insphere, and intersphere respectively.

Such an orthoscheme is shown for the cube in fig. 29:

$$O^R = \text{Radius circumference} = R$$

$$2^R = \text{radius insphere} = r$$

$$1^R = \text{radius intersphere} = r_i$$

Schlafli formulae are:

$$\cos \Phi = \frac{O_1 O_3}{O_0 O_3} = \frac{1^R}{O^R} = \cos \frac{\pi}{p} \cos \frac{\pi}{q}$$

$$\cos \psi = \frac{O_2 O_3}{O_1 O_3} = \frac{2^R}{1^R} = \csc \frac{\pi}{p} \cos \frac{\pi}{q}$$

$$\cos X = \frac{O_2 O_3}{O_0 O_3} = \frac{2^R}{O^R} = \cot \frac{\pi}{p} \cot \frac{\pi}{q}$$

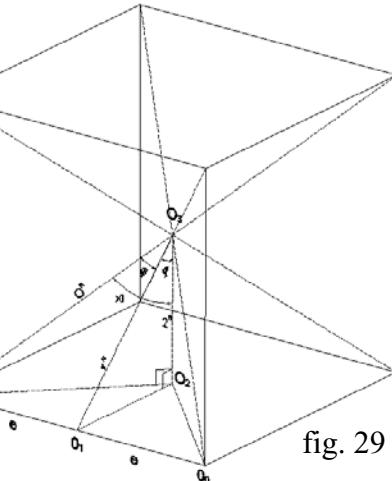


fig. 29

$$O^R = e \sin \frac{\pi}{q} \csc \frac{\pi}{h}$$

$$1^R = e \cos \frac{\pi}{p} \csc \frac{\pi}{h}$$

$$2^R = e \cot \frac{\pi}{p} \cos \frac{\pi}{q} \csc \frac{\pi}{h}$$

Where  $e$  is the semi edge length of the platonic polyhedron  
 $h$  is the number of lengths into which a great circle is divided by an edge  
 $q$  is the number of incident edges at a vertex  
 $p$  is the number of edges around a face

They naturally give exactly the same numerical results on those we have established.

For example, for the tetrahedron, we have

$$p=3, q=3$$

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<sup>14</sup> Kappraff. Op. Cit. p. 287

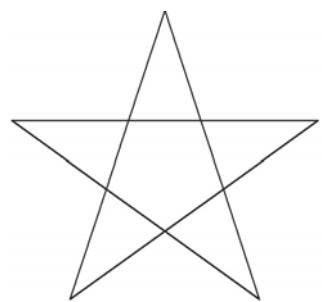


Fig. 30

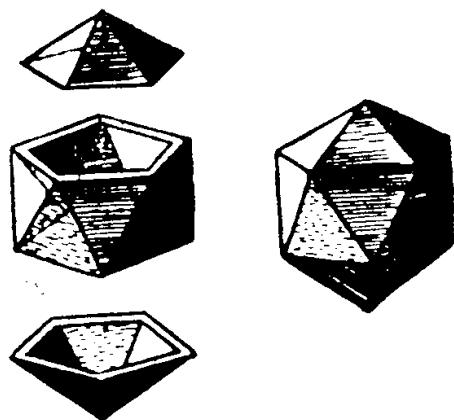


fig. 31A

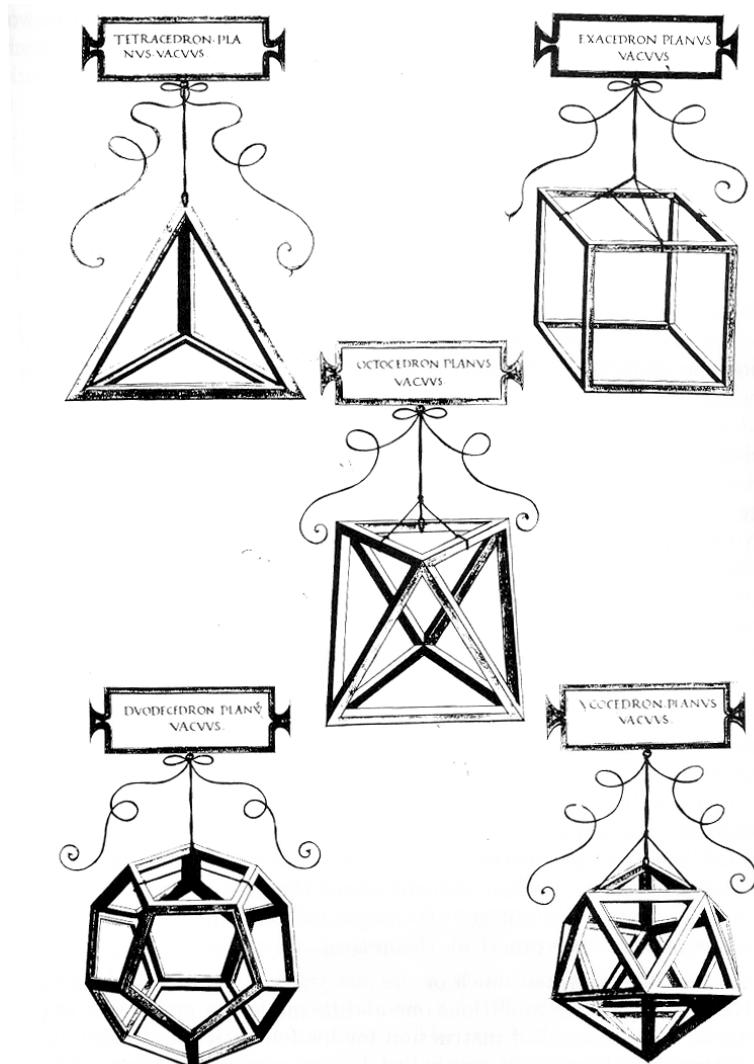


fig. 31B

$$\begin{aligned}\therefore \frac{1^R}{O^R} &= \frac{r_i}{R} = \cos \frac{\pi}{3} \csc \frac{\pi}{3} = \cos 60^\circ \csc 60^\circ \\ &= \left( \frac{1}{2} \right) \left( \frac{2}{\sqrt{3}} \right) = \frac{\sqrt{3}}{3}\end{aligned}$$

as previously found. Note however that the thinking remains palpably “solid” rather than “structural,” i.e., in terms of surfaces and volumes rather than radially from the center.

## Polyhedra & Regularity

Intuitively, a polyhedron can be naively defined as a three-dimensional region of space totally enclosed by a set of plane figures. If these plane figures are regular polygons (i.e., having all sides and all angles equal), the polyhedron is regular. Only triangles, squares and pentagons can give rise to regular polyhedra. For stellated polyhedra, we can add the pentagram or five-pointed star (fig. 30).

However, both the concepts of *polyhedron* and *regularity* have evolved with time. Some modern definitions of polyhedra are such that stellated polyhedra for instance are excluded<sup>15</sup>. The ancients seem to have thought of polyhedra as *solids* bounded by polygons. This appears also to be the case for Descartes (1596-1650) and other mathematicians such as Legendre (1752-1834). Judging from his drawings (fig. 31A), Kepler (1571-1630) seems to have thought of them as hollow surfaces. Cauchy (1789-1857) regarded them as potentially flexible deformable surfaces.

A more modern approach, often better suited to the interests of designers and structural engineers, takes a more topological view. Having perhaps its roots in the Renaissance drawings of Leonardo (1452-1519) (fig. 31B) and Wentzel Jamnitzer (1508-1585) (fig. 31C and 31D), it looks upon polyhedra as skeletons, considering as fundamental the relationships between edges rather than faces.

For our purpose, limited to the study of regular convex and stellated forms of potential utility in our design, an intuitive approach that selectively uses surfaces, volumes and topological approach seems best as long as it remains consistent. Our goal is more to look for the structurally useful rather than the mathematically all-encompassing for its own sake.

We alluded to a definition of regularity at the beginning. Again, as the definition of *polyhedron*, the concept of *regularity* has evolved over time. From Euclid down, regularity has involved equality of faces and faces being polygons with equal sides and angles. But this conception of regularity is incomplete. A case in point is that of the five so-called *deltahedra*, made up of congruent equilateral triangles as shown in fig. 31E.

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<sup>15</sup> Peter Cromwell: *Polyhedra*. Cambridge U.P., 1999, p. 209.

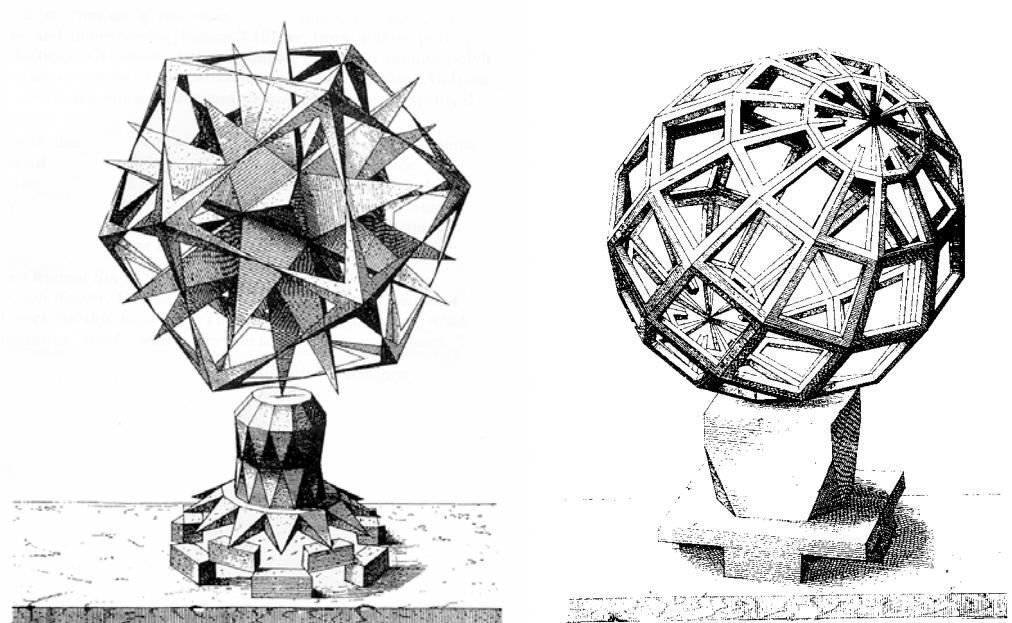


fig. 31C

fig. 31D

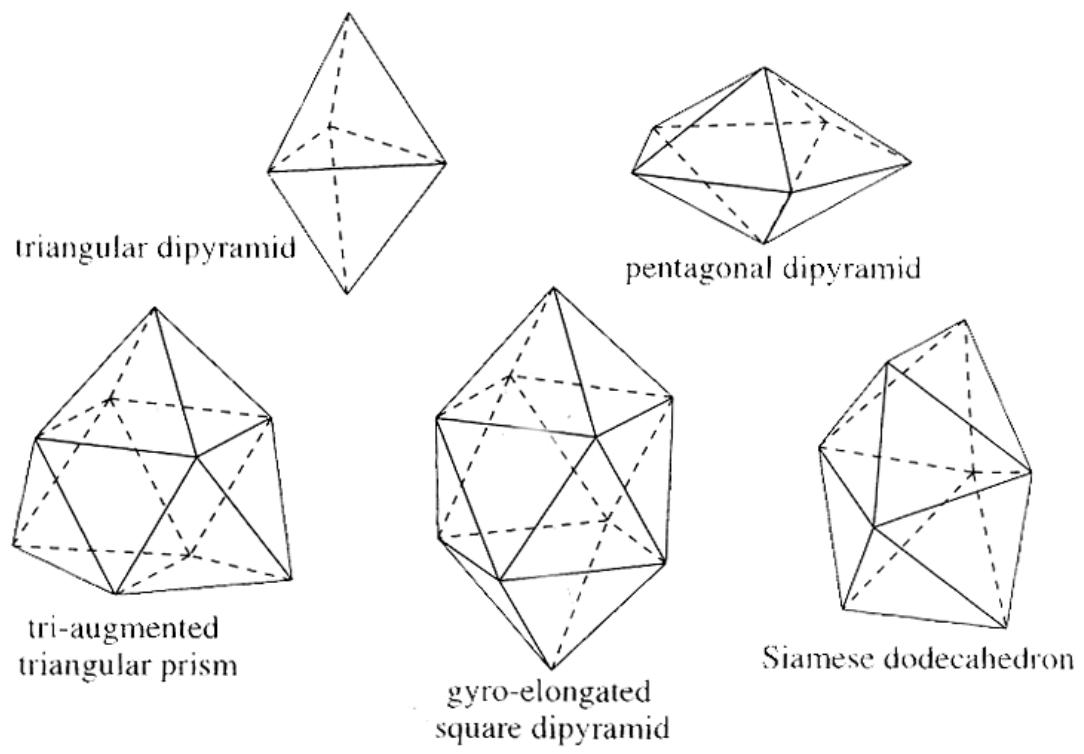


fig. 31E

They are not what is generally thought of as *regular*. The implicit assumption in Euclid's and subsequent definitions of regularity is that, to be regular, a polyhedron must be inscribable within a sphere, or alternatively, that the same number of faces will meet at each vertex.

Cromwell<sup>16</sup> cites as modern definition of a regular polyhedron one whose faces and *vertex figures*<sup>17</sup> are regular polygons. He also shows that if  $P$  is a *convex polyhedron* whose faces are congruent regular polygons, the following statements about  $P$  are equivalent:

1. All vertices of  $P$  lie on a sphere.
2. All the dihedral angles of  $P$  are equal.
3. All the vertex figures are regular polygons.
4. All the solid angles are congruent.
5. All the vertices are surrounded by the same number of faces.

A definition of regularity, which requires neither convexity nor the explicit statement regarding equal regular faces, is H.S.M. Coxeter's<sup>18</sup>:

*A polyhedron is regular if it has a circumsphere, an insphere and an intersphere.*

We would like to propose here another definition of regularity, which also requires neither convexity nor explicit statement concerning faces regularity. Our definition is that outlined on p. 3, namely that **to be regular, a polyhedron must be spherical and equimarialdian**.

As we shall establish, convex and stellated polyhedra have the same vertices and the same Maraldi angles.

A polyhedron will be **convex** if the network of lines (projections of the edges onto the sphere) joining a vertex to its most immediate equally distant neighbors is **locally planar**<sup>19</sup> and no other connection exists between any vertex and any other except to its immediate neighbors. It will be **stellated** if this network of lines connects non-immediate neighbors. It is then **locally non-planar**, so that lines do cross.

## Stellated Polyhedra

Plato and the ancient world knew the 5 regular convex polyhedra. Euclid's *Elements* concludes with a proof that there were in fact only five.

<sup>16</sup> P. Cromwell, op. cit. p. 77

<sup>17</sup> A vertex figure is the spherical polygon seen after scooping out the intersection of faces surrounding a vertex with a small sphere centered on that vertex.

<sup>18</sup> Quoted by Cromwell, op. cit. p. 77.

<sup>19</sup> Meaning that the segments between vertices do not cross. By "locally" we mean that the distance measured around a vertex, of at least two edges. More precisely, perhaps, because it seems paradoxical to speak of planar network on a sphere, the spherical surface involved in our *locality* must remain open (i.e., it cannot involve the whole entire surface of the sphere).

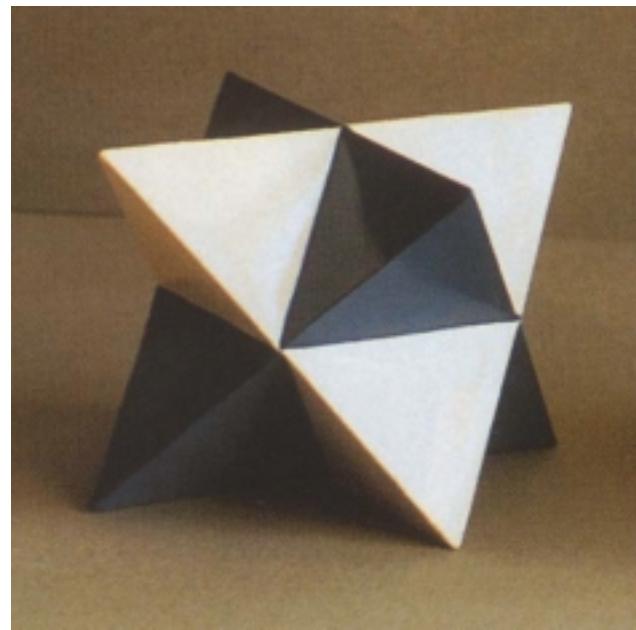


fig. 32A

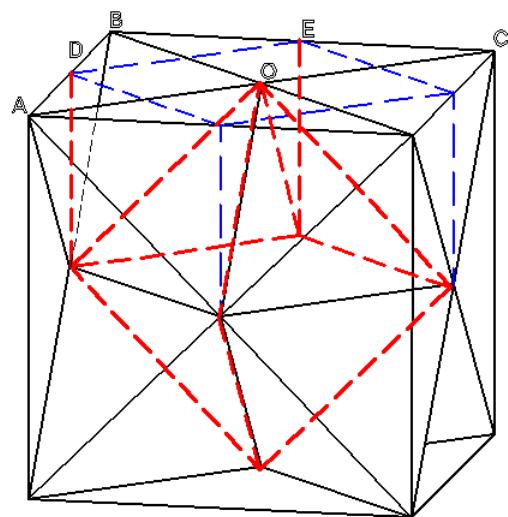


fig. 32B

Kepler (1571-1630) discovered in 1611 two regular stellated polyhedra. A stellated polyhedron is one in which either the faces or the vertices are star-shaped, as in the pentagram or five-pointed star. Such polyhedra are non-convex (i.e., they are concave). Convexity and non-convexity have been defined in the preceding section.

Kepler's discovery concerned the two regular star-*faced* polyhedra. Louis Poinsot (1810) rediscovered them and discovered in addition the two star-shaped *vertex* polyhedra. Cauchy (1811) proved that there are no regular star polyhedra other than these four bringing the total number of regular polyhedra to nine altogether.

Stellated polyhedra are obtained by extending the face planes (or the edges) of the platonic polyhedra. In the case of the tetrahedron and of the cube, no new regions of space are enclosed by the extensions. However, by extending the faces of the octahedron, eight regular tetrahedra will be formed on each of the faces of the octahedron. Their regularity is ensured by the symmetry of the figure. Together with the original octahedron, they constitute two large tetrahedra interpenetrating one another (fig. 32A).

This is the *stella octangula* of Kepler. Though sometimes referred to as the stellated octahedron, it is not however a regular stellated polyhedron. Being made up of two tetrahedra, it is simply a compound polyhedron rather than a new regular stellated polyhedron with its new features and properties. It corresponds in space to the pseudo-star-hexagon or shield of David in the plane and is therefore a sort of three-dimensional star of David.

We shall briefly study it here for its historical importance and its relation with the ad quadratum method.

### **1. Stella Octangula:**

Only the apexes of the two tetrahedra are considered vertices of this new figure, not the intersections at mid-point on the edges. The faces of the stella octangula are therefore the faces of the two interpenetrating tetrahedra, though for practical purposes, particularly when building models, the faces of the small tetrahedra formed on the faces of the original octahedron can be so considered.

The stella octangula is obviously inscribable into the cube that is the dual of the original octahedron where the vertices of the octahedron are the center points of the cube faces, the two tetrahedra edges being the diagonals of the cube faces (fig. 32B). The geometry of the stella octangula will be determined by that of the external pyramids affixed, so to speak, onto the faces of the octahedron.

Note that the circumsphere of the original octahedron is the intersphere of the two tetrahedra resulting of the extension of the face planes of the octahedron.

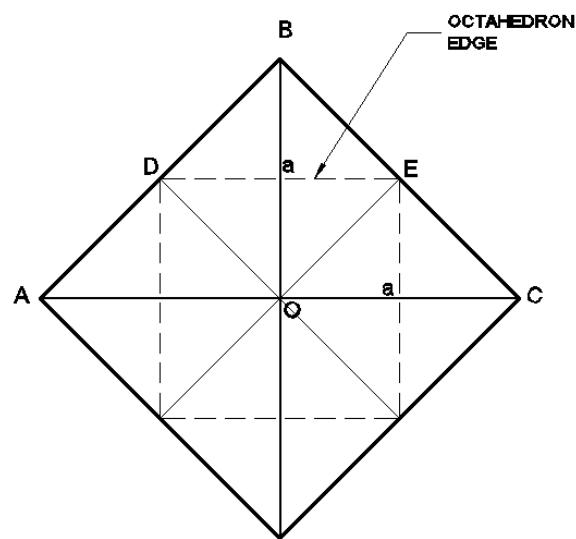


fig. 33

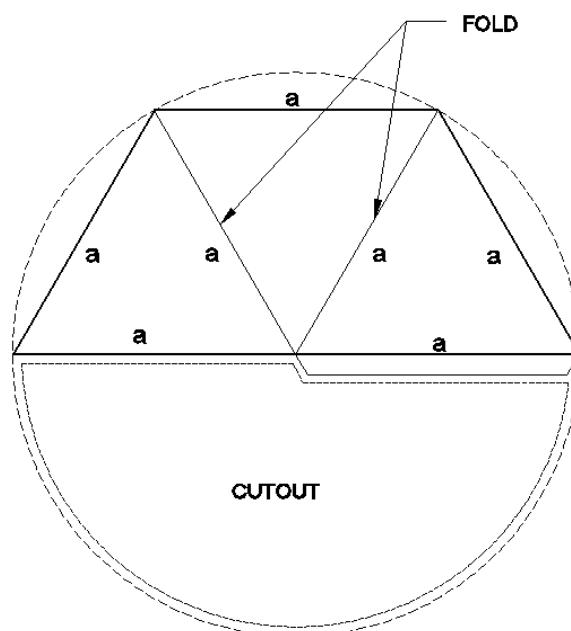


fig. 34

We shall now determine the diameter of the circumsphere of the new figure and the side of the cube circumscribing it.

A top view of the stella octangula (fig. 33) shows that the edge of one of the typical pyramids ( $OC$ ) is equal in length to the edge of the original octahedron( $DE$ ). Therefore the construction of the pyramids is straightforward. From the Ad Quadratum diagram measure  $a$ , the edge of the octahedron. Draw a circle (fig. 34) of radius  $a$  and measure  $a$  with a compass along the circumference three times. Draw chords and radii to obtain a half hexagon. This is the development of the pyramid. Cut out and fold. Repeat the operation 8 times (one for each pyramid). They can now be assembled to form the stella octangula.

Referring to fig. 33, if  $DE=a$  is the edge of octahedron, the edge of pyramid  $OC$  also equals  $a$ , then the edge of circumscribing cube is

$$AB = \sqrt{AO^2 + OB^2} = \sqrt{2a^2}$$

$$\therefore AB = a\sqrt{2}$$

Therefore, the circumsphere radius which is half the cube diagonal will be:

$$\frac{AB}{2}\sqrt{3} = \frac{a\sqrt{2}}{2}\sqrt{3} = a\sqrt{\frac{3}{2}}$$

The relationship between dihedral angles of the stella octangula and the original octahedron can readily be written.

We have:

$$\begin{aligned}\hat{O}_D &= \pi - \hat{C}_i \\ &= \hat{T}_i\end{aligned}$$

and

$$\hat{T}_D = \pi - \hat{T}_i \quad \therefore \quad \hat{T}_D = \pi - \hat{O}_D$$

or

$$\hat{O}_D = \pi - \hat{T}_D$$

Of course, the stella octangula can also be constructed from two intersecting internal tetrahedra.

## 2. The Process of Stellation

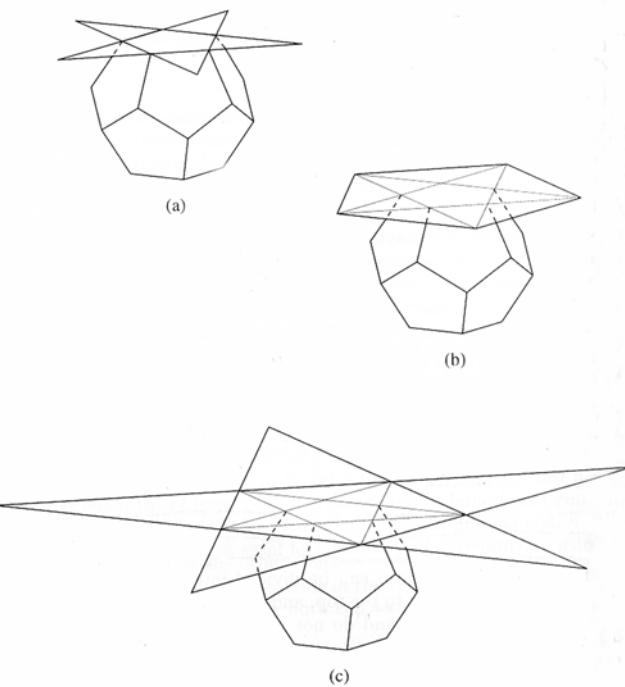


fig. 35A

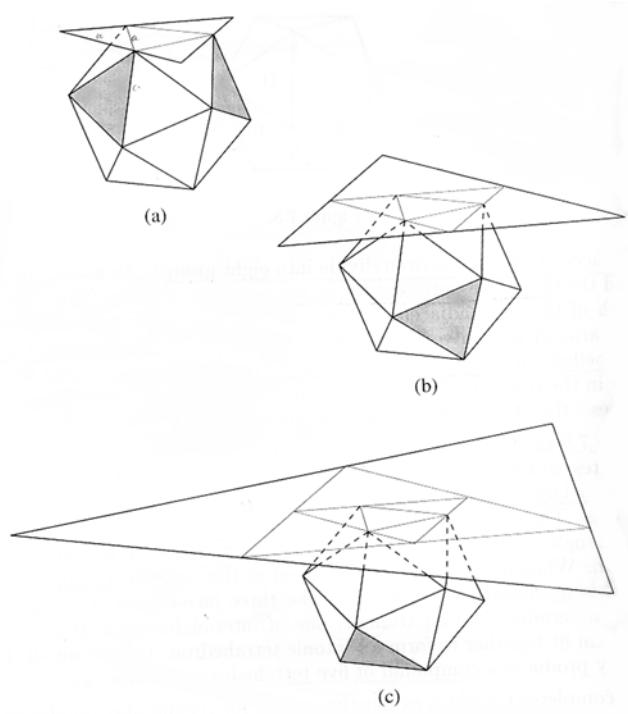


fig. 35B

The stellation process is one whereby stellated forms arise from convex forms. The process can originate from either face or edge extension. As we have seen, it only applies to the dodecahedron and the icosahedron since tetrahedron and cube do not enclose new spaces through extension and that the octahedron only gives rise to a compound (the stella-octangula).

Due to the principle of duality, the outcome of the extension of the dodecahedron and the icosahedron will alternate so that a particular figure may be said to have as kernel, or seed, either of these polyhedra depending on the starting point.

We want to restrict our study to that of *regular* stellated forms. The stellation process gives rise to many other forms besides the regular ones. There are, for example, 59 varieties of icosahedral stellations.<sup>20</sup> The four regular stellated polyhedra, the only possible ones (Cauchy), allow however a number of different viewpoints, some easier to visualize, some easier in model making, some easier for purposes of structural analysis.

The process of stellation can also be visualized directly in three-Dimension as one of addition of volumic cells such as pyramids onto the surfaces of the convex polyhedra. Of course, for the resulting figure to be regular, the only cells that can be added are those whose edges or faces are themselves extensions of the kernel polyhedron (edges or faces). Though it is not therefore really another method of stellation, it constitutes a convenient visualization and conceptualization tool.

We shall briefly consider the surface and edge extension process and their afferent 3-D approach to start with. However, most of our attention will be given to the process resulting from the dodecahedron and icosahedron common internal structure of three mutually perpendicular golden rectangles, since it is directly related to the concept of internal angle and our adquadratum method.

#### (a) Face Stellation<sup>21</sup>

Except for the tetrahedron, the face-planes of the regular convex polyhedra come in parallel pairs. But since only the dodecahedron and icosahedron generate regular stellated forms, we consider these two only. If one of the faces in one of the parallel pairs is chosen as base and the other as top, a regular star polyhedron will be the result of other faces extension forming a regular polygon in the plane of the base or top. For this, these other faces must be arranged symmetrically around an axis through the top and base center.

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<sup>20</sup> *The Fifty-nine Icosahedra*, by J.F. Petric, H.T. Flather, H.S.M. Coxeler, and P. Du Val, U. of Toronto Press, 1951.

<sup>21</sup> We follow here P. Cromwell op. cit. pp. 260-280.



fig. 35B(d)

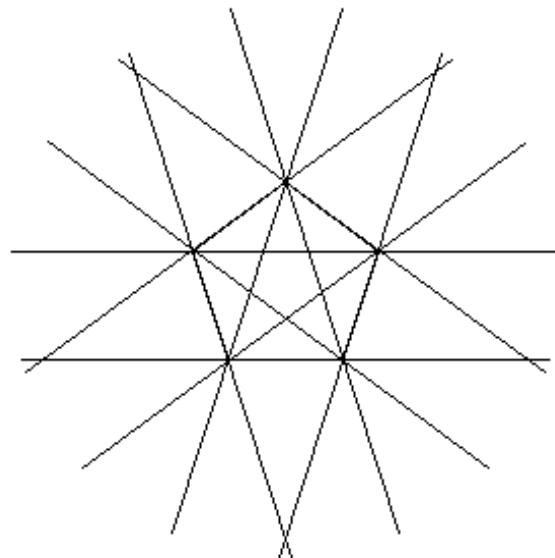


fig. 35B(e)

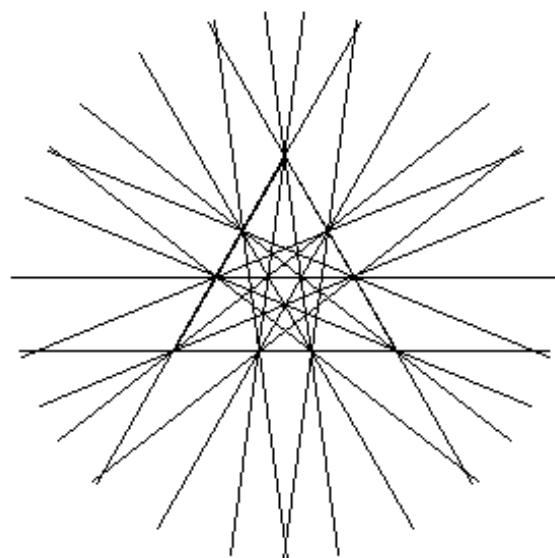


fig. 35B(f)

*i. Case of the Dodecahedron*

The faces of the convex dodecahedron can be divided into four groups: top, base, and the five faces adjacent to top and base respectively. Top and base being parallel do not meet. The five faces adjacent to the top are disposed symmetrically around the center of the top. Their extension cut the plane of the top to form the pentagram fig. 35A(a). Since any face can be chosen as top, twelve such pentagrams will result. They contribute the faces of the small stellated dodecahedron (SSD). This is the first step in the stellation process of the dodecahedra. If now the faces of the group adjacent to the base are extended, a pentagon circumscribing the pentagram, (fig. 35A(b)) just determined will first result and then a new pentagram (fig. 35A(c)). The pentagon (twelve of them altogether) is a face of the great dodecahedron (GD) while the pentagram is a face of the great stellated dodecahedron (GSD)

*ii. Case of the Icosahedron*

The case of the icosahedron is more complex from this view point since the faces will form eight groups: top, base, and the three faces adjacent to the top (i.e., having a common edge with a top) are three such groups (a total of 5 faces); followed in succession by the group of three faces adjacent to the base (total 8); the group of three faces adjacent to the right and the group of three faces adjacent to the left of the faces themselves adjacent to the base, respectively (total 14); finally, the two groups of three faces symmetrically disposed with respect to the top (total 20).

Obviously, with so many groups, many possibilities arise (59 in fact, as previously mentioned). Fig. 35 B shows some of the polygons arising from some of these groups. But out of all these possibilities, only one (fig. 35B(c)) gives rise to a regular stellated form: The great icosahedron (GI). Fig. 35B(a) gives a regular convex icosahedron while fig. 35B(b) will yield a compound of five tetrahedra (fig. 35B(d)).

The pattern of lines on the planes of the face generated by the cutting planes from the other faces determines a stellation pattern. These for the dodecahedron and the icosahedron are shown on fig. 35B(e) and 35B(f) respectively.

b) The Three-Dimensional Approach

More intuitive perhaps though less general, we shall use this approach when visualizing the construction of the stellated forms. It can be extended to include the inner (or Maraldian) pyramids that constitute the convex polyhedra built through the adquadratum method. Thus, each stellated form can be seen as made up of dipyrramids regularly distributed along Maraldi radii clustered around the center of the whole form.

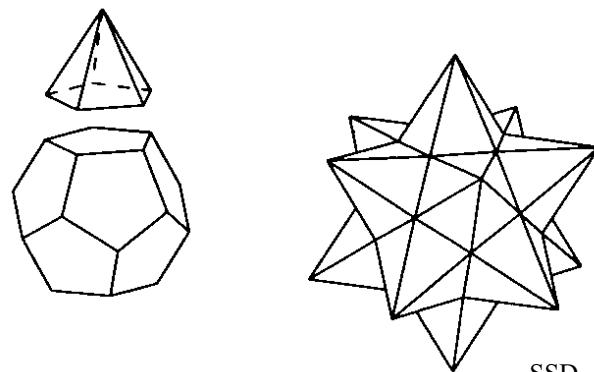


fig. 35C

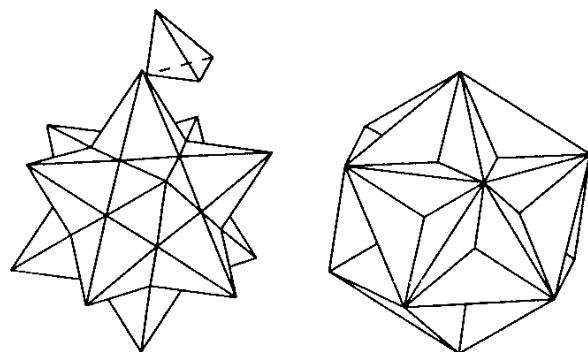


fig. 35D

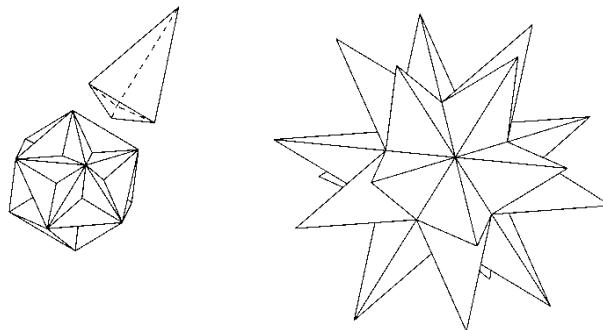


fig. 35E

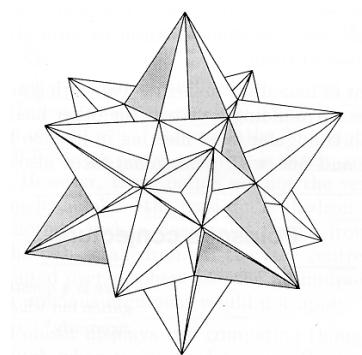


fig. 35F

These dipyramids in turn can be divided into sets of oppositely congruent tetrahedron similar to Schläfli's orthoschemes<sup>22</sup>. The case of the SSD (fig. 35C) is straightforward: pentagonal pyramids base to base divided by planes passing through the pyramids apexes and the vertices of the common pentagonal base.

For the GD (fig. 35D), it may be easier to consider the dipyramids as made up of a positive pyramid, part of the internal structure of the enveloping convex icosahedron and a *negative* pyramid representing a dimple.

The GSD (fig. 35E) will have the same internal pyramids as the GD, and an external pyramid with triangular basis built on an enveloping convex icosahedron.

The GI (fig. 35F) is somewhat more complicated. The dipyramids of its structure are made up of 12 pentagrammal external pyramids, base to base with the 12 pentagonal internal pyramids of a convex dodecahedron with, for each dipyramid, a set of five *negative* dimples to subtract from the internal pyramid.

All these forms are easily divided into orthoschemes. The geometry of the pyramids and dimples being known, formulas similar to Schläfli's can be derived for the stellated forms.

### 3. The Four Regular Stellated Polyhedra

We now proceed to the study of the regular stellated polyhedra. We start therefore from the regular convex dodecahedron and the icosahedron. Lengthening the edges of the dodecahedron till they meet gives rise to **the small stellated dodecahedron** (fig. 35C). This is the first step in the stellation process previously described (fig. 35A(a)). This figure can be visualized as a regular dodecahedron on the faces of which pentagonal pyramids would be attached. Generally attributed to Kepler, its image can however be seen in a marble marquetry design in the Basilica San Marco, Venice (fig. 36). This design, dated 1425-27, is ascribed there to Paolo Uccello, monk and geometer friend of Leonardo da Vinci. Having 12 faces (the intersecting pentagrammal stars), it justifies its dodecahedral name<sup>23</sup>, in spite of its 20 vertices relating it with the icosahedron.

If the 12 vertices (apexes of the pyramid) are joined together, we obtain the figure of the regular convex icosahedron enveloping, so to speak, the small stellated dodecahedron. If the process of expansion of the edges (or faces) is continued, enveloping dodecahedra and icosahedra will keep alternating in an ever-expanding pulsation of growth. This process of growth obeys a geometric progression, as will be shown in section 4.

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<sup>22</sup> Kapraff, op. cit. p. 287ft.

<sup>23</sup> These names of the stellated polyhedra were introduced by the British mathematician Arthur Cayley in 1859.



fig. 36

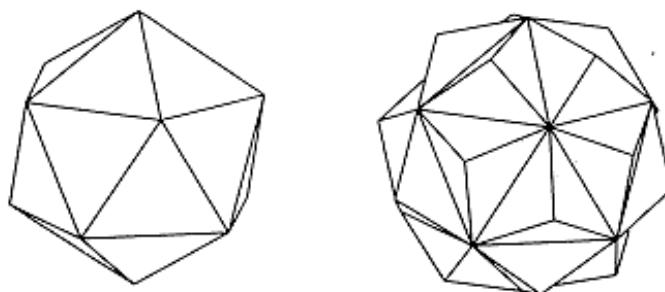
**A****B**

fig. 37

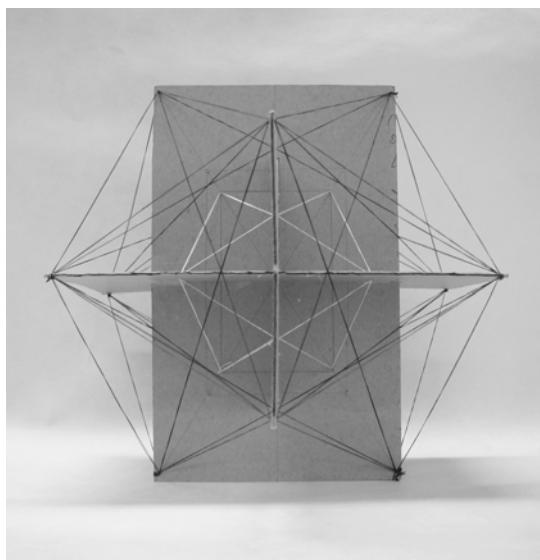


fig. 38

If the faces of the pyramids on the small stellated dodecahedron are extended, the gaps between the pyramids are eliminated and a trihedral dimple fills the gap between 3 consecutive pyramids. This corresponds to the second step of the stellation process (fig. 35A(b)). The resulting figure is **the great dodecahedron** (fig. 35D).

In this case the 12 intersecting pentagonal faces are highly visible as backplates to the three dimensional stars. We now continue with the enveloping icosahedron. This is the third step in stellation (fig. 35A(c)). Expanding the edges will give in the first step what is known as **the great stellated dodecahedron** (fig. 35E). It can be visualized as the original icosahedron on the faces of which, triangular pyramids would be attached. It has therefore 20 vertices (as the convex dodecahedron). If these vertices are joined together, we obtain the enveloping regular convex dodecahedron. As will be made clear subsequently, it has 12 faces made up of pentagrams.

Again, if the expansion process is allowed to proceed, the icosa-dodecahedral alternating growth pulsation will continue *ad infinitum*. Here the process of growth obeys a geometric progression of ratio 3. (see p. 77)

If the faces of the icosahedron (fig. 37A) are extended rather than the edges, a set of 20 short pyramids are built on each of the faces of the icosahedron resulting in the *triakis icosahedron* (fig. 37B), a figure that does not fully obey the rules of regularity.

