O2: Report - Methodology of integrating computational methods with sciences

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CHAPTER 1

Introduction

Attention: This document has online version which contains live code cells allowing for explorations without the need of installation of any software. It can be found at:

- http://visual.icse.us.edu.pl/methodology/

1.1 About iCSE4school project

Contemporary forms of use mathematical and natural sciences elements of information and communication technology in the work of a teacher is not only a legal obligation, but above all to diversify the lesson. A modern teacher can not imagine work without the use of modern information technology. Up-to-the-minute need is to improve their computer programming skills and programming languages. In everyday school work, the teacher uses computer technology primarily to create a variety of utility tests, diagnoses, research tools and surveys. With specialized commercial software, create invitations, posters, school informers and create lesson plans or school events.

Thanks to computer technology documentation of teachers' activities is conducted transparently, aesthetically and legibly. The own workplace is organized and continuously refined, also through the use of e-learning applications and programs or school-based software. All these programs and applications are characterized by a relatively intuitive and simple relationship to their method of construction, and therefore no change or performance in them other than those enabled by the manufacturer. From both the teacher's and the learner's point of view, it is equally important to address the problems identified in the core curriculum and beyond. A teacher who studies the core issues of subjects such as physics and chemistry encounters a lot of content, which requires the use of a complex account with elements of mathematical analysis. Commonly known methods of education include providing without demonstrating a relationship describing a given phenomenon or presenting graphs of the discussed relations without presenting equations describing the transformed curves. For a student who expects a full explanation of a given phenomenon and its interpretation, this is one of the most important obstacles in the assimilation and in the connection of theoretical considerations with the mathematical interpretation. Most applications of information technology in teaching come down to data retrieval or their introduction and automated processing. The programming language environment of Python and Sage is the release of the subject from the imposed constraints and the programming skills acquired by the learner during the learning process effectively develop his or her ability to conclude and evaluation during the above process.

The use of the proposed programming environment brings down complex dependencies and equations from the academic level to a level not exceeding the applicable science curriculum and is fully endorsed and recommended as a work tool by both the Polish Ministry of National Education and the Polish Ministry of Digitization.

The proposed methodology is the output of activities related to the "Computing in high school science education - icse4school" project implemented in 2014-2017 under the Erasmus Program, Action 2 of the Strategic Partnership. The project was implemented in international cross-sectoral cooperation, including higher education institutions (University of Silesia in Katowice, University of Augsburg, Simula School of Research and Innovation), 2 high schools (2 AZSO in Chorzów and 117 ZS - XXXIII in Warsaw) and NGO Foundation for the Development of the Silesian Interdisciplinary Center for Education and Interdisciplinary Studies - EDU RES. The main objective was to develop in a transnational consortium a methodology integrated with computer science of mathematics and physics in high schools on the basis of the pilot program in selected schools. The subject of the project was to teach Python programming using the Sage environment and use this skill to solve problems related to mathematics, physics, and computer science. The project was addressed to students of high schools. The need for ICSE4school project was due to the growing interest in programming and the knowledge gathered showing the lack of information on one of the most widely used Python programming languages. In addition, the observations of teachers - both academic and classroom leaders with extended curriculum in mathematics, computer science and physics in high schools indicated the helplessness of students and students in applying information technology and numerical methods in solving problems on the borderline of scientific issues.

Using our materials and without basic skills in Python programming is in large extent possible, but, we recommend that you start by learning the language and the SageMath environment available at SageMath.org. In the Sage environment, the student has the opportunity, apart from programming, to do exercises on the sample materials contained there, with the possibility of modifying the source texts.

The scripts developed in the project using Python are recommended for use in a variety of ways, depending on the needs and capabilities of students and teachers. Thus, they can be used directly during class activities or in the form of workshops, provided that the number of teaching hours is at the teacher's disposal. Materials or parts can also be used in home work using the innovative "flipped teaching" method, where pupils in homework analyze the underlying knowledge. This is a new topic and the lesson is devoted to consolidating and deepening the acquired knowledge and skills (as opposed to the classical methods whereby the basic knowledge is introduced during the lesson and the self-development of the material is done at home). Finally, our materials can be used to explore and broaden the knowledge and skills of students and students with special educational needs (gifted students and those who have the need for self-development or who need to use numerical methods for solving problems using a publicly accessible tool) learn outside the school for self-study, as well as by all those who are interested in repetition, consolidation and extension of knowledge.

The lesson scenarios from the iCSE4school project can also be used to work with students with less potential in the arts by presenting pieces of our materials by teachers, as well as in the form of workshop work, where students modify the source texts themselves, watching the effects of these changes and formulating appropriate conclusions. This way of using materials not only creates opportunities for teachers and young people, but also effectively uses classroom scenarios using Python in working with younger learners at earlier stages of education.

The materials from the iCSE4school project can be used in e-learning, blended-learning, and m-learning, using computers or other devices (eg smartphones, tablets); with access to the internet or off-line.

During the project students and teachers from two highschools worked on SageMath servers. They have created almost 3000 documents, some of them have been "published" and can be viewed:

- https://sage01.icse.us.edu.pl/pub/
- https://sage03.icse.us.edu.pl/pub/

Observing the students' efforts to create simple programs using Python to solve a problem during project implementation has confirmed the great potential of individualizing their work. The individually tailored level of individual needs, the pace of work, and the individual teacher's help influence the development of each student. An important factor in mobilizing pupils for action and learning is also the recent need for learning programming (coding) and awareness of the significant shortage of people able to program in the labor market. By encouraging the use of didactic materials developed in the iCSE4school project, we invite you to comment.

This methodology is a collaboration between representatives of the following institutions:

- High school teachers: Jolanta Drogoń, Łukasz Głaz, Krzysztof Jarczewski, Mirosław Malinowski, Justyna Matejczyk, Adam Ogaza, Krzysztof Oleś, Katarzyna Sikora, Hanna Stachera, Mariola Strojny,
- Jonas van den Brink, Vigdis Holta, Marie Roald, Freyja Jørgensen, from Simula School of Research and Innovation, Oslo
- Manuel Milling, Severin Wünsch and professor Gert Ingold from Uniwersity of Augsburg
- Marcin Kostur, Uniwersytet Śląski
- Magdalena Hampel, Joanna Klekowska i Marta Margiel from The Foundation EduRes

Authors!

1.2 School in digital era

In the modern school, the authority seems to play a lesser role, based on coercion, persuasion or teacher's authority. Interestingly, the scope and manner of assessing the competence (knowledge and skills) of the teacher are also changing. While the former teacher in the classroom was essentially the only and most important source of knowledge, nowadays, thanks to universal access to the Internet, his words can be immediately verified by the students and often challenged. The teacher's role of contemporary youth is to build authority on skills other than those closely related and only with knowledge. At present the teacher should also demonstrate openness and competence in using new technologies to acquire and verify knowledge. The widespread access to the Internet has made it not only the teacher's expertise to build authority, but the flexibility and ability to adapt to students' current needs and interests, including through joint exploration of themes and issues where they can often inspire teachers. One of the most important tasks of a teacher in digital reality is to show and teach his students how to use this reality; how to ensure the safety and privacy of the Internet, how to verify the information found there, what strategies and tools to use to organize messages encountered on the Internet, and how to supplement the knowledge learned at school with knowledge acquired thanks to digital sources.

Already in the early 1990s, the change in perceiving the role of the teacher described above was identified in Alison King's article, "From Sage on the Stage to a Guide to the Side" (King 1993). This article was about the style of teaching in American universities, and its purpose was to encourage lecturers to give up the function of giving the knowledge of the 'sage of the cathedral' to the 'side guide' that accompanies the students in gaining knowledge. Changing the role of a teacher or lecturer seems to perfectly match the contemporary needs of the Y generation, who value highly specific practical skills and the ability to learn what they are interested in right now or what they perceive as needed. teaching, teacher decisions, or predictions about what might be useful in the future. They also like to work in groups or teams and use new technologies that allow them to stay in touch with others outside of school hours.

Since pupils from Generation Y are referred to as digital natives (Prensky, 2001), people who feel online at home and do not know the world where there is no access, are much easier to motivate them to gain knowledge through new technologies. It is also much easier for a teacher (often as a digital immigrant, as Mark Prensky would say) to build his own authority if he is able to appreciate students' ability to use new technologies on the one hand, and on the other hand to show them how to acquire, verify and organize acquired knowledge with digital sources. In order to do so, he must give up his role as an infallible authority in favor of personal authority, often based on the so-called soft skills.

For memorizing, they are, for example, creating online fiches, creating digital notes (using text editors and online equivalents), co-writing by several or more pupils, etc. To demonstrate understanding of material, a student may conduct a blog with a summary of material from the lesson Also write comments on the statements of the teacher and other students in the forum. The application is manifested in the ability to edit the content of the network, use of computer programs and web, etc. Analyzing it, for example, tagging and categorizing digital content. The assessment is the ability to verify and test knowledge and programs, and creating it is programming, making videos, animations, podcasts, and publishing them on the web. Each of these stages can be assigned specific digital tools that will allow teachers and students to collaborate in the digital environment. However, we must remember that web applications and programs change and go out of use very quickly, so a teacher who wants to support his students in the useful use of new technology must keep his or her hand on the pulse and continue to formally and informally learn in this area.

1.2.1 Generation Y and its needs

The youth are so-called Y generation (Generation Y), also known as Millennials. According to Wikipedia, this is a generation of people born in Poland from 1984 to 1997, and in other countries, for example, in the USA, the baby boomers of the 1980s and 1990s. It is also called "the generation of Millennium", "next generation", "digital generation" and "generation of flip flops and iPods". Unlike the previous generation, called the X Generation, they "tamed" technological advances and actively use the media and digital technologies. The Millennials are considered to be a bold generation, open to new challenges, stigmatizing their way of learning the world and learning. Some of the characteristics of the Y generation are:

- actively and in all areas of their lives use technology and digital media;
- live in a "global village" thanks to Internet access they have knowledge of the world;
- characterized by high self-confidence;
- are well educated and willing to continue to develop;

According to a study conducted at the University of New Hampshire, they have a high opinion of their skills, belief in their own uniqueness, excessive expectations and strong aversion to criticism.

The change of learning characteristics the new generation in:

- technology (new devices have emerged);
- pedagogy (learning has become more individual)
- content (the content is shorter and the media has changed).

have generated the need to search for new methods and forms of teaching. E-learning, blended learning, m-learning, Flipped Teaching, and the project methodology (including WebQuest) are among the current and effective ways of matching. These are methods based on observation and action, and therefore very effective in understanding and memorizing new messages.

CHAPTER 2

Science Education with SageMath

2.1 Why SageMath?

2.1.1 Choosing the best ICT tool for a math or physics lesson

For many years, attempts have been made to use ICT tools and programming in science classes. Often the software solution has been is selected by professionals in one specialized field. Sometimes the choice is lobbied by the manufacturer of the given system. As a result, this leads to uncoordinated activities limited to individual subjects. A student learns in lessons informatics tools and languages not useful during other subjects. The lessons of physics and math are enriched with dedicated software that is not used in computer science. This is in inherently wrong procedure - we use the specialized tools to individual tasks. But, what if there is a general purpose tool and language which can be practically without a compromise used in a range of tasks in school education?

Let's consider what features a computer system should have to break the stereotype above? Let's find a solution at once having the following characteristics:

1. WIDE: The system should be based on popular and open language wide-ranging programming.

   Wide-ranging programming languages can be used make computer games as well as scientific applications or education. On the other hand, there are many so-called domain-specific languages created for a single application. Such languages do a great job, but they are not good at all other tasks. An example is the Matlab language, which despite its own popularity is not the solution adopted in teaching computer science. Languages like Python allow practically perform all tasks which are specific to Matlab, but their specificity allows admit them to teach computer science (see https://docs.scipy.org/doc/numpy-dev/user/numpy-for-matlab-users.html). Important feature of the system is avoiding vendor lock-in, which is often the case when using domain specyfic languages.

2. INTERACT: Programming language should allow for interactive work.

   Such a claim virtually eliminates compiled languages (C/C++). In order to use the computer system interactively, the most adequate seem to be languages with dynamic typing and introspection. This requirement is met by the most of domain specific languages provided by Computer Algebra Systems, but also many general-purpose languages such as Python.

3. FREE - The system should be widely available.

   Unrestricted access to the system is best exploited by the open software. In addition, the open software give the possibility of the insight into every algorithm used. It is important both in science and education. Availability is also related to the technical aspects of the installation software. Opportunity to work in “cloud” with only internet browser is very desirable feature of such a system.

4. POWER - The capabilities of the system should allow to deploy it uncompromisingly for all scientific subjects.

   Such a requirement eliminates languages that are not enough widespread, and do not have large enought set of implemetned tools and libraries. Python is an interesting example, because it is known for its ease of creating interfaces various libraries written in other languages. This feature is heavily exploited by the SageMath system which contains hundreds of scientific libraries linked by common interface in Python.

5. PROF - The system should allow a smooth transition from school computer lab to professional use in scientific research and industrial.

   There is no reason to teach at a “small” system school at school, and then study or work using the “big”-one. The best way to go would be to the even in primary school the same language and system, which is used by scientists, but of course limit this usage a small part of it. It saves a lot of time and effort halpe to make good habits right from the earliest period of study. It should be emphasized that often the cost of software licenses for proprietary systems are urging to use simpler and smaller one in schools. This problem does not exist when we relay on open software.

System	WIDE	INTERACT	FREE	POWER	PROF
SageMath	YES	YES	YES	YES	YES
Python/Scipy	YES	YES	YES	YES	YES
Mathematica	YES	YES	NO	YES	YES
C/C++	YES	NO	YES	YES/NO	YES
Geogebra	NO	YES	YES	NO	NO
Java	YES	NO	YES	YES/NO	YES

The above analysis shows that solutions based on Python meet all requirements. Moreover, Python is a language of increasing importance in computer science. Both the standard Python interpreter and the SageMath algebra system can provide similar capabilities. Definitely in the Maths or Physics class, SageMath will offer a shorter path to solution as a computer algebra system. But before we discuss these systems let’s answer the question of what is a computer algebra system?

What is a Computer Algebra System?

Computer Algebra System (CAS) is a computer program that supports symbolic calculations. Consider, for example, the following code in Python:

```
a = 23.0
b = 3.0
print ( (a/34+1/b)**2 )
```

This program will print an approximate value for the expression after Substituting the variables: math:*a = 23, b = 3*: 1.0197. If we do not perform the first two substitutions the interpreter will signal an error.

The situation is different in case of CAS system. Here only we inform the system that the variables: math: *a, b* will be symbols and we can Expand the algebraic expression containing these symbols. For example, executing:

```
var('a,b')
show( expand( (a/34+1/b)**2)  )
```

we will obtain: $\frac{1}{1156}\,a^2 + \frac{a}{17\,b} + \frac{1}{b^2}$

Modern computer algebra systems are not limited to the manipulation of mathematical formulas. As a rule, they are equipped with a numerical computing system and a rich set of visualization tools. As of today most of the possibilities of CAS systems are similar and the main differences are the programming language and the license for the software.

The proposed approach is based on SageMath, which is a free and open source software. This eliminates the cost of licenses. In addition, SageMath uses the popular Python language, which students can learn during IT lessons.

2.1.2 What is SageMath? (from Python to SageMath)

Python

Python has been developing since the nineties in the last century. However, its ubiquitous popularity has started in last decade. In the United States most programming projects is writen in this language. Python has extensive standard libraries and is characterized by the clear and concise syntax. Importantly Python supports different programming methods: procedural, object oriented and functional. Thanks to these advantages, Norway is the first country systematically introduced that programming language into schools.

Ecosystem Scipy

Python is a language widely used for scientific research and education. The most known set of tools is called scipy ecosystem. It contains:

- NumPy, the basic package for numerical calculation similar in the philosophy and functionality to well known Matlab software
- the SciPy library of numerical methods
- Matplotlib, a graphing package
- SymPy, symbolic computation library (CAS)

SageMath

SageMath is a complete Computer Algebra System. First version of SageMath was released on February 24, 2005 as free and open source software in accordance with the terms of the GNU General Public License. One can say that Sage is an "overlay" on Python, which integrates many specialized mathematical packages and hundreds of thousands unique line of code to add new features. Capabilities and the flexibility of SageMath is immeasurable, so it is worthwhile to implement the above programming language also in school. It is not without significance that this is an open source software and as such free. Teachers and students can access the platform at any time and place, if they only have access to the internet.