Finally **the great icosahedron** (fig. 35F) can be considered as made up of a convex dodecahedron on the faces of which pyramids with 5 pointed star bases are set up. There are 12 such pyramids giving 12 vertices (as in the convex icosahedron). It is generated like the small stellated dodecahedron by extending the edges of the convex dodecahedron.

These, therefore, are the four stellated regular polyhedra. Their geometry is perhaps best understood by reference to the three mutually perpendicular golden rectangles that constitute their inner structure. We have previously seen that the regular convex icosahedron can be considered as having its 12 apexes distributed at the corners of three such rectangles. By the duality principle, these same points will be centers for the faces of the convex dodecahedron. Since the stellated figures result from extending either faces or edges of the regular convex dodecahedron or icosahedron, the same inner structure of golden rectangles will obtain.

By joining the corners of these planes by a line so that this line connects a corner of a plane in a given quadrant to the corner of a perpendicular plane in the adjacent quadrant in a straight line tangent to an edge of the third plane, it will be seen that 5 lines issue (or converge) at each corner as shown (fig. 38), one of them being the length side of the large golden rectangle. The result is that we can clearly see in turn:

- *the small stellated dodecahedron* with its underlying regular convex dodecahedron

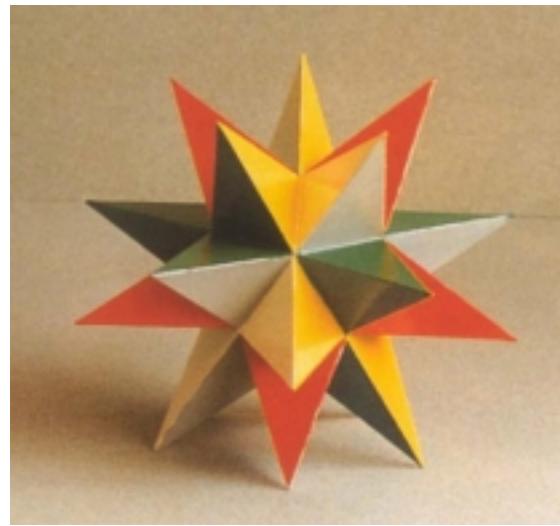


fig. 39

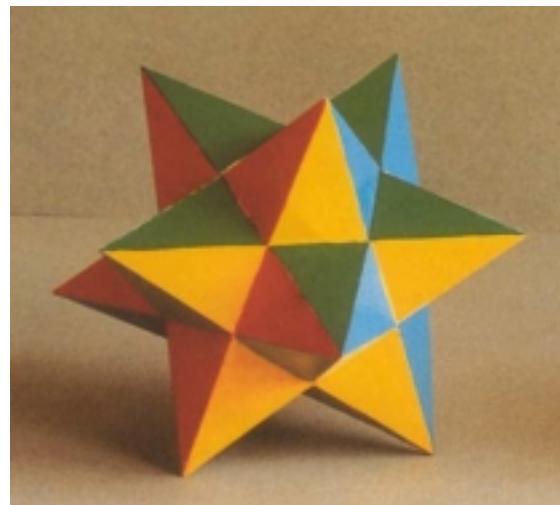


fig. 40

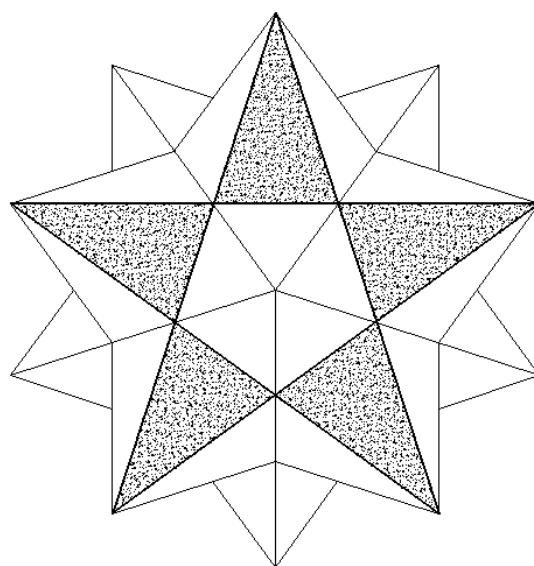


fig. 41

- *the great dodecahedron* by adding lines joining consecutive corners thus determining the enclosing regular icosahedron.
- *The great icosahedron* also appears as made up of five planes (equilateral triangles) at each corner of the golden rectangles, all having a common apex and intersecting one another so as to form a pentagram as their trace on the faces of the underlying convex dodecahedron. As will be seen in the study of the geometry of the great icosahedron, there are 20 such planes. Since we have 12 apexes determined by the 3 golden rectangles, we are indeed dealing with an icosahedron.
- *The great stellated dodecahedron* cannot be so readily seen on that model. As was previously explained, it can be visualized as a regular convex icosahedron on the faces of which triangular based pyramids are erected. Models of these pyramids can be built by extending the edges of the convex icosahedron (fig. 39). Adding them to the existing model reveals that the great stellated dodecahedron can be considered as made up of plane pentagrams. Each such pentagram has a pentagon at its center formed by the edges of the 5 faces of the icosahedron, these faces having a common apex.

Since there are 20 pyramids (apexes) and each pyramid has 3 sides, the number of plane pentagrams will be

$$\frac{20 \times 3}{5} = 12$$

Each of these plane pentagrams constitutes therefore a face of the stellated dodecahedron.

#### **a. *The small stellated dodecahedron (SSD)***

As we have seen, it can be visualized as 12 pyramids with pentagonal bases built onto each of the faces of the original dodecahedron (fig. 40). Alternatively it can also be seen as made up of 12 pentagrammal figures (5-pointed stars) as clearly shown on the same figure, joined by 3 at every of the 12 vertices of the underlying convex dodecahedron. Notice however that the vertices aren't vertices of the SSD. The vertices of the SSD are those corresponding to the points of the pentagrams where pentagrams come together by 5.

The stars having been stuck, so to speak, onto the faces of the dodecahedron, we want to determine the geometry of these pyramids and some of the properties of the small stellated dodecahedron, particularly in its relation to the Ad Quadratum method. To best understand this geometry it is easier to construct a model as previously shown on fig. 38. The diagrammatic view is as shown however on fig. 41 and represents the view of the small stellated dodecahedron along an axis passing through the vertices of 2 opposite pyramids (going through the centers of the opposite faces of the original dodecahedron

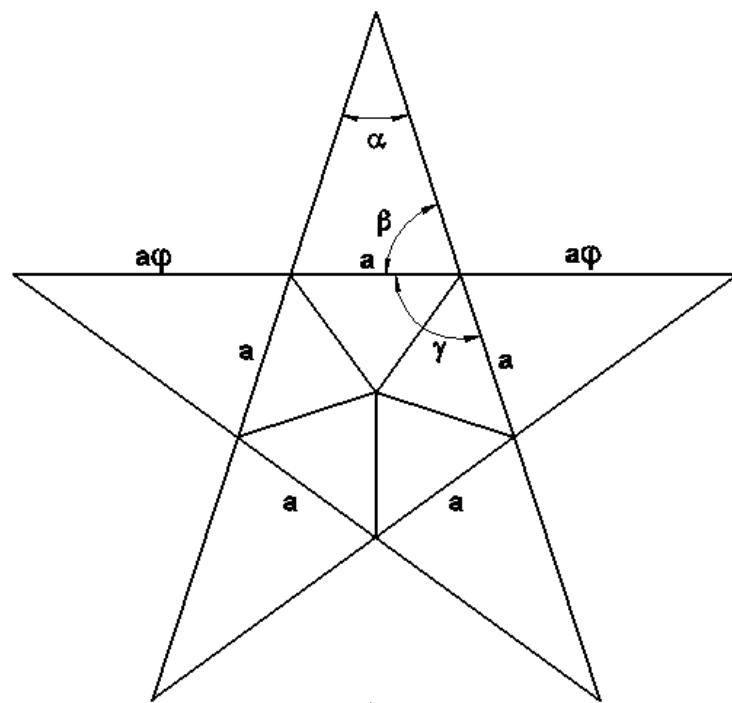


fig. 42

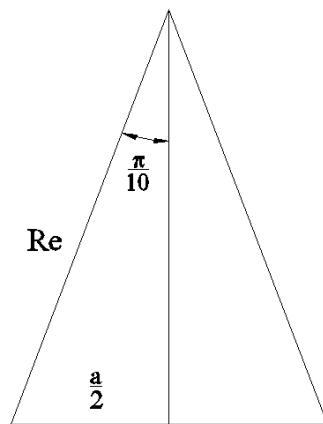


fig. 43

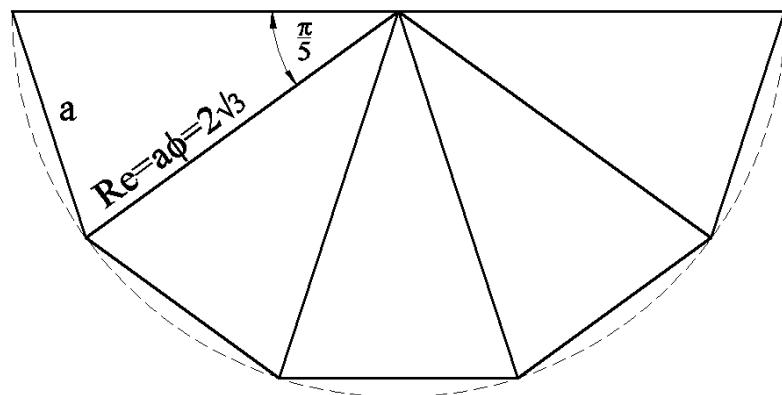


fig. 44

and therefore through the vertices of the icosahedron, dual of the original dodecahedron). What is then seen is a five-pointed star in true size, as shown in figure 42.

Note that each of the pyramids is made up of five isosceles triangles with base  $a$ , side of the pentagon.

We now show that these triangles are golden triangles of type 1. Since the angle at their base,  $\beta$  is

$$\beta = \pi - \gamma$$

with angle  $\gamma$ , angle of the pentagon, we can write

$$\gamma = \frac{3\pi}{5},$$

$$\therefore \beta = \pi - \frac{3\pi}{5} = \frac{2\pi}{5}$$

and therefore

$$\alpha = \pi - 2\beta,$$

or

$$\boxed{\alpha = \frac{\pi}{5}}$$

The pyramids having 5 faces with an apex angle of  $\frac{\pi}{5}$ , their development will be inscribed in a semi-circle of radius  $R_e$ , edge of the pyramid.

This radius  $R_e$  is easily calculated (fig. 43). We can write:

$$\sin \frac{\pi}{10} = \frac{a/2}{R_e}$$

$$\text{But } \sin \frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$$

$$\therefore \frac{a}{2R_e} = \frac{\sqrt{5}-1}{4}$$

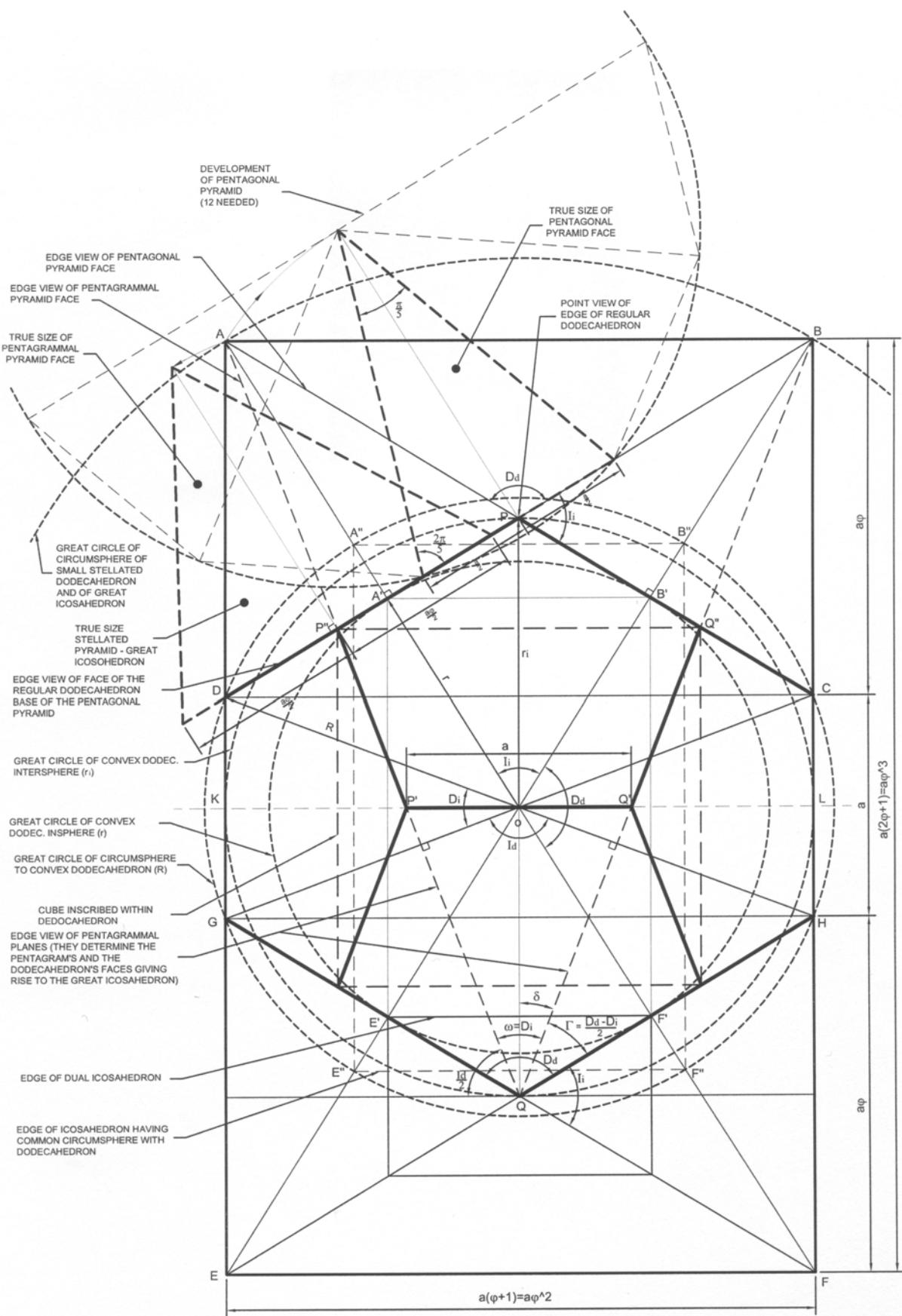
$$\text{or } \frac{a}{R_e} = \frac{\sqrt{5}-1}{2}$$

But

$$\frac{\sqrt{5}-1}{2} = \frac{1}{\phi},$$

where  $\phi$  is the golden ratio.

And we finally have



$$\frac{R_e}{a} = \phi$$

the well-known relation in pentagrams. It will be realized that each pyramid can be viewed as a folded pentagram constructed on the basis of the pentagonal face of the original dodecahedron.

Now  $a$  is the edge of the dodecahedron and we have seen that

$$a = \frac{\sqrt{3}}{3} (\sqrt{5}-1) R$$

where  $R$  is the circumradius of the dodecahedron.

$$\begin{aligned}\therefore R_e &= \frac{\sqrt{3}}{3} (\sqrt{5}-1) R \left( \frac{1+\sqrt{5}}{2} \right) \\ &= \frac{2}{3} \sqrt{3} R\end{aligned}$$

Now again, we will recall that  $\frac{2\sqrt{3}}{3} R$  is the edge of the cube having common circumsphere with the original dodecahedron.

This length is readily available on the Ad Quadratum diagram and for a circumsphere of radius 3, we have

$$R_e = 2\sqrt{3}$$

Having determined the construction of the pyramids (fig. 44) and therefore of the whole small stellated dodecahedron made up of these 12 pyramids, we now consider some geometric relationship in the figure as a whole.

The geometry of the S.S.D., like that of the convex regular dodecahedron and icosahedron, is governed by 3 mutually perpendicular golden rectangles. The internal or Maraldi angle of the stellated polyhedra will therefore be the same as for the convex polyhedra.

The rectangles are determined by the apexes of the four pyramids built on the opposite 2 pairs of contiguous faces of the original dodecahedron. We begin by looking at the original convex dodecahedron so that a pair of edges between 2 faces are seen as points  $P$  and  $Q$  (fig. 45).

The corresponding faces of the dodecahedron will be seen as edges ( $PD$ ,  $PC$ , and  $QG$ ,  $QH$ ), and so will the pyramids faces having a common edge corresponding to the point

view  $P$  and  $Q$  of the convex dodecahedron edges. The angle between the faces of the dodecahedron is the dihedral angle  $\hat{D}_D$  and since

$$\hat{D}_D = \pi - \hat{I}_i,$$

it follows that the dihedral angle  $C\hat{P}B$  between the base and the side of pyramids is also  $\hat{I}_i$  which implies that rectangle  $ABCD$  is indeed a golden rectangle.

Drawing the diagonals  $AF$  and  $BE$  in rectangle  $ABFE$  we see that  $ABFE$  is also a golden rectangle ( $A\hat{O}B = \hat{I}_i$ ), which implies that  $DCFE$  is a square, and  $EFGH$  a golden rectangle.

Since this geometry holds for any two opposite pairs of pyramids, it can therefore be seen that, by the effect of symmetry, we shall obtain 2 other such golden rectangles mutually perpendicular to  $ABFE$ .

Note that construction of the pyramids can be obtained directly from this figure through descriptive geometry: Having an edge view of one of the faces of the pyramid, we have the true height of the triangle constituting that face. A perpendicular at its base with a length of  $\frac{a}{2}$  marked off on either side of the perpendicular determines the face of the pyramid. (One can check, apex  $= \frac{\pi}{5}$  )

It can also be remarked here (as already noted) that the SSD can be considered as made up of 12 pentagrams connected 5 together at each corner of the 3 golden rectangles. We verify that there are 12 such planes (dodecahedron). Since each pyramid apex is a vertex of the SSD there are  $5 \times 12 = 60$  pyramidal sides altogether, and since each pentagram plane contributes 5 pyramidal sides there are  $60:5=12$  pentagrammal planes. An obvious conclusion given the fact that the pyramids were obtained in the first place by extending the 12 faces of the original convex dodecahedron.

The circumscribing half circle of the development  $\left(\frac{\pi}{5} \times 5 = \pi\right)$  can now be drawn and the development of the pyramid performed as shown on fig. 44.

The circumsphere to the SSD passes through the corners  $A, B, E, F$  of that golden rectangle. The radius of the circumsphere is therefore  $AO$ , which can be obtained directly from the diagram when the golden rectangle is established.

$AD$  is the true length of the pyramid edge, and since

$$AD = R_e$$

we can write

$$AD = a\phi$$

from

$$\frac{R_e}{a} = \phi$$

Since  $ABCD$  is a golden rectangle,

$$\begin{aligned} AB &= \phi AD \\ \therefore AB &= a\phi^2 \\ \text{But } \phi^2 &= (\phi + 1) \\ \therefore AB &= a(\phi + 1) = a\phi + a \\ \text{or } AB &= AD + a \end{aligned}$$

Where  $a$  is the edge of the convex dodecahedron and  $AD$  the edge of the cube inscribed in the same circumsphere, as previously seen.

$ABCD$  can therefore be constructed directly from measurements taken from the Ad Quadratum diagram (fig. 13).  $ABEF$  can therefore be obtained by adding a square  $DCFE$ .

Note also that  $ABEF$  could also be completed by remarking that  $DG = CH = a$  and that  $GHEF$  is another golden rectangle identical to  $ABCD$ . (A more complete analysis of the construction of the golden rectangle based on the adquadratum method is given below under Section 4: *Construction of the Golden Rectangles Determining the Structure of the Dodecahedra and Icosahedra.*)

We also verify that  $AE = AB\phi = a(\phi + 1)\phi = a\phi^3 = a(2\phi + 1)$ , which can also be seen directly on fig. 45, since

$$\begin{aligned} AE &= BF = BC + CH + HF \\ &= a\phi + a + a\phi = a(2\phi + 1) \end{aligned}$$

We now proceed to calculate the ratio of the SSD circumsphere radius  $R_s$  to the original dodecahedron circumsphere radius  $R$ .

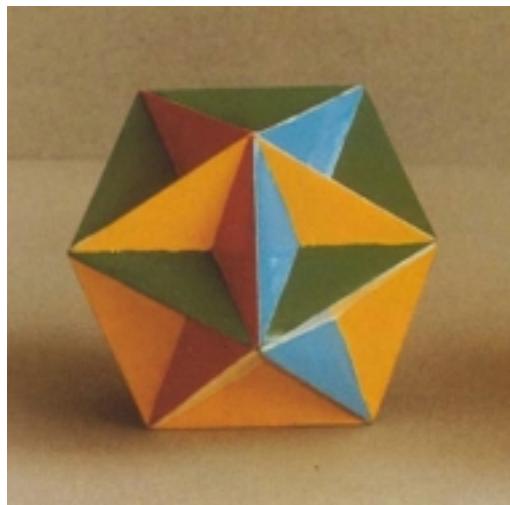


fig. 46A

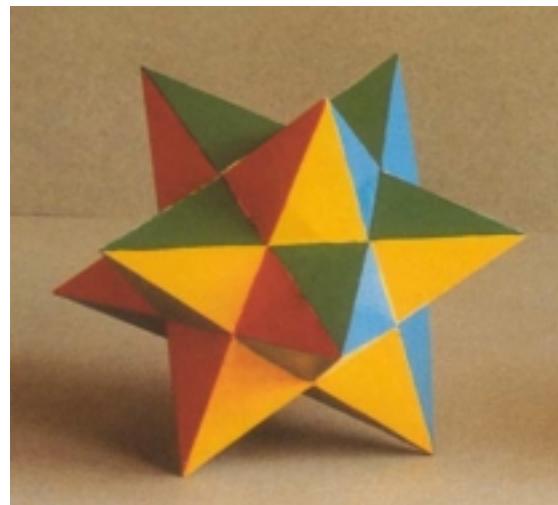


fig. 46B

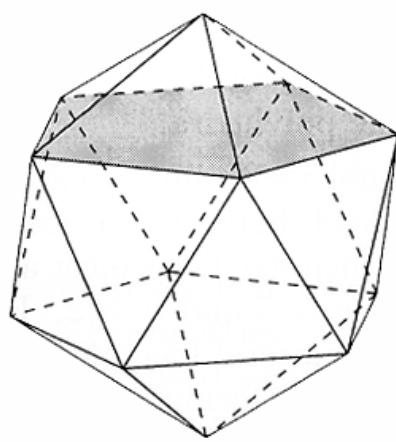


fig. 47A

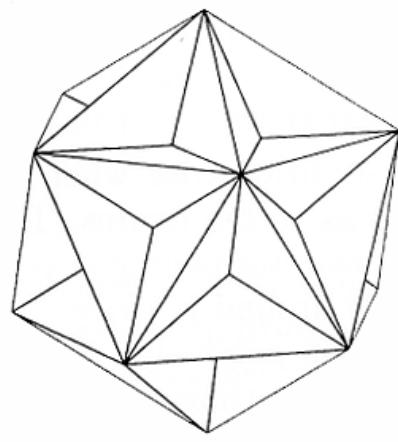


fig. 47B

We have:

$$R_s = OB \text{ and } OB = \sqrt{OL^2 + BL^2}$$

$$\text{Now } OL = \frac{a(1+\phi)}{2} = \frac{a\phi^2}{2}$$

$$\text{and } BL = a(1+\phi) \frac{\phi}{2} = \frac{a\phi^3}{2}$$

$$\therefore OB = \frac{a}{2}\phi^2\sqrt{1+\phi^2}$$

$$\text{with } a = \frac{\sqrt{3}}{3}(\sqrt{5}-1)R = \frac{\sqrt{3}}{3}\frac{2}{\phi}R$$

$$\therefore OB = \frac{\sqrt{3}}{3}\phi\sqrt{1+\phi^2}R$$

$$= \frac{\sqrt{3}}{3}\phi\sqrt{\phi+2}R$$

$$\therefore \frac{R_s}{R} = \phi\sqrt{\frac{\phi+2}{3}} \quad (\text{or } \frac{R_s}{R} = 1.776)$$

Noting that  $\cos\frac{Di}{2} = \frac{\phi}{\sqrt{3}}$  and  $\sin\frac{I_i}{2} = \frac{1}{\sqrt{\phi+2}}$ , we can write  $\frac{R_s}{R} = \frac{\cos\frac{D_i}{2}}{\sin\frac{I_i}{2}}$ , which we can

define as a growth factor  $g_{d_1}$ . If instead, we consider the ratio of  $R_s$  to the insphere radius as will be established for the GSD, we can define a new growth factor  $g'_d$ :

$$g'_d = \frac{R_s}{r} = \frac{r+h}{r} = \frac{R\cos\frac{D_i}{2}\cos\frac{I_i}{2} + a\phi\cos\frac{I_i}{2}}{R\cos\frac{D_i}{2}\cos\frac{I_i}{2}} = 1 + \frac{a\phi}{R\cos\frac{D_i}{2}}$$

with  $\cos\frac{D_i}{2} = \frac{\phi}{\sqrt{3}}$  and  $\frac{a}{R} = \frac{\sqrt{3}}{3}(\sqrt{5}-1)$  it comes  $\boxed{g'_d = \frac{R_s}{r} = \sqrt{5}}$ .

### b. The Great dodecahedron: (GD)

The same basic geometry will obtain for the GD as for the SSD since the GD can be considered as made up on the basis of the SSD by extending the planes of the stellated faces so as to fill the gaps between the star branches, thus forming a convex pentagram (fig. 46A and 46B). Alternatively, it can be viewed as made up of twelve intersecting pentagonal planes and the figure of the enveloping icosahedron clearly appears on fig. 47A and 47B.

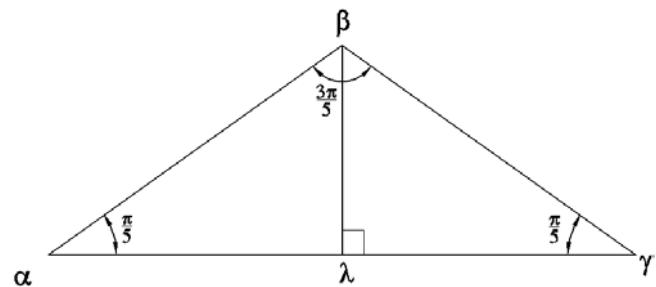


fig. 48

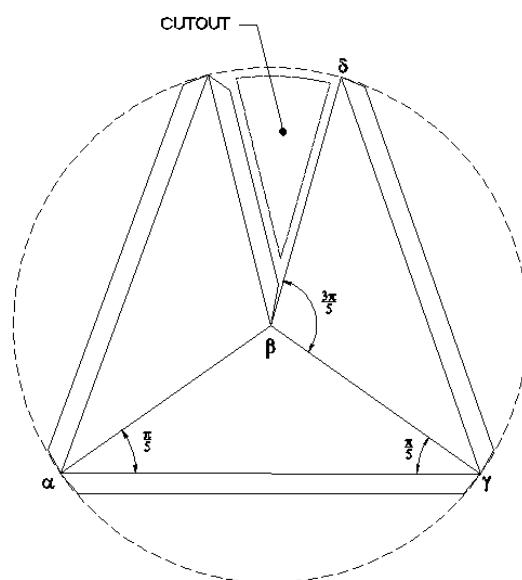


fig. 49

Fig. 46A shows such a pentagonal plane inscribed in the enveloping icosahedron. The 12 possibilities of arranging these planes in the icosahedron yield the GD. The GD can therefore be considered as made up of 20 trihedral dimples with the equilateral base and formed of isosceles triangles whose sides are the edges of the pyramids of the SSD and therefore equal to  $a\phi$  (with  $a\phi = \frac{2}{\sqrt{3}}R$ , the edge of the cube and  $a$ , the edge of the convex dodecahedron), and whose base is the distance between 2 consecutive vertices of the SSD.

From fig. 45, it is therefore seen that the distance between 2 consecutive vertices of the SSD is equal to  $AB = a(\phi + 1)$ .

Considering such a triangle  $\alpha\beta\gamma$  (fig. 48) :

$$\begin{aligned}\alpha\beta &= \beta\gamma = a\phi \\ \alpha\gamma &= a(\phi + 1) \\ \text{drop } \beta\lambda \text{ perpendicular to } \alpha\gamma \\ \alpha\lambda &= \frac{a(\phi + 1)}{2} \\ \therefore \cos\alpha &= \frac{\alpha\lambda}{\alpha\beta} = \frac{a(\phi + 1)}{2a\phi} = \frac{\phi^2}{2\phi} = \frac{\phi}{2} = \frac{1 + \sqrt{5}}{4} \\ \therefore \alpha &= \frac{\pi}{5} \quad (36^\circ) \\ \therefore \beta &= \frac{3\pi}{5} \quad (108^\circ)\end{aligned}$$

Triangle  $\alpha\beta\gamma$  is a golden triangle type 2.

The dimples are readily constructed from the Ad Quadratum diagram since

$\alpha\beta$ =edge of cube

$$\left( a\phi = \frac{2R}{\sqrt{3}} \right)$$

$\alpha\gamma$ =edge of cube+edge of original dodecahedron ( $a\phi + a$ ) taken directly from the diagram (fig. 13).

Therefore, (fig. 49) to build the dimple that fits between 3 consecutive pyramids, draw a circle of radius equal to the edge of the cube and mark off along the circumference the length of 3 chords each equal in length to  $\alpha\gamma$ . Cut out and fold as indicated. When glued along  $\beta\delta$  one obtains an element of a dimple.

20 such glued together along the 3 edges (such as  $\alpha\gamma$ ) of their base will yield the GD.

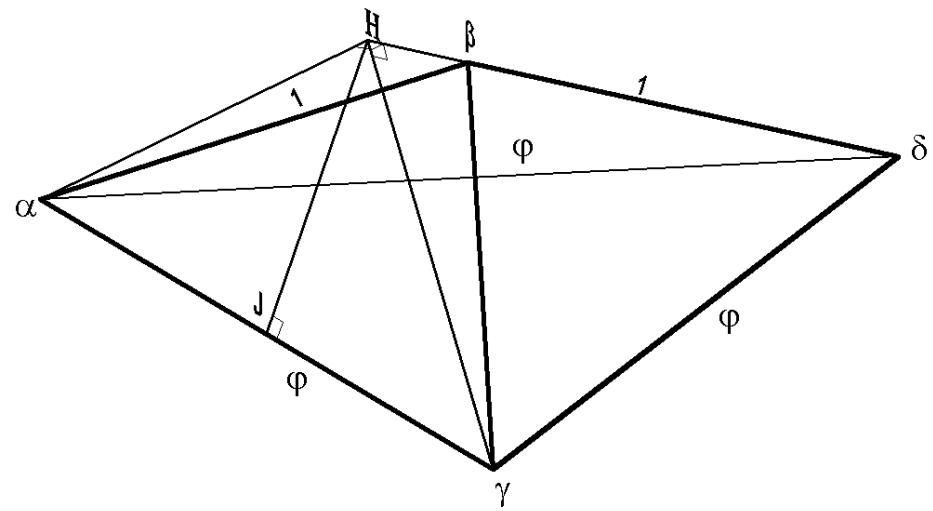


fig. 50

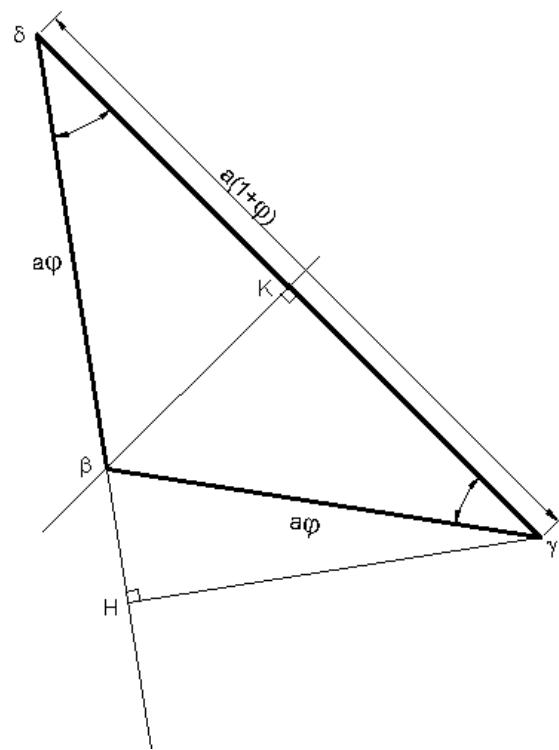


fig. 51

The inner geometry of the GD will be similar to that of the SSD.

Note that the 20 dimple centers correspond to the apexes of an inner dodecahedron and that the GD can be considered as a concave icosahedron where each triangle face of the regular icosahedron is replaced by trihedral dimple.

We also mentioned that the GD can be considered as made up of 12 pentagonal planes.

Considering one such plane as the basis, five other planes will cut it along traces forming a pentagram on its surface. Together these 6 planes determine half the structure of the GD. A similar set, symmetrically positioned, will complete the structure. These planes are made very visible on the model of fig. 46A and on fig. 46B.

Given the pentagonal form of the plane (constructed ad quadratum as shown later: fig. 53), the GD will be determined by the knowledge of the dihedral angle between such intersecting planes.

This dihedral angle is easily found from the geometry of the dimples just established.

The base  $\alpha\gamma\delta$  is an equilateral triangle of side  $a(\phi+1)$  (fig. 50).

$\beta$  is the peak of the dimple. The sides of the pyramidal dimple such as  $\alpha\beta$  are equal to  $a\phi$ .

Dropping the perpendiculars  $\gamma H$  and  $\alpha H$  on the extension of  $\beta\delta$ , we see that the dihedral angle  $\hat{G}$  between planes  $\alpha\beta\delta$  and  $\gamma\beta\delta$  is  $\alpha\hat{H}\gamma$ .

Now, dropping perpendicular  $HJ$  on  $\alpha\gamma$ , we can write in triangle  $JH\gamma$ :

$$\sin \frac{\hat{G}}{2} = \frac{J\gamma}{H\gamma}$$

and:

$$J\gamma = \frac{\alpha\gamma}{2} = \frac{a}{2}(\phi+1)$$

To find  $H\gamma$ , consider triangle  $\beta\delta\gamma$ , reproduced in fig. 51 with perpendiculars  $\gamma H$  and  $\beta K$  and in true size.

$$\delta K = \frac{a}{2}(1+\phi)$$

$$\therefore \cos \delta = \frac{\frac{a}{2}(1+\phi)}{a\phi} = \frac{(1+\phi)}{2\phi}$$

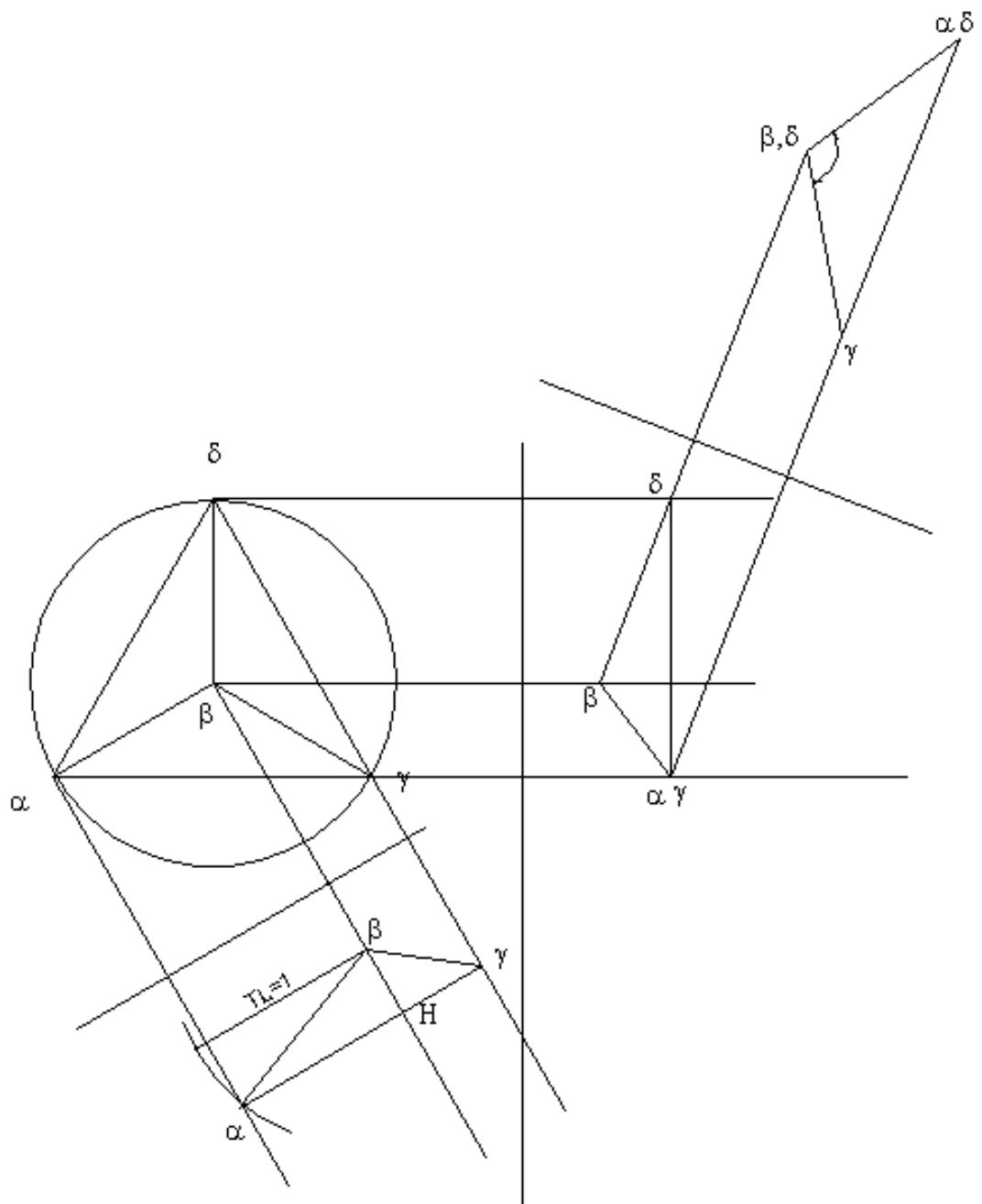


fig. 52

$$= \frac{\phi^2}{2\phi} = \frac{\phi}{2}$$

Now

$$H\gamma = \delta\gamma \sin \delta$$

And

$$\begin{aligned}\sin \delta &= \sqrt{1 - \cos^2 \delta} = \sqrt{1 - \frac{\phi^2}{4}} \\ &= \frac{1}{2} \sqrt{3 - \phi}\end{aligned}$$

$$\therefore H\gamma = \frac{a}{2} (1 + \phi) \sqrt{3 - \phi}$$

and

$$\sin \frac{\hat{G}}{2} = \frac{J\gamma}{H\gamma}$$

$$\sin \frac{\hat{G}}{2} = \frac{\frac{a}{2} (1 + \phi)}{\frac{a}{2} (1 + \phi) \sqrt{3 - \phi}} = \frac{1}{\sqrt{3 - \phi}}$$

$$\sin \frac{\hat{G}}{2} = 0.85065$$

$$\therefore \frac{\hat{G}}{2} = 58^\circ 28$$

so that

$$\boxed{\hat{G} = 116^\circ 56}$$

Such is the dihedral angle of the pentagonal planes, of course the same as the dihedral angle of the convex dodecahedron already calculated.

Naturally, a complete ruler and compass descriptive geometry construction is possible starting from the top view of a dimple to find the point view of  $\beta\delta$  (or any other edge such as  $\beta\alpha$  or  $\beta\gamma$ , and therefore the true size of  $\hat{G}$ . The construction is shown on fig. 52.

### **c. The Great Stellated Dodecahedron: GSD (Kepler)**

If the edges of the convex icosahedron enveloping the SSD and GD are extended, they form triangular based pyramids on each of the 20 faces of this icosahedron (fig. 53). The geometry of these pyramids, as in the case of the SSD, determines the geometry of the

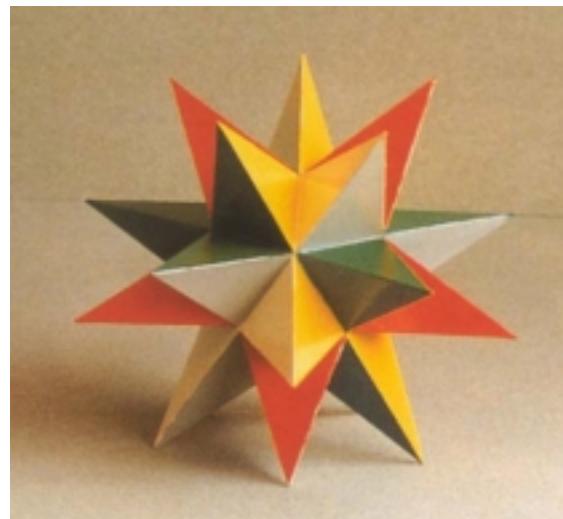


fig. 53

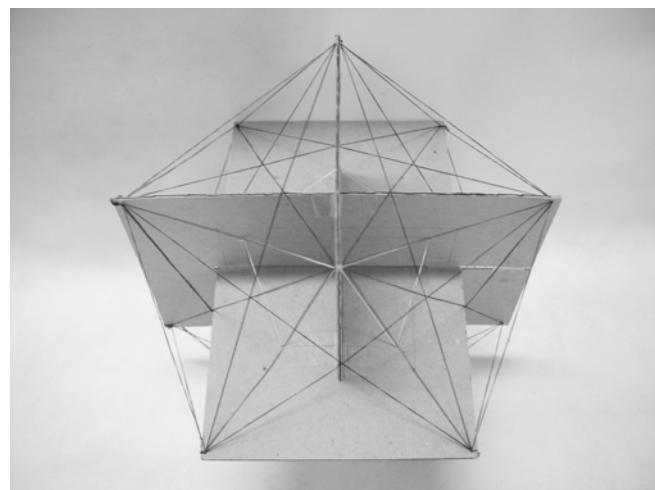


fig. 54

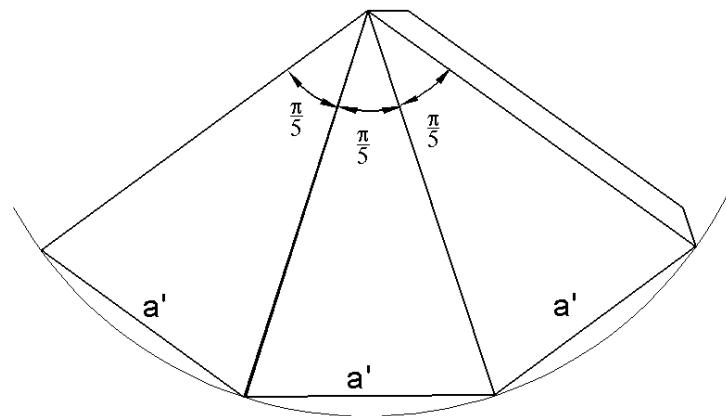


fig. 55

GSD. Since the apexes of the pyramids can be projected towards the center of the icosahedron through the center of its faces, we see that the Maraldi angle of the GSD will be the same as that of the convex dodecahedron.

If we look at the convex icosahedron along a line of sight coinciding with an axis going through polar opposite vertices, we see the figure of a plane pentagon (fig. 54). The perimeter of that pentagon is made up of the edges of the icosahedron seen in true length. Their extension will therefore show the sides of the pyramids in true size, and as in the case of the SSD, it is seen that these pyramid sides are constituted of golden triangles of type 1. (base  $a'$ , sides  $a'\phi$ , angles at base  $\frac{3\pi}{5}$ , apex angle  $\frac{\pi}{5}$ ,  $a'$  being the edge of the icosahedron enveloping SSD and GD, i.e., edge of GD with  $a' = a(\phi+1)$  and  $a$  edge of the original dodecahedron.)

Each pyramid being made up of three sides, its development is readily obtained in a circular section of radius  $a'\phi$  (fig. 55). Taking the base  $a'$  from fig. 45 (width of the Great Golden Rectangle  $ABEF$ ),  $a'\phi$  is readily obtained as the length of that same rectangle.

The geometry of the GSD is easily determined by looking at the convex icosahedron as shown in fig. 56 where the top and bottom edges  $CC'$  and  $VV'$  are seen as points. Then edges  $AB$ ,  $PL$  and  $FM$  are seen in true length and  $PC$ ,  $CF$ ,  $MV$ ,  $LV$  are the edge views of the 2 top and bottom faces respectively. (For convenience of pagination fig. 56 has been turned by  $90^\circ$ .)

In the two pyramids built on the upper faces,  $PN$  and  $FE$  will be true length edge ( $= a'\phi$ );  $N\bar{C}\bar{C}'$  and  $E\bar{C}\bar{C}'$  will be respectively the edge view of one of the pyramidal sides of the left and right pyramids. From the geometry of the icosahedron, we indicate on the figure the values of the angles and lengths.

We now want to calculate the radius  $R_G$  of the circumsphere of the GSD:

We have

$$R_G = OE \text{ with } OE = \frac{OD}{\sin \frac{D_i}{2}}$$

Now

$$OD = \frac{a'\phi}{2} \text{ and } \sin \frac{D_i}{2} = \frac{1}{\phi\sqrt{3}}$$

So that

$$\begin{aligned} R_G &= \frac{a'\phi}{2} (\phi\sqrt{3}) \\ &= a' \frac{\sqrt{3}}{2} \phi^2 \\ &= a' \frac{\sqrt{3}}{2} (\phi+1) \end{aligned}$$

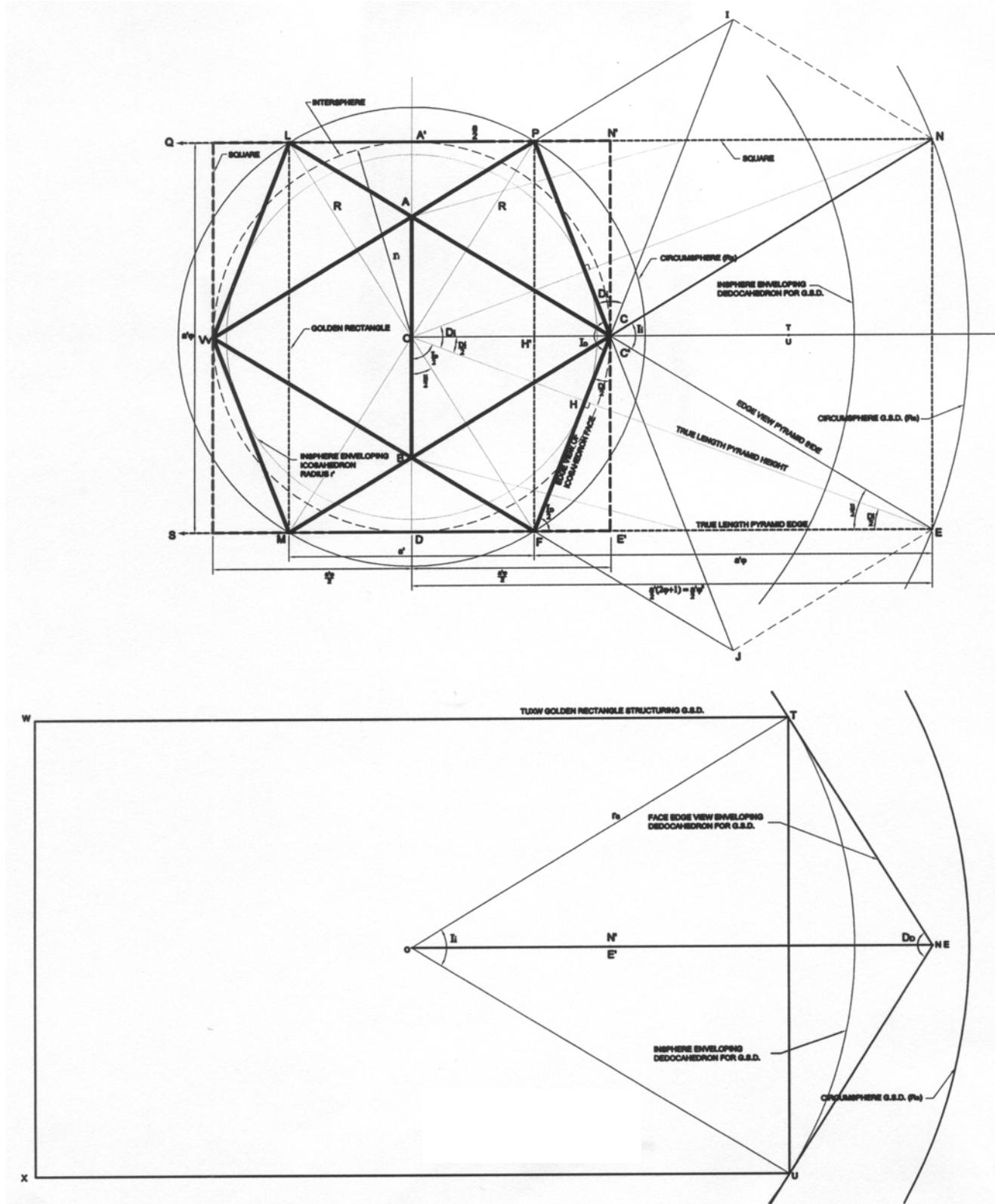


fig. 56

The circumsphere radius of the enveloping icosahedron  $R_s$  is:

$$\begin{aligned} R_s = OF &= \sqrt{OD^2 + DF^2} = \sqrt{\frac{a'^2\phi^2}{4} + \frac{a'^2}{4}} \\ &= \frac{a'}{2}\sqrt{\phi+2} \end{aligned}$$

We can therefore define a “growth factor”  $g_d|_3$  from  $GD$  to  $GSD$ :

$$g_d|_3 = \frac{R_G}{R_S} = \frac{\frac{a'\sqrt{3}}{2}(\phi+1)}{\frac{a'}{2}\sqrt{\phi+2}}$$

or

$$g_d|_3 = \sqrt{3} \frac{(\phi+1)}{\sqrt{\phi+2}} = 2.3839$$

which can be rewritten, remarking that

$$\frac{\phi}{\sqrt{\phi+2}} = \cos \frac{I_i}{2} \text{ and } \frac{1}{\sqrt{\phi+2}} = \sin \frac{I_i}{2}$$

$$g_d|_3 = \sqrt{3} \left( \cos \frac{I_i}{2} + \sin \frac{I_i}{2} \right)$$

or alternatively:

$$g_d|_3 = \sqrt{6} \sin \left( \frac{I_i}{2} + \frac{\pi}{4} \right)$$

In the study of the SSD (p. 65), we have established that the ratio of the circumspheres  $R_s$  to  $R$ , i.e. that of the SSD to that of the original or kernel dodecahedron was:

$$\frac{R_s}{R} = \frac{\phi}{\sqrt{3}} \sqrt{\phi+2} = 1.776$$

Now, the growth factor of the second stellation  $g_d|_2$  from SSD to GD is  $g_d|_2 = 1$ , since SSD and GD have the same circumsphere. We have also shown that the growth factor from GD to GSD  $g_d|_3$  is:

$$g_d|_3 = \frac{R_G}{R_S} = \sqrt{3} \frac{(\phi+1)}{\sqrt{\phi+2}}$$

so that the overall growth factor from  $R$  to  $R_G$  is:

$$\begin{aligned} G_d &= \frac{R_G}{R} = \frac{R_G}{R_S} \times \frac{R_S}{R} = \left[ \sqrt{3} \frac{(\phi+1)}{\sqrt{\phi+2}} \right] \left[ \frac{\phi}{\sqrt{3}} \sqrt{\phi+2} \right] \\ &= \phi(\phi+1) \\ &= \phi^3 = 2\phi + 1 = 2 + \sqrt{5} = 4.236 \end{aligned}$$

A complete stellation process from kernel dodecahedron to GSD is therefore governed by a growth factor  $\phi^3$ .

The stellation process of the dodecahedron normally stops there, but a new process can be started on the basis of the enveloping convex dodecahedron in a geometric progression of ratio  $\phi^3$  *ad infinitum*.

If we go from step to step, i.e., from kernel dodecahedron to icosahedron, enveloping GD and SSD to dodecahedron enveloping GSD, we have a rhythmical pulsation governed in turn by  $g_d|_1$  and  $g_d|_3$ .

However, if instead of considering the growth from circumsphere to circumsphere, we take it from the face of the kernel icosahedron to the peak of the pyramid of the GSD, then it is the ratio of the icosahedron insphere radius to the GSD circumsphere radius that we must examine.

The height of the pyramid:

$$HE = h = a' \phi \cos \frac{D_i}{2}$$

Now  $R_G$  can also be written

$$R_G = OH + HE$$

with  $OH=r'$  = insphere radius of icosahedron enveloping SSD and GD.

$$\therefore R_G = r' + h$$

But we have previously established that

$$\frac{r}{R} = \cos \frac{D_i}{2} \cos \frac{I_i}{2} = \frac{r'}{R_S}$$

We can therefore define a new growth factor  $g'_d|_3$  as:

$$g'_d|_3 = \frac{R_G}{r'} = \frac{r' + h}{r'} = \frac{R_S \cos \frac{D_i}{2} \cos \frac{I_i}{2} + a' \phi \cos \frac{D_i}{2}}{R_S \cos \frac{D_i}{2} \cos \frac{I_i}{2}}$$

where  $R_s$  is the circumsphere of the enveloping icosahedron.

$$\therefore g'_d|_3 = 1 + \frac{a'\phi}{R_s \cos \frac{I_i}{2}}$$

$$= 1 + \frac{a'}{R_s} \sqrt{\phi+2}$$

$$\text{but } \frac{a'}{R_s} = \frac{\sqrt{10}}{5} \sqrt{5-\sqrt{5}} \text{ for the icosahedron}$$

now

$$\sqrt{\phi+2} = \sqrt{\frac{5+\sqrt{5}}{2}}$$

so that

$$\frac{a'}{R_s} \sqrt{\phi+2} = \frac{\sqrt{10}}{5\sqrt{2}} \sqrt{5-\sqrt{5}} \sqrt{5+\sqrt{5}} = 2$$

∴

$$g'_d|_3 = 1 + 2 = 3$$

or

$$R_G = 3r'$$

We now look at the growth from kernel dodecahedron circumsphere  $R$  to  $r'$  insphere radius of enveloping icosahedron of SSD and GD.

We have established

$$\frac{R_s}{R} = \frac{\phi}{\sqrt{3}} \sqrt{\phi+2}$$

On the other hand, insphere to circumsphere radius ratio for icosahedra and dodecahedra gives:

$$\begin{aligned} \frac{r'}{R_s} &= \cos \frac{D_i}{2} \cos \frac{I_i}{2} \\ \therefore \frac{r'}{R} &= \frac{R_s \cos \frac{D_i}{2} \cos \frac{I_i}{2}}{R} = \frac{\left( R \frac{\phi}{\sqrt{3}} \sqrt{\phi+2} \right) \left( \cos \frac{D_i}{2} \cos \frac{I_i}{2} \right)}{R} \end{aligned}$$

and, noticing that

$$\cos \frac{D_i}{2} = \frac{\phi}{\sqrt{3}} \text{ and } \cos \frac{I_i}{2} = \frac{\phi}{\sqrt{\phi+2}}$$

$$\frac{r'}{R} = \frac{\phi}{\sqrt{3}} \left( \frac{\phi^2}{\sqrt{3}} \right) = \frac{\phi^3}{3}$$

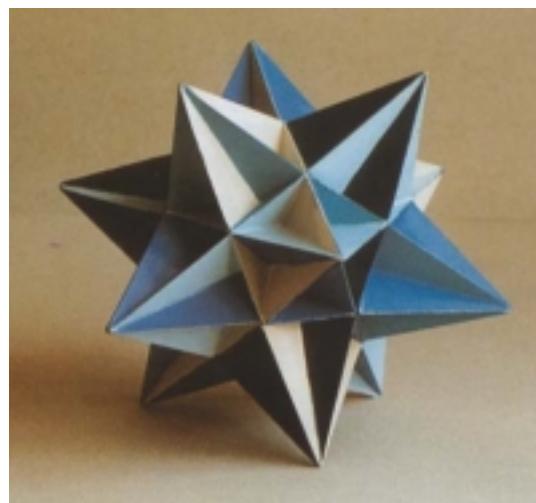


fig. 57

$$= \frac{2\phi+1}{3} = 1.412$$

And since

$$\frac{R_G}{r'} = 3$$

We verify that

$$\frac{R_G}{R} = \frac{R_G}{r'} \times \frac{r'}{R} = 3 \left( \frac{\phi^3}{3} \right) = \phi^3$$

as established p.76.

#### **d. The Great Icosahedron (GI) (Poinsot)**

As was pointed out in section 2.b., the case of the stellation pattern of the icosahedron is more complex than that of the dodecahedron since all but one of its 59 stellations is regular. This is the one of interest to us here and its generation by means of face-stellation is illustrated on fig. 35. B (c). Designating by  $a|_{ic}$ , the edge of the kernel icosahedron, reserving  $a|_{ic}$  for the edge of the icosahedron having common circumsphere with the convex dodecahedron kernel of the SSD, GD, and GSD just studied, the edge of the GI will be seen from fig. 35B(c) to be

$$a|_{GI} = 2^3 a|_{ic} = 8a|_{ic}$$

If we therefore took for kernel the icosahedron of edge  $a|_{ic}$ , the figure would be quite out of scale in our study. However, as we mentioned earlier, the Great icosahedron can be visualized as a regular dodecahedron on the pentagonal faces of which pentagrammal (five pointed star basis) pyramids would be erected. Such a figure will have 12 vertices, and, as we shall see, can be considered as having 20 faces, qualifying it as an icosahedron (fig. 57).

Starting from the regular convex dodecahedron as in the case of the small stellated dodecahedron (SSD), the Great icosahedron (GI) will be very similar geometrically to the SSD. In particular, the GI having 12 vertices evenly distributed in space around its center, will have the same inner structure as the SSD made up of three mutually perpendicular golden rectangles, the corners of which are the apexes of the 12 pyramids. The pyramids in both cases will have the same height and the external edge of the pyramids will be the same, namely,  $a\phi$  in both cases,  $a$  being here the edge of the kernel dodecahedron inscribed in the circumsphere of radius  $R$ .

Therefore, the edge of the pentagonal face of the dodecahedron will constitute the ridge of a dimple since the inner edge of the pyramid will extend within beyond the dodecahedron faces. The 12 vertices will then be seen as arising out of pentagonal dimples, as in the case of the GSD the vertices were seen as arising out of triangular dimples.

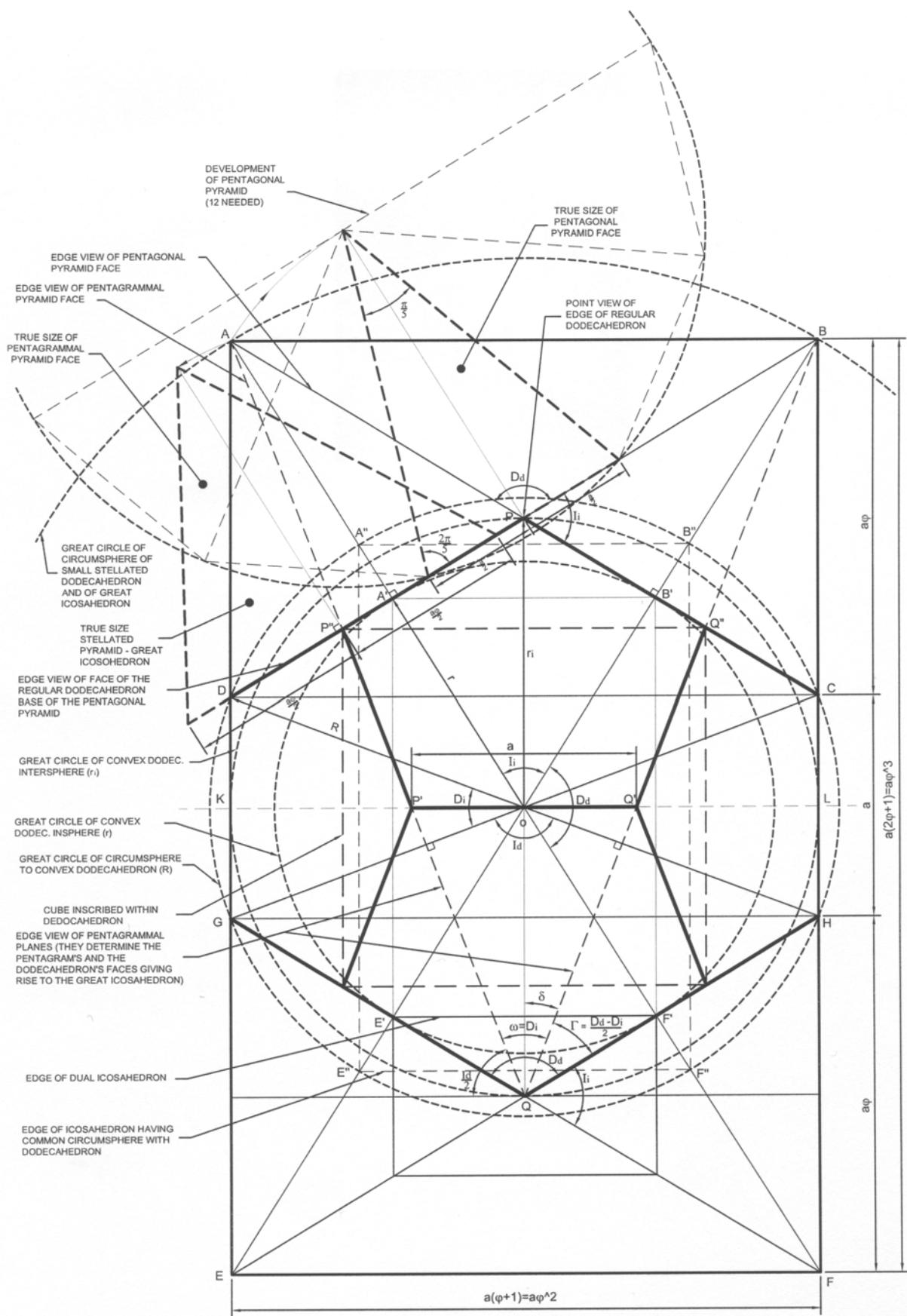


fig. 45

The intersection of these pyramids with the faces of the convex dodecahedron determine pentagrams. Since the side of the pentagram is also the diagonal of the pentagon, that diagonal will have the same length  $a\phi$ . It follows that the portion of the pyramids of the GI above the faces of the dodecahedron can be geometrically considered as made up of five intersecting equilateral triangles with a common apex and a pentagrammal base. Visually and externally, they appear as indented SSD pyramids.

To understand the geometry of the GI we start with a view of the underlying regular dodecahedron, such that top and bottom edges appear as points ( $P$  and  $Q$  on fig. 45). Another edge,  $P'Q'$ , will then be seen to be horizontal and in true length in the middle of the figure. As in the case of the SSD, the figure is framed by the great golden rectangle  $ABEF$ , which also appears in true size.

The bottom edge of the regular dodecahedron being seen as a point at  $Q$ , can be considered the point view of the line of intersection of two planes (seen on edge  $QA$  and  $QB$ ) whose traces at  $Q''$  and  $P''$  on the faces of the regular dodecahedron facing the viewer (edge views:  $PC$  and  $PD$ ) constitute sides of the pentagram, basis of the GI pyramid on that face. The edge view of these planes follows along the side of the visible upper face of the regular dodecahedron ( $PC$  and  $PD$ ) and further to the upper corners of the great golden rectangle (at points  $A$  and  $B$ ).

Each of these planes, viewed in true size, will be seen to be equilateral triangles with side equal to ( $a\phi + a + a\phi = a(1 + 2\phi)$ ) as shown on fig 49. Note that the three equilateral triangles with sides  $a\phi$  in each corner of the planes belong to the pyramids of the GI, their sides being the edges of the pyramids, and  $a$ =edge of convex dodecahedron.

Since these pyramids are made up of 5 such intersecting triangles (basis disposed along the lines of the pentagram) there must be  $5 \times 12 = 60$  such triangles altogether for the GI.

Given that each plane contributes 3 of them, it follows that the number of planes within the GI must be  $\frac{60}{3} = 20$ . These 20 planes meeting at 12 apexes are the faces of the GI.

The scale factor caused by our choosing the dodecahedron of edge  $a$ , designated  $a|_{\text{dod}}$  in this context, circumsphere  $R$ , rather than the icosahedron of edge  $a|_{ic}$  can now be evaluated.

We begin by noting that for convex polyhedra of common circumsphere we can write:

$$\begin{aligned}\frac{a|_{ic}}{a|_{\text{dod}}} &= \frac{\frac{\sqrt{10}}{5}\sqrt{5-\sqrt{5}}}{\frac{\sqrt{3}}{2}(\sqrt{5}-1)} \\ &= \sqrt{3} \cos \frac{I_i}{2} = 1.4733\end{aligned}$$

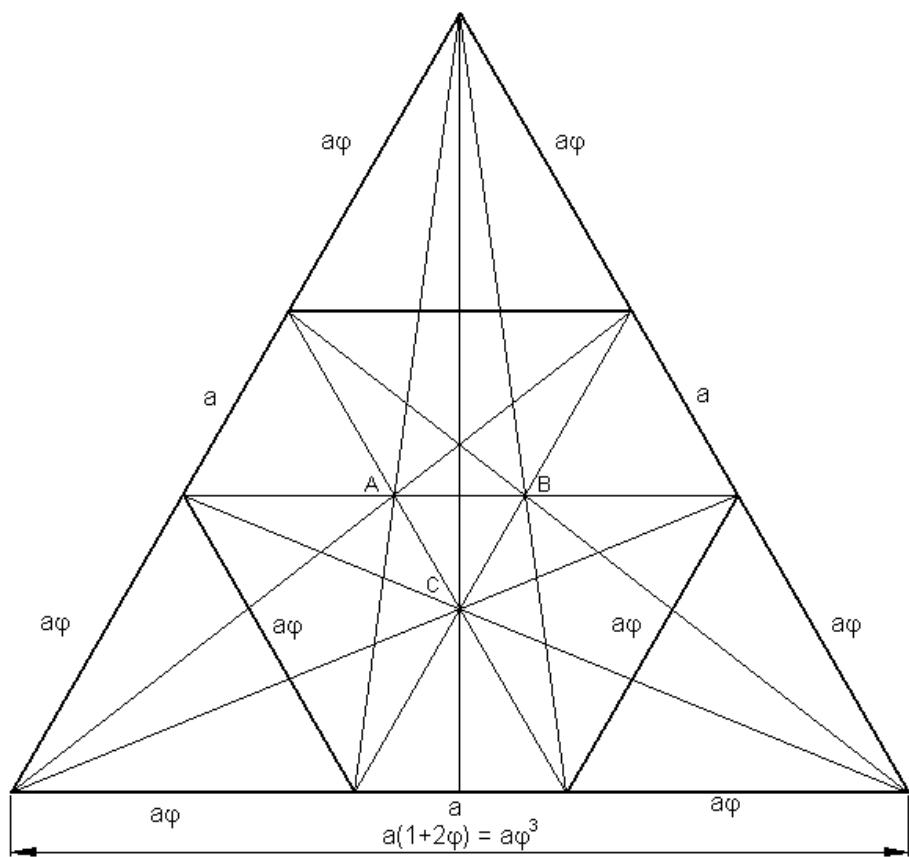


fig. 58

Now, as has just been established, the edge of the GI as shown in fig. 45 and fig. 58 is:

$$a|_{\text{dod}}(1+2\phi)$$

which can be written:

$$a|_{\text{dod}}\phi^3$$

The scale factor to consider is therefore

$$S_f = \frac{8a|_{ic}}{a|_{\text{dod}}\phi^3}$$

or, replacing from above:

$$S_f = \frac{8}{\phi^3} \sqrt{3} \cos \frac{I_i}{2}$$

with  $\cos \frac{I_i}{2} = \frac{\phi}{\sqrt{\phi+2}}$

it comes  $S_f = \frac{8\sqrt{3}}{\phi^2 \sqrt{\phi+2}}$

After some calculations and reduction we have:

$$S_f = \frac{8\sqrt{3}}{\sqrt{11\phi+7}} = 2.7826$$

This is the number we should multiply measurements on say fig. 45 to find the actual dimension on a model having for kernel the icosahedron of edge  $a|_{ic}$ .

The question is what is the size of the original kernel icosahedron that would give rise to the GI built on the dodecahedron we started with.

We have designated its edge as  $a|_{ic}$  so that we can write:

$$8a|_{ic} = a|_{\text{dod}}\phi^3$$

or  $a|_{ic} = \frac{\phi^3}{8}a|_{\text{dod}}$

$$= \frac{(2\phi+1)}{8}a|_{\text{dod}}$$

$$= \left(\frac{\phi}{4} + \frac{1}{8}\right)a|_{\text{dod}} = 0.5295a|_{\text{dod}}$$

For example, for  $R = 3$ ,  $d|_{\text{dod}} = 3 \times 2 \sin \frac{D_i}{2} = 2.1408$   
and  $d|_{ic} = 1.1335$ .

We can now also calculate the growth factor of the regular icosahedra stellation process. We start from the convex icosahedron inscribed in the circumsphere of radius  $R$ . The first *regular* stellation is as shown fig. 35B(c) and produces a GI with a circumsphere we will designate  $R_I$ .

A growth factor  $G_i$  can therefore be defined as:

$$G_i = \frac{R_I}{R}$$

Let  $R'_I$  be the circumsphere radius of the *scaled down* GI represented by  $gi$  of fig. 45. We see on that figure that both  $gi$  and SSD have a common circumsphere of radius  $R_s$ , so that

$$R'_I = R_s$$

On the other hand, using the scaling factor  $S_f$  we can write:

$$R_I = R'_I S_f$$

But since we have also shown (p. 65) that

$$\frac{R_s}{R} = \phi \sqrt{\frac{\phi+2}{3}}$$

we can write:

$$\frac{R_I}{R} = \frac{R_s S_f}{R} = \left( \phi \sqrt{\frac{\phi+2}{3}} \right) \left( \frac{8\sqrt{3}}{\phi^2 \sqrt{\phi+2}} \right)$$

or finally:

$$\begin{aligned} G_i &= \frac{8}{\phi} = 8(\phi - 1) \\ &= 4.9443 \end{aligned}$$

The figure enveloping the GI is a convex icosahedron so that the process can start again afresh from that point. We have therefore a geometric series of ratio  $\frac{8}{\phi}$  for the expansion of GI.

To get a clearer understanding of the GI geometry, i.e., of the relationship between the 20 planes in space, we consider the dihedral angle  $w$  between two such planes, intersecting along the edge of the regular dodecahedron, and seen on edge on fig. 45.

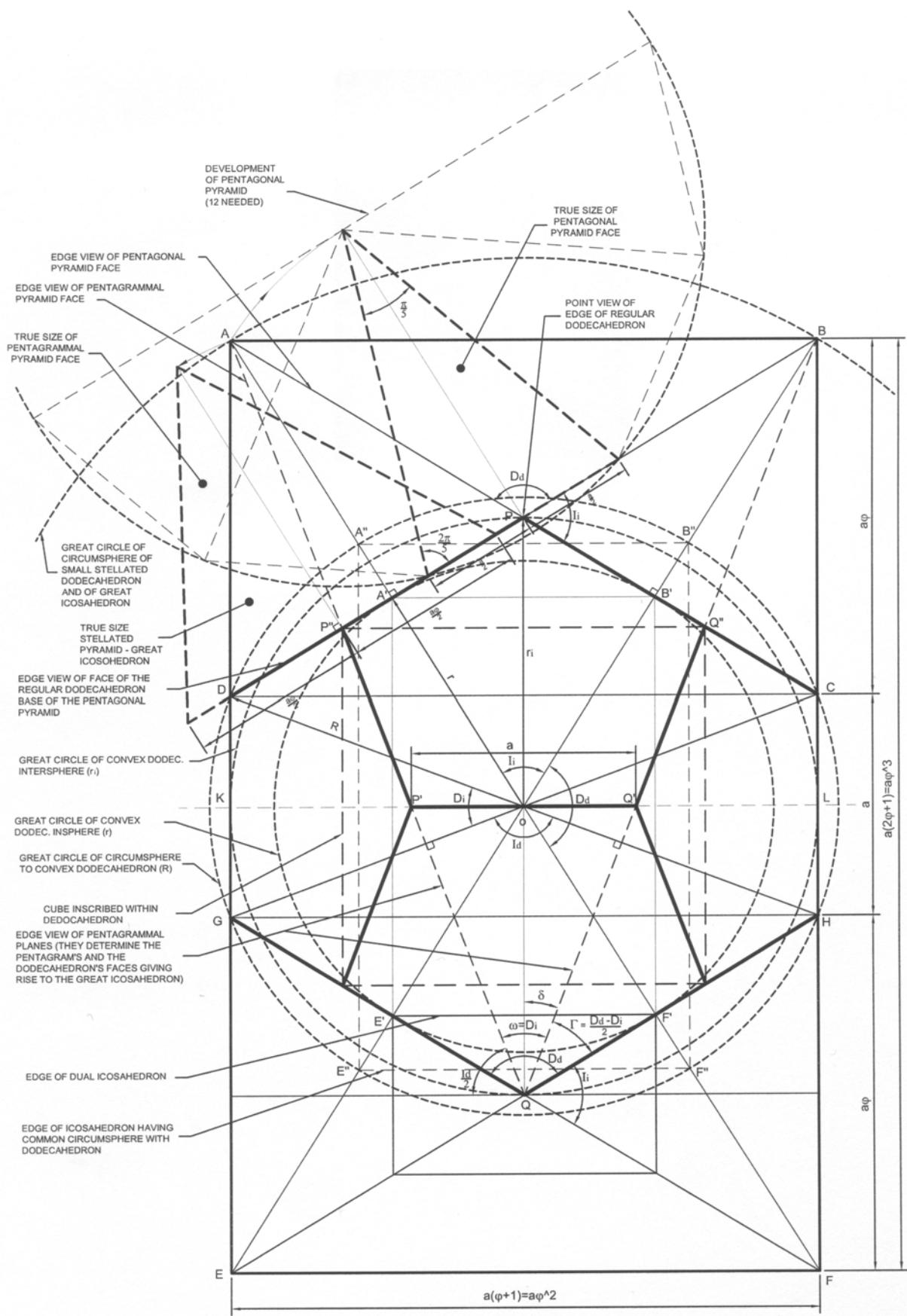


fig. 45

We can write

$$w = 2\delta$$

and remembering that

$$(OQ = r_i)$$

we have

$$\tan \delta = \frac{a/\sqrt{2}}{r_i}$$

where  $a$  = the edge of regular convex dodecahedron  
and  $r_i$  = radius of intersphere of that same dodecahedron.

But as we previously established, for the dodecahedron,

$$\frac{r_i}{R} = \frac{\text{intersphere radius}}{\text{circumsphere radius}}$$

$$\frac{r_i}{R} = \cos \frac{\hat{D}_i}{2} = \frac{\phi}{\sqrt{3}}$$

with  $\hat{D}_i$  = regular convex dodecahedron internal angle.

$$\therefore \tan \delta = \frac{a/\sqrt{2}}{R \cos \frac{\hat{D}_i}{2}} = \frac{a\sqrt{3}}{2\phi R}$$

However, we have also seen that for the dodecahedron:

$$a = \frac{\sqrt{3}}{3} (\sqrt{5} - 1) R$$

$$= \frac{\sqrt{3}}{3} \frac{2}{\phi} R$$

$$\therefore R = \frac{3\phi}{2\sqrt{3}} a = \frac{\sqrt{3}}{2} \phi a$$

$$\therefore \tan \delta = \frac{1}{\phi^2} = 2 - \phi$$

since

$$2 = \phi + \frac{1}{\phi^2}.$$

Making use of the identity

$$\tan w = \tan 2\delta = \frac{2 \tan \delta}{1 - \tan^2 \delta}$$

We have

$$\tan w = \frac{\frac{2}{\phi^2}}{1 - \frac{1}{\phi^4}} = \frac{2}{2\phi - 1}$$

With

$$\phi = \frac{1 + \sqrt{5}}{2}$$

$$\tan w = \frac{2}{\sqrt{5}} \quad (\tan w = 0.8944)$$

$$\therefore w = 41^\circ 81.$$

This is the angle we had previously determined to be the internal angle for the regular convex dodecahedron  $\hat{D}_i$ , whose sin was found to be  $\frac{2}{3}$ .

Given the following identity:

$$\frac{1}{\cos \delta} = \sec \delta = \sqrt{1 + \tan^2 \delta}$$

we indeed verify that

$$\begin{aligned} 1 + \tan^2 \delta &= 1 + \left( \frac{1}{\phi^2} \right)^2 \\ &= 1 + (2 - \phi)^2 \\ &= 1 + 4 - 4\phi + \phi^2 \\ &= 5 - 4\phi + (\phi + 1) \\ &= 3(2 - \phi) \\ &= \frac{3}{\phi^2} \\ \therefore \cos \delta &= \frac{\phi}{\sqrt{3}} \end{aligned}$$

as we previously established for  $\frac{D_i}{2}$ .

$$\cos \frac{\hat{D}_i}{2} = \frac{\phi}{\sqrt{3}}$$

$$\therefore \delta = \frac{\hat{D}_i}{2} \text{ and therefore } w = \hat{D}_i.$$

(As we have  $\sin \hat{D}_i = \frac{2}{3}$ , we also have  $\sin w = \frac{2}{3}$ .)

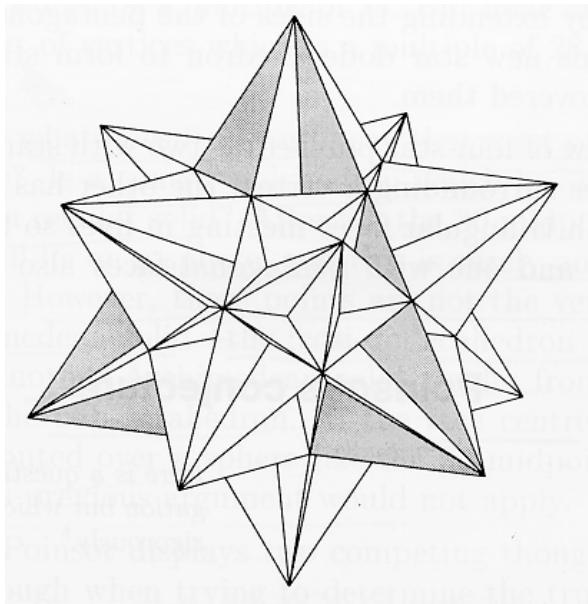


fig. 35F

The structure of this scaled G.I. can therefore be considered as made up of 20 equilateral triangles (sides  $a(1+2\phi)$ ) equal to the length of the great golden rectangle  $ABFE$  of fig. 45.

The corners of such triangles will be gathered together in groups of five at the 12 corners of 3 mutually perpendicular golden rectangles. The sides opposite to a given corner will form a pentagram. These five planes will therefore intersect one another as seen on fig. 35F).

Each side of the pentagram, forms the hinge for another similar plane (equilateral triangle set at a dihedral angle  $w = \hat{D}_i$ ).

Fig. 45 indicates that the lines  $OG$  and  $OH$  to these planes from center  $O$  form an angle equal to  $\hat{I}_D$ , the regular icosahedron dihedral angle. Now, since  $\hat{I}_D + \hat{D}_i = \pi$ , by symmetry of the figure, the angles of  $OG$  and  $OH$  with the respective planes are therefore equal to  $\frac{\pi}{2}$ , i.e., they are perpendicular to the planes showing the relationship between the geometry of the convex and of the great icosahedra.

The scaled GI, the SSD and the GD have the same circumsphere. They also have the same kernel: the convex dodecahedron. Their growth factor will therefore be the same. That of the GI stellated on the icosahedron of circumsphere  $R$  will have however a different growth factor of  $\frac{8}{\phi}$  as previously shown.

#### **4. Construction of the Golden Rectangles Determining the Structure of Dodecahedra and Icosahedra**

The three mutually perpendicular golden rectangles provide a unified, visually coherent reference system that accounts for all regular polyhedra and “explains” their shape and interdependence. It is directly related to the internal or Maraldi angles that we have been developing with their implicit spherical coordinate system as well as the Cartesian coordinate planes that the golden rectangles naturally constitute.

We therefore begin by considering the construction of the golden rectangles, structure of the dodecahedron and icosahedron. As we have seen, that structure applies both to convex and stellated forms.