Ecosystem Scipy vs SageMath

The SageMath computer algebra system is a huge collection of tools and it includes, among other things, tools from the Scipy ecosystem. The essential difference is, however, a common interface for all tools. Way using SageMath is optimized for interactive work and convenience of mathematicians. Running SageMath one has a Python 2.7 interpreter available with two key differences:

1. **Each command is processed by the so-called preparser before** will be sent to Python interpreter. Preparser changes input in the following way:
   - replaces the power of *2^3* to the Python syntax *2**3*
   - literals like e.g.: *1* and *1.0* are transformed to constructors: *Integer(1)* and *RealNumber(1.0)* respectively
2. About 2000 useful functions are automatically loaded like *plot*,'simplify', etc., and a symbolic variable x is predefined.

Therefore, for example, to solve a square equation in SageMath, it is enough write *solve(x^2 + 2*x + 1 == 0, x)* and we will get the answer. The same can be accomplished in "pure" Python but one needed to load the appropriate modules and define the variable and only then proceed to the proper computations.

These advantages of SageMath have prompted us to apply this system in physics, mathematics and chemistry classes. However, it should be noted that usingf SageMath is **actually Programming in Python** and if the students get this skill during IT lessons then there will be only a small threshold to be overcomed for effective application of SageMath system for e.g. mathematics or physics. As a result, the solution is based on the SageMath system will provide a very efficient tool with very small overhead and will reuse potential students skills in Python programming.

2.2 SageMath in nutshell

2.2.1 Rich and fast scientific calculator

Practically, every important function, a mathematical formula, is already implemented in SageMath. Below are some instructions that can be used in high school:

- absolute value - `abs`,
- factoring - `factor`,
- factorial - `factorial`,
- Newton's symbol - `binomial`,
- solve the equation - `solve`,
- graph a function - `plot`,
- next prime number - `next_prime`,
- the greatest common divisor `gcd`
- the least common multiple: `lcm`,
- derivative - `diff`,
- integral - `integrate`.

The first example shows Sage's capabilites in arithmetics. You can use it to check for calculations by hand, homework assignments by students. If the teacher learns the programming language above, it can create code that will allow you to solve the "step-by-step" calculation.

```
print "(4/3+5/5)-(5/2-4/6) =", (4/3+5/5)-(5/2-4/6)
print "(3^15-3^13)/(3^13+3^14) =", (3^15-3^13)/(3^13+3^14)
print "1001 =", factor(1001)
print "(sqrt(8)-sqrt(2))^2 =", (sqrt(8)-sqrt(2))^2
print "5! =", factorial(5)
print  gcd(354, 222)
```

Executing above code one obtains:

```
(4/3+5/5)-(5/2-4/6) = 1/2
(3^15-3^13)/(3^13+3^14) = 2
1001 = 7 * 11 * 13
(sqrt(8)-sqrt(2))^2 = 2
5! = 120
6
```

SageMath is equipped with arbitrary precision arithmetics and, for example, can approximate numbers with any precision. These possibilities we used in our projects, first of all in RSA encryption and in the chapter on approximations of irrational expressions.

```
show(sqrt(2), "=", N(sqrt(2), digits=60))
show(pi, "=", N(pi, digits=60))
show(2^168+5^80)
```

Executing above code one obtains:

$$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694807317668$$
$$\pi = 3.14159265358979323846264338327950288419716939937510582097494$$
$$82718435399721924198287929350313460725034243008818892481$$

2.2.2 Logical expressions

SageMath and also the Python language, allows you to perform oparations on logical expressions. It can turn out to be useful in many areas. For example, consider the puzzle:

Riddles about liars

There are two kinds of people - one always lies and one says truth. Ala and Bolek belong to one of these categories. ala she said: Bolek and me are liars. Who is a liar and who he tells the truth?

By using SageMath we can accept the following interpretations: let a will be true if Ala is truthful and b will be true if Bolek is truthful. Then we can write in Sage:

```
f = propcalc.formula("a&(~a&~b) | ~a&(~(~a&~b))")
show(f)
print(f.truthtable())
```

Executing above code one obtains:

```
a       b       value
False   False   False
False   True    True
True    False   False
True    True    False
```

You can see that the only solution is the one where Ala lies and Bolek speaks the truth.

2.2.3 Algebraic expressions

One of the most important possibilities that can be used in the classroom Mathematics, physics and chemistry are operation not only on on numbers, but also on symbolic variables. SageMath is excellent with symbolic calculations, i.e. it can perform calculations, transformations on algebraic expressions. Thus we can modify the form of the formula, express one variable with the help of others, derive general solutions to equations. Below are shown simple examples of shortened multiplication patterns and expressions measurable.

Patterns of shortened multiplication.

```
var('a','b')
expr1 = (a+b)^2
expr2 = (a-b)^2
expr3 = (a+b)*(a-b)
show (expr1, "=", expr1.canonicalize_radical())
show (expr2, "=", expr2.canonicalize_radical())
show (expr3, "=", expr3.canonicalize_radical())
a=sqrt(3)
b=2
expr1=(a+b)^2
expr2=(a-b)^2
expr3=(a+b)*(a-b)
show (expr1, "=", expr1.canonicalize_radical())
show (expr2, "=", expr2.canonicalize_radical())
show (expr3, "=", expr3.canonicalize_radical())
```

Executing above code one obtains:

$$(a+b)^2 = a^2 + 2\,ab + b^2$$

$$(a-b)^2 = a^2 - 2\,ab + b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

$$\left(\sqrt{3}+2\right)^2 = 4\,\sqrt{3} + 7$$

$$\left(\sqrt{3}-2\right)^2 = -\,4\,\sqrt{3} + 7$$

$$\left(\sqrt{3}+2\right)\left(\sqrt{3}-2\right) = -\,1$$

Conversion of the form of algebraic expressions

```
var('n')
expr = n^3-(n-1)^3
show ("n=2")
show(expr," = ", expr.canonicalize_radical()," = ",expr.substitute(n
  ↪ = 2))
```

Executing above code one obtains:

$$n = 2$$

$$-(n-1)^3 + n^3 \;=\; 3\,n^2 - 3\,n + 1 \;=\; 7$$

Substitutions in symbolic expressions.

```
var('z')
expr = (z^2+3*z)/z
show (expr)
show (expr.canonicalize_radical())
show (expr.subs(z=x+1))
show (expr.subs(z=2))
```

Executing above code one obtains:

$$\frac{z^2 + 3\,z}{z}$$

$$z + 3$$

$$\frac{(x+1)^2 + 3\,x + 3}{x + 1}$$

$$5$$

Simplifying expressions containing trigonometric functions requires method . `trig_simplify`. For example, to use trigonometric identities:

```
( sin(x)^2+cos(x)^2 ).trig_simplify()
```

If we want to prove the trigonometric identity better to use `bool` than trying to simplify expressions.

```
expr = (2*sin(x)^2-1)/(sin(x)*cos(x)) == tan(x)-cot(x)
show(expr)
bool(expr)
```

Executing above code one obtains:

$$\frac{2\,\sin(x)^2 - 1}{\cos(x)\sin(x)} = -\cot(x) + \tan(x)$$

$$True$$

2.2.4 Solving equations and systems of equations

Many problems in physics and math lead to a equation or a system of equations which has to be solved. Of course no tool should replace the student's self-solving skills, but can be very useful for exercises, checking the results, or too solving too complex equations. SageMath allows for solving difficult equations and systems of equations with one instruction - `solve`.

Here are some examples that demonstrate use this functionality in two cases: quadratic eqaution and system of eqautions.

Quadratic equation.

```
var('a','b','c')
r_kwadr = a*x^2 + b*x + c == 0
show(solve(r_kwadr, x))
a = 1
b = 4
c = -5
r_kwadr = a*x^2 + b*x + c == 0
show (solve(r_kwadr, x))
```

Executing above code one obtains:

$$\left[x = -\frac{b + \sqrt{b^2 - 4\,ac}}{2\,a}, x = -\frac{b - \sqrt{b^2 - 4\,ac}}{2\,a}\right]$$

$$[x = (-5)\,, x = 1]$$

System of two equations

```
var('x','y')
solve([x-3*y==2, x-2*y==8],x,y)
```

Executing above code one obtains:

$$[[x == 20, y == 6]]$$

2.2.5 Visualisation

Visualization is a very important aspect of learning especially for the present generation of young people. Sage allows graphing functions in a simple way. So we can quickly present solutions on the plot or draw interesting functions during lessons. Students can modify existing code program and analyze the functions. This can be used not only in mathematics but also in other science subjects.

The following program address a classical problem of calculating zeros of a quadratic function. Short program allows not only for obtaining numerical result but also for its graphical visualization.

```
a = 1
b = 3
c = 2
d = b*b - 4*a*c
f(x) = a*x*x + b*x + c
if d < 0:
    print "No real solution!"
    xmin,xmax =-5, 5
    x1,x2 = 0,0

if d > 0:
    x1 = float((-b-sqrt(d))/(2*a))
```

```
    x2 = float((-b+sqrt(d))/(2*a))

    print "x1=", x1, ", ", "x2=", x2

    if x1<x2:
        xmin,xmax = x1-2,x2+2
    else:
        xmin,xmax = x2-2,x1+2

p1 = point((x1,0), color="red", size=35)
p2 = point((x2,0), color="red", size=35)
p3 = point((0, c), color="green", size=35)
q = plot(f(x),(x,xmin,xmax))
show(p1+p2+p3+q, figsize=4)
```

Executing this code one obtains the plot Fig. 2.1.

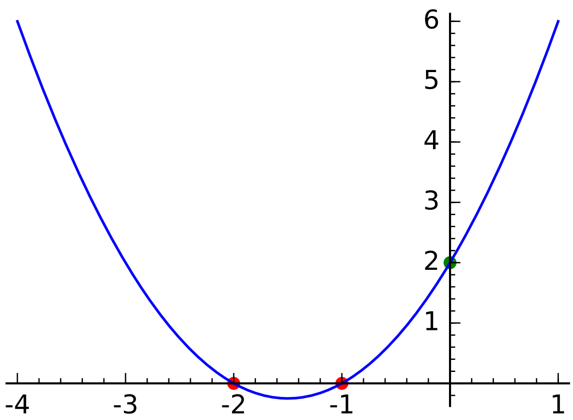


Fig. 2.1: Parabola and its zeros.

The command `region_plot` it is possible to visualize solution to inequalities.

```
var('x','y')
g1 = -x^2/4+1*x
g2 = 0.25*x
f1 = plot(g1, (x,-0.4,4.5), linestyle="--")
f2 = plot(g2,(x,-0.4,4.5), linestyle="-", color="green")
rp = region_plot([y<g1,y>=g2],(x,-0.3,4.5),(y,-1,1.2),
 ↪  incol="khaki")
show(f1 + f2 + rp, figsize=5)
```

Executing this code one obtains the plot Fig. 2.1.

```
var('x','y')
g1 = -x-2
g2 = -x+2
```

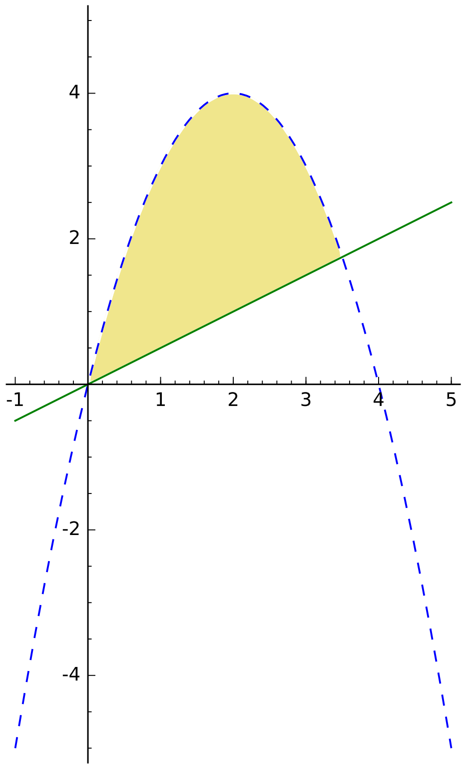


Fig. 2.2: Visualization of inequality *region_plot*

```
g3 = x-2
g4 = x+2
f1 = plot(g1, (x,-2.5,2.5), linestyle="--")
f2 = plot(g2, (x,-2.5,2.5), linestyle="--", color="royalblue")
f3 = plot(g3, (x,-2.5,2.5), linestyle="-", color="green")
f4 = plot(g4, (x,-2.5,2.5), linestyle="-", color="lightgreen")
rp = region_plot([y>g1,y<g2,y>=g3,y<=g4],\
      (x,-2,2),(y,-2,2), incol="khaki")
show(f1 + f2 + f3 + f4 + rp, figsize=5,ymax=3,ymin=-3)
```

Executing this code one obtains the plot Fig. 2.3.

In SageMath we can algorithmically create a formula of the function. Let us imagine that we want to plot a following expression:

$$f(x) = \sum_{i=0}^{N} \sin(\omega_i x)$$

for large values of of N. It is clear that it it very difficult task without a computer. In SageMath we can easily make use of a loop and construct above sum. In physics, many wave and aoustic phenomema are connecteed with sums of signals of various frequencies and therefore plotting such function is not only an academic excercise. It can serve as an example of e.g. a wave packet.

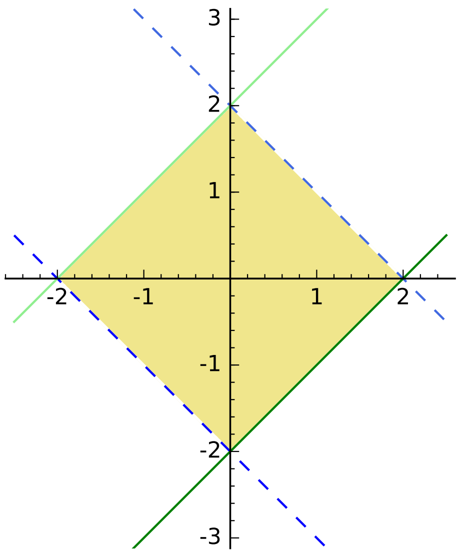


Fig. 2.3: Visualization of inequality *region_plot*

```
f = sum([sin(w*x) for w in srange(0.9,1.101,0.02)])
plot(f,(x,-200,200),figsize=(10,2),thickness=0.5)
```

Executing this code one obtains the plot: Fig. 2.4.

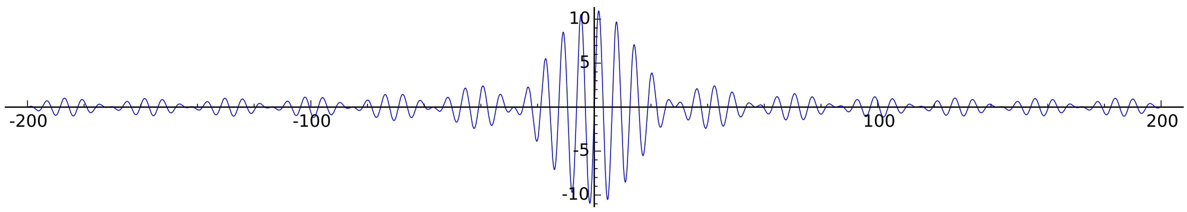


Fig. 2.4: Wave packet visualization.

Another interesting example is a graphical presentation of vector fields. It can have potential applical for physics lessons. Below, we present a plot of magnetic field of a magnetic dipole. It is possible to draw in 3d, but here we plot a section at $x = 0$:

```
var('x y z',domain='real')
m = 1
r = sqrt(x^2+y^2+z^2+1e-6)
Bx = 3*m*x*z/(r^5)
By = 3*m*y*z/(r^5)
Bz = 3*m*z^2/(r^5)-m/r^3
B = vector( [Bx,By,Bz])
Bmod = B.subs(x==0)[1:].norm()
plot_vector_field(B.subs(x==0)[1:]/Bmod,(y,-2,2),(z,-2,2))
```

Executing this code one obtains the plot: Fig. 2.5.

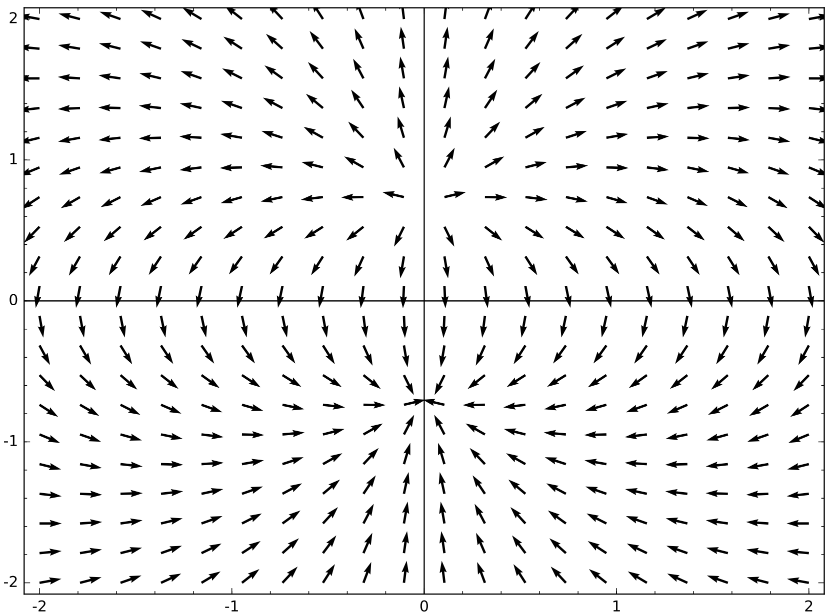


Fig. 2.5: Vector field from dipole.

2.2.6 Interaction

Large educational values have computer programs able to produce animation of a given phenomena and/or allowing for dynamical change some parameters. This class of programms are, most often made in Flash or javascript technologies, are an attractive digital help in teaching. Usually, however, the student is limited to interacting with such a program. SageMath lets you go one step further - it makes it very easy to create these elements. Using relatively simple functions student can create an interactive application that can illustrate a given problem.

We will illustrate the following problem:

Intersections

How many solution has the equation $x^2 = x - a$ as a function of parameter $a \in (0, \frac{1}{2})$?

In SageMath we can easily plot as line $y = x - a$ and parabola $y = x^2$ and grafically inspect if they have intersections. We can also calculate analytically (using Sage or not) solution and plot them in the same figure. If we then use decorator `@interact` SageMath will generate us an interactive application:

```
@interact
def fun(a=slider(0,1/2,0.01)):
    p = plot([x^2,x-a],(x,-1,1),figsize=5,ymax=1,ymin=-1)
    assume(x,'real')
    pkt = [(x.subs(s),x.subs(s)-a) for s in solve(x^2==x-a,x)]
    if pkt:
        p += point(pkt,size=40,color='red')
    else:
        print "No roots"
```

```
    show(p)
```

The above code will produce interactive element Fig. 2.6.

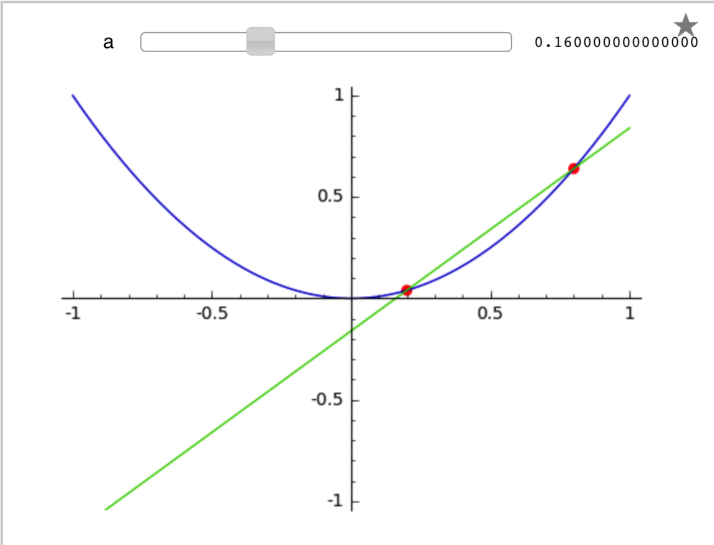


Fig. 2.6: An interactive illustration of an equation with parameter: $x^2 = x - a$.

2.2.7 SageMath = interdisciplinarity

In summary, SageMath gives an excellent tool following operations in a classroom:

1. Quick and accurate calculations with arbitrarily large numbers.
2. Manipulate algebraic expressions, Solving equations in complex numbers, solving systems of equations and inequalities.
3. Visualization of solutions on graphs, drawing graphs of functions.
4. Calculation of derivatives, integrals, and many other mathematical operations.

Moreover SageMath is a very good and rich programming tool, thanks to which we can combine computer science with mathematics, physics, chemistry.

CHAPTER 3

Bavarian final secondary-school examinations in mathematics

In this chapter we present numerous examples how SageMath computer algebra system can be usefull in problem solving. We have choosen matriculation exams with mathematics which have been conducted in years 2013-2016 in Bavaria (Germany). Many tasks one can solve just by typing `solve (..)` which demonstrates usability of the software, but at the same time is not very educational. Therefore in some problem the simulation and visualization are used as addtional tools helping in understanding the underlying theory. As a result, the following materials may serve as practical "cookbook" on SageMath.

> **Attention:** It is strongly recommended to use the interactive version available at:
>
> - http://visual.icse.us.edu.pl/methodology/

3.1 Analysis

3.1.1 Maximal domain and tangent of the square root function

Problem

We are given the function $g : x \to \sqrt{3x+9}$ with maximal domain D.

1. Determine D and state the zero of g.
2. Determine the equation of the tangent at the graph of g at point $P(0|3)$.

Solution of part a

By solving the condition $\sqrt{3x+9} = 0$ for x, one obtains the zero at $x = -3$.

This result can easily be checked by means of Sage:

```
sage: g(x) = sqrt(3*x+9)
sage: solve(g == 0, x)
```

The domain is obtained by the requirement that the argument of the square root be larger or equal zero. This is the case if $3x + 9 \geq 0$ or $x \geq -3$. Thus the domain is obtained as $D = [-3, \infty[$. The result is illustrated by the graph of the function $g(x)$.

```
sage: plot(g, (-3, 3), figsize=(4, 2.8))
```

Solution of part b

In order to determine the equation of the tangent at the point $P(0|3)$, we need to evaluate the derivative of g at this point. We find

$$\frac{\mathrm{d}g}{\mathrm{d}x} = g'(x) = \frac{3}{2\sqrt{3x+9}}.$$

As a consequence, $g'(0) = \frac{1}{2}$.

```
sage: dg = g.derivative()
sage: print "Derivative of g(x) : ", dg
sage: print "Derivative at x = 0: ", dg(0)
```

Because of $g(0) = 3$, the tangent h at point P is given by

$$h(x) = \frac{1}{2}x + 3\,.$$

This result can be graphically verified by means of Sage. The function $g(x)$ is represented in blue while the tangent $h(x)$ is displayed in red.

```
sage: pg = plot(g, (-3, 5), color='blue')
sage: h(x) = dg(0)*x+g(0)
sage: ph = plot(h, (-3,5), color='red')
sage: show(pg+ph, figsize=(4, 2.8))
```

3.1.2 Function and its codomain

Problem

Name a term of a function defined in $\mathbb{R}$ with the codomain

1. $W = [2; +\infty[$
2. $W = [-2; 2]$

Solution of part a

A possible solution is given by

$$f(x) = x^2 + 2.$$

Solution of part b

An example for a function limited from above and below is the sine function. However, its codomain is given by $W = [-1; 1]$. In order to obtain the required codomain, we multiply with 2. Thus we arrive at a possible solution

$$g(x) = 2\sin(x).$$

Both solutions can be graphically represented by Sage and we can check the codomains.

```
sage: f(x) = x**2+2
sage: g(x) = 2*sin(x)
sage: pf = plot(f, (-3, 3), color='blue')
sage: pg = plot(g, (-4, 4), color='red')
sage: show(pf+pg, figsize=(4, 2.8))
```

3.1.3 Nonlinear equation

Problem

State for $x \in \mathbb{R}^+$ the solutions of the following equation:

$$(\ln x - 1) \cdot (e^x - 2) \cdot \left(\frac{1}{x} - 3\right) = 0$$

Solution

The zeros of the function are obtained by determining the zeros of the three factors.

The requirement $\ln x - 1 = 0$ implies $\ln x = 1$. Applying the exponential function to both sides, one finds $\mathrm{e}^{\ln x} = \mathrm{e}^1 = \mathrm{e}$. The logarithm being the inverse of the exponential function, $\mathrm{e}^{\ln x} = x$ holds. Therefore, we find as a first zero $x_1 = \mathrm{e}$.

From the second factor, we find $\mathrm{e}^x = 2$ to which we apply the logarithm on both sides. In analogy to the previous reasoning, we have $\ln \mathrm{e}^x = x$. The second zero thus follows as $x_2 = \ln 2$.

By simply solving for x, the last factor yields the zero $x_3 = \frac{1}{3}$.

This result can easily be verified by means of Sage

```
solve((ln(x)-1) * (exp(x)-2) * (1/x-3) == 0, x)
```

3.1.4 Function and its aniderivative

Problem

Figure 1 displays the graph G_f of a function f defined in $\mathbb{R}$. Sketch in figure 1 the graph of the integral function $F : x \mapsto \int\limits_1^x f(t)\mathrm{d}t$ defined in $\mathbb{R}$. Consider with appropriate precision in particular the zeros and extrema of F as well as $F(0)$.

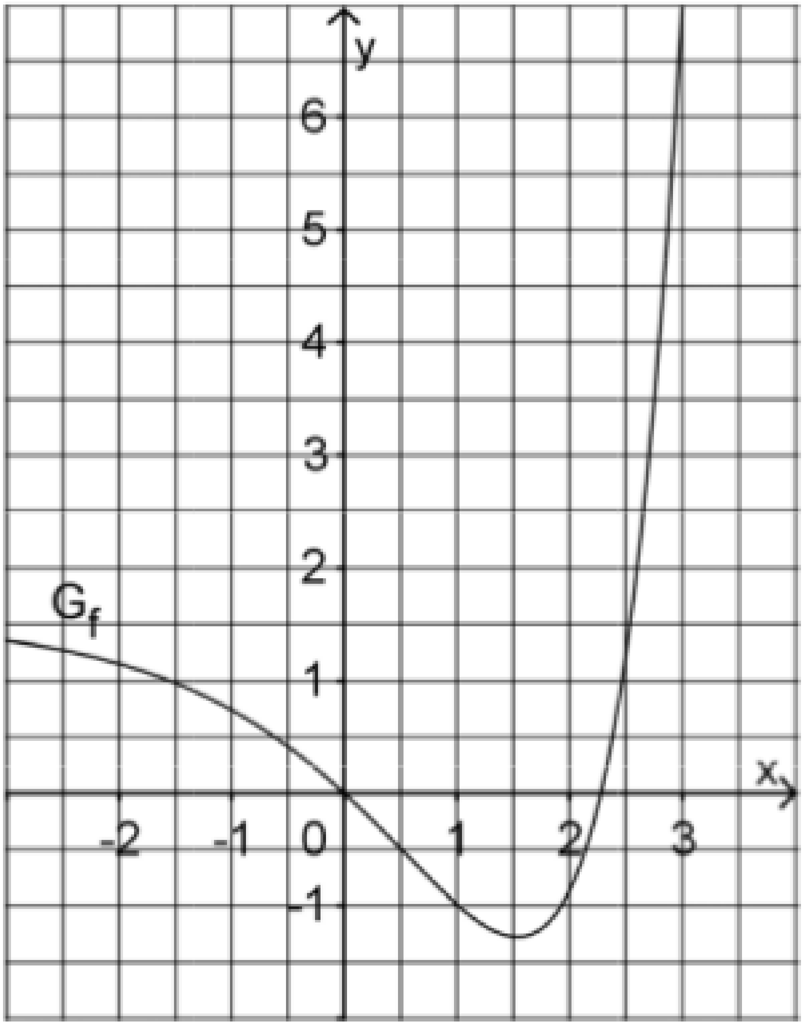


Abb. 1

Solution

If a function possesses a zero with a change of sign, the corresponding antiderivative possesses a local extremum at this point. It is a maximum if the slope of the function is negative. If, on the other hand, the slope is positive, the antiderivative has a minimum at that point.

Considering the function given in the problem text, one finds zeros of $f(x)$ at $x_1 = 0$ and $x_2 \approx 2.25$. In the first case, the slope is negative so that $F(0)$ represents a local maximum of the antiderivative. At the other point x_2 the slope of f is positive so that $F(x_2)$ is a local minimum.

Another property of $F(x)$ follows from the lower limit of integration at $t = 1$. As a consequence, $F(1) = 0$. Finally, from counting squares, $F(0) = -\int_0^1 f(t)\mathrm{d}t$ can be estimated to equal $\frac{1}{2}$.

By means of Sage, we can carry out the integration provided the function f is known. We choose

$$f(x) = \frac{49}{5}\frac{x(4x-9)}{(2x-9)^2},$$

which possesses the properties used in the above reasoning. In addition, its qualitative form resembles that given in figure 1.

```
sage: var('t')
sage: f(x) = 49/5*x*(4*x-9)/(2*x-9)^2
sage: assume(1 < x, 2*x-9 < 0)
sage: F(x) = integrate(f(t), t, 1, x)
sage: ranges = {'xmin': -3, 'xmax': 4, 'ymin': -2, 'ymax': 7}
sage: show(plot(f, color='blue', **ranges)+plot(F, color='red', **ranges),
            figsize=(3.5, 4.5))
```

3.1.5 Properties of the graph of the function

Problem

1. We are given the function $f : x \mapsto 2x \cdot e^{-0.5x^2}$ defined in $\mathbb{R}$. Figure 2 displays the graph G_f of f.

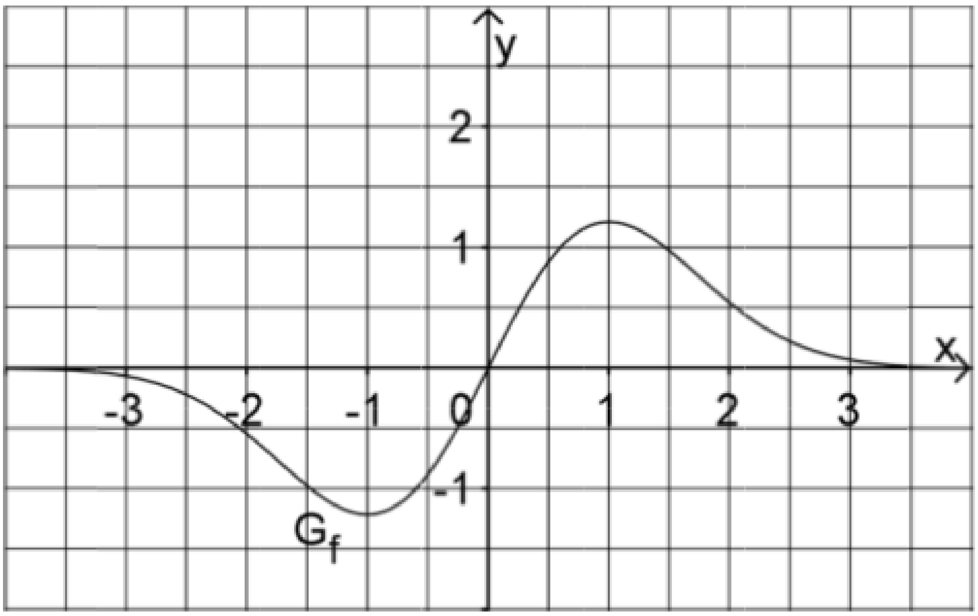


Abb. 2

   (a) Demonstrate by calculation that G_f is point-symmetric with respect to the origin. On the basis of the function term of f make plausible that $\lim_{x\to+\infty} f(x) = 0$.