##### *a. Case of Convex Polyhedra*

First consider the case where the icosahedron has its vertices at the center of the faces of the dodecahedron (dual) (fig. 25B and 26). This is the situation which obtains when we address the generation of the stellated polyhedra as previously described in section 2. Here, referring to fig. 45, both convex icosahedron and dodecahedron can be taken as seeds of the expansion resulting in the generation of the stellated figures. Then the set of

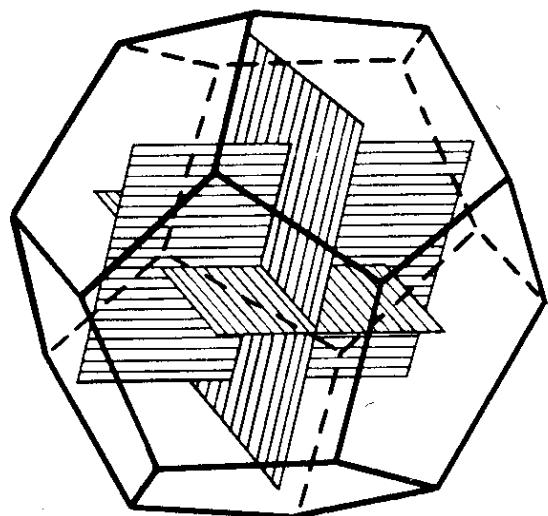


fig. 25B

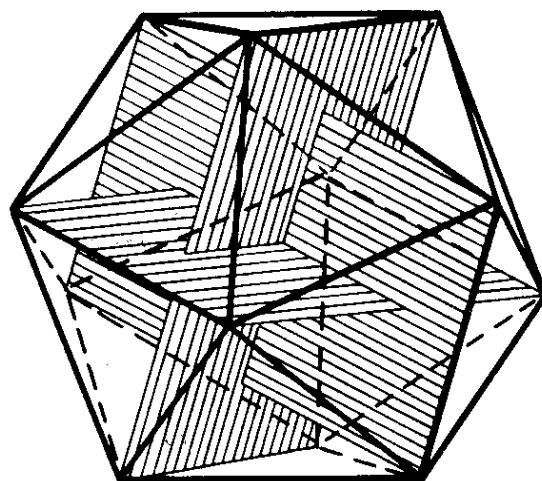


fig. 26

the three mutually perpendicular golden rectangles is common to both forms and the icosahedron's circumsphere radius will be equal to the dodecahedron insphere radius.

$$\text{i.e., } R|_{\text{icos.}} = r|_{\text{dodec.}} = OA' \quad (\text{fig. 45})$$

One of the Golden rectangles appears as  $A'B'F'E'$ .

The construction of the golden rectangle derives directly from the ad-quadratum figure (fig. 59) and results from the relation previously established for the dodecahedron:

$$r = R \cos \frac{\hat{D}_i}{2} \cos \frac{\hat{I}_i}{2}$$

Referring to fig. 59, based on fig. 13:

Bisect  $\hat{D}\hat{O}R$  to determine  $D'$

$$OD' = R \cos \frac{\hat{D}_i}{2}$$

Rotate  $OD'$  to  $OD''$  on  $OR$ .

Bisect  $\hat{W}\hat{O}R$  with  $OW'$  and project  $D''$  on  $OW'$  at  $D'''$

$$\text{Then } OD''' = R \cos \frac{\hat{D}_i}{2} \cos \frac{\hat{I}_i}{2}$$

Therefore

$$OD''' = r$$

Draw circle, radius  $OD'''$  and mark points  $W'', R', S', U'$ .

Rectangle  $W''R'S'U'$  is the golden rectangle desired and is the same as rectangle  $A'B'F'E'$  of fig. 45.

Second, consider the case where both icosahedron and dodecahedron have the same circumsphere.

$$\text{i.e., } R|_{\text{icos.}} = R|_{\text{dodec.}} = R$$

as we have assumed in our study of the convex polyhedra. It has been shown that, in that case, they also have a common insphere, i.e.,  $r|_{\text{icos.}} = r|_{\text{dodec.}}$ .

The sets of golden rectangles establishing the structure of these polyhedra will then be of different size though they may have the same orientation.

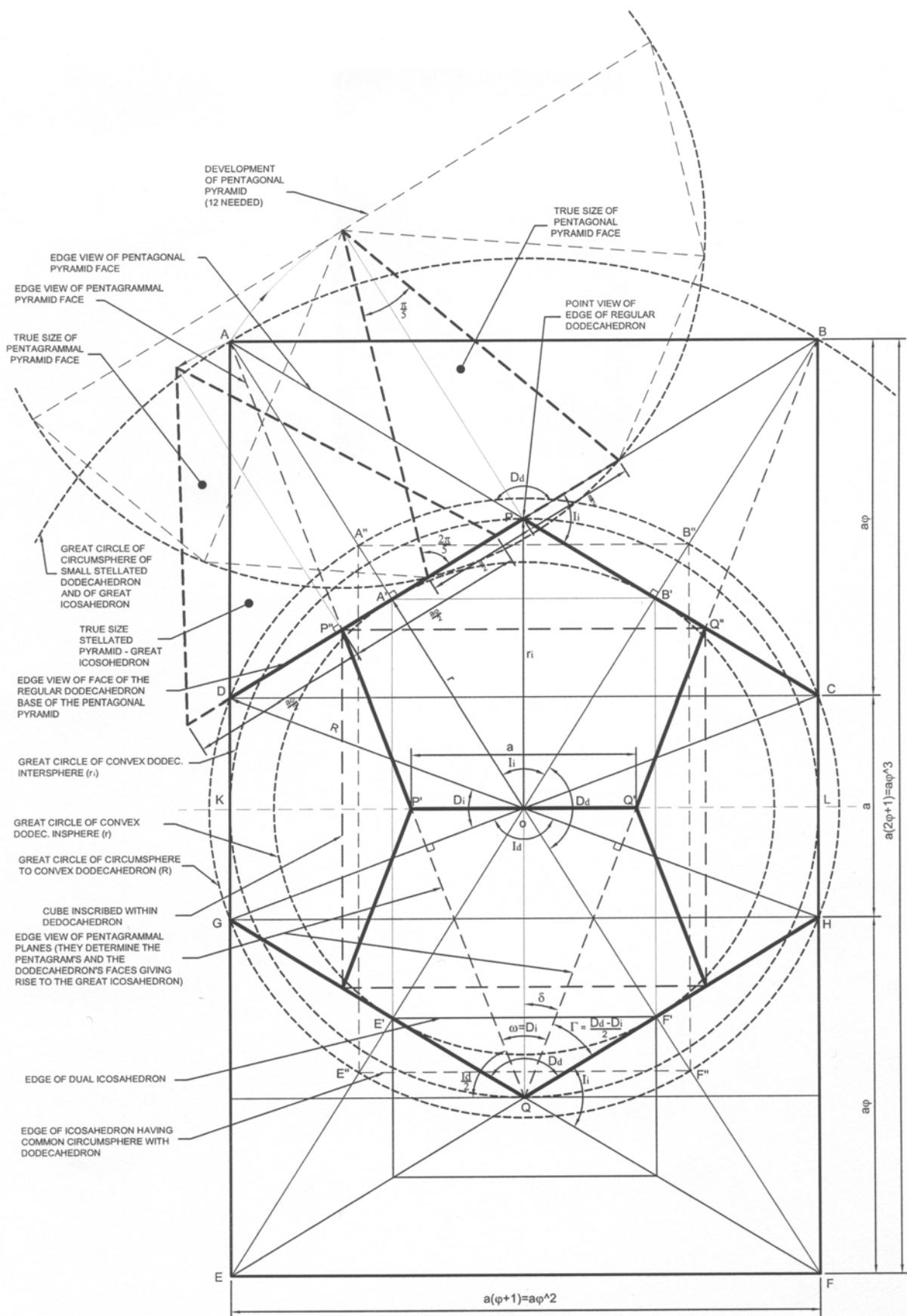


fig. 45

- For the dodecahedron, the golden rectangles will be identical to those of case (a).
- For the icosahedron, the half diagonal will have the value  $R$  and the width of the golden rectangle will equal in length the edge of the icosahedron. The golden rectangle appears as  $A''B''F''E''$  on fig. 45. The construction also follows immediately from the Ad-Quadratum diagram (fig. 59):

Extend  $WO$  to  $S''$  and join  $W$  to  $U$  to  $S''$  to  $R$ .  $WRS''U$  is the golden rectangle desired.

We now turn our attention to the stellated polyhedra.

### **b. Case of stellated polyhedra**

We want to construct the sets of three mutually perpendicular golden rectangles, which determine the structure of the small stellated dodecahedron (SSD), the great stellated icosahedron (GSI), the great dodecahedron (GD) and the great stellated dodecahedron (GSD).

- *Case of SSD:*

The SSD can be viewed as previously explained at the beginning of section 2 as made up of a basic convex dodecahedron on the faces of which pentagonal pyramids have been erected. If the convex dodecahedron we started with has a circumsphere of radius  $R$ , then the width of the golden rectangle structuring the SSD, namely  $ABFE$  on fig. 45, is equal to the diameter of the dodecahedron intersphere (radius  $r_i$ ) as can be seen from the figure, so that as previously established:

$$\frac{r_i}{R} = \cos \frac{\hat{D}_i}{2}$$

$$\text{i.e., width} = 2r_i = 2R \cos \frac{\hat{D}_i}{2}$$

Now  $R \cos \frac{\hat{D}_i}{2}$  is already available from the previous construction (fig. 59, as  $OD''$ ), so that the required golden rectangle can be also constructed ad-quadratum as shown on that figure.

We verify that, as we have seen

$$\cos \frac{\hat{D}_i}{2} = \frac{\phi}{\sqrt{3}}$$

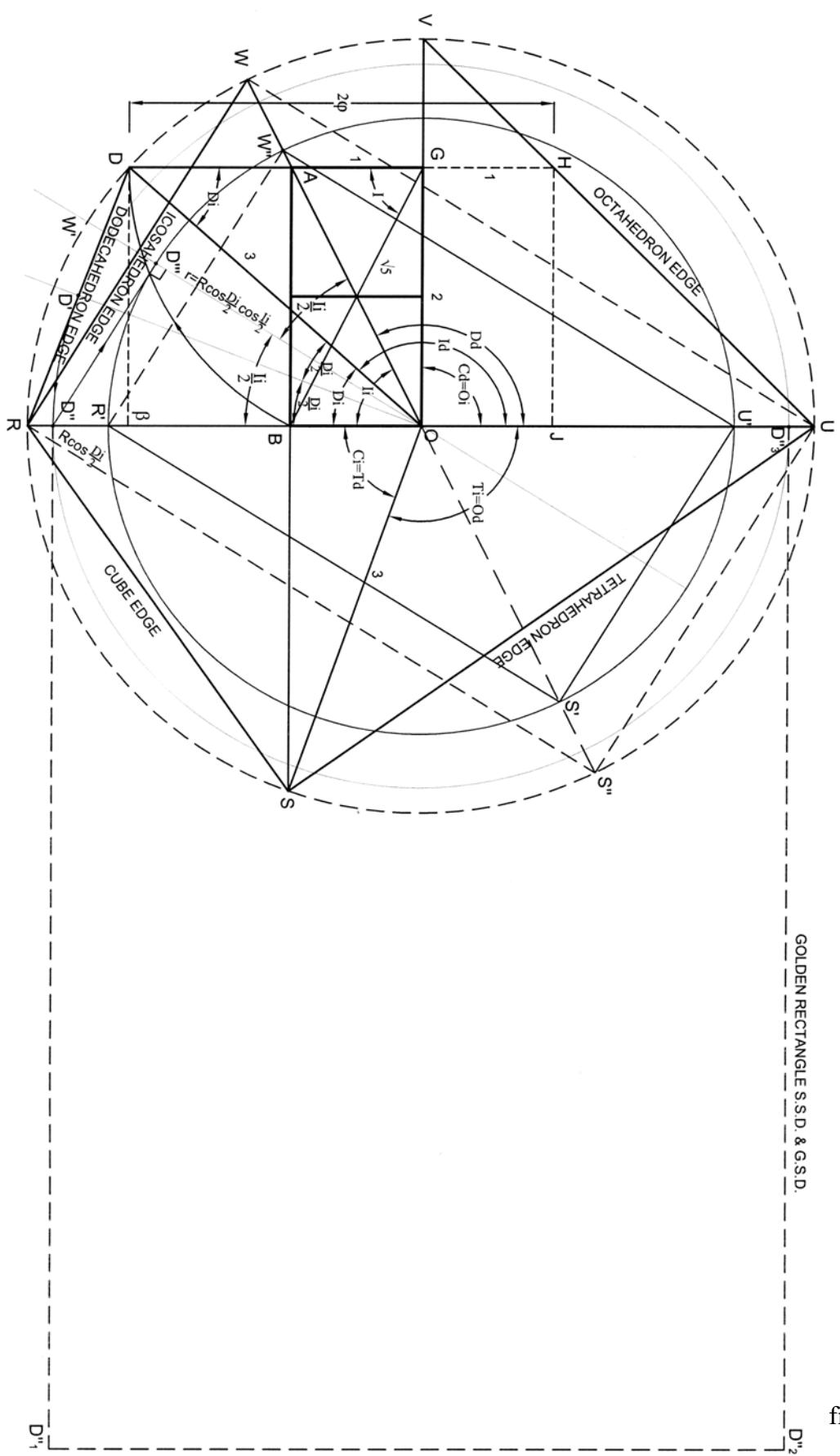


fig. 59

so that

$$2R \cos \frac{\hat{D}_i}{2} = 2R \frac{\phi}{\sqrt{3}}.$$

For the dodecahedron:

$$\begin{aligned} R &= \frac{a\sqrt{3}(1+\sqrt{5})}{4} \\ &= \frac{a\sqrt{3}\phi}{2} \end{aligned}$$

giving finally

$$\begin{aligned} 2R \cos \frac{\hat{D}_i}{2} &= 2 \left( \frac{a\sqrt{3}}{2} \right) \frac{\phi^2}{\sqrt{3}} \\ &= a\phi^2, \end{aligned}$$

as was established from fig. 45 for the width of the great golden rectangle.

- *Case of the GD:*

Since the G.D. can be considered as obtained by extending the faces of the pyramids of the SSD, the same golden rectangles as in the previous case will hold. In fact, the length of the top edges of the five pointed stars that appear on the GD faces are equal to the width of the golden rectangle as fig. 45 clearly shows and forms the edges of an enveloping convex icosahedron. This is seen as distance AB between the vertices of the pyramids. It also happens to be equal to the kernel convex dodecahedron intersphere diameter.

- *Case of the GSD:*

From the study of the geometry of the GSD done under section 3C above, we see that the GSD will have a circumsphere of radius  $R_G$ ,

$$\begin{aligned} R_G &= \phi^3 R \\ \text{with } &= (2\phi+1)R \end{aligned}$$

The GSD has the same Maraldi angle as the convex dodecahedron as previously explained, i.e., its 20 vertices are projections of the convex dodecahedron vertices onto the sphere of radius  $R_G$ . We can also remark that by joining the GSD vertices, the enveloping dodecahedron appears.

The golden rectangles that govern the structure of the GSD will therefore be inscribed into a great circle of the sphere of radius  $R_G$ . The diagonals of such a rectangle form the angle  $I_i$  at the center. The width  $W$  and length  $L$  of the rectangle can therefore be written:

$$W = 2R\phi^3 \sin \frac{I_i}{2} = \frac{2}{\sin \frac{I_i}{2}} R$$

$$L = 2R\phi^4 \sin \frac{I_i}{2} = \frac{2\phi}{\sin \frac{I_i}{2}} R$$

or:

$$W = 2 \csc \frac{I_i}{2} R = 3.804 R$$

$$L = 2\phi \csc \frac{I_i}{2} R = 6.155 R$$

$$\begin{aligned} \text{For } R=3 \\ W &= 11.412 \\ L &= 18.465 \end{aligned}$$

- *Case of the GI:*

The great golden rectangle common to SSD and gi is *ABEF* on fig. 45. The sides are  $a_d \phi^2$  and  $a_d \phi^3$  respectively.

On the other hand, we have established (p. 83) the scaling factor  $S_f$  between gi and GI:

$$S_f = \frac{8\sqrt{3}}{\phi^2 \sqrt{\phi+2}}$$

We can therefore write for the GI golden rectangle:

$$\begin{aligned} L &= a_d \phi^3 S_f \\ &= a_d \frac{\phi 8\sqrt{3}}{\sqrt{\phi+2}} \end{aligned}$$

$$a_d = \frac{2}{\sqrt{3}\phi} R$$

but

$$\therefore L = \frac{2\phi}{\sqrt{3}\phi} \frac{8\sqrt{3}}{\sqrt{\phi+2}} R$$

$$L = \frac{16}{\sqrt{\phi+2}} R = 16 \sin \frac{I_i}{2} R = 8.411 R$$

and

$$W = \frac{16}{\phi \sqrt{\phi+2}} R = 16(\phi-1) \sin \frac{I_i}{2} R = 5.198 R$$

## Alternative Methods of Generating the Regular Polyhedra

### 1. Historical Perspective

The word *generation* is telling since it implies a living process, a transmission of the *genus* from one generation to the next, from one form to the next, as we read in the biblical account the succession of the generations: one individual *begetting* another who in turn *begets* others, down.

Indeed, this is very much the way these figures have been thought of from antiquity to the Renaissance and beyond. Kepler<sup>24</sup> speaks of them in terms of sexual relations:

*"The cube is the outermost and most spacious, because it is the first-born and, in the very form of its generation, embodies the principle of all the others. These shapes join together in two noteworthy types of matings because they have different sexes. For among the first group of three, the cube and the dodecahedron are male, and among the second group the octahedron and the icosahedron are female. To this is added the bachelor or androgynous, the tetrahedron, because it is inscribed in itself [i.e. it mates with itself]. The female solids are inscribed inside the males and are as it were subject to them, and have the characteristics of the feminine as opposed to the masculine sex, namely, that [when they are nested inside male shapes] their angles are opposite the [male] faces."*

This is certainly a less dry way to state the duality principle than our more modern mathematical way.

That forms beget one another is another aspect of the natural cycles of changes that govern creation in the ancient and classical view. Plato had spoken of it in the Timaeus:

*"In the first place, we see that what we just now called water, by condensation, I suppose, becomes stone and earth, and this same element, when melted and dispersed, passes into vapor and air....and thus generation appears to be transmitted from one to the other in a circle."*

Later, he had identified these elements (Tim. 55ff) with the very *platonic* forms Kepler will use as a structure for his world system. Kepler considers the cube as first in rank for it generates the tetrahedron by subtraction and the dodecahedron by addition. The octahedron results from the tetrahedron mating with itself (as in the stella octangula); the icosahedron is born of the dodecahedron by subtraction (chopping of vertices), giving rise to faces.

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<sup>24</sup> From Kepler, Harmonices Mundi Books, 180 ff, quoted in George Hersey, *Architecture and Geometry in the Age of the Baroque*, Chicago U.P. 2000, p. 90.

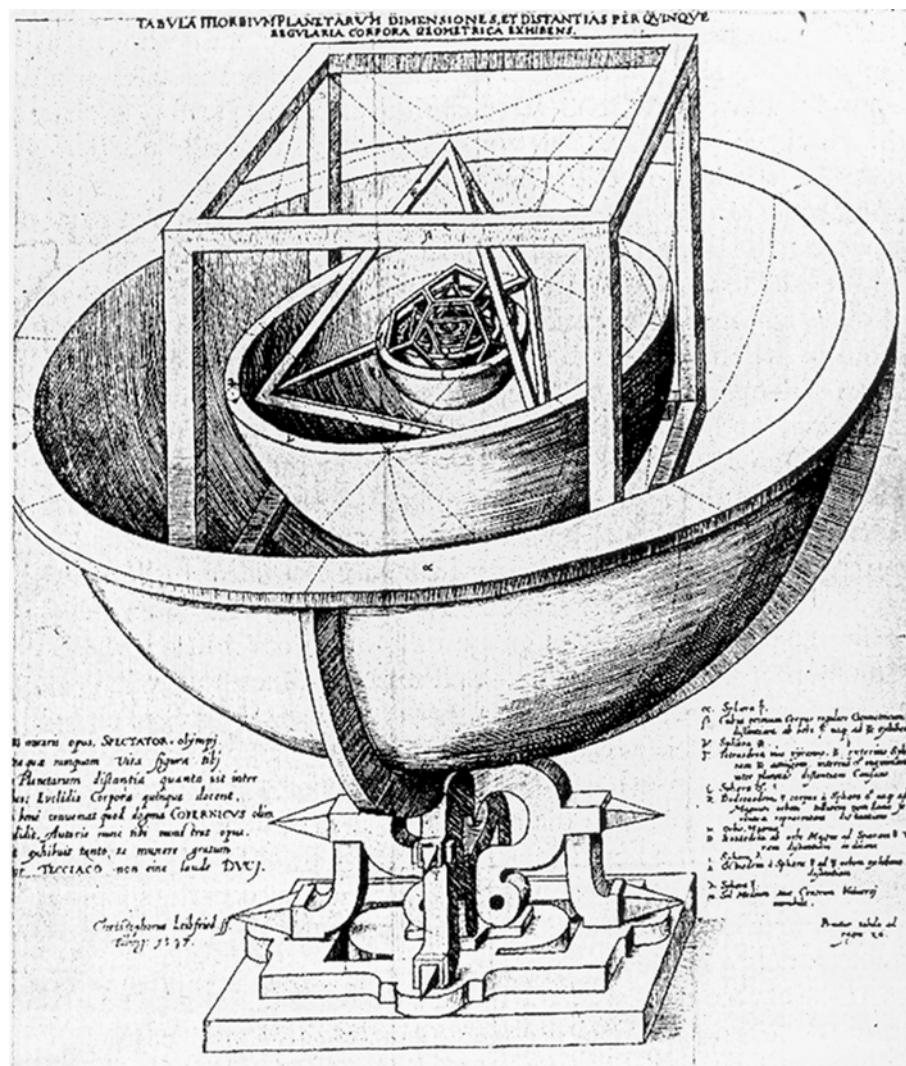


fig. 60

As Fernand Hallyn<sup>25</sup> points our, “Kepler based the order of the polyhedrons on a *genealogical* metaphor that links their “nobility” to their more or less direct descent from the sphere and permit them to be sexually differentiated and enter into marriages.”

Galileo was more matter-of-fact and ironical in his approach: “As for me, never having read the pedigrees and patents of nobility of shapes....I believe that there are none which are noble and perfect or any that are ignoble and imperfect, except in so far as for building walls the square shape is more perfect than the circular, while for rolling or for moving wagons I deem the circular more perfect than the triangular.”

In Kepler’s view however, the regular polyhedra were at the very core of Creation and presided at the organization of the world system (fig. 60). Their generation was therefore not just a matter of geometry, physics or even metaphysics, but of theology, indeed of reading the very mind of God:

*“It is my intention, dear reader, to demonstrate in this little work that with the Creation of this mobile universe and the arrangement of the heavens, God the Great Creator had in mind these five regular bodies that have been so famous from Pythagoras and Plato to our days, and that he caused the number of the heavens, their proportions, and the system of their motions to conform to the motions of these bodies.”<sup>26</sup>*

As we shall see later (*Music and World View, Adquadratum and Astronomy*), Kepler proposed an elaborate system made famous through his “planetarium”, showing the imbrication of the planetary orbits within the first platonic forms (see fig. 60).

Plato had claimed<sup>27</sup> that the smallest constituents of the elements were composed of the five regular polyhedra. As Hallyn<sup>28</sup> put it: “Kepler’s accomplishment in *Mysterium Cosmographicum* is to have transformed the cosmogonic function of the five solids by transferring it from the creation of matter to the construction of cosmic space.”

Throughout the ages, polyhedra have been viewed through the prism, so to speak, of the culture within which they have been studied. Their study played a major role during the Renaissance in the development of perspective. The work of Dürer, Wenzel Jamnitzer, Piero della Francesca, Barbaro, Pacioli, and others gave prime place to their two dimensional representation that gave impetus to projective geometry. Besides the purely mathematical properties that have emerged there has been a fascination with ways to construct and represent these platonic forms. The metaphors used to explain their generation have evolved but have always naturally been related to the general worldview

<sup>25</sup> Fernand Hallyn: *The Poetic Structure of the World – Copernicus and Kepler*, Zone Books, New York, 1990, p. 199.

<sup>26</sup> Kepler: *Mysterium Cosmographicum* – Introduction.

<sup>27</sup> Plato: *Timaeus*, 55ff.

<sup>28</sup> Hallyn: op. cit. p. 197.

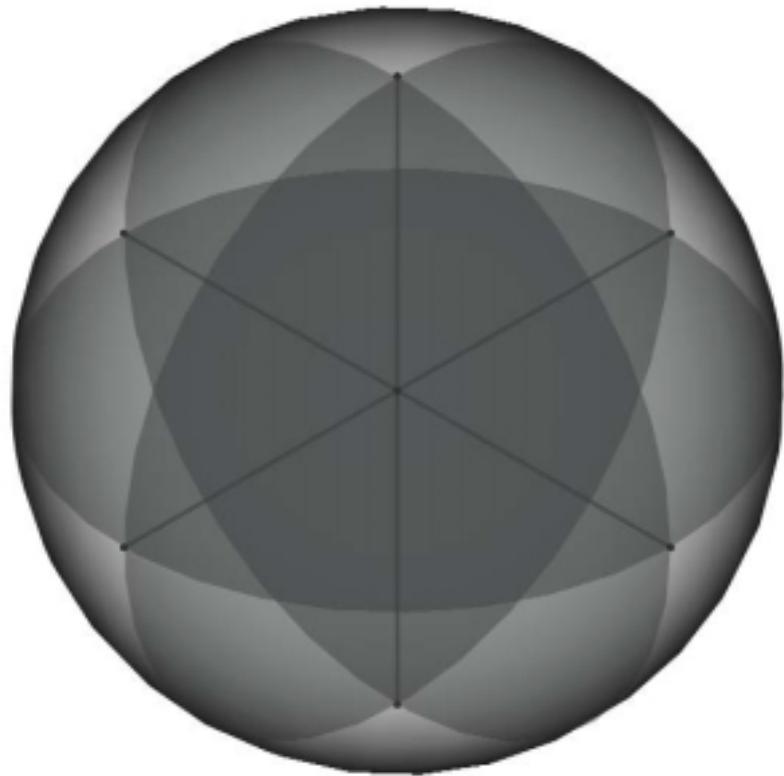


fig. 61

held by their authors. From the divine artificer in Plato to sexual reproduction in Kepler, we come to our own time of Big Bang and cosmic explosion. It is therefore in this cultural perspective that implicitly, we have considered the method of generating the regular polyhedra, namely, that resulting from a point in Euclidean space radiating along straight lines at angles determined by the polyhedra internal angles. These radii on impacting a sphere whose center is at the issuing point, mark the vertices of the polyhedra on the sphere, which then becomes the circumsphere to these figures. The regular polyhedra, convex and stellated, can then be materialized by joining the respective vertices of the figures by straight lines, i.e., establishing relationships between points spherically distributed in space, thus forming the edges of the polyhedra.

The new definition of regularity that we have proposed<sup>29</sup> allows us to locate directly on the sphere the vertices of all regular forms from our adquadratum construction without explicit knowledge of the number of sides in the faces of the polygons making up each polyhedron. Then, following simple topological rules, the edges of all the forms involved, convex and stellated, can be determined. Several alternatives to locate the vertices present themselves, some of which we examine in turn.

## **2. Method of Vertices Location on the Sphere via the Internal Angle :**

### *a. With the Three Mutually Perpendicular Golden Rectangles*

The three mutually perpendicular golden rectangles underlying the structure of the dodecahedron and icosahedron as well as that of the regular stellated forms constitutes in fact a Cartesian frame of reference at the heart of our sphere. Their construction within the sphere is straightforward with the knowledge of the internal angle of the icosahedron obtained from the adquadratum construction, since that angle is the angle of their diagonals.

To construct the golden rectangles, we draw a great circle on the sphere establishing the plane of one of them (the octahedron edge obtained from the adquadratum diagram can be used for that purpose) (fig. 61). From a point at random on that circle, we draw another great circle, which will be perpendicular to the first. Then from either of the two points of intersection of these two circles with the same compass opening, namely the octahedron edge, draw a third circle which will also be perpendicular to the two previous circles, establishing three mutually perpendicular planes with intersections having a common point at the center of the sphere.

Having the three mutually perpendicular planes within the sphere, a golden rectangle can be traced on each one of them as shown before in section 4a above (p. 89ff., and fig. 59). The corners of these golden rectangles mark the 12 vertices of the icosahedron on the sphere.

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<sup>29</sup> See p. 3 and 43.

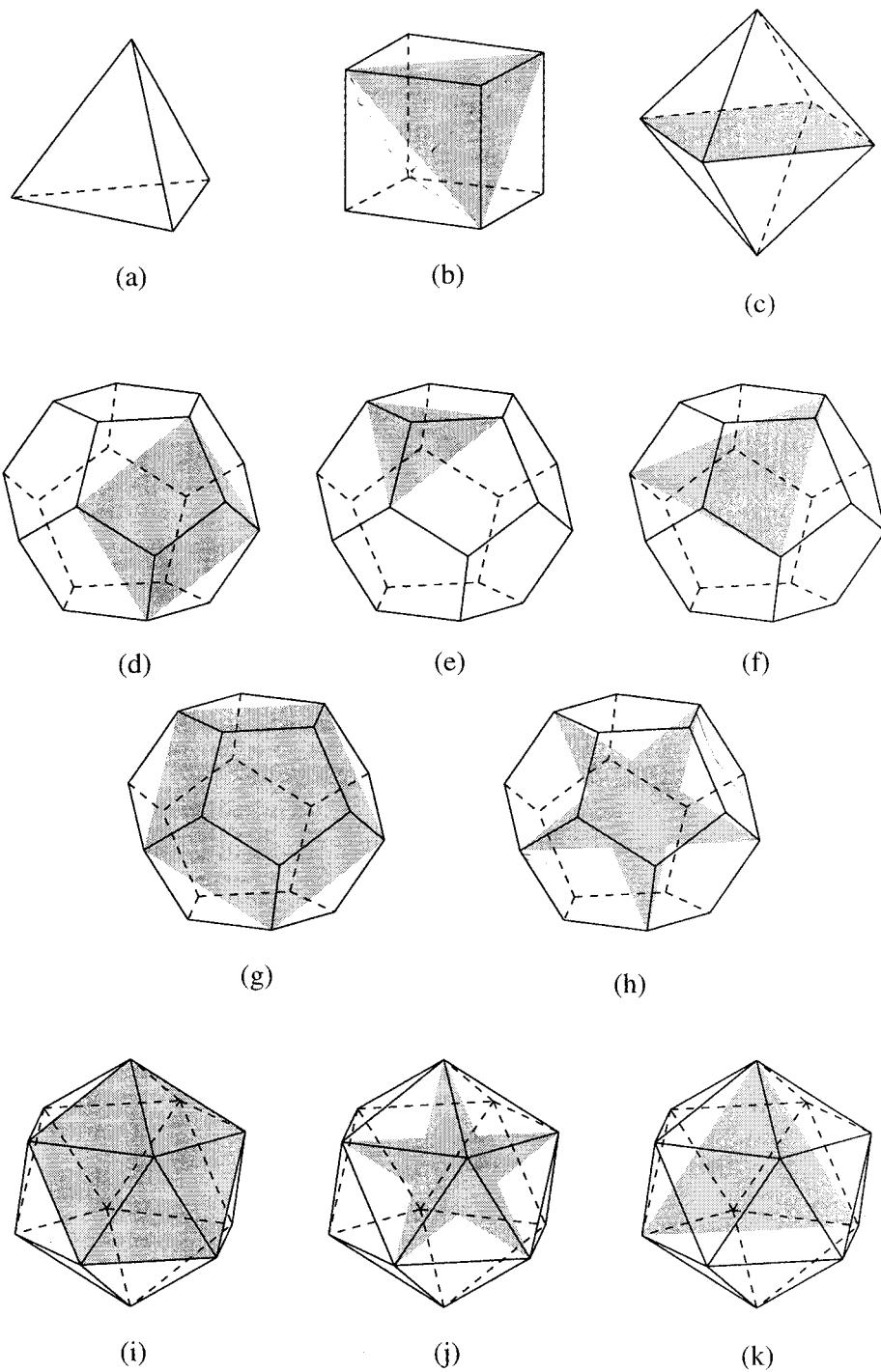


fig. 62

By joining each such vertex to its immediate neighbors (all five of them) through straight lines, we obtain a convex icosahedron. By joining them through arcs of great circles we obtain on the sphere a “planar” network.

We now join each vertex to its five *second immediate* neighbors by straight lines. Connecting these vertices similarly through arcs of great circles will yield a *non-planar* network where lines cross one another at points that are not vertices.

We now consider the polygons formed by grouping of these latter straight lines just drawn.

If we take two consecutive lines issuing from a given vertex we see them subtending the edge of the icosahedron (fig. 62(j)), and forming pentagrams within the pentagonal facets of the convex icosahedron just described (fig. 62(i)). A facet is a plane polygon (not a face) spanning the vertices of a polyhedron. Thus by joining the vertices diametrically opposed on the face of a cube a triangular facet (side of a tetrahedron) appears in the cube (fig. 62(b)). In our present case, these pentagrammal facets (12 of them altogether) (fig. 62(j)) are sides of the SSD. They come together in groups of 5 at each vertex.

If now instead of taking consecutive lines issuing from a vertex we take the two lines subtending the length of the golden rectangle with which they form all equilateral facet (fig. 62(k)) we shall see at each vertex, five such facets intersecting one another to form the GI. With 12 vertices and 5 planes at each, we have a total of 60, but since each contributes to 3 vertices, the number of planes is

$$\frac{60}{3} = 20.$$

We will also notice that the non-vertex crossing points serve as vertices for a small convex dodecahedron on the faces of which pentagonal pyramids would have been erected to yield the SSD already determined. Alternatively, it can also be considered as that yielding a GI by erecting on its faces pentagrammal pyramids resulting from the intersection of the triangular facets just established.

The combinations of lines resulting from the SSD and the original convex icosahedron is seen to yield the GD which can also be viewed as resulting from the intersection of 12 pentagonal facets within the icosahedron (fig. 62(i)).

Now take the convex dodecahedron such as the one just mentioned and consider two parallel faces and the five vertices in their immediate neighborhood. These five vertices are in a plane parallel to the bases. They form either a pentagonal facet (fig. 62(g)) or a pentagrammal one (fig. 62(h)) when joined together. Note that the edge of the pentagon which is a diagonal of the original pentagon is equal to the edge of the cube inscribed in the same sphere as the original dodecahedron.

Note also that three such pentagrams meet at each of the 20 vertices of the convex dodecahedron to yield a GSD.

So far, by means of the three mutually perpendicular golden rectangles we have established the convex icosahedron and dodecahedron as well as the four stellated regular polyhedra.

The octahedron has been implicitly established when we constructed adquadratum the three mutually perpendicular planes within the sphere. Their mutual intersections two at a time with the sphere at 6 different points are the vertices of the octahedron.

For the cube, on one of the three planes we draw through the center of the sphere two diameters forming angle  $\hat{C}_i$ . Four vertices are thereby determined at the contact point with the sphere. Drawing circles of radius equal to the adquadratum value of the cube edge on the sphere from each of these four vertices will yield four new vertices at their intersection. The total of 8 vertices is then available for the cube.

The tetrahedron will then be easily constructed by selecting one of the cube vertices as a starting point and further selecting its three *second immediate* neighbors and joining these 4 vertices together.

#### *b. By Individual Polyhedra*

The three golden rectangles approach yields a top-down generation of the regular polyhedra so to speak. We go from the complex to the simple, from the icosahedron and dodecahedron and their stellated forms to the cube and the tetrahedron. In the present method, we go from the simplest to the more complex, starting with the individual tetrahedron. We make no direct reference to the golden rectangle structure.

Given are the sphere of Radius  $R$  and the adquadratum diagram providing us with the internal angle and therefore the edge of each polyhedron inscribed in the sphere of radius  $R$ .

- *For the Tetrahedron*, pick a point at random on the sphere. One can imagine doing this either externally or internally to the sphere. From that point as center, draw on the surface of the sphere the circle having for radius the chord of the tetrahedron internal angle (edge of the tetrahedron) as given on the adquadratum diagram. From another point at random on this circle and with the same compass opening, draw another circle on the sphere. It will cut the first circle at two points. These and the two previous points are the vertices of the tetrahedron. The six straight line segments joining the 4 vertices together are the edges of the tetrahedron. Joining the vertices to the center of the sphere gives form to the Maraldian pyramids of the tetrahedron.

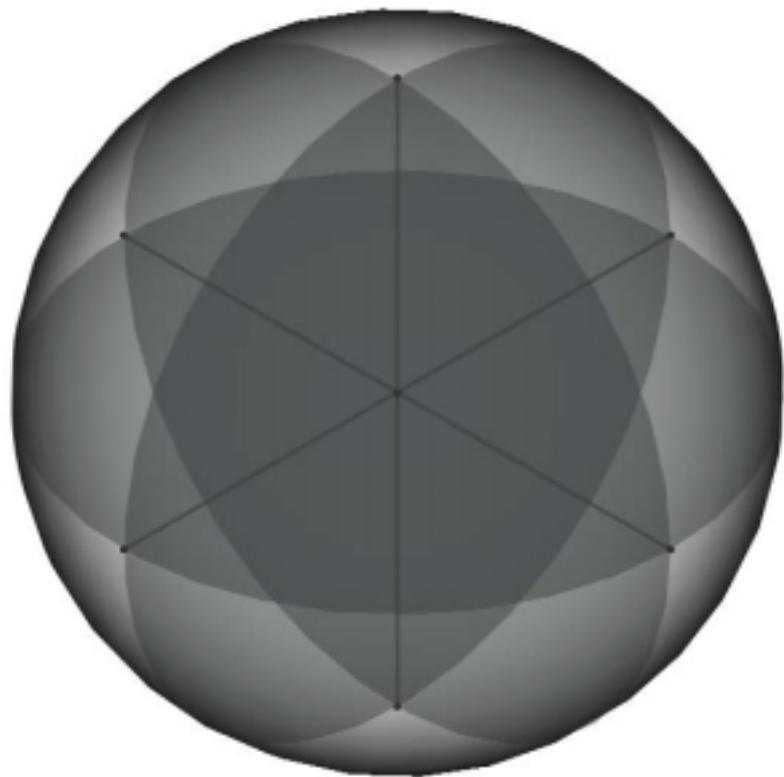


fig. 61

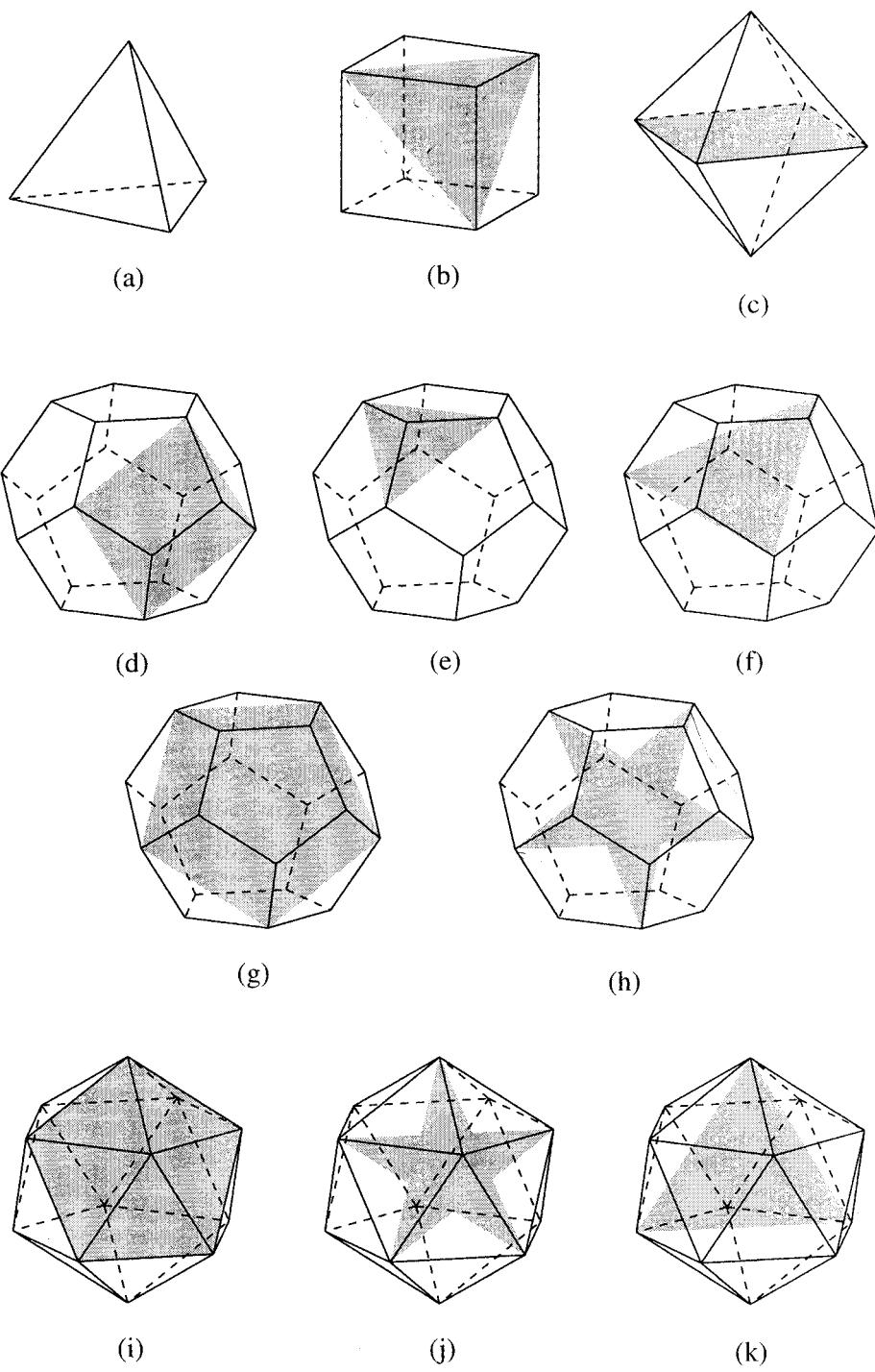


fig. 62

- *For the cube*, select a pole on the sphere and its antipole (point diametrically opposite). This can be done by drawing two great circles perpendicular to one another as shown in the previous section. There from the poles draw two circles with compass set at the adquadratum value of cube edge. From an arbitrary point on one of these circles and the same compass setting mark off a point on the *other* circle and from this last point mark off another point of the *previous* circle and move alternatively from one circle to the other 6 times altogether until one is back at the starting point (cycle completed). The two poles and the 6 points marked off on the two circles are the 8 vertices of the cube. Joining each vertex to its nearest neighbors through straight line segments will yield the Maraldian pyramids.