   (b) Determine by calculation the position and nature of the extrema of G_f.

   (c) Determine the mean rate of change m_S of f in the interval $[-0.5; 0.5]$ as well as the local rate of change m_T of f at $x = 0$. Determine by how many percent m_S differs from m_T.

(d) The graph of f, the x-axis and the straight line $x = u$ with $u \in \mathbb{R}^+$ enclose for $0 \leq x \leq u$ a region with area $A(u)$.

Demonstrate that $A(u) = 2 - 2e^{-0.5u^2}$. Give $\lim\limits_{u \to +\infty} A(u)$ and interpret the result geometrically.

(e) The line h through the origin defined by $y = \frac{2}{e^2} \cdot x$ together with G_f completely encloses for $x \geq 0$ a region with area B.

Determine the x-coordinates of the three intersections of the straight line h with G_f and draw the straight line into figure 2. Calculate B.

2. In the following, the family of functions $g_c : x \mapsto f(x) + c$ defined in $\mathbb{R}$ with $c \in \mathbb{R}$ will be considered.

   (a) State without any further calculation the coordinates of the maximum of the graph of g_c as well as the behavior of g_c for $x \to +\infty$ as a function of c.

   (b) The number of zeros of g_c depends on c. Give for each of the following cases a possible value of c so that the respective condition holds:

   α) g_c has no zero.

   β) g_c has exactly one zero.

   γ) g_c has exactly two zeros.

   (c) Argue by means of an appropriate sketch that $\int\limits_0^3 g_c(x)\mathrm{d}x = \int\limits_0^3 f(x)\mathrm{d}x + 3c$.

3. The number of children to which a woman gives birth during her life is described by the so-called birth rate which is statistically determined each year.

   The function $g_{1.4} : x \to 2x \cdot e^{-0.5x^2} + 1.4$ for $x \geq 0$ exemplarily describes the time evolution of the birth rate in a European country. Here, x is the number of decades passed since 1955 (i.e., $x = 1$ corresponds to the year 1965) and $g_{1.4}(x)$ is the birth rate. For the population to be approximately constant in the long run in that contry, a birth rate of approximately 2.1 is required.

   (a) Sketch the graph of $g_{1,4}$ in figure 2 and deduce with reasonable accuracy in which time period the birth rate is at least 2.1.

   (b) What is to be expected on the basis of this model for the future evolution of the population? Justify your answer.

   (c) In the time period considered, there exists a year in which the birth rate decreases most. Make use of figure 2 to give an approximate value for that year. Describe how on the basis of this model one could demonstrate by calculation that the decrease of the birth rate becomes continuously weaker from that year on.

Part 1

Solution of part a

In order to prove the point-symmetry of G_f we need to demonstrate $f(x) = -f(-x)$ or, equivalently, $f(x) + f(-x) = 0$.

$$\begin{aligned} f(x) + f(-x) &= 2x \cdot e^{-0.5x^2} + 2(-x) \cdot e^{-0.5(-x)^2} \\ &= 2x \cdot e^{-0.5x^2} - 2x \cdot e^{-0.5x^2} \\ &= 0 \end{aligned}$$

This result can easily be checked by means of Sage:

```
sage: f(x) = 2*x*exp(-1/2*x**2)
sage: print("f(x) + f(-x) = " + str(f(x) + f(-x)))
```

Solution of part b

In order to identify the extrema of G_f, one needs to determine the zeros of the derivative of f.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(2x \cdot e^{-0{,}5x^2}\right) \\ &= 2 \cdot e^{-0{,}5x^2} + 2x \cdot e^{-0{,}5x^2} \cdot (-x) \\ &= 2 \cdot e^{-0{,}5x^2}\left(1 - x^2\right) \end{aligned}$$

By means of Sage, the derivative may be obtained as follows:

```
sage: df(x) = derivative(f(x),x)
sage: print("Derivative of f(x): " + str(df(x).full_simplify()))
```

In order to determine the extrema, the equation

$$2 \cdot e^{-0.5x^2}(1 - x^2) = 0$$

must be solved. As the exponential function does not possess zeros, all zeros result from $1 - x^2 = 0$. We thus find the solutions $x_1 = -1$ and $x_2 = 1$.

The solutions can be confirmed with Sage.

```
sage: nstn = solve(df(x)==0, x)
sage: print("Zeros of the derivative of f: " +repr(nstn))
```

In order to identify the nature of the extrema, one needs to consider the second derivative of f at the x values determined above. For a positive second derivative , one has a minimum, while for a negative second derivative, the extremum is a maximum. By means of the derivative of f already determined above, one obtains for the second derivative

$$\begin{aligned} f''(x) &= \frac{d^2}{dx^2} f(x) \\ &= \frac{d}{dx}\left(2 \cdot e^{-0.5x^2}\left(1 - x^2\right)\right) \\ &= -2x \cdot e^{-0.5x^2}\left(3 - x^2\right) \end{aligned}$$

Inserting x_1 and x_2 yields:

$$\begin{aligned} f''(x_1) = f''(-1) &= -2 \cdot (-1) \cdot e^{-0.5\cdot(-1)^2}\left(3-(-1)^2\right) \\ &= 4 \cdot e^{-0.5} > 0 \\ f''(x_2) = f''(1) &= -2 \cdot 1 \cdot e^{-0.5\cdot 1^2}\left(3-1^2\right) \\ &= -4 \cdot e^{-0.5} < 0 \end{aligned}$$

Therefore, x_1 is a minimum while x_2 is a maximum. This result is confirmed by figure 2.

By means of Sage, one obtains the second derivative and its values at the points x_1 and x_2 as follows:

```
sage: ddf(x) = derivative(df(x),x)
sage: print("Second derivative of f(x): " +
 ↪ str(ddf(x).full_simplify()))
sage: print("ddf(-1) = " + str(ddf(-1)))
sage: print("ddf(1) = " + str(ddf(1)))
```

Solution of part c

The mean rate of change m_S of f in the interval $[-0.5; 0.5]$ is obtained as the difference of the function values at the borders of the inervall divided by the length of the interval. Emplying the point-symmetry with respect to the origin, one finds

$$\begin{aligned} m_S &= \frac{f(0.5)-f(-0.5)}{0.5-(-0.5)} \\ &= 2f(0.5) \\ &= 4 \cdot (0.5) \cdot e^{-0.5\cdot(0.5)^2} \\ &= 2 \cdot e^{-0.125} \\ &\approx 1.76 \end{aligned}$$

In Sage, one can easily define a function which determines the mean slope for two intervall borders.

```
sage: def ms(x1,x2):
sage:     return (f(x2)-f(x1))/(x2-x1)
sage: print("Mean rate of change between -0.5 and 0.5: %4.2f" %
 ↪ ms(-0.5, 0.5))
```

The local rate of change m_T at $x = 0$ is the slope in that point. The derivative has already been determined in the previous subproblem. Thus we find:

$$m_T = f'(0) = 2 \cdot e^{-0.5\cdot 0}(1-0) = 2$$

```
sage: print("Local rate of change at x=0: " + str(df(0)))
```

The deviation in percent thus amounts to

$$\left(\frac{m_S}{m_T} - 1\right) = e^{-0.125} = 0.882 = 88.2\%.$$

```
sage: print("Deviation in percent between mean and local rate of
  ↪  change: %4.1f%%" % (100*ms(-0.5,0.5)/df(0)))
```

Solution of part d

The enclosed area $A(u)$ can be computed as integral over $f(x)$ from the lower limit 0 to the upper limit u:

$$A(u) = \int_0^u f(x)\mathrm{d}x$$

The area must vanish for $u = 0$.

$$A(0) = 2 - 2e^{-0.5\cdot 0^2} = 2 - 2 = 0$$

Taking the derivative of $A(u)$ with respect to u, one finds:

$$\frac{\mathrm{d}}{\mathrm{d}u}A(u) = \frac{\mathrm{d}}{\mathrm{d}u}\left(2 - 2e^{-0.5u^2}\right) = 2u \cdot e^{-0.5u^2} = f(u)$$

These two properties prove that $A(u)$ is the definite integral of $f(x)$ on the interval $[0; u]$.

The integration can be carried out directly by means of Sage.

```
sage: var('u')
sage: assume(u > 0)
sage: a(u) = f.integral(x, 0, u)
sage: print("A(u) = "+str(a(u)))
```

The limit of $A(u)$ for $u \to +\infty$ is found as:

$$\lim_{u\to+\infty} A(u) = \lim_{u\to+\infty}\left(2 - 2e^{-0.5u^2}\right) = 2 - 0 = 2$$

By means of Sage the limit is obtained as follows:

```
sage: print("A(inf) = " + str(limit(a(u), u=Infinity)))
```

The result implies that the area enclosed by the graph between 0 and $+\infty$ is finite and equals 2.

Solution of part e

The first intersection of the straight line $y = \frac{2}{e^2} \cdot x$ with G_f is found at $x_1 = 0$, since both functions pass through the origin. Additional intersections are obtained by equating the two functions.

$$\begin{aligned}
\frac{2}{e^2} \cdot x = 2x \cdot e^{-0.5x^2} &\left| \cdot \frac{e^2}{2x} \right. \\
1 = e^{-0.5x^2+2} &|\ln() \\
0 = -0.5x^2 + 2 \qquad &|-2 \\
4 = x^2 &|\sqrt{} \\
x_{2/3} = \pm 2 &
\end{aligned}$$

With the help of Sage, the intersections can be obtained as follows:

```
sage: h(x) = x * 2 / e^2
sage: solve(f(x) == h(x), x)
```

The intersection points can be identified graphically if both functions are drawn in the same coordinate system.

```
sage: pf = plot(f, (0, 2), color='blue', fill=h, fillcolor='yellow')
sage: ppf = plot(f, (-4, 0), color='blue')
sage: pppf = plot(f, (2, 4), color='blue')
sage: ph = plot(h, (-4, 4), color='red')
sage: b = text("B",(1, 0.7))
sage: show(pf + ph + ppf + pppf + b, aspect_ratio=1)
```

The enclosed area B results from taking the difference between the area $A(2)$ under the function f and the triangular area below the straight line

$$\begin{aligned}
B &= A(2) - \frac{1}{2} \cdot 2 \cdot \frac{2}{e^2} \cdot 2 \\
&= 2 - 2e^{-2} - 4e^{-2} \\
&= 2 - 6e^{-2} \\
&\approx 1.19.
\end{aligned}$$

The integral can easily be evaluated by means of Sage.

```
sage: b = a(2)-integral(h(x), x, 0, 2)
sage: print(u"The area B is given by: " + str(b) + u"  %4.2f" % b)
```

Part 2

Solution of part a

The family of functions g_c is displaced vertically with respect to $f(x)$ by a constant c. Therefore, the maximum of the graph remains at the same x coordinate. For $f(x)$, the maximum was determined in problem 1b) and found at $x = 1$ and $f(1) = \frac{2}{\sqrt{e}}$. The y-value of the maximum of g_c correspondingly is found as

$$g_c(1) = f(1) + c = \frac{2}{\sqrt{e}} + c.$$

Thus the coordinates of the maximum are given by $\left(1; \frac{2}{\sqrt{e}} + c\right)$.

For a fixed value of c, Sage determines the maximum numerically. Here, we choose $c = 3$.

```
sage: c = var('c')
sage: gc(c, x) = f(x) + c
sage: hy, hx = find_local_maximum(gc(3), -30, 30)
sage: print("The maximum for c=3 is found at: (%4.2f,%4.2f)" % (hx,
↪  hy))
```

As $f(x)$ vanishes in the limit $x \to +\infty$, we find $\lim\limits_{x\to+\infty} g_c(x) = c$.

```
sage: print(u"g_c(c, inf) = " + str(limit(gc(c, x), x=Infinity)))
```

Solution of part b

For g_c not to have a zero, a positive or negative value of c must be chosen such that its absolute value is larger than the absolute value of the minimum or maximum, respectively, of the graph, e.g., $c = 2$. For exactly one zero, c can be chosen equal to the negative value of the maximum or the positive value of the minimum of $f(x)$. From problem 1b) $c = \pm\frac{2}{\sqrt{e}}$ follow as solutions. Furthermore, $c = 0$ can be chosen. In this case, $f(x)$ only possesses a zero at the origin. For all other values of c (smaller than the absolute value of the maximum and minimum and different from zero), g_c possesses two zeros.

The following diagram $c \in \{0; 1; \frac{2}{\sqrt{e}}; 2\}$ displays graphs of functions with a different number of zeros.

```
sage: pg0 = plot(gc(0, x), (-4, 4), color='blue')
sage: pg1 = plot(gc(1, x), (-4, 4), color='red')
sage: pgtp = plot(gc(2/sqrt(e), x), (-4, 4), color='purple')
sage: pg2 = plot(gc(2, x), (-4, 4), color='green')
sage: show(pg0 + pg1 + pgtp + pg2, aspect_ratio=1)
```

The zeros can numerically be determined by means of Sage, if one provides an interval in which not more than one zero is expected. The total interval to be examined must therefore be divided into sufficiently small subintervals in order to find all zeros. In the following example, the interval $[-5, 5]$ is divided into a selectable number of subintervals. It is interesting to choose c close to the value for which only one zero exists, e.g. $c = \pm 1.2$. Then, the number of found zeros depends on the subintervals chosen.

```
sage: from numpy import linspace

sage: def my_find_root(f, a, b, n):
...       """find zeros of the function f in the interval [a, b] by
...       subdivision into n subintervals of equal size
...
...       """
...       roots = set()
...       limits = linspace(a, b, n+1)
...       for x0, x1 in zip(limits[:-1], limits[1:]):
...           try:
...               r = find_root(f, x0, x1)
...               roots.add(str(r))
...           except RuntimeError: # No zero was found in this interval
...               pass
...       zeros = "{" + ", ".join(roots) + "}"
...       print("Zeros of " + str(f) + ": " + zeros)

sage: @interact
sage: def _(c=slider(-2, 2, 0.1, 0),
...          n=slider(1, 80, 1)):
...       my_find_root(gc(c), -5, 5, n)
```

Solution of part c

It is straightforward to derive the given formula by exploiting the linearity of the integration:

$$\int_0^3 g_c(x)\mathrm{d}x = \int_0^3 (f(x)+c)\mathrm{d}x = \int_0^3 f(x)\mathrm{d}x + \int_0^3 c\mathrm{d}x = \int_0^3 f(x)\mathrm{d}x + 3c$$

A sketch visualizing this relation can easily be produced with Sage. The green rectangle has the size $3c$. The yellow area corresponds to the integral over $f(x)$.

```
sage: c = 1
sage: pg = plot(gc(c, x), (0, 3), color='red', fill=c, fillcolor='yellow')
sage: pgl = plot(gc(c, x), (-1, 0), color='red')
sage: pgr = plot(gc(c, x), (3, 4), color='red')
sage: gtext = text(r"$g_1(x)$", (2, c + 0.8), fontsize=14)
sage: pc = plot(c,(0, 3), color='white', fill=True, fillcolor='lightgreen')
sage: ftext = text(r"$\int_0^3 f(x) \mathrm{d}x$",(1, c + 0.5), fontsize=14)
sage: ctext = text(r"$c=" + str(c) + r"$",(-0.5, c), fontsize=14)
```

```
sage: c3text = text(r"$3c$",(1, c/2), fontsize=14)
sage: show(pgl + pg + pgr + gtext+ pc + ftext + ctext + c3text, aspect_ratio=1, xmax=4)
```

Part 3

Solution of part a

In order to find the starting and end points of the interval in which $g_{1.4}(x) > 2.1$, we must solve the following equation:

$$g_{1.4}(x) - 2.1 = 0$$

Based on the previously solved problems we know that the maximum of $g_c(x)$ is situated at $x = 1$. Therefore, the starting point of the interval must have a smaller value of x. On the other hand, the end point must have a larger value of x. We determine the limit numerically by means of Sage.

```
sage: startx = find_root(gc(1.4)-2.1, -1, 1)
sage: endx = find_root(gc(1.4)-2.1, 1, 3)
sage: print("In the interval [" + str(startx) + ", " + str(endx) + "] gc(1.4, x) is larger than 2.1")
sage: pg14l = plot(gc(1.4, x), (-4, startx), color='red')
sage: pg14 = plot(gc(1.4, x), (startx, endx), fill=2.1, fillcolor='yellow', color='red')
sage: pg14r = plot(gc(1.4, x), (endx, 4), color='red')
sage: show(pg0 + pg14l + pg14 + pg14r, aspect_ratio=1)
```

Solution of part b

As the solution to the previous subproblem indicates, the birth rate within the given model at the latest in 1975 was below the value where the population remains constant in the long run. Therefore a decrease of the population is also to be expected for the future.

Solution of part c

The point of the strongest decrease of the birth rate is given by the minimum of the derivative of the birth rate. As $g_c(x)$ is only vertically shifted by a constant with respect to $f(x)$, the derivatives of the two functions agree. The first two derivatives of $f(x)$ have been evaluated in subproblem 1b). The minimum of the derivative of the birth rate is found by determining the zeros of $f''(x)$:

$$f''(x) = 2x \cdot e^{-0.5x^2}\left(x^2 - 3\right) \stackrel{!}{=} 0$$
$$\rightarrow x_1 = 0, x_{2/3} = \pm\sqrt{3}$$

This result can also be reproduced by means of Sage:

```
sage: solve(ddf(x) == 0, x)
```

As the model is only valid for $x \geq 0$, the zero at $x_3 = -\sqrt{3}$ has to be excluded. $x_1 = 0$ cannot correspond to a global minimum, because according to subproblem 1c) the birth rate increases at this point. Furthermore, we know that negative values for the derivative exist. As a consequence $x_2 = \sqrt{3}$ is the minimum of the derivative which we are looking for. It corresponds to the year 1972.

For the decrease of the birth rate to become continuously weaker beyond this point in time, the derivative $g'(x)$ for $x > \sqrt{3}$ must be negative. According to the derivative of $f(x)$ determined in subproblem 1b), this is indeed the case.

3.1.6 Properties of the logarithm

Problem

State for the function f with $f(x) = \ln(2013 - x)$ the maximal domain $\mathbb{D}$, the behavior of f at the borders of $\mathbb{D}$ as well as the intersections of the graph of f with the coordinate axes.

Solution

The domain of the natural logarithm $\ln(x)$ is $(0, \infty)$. As a consequence, the domain of $\ln(-x)$ is $(-\infty, 0)$. Adding a number to the argument of the logarithm, the same needs to be done for the domain. Thus, the domain of $f(x) = \ln(2013 - x)$ is given by $\mathbb{D} = (-\infty, 2013)$.

By means of Sage, we obtain an idea of the function graph.

```
sage: f(x) = log(2013-x)
sage: plot(f(x), x, (-2014, 2014), figsize=(4, 2.8))
```

At the borders of $\mathbb{D}$ one finds the following behavior:

$$\lim_{x\to-\infty} \ln(2013 - x) = \lim_{x\to-\infty} \ln(-x) = \lim_{\tilde{x}\to+\infty} \ln(\tilde{x}) = +\infty$$

This result can be confirmed by means of Sage by inserting $-\infty$ for x.

```
sage: print("f(-inf) = " + str(f(-infinity)))
```

For x going to 2013, the argument of the logarithm goes to 0. In this case, the logarithm goes to $-\infty$.

$$\lim_{x\to 2013^-} \ln(2013 - x) = \lim_{\tilde{x}\to 0^+} \ln(\tilde{x}) = -\infty$$

```
sage: print("f(2013) = " + str(f(2013)))
```

The intersection with the y-axis is obtained by setting $x = 0$:

$$f(0) = \ln(2013 - 0) = \ln(2013) \approx 7.61$$

This result is in agreement with the function graph shown above.

```
sage: print("f(0) = " + str(f(0)) + " = " + str(f(0).n(12)))
```

The logarithm intersects the x-axis if its argument equals 1:

$$2013 - x \stackrel{!}{=} 1 \rightarrow x = 2012$$

This result can be confirmed with Sage:

```
sage: solve(f(x) == 0, x)
```

3.1.7 Properties of the x sin(x) function

Problem

The graph of the function $f : x \mapsto x \cdot \sin x$ defined in $\mathbb{R}$ passes through the origin. Determine $f''(0)$ and describe how the graph of f is bent close to the origin.

Solution

The first derivative of $f(x)$ is given by:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x \cdot \sin x) = \sin x + x \cdot \cos x$$

Taking another derivative, one finds:

$$\begin{aligned} f''(x) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2} x \cdot \sin x \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(\sin x + x \cdot \cos x) \\ &= \cos x + \cos x + x \cdot (-\sin x) \\ &= 2 \cdot \cos x - x \cdot \sin x \end{aligned}$$

For $x = 0$ one obtains $f''(0) = 2 \cdot \cos 0 - 0 \cdot \sin 0 = 2$.

This result can be verified by means of Sage:

```
sage: f(x) = x*sin(x)
sage: df(x) = derivative(f, x)
sage: ddf(x) = derivative(df, x)
sage: print("ddf(x) = " + str(ddf(x)))
sage: print("ddf(0) = " + str(ddf(0)))
```

A positive second derivative indicates that the function bends to the left as is the case for f at $x = 0$.

This behavior can also be seen by plotting the function graph.

```
sage: pf = plot(f(x), (-6, 6))
sage: show(pf, aspect_ratio=1)
```

3.1.8 Intersection point

Problem

We are given the functions $g : x \mapsto e^{-x}$ and $h : x \mapsto x^3$ defined in $\mathbb{R}$.

1. Illustrate by means of a sketch, that the graphs of g and h possess precisely one intersection.
2. Determine an approximate value x_1 for the x-coordinate of this intersection by carrying out the first step of the Newton method for the function $d : x \mapsto g(x) - h(x)$ defined in $\mathbb{R}$ using as initial value $x_0 = 1$.

Solution of part a

The graphs may easily be drawn by means of Sage:

```
sage: g(x) = exp(-x)
sage: h(x) = x**3
sage: pg = plot(g, color='blue')
sage: ph = plot(h, color='red')
sage: show(pg + ph, ymax=1.5, aspect_ratio=1, figsize=4)
```

Solution of part b

For an appropriately chosen initial value, the Newton method computes an approximate value for the zero nearby. In order to determine the zero of the function $f(x)$, one needs to evaluate in each step

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} .$$

For the first step one finds together with $f(x) = d(x) = g(x) - h(x)$ and $x_0 = 1$:

$$x_1 = 1 - \frac{g(1) - h(1)}{g'(1) - h'(1)} = 1 - \frac{e^{-1} - 1}{-e^{-1} - 3} \approx 0{,}812$$

By means of Sage, several steps of the Newton method can be executed. As can be seen, already after five steps does the Newton method yield a result agreeing up to 12 digits with the zero determined numerically by Sage.

```
sage: f(x) = g(x) - h(x)
sage: df(x) = derivative(f, x)
sage: zero_approx = 1
sage: newton(x) = x - f(x) / df(x)
sage: for i in range(5):
...       zero_approx = newton(zero_approx)
...       pretty_print(html("$x_%i = %s$" % (i+1,
 ↪  str(float(zero_approx)))))
sage: pretty_print(html("Zero obtained numerically by Sage:
 ↪  $x_\mathrm{S} = %s$"
...                     % str(find_root(f(x), -1, 1))))
```

3.1.9 Integrals of semicircle

Problem

Figure 1 displays the graph G_f of the function f with the domain $[-2; 2]$. The graph consists of two half-circles with centers at $(-1|0)$ and $(1|0)$, respectively, and a radius of 1. Consider the antiderivative $F : x \mapsto \int_0^x f(t)\mathrm{d}t$ defined in $[-2; 2]$.

1. State the values of $F(0)$, $F(2)$, and $F(-2)$.
2. Add a sketch of the graph of F to figure 1.

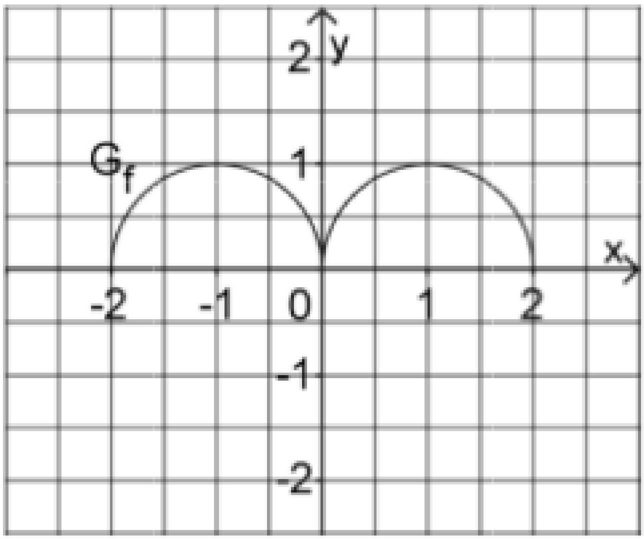


Abb. 1

Solution of part a

In view of the limits of integration of F one finds for $F(0)$:

$$F(0) = \int_0^0 f(t)\mathrm{d}t = 0.$$

For $F(2)$ one integrates over the area of a half-circle with radius 1. With the area of a circle of radius r given by πr^2, one obtains

$$F(2) = \frac{\pi}{2}.$$

As a consequence of the symmetry of the function $f(x)$, its antiderivative is antisymmetric:

$$F(-x) = \int_0^{-x} f(t)\mathrm{d}t = -\int_{-x}^{0} f(t)\mathrm{d}t = -\int_0^{x} f(-t)\mathrm{d}t = -\int_0^{x} f(t)\mathrm{d}t = -F(x)$$

Correspondingly, $F(-2) = -\frac{\pi}{2}$.

These results may be checked by means of Sage:

```
sage: f1(t) = sqrt(1 - (t - 1)^2)
sage: f2(t) = sqrt(1 - (t + 1)^2)
sage: f = Piecewise([[(-2, 0), f2], [(0, 2), f1]], t)
sage: print("F(0) = " + str(integrate(f, t, 0, 0)))
sage: print("F(2) = " + str(integrate(f, t, 0, 2)))
sage: print("F(-2) = " + str(integrate(f, t, 0, -2)))
```

Solution of part b

The function being defined piecewise, it is best to carry out the symbolic integration in Sage piecewise as well. Then we are able to display the function f and its antiderivative.

```
sage: x = var('x')
sage: assume(x > 0)
sage: F1(x) = integrate(f1, t, 0, x)
sage: forget()
sage: assume(x < 0)
sage: F2(x) = integrate(f2, t, 0, x)
sage: F = Piecewise([[(-2, 0), F2], [(0, 2), F1]], x)
sage: pf = plot(f)
sage: pF = plot(F, color='red')
sage: show(pf + pF, aspect_ratio=1, figsize=4)
```

3.1.10 Asymptotes

Introduction

We are given the function $f : x \mapsto \frac{1}{2}x - \frac{1}{2} + \frac{8}{x+1}$ with domain $\mathbb{R}\backslash\{-1\}$. Figure 2 displays the graph G_f of f.

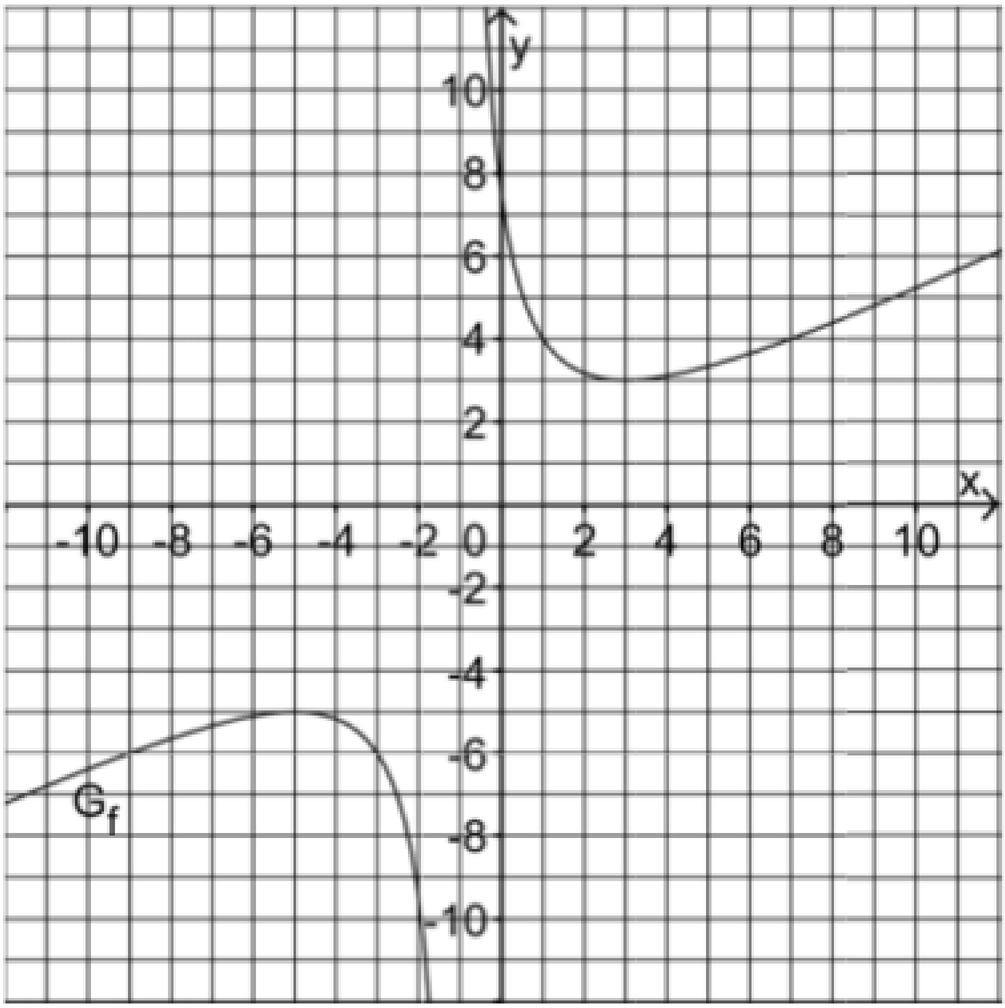


Abb. 2

Problem 1

Problem 1

1. State the equations of the asymptotes of G_f and demonstrate by means of a calculation that G_f does not intersect its inclined asymptote. Add a sketch of the asymptotes to figure 2.
2. Calculate the position and nature of the extrema of G_f.

Solution of part a

At $x = -1$ the function $f(x)$ possesses a pole. Accordingly, the first asymptote is described by the equation $x = -1$. The second asymptote ist obtained by considering $f(x)$ for very large absolute values of x. In this case, the term $\frac{8}{x+1}$ goes to zero. Therefore, the function approaches the straight line $g(x) = \frac{1}{2}x - \frac{1}{2}$.

The following code asks Sage to identify the poles of the function which are then displayed by grey dashed lines. In this way, the vertical asymptote $x = -1$ is accounted for. In addition, the asymptote for large values of $|x|$ must be depicted.

```
sage: ranges = {'xmin': -10, 'xmax': 10, 'ymin': -10, 'ymax': 10}
sage: f(x) = x/2- 1/2 + 8/(x+1)
sage: pf = plot(f, detect_poles="show", **ranges)
sage: asymptote = x/2 - 1/2
sage: pasymp = plot(asymptote, color='green', **ranges)
sage: show(pf + pasymp, aspect_ratio=1, figsize=4)
```

An intersection of the function with its asymptote can only exist provided that

$$f(x) - g(x) = \frac{8}{x+1} = 0$$

possesses a solution. However, this is obviously not the case. Accordingly, Sage does not find an intersection.

```
sage: solve(asymptote == f, x)
```

Solution of part b

In order to determine the positions of the extrema, the derivative of f must be set equal to zero. The nature of the extrema can be deduced from the sign of the second derivatives at these points.