Alternatively, since cube and tetrahedron of common circumsphere may be considered as sharing 4 vertices, from these 4 vertices, circles may be drawn on the sphere with radii corresponding to the chord subtending the cube internal angle and obtained from the adquadratum diagram. Their intersections will yield the 4 additional vertices needed to define the cube.

As previously mentioned, joining each vertex to its nearest neighbors and to the sphere through straight line segments will yield the cube.

- *For the octahedron*, start with one pole and an adquadratum compass setting corresponding to the octahedron edge, as in establishing the three mutually perpendicular golden rectangles. The circle described on the sphere will be a great circle. Then from a point at random on that circle, draw another circle cutting the first at two points. Finally from one of these last two points draw a third circle cutting the other two circles. The three circles will be mutually perpendicular. This new circle will pass through already determined points and will set two new points which together with the previously established points will locate the 6 vertices of the octahedron. (fig. 61)

Again, joining each vertex to its 4 immediate neighbors by straight line segments will yield the octahedron. Connection with the sphere center gives the Maraldian pyramid.

- *For the icosahedron*, the same method as that used for the cube will work. First select diametrically opposed points on the sphere and from each as center draw circles with compass set at adquadratum value of icosahedron edge. Then from a point at random on one of the circles strike a point on the *other circle* with the same compass setting and from this new point strike another point on the *previous* circle. Keep alternating from circle to circle. After 10 strikes, one is back to the starting point. These 10 points plus the two original poles are the twelve vertices of the icosahedron. Joining each vertex to its five immediate neighbors will yield the icosahedron; joining them to the center of the sphere, one obtains the three Maraldian pyramids. Joining them to their 5 second immediate neighbors will yield the SSD (fig. 62(j)) and the GI (fig. 62(k)). One reads the

SSD or the GI depending on the grouping of connecting lines between vertices lying in the same planes (pentagram, p. 62(j)) or triangle (fig. 62(k)).

Alternatively, one can proceed step by step. Starting from a point at random with a compass opening corresponding to the internal angle draw a complete circle on the sphere. From any point on that circle, strike arcs around with the same compass opening, to divide the circle in five equal segments. Then from each of these points, draw circles on the sphere. Their intersections will yield the five new vertices. Two circles drawn from any of these new vertices will yield the 12<sup>th</sup> vertice, opposite pole of the original vertex.

- *For the dodecahedron*, we begin by remarking that, as we already saw with our adquadratum construction,

$$d|_{\text{cube}} = a|_{\text{dod}} \phi$$

i.e., the internal angle of the cube subtends the diagonal of the pentagonal face of the dodecahedron. In other words, all vertices of the cube belong also to the dodecahedron (as evidenced from the Euclidean construction of the dodecahedron based on the cube). To build our dodecahedron, we start with the cube vertices as center of circles of radius  $d|_{\text{dod}}$ , taken from the adquadratum diagram. These circles intersect in pairs at two points (spherical vesica pisces). To qualify as a vertex the point must be distant from its closest neighbors by  $d|_{\text{dod}}$ . Only one of the 2 points will satisfy the condition. However, two such points will be generated for each face of the cube for a total of 12 new vertices which, added to the 8 of the cube brings the grand total to 20, the number of vertices of the dodecahedron.

Having all the vertices of the convex polyhedra it is possible to build all the stellated forms.

### 3. Method of Interference:

We now turn to what might be called the method of interference. We start again from the five sets of radial directions. But this time, instead of considering the pyramids or cells of space delimited by the planes formed by pairs of radii, we examine the case where we have, for each set of radii respectively, a set of spheres of diameter equal to that of the circumsphere with centers located on the radii and all equidistant from the center of the circumsphere. We next proceed to move these spheres towards the center of the circumsphere at an equal rate so that, eventually, they will first touch the circumsphere together, and as the movement proceeds, interfere with it, thus forming circular planes perpendicular to the axes of progression. As the progression continues further these circular planes will in turn interfere with one another thus forming dihedral angles (all equal in a given polyhedral form) whose lines of intersection will come to form the edges of the respective polyhedra when these edges meet together and thus form vertices. These vertices will just lie on the circumsphere and will be the vertices of the respective

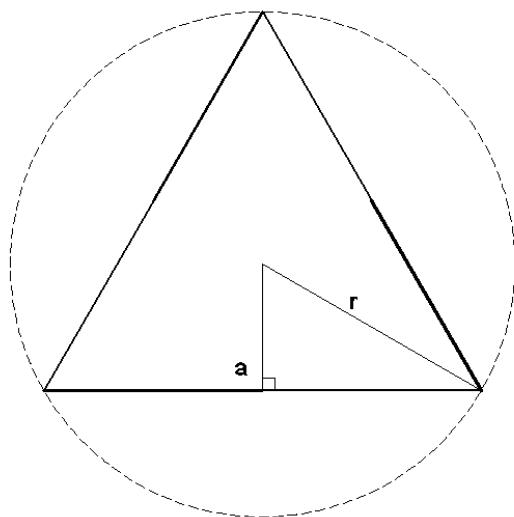


fig. 63A

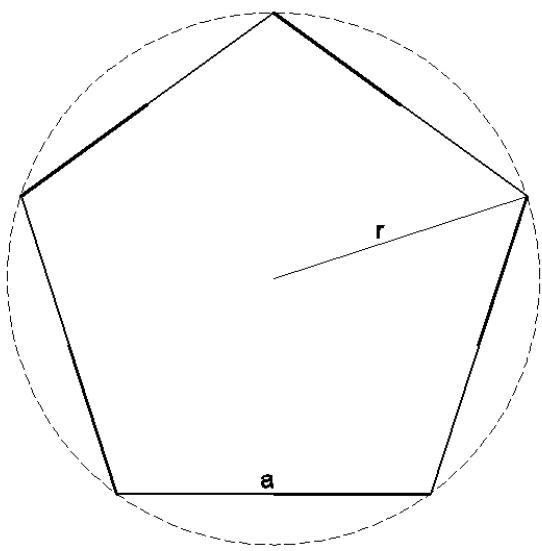


fig. 63B

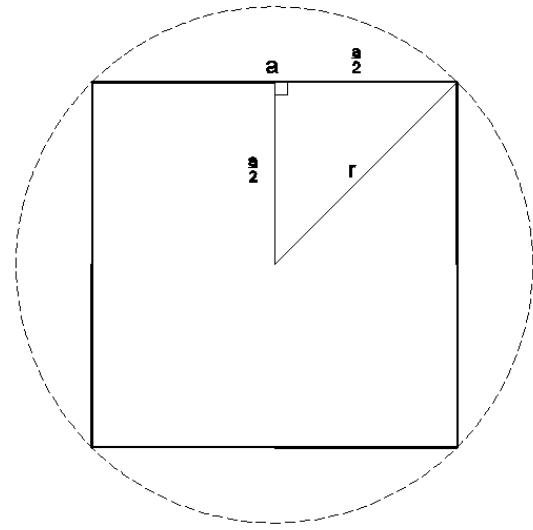


fig. 63C

platonic forms. They will also lie on the circular planes on which the lines between vertices (intersection of the dihedral angles) will be the edges of the faces of the polyhedra.

A point to notice is that radial axes along which the spheres are moving are the radii on which lie the apexes of the dual of the form generated.

Thus for the generation of the cube, the axes will be the radial lines of the octahedron and conversely. Similarly, for the dodecahedron and the icosahedron. For the tetrahedron, since the tetrahedron is its own dual, it will be the radial lines of a tetrahedron, oriented as shown in Kepler's *stella octangula*.

We now proceed to calculate the size of the circular interference planes necessary to build the faces of the five platonic forms. We shall subsequently establish a purely geometric process based on the ad quadratum method. The assumption is that all platonic forms will have a common circumsphere. In each case,  $r$  will present the radius of the circular plane bearing the face of a particular polyhedron while  $a$  will be the edge of that face.

$R$  represents the Radius of the common circumsphere. The number of circular planes required per polyhedron will naturally be equal to the number of faces of that polyhedron. With circles cut out, the polyhedra models can be assembled by sliding the circles through the slots as shown on fig. 63.

♦ *Tetrahedron (triangle)* (fig. 63A)

As we have previously seen:

$$a = \frac{2}{3}\sqrt{6}R$$

or  $a = 1.633R$

And as established by Euclid<sup>30</sup>

$$r = \frac{2\sqrt{2}}{3}R$$

$r = 0.943R$

so that

$$\frac{a}{r} = \sqrt{3}$$

for  $R=3$

$$a = 2\sqrt{6} = 4.900$$

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<sup>30</sup> Heath, Thomas: op. cit. p. 251

$$r = 2\sqrt{2} = 2.828$$

(We consider the case of  $R=3$  throughout since it is that corresponding to the ad quadratum construction based on the square of unit side. It also simplifies expressions due to the presence of the factor  $\frac{1}{3}$  in many cases and provides a useful scale for our models.)

- ♦ *Cube (square)* (fig. 63B)

Referring to figure (63B) we can write

$$\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2 = r^2$$

$$\therefore r = \frac{a\sqrt{2}}{2}$$

But

$$a = \frac{2\sqrt{3}}{3} R \text{ so that } r = \frac{\sqrt{6}}{3} R$$

$$a = 1.155 R \quad r = 0.816R$$

For  $R=3$ ,

$$a = 2\sqrt{3} = 3.464$$

$$r = \sqrt{6} = 2.449$$

- ♦ *Octahedron: (triangle)*

$$a = \sqrt{2}R \quad \text{as seen previously.}$$

The geometry of the triangle is the same as for the tetrahedron and will be the same for the icosahedron. So that we can write directly by reference to fig (63A):

$$r = \frac{\sqrt{3}}{3} a$$

and therefore:

$$r = \frac{\sqrt{6}}{3} R$$

(the same as for the cube and half the side of the tetrahedron)  
and we check that indeed

$$\frac{a}{r} = \sqrt{3}$$

For  $R=3$

$$a = 3\sqrt{2} = 4.243$$

$$r = \sqrt{6} = 2.449$$

- ♦ *Icosahedron: (triangle)*

Here again

$$r = \frac{\sqrt{3}}{3} a$$

But this time

$$a = \frac{\sqrt{10}}{5} \sqrt{5 - \sqrt{5}} R$$

$$a = 1.05146R$$

So that:

$$r = \sqrt{\frac{2(5 - \sqrt{5})}{15}} R$$

$$r = 0.6070R$$

$$\text{For } R=3, \quad a=3.154$$

$$r = 1.821$$

- ♦ *Dodecahedron: (pentagon) (fig. 63C)*

Here also, we know that

$$a = \frac{\sqrt{3}}{3} (\sqrt{5} - 1) R$$

$$a = 0.714R$$

But we also know that Aristaeus, *Hypsicles dixit*<sup>31</sup>, proved that

$$r|_{dodi.} = r|_{isoc.}$$

Which follows from the fact we have proved earlier that both dodecahedron and icosahedron have a common insphere if they have a common circumsphere since tangent planes to the insphere will cut similar circles on the circumsphere in both cases.

We can therefore write directly:

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<sup>31</sup> Heath, Thomas: op. cit. p. 254

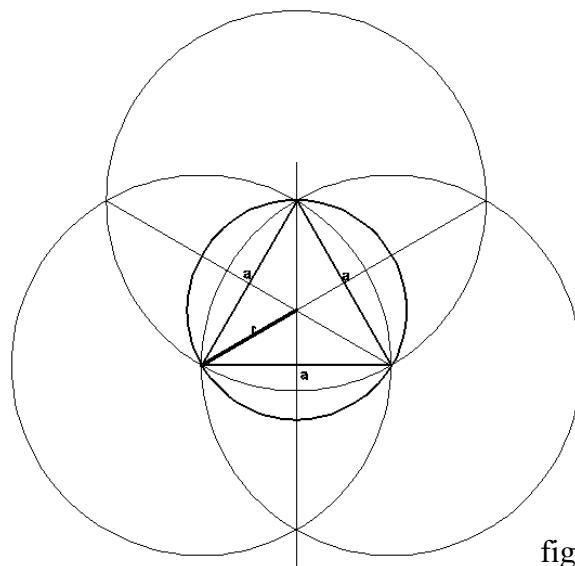


fig. 64

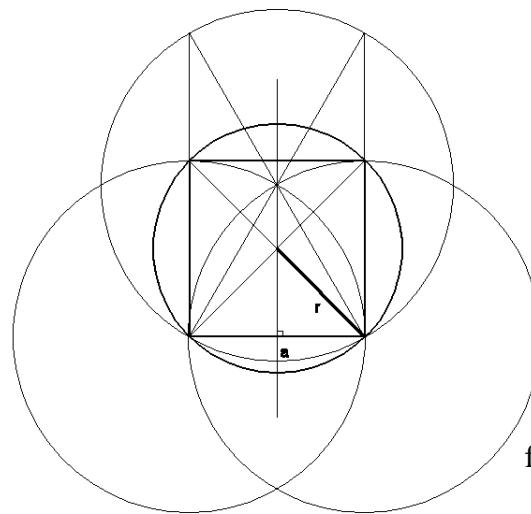


fig. 65

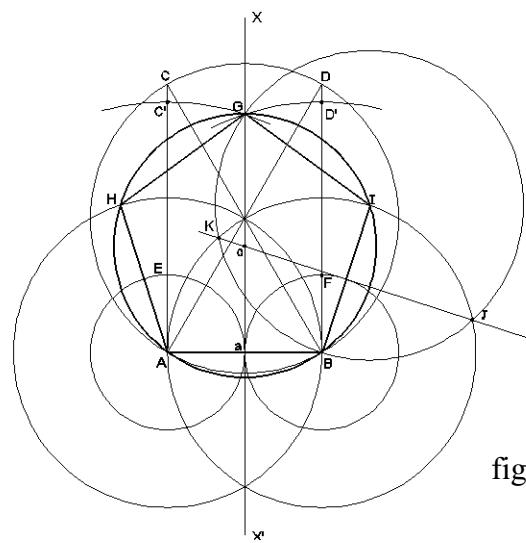


fig. 66

$$r = \sqrt{\frac{2(5-\sqrt{5})}{15}} R$$

$$r=0.607R$$

$$\begin{aligned} \text{For } R=3, \quad a &= 2.141 \\ r &= 1.821 \end{aligned}$$

As also mentioned before, all these results can be obtained directly geometrically from the Ad Quadratum method, which we now consider.

#### **4. Results of Interference Method Through Ad Quadratum:**

For all polyhedra whose faces are triangles, namely the tetrahedron, the octahedron and the icosahedron, it is simply a matter of finding the circle circumscribing an equilateral triangle of given side  $a$ .

The appropriate measure of  $a$  is picked off directly from the *ad quadratum* diagram (fig. 13) for each polyhedron. The circumscribing circle can then be constructed as shown on figs. 64, 65 and 66 where  $a$  represents alternatively the edge of the tetrahedron, the octahedron and the icosahedron and  $r$  the radius of the circular plane as shown on figs. 63A, B and C respectively.

For the cube, we find the circle circumscribing the square of side  $a$  as shown by the construction of fig. 65.

For the dodecahedron, we find the circle circumscribing the pentagon of side  $a$  shown by the construction on fig. 66.

We begin the construction as for building the triangle (fig. 64), extending the sides of the triangle as when building the square (fig. 65). Erect  $AC$  and  $BD$ . Then draw circles of radius  $\frac{a}{2}$  centered on  $A$  and  $B$ .

Swing  $EB$  and  $FA$  to  $C'$  and  $D'$  on  $AC$  and  $BD$  respectively. Then swing  $AC'$  and  $BD'$  to  $G$  on axis  $XX'$ . From  $G$  with radius  $a$  mark off  $H$  and  $I$  on circles of radius  $a$  centered on  $A$  and  $B$  respectively.  $ABIGH$  is the pentagonal face of the dodecahedron of edge  $a$ . To determine the center of the circumscribing circle to the pentagon, draw circle of radius  $a$  centered on  $I$ . This circle intersects the circle radius  $a$ , centered on  $B$  at  $J$  and  $K$ . Join  $J$  and  $K$ . Line  $JK$  intersects axis  $XX'$  at  $O$ , center of pentagon. Then with radius  $OG$  (or  $OI$ , etc....) draw circle circumscribing the pentagon, which completes the construction.

#### **5. Method of the Six Directions of Space:**

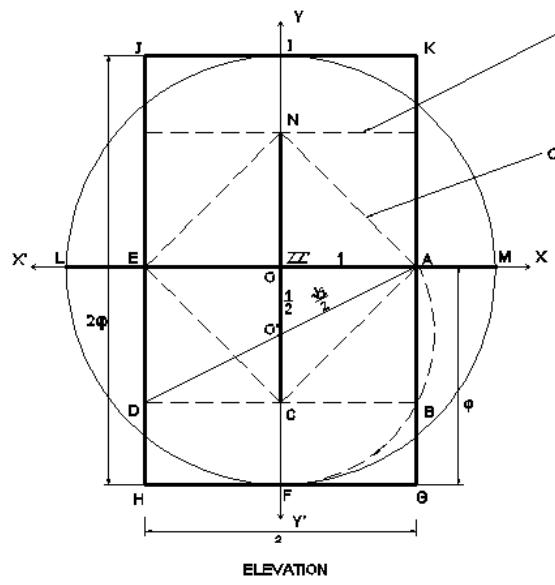
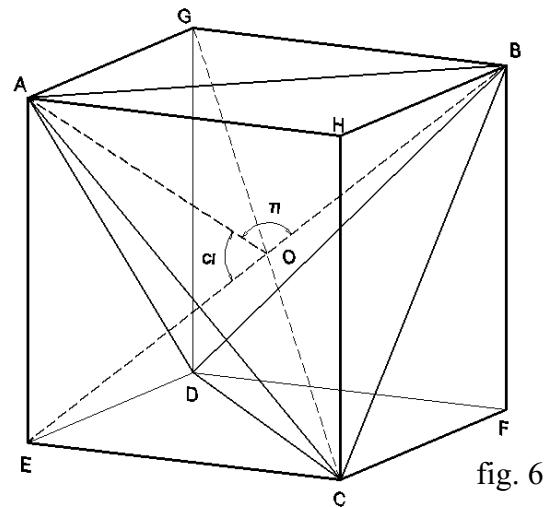
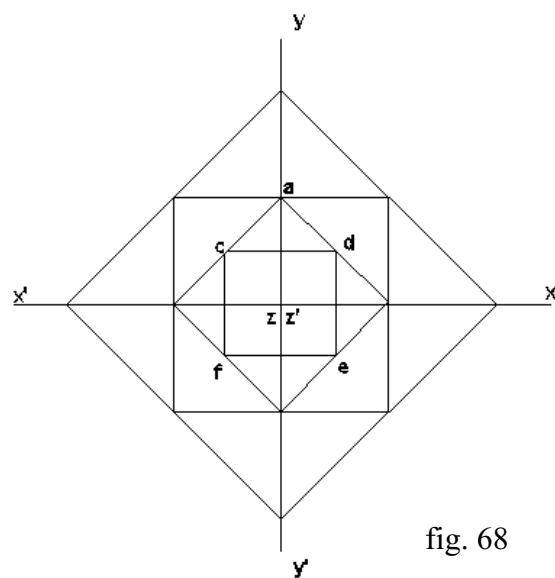
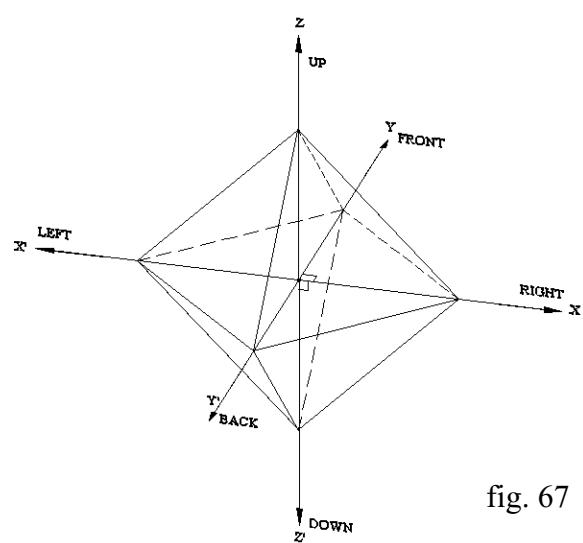
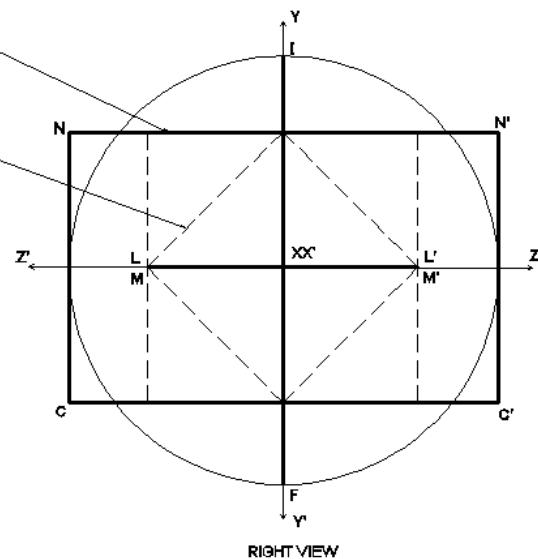


fig. 69



This method is based on the simplest intuitive approach to the generation of the regular polyhedra.

In the most immediate apprehension of Euclidean space one is aware of the front, back, left, right, up and down directions. These directions determine in fact a Cartesian system of coordinates. We therefore begin with a set of three mutually perpendicular axes, each extending in both directions away from their common origin.

By measuring equal distances along each of these directions, an octahedron is generated (fig. 67). Imagining now other such octahedra located along the 6 directions, and as in the previous method of interference, letting them be moved along the axes at an equal rate they will eventually interfere with the original octahedron. As the movement progresses, a smaller cube ( $c, d, e, f$ ), the faces of which will be perpendicular to the axes will be formed as shown in fig 68, where the view is along the  $ZZ'$  direction of fig. 67.

Alternatively, one can generate the cube directly by erecting planes perpendicular to the axes at each of the vertices of the original octahedron. This cube is the dual of the octahedron.

Once we have the cube, the tetrahedron can be generated by partitioning the cube as shown on fig. 6.

The next step is to generate the icosahedron and the dodecahedron. For this we remark that the structure of these two polyhedra is governed by the existence of three mutually perpendicular golden rectangles, the corners of which constitute the vertices of the icosahedron or the face centers of the dodecahedron.

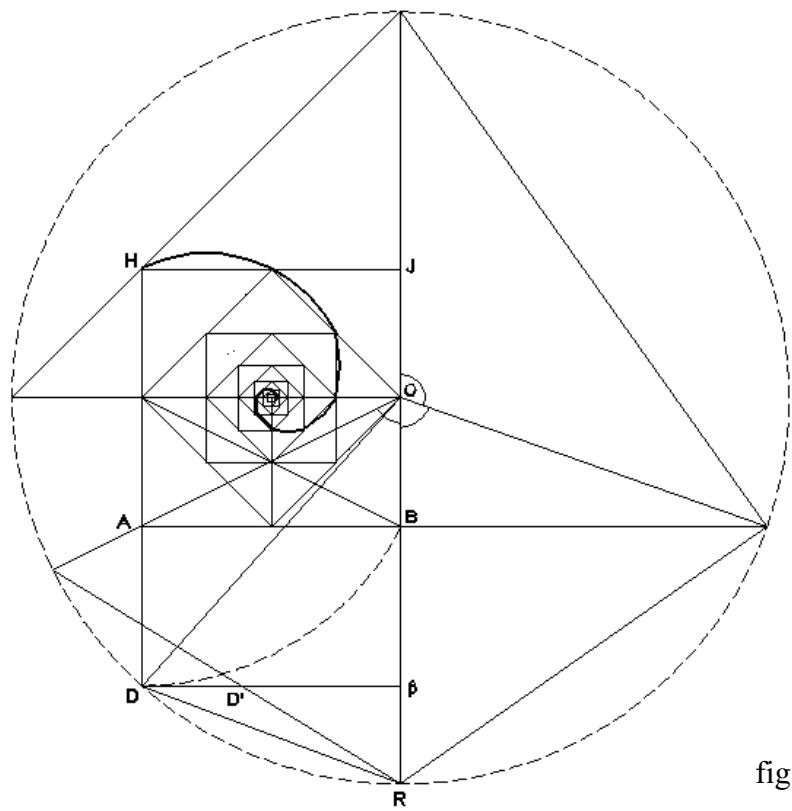
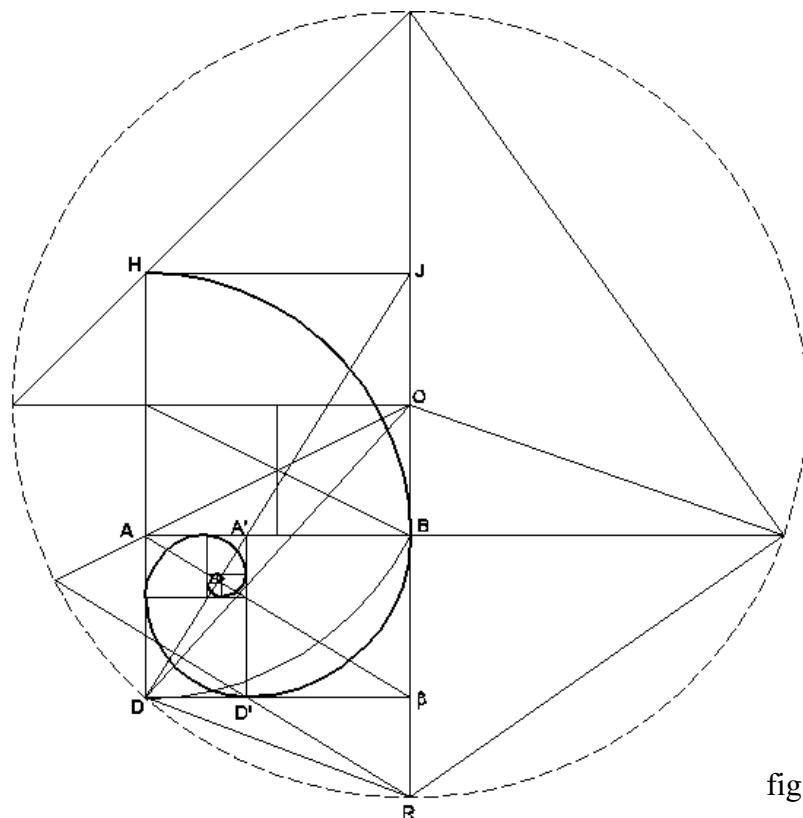
Given the 6 directions of space determining the mutually perpendicular planes of a Cartesian system of coordinates, it is sufficient to trace out on these planes golden rectangles the centers of which coincide with the origin of the coordinate system. To this end (see fig. 69) and using the *ad quadratum* method, draw squares of unit side ( $OABC$  and  $OCDE$ ) in two of each adjacent quadrants of a given plane (say  $OXY'$  and  $OX'Y'$ ). Then draw the diagonal  $DA$  of the double square. With center  $O'$ , swing  $O'A$  down to  $F$ .

$$\text{Then } OF = \frac{1 + \sqrt{5}}{2} = \phi$$

and since  $OA = 1$  by construction, it follows that  $OAGF$  and  $OFHE$  are golden rectangles. Now swinging  $OF$  to  $I$ , we can build the golden rectangle  $GHJK$  centered on the origin. The other 2 mutually perpendicular golden rectangles can then be built in their respective planes and then appear as edges, one as full length  $LM$ , the other as full width  $NC$ , in the elevation and right view respectively.

As we have previously seen the three mutually perpendicular golden rectangles constitute also the structure of the four stellated polyhedra. Combined with the view from the center involving the internal angles, we have here a most potent instrument to visualize

the structure of all regular polyhedra, stellated or not, since it is a simple matter to construct on the planes of the golden rectangles the octahedron and the cube as previously described.



## **Ad Quadratum method and the Generation of the Spirals.**

### **1. The Golden spiral:**

Considering the *ad quadratum* diagram (reproduced in fig. 70), we see that, by construction, rectangle  $HJD\beta$  has for sides  $HD = J\beta = 1 + \sqrt{5}$  and  $HJ = D\beta = 2$ .  $HJD\beta$  is therefore a golden rectangle.

By successive square subtraction such as  $ABJH$ ,  $A'B\beta D'$ , etc., out of the golden rectangles the golden or logarithmic spiral can be traced.

### **2. The Growth spirals:**

Now, considering the spiral of growth of the square (cube face) on fig. 71, it is seen that as the angle of turn increases by  $\frac{\pi}{4}$  in arithmetic progression, the radius grows in geometric progression of ratio  $\sqrt{2}$ . That growth spiral is therefore also a logarithmic spiral. Similar spirals can be constructed for each side of the square, giving rise to lobes as shown on fig. 72 and 72A.

Based on the equilateral triangle (face of the tetrahedron, the octahedron and the icosahedron) or the regular pentagon (face of the dodecahedron), other spirals of growth as shown on fig. 73 can be traced.

It is these spirals of growth that we see projected on the inner surfaces of the pyramids making up the platonic forms as shown on the plate showing the evolution of these platonic forms (fig. 74).

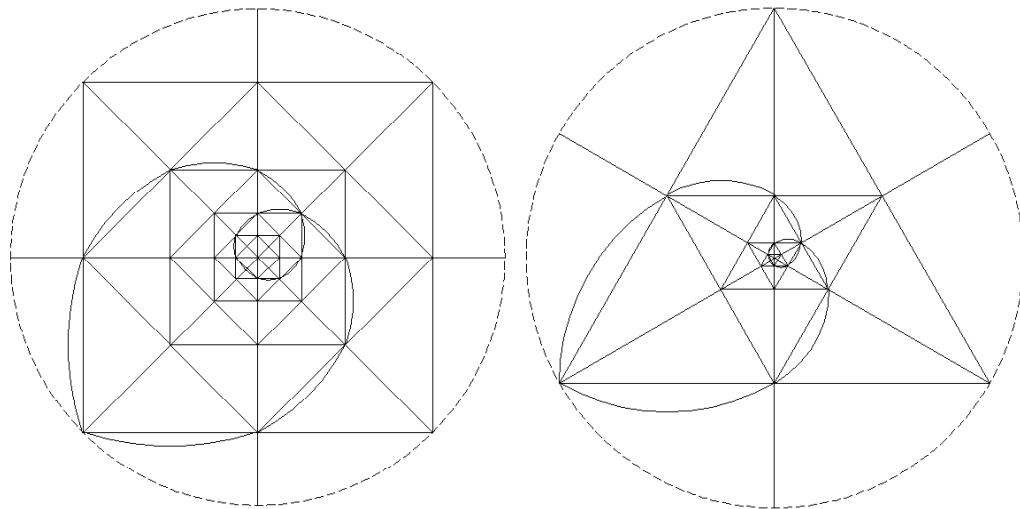


fig. 72

fig. 73



fig. 72A

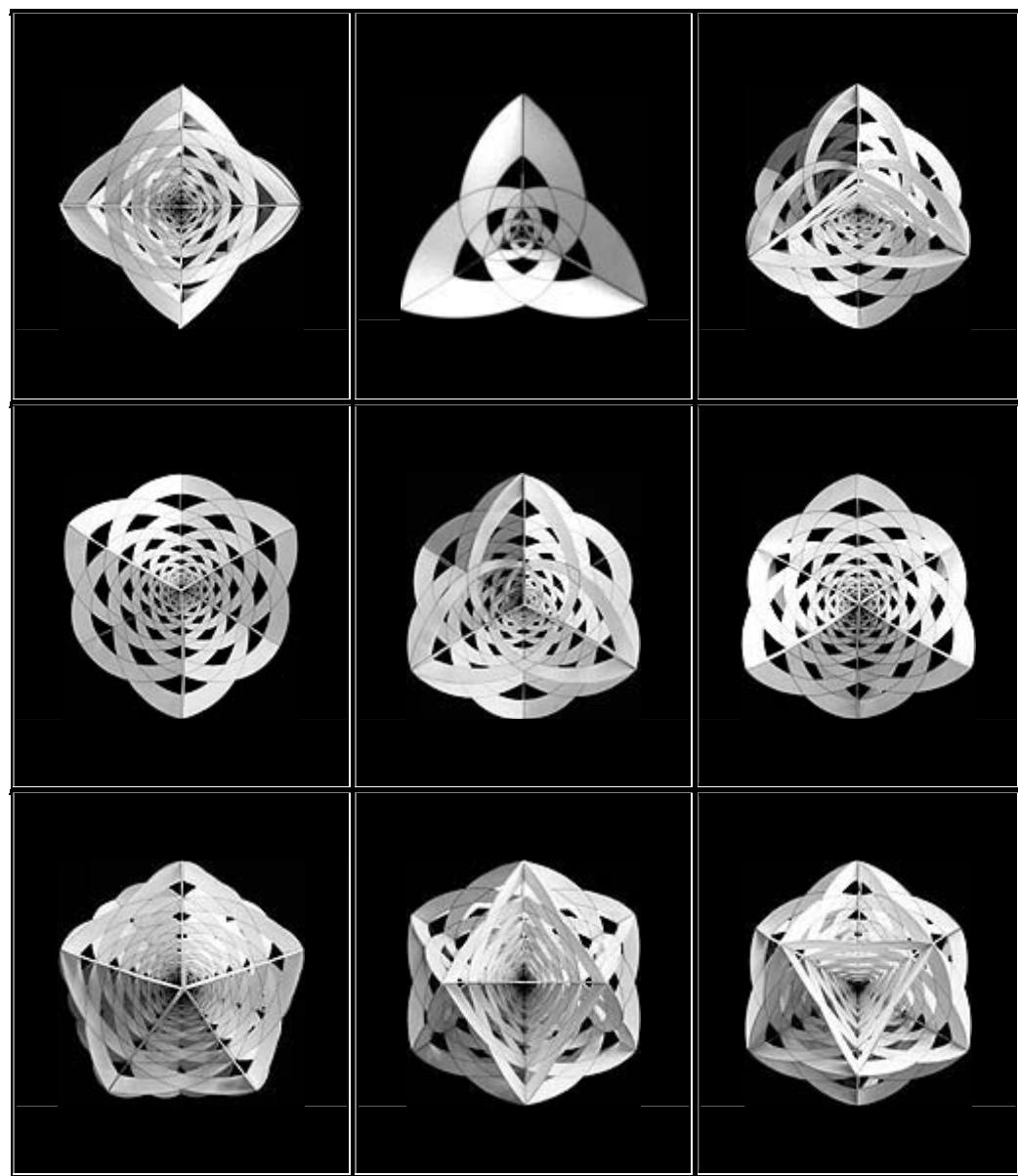


fig. 74

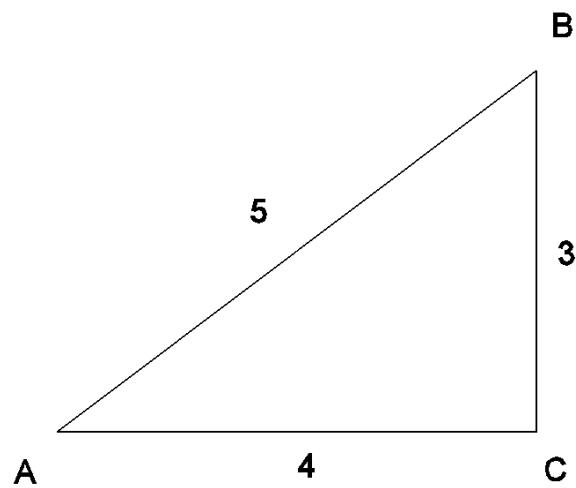


fig. 75

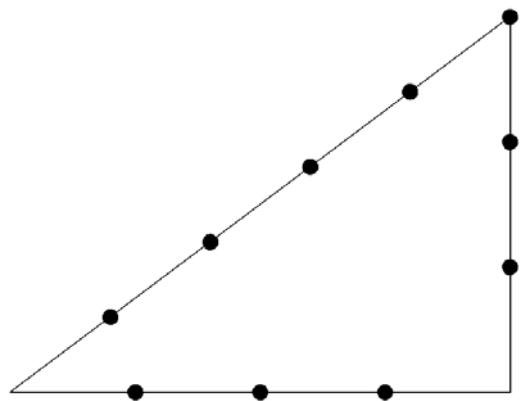


fig. 76

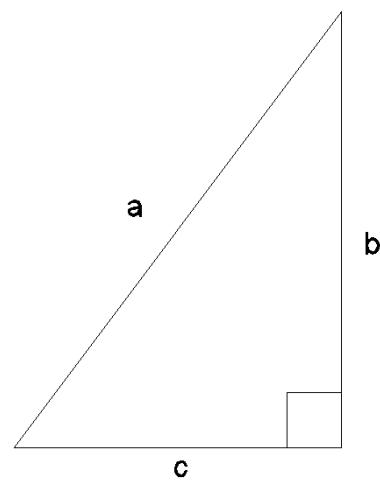


fig. 77

### Ad Quadratum and the Pythagorean Triples

*Triples* in general are sets of three numbers such that the sum of the squares of the first two is equal to the square of the third.

$$\begin{array}{ll} \text{Thus} & 3^2 + 4^2 = 5^2 \quad (9 + 16 = 25) \\ \text{Or} & 6^2 + 8^2 = 10^2 \quad (36 + 64 = 100) \\ & (1.5)^2 + 2^2 = 2.5^2 \quad (2.25 + 4 = 6.25) \\ & (-4)^2 + 3^2 = 5^2 \quad (16 + 9 = 25) \\ & 2^2 + 3^2 = (\sqrt{13})^2 \quad (4 + 9 = 13) \end{array}$$

This is the direct result of the Pythagorean theorem.

The fact that a triple represents the sides of a right angle triangle is very useful.

A convention for representation is that the numbers are given preceded by the designation of the angle (if a geometric representation is intended) followed by the base, the height and the hypotenuse.

e.g. A) 4,3,5 or B) 3,4,5, as seen on fig. 75, whether we consider A or B to be angles of import. This has the advantage of allowing trigonometric ratios to be readily available.

$$\text{e.g. } A) 4,3,5 \text{ implies } \cos A = \frac{4}{5}, \sin A = \frac{3}{5}, \tan A = \frac{3}{4}.$$

If the angle is at the origin of a rectangular coordinate system and the base along the positive X-axis, then in our example, 4 and 3 become coordinates of B, the other extremity of the hypotenuse, combining Cartesian and polar description.

The internal angles of the Platonic forms can therefore be identified thus:

Cube:  $\hat{C}_i)1,2\sqrt{2},3$ ; Tetrahedron:  $\hat{T}_i)-1,2\sqrt{2},3$ ; Octahedron:  $\hat{O}_i)1,1,\sqrt{2}$ ; Icosahedron:  $\hat{I}_i)1,2,\sqrt{5}$ ; Dodecahedron  $\hat{D}_i)\sqrt{5},2,3$ .

*Pythagorean triples* are restricted to whole numbers, however, so that these latter triples containing irrational numbers such as  $\sqrt{2}$  or  $\sqrt{5}$  would not qualify as Pythagorean.

Pythagorean triples are limited to squares. This is a consequence of Fermat's famous last theorem which states that the equation  $z^u = x^u + y^u$  has no whole-number solutions for any power  $u$  greater than 2.

Though stated by Fermat around 1660, the theorem remained a conjecture for about 333 years, when Andrew Wiles of Princeton finally proved it in 1994. It is not the place to

elaborate on this here.<sup>32</sup> As far as Pythagorean triples are concerned, the question is: *do there exist right angle triangles of which all sides are representable by whole numbers?*

As we have seen, the triple     $x=3, y=4, z=5$   
                     i.e.,     $3^2 + 4^2 = 5^2$

which goes back to ancient Egyptians and Babylonians, is such an example. The 3, 4, 5 triangle was used by architects and builders in chain links or ropes (fig. 76) of 12 units to construct right angles in an application of the converse of Pythagoras Theorem:

This converse of Pythagoras Theorem can be stated as:

If  $z^2 = x^2 + y^2$ , then the angle opposite  $z$  is a right angle. This statement forms Euclid's Proposition I-48 in the Elements, and Euclid proved that there are an infinite number of solutions.

For algebraic proof, we follow Neugebauer.<sup>33</sup>

Consider the triangle  $a, b, c$ . (fig. 77)

Let

$$\begin{aligned} a &= u+v \\ b &= u-v \end{aligned} \quad \left. \right\} \quad (1)$$

Where  $u$  and  $v$  are any natural numbers so that  $a$  and  $b$  can also range over the whole field of natural numbers. Then, for  $a, b, c$  to be Pythagorean (i.e., right angled with  $a, b, c$  whole numbers), the following relation must obtain:

$$a^2 = b^2 + c^2$$

Which implies that:

$$c^2 = 4uv \quad (2)$$

Since:

$$\begin{aligned} a^2 &= u^2 + v^2 + 2uv \\ b^2 &= u^2 + v^2 - 2uv \end{aligned}$$

So if  $u$  and  $v$  are integers,  $a$  and  $b$  will also be integers. But  $c$  will be integer only if  $\sqrt{uv}$  is an integer. This will be the case if we assume  $u$  and  $v$  to be squares of integers such as

$$u = s^2 \quad v = t^2$$

So if  $s$  and  $t$  are arbitrary integers with  $b$  positive, it follows that:

$$u > v \quad \therefore \quad s > t$$

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<sup>32</sup> see for instance Keith Devlin: The Language of Mathematics: Making the Invisible Visible. W.H. Freeman & Co., New York, 1998.

<sup>33</sup> O. Neugebauer: The Exact Sciences in Antiquity. Dover, NY 1969, p. 41

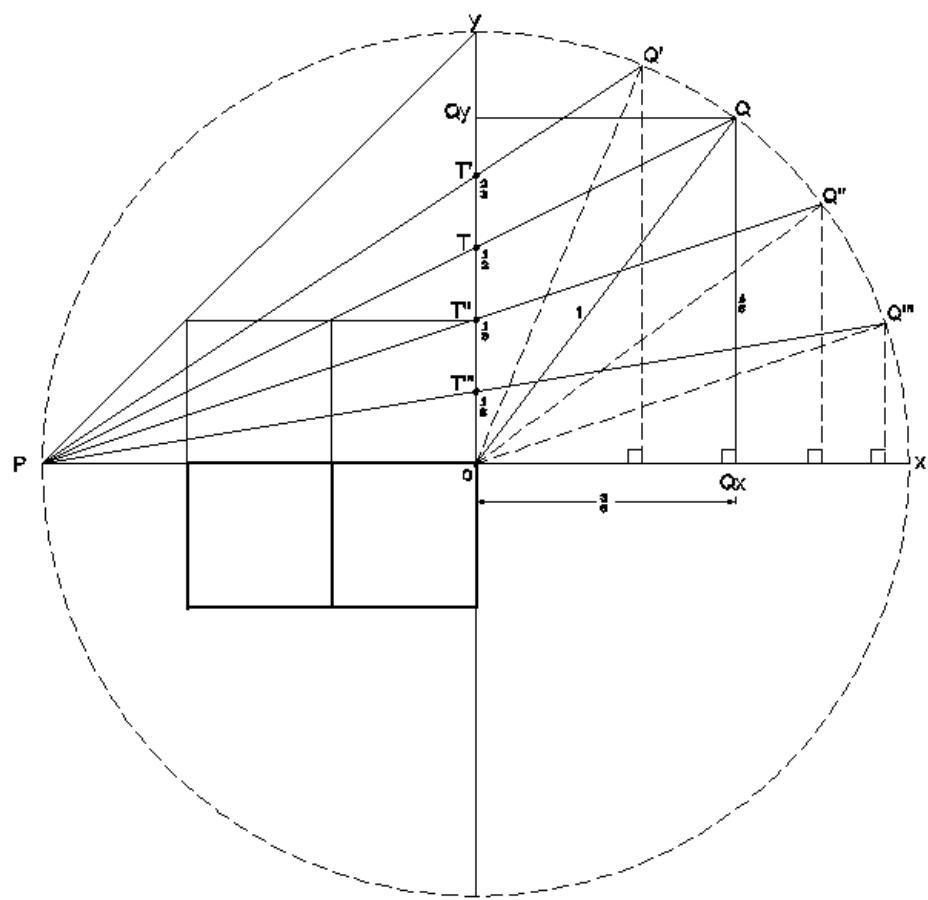


fig. 78

Finally, from (1) and (2) we see that we can write:

$$a = s^2 + t^2; \quad b = s^2 - t^2; \quad c = 2st;$$

The rest follows, with  $s$  and  $t$  varying over all natural numbers and such that:

1.  $s > t$
2.  $s$  and  $t$  have no common factor
3. One of  $s, t$  is even, the other odd.

The ***ad quadratum method*** allows for the direct construction of all Pythagorean triangles as shown in fig 78.

To facilitate and generalize the solution, note that the equation  $x^2 + y^2 = z^2$  can be normalized by dividing through by  $z^2$  so that:

With  $\frac{x}{z} = \bar{x}$  and  $\frac{y}{z} = \bar{y}$

We can write:  $\bar{x}^2 + \bar{y}^2 = 1$

If the solution is  $x=a, y=b, z=c$  with  $a, b, c$  whole numbers, then

$$\bar{x} = \frac{a}{c}, \bar{y} = \frac{b}{c},$$

with  $\frac{a}{c}$  and  $\frac{b}{c}$  both rational numbers.

For instance, starting with

$$(3)^2 + (4)^2 = (5)^2,$$

we have

$$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1$$

Now  $\bar{x}^2 + \bar{y}^2 = 1$  is the equation of a circle, of radius unity and center at the origin in a Cartesian  $\bar{x}, \bar{y}$  coordinate system. (see fig. 67 and 78; the coordinates have been relabeled  $x$  and  $y$  to conform to standard notation.)

The problem is to find points such as  $Q$  on the circle such that triangles  $OQQ_x$  form a Pythagorean Triple, i.e., the coordinates of  $Q$  (i.e.,  $Q_x$  and  $Q_y$ ) must both be rational.

Considering the triangles  $PTO$  and  $PQQ_x$ , we have:

$$\frac{OT}{OP} = \frac{QQ_x}{PQ_x}$$

Or, since  $OP=1$ ,  $QQ_x = y$  and  $PQ_x = 1+x$

$$OT = \frac{y}{1+x}$$

So that, if  $OT$  is a rational fraction, so will  $\frac{y}{1+x}$ , and if  $x$  is even,  $y$  will be odd and conversely, also  $OT \leq 1$ . All the Pythagorean triangles can therefore be established in the *oxy* quarter by choosing point  $T$  such that  $OT$  is any rational fraction between 0 and 1.

## Ad Quadratum and Music

### Music and World View:

Pythagoras is generally credited for introducing mathematics into music, i.e., associating intervals between pitches with ratios of numbers representing frequencies or lengths of resonating devices such as taut strings or wind pipes.

However, the idea is, in all probabilities, much more ancient. Sumerian, Babylonian, Egyptian, Chinese civilizations show evidence that music, very early, was related to the concept of number<sup>34</sup>. By Plato's time music had evolved into two distinct and virtually separate branches – one practical, having to do with musical performance, the other theoretical, consisting of the study of the mathematical relationships between tones, or more generally proportion.

The relationship between numbers and acoustical phenomena seems to have given the ancients the notion that all other phenomena were also governed by numbers<sup>35</sup>. Music being the most advanced science of the time, most other aspects of human experience were cast according to its model, as in our own age, physics has been used as a paradigm for most other branches of knowledge, a function that biology now seems poised to take over.

It is the existence of such paradigms, which provides a degree of unity and coherence to the world views that dominate historical periods. For instance, the dominance of musical models<sup>36</sup> from antiquity to well into the 17<sup>th</sup> Century in the mentalities of philosophers, theologians, astronomers and astrologers as well as mathematicians, architects and physicians made plausible the concept of universal harmony. Boethius (c. 480-524) is among the first to have explicitly presented this idea through his division of music in three types: *musica mundana*, or music of the universe, describing the movement of the celestial spheres; *musica humana*, or music of the human being, expressing the correspondence between soul and body; and *musica instrumentalis*, instrumental (and vocal) music. Only the last is concerned with acoustic phenomena while the other two relate to proportions and intellectual harmony.

The human mind being so designed that it seems perpetually in search of a unifying viewpoint to conceptualize its experience, found in the musical model an enduring and

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<sup>34</sup> McClain, Ernest G. *The Pythagorean Plato*. Nicholas-Hays, York Beach (Maine) 1978.

<sup>35</sup> Dae-Am Yi. *Musical Analogy in Gothic and Renaissance Architecture*. Unpublished Ph.D. thesis, University of Sydney, 1991.

<sup>36</sup> "The eternal universe, originally without order and in all that pertained to it formless and devoid of all things that are most clearly distinguished according to the categories of quality and quantity and the rest, was rendered discrete by number as the most sovereign and artistic form, and its elements were given a most distinctive organization and it came to partake of harmonious variation and perfect congruity in accordance with its affinity for and imitation of the unique properties of music." De Falco 44.7, [Iamblich] Theologoumena Arithmeticæ. Pp. 33-34, Leipzig, 1932. To Robert Fludd (*Utriusque Cosmi*, 1619) the monochord is "the most exact symbol of the nature of the world and the figure of truth itself" (Amman: *The Musical Theory and Philosophy of Robert Fludd*, pp. 223-224) and to Kepler it is all part of a Universal harmony (*Harmonices Mundi*).

fruitful paradigm. For more than two millennia this paradigm allowed what came to be known as the *quadrivium* (the four-way), i.e., a curriculum of arithmetic, geometry, music, and astronomy to be presented as a coherent system, providing a world view and a framework for teaching as well as for scientific, metaphysical, philosophical and theological speculations.

In this, arithmetics was seen as the study of number, pure and in itself, number at rest so to speak, the basis underlying all other sciences. In turn, Geometry was the study of number in space (i.e., magnitude at rest); music, number in time and astronomy, number in time and space, or in our current idiom, magnitude in motion or kinematics, from which naturally arose the notion of harmony of the spheres.

The approach is not unlike that taken in classical science where algebra is the study of quantity and relation between quantities, geometry, of quantity in space such as length, giving rise to surfaces, volume, and their relations, while kinematics adds to it the element of time. Statics substitutes mass for time; dynamics combines length, time, and mass; thermodynamics adds the notion of temperature, and electrodynamics, the notion of electrical charge, so that all phenomena studied in these branches of science are reducible to expressions involving only L (length), T (time), M (mass), C (degree Celsius), and e (electron charge) as the case may be.

In the ancient viewpoint, therefore, number and proportion, i.e., music, pervaded all learning and formed the basis of a rational system establishing correspondences, resonances, and analogies between phenomena. The relative distances of planets from one another were thought to be the same as those of the notes of musical scale; Ptolemy (2<sup>nd</sup> century AD) used musical ratios to correlate the intellectual, perceptive and animating functions in Man, thus:

“The octave is attuned to the intellectual part, since in each of these there is the degree of simplicity; the fifth to the perceptible part; and the fourth to the animating part. For the fifth is closer to the octave than is the fourth, since it is more concordant...”<sup>37</sup>

### **Music and the Platonic forms:**

Musical tones and their afferent string length on a monochord are associated with simple numerical ratios as those we have seen in the determination of the inner structure of the platonic forms, lending plausibility to the analogy between geometry and music.

For example, the most fundamental ratio in music is the octave defined by a doubling (or halving) of frequency or a halving (or doubling) of string length on a monochord, respectively. As we have seen, this 2:1 ratio is typical of the icosahedron internal angle.

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<sup>37</sup> Ptolemy, Harmonica III,V, tr. in Andrew Barker (ed), Greek Musical Writings. Cambridge, 1989, vol. II, p. 375.

Similarly, the octahedron is characterized by the ratio 1:1, which represents unison in music, two instruments or voices sounding together at the same pitch. In turn, the cube internal angle ratio is 1/3. Musically, if we divide the string length by 3 and therefore multiply the frequency by 3 we have the twelfth (i.e., an octave plus a fifth or the G of the upper octave G'). For the Dodecahedron, characterized by the ratio 2/3 (or 3/2) for its internal angle we obtain musically 3/2 (in pitch) indicative of the fifth (G).

Given the regularity of the platonic forms, other simple ratios characteristic of pure musical tones will appear quite naturally.

For instance, consider the various whole numbers characterizing the different Platonic forms:

	Tetrah.	Cube	Octah.	Dodec.	Icosa.
# of faces	4	6	8	12	20
# of vertices	4	8	6	20	12
# of edges to a face	3	4	3	5	3
# of edges to a vertex	3	3	4	3	5
# of faces at a vertex	3	3	4	3	5
Total # of edges	6	12	12	30	30

These numbers give rise within each form to musical ratios in addition to those just mentioned previously, so that we have:

- ♦ For the tetrahedron:

$$\frac{4}{4} = 1 \text{ (unisson)}; \frac{4}{3} \text{ (fourth)}; \frac{6}{4} = \frac{3}{2} \text{ (fifth)}; \frac{6}{3} = 2 \text{ (octave)}$$

- ♦ For the cube and the octahedron:

$$\frac{3}{3} = 1 \text{ (unisson)}; \frac{4}{3} \text{ (fourth)}; \frac{6}{4} = \frac{3}{2} \text{ (fifth)}; \frac{6}{3} = \frac{8}{4} = 2 \text{ (octave)}$$

$$\frac{12}{4} = 3 \text{ (twelfth: octave + fifth)}; \frac{8}{3} \text{ (thirteenth = octave + sixth)}$$

$$\frac{12}{3} = 4 \text{ (double octave)}$$

- ♦ For the dodecahedron and the icosahedron we obtain two sets of ratios, one pertaining to the Pythagorean tuning and the other to the so-called Just intonation. These two scales will be explained subsequently.

In the Pythagorean tuning we have:

$$\frac{5}{5} = \frac{3}{3} = 1 \text{ (unisson)}$$

$$\frac{30}{20} = \frac{3}{2} \text{ (fifth)}$$

$$\frac{20}{5} = \frac{12}{3} = 4 \text{ (double octave)}$$

And in the Just intonation tuning:

$$\frac{20}{12} = \frac{5}{3} \text{ (sixth)}$$

$$\frac{30}{5} = 6 \text{ (double octave + fifth)}$$

$$\frac{30}{3} = 10 \text{ (triple octave + third)}$$

We should remark that these numerical ratios characteristic of musical tones do not have anything to do with the acoustical properties that bodies made of say metal in the form of the platonic solids would exhibit. The ratios or proportions we are considering are purely mathematical in nature. They exist in the platonic forms and therefore in the design of the millennium sphere as they are contained in the geometry of Gothic Cathedrals or Renaissance buildings, silently resonating in mind, not in air.

We now turn our attention to the construction of the Pythagorean and Just intonation scales. We shall later consider their relation to the ad quadratum construction. Both scales were known to the Ancients. However, just intonation did not come in musical use till the Renaissance with the development of polyphonic music.

*Pythagorean scale:*

The remarkable thing about the Pythagorean musical scale and the adquadratum method of generating the platonic forms is that they both rely on the first three integers 1, 2, and 3.

The Pythagorean scale is based on the pure fifth (3:2) and derived in two stages<sup>38</sup>.

In the first movement, starting from an arbitrary note (referred to as c) and adding five fifth above and one below, we obtain

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<sup>38</sup> Dae-Am Yi, op. cit.

**F G A B / c d e f g a b / c' d' e' f' g' a' b' / c" d" e" f" g" a" b"**

The frequency ratio of a fifth is  $\frac{3}{2}$  so that the above sequence of fifths spanning more than three octaves can be rewritten, assigning 1 to c:

F	c	g	d'	a'	e"	b"
$\left(\frac{3}{2}\right)^{-1}$	1	$\frac{3}{2}$	$\left(\frac{3}{2}\right)^2$	$\left(\frac{3}{2}\right)^3$	$\left(\frac{3}{2}\right)^4$	$\left(\frac{3}{2}\right)^5$
$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{8}$	$\frac{81}{16}$	$\frac{243}{32}$

In the second movement of building the scale, this set of notes is compressed within the span of an octave. The first step in this process is to rearrange the notes according to note name, regardless of actual pitch so that we obtain:

c	d'	e"	F	g	a'	b"
1	$\frac{9}{4}$	$\frac{81}{16}$	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{27}{8}$	$\frac{243}{32}$

The next step is to shift the notes by the appropriate number of octaves so that they all fit within a single octave (1 to 2).

For example, for b", ratio  $\frac{243}{32}$ , to come within the octave c c' (1 to 2), it must be shifted down two octaves (i.e., multiplied by  $\frac{1}{(2)^2} = \frac{1}{4}$ ).

$$\text{Then } b = \frac{243}{32} \times \frac{1}{4} = \frac{243}{128}$$

Similarly F, ratio  $\frac{2}{3}$ , has to be lifted up by one octave to come within the 1 to 2 octave:

$$f = \frac{2}{3} \times 2 = \frac{4}{3}$$

When all notes have been dealt with, we end up with the following scale:

c	d	e	f	g	a	b	c'
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

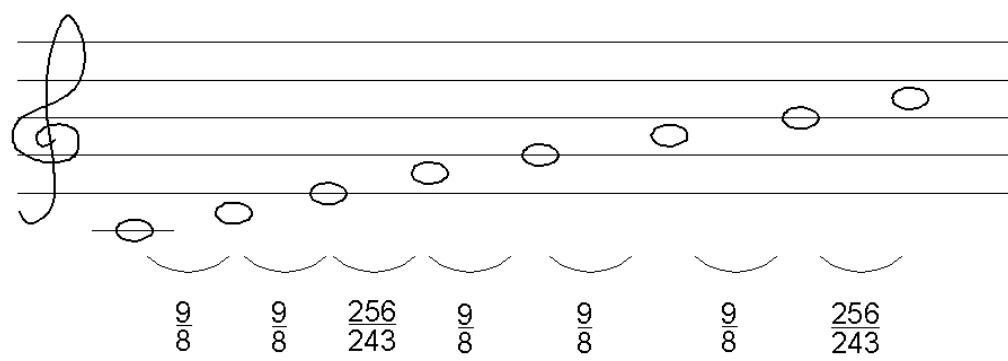


fig. 79

Which gives the intervals with respect to the starting note c, so that the second or tone (d-c) is  $\frac{9}{8}$ , the third (e-c) is  $\frac{81}{64}$ , the fourth (f-c) is  $\frac{4}{3}$ , the fifth (g-c) is  $\frac{3}{2}$ , the sixth (a-c) is  $\frac{27}{16}$ , the seventh (b-c) is  $\frac{243}{128}$ , and the eighth or octave (c'-c) is 2.

To establish the intervals between the notes, it is sufficient to calculate the frequency ratio between them. Thus:

$$\text{d to c: } \frac{9}{8} : 1 = \frac{9}{8}$$

$$\text{e to d: } \frac{81}{64} : \frac{9}{8} = \frac{9}{8}$$

$$\text{f to e: } \frac{4}{3} : \frac{81}{64} = \frac{256}{243}$$

$$\text{g to f: } \frac{3}{2} : \frac{4}{3} = \frac{9}{8}$$

$$\text{a to g: } \frac{27}{16} : \frac{3}{2} = \frac{9}{8}$$

$$\text{b to a: } \frac{243}{128} : \frac{27}{16} = \frac{9}{8}$$

$$\text{c' to b: } 2 : \frac{243}{128} = \frac{256}{243}$$

The Pythagorean scale consists therefore of five whole tones  $\left(\frac{9}{8}\right)$  and two semitones  $\left(\frac{256}{243}\right)$ , also called by their Greek names as *leimma* or *diesis*.

The Pythagorean scale can therefore be represented on the staff as shown on fig. 79.

In Pythagorean tuning only the octave (2:1), the fifth (3:2) and the fourth (4:3) were considered to be consonances.

The difference between a whole tone  $\left(\frac{9}{8}\right)$  and the semitone  $\left(\frac{256}{243}\right)$  is

$$\frac{9}{8} : \frac{256}{243} = \frac{2187}{2048} = 1.06787$$

The double process used in establishing the Pythagorean scale (addition by fifths and reduction by octaves) leads to a problem since it is impossible to have the two processes converge to a single note, i.e., the fifths expanding in either direction never meet the octave.

Mathematically, if the fifth added x times and the octave added y times would meet, one could write:

$$\left(\frac{3}{2}\right)^x = \left(\frac{2}{1}\right)^y \text{ or } 3^x = 2^{x+y}$$

with  $x$  and  $y$ , both integers.

## Taking the logs

$$\begin{aligned}x \log 3 &= (x+y) \log 2 \\ \text{or } x(\log 3 - \log 2) &= y \log 2 \\ \therefore \frac{x}{y} &= \frac{\log 2}{\log 3 - \log 2} = 1.7095\dots\end{aligned}$$

Clearly therefore if either  $x$  or  $y$  is an integer, the other one is not, which is contrary to what is required,

e.g., when  $y=7$      $x=11.966$

The closest integers are 7 and 12.

The difference between the two pitches, i.e., the one corresponding to the note obtained after 7 octaves and that after 12 fifth is

$$(2)^7 : \left(\frac{3}{2}\right)^{12} = \frac{2^{19}}{3^{12}}$$

$$= \frac{524288}{531441} = 0.98654 < 1$$

This is known as the Pythagorean comma. As long as a melody evolves within an octave, or nearly so, as in Gregorian chants for example, there is no problem. Difficulties arise in polyphonic music when voices are separated by 2 or more octaves: if the interval between notes is maintained as the voices move across octaves, then the octave notes will not coincide or conversely if the octaves are maintained, the interval between notes will not coincide.

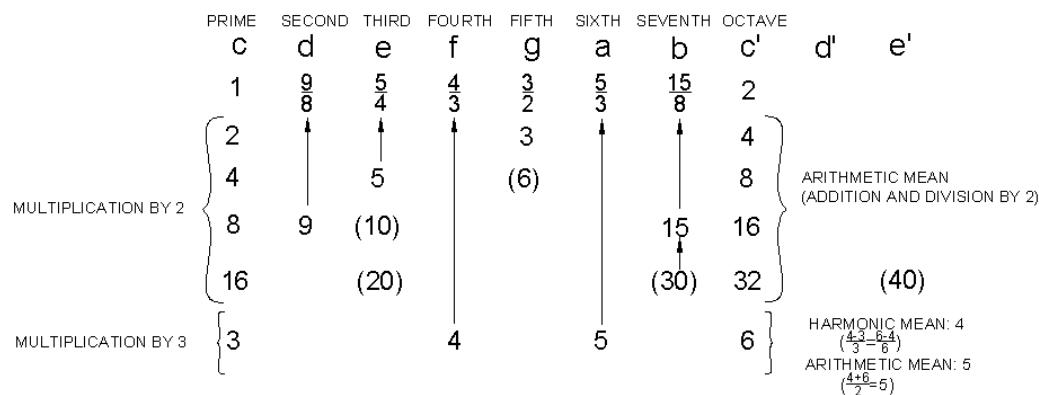


fig. 80

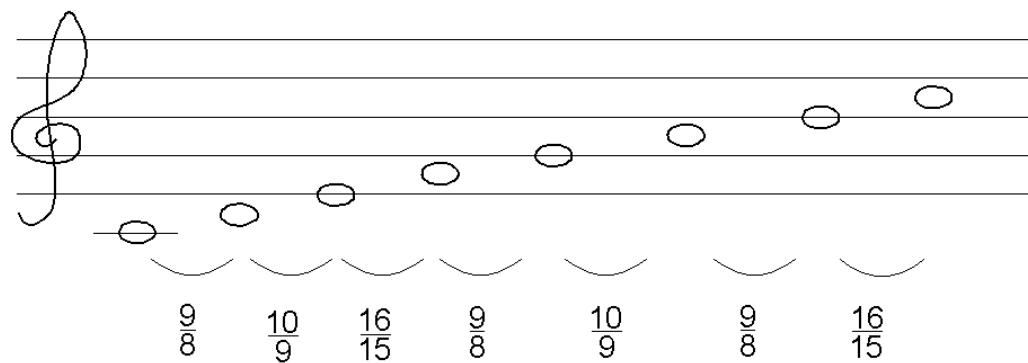


fig. 81

In either case, consonance is lost. The resolution was eventually found in the Tempered scale popularized by J.S. Bach in the early 18<sup>th</sup> century. The Ancients knew of it<sup>39</sup> but they also knew the importance of finding one's limits and always kept their melodies within proper bounds so as to not have to tamper with either intervals or octaves. The problem was solved by spreading the comma across the octave by dividing the octave scale in 12 equal tones.

Pythagorean tuning had another problem and that was the discordance of the third  $\left(\frac{81}{64}\right)$ .

Remember that only the fourth and the fifth, together with the octave and unisson were considered as concordances by the Pythagoreans.

To remove this discordance, just intonation, which replaced the Pythagorean third  $\left(\frac{81}{64}\right)$

by the natural third  $\left(\frac{5}{4}\right)$ , was suggested. It was not adopted however until the

Renaissance since the third was avoided in ancient and early medieval music as was pointed out earlier.

We shall return to Pythagorean tuning and its relation with the adquadratum construction but before we shall address the design of the just intonation scale.

#### *Just Intonation:*

As the Pythagorean scale was shown to originate out of the play of numbers 1, 2, and 3 through the prime ( $c=1$ ), the octave ( $c'=2$ ) and the fifth ( $g=\frac{3}{2}$ ), the just intonation scale can be constructed through multiplication or division out of numbers 1, 2, and 3 equally, the natural third  $\left(\frac{5}{4}\right)$  appearing as a consequence of these multiplications and divisions

by 2 and 3, as shown on fig. 80. This is interesting in the sense that the just intonation is usually presented as the result of combination of the natural fifth and the natural third.

(e.g.,  $N = \left(\frac{3}{2}\right)^m \left(\frac{5}{4}\right)^n$ ).

Since we can show that only 1, 2, and 3 are involved, it brings Just intonation within the scope of the adquadratum construction.

Acoustically, the operation involves the half-cut and the third cut, which can be done aurally or visually on a monochord. The first line of fig. 80 gives the intervals to the prime; it constitutes the c-major scale.

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<sup>39</sup> In fact McClain (op. cit. p. 5) claims that Plato's Republic embodies, from a musician's perspective, a treatise on equal temperament.

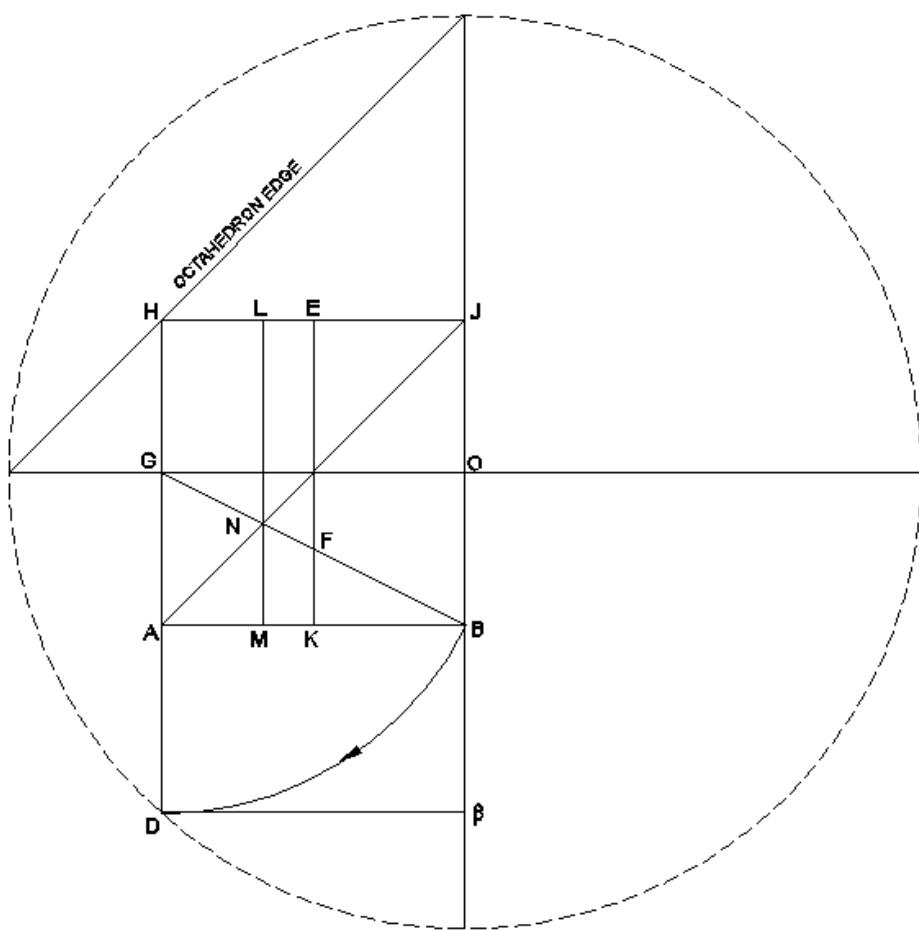


fig. 82

The interval between the notes can be obtained as for the Pythagorean scale by calculating the ratios:

d to c	$\frac{9}{8}$
e to d	$\frac{5}{4} : \frac{9}{8} = \frac{40}{36} = \frac{10}{9}$
f to e	$\frac{4}{3} : \frac{5}{4} = \frac{16}{15}$
g to f	$\frac{3}{2} : \frac{4}{3} = \frac{9}{8}$
a to g	$\frac{5}{3} : \frac{3}{2} = \frac{10}{9}$
b to a	$\frac{15}{8} : \frac{5}{3} = \frac{45}{40} = \frac{9}{8}$
c' to b	$2 : \frac{15}{8} = \frac{16}{15}$

The Just intonation has therefore two different whole tones namely  $\frac{9}{8}$  (major) and  $\frac{10}{9}$  (minor) and the semitone is  $\frac{16}{15}$ . Since the minor scale starts on *a*, we see that the interval of the third in that scale will be  $\frac{6}{5}$ , and that of the sixth  $\frac{8}{5}$ .

On the staff, the Just Intonation is as shown on fig. 81:

### Adquadratum construction of the Pythagorean consonances:

Ptolemy in his *Harmonica* gives us a construction for the Pythagorean consonances based on the square known as the Heliconian Square<sup>40</sup>.

The square *ABJH* shows the frequency ratios (i.e., inverse string length ratios) in Pythagorean tuning by its geometrical proportions. As seen on fig. 82, this square results directly from the adquadratum construction.

Triangles *GAN* and *JBN* are similar, so that:

$$\frac{GA}{JB} = \frac{AM}{MB} = \frac{1}{2}$$

Therefore

$$2AM = MB \text{ or } AM = \frac{1}{3} AB,$$

letting

$$AM + MB = 1$$

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<sup>40</sup> Dae-Am Yi, op. cit. p. 14

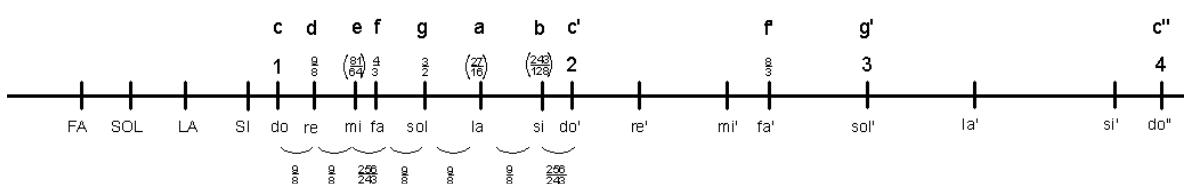


fig. 83

We can also write:

$$\frac{MN}{ML} = \frac{1}{3} \text{ and } \frac{LN}{LM} = \frac{2}{3}$$

And the following ratios can therefore be read off the graph.

$$\frac{JH}{LH} = \frac{3}{1} \text{ A tripling of frequency} \Rightarrow \text{a twelfth (g') (sol')} \\ = \text{octave + fifth}$$

$$\begin{aligned} & (7+5) \text{ in terms of position} \\ & \left( 2 \times \frac{3}{2} = 3 \right) \text{ in terms of frequencies.} \end{aligned}$$

$$\frac{HA}{HG} = \frac{2}{1} \text{ The octave ratio (c') (do')}$$

$$\frac{LM}{LN} = \frac{3}{2} \text{ (ascending fifth) (g) (sol)}$$

$$\frac{EK}{EF} = \frac{4}{3} \text{ (ascending fourth) (f) (la)}$$

$$\frac{LN}{FK} = \frac{\sqrt[2]{3}}{\sqrt[1]{4}} = \frac{2}{3} \times \frac{4}{1} = \frac{8}{3} \text{ (octave + fourth) (f') (fa')}$$

$$\frac{EK}{FK} = \frac{4}{1} = 4 \text{ (double octave) (c'') (do'')}$$

All elements to build the Pythagorean scale are therefore at hand. The graphical scale shown on fig. 83 indicates the ratios just obtained. Those missing, shown in brackets, are

easily obtained since the value of the tone  $\left(\frac{9}{8}\right)$  is known thus:

$$e = \frac{9}{8} \times \frac{9}{8} = \frac{81}{64}; \quad a = \frac{3}{2} \times \frac{9}{8} = \frac{27}{16}; \quad b = \frac{27}{16} \times \frac{9}{8} = \frac{243}{128}$$

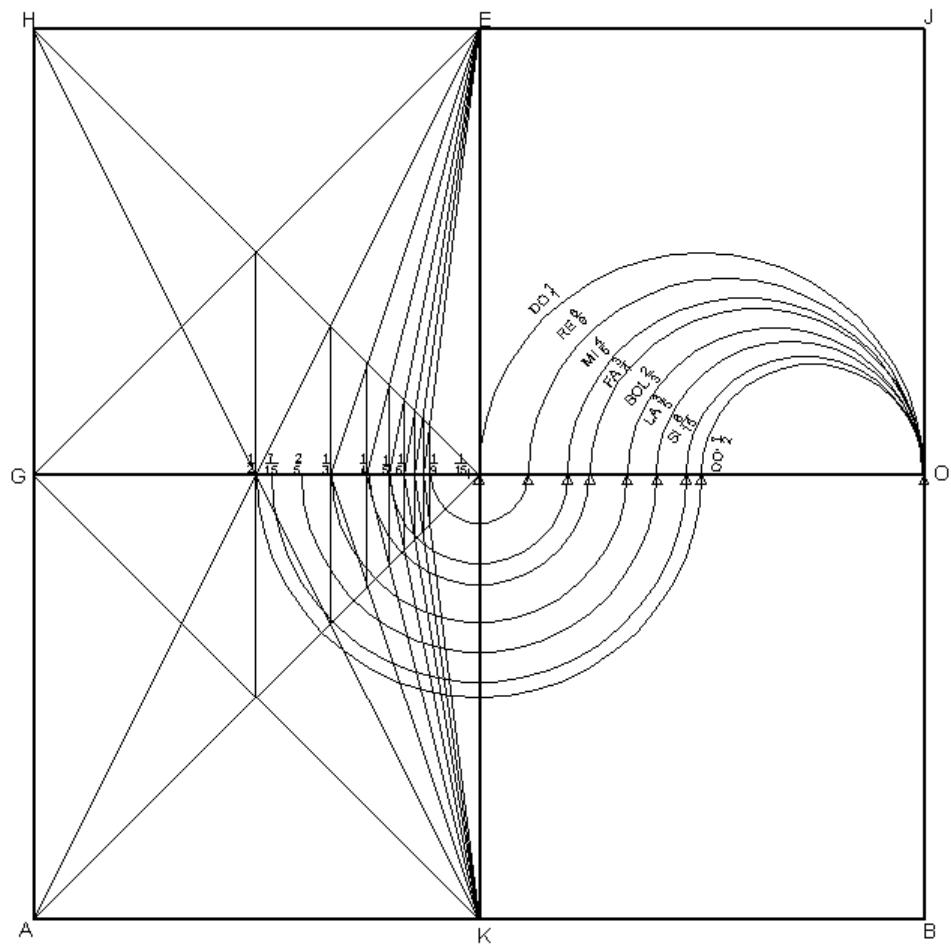


fig. 84

### Ad Quadratum and the Just Intonation Scale:

As we have seen, the frequency ratios of the just intonation scale have been shown to be:

c	d	e	f	g	a	b	c'
do	re	mi	fa	sol	la	si	do'
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

Since frequencies vary in inverse ratios to string lengths on a monochord (assuming constant tension in the string), the string length ratios can be readily obtained as:

$$1 \quad \frac{8}{9} \quad \frac{4}{5} \quad \frac{3}{4} \quad \frac{2}{3} \quad \frac{3}{5} \quad \frac{8}{15} \quad \frac{1}{2}$$

The Ad Quadratum method allows a geometrical determination of the movable bridge positions on the monochord as shown on fig. 84.

We again consider square *ABJH* of figure 71 and 82, that we have isolated and enlarged on figure 84.

The left hand side of the figure, namely the double square *HEKA* shows the construction of the harmonic series  $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ , etc....

i.e., the series of the inverses. This series is harmonic in the sense that any term is the harmonic mean of its two adjoining terms<sup>41</sup>.

The string lengths are obtained by a simple compass swings. e.g., by swinging  $\frac{1}{9}$  around to the right, the segment left to the right between the compass lead and point 0 will be  $\frac{8}{9}$ , similarly for  $\frac{4}{5}, \frac{3}{4}, \frac{2}{3}$ , and  $\frac{1}{2}$ .

<sup>41</sup> e.g., if c is the harmonic mean between a and b then  $c = a \left[ 1 + \frac{b-a}{b+a} \right] = b \left[ 1 - \frac{b-a}{b+a} \right]$

For  $\frac{3}{5}$  and  $\frac{8}{15}$ , having respectively  $\frac{1}{5}$  and  $\frac{1}{15}$  from the harmonic construction, it is necessary to just measure out  $\frac{2}{5}$  and  $\frac{7}{15}$  respectively on the left with the compass and swing the measurement to the right to find the proper location.

### *Conclusions:*

The generating process for both the Pythagorean and the Just intonation scales starts with the prime (one) and uses only 2 and 3 – under the form of the fifth  $\left(\frac{3}{2}\right)$  for Pythagorean tuning and the half-cut and third-cut for Just intonation. Number 4 comes out of the generating process through inversion of the F, in Pythagorean tuning, to bring it within the octave 1 to 2, and through the third cut (harmonic mean) is Just intonation.

Similarly, in the adquadratum method for generating the platonic forms, only 1, 2, and 3 are used with 4 appearing only as a result of the tetrahedron construction, measuring its height within a circumsphere of radius 3 and encompassing an insphere of radius 1. This, in turn, gives rise to number 9 (ratio of the areas of the circumsphere to the insphere) and 27 (ratio of their volume) so that all the numbers out of which Plato compounded the world soul, namely 1, 2, 3, 4, 8, 9, and 27 are to be found in the generation of the tetrahedron, the simplest of the platonic forms.