We find by taking the derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{2} - \frac{8}{(x+1)^2}$$
$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{16}{(x+1)^3}$$

The first derivative vanishes at $x + 1 = \pm 4$, i.e., at $x_1 = -5$ and $x_2 = 3$. In the first case, the second derivative is negative and we have a maximum. In contrast, in the second case the second derivative is positive and we have a minimum.

These results may be confirmed by Sage.

```
sage: df = derivative(f)
sage: ddf = derivative(df)
sage: print "f'(x)  = ", df
sage: print "f''(x) = ", ddf
sage: for extremum in solve(df == 0, x):
sage:     x = extremum.right()
sage:     print "Second derivative for the extremum at x=%s: %s" % (x, ddf(x))
```

Problem 2

Problem 2

Figure 2 suggests that G_f is symmetric with respect to the intersection $P(-1|-1)$ of its asymptotes. In order to prove this symmetry of G_f, one may consider the function g the graph of which is obtained from G_f by a shift in the positive x direction by 1 and in the positive y direction by 1.

1. Determine the function term of g. Then demonstrate the point symmetry of G_f by proving that the graph of g is point-symmetric with respect to the origin.

   (Partial result: $g(x) = \frac{1}{2}x + \frac{8}{x}$*)*

2. Demonstrate that $\int\limits_0^4 f(x)\mathrm{d}x = 2 + 8 \cdot \ln 5$. Then determine without any further integration the value of the integral $\int\limits_{-6}^{-2} f(x)\mathrm{d}x$; illustrate your procedure by appropriate additions to figure 2.

Solution of part a

A positive shift in x direction by 1 is obtained by replacing x by $x - 1$. Adding 1 leads to the required shift in the y direction. The function g is then obtained in agreement with the partial result stated in the problem text:

```
sage: g(x) = f(x-1) + 1
sage: print(g)
```

The point symmetry of g with respect to the origin is a consequence of the presence of only linear or inversely linear terms. It can be demonstrated by means of Sage as follows:

```
sage: print "g(x) = ", g(x)
sage: print "-g(-x) = ", -g(-x)
sage: if g(x) == -g(-x):
sage:     print "g(x) is point-symmetric."
sage: else:
sage:     print "g(x) is not point-symmetric."
```

Solution of part b

The antiderivative of the function f is

$$F(x) = \frac{x^2}{4} - \frac{x}{2} + 8\ln(|x+1|).$$

Inserting the limits of integration, one finds the stated result

$$\int_0^4 f(x)\mathrm{d}x = F(4) - F(0) = 2 + 8\ln(5)\,.$$

By means of Sage, one obtains accordingly:

```
sage: F = f.integrate(x)
sage: print "antiderivative F = ", F
sage: pretty_print(html("$\int_0^4 f(x)\mathrm{d}x = $" +
  ↪ str(F(4)-F(0))))
```

As a consequence of the point symmetry $g(-x) = -g(x)$ and the relation $g(x) = f(x-1)+1$ it follows that

$$f(x) = -f(-x-2) - 2\,.$$

The second integral to be evaluated can then be determined as follows:

$$\begin{aligned}
\int_{-6}^{-2} f(x)\mathrm{d}x &= -\int_{-6}^{-2} [f(-x-2)+2]\mathrm{d}x \\
&= -\int_{2}^{6} [f(x-2)+2]\mathrm{d}x \\
&= -\int_{0}^{4} [f(x)+2]\mathrm{d}x \\
&= -\int_{0}^{4} f(x)\mathrm{d}x - 8 \\
&= -8\ln(5) - 10\,.
\end{aligned}$$

Here, we first took the mirror image of the interval of integration and then shifted it.

The resulting value can be confirmed by Sage:

```
sage: f.integrate(x, -6, -2)
```

The procedure can be displayed graphically by Sage. The area of size $2 \cdot 4 = 8$ displayed in red represents the correction to the integral following from the point symmetry with respect to $y = -1$.

```
sage: pf = plot(f, exclude=[-1], xmin=-10, xmax=10, ymin=-10,
  ↪  ymax=10)
sage: pf1 = plot(f, -6, -2, fill=-2)
sage: pf2 = plot(f, 0, 4, fill='axis')
sage: rect = polygon([[-6, 0], [-2, 0], [-2, -2], [-6, -2]],
  ↪  color='red')
sage: show(pf + pf1 + pf2 + rect, aspect_ratio=1, figsize=4)
```

Problem 3

Problem 3

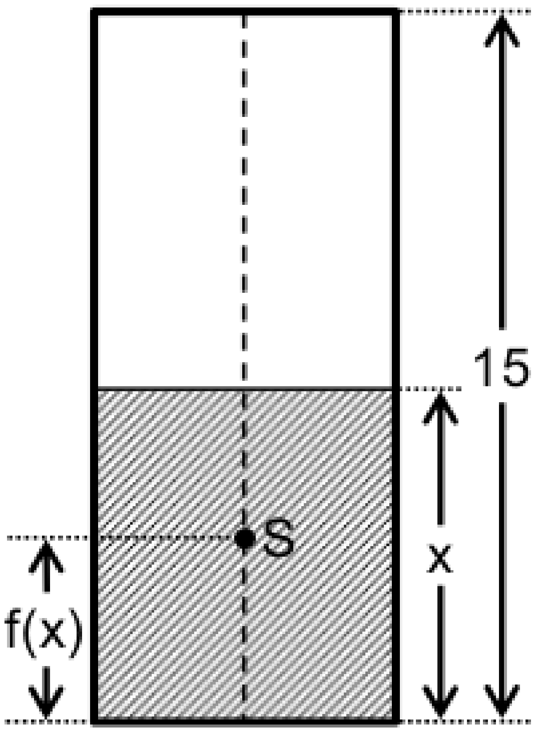


Abb. 3

A vertically standing beverage can is of the form of a straight cylinder. The position of the joint center of mass S of the can and the liquid contained in it depends on the filling height of the liquid above the bottom of the can. For a completely filled can, the filling height amounts to 15 cm.

The function f considered so far, yields for $0 \leq x \leq 15$ the height of S in centimetres with respect to the bottom of the can. Here, x is the filling height in centimetres (cf. figure 3).

1. Determine $f(0)$ und $f(15)$. Interprete the two results within the context.
2. The initially empty can is slowly filled with a liquid until the maximal filling height of 15 cm is reached. Use figure 2 to describe the motion of the center of mass S during the filling process. Within the context, explain the meaning of the fact that the coordinates x and y of the minimum of G_f agree.
3. For which filling heights x is the center of mass S situated at a height of at most 5 cm? First answer this question approximately with the help of figure 2 and then by means of a calculation.

Solution of part a

Inserting the given arguments into the function f, we find

$$f(0) = f(15) = \frac{15}{2}\,.$$

For a completely empty can ($x = 0$) the center of mass S agrees with the center of mass of the can. For a homogeneous distribution of weight, S is thus situated in the midpoint of the can at a height of $7.5\,\mathrm{cm}$. If the can is completely filled, the center of mass of the liquid corresponds to the center of mass of the empty can. Therefore, the joint center of mass agrees with the center of mass of the empty can.

Solution of part b

Filling the empty can with the liquid, the center of mass is first lowered. At a filling height of $3\,\mathrm{cm}$ the surface reaches the center of mass. Beyond this filling height, the center of mass rises again.

Solution of part c

From figure 2 we can estimate the region in which the center of mass is below 5 cm to $0.5 < x < 9.5$. The exact limits for x are obtained by solving the equation

$$f(x) = \frac{1}{2}x - \frac{1}{2} + \frac{8}{x+1} = 5$$

for x.

By means of Sage, we can calculate the exact values:

```
sage: result = solve(f(x) == 5, x)
sage: print "Solution of the equation " + str(f) + " = 5 :" +
  ↪ repr(result)
sage: print "This corresponds approximately to: x = %5.3f and x =
  ↪ %5.3f." % (result[0].right().n(), result[1].right().n())
```

The same result

$$x_{1,2} = 5 \pm 2\sqrt{5}$$

is obtained by solving the quadratic equation obtained from the above equation

$$x^2 - 10x + 5 = 0\,.$$

3.1.11 Extremum of the function

Problem

Given the function $f : x \mapsto \frac{x}{\ln(x)}$ in the domain $\mathbb{R}^+ \backslash \{1\}$, determine position and nature of the extrema of the graph of f.

Solution

We first employ Sage to obtain an overview of the given function.

```
sage: p = plot(x/ln(x), xmin=0, xmax=10, ymin=-2, ymax=5,
  ↪  color='blue', detect_poles='show')
sage: p.show(figsize=(4, 2.8))
```

The extrema of the function are found by setting the derivative equal to zero. The derivative is obtained as

$$f'(x) = \frac{\ln(x) - 1}{\ln(x)^2}$$

which can easily be verified by means of Sage:

```
sage: f(x) = x/ln(x)
sage: df(x) = f.derivative(x).log_simplify()
sage: print df(x)
```

Setting the derivative equal to zero yields the only extremum at $x_0 = e$ with $f(x_0) = e$.

```
sage: x0 = solve(df(x)==0, x)[0].lhs()
sage: print 'extremum at (', x0, '|', f(x0), ')'
```

In order to determine the nature of the extremum, we calculate the second derivative of the function $f(x)$

$$f''(x) = \frac{-\ln(x) + 2}{x \ln(x)^3}$$

and evaluate it at the extremum

$$f''(x_0) = \frac{1}{\mathrm{e}} > 0$$

This result can again be verified by means of Sage:

```
sage: ddf(x) = df.derivative(x).log_simplify()
sage: print ddf(x)
sage: print ddf(x0)
```

We conclude that the extremum is a minimum in agreement with the graph of the function obtained above.

3.1.12 Zeros and antiderivative

Problem

You are given the function f defined on $\mathbb{R}$ with $f(x) = e^x \cdot (2x + x^2)$.

1. Determine the zeros of the function f.
2. Show that the function F defined on $\mathbb{R}$ with $F(x) = x^2 \cdot e^x$ is an antiderivative of f. Find another antiderivative G of f for which $G(1) = 2e$.

Solution of part a

As the exponential function always yields positive values, the zeros of f are given by the zeros of $2x + x^2 = x(2 + x)$. By means of this factorization, the two zeros $x_1 = 0$ und $x_2 = -2$ can easily be read off.

The result can also be reproduced with the help of Sage:

```
sage: f(x) = exp(x)*(2*x+x^2)
sage: solve(f(x)==0,x)
```

Solution of part b

In order to demonstrate that F is an antiderivative of f, we need to prove that f is the derivative of F. By means of the product rule we find

$$\frac{\mathrm{d}F}{\mathrm{d}x} = 2x \cdot \mathrm{e}^x + x^2 \cdot \mathrm{e}^x = \mathrm{e}^x \cdot (2x + x^2) = f(x).$$

This result can be verified by Sage as follows:

```
sage: F(x) = x^2*exp(x)
sage: (F(x).derivative()-f(x)).simplify_rational()
```

The second antiderivative G can differ from the antiderivative F only by a constant. It therefore is of the form

$$G(x) = x^2 \cdot \mathrm{e}^x + c.$$

```
sage: G(x, c) = F(x)+c
sage: (G(x, c).derivative(x)-f(x)).simplify_rational()
```

We use the requirement $G(1) = 2e$ to determine the constant c, yielding $c = \mathrm{e}$:

```
sage: solve(G(1, c)==2*e, c)
```

3.1.13 Properties of sin(ax) +c

Problem

Given are the functions $g_{a,c} : x \mapsto \sin(ax) + c$ defined on $\mathbb{R}$ with $a, c \in \mathbb{R}_0^+$.

1. For each of the following two properties find a possible value for a and a possible value for c such that the corresponding function $g_{a,c}$ has this property.

1. The function $g_{a,c}$ has the codomain $[0; 2]$.
2. The function $g_{a,c}$ contains exactly three zeros in the interval $[0; \pi]$.

2. Determine as a function of a the possible values of the derivative of $g_{a,c}$.

Solution of part a

The parameter a determines the period of the sine but does not influence the function's codomain. The parameter c, on the other hand, shifts the function's codomain by a constant value.

1. The codomain of the sine function $\sin(ax)$ is given by $[-1; 1]$. A constant shift by $c = 1$ changes the codomain as specified in the requirement. For the parameter a an arbitrary nonzero value can be chosen, e.g. $a = 1$. We check our statement with the help of Sage:

```
sage: g(x, a, c) = sin(a*x)+c
sage: plot(g(x, 1, 1), (-2*pi, 2*pi), figsize=(4, 2.5))
```

2. The number of the function's zeros in the interval $[0; \pi]$ can be adjusted by means of the parameter a. It is necessary, however, that the codomain of the function includes zero. Choosing $c = 0$, half a period of the sine function with $a = 1$ fits into the given interval, thus leading to only two zeros. In contrast, choosing $a = 2$, a full period of the sine function with exactly three zeros lies within the interval. Again, we check our result with Sage:

```
sage: plot(g(x, 2, 0), (0, pi), figsize=(4, 2.5))
```

Solution of part b

First, we need to determine the derivative of the function $g_{a,c}(x)$. By means of the chain rule, we obtain

$$\frac{\mathrm{d}g_{a,c}(x)}{\mathrm{d}x} = a\cos(ax).$$

We remark that the derivative is independent of the parameter c. The cosine function possesses the codomain $[-1;1]$ but, in addition, is compressed or stretched because of the amplitude a. The codomain of the derivate thus results in $[-a;a]$.

In order to obtain a general expression for the derivative, in Sage we formally introduce the parameters a and c as additional variables of the function. Then, we plot the derivative $g'_{a,c}(x)$ for a few values of the parameter a. The different amplitudes and periods can easily be read off.

```
sage: dg(x, a, c) = g.derivative(x)
sage: print 'dg/dx =', dg(x, a, c)
sage: p1 = plot(dg(x, 0.5, 0), (-2*pi,2*pi), color='blue')
sage: p2 = plot(dg(x, 1, 0), (-2*pi,2*pi), color='red')
sage: p3 = plot(dg(x, 2, 0), (-2*pi,2*pi), color='green')
sage: (p1 + p2 + p3).show(figsize=(4, 2.5))
```

3.1.14 The construction of periodic functions

Problem

For each requirement, find a corresponding term of a periodic function defined in $\mathbb{R}$.

1. The graph of the function g is obtained from the graph of the function $x \mapsto \sin(x)$ defined in $\mathbb{R}$ by reflection at the y-axis.
2. The function h has the codomain $[1;3]$.
3. The function k possesses the period π.

Solution of part a

If a function g is to be obtained by reflection of the function f at the y-axis, we have $g(x) = f(-x)$. In our case, we obtain $g(x) = \sin(-x)$. In view of the point symmetry of the sine function with respect to the origin the function g is of the form $g(x) = -\sin(x)$.

We can easily check this result with the help of Sage by plotting in one graph the sine function (blue) and its reflection (red).

```
sage: f(x) = sin(x)
sage: g(x) = -sin(x)
sage: p1 = plot(f(x), x, (-2*pi,2*pi), color='blue')
sage: p2 = plot(g(x), x, (-2*pi,2*pi), color='red')
sage: (p1+p2).show(figsize=(4, 2.5))
```

Solution of part b

In this part, we again make use of the sine function. The codomain of the sine function with amplitude 1 is given by $[-1;1]$. In order to obtain the required codomain, we can shift the sine function together with its codomain by a constant of 2 towards the top and thus obtain $h(x) = \sin(x) + 2$.

We check the codomain of the function by means of Sage:

```
sage: h(x) = sin(x) + 2
sage: plot(h(x), x, (-2*pi,2*pi), figsize=(4, 2.5))
```

Solution of part c

The period of the sine function can be adjusted by means of a parameter a in front of the function's argument. Our function will thus be of the form $k(x) = \sin(ax)$. The period for $a = 1$ is 2π. The requested period is only half as long so that the parameter a needs to be doubled. We thus obtain $k(x) = \sin(2x)$.