The numbers 1, 2, 3, and 4 whose sum is 10 form the celebrated Tetrakty of the Pythagoreans. Out of these simple numbers, as we have seen, the consonant ratios of the octave (1:2), the fourth (3:4), the fifth (3:2), the double octave (1:4) and the twelfth, i.e., the octave plus a fifth (1:3) can be formed.

The Tetrakty, furthermore, symbolized the universal structure, starting in unity – one – the point, moving to 2, the one-dimensional line; 3, the two dimensional triangle, and 4, the tetrahedron, the first three dimensional form. It eventually returns through its sum to unity, Ten. In this respect it is amusing to indulge in a bit of Pythagorean numerology by remarking that if we sum up all the elements of the five Platonic forms as listed on p. 130, we obtain a total of 244, the integers of which sum up to 10, i.e., again unity, while their product comes to 32 whose product is 6, the first perfect number, since  $1 \times 2 \times 3 = 1 + 2 + 6$ , ha!

No wonder the Ancients were awed by such perfection. As Theon of Smyrna<sup>42</sup> puts it:

*“Unity is the principle of all things and the most dominant of all that is: all things emanate from it and it emanates*

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<sup>42</sup> Theon of Smyrna: *Mathematics Useful for Understanding Plato*. Tr. By R&D Lawlor, San Diego 1979.

*from nothing. It is indivisible and it is everything in power. It is immutable and it never departs from its own nature through multiplication ( $1 \times 1 = 1$ ). All that is intelligible and cannot be engendered exists in it: the nature of ideas, God himself, the soul, the beautiful and the good, and every intelligible essence, such as beauty itself, for we conceive of each of these things as being one and as existing in itself.”*

“Greek musical theory is founded on the so-called ‘musical proportion’, which Pythagoras reputedly brought home from Babylon,” writes McClain<sup>43</sup>. “It is this proportion which exemplifies the science Plato labels *Stereometry* (the gauging of solids),

*‘a device of God’s contriving which breeds amazement in those who fix their gaze on it and consider how universal nature molds form and type...a gift from the blessed choir of the Muses to which mankind owes the boon of the play of consonance and measure with all they contribute to rhythm and melody.’*

(--Epinomis 990-991.)

### **The Aural and the Visual – Esthetic of Proportion:**

For the Ancients, from Pythagoras and Plato down, the appreciation of beauty was linked with that of proportion however perceived. It was an intellectual experience irrespective of its origin in the sensorium. But it is Augustine (354-430) who developed the concept further in his theory of aesthetics with the notion of *synesthesia*. Here, visual and aural sensations are combined since they, as well as other motions, all depend on numbers. It is these numbers, appreciated by the mind, that are ultimately the source of the experience.

*“Whether they are considered in themselves or applied to the laws of figures, or of sound, or of some other motion, numbers have immutable rules not instituted by man but discovered through the sagacity of the ingenuous.”<sup>44</sup>*

To Augustine, music is *scientia bene modulandi* i.e., the science of good modulation

*“concerned with the relating of several musical units according to a module, a measure, in such a way that the*

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<sup>43</sup> McClain, Ernest: The Pythagorean Plato. P. 8. Nicholas Hays 1978.

<sup>44</sup> Augustine: De Doctrina Christiana, II XXXViii, 56. tr. D.W. Robertson, Jr., Indianapolis, 1958, p. 73.

*relations can be expressed in simple arithmetical ratios....1:1, 1:2, 2:3, 3:4.*"<sup>45</sup>

*"Beautiful things please by proportions...equality is not found only in sounds for the ear and in bodily movements, but also in visible forms..."* says Augustine.<sup>46</sup>

Centuries later, Leonardo da Vinci would write<sup>47</sup>:

*"I give the degrees of the objects seen by the eye as the musician does the notes heard by the ear."*

...thereby introducing perspective as an analogy to musical harmony, and Kepler would add<sup>48</sup>:

*"The mathematics of the senses, if they are recognized, excite an intellectual mathematics, previously present to the inner man, such that there actively lies within the soul what beforehand was hidden beneath the veil of potentiality."*

This visible form of harmony manifested through modules is evident for instance in the simple arrangements of a row of evenly spaced columns or trees (or the ties of a railroad) perceived as a harmonic series (see fig. 85). Here a constant frequency on the ground, mathematically expressed as an arithmetic series (the equal spacing between trees is additive) is transformed through the eye or the pinhole of a *camera oscura* by the effect of perspective (projection on the retina or the plate of the camera) into an harmonic series (each term, i.e., length of the column, tree or tie appears to be the harmonic means of its two adjoining terms) i.e., the frequency appears to get compressed more and more as the distance between items seems to keep on decreasing.

Referring to fig. 85, we assume for simplicity's sake unit distance between columns (or trees or ties), i.e.,  $O'A' = A'B'$  etc., columns two units tall, i.e.,  $AA' = 2O'A'$  etc., and observation point  $O$  (pinhole) at midheight, i.e.,  $OO' = \frac{AA'}{2}$ . Now consider triangles  $OAA'$  and  $OPP'$ ;  $OBB'$  and  $OP_B P'_B$ ;  $OCC'$  and  $OP_C P'_C$ ;  $ODD'$  and  $OP_D P_D$ ; they are respectively similar. We can therefore write in turn:

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<sup>45</sup> Von Simson, Otto, *The Gothic Cathedral*. Princeton/Bollingen, 1974. p. 21.

<sup>46</sup> De Musica VI, xiii, 38.

<sup>47</sup> *The Notebooks of Leonardo da Vinci*. Jean Paul Richter; 102 Vol. I. Dover, NY, 1970

<sup>48</sup> Kepler, J. *Gesammelte Werke*, W. von Dyck et. Al. Ed. Munich, Beck, 1938. p. 226, Vol. 6. Quoted in Hallyn, op. cit. p. 171.

$$\begin{aligned}
 \frac{AA'}{OA_1} = \frac{2}{1} &= \frac{PP'}{OP_1} \quad \therefore \quad PP' = 2(1) \\
 \frac{BB'}{OB_1} = \frac{2}{2} &= \frac{P_BP'_B}{OP_1} \quad \therefore \quad P_BP'_B = 2\left(\frac{1}{2}\right) \quad (OP_1 = 1) \\
 \frac{CC'}{OC_1} = \frac{2}{3} &= \frac{P_C P'_C}{OP_1} \quad \therefore \quad P_C P'_C = 2\left(\frac{1}{3}\right) \\
 \frac{DD'}{OD_1} = \frac{2}{4} &= \frac{P_D P'_D}{OP_1} \quad \therefore \quad P_D P'_D = 2\left(\frac{1}{4}\right)
 \end{aligned}$$

Now, projections on plane  $PP'$  are inverted and can be considered as projection on the plate of a *camera oscura* or the retina (assuming it flat). Different focal distances can be

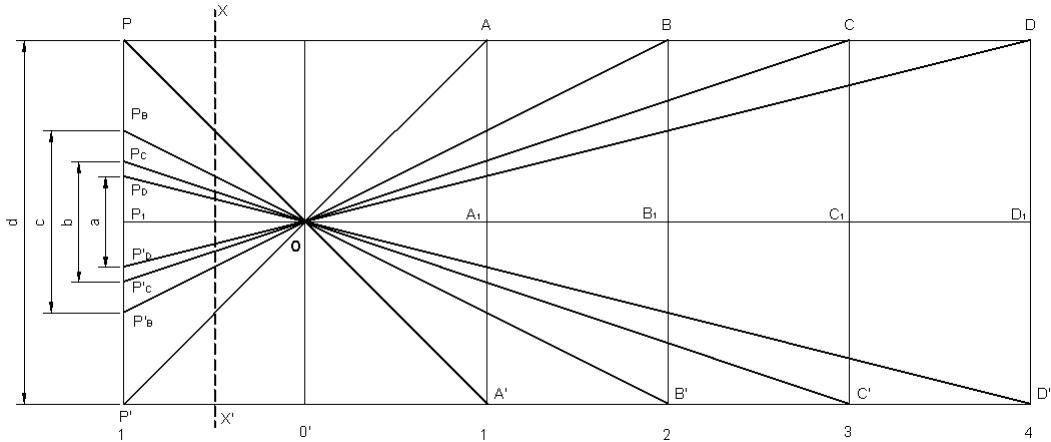


fig. 85

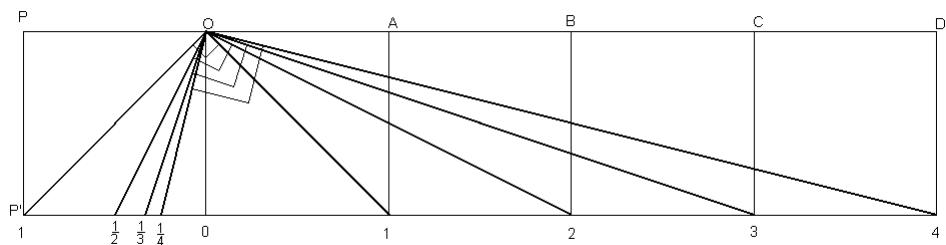


fig. 86

accounted for by other projection planes such as  $xx'$ . The geometry remains the same. It is therefore obvious that a row of columns of same heights  $AA'$ ,  $BB'$ , etc., equally spaced on the ground (i.e., in arithmetical series from a point of origin  $OA_1 = 1$ ,  $OB_1 = 2$ ,  $OC_1 = 3$ , etc.) appears on the projection plane  $PP'$  as an harmonic series:  $K\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots\right)$  where  $K$  is an arbitrary factor depending on the height of the columns with respect to their spacing.<sup>49</sup>

Note (fig. 86) that the product of the distance of a point on the ground from the point of observation, i.e., 1, 2, 3, ....n...and the corresponding projections of the half column height on the projection plane  $1, \frac{1}{2}, \frac{1}{3} \dots \frac{1}{n} \dots$  have a geometric mean of unity.

$$\text{i.e., } \sqrt{1 \times 1} = 1, \quad \sqrt{2 \times \frac{1}{2}} = 1, \quad \sqrt{n \times \frac{1}{n}} = 1$$

Taking an analogical viewpoint dear to the medieval, one could say that one, the image of God in mind, remains ever the same and equal to unity irrespective of position. The geometric mean constitutes therefore some kind of an invariant connecting a point to its projection.

Also, as seen before in our consideration of the relationship between string length in the monochord and frequency, it is apparent that visually the harmonic series  $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$  generates the string length series required to produce the Pythagorean consonances or “fixed tones” of the third  $\left(\frac{3}{4}\right)$  the fifth  $\left(\frac{2}{3}\right)$  and the octave  $\left(\frac{1}{2}\right)$  as shown on fig. 87, or indeed the whole Just intonation scale as shown on fig. 84, justifying Leonardo’s remark quoted earlier.

An interesting point made by McClain<sup>50</sup> is that,

*“for Plato, sight – not hearing – is the cause of the highest benefit to us.” (Timaeus 470). He speaks of those who fix their gaze on his example – ‘The constant revolution of potency and its converse’ – using words which warn us that number and tone must be translated into geometric imagery, and revealing that his own primary image is the circle, purest embodiment of the notion of cycle.”*

(see circle of tones in fig. 87.)

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<sup>49</sup> Of course slight complications occur when the height of the observation point changes with respect to the columns, but we are not here writing a treatise on perspective.

<sup>50</sup> McClain, op. cit. p. 9.

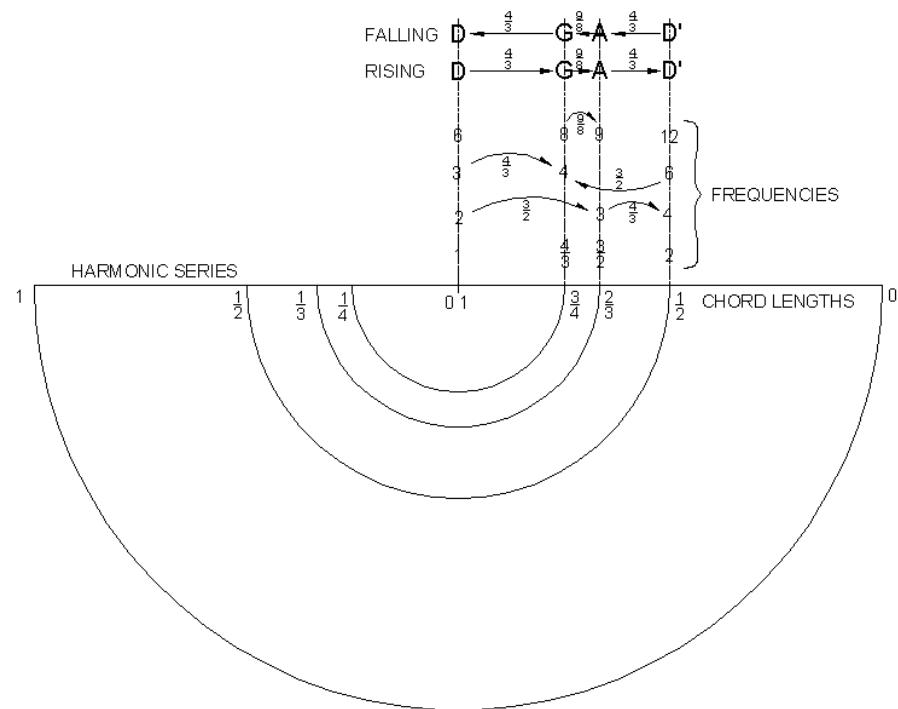
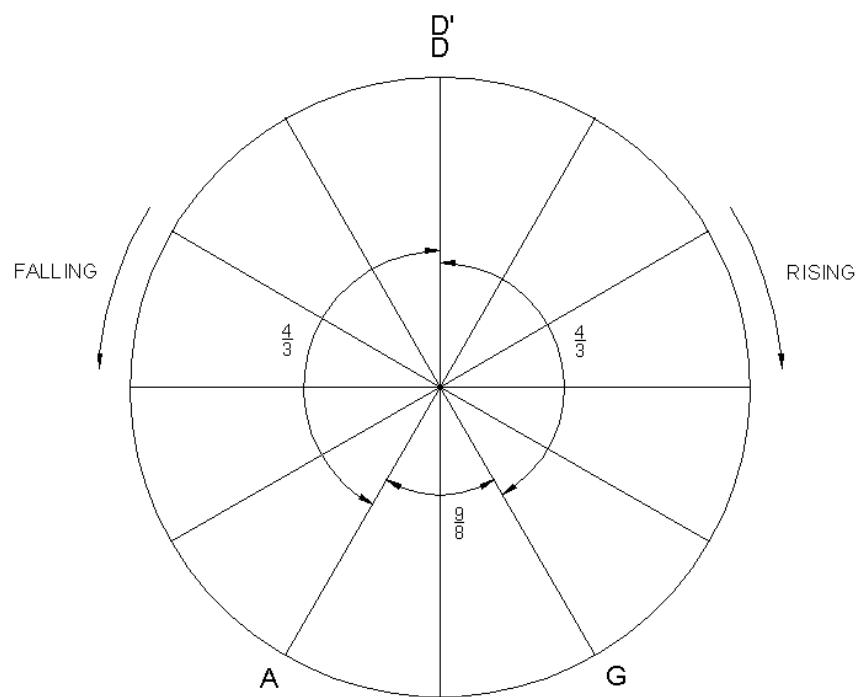


fig. 87

Notwithstanding Plato, there is an interesting observation that can be made here to the effect that the ear works in a somewhat opposite direction to the eye. Whereas an octave in a low frequency range or a higher one such as that of a tenor or a soprano appear to span the same aural (or acoustical) space and be additive (one sitting on top of the other) being then perceived as an arithmetical progression, their production results from a geometric progression through doubling of frequency across each octave.

$$\begin{array}{cccccc} \text{i.e., octave 1} & \text{octave 2} & \text{octave 3} \\ (2)^0 & (2)^1 & (2)^2 & (2)^3 \dots & (2)^n \\ 1 & 2 & 4 & 8 & \dots \end{array}$$

As the octaves go up, more and more frequencies are compressed within the corresponding octave. Here, the relationship between what is heard and what is produced is therefore logarithmic. This is the result of the design of the inner ear. This fact has the interesting result of increasing considerably the range of audible frequencies for the same ear size. It is a device that maintains virtually the same discriminatory tone sensitivity within a wide range.

So, as the ear perceives linearly (or additively) what is physically harmonic, the eye, as we have seen, perceives harmonically what is physically linear. It is as if eye and ear worked together in harmony to present the mind with an image of external “reality” unitive, corrected and compensated for the deformations inherent in their very design. Again, the Ancient and Augustine may have had a point: what matters is the total experience, when all the senses work together “synesthetically.”

Another point worthy of notice is that whereas sight spans approximately one octave of frequencies in the electromagnetic field, hearing spans close to 10 octaves of sound frequencies, making the ear a more discriminative instrument than the eye. It may also be remarked here that certain traditions such as the Indian tradition (Vedic) emphasize hearing over sight, contrary to the Greek. It is also a fact of experience that sight is more the instrument of the intellect and hearing that of the heart. Hence the importance of chant in spiritual work, but perhaps also a clue to the difference of spirituality between the East and the West and of the efficacy in bringing the two together through iconic and chant worship as in the Christian Orthodox tradition.



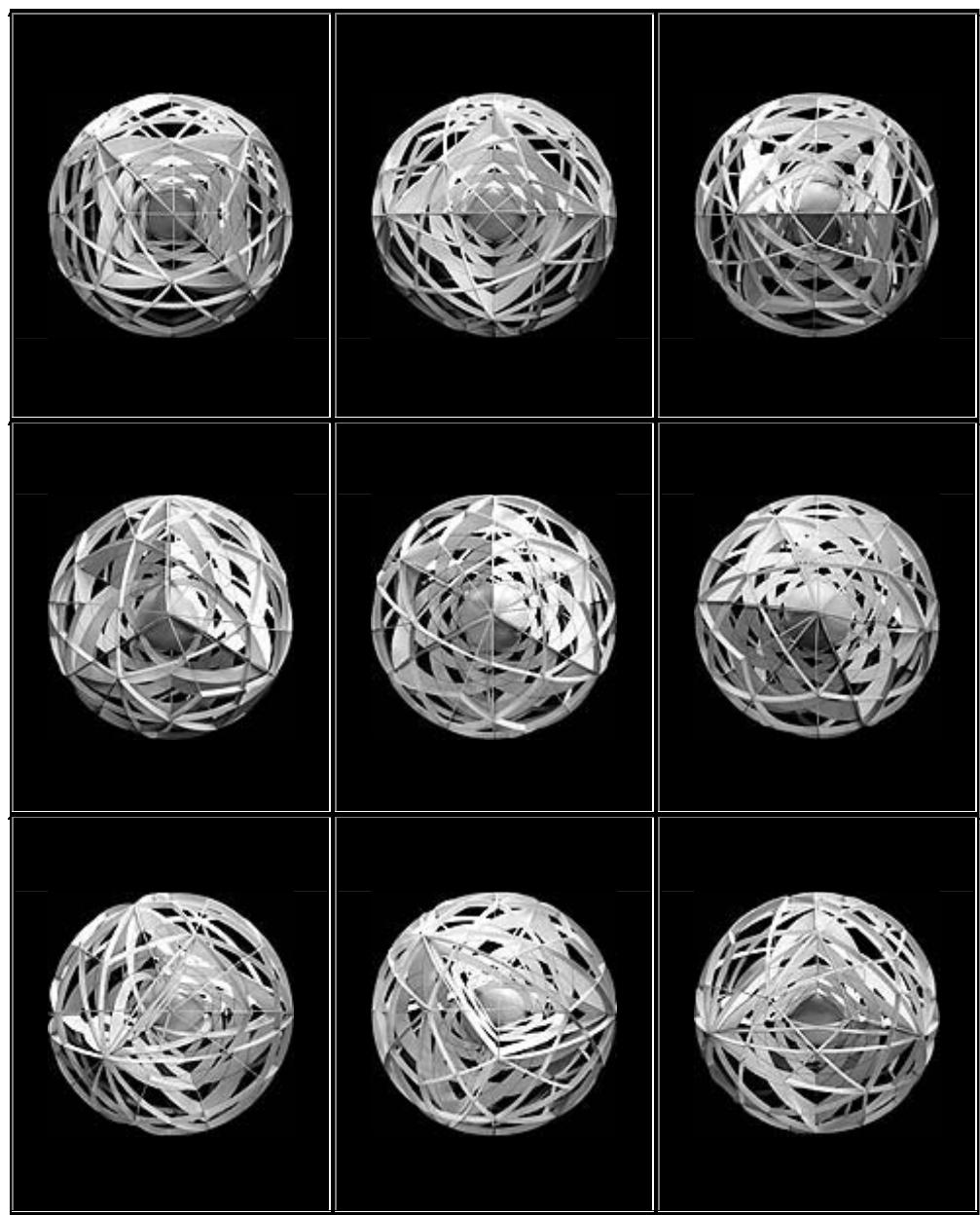


fig. 88

## Ad Quadratum, the Millennium Sphere, and Astronomy

As mentioned in the section on Music and World View<sup>51</sup>, the ratios that the ad quadratum method establishes and which are at the basis of musical structures find echoes in the way the cosmos was understood in ancient as well as classical times. This represents an unbroken tradition of nearly two millennia of cosmological thinking in musical terms.

As a sample of such thinking we may look at Johannes Scotus Erigena (c. 810-880). Though bold in his philosophical and theological propositions, Scotus was not an original thinker in either astronomy or music, but he used his knowledge of both to frame his philosophical and theological reflection particularly in connection with the process of creation. He can also be taken as perhaps among the best-educated minds of the early Middle Ages<sup>52</sup>.

Music for him is

*"That discipline which in the light of reason contemplates the harmony of all things, whether they be in movement or in a status of discernable perseverance, and combined in natural proportion."*<sup>53</sup>

The harmony of heaven as well as of the world depends on mathematical proportions. From earth to sun there is one octave and from sun to firmament another. Between earth and moon there is a whole tone. Each planet corresponds to one tone:

*"The ethereal circle turned the starry sky, which went in circles round the world. In manifold advance the consonant crowd of planets moved, emitting sweet whole-tones, six in number, [with] seven intervals and eight tones."*<sup>54</sup>

Basically this same scheme will obtain till Copernicus and Kepler. The ad quadratum method that gives the musical ratios supplies therefore the basic cosmological information that these systems of astronomy embodied.

The Millennium Sphere (fig. 88), imbricated growth spirals of tetrahedron and cube, complemented by arcs of great circles circumscribing octahedron, icosahedron, and dodecahedron, though not a strict representation of the heavens, is however more than symbolic. Its spirals of growth literally let polyhedra of all sizes fit within its structure so that for instance Kepler's nested platonic forms of the orbits of the planets can find their

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<sup>51</sup> See also above under 1. *Historical Perspective* in the section on *Alternative Methods of generating the Platonic Forms*.

<sup>52</sup> Some of luminaries succeeding him will be Gerbert d'Aurillac (c. 940-1003), who became pope as Sylvester II, and Fulbert de Chartres (d. 1020).

<sup>53</sup> Quoted in Nils L. Wallis: *Geometry, Arithmetic and Musical Creation in Window on Creativity and Invention*, Lomond Publications, 1988, pp. 89-110.

<sup>54</sup> Ibid.

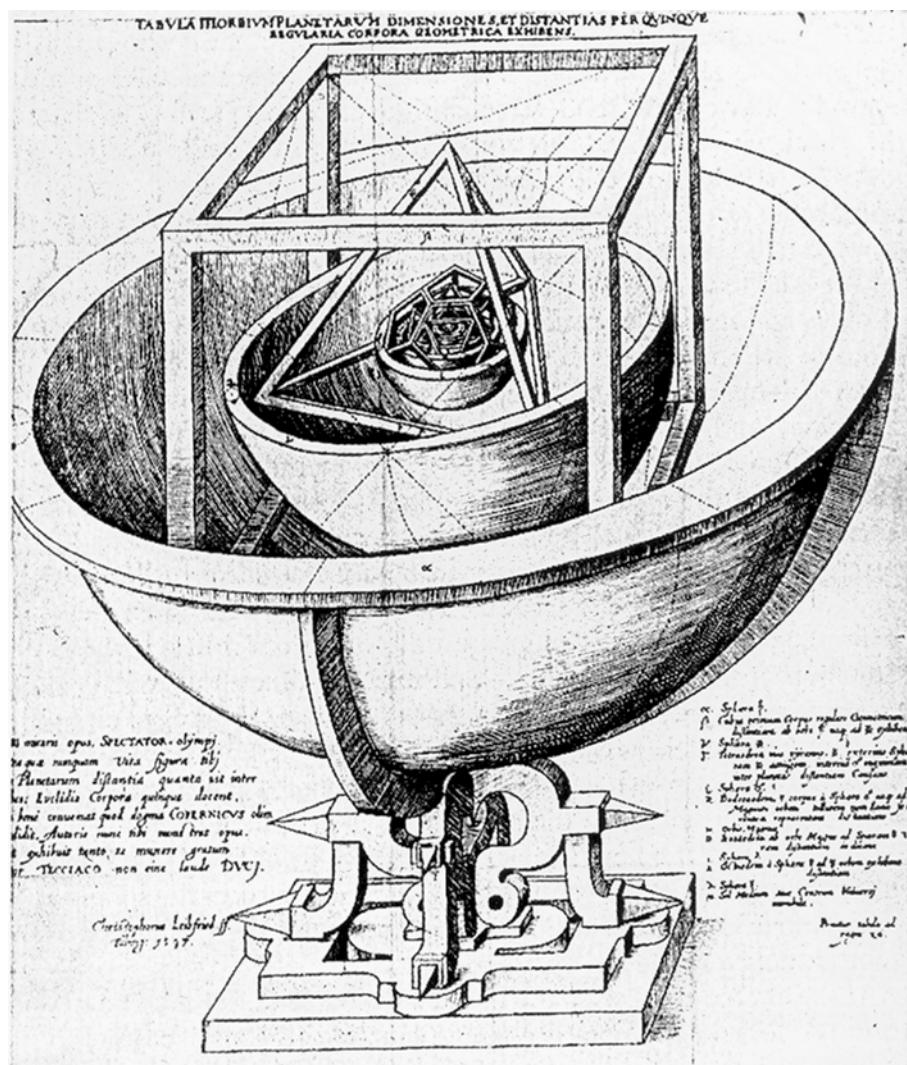


fig. 60

own positions in it. Kepler's idea, illustrated on his famous *Planetarium* (fig. 60) published in his *Mysterium Cosmographicum*, was that the solar system was so organized that the spheres of the orbits of the planets and the platonic forms constituted a concentric, nested set, each touching its inner and outer neighbors. Kepler found in this an explanation for the number of the planets (6 known at the time): "There could only be 6 planets because there were only five regular solids." In his preface Kepler further writes:

"The Earth is the circle which is the measure of all.  
Construct a dodecahedron round it. The circle  
surrounding that will be Mars. Round Mars construct a  
cube. The circle surrounding that will be Saturn. Now  
construct an icosahedron inside the Earth. The circle  
inscribed within that will be Venus. Inside Venus inscribe  
an octahedron. The circle inscribed within that will be  
Mercury. There you have the explanation of the number of  
planets."<sup>55</sup>

His theory agreed well with Copernicus's figures and he attributed the discrepancies to minor errors in Copernicus's values for the eccentricities and radii of the orbits.

The Millennium Sphere with its nested polyhedra is therefore a monument to Keplerian astronomy as well as to more ancient astronomy with its small central sphere capsule symbolic of the central Earth theory surrounded by the orbs of the planets and the sphere of the stars.

Of course, Kepler's scheme, though a new version of the old nested-spheres, was radically different through its reversal of Earth and Sun positions, but the fundamental concepts remained within the same framework. For him, music was still the coordinating force of the cosmos. In his five volumes on world harmonies (*Harmonices Mundi*, 1619) he explains:<sup>56</sup>

"Pour air in the heaven, and a real and true music will  
sound. There is a 'spiritual harmony' (*concentus  
intellectualis*) that gives pleasure and amusement to pure  
spiritual beings and in a certain sense even to God himself,  
not less than to man with his ear devoted to musical  
chords."

Further, he adds:

"The heavenly motions are nothing but a continuous song  
for several voices, to be perceived by the intellect, not the  
ear; a music which, through discordant tensions, through

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<sup>55</sup> Quoted in James Evans: *Ancient Astronomy*, Oxford U.P. 1998, (tr. Duncan p. 69)

<sup>56</sup> Johannes Kepler, *Harmonices Mundi*, Book V, Chapter 7.

*syncopations and cadenzas as it were, progresses toward certain pre-designed six-voiced cadences, and thereby sets landmarks in the immeasurable flow of time.*"<sup>57</sup>

In 1952 the composer Paul Hindemith, who based his opera *Die Harmonie der Welt* on the life of Kepler, wrote:

*"The science of music deals with the proportions objects assume in their quantitative and spatial, but also in their biological and spiritual, relations. Kepler's three basic laws of planetary motion...could perhaps not have been discovered without a serious backing in music theory. It may well be that the last word concerning the interdependence of music and the exact sciences has not yet been spoken."*<sup>58</sup>

The Millennium Sphere as an artifact is therefore emblematic of the universe or Cosmos as conceived through a long tradition. As such, it inscribes itself in the ancient practice of *sphairopoia*<sup>59</sup> or sphere-making.

Thales of Miletus (6<sup>th</sup> C. B.C.) is said to have been the first to represent the heavens with a sphere. Plato in the tenth book of the Republic and in the Timaeus describes the universe as if he had a mechanical model under his eyes and Theon of Smyrna<sup>60</sup> (c. 115-140 AD), commenting on Plato, explicitly says that he himself "constructed a sphere according to his (Plato's) explanations." Together with *gnomics* (the making of sundials) and *dioptrics* (the design and use of sighting instruments), *sphairopoia* was a recognized branch of technical writings on mechanics. It reached a very high level with Archimedes (c. 250 BC). Spheres in the image of the heavens led to the development of orreries and armillary spheres showing the movements of the planets with respect to a fixed central earth. Armillary spheres became very popular during the Renaissance, at the time of the voyages of exploration.

From a more modern viewpoint, the Millennium Sphere with its inner spiral arms serpentine within its volume is a reminder of the countless galaxies of the Universe as we now conceive it; its criss-crossing arcs circling around its center another reminder of the intense human presence surrounding the Earth both on its surface and in space: routes of ships and planes, orbits of satellites, crafts of all sorts shuttling round the planet to build wealth and knowledge at an ever-rising pitch as swarms of bees around a pollen laden flower.

<sup>57</sup> Ibid. It may be noted here that Kepler was a contemporary of Palestrina and Vittoria, the Renaissance polyphonists and was himself a musician well-acquainted with the music of his time.

<sup>58</sup> Quoted in J. Rodgers & W. Ruff: *Kepler's Harmony of the World: A Realization of the Ear*. American Sci. vol. 67, May-June 1979 pp. 286-292.

<sup>59</sup> James Evans op.cit., p. 80 ff.

<sup>60</sup> Theon of Smyrna: *Mathematics Useful for Understanding Plato*. Tr. R. and D. Lawlor, San Diego, 1979, p. 95.

## The Millennium Sphere and the Liberal Arts Curriculum:

The early Pythagoreans were reputedly the first to link arithmetic, geometry, music and astronomy together, and to organize their teaching into a curriculum<sup>61</sup>, on which the earliest statement came down to us through Archytas, a contemporary of Plato (4<sup>th</sup> C. BC). That four-fold part of the curriculum to be known later as the *quadrivium* became canonical in the schools of Greece and, through Rome and Boethius (470-525 AD), it re-emerged in the West with the Carolingian Renaissance. Scotus among others, was one of its proponents. In a document of the ninth Century, the *Musica Enchiriadis* and its commentary by an anonymous author, the *Scholia Enchiriadis*, we find a good example of how early medieval school men thought about these matters. In a passage of the *Scholia*, we find the following “dialog” in catechismal form between a teacher and his student:<sup>62</sup>

**Pupil:** How was harmony born of arithmetic as from a mother? And what is harmony, and what is music?

**Teacher:** We regard harmony to be a mixed symphonie of different sounds, altogether dependent on the theory of numbers, like all the other mathematical disciplines, it is only through numbers that we understand it.

**Pupil:** Which are the mathematical disciplines?

**Teacher:** Arithmetic, geometry, music, and astronomy.

**Pupil:** What is mathematics?

**Teacher:** A doctrinal science.

**Pupil:** Why doctrinal?

**Teacher:** Because it deals with abstract quantities.

**Pupil:** What are abstract quantities?

**Teacher:** Abstract quantities are those embraced only by the intellect because they lack material, i.e., physical admixture. And further: multitudes, magnitudes, their opposites, forms, similarities, relations and many other things ... change when connected with physical substance. These quantities are each directly treated in arithmetic, in music, in geometry and in astronomy. It is thus because these four disciplines are not skills of human invention but important researches in holy works; and they support in the most wondrous way acute minds in the understanding of Creation.

This four-fold mathematical curriculum, together with the *trivium* of grammar, rhetoric, and dialectic, flourished in the medieval universities and, with it, formed the curriculum of the seven liberal arts, providing the foundations for higher studies in philosophy and theology. The ad quadratum diagram can therefore be conceived of as an accurate though stylized compendium of the quadrivium since it embodies some of the

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<sup>61</sup> David Wagner, Ed.: *The Seven Liberal Arts in the Middle Ages*. Indiana U.P. Bloomington, 1983.

<sup>62</sup> Quoted in Nils Wallis' article in David Wagner, Op. cit.

fundamental elements of the four mathematical sciences.

A simple *ad triangulum* diagram can also be derived to give us the essential structure of the trivium as we shall presently see.

These ad quadratum and ad triangulum diagrams are doubly important and symbolic to us for medieval architects and master-masons relied on similar constructions for their designs.

The Millennium sphere crystallizes these constructs in its structure.

The ad quadratum and ad triangulum construction are therefore literally ex-planations, i.e., renderings into planes of multidimensional events or structures. The Trivium (which received its name like the quadrivium only much later with Boethius) has roots going back to at least the fifth century B.C.<sup>63</sup> With the growth of democracy, oratory in fifth century Greece had assumed an increased preponderance. Sophists, the speech writers, coaches, and handlers of the day, more intent like those of today on persuasion than on truth, opened schools of rhetoric with claims to teach oratorical skills, with Isocrates, the first on record, in the fourth Century B.C. It is in fact in conscious opposition to their influence and to counter their moral relativism that Plato founded his academy.

*“For Plato, words express the essence of things grasped in thought as concepts. Words are combined into sentences (or prepositions) reflecting the necessary connection in reality. This, in opposition to the sophists’ view that language was merely conventional.”<sup>64</sup>*

However, it is with the Stoics (3<sup>rd</sup> C. B.C.) that the trivium as such (though not in name yet) emerges. For them logic which they divided into dialectic, grammar, and rhetoric formed a unity and a branch of philosophy along with physics and ethics. Rhetoric sought to discover linguistic means of persuasion for all arguments, strategies that would work in any disputation while grammar explained the structure of language. Crates, a stoic of the first century B.C. is credited with writing the first systematic Greek grammar. We may note here that in India, Pānini had conceived with his Sanskrit generative and prescriptive grammar a much more elaborate system at least three hundred years before.

From the Greeks to the Latins through Martianus Capella and Pliny the Elder (first Century A.D.) to the encyclopedists of the fourth and fifth centuries (Donatus, Chalcidius, Boethius) and later, Isidore of Seville (d. 636), one reaches the ninth Century Carolingian Renaissance with Alcuin and Scotus, and from them, through the Cathedral Schools such as Chartres, the 12<sup>th</sup> century renaissance and the foundation of the universities. Here the liberal arts are fully developed, mastered and clearly become preparatory to the study of philosophy and theology. This foundation of the liberal arts

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<sup>63</sup> David Wagner: op. cit.

<sup>64</sup> ibid.

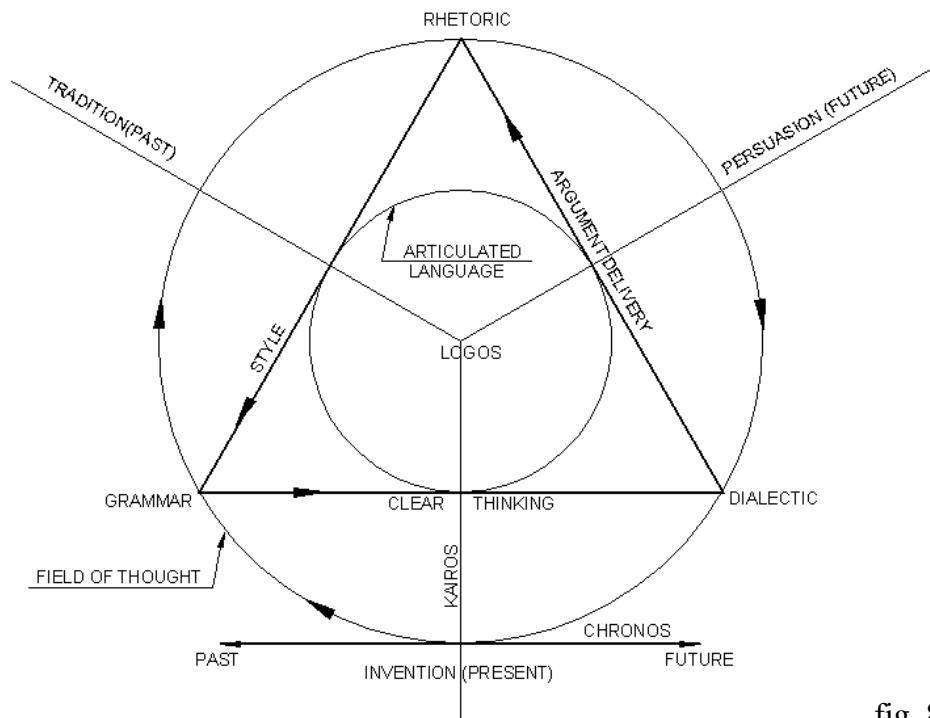


fig. 89

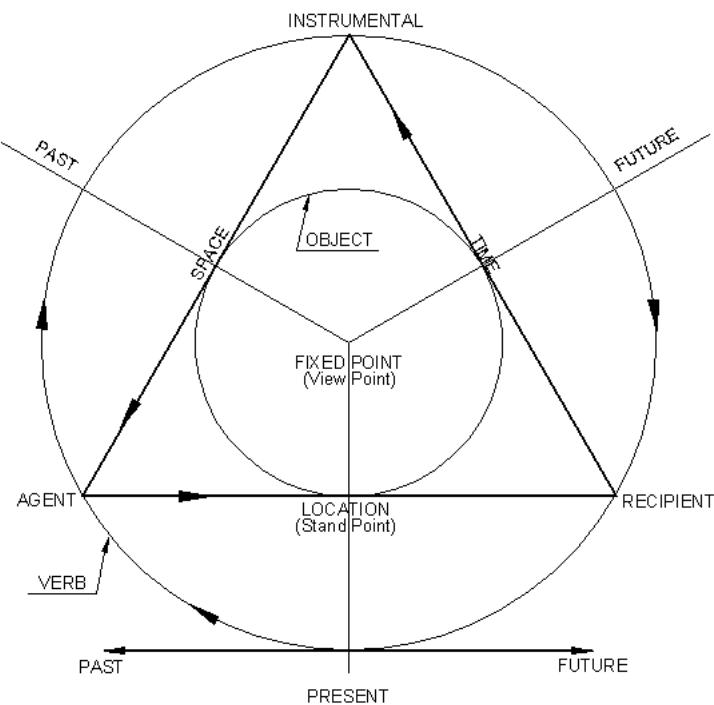


fig. 90

will continue its tradition more or less unbroken up to the 18<sup>th</sup> century in European universities.

Having completed our rapid historical survey of the trivium, we now consider it in relation with the ad triangulum construction.

The simplest form to emerge from the 1, 2, 3 explosion in space is the tetrahedron and with it, the figure of the triangle. It is that triangle, the equilateral triangle, which we shall take as our basic figure in the ad triangulum construction. It can also be constructed symbolically in our *ex-planation* as issuing from the point of no dimension, the One, radiating into a circle, the Two, within which the Three, appearing as a triangle, sets by its form the laws, constraints, and limitations that give shape to Creation. As we shall see, from it will emerge a simple structure that can be taken as symbolic of not only the trivium but in fact of any event.

For the Greeks, language had its source in the logos (fig. 88), as for the Christians it was in the Word, that Word that was in the beginning....It is therefore represented by a dimensionless point, the one. Expanding into a circle, the Two, it forms within itself the Three under the form of the triangle within which another circle inscribes itself. This, then, is the fundamental structure we will use to symbolically represent the structure of the trivium. The large circumscribing circle will represent the field of thought. The vertices of the inscribed triangle will represent respectively Grammar, Dialectic and Rhetoric; the circle constrained within the triangle, articulated language; the sides of the triangle, constraints which will in turn represent:

- Between Grammar and Rhetoric – style
- Between Grammar and dialectic – clear thinking
- Between rhetoric and dialectic – delivery and argument

The radial directions perpendicular to the triangle sides represent:

- Between grammar and rhetoric – tradition (past)
- Between grammar and dialectic – invention (present)
- Between rhetoric and dialectic – persuasion (future)

The radius of the inscribed circle is symbolic of the depth of reasoning and along the various directions, its degree of clarity, its quality of style and its persuasiveness.

As previously mentioned, the same basic scheme applies to any situation for it is based on the structure of the sentence which governs the articulation of thought. Pānini, the Sanskrit grammarian of the fifth century B.C., introduced the idea of *factors of the action* (Kāraka) of which he identified six as necessary and sufficient to characterize the relationships existing in any action.

The action itself is represented by the verb. On our geometrical symbol (fig. 90) this is the large circumscribing circle. The whole diagram is the sentence standing for the

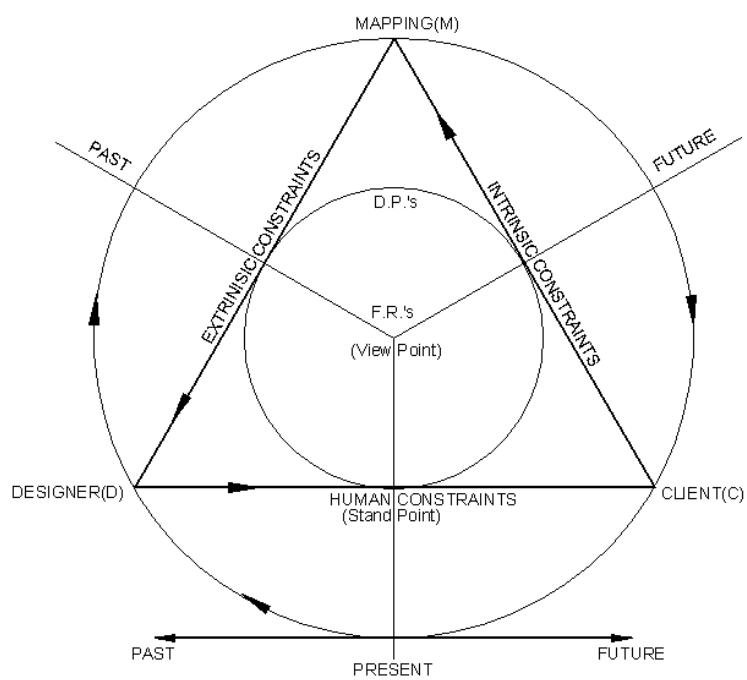


fig. 91

whole action being described. He then listed successively:

- The source of the action or the fixed point in relation to moving away which we shall place at the center of the circle;
- The object of the action or item directly reached by the action of the agent which we represent as the inner circle;
- The recipient of the action, i.e., the item one has in view through the object. The triangle vertex to the right will be its representation;
- The most effective means of achieving the action represented by the triangle vertex at the top;
- The location, and constraints on the action represented by the sides of the triangle;
- The agent or independent factor that does not depend on any other but carries within its/his/her self all the knowledge necessary for the action, and is represented by the vertex on the left of the triangle.

Note that Pānini introduces non-linguistic features in his depiction of syntactic meaning to establish a correspondence between linguistic construction and non-linguistic fact. In other words, he establishes a *viewpoint* from which the action may be expressed linguistically, i.e., thought out. As an example, we consider the process of design, also viewed as an action and formulated according to the same scheme.

In essence, *designing is providing a set of prescriptive rules for reorganizing the elements of the environment according to some purpose*. Purpose is determined through a dialog between a “designer” and a “client”. Design is therefore based on a dialectical process founded on language.

Designer and client may be individuals or groups. They may indeed be the same individual. Design is a reflective, iterative, feedback process.

From the dialog arises common goals (Functional Requirements (FR's)) which are refined and made more precise through iterations (fig. 91). This action takes place in the present. It establishes the *viewpoint*. Through reflection, inspiration, experience, discussion (the mapping (M)), tentative solutions (design parameters(DP's)) are proposed which satisfy the requirements. Through a similar dialectical process, designer (D) and client (C) recognize the constraints. Constraints are of three kinds:

1. Human: resulting from the dialectical relationship present between designer and client. Being in consciousness, these constraints always manifest in the *present* of the action.
2. Extrinsic: resulting from the state of the world where the action takes place (law physical and human, regulations, traditions, history, etc.). They are governed by the *past*.
3. Intrinsic: i.e., imposed by the very nature of the proposed design. They determine its *future*.

This determines the envelope of design and establishes the *standpoint* from which the designer can act.

Thus, the functional requirements (FR's, fig. 91) constitute the point of departure, the design parameters (DP's) are the object, the client , the recipient of the action (design); the method of approach, i.e., the mapping of the concepts into the physical reality, is the means of achieving the action (design), and through optimization becomes the most effective means of achieving this result. The designer or design team is the agent, and finally, the constraints, extrinsic, intrinsic, and human, contain and localize the design for a particular time and place.

Implicit therefore in the structure of the millennium sphere are not only some of the exact relationships that the sciences of the quadrivium established, but the very structure of the trivium and of the process through which the millennium sphere came to be. More than a mere image, it takes on true iconic value, becoming a window opened on the creative process and the mystery and power of the Word.

## Appendices

Table 1 and Table 2:

Due to the high symmetry existing in the regular polyhedra, the values of the trigonometric ratios of the internal angle and its half often occur in the study of their geometry. We have thought helpful to list in table 1 and 2 the sine, cosine, and tangent of these angles. These values apply *mutatis mutandis* to the stellated forms as well.

Table 3:

Table 3 gives the relationships between edges of the regular convex polyhedra and their value as a function of the common circumradius.

Table 4:

Table 4 summarized the study of the growth of the stellated figure.

Circumradius and Edge Relationships and Edge Relationships

TABLE 1  
TRIGONOMETRIC PROPERTIES OF THE INTERNAL ANGLES

FORM	Internal or Maraldi Angle	Dihedral Angle	Remarks
Tetrahedron $T_i = 109^\circ 47$	$\sin T_i = \frac{2}{3}\sqrt{2}$ $= 0.9428$	$\cos C_i = -\frac{1}{3}$ $= -0.3333$	$\tan T_i = -2\sqrt{2}$ $= -2.8284$
Cube $C_i = 70^\circ 53$	$\sin C_i = \frac{2}{3}\sqrt{2}$ $= 0.9428$	$\cos C_i = \frac{1}{3}$ $= 0.3333$	$\tan C_i = 2\sqrt{2}$ $= 2.8284$
Octahedron $O_i = 90^\circ$	$[\sin O_i = 1]$	$\cos O_i = 0$	$\tan O_i = \infty$
Dodecahedron $D_i = 41^\circ 81$	$\sin D_i = \frac{2}{3}$ $= 0.66666$	$\cos D_i = \frac{\sqrt{5}}{3}$ $= 0.7453$	$\tan D_i = \frac{2}{\sqrt{5}}$ $= \frac{2\sqrt{5}}{5}$ $= 0.8944$
Icosahedron $I_i = 63^\circ 44$	$\sin I_i = \frac{2}{\sqrt{5}}$ $= \frac{2\sqrt{5}}{5}$ $= 0.8944$	$\cos I_i = \frac{1}{\sqrt{5}}$ $= \frac{\sqrt{5}}{5}$ $= 0.4472$	$\tan I_i = 2$ $I_D = \pi - D_i$ $= 138^\circ 19$

TABLE 2 TRIGONOMETRIC PROPERTIES OF THE HALF-INTERNAL ANGLE

FORM	sin	cos	tan		Remarks
Tetrahedron $\frac{T_i}{2} = 54^\circ 74$	$\sin \frac{T_i}{2} = \sqrt{\frac{2}{3}} = 0.8165$	$\cos \frac{T_i}{2} = \sqrt{\frac{1}{3}} = 0.5773$	$\tan \frac{T_i}{2} = \sqrt{2} = 1.4142$	$\sin \frac{T_i}{2} = \cos \frac{C_i}{2}$ $\sin \frac{C_i}{2} = \cos \frac{T_i}{2}$ $\tan \frac{T_i}{2} = 2 \tan \frac{C_i}{2}$	
Cube $\frac{C_i}{2} = 35^\circ 27$	$\sin \frac{C_i}{2} = \sqrt{\frac{1}{3}} = 0.5773$	$\cos \frac{C_i}{2} = \sqrt{\frac{2}{3}} = 0.8165$	$\tan \frac{C_i}{2} = \sqrt{\frac{1}{2}} = 0.7071$	$\tan \frac{C_i}{2} = \sin \frac{O_i}{2}$ $= \cos \frac{O_i}{2}$	$\tan \frac{C_i}{2} = \sin \frac{\pi}{4}$ $= \cos \frac{\pi}{4}$
Octahedron $\frac{O_i}{2} = 45^\circ$	$\sin \frac{O_i}{2} = \sqrt{\frac{1}{2}} = 0.7071$	$\cos \frac{O_i}{2} = \sqrt{\frac{1}{2}} = 0.7071$	$\tan \frac{O_i}{2} = 1$	$\sin \frac{O_i}{2} = \cos \frac{O_i}{2}$ $= \tan \frac{C_i}{2}$	$\frac{O_i}{2} = \frac{\pi}{4}$
Dodecahedron $\frac{D_i}{2} = 20^\circ 90$	$\sin \frac{D_i}{2} = \frac{1}{\phi\sqrt{3}}$ $= \frac{\sqrt{3}}{3}(\phi - 1)$ $= \frac{\sqrt{3}}{3}\left(\frac{\sqrt{5}-1}{2}\right)$ $= 0.3568$	$\cos \frac{D_i}{2} = \frac{\phi}{\sqrt{3}}$ $= \frac{1+\sqrt{5}}{2\sqrt{3}}$ $= 0.9342$	$\tan \frac{D_i}{2} = \frac{1}{\phi^2}$ $= (2-\phi)$ $= \frac{3-\sqrt{5}}{2}$ $= 0.3819$	$\cos \frac{D_i}{2} = \phi^2 \sin \frac{D_i}{2}$ $= \frac{\sin D_i}{\tan \frac{D_i}{2}}$ $= \frac{2}{\tan \frac{D_i}{2}}$	$\phi^2 = \frac{1}{\tan \frac{D_i}{2}} = \frac{\cos \frac{D_i}{2}}{\sin \frac{D_i}{2}}$ $\sin \frac{D_i}{2} = \frac{2\sqrt{3}}{3} \sin \frac{\pi}{10}$
Icosahedron $\frac{I_i}{2} = 31^\circ 72$	$\sin \frac{I_i}{2} = \frac{1}{\sqrt{1+\phi^2}}$ $= \frac{1}{\sqrt{\phi+2}} = \sqrt{\frac{5-\sqrt{5}}{10}}$ $= 0.5257$	$\cos \frac{I_i}{2} = \frac{\phi}{\sqrt{1+\phi^2}}$ $= \frac{\phi}{\sqrt{\phi+2}} = \frac{1+\sqrt{5}}{\sqrt{2}(5+\sqrt{5})}$ $= 0.8506$	$\tan \frac{I_i}{2} = \frac{1}{\phi}$ $= (\phi - 1)$ $= 0.6180$	$\cos \frac{I_i}{2} = \phi \sin \frac{I_i}{2}$ $\tan \frac{I_i}{2} = \phi \tan \frac{I_i}{2}$ $\tan \frac{D_i}{2} + \tan \frac{I_i}{2} = 1$	$\phi = \frac{1}{\tan \frac{I_i}{2}} = \frac{\cos \frac{I_i}{2}}{\sin \frac{I_i}{2}}$ $\phi = 2 \sin \frac{3\pi}{10}$ $= 2 \cos \frac{\pi}{5}$

TABLE 3 REGULAR CONVEX POLYHEDRA METRICS

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
$R = a_T \frac{\sqrt{6}}{4}$	$R = a_c \frac{\sqrt{3}}{2}$	$R = a_o \frac{\sqrt{2}}{2}$	$R = \frac{a_d}{4} \sqrt{3} (1 + \sqrt{5})$ $= a_d \frac{\sqrt{3}}{2} \phi$	$R = \frac{a_i}{4} \sqrt{2(5 + \sqrt{5})}$ $= \frac{a_i}{2} \sqrt{\phi + 2}$
$a_T = \frac{2}{3} \sqrt{6}R$	$a_c = \frac{2\sqrt{3}}{3} R$	$a_o \sqrt{2}R$	$a_d = \frac{\sqrt{3}}{3} (\sqrt{5} - 1)R$ $= \frac{2\sqrt{3}}{3} (\phi - 1)R$	$a_i = \frac{\sqrt{10}}{5} \sqrt{(5 - \sqrt{5})}R$ $= \frac{2}{\sqrt{\phi + 2}} R$
$a_T = 2 \sin \frac{T_i}{2} R$	$a_c = 2 \sin \frac{C_i}{2} R$	$a_o = 2 \sin \frac{O_i}{2} R$	$a_d = 2 \sin \frac{D_i}{2} R$	$a_i = 2 \sin \frac{I_i}{2} R$
$a_T = a_c \sqrt{2}$	$a_c = a_d \phi$ $= a_T \frac{\sqrt{2}}{2}$	$a_o = a_T \frac{\sqrt{3}}{2}$ $= a_c \sqrt{\frac{3}{2}}$	$a_d = a_c \frac{1}{\phi} = a_c (\phi - 1)$ $= a_c \tan \frac{I_i}{2}$	$a_i = a_c \sqrt{3} \sin \frac{I_i}{2}$
$A_T = \frac{a_T^2}{4} \sqrt{3} = \frac{2}{3} \sqrt{3} R^2$ (face area)	$A_c = a_c^2 = \frac{4}{3} R^2$	$A_o = \frac{a_o^2}{4} \sqrt{3} = \frac{3}{2} R^2$	$A_d = 0.8760 R^2$	$A_i = 0.4787 R^2$
$r_T = \frac{R}{3}$ (insphere radius)	$r_c = \frac{\sqrt{3}}{3} R$	$r_o = \frac{\sqrt{3}}{3} R$	$r_d = 0.794 R$ $= \cos \frac{D_i}{2} \cos \frac{I_i}{2} R$	$r_i = 0.794 R$ $= \cos \frac{D_i}{2} \cos \frac{I_i}{2} R$
$V_t = \frac{8}{27} \sqrt{3} R^3$ (volume)	$V_c = \frac{8}{9} \sqrt{3} R^3$	$V_o = \frac{4}{3} \sqrt{3} R^3$	$V_d = 1.606 \sqrt{3} R^3$	$V_i = 1.463 \sqrt{3} R^3$
$\frac{V_t}{V_s} = \frac{2}{9} \frac{\sqrt{3}}{\pi} = 12.235\%$ ( $V_s$ =circumsphere volume)	$\frac{V_c}{V_s} = \frac{2}{3} \frac{\sqrt{3}}{\pi} = 36.705\%$	$\frac{V_o}{V_s} = \frac{\sqrt{3}}{\pi} = 55.133\%$	$\frac{V_d}{V_s} = (1.606) \frac{3}{4} \frac{\sqrt{3}}{\pi}$ $= 66.406\%$	$\frac{V_i}{V_s} = (1.463) \frac{3}{4} \frac{\sqrt{3}}{\pi}$ $= 60.495\%$

**Table 4:****Table of Stellation Growth Factors**

Dodecahedron:

First stellation (dod. $\rightarrow$ SSD)

$$g_d|_1 = \frac{R_s}{R} = \frac{\phi}{\sqrt{3}} \sqrt{\phi+2} = \frac{\cos \frac{D_i}{2}}{\sin \frac{I_i}{2}} = 1.777$$

Second stellation (SSD $\rightarrow$ GD)

$$g_d|_2 = 1 \quad (\text{same circumsphere})$$

Third stellation (GD $\rightarrow$ GSD)

$$\begin{aligned} g_d|_3 &= \frac{R_G}{R_S} = \sqrt{3} \frac{(\phi+1)}{\sqrt{\phi+2}} = \phi \frac{\sin \frac{I_i}{2}}{\sin \frac{D_i}{2}} = 2.3839 \\ &= \frac{\cos \frac{I_i}{2}}{\sin \frac{I_i}{2}} \end{aligned}$$

Overall Growth Factor (dod. $\rightarrow$ GSD)

$$\begin{aligned} G_d &= \frac{R_G}{R} = \frac{R_G}{R_S} \times \frac{R_S}{R} = \phi^3 \\ &= \frac{1}{\tan \frac{I_i}{2} \tan \frac{D_i}{2}} = (2\phi+1) = 2 + \sqrt{5} = 4.2360 \end{aligned}$$

From insphere convex dod. (& icos.) radius  $r$  to SSD (& GD) circumsphere  $R_s$

$$g'_d = \frac{R_s}{r} = \sqrt{5} = 2.2360$$

From enveloping icosahedron insphere  $r'$  to GSD circumsphere  $R_G$

$$g'_d|_3 = \frac{R_G}{r'} = 3$$

From  $r$  to  $R_G$

$$\begin{aligned}
 \frac{R_G}{r} &= \frac{R_G}{R_S} \times \frac{R_S}{r} = \sqrt{3} \frac{(\phi+1)}{\sqrt{\phi+2}} \sqrt{5} = \sqrt{5} \left( \sin \frac{I_i}{2} \right) \left| \frac{1}{\sin \frac{D_i}{2}} + \sqrt{3} \right| \\
 &= \sqrt{3} \sqrt{5} \left[ \cos \frac{I_i}{2} + \sin \frac{I_i}{2} \right] \\
 \therefore \frac{R_G}{r} &= \phi \sqrt{5} \frac{\sin \frac{I_i}{2}}{\sin \frac{D_i}{2}} = 5.330
 \end{aligned}$$

Icosahedron:

First regular stellation (convex icosahedron, circumsphere  $R \rightarrow GI$ )

$$G_I = \frac{8}{\phi} = 8(\phi - 1) = 4.944$$

## Circumradius and Edge Relationships

The relationships given on page 7 are derived here for each of the five regular convex polyhedra.

### 1. Cube:

$$AB = a$$

$$AC = a\sqrt{2}$$

$$BC = 2R = \sqrt{(a\sqrt{2})^2 + a^2} = a\sqrt{3}$$

$$\therefore R = \frac{a}{2}\sqrt{3}$$

$$\text{and } a = \frac{2\sqrt{3}}{3}R$$

fig. 1Ap

### 2. Tetrahedron:

The tetrahedron having the same circumsphere as the cube and therefore the same circumradius is  $ABCD$ . Let the edge of the tetrahedron be  $a_T$ .

$$a_T = a_C\sqrt{2} \text{ where } a_C = \text{cube edge.}$$

Since  $R$  is the same as that of the cube

$$R = \frac{a_C}{2}\sqrt{3}$$

fig. 2Ap

Replacing  $a_C$  by

$$a_C = \frac{a_T}{\sqrt{2}} = a_T \frac{\sqrt{2}}{2}$$

$$R = a_T \frac{\sqrt{6}}{4} \equiv a \frac{\sqrt{6}}{4}$$

$$a = \frac{2}{3} \sqrt{6} R$$

### 3. Octahedron:

$$AO = R$$

$$AB = a$$

$$\therefore a = \sqrt{2}R$$

$$\text{and } R = \frac{a}{2} \sqrt{2}$$

fig. 3Ap

### 4. Dodecahedron:

fig. 4Ap

Euclid constructed the dodecahedron on the basis of the cube as detailed by Heath (*op. cit.* p. 253). Indeed, consideration of a dodecahedron model (fig. 5 Ap.) readily shows that a dodecahedron can be seen as a cube on the faces of which a roof-like structure would be erected, the edges of the cube being the diagonals of the pentagonal faces of the dodecahedron. Further examination will show that, at each vertex, two such diagonals converge to form, with those of the other two pentagonal faces constituting that vertex, two sets of three mutually perpendicular lines determining the corners of the two cubes that can be traced on the dodecahedron model. One such set is  $AB, CB, DB$ . To prove that they are mutually perpendicular, *i.e.*, to show that  $ABDEHCGH$  is a cube, we note that  $ABDE$  must be a square since it has:

- 4 equal sides (diagonals of the pentagonal faces of the dodecahedron)
- 4 equal angles by symmetry
- furthermore in a quadrilateral figure the sum of the angles is  $2\pi$ , each angle is therefore equal to  $\frac{\pi}{2}$

fig. 5Ap

The same will obviously obtain for the other faces such as  $BCHD$  etc., making  $ABDEHCGH$  a cube. The dodecahedron will include five such cubes.

This results immediately from the fact that there are five diagonals in a pentagonal face. The total number of diagonals in the whole dodecahedron is therefore  $5 \times 12 = 60$  and since each cube has 12 edges, the number of cubes must be  $\frac{60}{12} = 5$ .

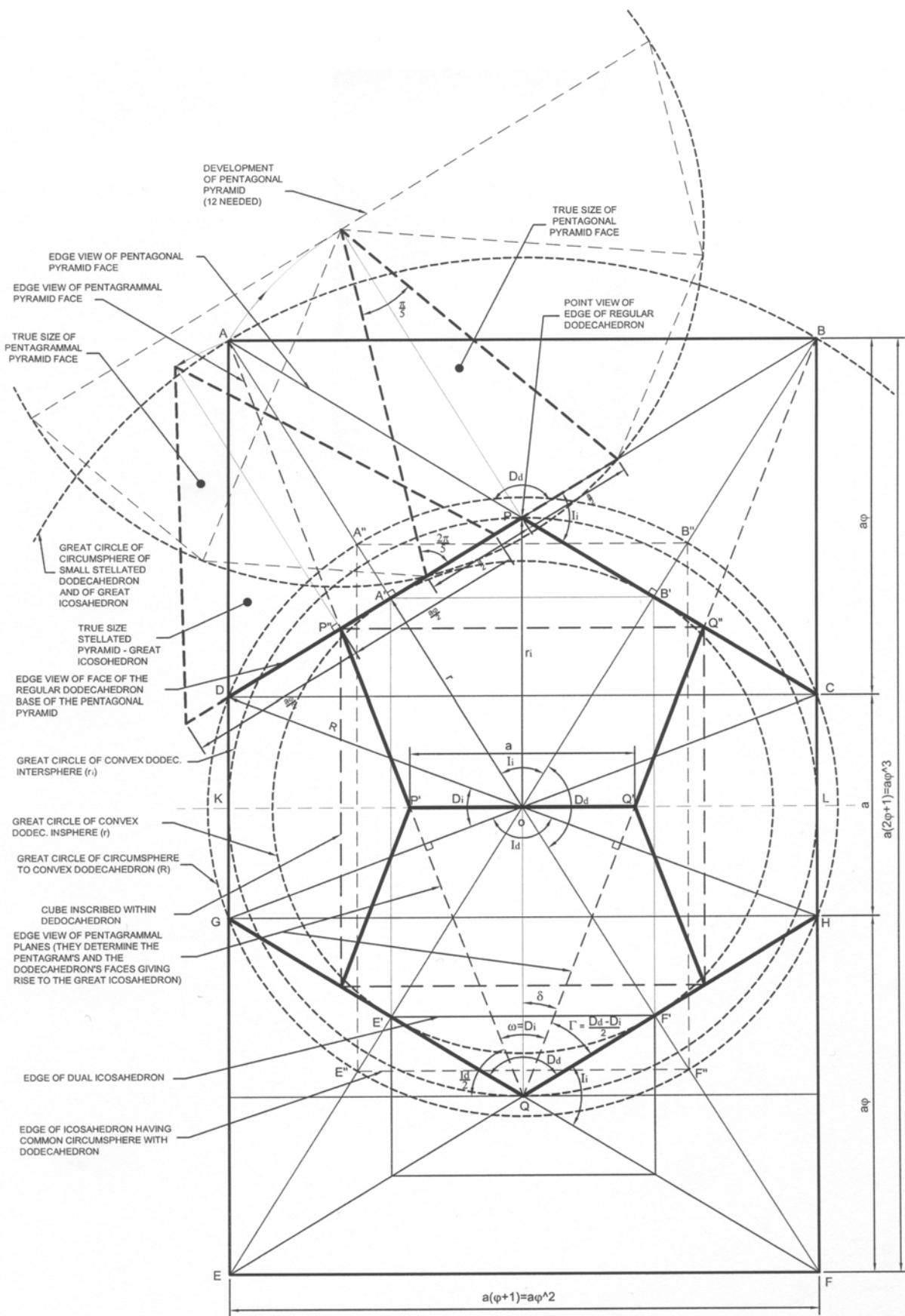


fig. 45

One can also reason that there are 20 vertices in the dodecahedron while each cube has 8. Each vertex of the dodecahedron is common to a pair of cube corners. That pair of cubes has another common corner diametrically opposite.

The first cube inscribed in the dodecahedron will therefore leave  $20 - 8 = 12$  free vertices.

The second cube having a pair of corners common with the first one will occupy  $8 - 2 = 6$  new vertices, leaving  $12 - 6 = 6$  free vertices.

The third cube will leave  $6 - 2 = 4$  free vertices

the fourth  $4 - 2 = 2$

and the fifth  $2 - 2 = 0$

so that the total number of cubes will be 5 as determined previously.

Now, letting  $a_c$  be the edge of the cube (diagonal of the dodecahedron pentagonal face) and  $a_d$  the edge of the dodecahedron (side of the pentagon), and  $\phi$  of the golden ratio, we can write:

$$a_c = a_d \Phi$$

$$\text{or} \quad a_d = \frac{a_c}{\Phi}$$

$$\text{since} \quad a_c = \frac{2\sqrt{3}}{3} R$$

$$\text{we have} \quad a_d = \frac{2\sqrt{3}}{3\Phi} R$$

$$\text{or} \quad a_d = \frac{\sqrt{3}}{3} (\sqrt{5} - 1) R$$

$$\therefore R = \frac{3\Phi}{2\sqrt{2}} a_d$$

$$= \sqrt{3} \left( \frac{1+\sqrt{5}}{4} \right) a_d$$

A more analytical method can of course be established. Referring to fig. 45, it will be observed that the intersphere, radius  $r_i$  (as described on p. 35), passes through points  $P$  and  $L$ .

$$\therefore OP = OL = r_i$$

Furthermore  $CL = \frac{a}{2}$

and  $PC$  is the edge view of a pentagonal face of the dodecahedron showing the height of the pentagon in true length (see figs. 6 Ap. and 7 Ap.)

$$PC = h$$

Point  $C$  is on the circumsphere so that

$$OC = R$$

$$PC' = r_i - \frac{a}{2}$$

$$\therefore h^2 = r_i^2 + \left( r_i - \frac{a}{2} \right)^2 \quad (1)$$

fig. 6Ap

As can be seen from fig. 66 (p. 113),  $CD$  on fig. 7 Ap. is the pentagon diagonal

$$\therefore CD = a\Phi = a \left( \frac{1+\sqrt{5}}{2} \right)$$

so that

$$PC^2 = a^2\Phi^2 - \frac{a^2}{4}$$

fig. 7Ap

$$= a^2(\Phi^2 - \frac{1}{4})$$

$$\therefore PC = h = a\sqrt{\Phi^2 - \frac{1}{4}}$$

$$= a\sqrt{\Phi + \frac{3}{4}}$$

$$h = \frac{a}{2}\sqrt{5+2\sqrt{5}} \quad (2)$$

Replacing in (1) by (2)

$$r_i^2 + \left(r_i - \frac{a}{2}\right)^2 = \frac{a^2}{4}(2\sqrt{5} + 5)$$

or, rearranging:

$$r_i^2 - \frac{a}{2}r_i - \frac{a^2}{4}(\sqrt{5} + 2) = o \quad (3)$$

Solution to this quadratic will yield  $r_i = f(a)$  which, introduced into (4), below, will in turn yield the sought-for relation between  $R$  and  $a$ .

$$R = \sqrt{r_i^2 + \frac{a^2}{4}} \quad (4)$$

$$\text{Let } (\sqrt{5} + 2) = \gamma$$

$$\text{Then } r_i^2 - \frac{a}{2}r_i - \frac{a^2}{4}\gamma = o$$

$$\therefore r_i = \frac{1}{2} \left[ \frac{a}{2} + \sqrt{\frac{a^2}{4} + 4 \frac{a^2}{4}\gamma} \right]$$

$$r_i = \frac{a}{4} [1 + \sqrt{1 + 4\gamma}]$$

$$\text{but } \sqrt{1+4\gamma} = \gamma$$

$$\therefore r_i = \frac{a}{4}[1 + \gamma]$$

$$r_i = \frac{a}{4}[\sqrt{5} + 3] \quad (5)$$

That  $\sqrt{1+4\gamma} = \gamma$  may appear strange at first but this is nevertheless easily shown by expansion in terms of  $\Phi$  where

$$\Phi = \frac{1+\sqrt{5}}{2}$$

$$\text{Then } \sqrt{5} + 1 = 2\Phi$$

$$\text{and } \sqrt{5} + 2 = 2\Phi + 1$$

$$\text{or } \gamma = 2\Phi + 1 \quad (6)$$

$$\text{so that } 1 + 4\gamma = 1 + 4(2\Phi + 1)$$

$$= 5 + 8\Phi$$

We recognize 5 and 8 as consecutive terms in the Fibonacci series and recalling that the powers of  $\Phi$  are expanded in terms of the successive pairs of terms of that series

$$\text{i.e. } \Phi^n = f_{n-1} + f_n \Phi$$

$$\text{we see that } 5 + 8\Phi = \Phi^6$$

since 8 is the 6<sup>th</sup> term of the Fibonacci series and 5 the preceding term.

$$\therefore \sqrt{1+4\gamma} = \sqrt{\Phi^6} = \Phi^3$$

$$\text{But } \Phi^3 = \Phi^2 \Phi = (\Phi + 1)\Phi$$

$$= \Phi^2 + \Phi$$

$$= (\Phi + 1) + \Phi$$

$$= 2\Phi + 1$$

$$\therefore \sqrt{1+4\gamma} = 2\Phi + 1 \quad (7)$$

and from (6) we see that

$$\gamma = \sqrt{1+4\gamma} \quad Q.E.D.$$

It can also be verified that if we assume

$$\sqrt{1+4\gamma} = \gamma$$

$$\text{then } 1+4\gamma = \gamma^2$$

$$\text{or } \gamma^2 - 4\gamma - 1 = 0$$

$$\text{and therefore } \gamma = 2 \pm \sqrt{5}$$

as per our assumption, so that we can write directly, taking the positive root

$$r_i = \frac{a}{4}[1 + \gamma]$$

$$= \frac{a}{4}[\sqrt{5} + 3]$$

Returning now to equation (4) and replacing with (5), we obtain in turn:

$$R = \sqrt{r_i^2 + \frac{a^2}{4}}$$

$$r_i^2 = \frac{a^2}{16} [ \sqrt{5} + 3 ]^2$$

$$= \frac{a^2}{8} [ 7 + 3\sqrt{5} ]$$

$$r_i^2 + \frac{a^2}{4} = \frac{a^2}{4} \left[ \frac{7 + 3\sqrt{5}}{2} + 1 \right]$$

$$= \frac{3a^2}{4} \left[ \frac{3 + \sqrt{5}}{2} \right]$$

$$\therefore R = \sqrt{\frac{3a^2}{4} \left[ \frac{3 + \sqrt{5}}{2} \right]}$$

$$= \frac{a}{2} \sqrt{3} \sqrt{\frac{3 + \sqrt{5}}{2}}$$

again, expanding in terms of  $\phi$

$$\frac{3 + \sqrt{5}}{2} = \frac{1 + \sqrt{5}}{2} + \frac{2}{2} = \Phi + 1 = \Phi^2$$

$$\therefore R = \frac{a}{2} \sqrt{3} \sqrt{\Phi^2}$$

$$R = \frac{a}{2} \sqrt{3} \Phi \quad (8)$$

$$= \frac{a}{2} \sqrt{3} \left( \frac{1+\sqrt{5}}{2} \right)$$

$$R = \frac{a\sqrt{3}(1+\sqrt{5})}{4}$$

(9)

Alternatively, from (8) we obtain

$$a = \frac{2R}{\sqrt{3}\Phi} = 2 \frac{\sqrt{3}}{3} \frac{1}{\Phi} R$$

But  $\frac{1}{\Phi} = \frac{\sqrt{5}-1}{2}$

$$\therefore a = \frac{\sqrt{3}}{3} (\sqrt{5}-1) R$$

(10)

Note that, from fig. 45, one can write directly

$$\frac{a}{2} = R \sin \frac{D_i}{2}$$

and since we established that

$$\sin \frac{D_i}{2} = \frac{1}{\sqrt{3}\Phi}$$

equations (9) and (10) follow immediately.

However, the trigonometric functions of the internal angles of the polyhedra were established assuming the  $a$  and  $R$  relationships as established (p. 7). Our last result therefore confirms the previous developments regarding these functions.

### 5. Icosahedron:

Referring to fig. 56, it will be seen that the intersphere (*c.f.* p. 35) passes through points  $A'$  and  $C$  so that:

$$OA' = OC = r_i$$

Furthermore

$$A'P = \frac{a}{2}$$

and  $PC$ , height of a triangular face of the icosahedron is:

$$PC = a \frac{\sqrt{3}}{2}$$

Point  $P$  is on the circumsphere so that:

$$OP = R$$

We can therefore write:

- in triangle  $OA'P$ :

$$(OA')^2 + A'P^2 = OP^2$$

$$\text{or } r_i^2 + \frac{a^2}{4} = R^2 \quad (11)$$

- in triangle  $PH'C$ :

$$PH'^2 + H'C^2 = PC^2$$

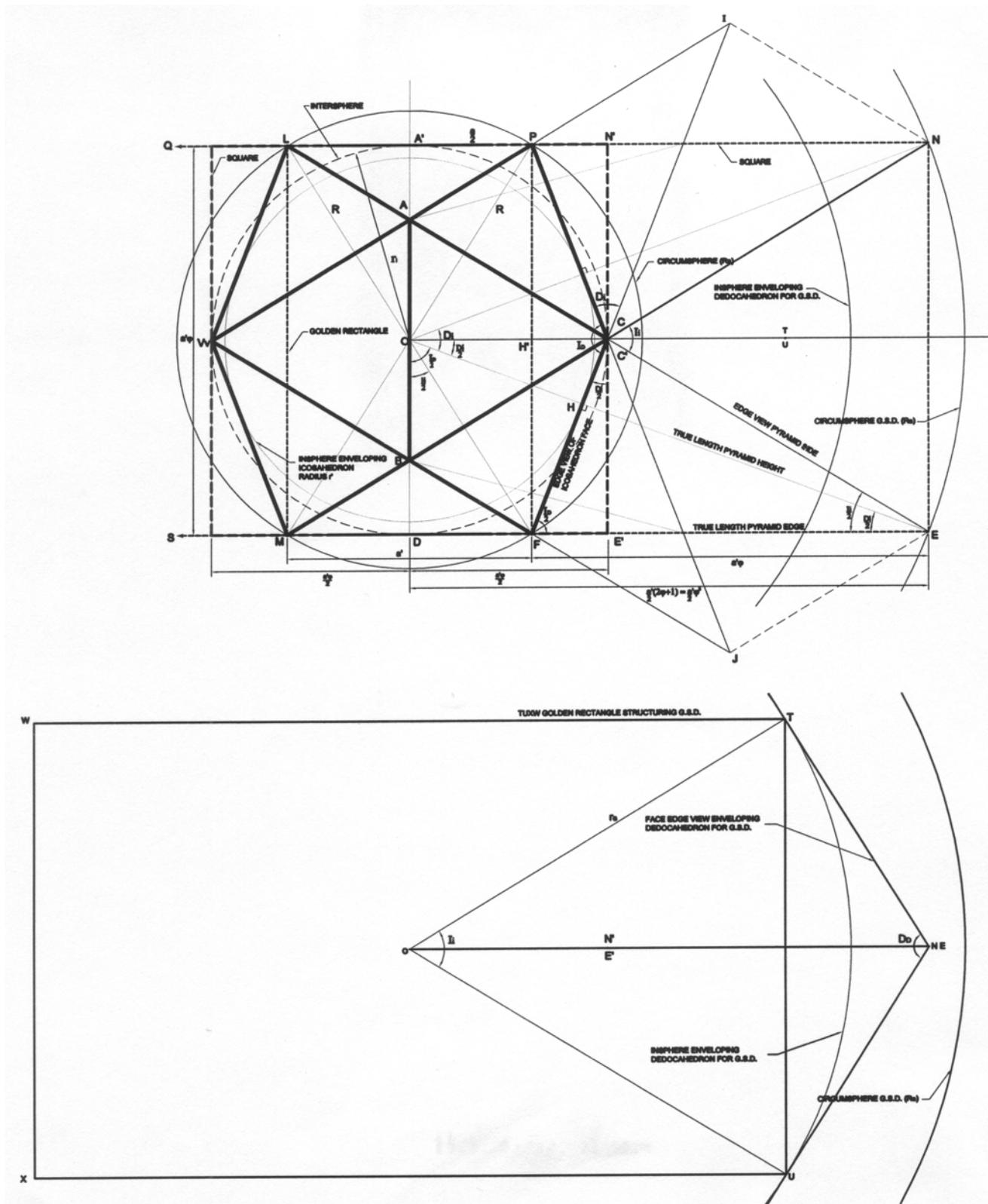


fig. 56

$$\text{or} \quad r_i^2 + \left(r_i - \frac{a}{2}\right)^2 = \frac{3}{4}a^2$$

$$\text{i.e.} \quad r_i^2 - \frac{a}{2}r_i - \frac{a^2}{4} = 0$$

$$\therefore \quad r_i = \frac{a}{4}(1 + \sqrt{5})$$

$$\text{or} \quad r_i = \frac{a}{2}\Phi$$

We can remark in passing that the ratio of  $r_i$  to  $\frac{a}{2}$  is  $\Phi$  and therefore that the rectangle  $A'PH'O$  is a golden rectangle and that consequently the same will obtain for  $LPFM$ , as shown in fig. 26. Similarly, the duality of icosahedron and dodecahedron ensures that rectangle  $A'B'F'E'$  on fig. 45 is a golden rectangle, again as shown on fig. 25B.

Replacing in (11) and taking the square root:

$$R = \sqrt{r_i^2 + \frac{a^2}{4}}$$

$$= \frac{a}{2}\sqrt{\Phi^2 + 1}$$

$$\text{or} \quad R = \frac{a}{2}\sqrt{\Phi + 2}$$

$$\text{with} \quad \sqrt{\Phi + 2} = \sqrt{\frac{5 + \sqrt{5}}{2}}$$

and finally 
$$R = \frac{a}{4} \sqrt{2(5 + \sqrt{5})} \quad (12)$$

Therefore 
$$a = \frac{4}{\sqrt{2(5 + \sqrt{5})}} R$$

Multiplying top and bottom by  $\sqrt{5 - \sqrt{5}}$ :

$$a = \frac{\sqrt{10}}{5} \sqrt{5 - \sqrt{5}} R \quad (13)$$

Again, similarly to the case of the dodecahedron, we have for the icosahedron, directly from fig. 56

$$DF = OF \sin \frac{I_i}{2}$$

or 
$$\frac{a}{2} = R \sin \frac{I_i}{2}$$

And, remarking that

$$\sin \frac{I_i}{2} = \frac{1}{\sqrt{\Phi + 2}}$$

we readily obtain equations (12) and (13), again confirming our previous results.

**Figure Credit**

Figs 2; 3; 19A through 23A; 72A: Models and Slides by Astrid Fitzgerald.

Figs. 25B and 26 from H.E. Huntley: *The Divine Proportions*, Dover, New York 1970

Fig. 29 after J. Kappraff: *Connections*, McGraw-Hill, New York 1992

Figs. 31; 32; 35; 36; 37; 39; 40; 41; 46; 47; 53; 57; 60; 62 from P. Cromwel: *Polyhedra*, Cambridge 1997

Figs. 74 and 88: Models and photos by Kaetan Hanansen

All other figures drawn originally by the author and redone on the computer by Marcin Balicki, who was also responsible for the overall book design.