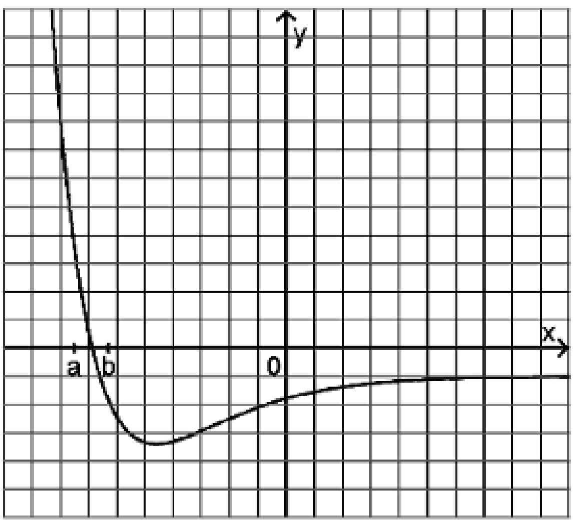
The period of the function can be checked by means of Sage. In the plot, one period is represented in red.

```
sage: k(x) = sin(2*x)
sage: p1 = plot(k(x), x, (-2*pi, 2*pi), color='blue')
sage: p2 = plot(k(x), x, (0, pi), color='red')
sage: (p1+p2).show(figsize=(4, 2.5))
```

3.1.15 Graphing the antiderivative

Problem

The figure depicts the graph of a function f.



1. Describe the shape of the graph of an antiderivative of f for $a \le x \le b$.
2. Sketch in the figure the graph of an antiderivative of f in the full range represented.

Solution of part a

We first sketch the antiderivative of the form

$$F(x) = \int_a^x f(x')\mathrm{d}x'.$$

For $x = a$, the antiderivative thus vanishes. For $x > a$, the antiderivative starts to grow initially more strongly, then more weakly until the zero $x = x_0$ of f is reached. Then, the antiderivative decreases, first slowly and then more strongly until $x = b$ is reached.

In order to treat this problem with the help of Sage, we first need to find a function which resembles the one displayed in the figure. This is more or less the case for the so called Lennard-Jones potential, a function which in physics describes the binding energy between certain atoms. A possible form of the function is

$$f(x) = \frac{a}{x^{12}} - \frac{b}{x^6},$$

where a and b are constants. In order to obtain a form close to the one given in this problem, we shift and compress the function accordingly. For the use with Sage, we employ the function

$$f(x) = \frac{1}{(x/3 + 1.5)^{12}} - \frac{1}{(x/3 + 1.5)^6} - \frac{1}{5}.$$

```
sage: f(x) = 1/(x/3+1.5)^12-1/(x/3+1.5)^6-0.2
sage: plot(f(x), x, (-2, 2), ymax=1, figsize=(4, 2.8))
```

To obtain the shape of the antiderivative in the range surrounding the zero of the function, we determine the definite integral of the function from a to x where x runs from $a = -1.8$ to $b = -1.4$.

```
sage: a = -1.8
sage: b = -1.4
sage: F(x) = integral(f(x), x)
sage: sf(x) = F(x) - F(a)
sage: plot(sf(x), (a, b), figsize=(4, 2.8))
```

Solution of part b

In this part of the problem we consider an antiderivative which vanishes at the point where the representation of the function in the figure begins, i.e., we consider the function

$$F(x) = \int_c^x f(x')\mathrm{d}x',$$

with c being the left-most point where the function is depicted. Like in the first part of the problem, our antiderivative is growing initially rather strongly. The growth then slows down and, at the zero of f, changes its sign. Then, the antiderivative decreases strictly monotonic and beyond a certain point remains negative. The strongest decrease of the antiderivative is found at the minimum of the function f. Afterwards, the antiderivative decreases more slowly and approaches a straight line with a negative slope when reaching the end of the range represented.

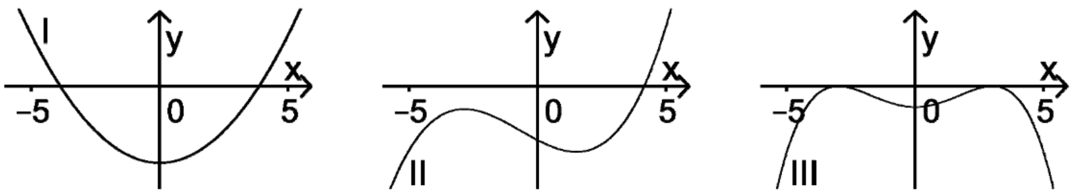
We proceed with Sage like in the previous part of the problem, simply adjusting the antiderivative and the range where it is represented.

```
sage: c = -2
sage: d = 2
sage: sf(x) = F(x) - F(c)
sage: plot(sf(x), (c, d), figsize=(4, 2.8))
```

3.1.16 Inflection points

Problem

The graph of a function $g : x \mapsto g(x)$ defined in $\mathbb{R}$ possesses two turning points for $-5 \leq x \leq 5$. Decide which of the graphs I, II, and III corresponds to the second derivative g'' of g. Justify your choice.



Solution

The second derivative of a function allows to deduce the curvature. For a function g possessing two turning points in the interval described here, the second derivative g'' must exhibit two zeros with changes of sign, i.e. transitions from a curvature to the left ($g'' > 0$) to a curvature to the right ($g'' < 0$) or vice versa. Only graph I fulfills this requirement.

With Sage, we will now try to construct possible forms of g based on the given second derivatives. The second derivatives have the form of polynomials of second (I), third (II), and fourth (III) order. In terms of their zeros, we can construct polynomials with similar behavior:

$$g_I''(x) = (x+4)(x-4) = x^2 - 16$$
$$g_{II}''(x) = (x+4)(x+1{,}5)(x-4) - 50$$
$$g_{III}''(x) = -(x+3)^2(x-3)^2$$

In the following function graphs, the zeros corresponding to turning points are marked by red points.

```
sage: def turning_points(f):
sage:     df = f.diff()
sage:     return [r[0] for r in f.roots() if r[0] in RR and df(r[0]) !=0]

sage: dd_i(x) = x^2 - 16
sage: p = plot(dd_i(x), x, (-6, 6), figsize=(4, 2.8))
sage: tps = point([(x, 0) for x in turning_points(dd_i)], size=30, color='red')
sage: show(p+tps)
```

```
sage: dd_ii(x) = (x+4) * (x+1.5) * (x-4) - 50
sage: p = plot(dd_ii(x), x, (-6, 6), figsize=(4, 2.8))
sage: tps = point([(x, 0) for x in turning_points(dd_ii)], size=30, color='red')
sage: show(p+tps)
```

```
sage: dd_iii(x) = -(x+3)^2 * (x-3)^2
sage: p = plot(dd_iii(x), x, (-6, 6), figsize=(4, 2.8))
sage: tps = point([(x, 0) for x in turning_points(dd_iii)], size=30, color='red')
sage: show(p+tps)
```

Finally, we confirm our considerations by integrating the functions twice. Red points in the graphs mark the turning points.

```
sage: d_i(x) = integrate(dd_i(x), x)
sage: g_i(x) = integrate(d_i(x), x)
sage: p = plot(g_i(x), x, (-8, 8), figsize=(4, 2.8))
sage: tps = point([(x, g_i(x))  for x in turning_points(dd_i)], size=30, color='red')
sage: show(p+tps)
```

```
sage: d_ii(x) = integrate(dd_ii(x), x)
sage: g_ii(x) = integrate(d_ii(x), x)
sage: p = plot(g_ii(x), x, (-6, 8), figsize=(4, 2.8))
sage: tps = point([(x, g_ii(x))  for x in turning_points(dd_ii)], size=30, color='red')
sage: show(p+tps)
```

```
sage: d_iii(x) = integrate(dd_iii(x), x)
sage: g_iii(x) = integrate(d_iii(x), x)
```

```
sage: p = plot(g_iii(x), x, (-6, 6), figsize=(4, 2.8))
sage: tps = point([(x, g_iii(x))  for x in turning_points(dd_iii)],
 ↪  size=30, color='red')
sage: show(p+tps)
```

3.1.17 Optimisation problem in analytical geometry

Problem

Consider all rectangles in a coordinate system (cf. figure 1) which fulfill the following conditions:

1. Two sides lie on the coordinate axes.
2. One vertex lies on the graph G_f of the function $f : x \mapsto -\ln x$ with $0 < x < 1$.

Figure 1 depicts such a rectangle.

Among the considered rectangles there exist one with the largest area. Calculate the side lengths of this rectangle.

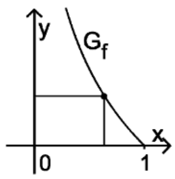


Abb. 1

Solution

The area of the considered rectangles is obtained as product of height and width. The width is given by the x-value of the vertex lying on the function's graph. The height corresponds to its y-value, i.e. $f(x)$. The area of a rectangle with a given value for x is therefore obtained as

$$A(x) = xf(x) = -x\ln(x).$$

In order to determine the the rectangle with the biggest area, we need to find the extremum of $A(x)$ which is obtained from

$$A'(x_0) = -\ln(x_0) - 1 = 0.$$

Because of $A''(x) = -1/x < 0$ this extremum is always a maximum. It lies at $x_0 = 1/\mathrm{e}$ with $f(x_0) = 1$. The corresponding rectangle therefore has a width of $1/\mathrm{e}$, a height of 1 and circumscribes an area $1/\mathrm{e}$.

We can determine the rectangle with maximum area also with the help of Sage by asking for a zero of the derivative of the area.

```
sage: f(x) = -ln(x)
sage: a(x) = x*f(x)
sage: da(x) = a.derivative(x).log_simplify()
sage: x0 = solve(da(x)==0, x)[0].rhs()
sage: print 'width :', x0
sage: print 'height:', f(x0)
sage: print 'area  :', a(x0)
```

The following two figures show the rectangle under the graph G_f on the left and the corresponding area as a green point in the right figure as a function of the variable width x_0.

```
sage: @interact
sage: def _(x0=slider(0.1, 1.)):
sage:     f(x) = -ln(x)
sage:     a(x) = -x*ln(x)
sage:     p1 = plot(f(x), x, (0.1, 1), color='blue')
sage:     p1 = p1+polygon([(0, 0), (0, f(x0)), (x0, f(x0)), (x0,
  ↪  0)], color = 'green')
sage:     p2 = plot(a(x), x, (0, 1), color='blue')
sage:     p2 = p2+point((x0, a(x0)), size=40, color='green')
sage:     G = graphics_array([p1, p2], nrows=1)
sage:     G.show(figsize=[7, 3])
```

3.1.18 Properties of the square root function

Problem 1

Given the function $f : x \mapsto 2 - \sqrt{12 - 2x}$ with a maximal domain $\mathbb{D}_f =]-\infty; 6]$, we denote the graph of f by G_f.

1. Calculate the coordinates of the intersection points of G_f and the coordinate axes. Determine the behavior of f in the limit $x \to -\infty$ and calculate $f(6)$.
2. Determine the derivative f' of f and indicate the maximal domain for f'. Determine $\lim\limits_{x\to 6} f'(x)$. Which property arises from this result?
3. Describe the monotonic behavior of G_f and state the codomain of f.
4. Determine $f(-2)$ and draw G_f in a coordinate system taking into account the previous results. (Space needed in view of the following tasks: $-3 \leq y \leq 7$).
5. The function f can be inverted in $\mathbb{D}_f$. Give the domain for the inverse function f^{-1} of f and prove that $f^{-1}(x) = -\frac{1}{2}x^2 + 2x + 4$.

Solution of part 1a

First, we make use of Sage to obtain an overview of the given function.

```
sage: f(x) = 2 - sqrt(12 - 2*x)
sage: p1 = plot(f(x), x, (-7,6), figsize=(4, 2.8))
sage: show(p1, gridlines=True)
```

The intersection of f and the y-axis is located at $(0|f(0))$ with

$$f(0) = 2 - 2\sqrt{3}.$$

This can be checked using Sage:

```
sage: print "intersection with the y-axis ", f(0), u"\u2248",
  ↪  f(0).n(digits=4)
```

The intersection of f and the x-axis is given by the zero of the function f:

$$f(x) = 2 - \sqrt{12 - 2x} \overset{!}{=} 0$$

Solving the equation for x leads to

$$x = 4,$$

which is confirmed by Sage:

```
sage: solve(f(x) == 0, x)
```

Thus, we obtain exactly one intersection of f and the x-axis located at $(4|0)$.

The two intersections with the coordinate axes are consistent with the graph of f obtained earlier by means of Sage.

For the behavior of $f(x)$ in the limit $x \to -\infty$, we find

$$\begin{aligned}\lim_{x\to-\infty} 2 - \sqrt{12 - 2x} &= \lim_{x\to\infty} 2 - \sqrt{12 + 2x}\\ &= -\lim_{x\to\infty} \sqrt{2x}\\ &= -\infty.\end{aligned}$$

This can be verified with Sage:

```
sage: limitval = limit(f(x), x=-infinity)
sage: html("$\lim_{x=-\infty} f(x) = %s$" % latex(limitval))
```

The value $f(6)$ can be determined as

$$f(6) = 2 - \sqrt{12 - 2 \cdot 6} = 2.$$

```
print "f(6) =", f(6),
```

This result can also be obtained from the graph of the function.

Solution of part 1b

The derivative of f can be calculated by using the chain rule:

$$f'(x) = -\frac{1}{2}(12-2x)^{-\frac{1}{2}} \cdot (-2) = \frac{1}{\sqrt{12-2x}}.$$

The same result is obtained by means of Sage:

```
sage: df = derivative(f,  x)
sage: print "The derivation of f is:", df
sage: p2 = plot(df(x), x, (-7,6), figsize=(4, 2.8))
sage: show(p2)
```

Compared to the domain of f, the domain of the derivative is obtained by taking out the zero of the denominator at $x_0 = 6$, i.e. $\mathbb{D}_{f'} =]-\infty; 6[$.

The limit $\lim_{x\to 6} f'(x)$ can be calculated as

$$\lim_{x\to 6} f'(x) = \lim_{x\to 0} \frac{1}{\sqrt{x}} = \infty,$$

This result can read off the graph of $f'(x)$ or determined by means of Sage:

```
sage: limitval = limit(df(x), x=6)
sage: html("$\lim_{x=6} f'(x) = %s$" % latex(limitval))
```

Solution of part 1c

In order to determine the monotonic behavior of f, we have to analyze its derivative f'. From the graph of f' we conclude that f' is positive in every point, which means that f is monotonically increasing in its entire domain.

A different argumentation is based on the function term of f': The square root in the denominater is a continuous and positive function in the domain of f'. Together with the constant 1 in the numerator, we obtain a strictly positive derivative f'.

According to part a, the codomain of f starts at $-\infty$. At the right end $x = 6$ of the domain, f takes on the value 2. Due to the fact that f is continuous and monotonically increasing in its domain, the codomain of f is found as

$$\mathbb{W}_f =]-\infty; 2]$$

Solution of part 1d

We obtain

$$f(-2) = -2.$$

```
sage: print f(-2)
```

The graph of f has already been drawn in part a.

Solution of part 1e

Domain and codomain of an inverse function f^{-1} are interchanged as compared to domain and codomain of the corresponding function f:

$$\mathbb{D}_{f^{-1}} = \mathbb{W}_f$$
$$\mathbb{W}_{f^{-1}} = \mathbb{D}_f.$$

In our case, we obtain the domain of definition

$$\mathbb{D}_{f^{-1}} =]-\infty; 2]$$

The inverse function of f can be calculated by solving the equation

$$f(y) = x$$

for y. We obtain

$$\begin{aligned} &2 - \sqrt{12 - 2y} = x \Leftrightarrow \\ &12 - 2y = (x-2)^2 \Leftrightarrow \\ &y = f^{-1}(x) = -\frac{1}{2}x^2 + 2x + 4, \end{aligned}$$

which can be verified with Sage:

```
sage: var('y')
sage: assume(y<2)
sage: solve(f(x) == y, x)
```

Problem 2

The graph of the function $h : x \mapsto -\frac{1}{2}x^2 + 2x + 4$ with the domain of definition $\mathbb{R}$ is the parabola G_h. The graph of the inverse function from part 1e is a part of G_h.

1. Calculate the coordinates of the intersection points of the curve G_h and the bisector of the first and third quadrant w given by the term $y = x$
2. Draw the parabola G_h – including the apex – on the interval $-2 \leq x \leq 4$ into your graph from part 1d. Taking the mirror image of this part of G_h with respect to the angle bisector w leads to a heart-shaped figure. Complete your graph accordingly.

Solution of part 2a

The intersection points of these two graphs can be calculated by equating their function terms:

$$-\frac{1}{2}x^2 + 2x + 4 = x$$
$$-\frac{1}{2}x^2 + x + 4 = 0$$
$$x_{1/2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot \left(-\frac{1}{2}\right) \cdot 4}}{2 \cdot \left(-\frac{1}{2}\right)}$$
$$x_1 = -2, \quad x_2 = 4$$

The y-coordinates of the intersection points are equal to their x-coordinates, due to the function term $y = x$ of the angle bisector:

$$y_1 = -2, \quad y_2 = 4.$$

Using Sage, we can solve the problem analytically

```
sage: h(x)=-1/2*x^2+2*x+4
sage: w(x) = x
sage: for solution in solve(h(x) == w(x), x, solution_dict=True):
sage:     print "(", solution[x], "|", solution[x], ")"
```

as well as graphically:

```
sage: p3 = plot(h(x), x, (-3, 6), fill=w, fillcolor = 'red')
sage: p4 = plot(w(x), x, (-3, 6), color='green')
sage: show(p3+p4, aspect_ratio=1, figsize=4)
```

Solution of part 2b

The upper half of the heart-shaped figure can already be recognized in the graph of part 2a. Its full form can be obtained by reflection. A reflection of a function at the angle bisector $y = x$ can be implemented by reversing this function. The inverse function of h is f, as we already know from part 1e. However, this inversion only holds within the domain $]-\infty; 2]$ (compare part 1e). For $x > 2$ we can use

$$f_2 = 2 + \sqrt{12 - 2x}$$

as inverse function of h. The only difference between f and f_2 is the sign in front of the square root. In order to obtain the heart-shaped figure by reflection, we will proceed as follows:

- We draw G_h between the two intersection points $x_1 = -2$ and $x_2 = 4$.
- Further, we draw the inverse function f of h from the intersection point (-2|-2) to the maximum of its domain at $x_{\max} = 6$, which represents the reflection of h at w, starting at the intersection point (-2|-2) and ending at the apex (2|6).

- Finally, we draw the inverse function f_2 of h from the intersection point (4|4) to the maximum of its domain $x_{\max} = 6$, which represents the reflection of h at w, starting at the apex (2|6) and ending at the intersection point (4|4).

```
sage: f2(x) =  2 + sqrt(12-2*x)
sage: p5 = plot(h(x), x, (-2, 4))
sage: p6 = plot(f(x), x, (-2, 6))
sage: p7 = plot(f2(x), x, (4, 6))
sage: show(p5+p6+p7, aspect_ratio=1, figsize=4)
```

Problem 3

The heart-shaped figure obtained in problem 2 represents a model for the leaf depicted in the figure below. One length unit in the coordinate system used in part 1d shall correspond to 1 cm in reality.

1. Calculate the area delimited by G_h and the angle bisector w. Use this result to determine the area of the leaf, based on our model.
2. Determine the term of the tangent to G_h at the point $(-2\,|h(-2))$. Calculate the angle between the two leaf edges at the leaf apex.
3. The current model does not describe well enough the shape of the upper leaf edge near the leaf apex. Therefore, we will use the graph G_k of a third order polynomial k in order to describe the upper leaf edge in the interval $-2 \leq x \leq 0$. The function k has to fulfill the following conditions (k' and h' are the derivatives of k and h):

$$\begin{aligned} \text{I} \qquad k(0) &= h(0) \\ \text{II} k'(0) &= h'(0) \\ \text{III} k(-2) &= h(-2) \\ \text{IV} k'(-2) &= h'(-2) \end{aligned}$$

   Explain, why the conditions I, II and III are reasonable. Make plausible that the condition IV leads to a better description of the leaf edge near the leaf apex as compared to the first model.

Solution of part 3a

First, we want to calculate the area of the red region from part 2a. This can be done by subtracting the integrals of the functions $h(x)$ and $w(x)$ in the interval $]-2;4[$:

$$\begin{aligned}\int_{-2}^{4} (h(x) - w(x))\ \mathrm{d}x &= \int_{-2}^{4} \left(-\frac{1}{2}x^2 + 2x + 4 - x\right) \mathrm{d}x \\ &= \int_{-2}^{4} \left(-\frac{1}{2}x^2 + x + 4\right) \mathrm{d}x \\ &= \left[-\frac{1}{6}x^3 + \frac{1}{2}x^2 + 4x\right]_{-2}^{4} \\ &= 18\end{aligned}$$

This result can be verified with Sage:

```
sage: print "The content of the red area is:", integrate(h(x)-w(x),
  ↪  x, -2, 4)
```

Having obtained the heart-shaped figure by reflection of the red region at the angle bisector w, the area enclosed by the heart-shaped figure is twice the red area. In view of the specified length scale, we obtain:

$$A_{\text{Leaf}} = 36\text{cm}^2$$

Solution of part 3b

In order to determine the term of the tangent at the point $(-2\,|h(-2)) = (-2| - 2)$, we first have to calculate the slope of the function h at the point -2. Using

$$h'(x) = -x + 2$$

we obtain

$$m = h'(-2) = 4.$$

Inserting the point $x = -2, y = -2$, the equation of the tangent $y = m \cdot x + t$ becomes

$$y = 4x + 6.$$

Using Sage, we can obtain this equation directly from the conditions that the tangent has to include the specified point and that the slope of the tangent has to equal the slope of the function $h(x)$ at this point.

```
sage: m, t = var('m t')
sage: y(x) = m*x+t
sage: dh = derivative(h, x)
sage: dy = derivative(y, x)
sage: solution = solve([y(-2)==h(-2),
...                     dy(-2)==dh(-2)], m, t, solution_dict=True)[0]
sage: y(x) = y.subs(solution)
sage: print 'Equation of the tangent: y = %sx+%s' % (solution[m],
  ↪ solution[t])
```

Further, we use Sage to draw the tangent into our figure.

```
sage: p8 = plot(h(x), x, (-3, 0))
sage: p9 = plot(y(x), x, (-3, 0), color='green')
sage: show(p8+p9, figsize=(4, 2.8))
```

The figure already indicates that the angle, based on our model, is considerably larger than the angle on the picture of the leaf. We can calculate the angle between the angle bisector and the tangent, based on their slopes $m_w = 1$ and $m_t = 4$ using the formula

$$\alpha = \arctan\left(\frac{m_t - m_w}{1 + m_w m_t}\right) = \arctan\left(\frac{4-1}{1+4}\right) \approx 30,9$$

The angle between the two edges is twice as large, i.e. approximately 62.

Solution of part 3c

The conditions I and III are necessary for a continuous insertion of G_k. Condition II ensures that the transition from h to k is smooth. Condition IV leads to a smaller angle between the leaf edges at the point (-2|-2) and therefore to a sharper leaf apex.

The problem did not ask for the exact solution of k, but Sage will work this out for us:

```
sage: a, b, c, d = var('a b c d')
sage: k(x) = a*x^3+b*x^2+c*x+d
sage: dk = derivative(k, x)
sage: equations = [k(0)==h(0),
...                dk(0)==dh(0),
...                k(-2)==h(-2),
...                dk(-2)==1.5]
sage: solutions = solve(equations, a, b, c, d,
  ↪ solution_dict=True)[0]
sage: k = k.subs(solutions)
sage: print k
```

We can use these parameters to plot the leaf according to the new model. The red curve is the new function k.

```
sage: p10 = plot(h(x), x, (0, 4))
sage: p11 = plot(k(x), x, (-2, 0), color='red')
sage: p12 = plot(f(x), x, (-2, 6))
sage: p13 = plot(f2(x), x, (4, 6))
sage: p14 = plot(h(x), x, (-2, 0), linestyle=':')
sage: show(p10+p11+p12+p13+p14, aspect_ratio=1, figsize=4)
```

Obviously, the new model fits the shape of the leaf better than the old model.

3.1.19 Rational function

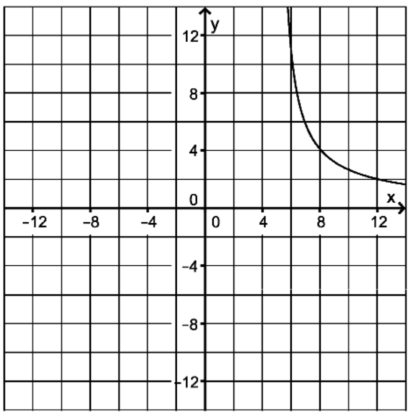
Problem 1

Problem 1

Given the function f with

$$f(x) = \frac{20x}{x^2 - 25}$$

and a maximal domain D_f, the figure shows part of its graph G_f.



1. Prove that $D_f = \mathbb{R}\backslash\{-5; 5\}$ and that G_f is symmetric relative to the point $(0|0)$. Specify the zeros of f and the equations of the three asymptotes of G_f.
2. Prove that the slope of G_f is negative in every point. Calculate the angle between G_f and the x-axis at their intersection point.
3. Draw the missing parts of the graph into the figure above, considering the results obtained so far.
4. The function $f^* : x \mapsto f(x)$ with the domain $]5; +\infty[$ differs from the function f only in terms of its domain. Explain, why the function f is not invertible, incontrast to the function f^*. Draw the graph of the inverted function f^* into the figure.
5. The graph of f, the x-axis and the lines given by the $x = 10$ and $x = s$ with $s > 10$ enclose an area with the content $A(s)$. Determine $A(s)$.

6. Determine s so that the content of the area from problem 1e equals 100.
7. Determine the behavior of $A(s)$ in the limit $s \to \infty$.

Solution of part 1a

First we will complete the graph of f with the help of Sage:

```
sage: f(x)=20*x/(x^2-25)
sage: plot(f(x), x, (-10, 10), exclude=(-5, 5), ymax=12, ymin=-12, figsize=4, aspect_ratio=1)
```

The graph already indicates that the points $x_1 = -5$ and $x_2 = 5$ are to be excluded from the domain, because they correspond to the poles of f. We can verify this by inspecting the factorized form of f

$$f(x) = \frac{20x}{(x-5)(x+5)},$$

and reading off the zeros of the denominator. The zero of the function is identical to the zero of the numerator, which is located at $x_0 = 0$. The symmetry with respect to the origin corresponds to the condition $f(-x) = -f(x)$, which is obviously fulfilled:

$$f(-x) = \frac{20(-x)}{(-x)^2 - 25} = -\frac{20x}{x^2 - 25} = -f(x)$$

The two asymptotes resulting from the poles are given by the equations $x = -5$ and $x = 5$. The function f vanishes in the limits $x \to -\infty$ and $x \to \infty$, because the order of the polynomial in the denominator exceeds the order of the polynomial in the numerator. Therefore, the third asymptote corresponds to the x-axis, given by the equation $y = 0$.

We draw the asymptotes into the graph of f.

```
sage: g(x) = 0
sage: p1 = plot(f(x), x, (-20, 20), exclude=(-5, 5), ymax=12, ymin=-12)
sage: p2 = line([[-5, -13], [-5, 13]], color = 'red')
sage: p3 = line([[5, -13], [5, 13]], color = 'red')
sage: p4 = plot(g(x), x, (-20, 20), aspect_ratio=1, color = 'red')
sage: show(p1+p2+p3+p4, aspect_ratio=1, figsize=4)
```

Solution of part 1b

First, we calculate the derivative of f using the quotient rule:

$$f'(x) = \frac{(x^2-25)\cdot 20 - 20x \cdot 2x}{(x^2-25)^2} = \frac{-20x^2 - 500}{(x^2-25)^2}.$$

As we can see, the numerator is always negative while the denominator is always greater or equal zero. Therefore, the derivative $f'(x)$ is – except for the zeros of the denominator, which are not included in the domain – always negative.

The graph of $f'(x)$ confirms this result.

```
sage: df = derivative(f, x)
sage: p5 = plot(df(x), x, (-20, 20), ymax=12, ymin=-12)
sage: show(p5, aspect_ratio=1, figsize=4)
```

In order to determine the angle between G_f and the x-axis at their intersection point, we calculate the slope of f at the point $x = 0$:

$$f'(0) = \frac{-500}{(-25)^2} = -\frac{4}{5}$$

The angle is then given by

$$\alpha = \arctan\left(-\frac{4}{5}\right) = -38,7.$$

We check the result with Sage and draw the tangent of f at the point $x = 0$.

```
sage: m = df(0)
sage: print u"\u03b1 =", RDF(180/pi*arctan(m))
sage: w(x) = m*x
sage: p6 = plot(f(x), x, (-4, 4), ymax=5, ymin=-5)
sage: p7 = plot(w(x), x, (-4, 4), color='green')
sage: show(p6+p7, aspect_ratio=1, figsize=4)
```

Solution of part 1c

The graph was already completed in part 1a.

Solution of part 1d

The function f assigns multiple x-values to the same y-values and is therefore not injective on its complete domain. Thus, the function f is not invertible.

f^* on the other hand is injective on its complete domain and is therefore invertible. The reversed function f^{*-1} can be calculated by solving $y = f^*(x)$ for x. We obtain the quadratic equation

$$x^2y - 20x - 25y = 0$$

for x, which is solved by

$$x_{1/2} = \frac{5\left(2 \pm \sqrt{y^2+4}\right)}{y}.$$

These solutions can also be obtained with Sage:

```
sage: y = var('y')
sage: solve(f(x)==y, x)
```

Due to the domain of f^*, f^{*-1} can only take values ≥ 5. Thus, we have to choose the plus sign in the solution given above. The inverse function then reads

$$f^{*-1} = \frac{5\left(2+\sqrt{x^2+4}\right)}{x}.$$

We will use Sage to draw f^* and f^{*-1}.

```
sage: f_inv(x) = 5*(sqrt(x^2 + 4) + 2)/x
sage: p8 = plot(f(x), x, (5, 20))
sage: p9 = plot(f_inv(x), x, (0, 20))
sage: show(p8+p9, aspect_ratio=1, ymax=20, figsize=4)
```

Solution of part 1e

Due to the fact that f is always positive for $x > 10$, the described area can be calculated as the integral of the function f from 10 to s:

$$A(s) = \int_{10}^{s} f(x)\mathrm{d}x.$$

We note that the numerator of f corresponds to the derivative of the denominator up to a constant factor. Therefore, the antiderivative can be expressed in terms of a logarithm:

$$A(s) = \int_{10}^{s} \frac{20x}{x^2-25}\mathrm{d}x = 10\int_{10}^{s} \frac{2x}{x^2-25}\mathrm{d}x = 10\,\ln(x^2-25)\big|_{10}^{s} = 10\ln\left(\frac{s^2-25}{75}\right)$$

Sage confirms this solution:

```
sage: from sage.symbolic.integration.integral import
  ↪ definite_integral
sage: s = var('s')
sage: assume(s > 10)
sage: A(s) = definite_integral(f(x), x, 10, s)
sage: print "The area is given by A(s) =", A(s)
```

In the following figure, the area is highlighted in yellow. The value of s can be changed dynamically and the content of the area is calculated and presented under the figure.

```
sage: @interact
sage: def _(s=slider(10.1, 19.9, 0.1)):
...        p10 = plot(f(x), x, (5, 10))
...        p11 = plot(f(x), x, (10, s), fill = 0, fillcolor='yellow')
...        p12 = plot(f(x), x, (s, 20))
...        show(p10+p11+p12, aspect_ratio=1, ymax=10, figsize=4)
...        print "Area of the yellow region:", float(A(s))
```

Solution of part 1f

The area with the content 100 can be calculated by solving the equation

$$A(s) = 100$$

with Sage. At this point, it is important that we had specified $s > 10$ in the code above. The requested value of s is found as

```
sage: print float(solve(A(s) == 100, s)[0].right())
```

Solution of part 1g

In the limit $x \to \infty$ the logarithm goes to infinity. Therefore, the content of the area $A(s)$ also goes to infinity in this case.

We verify this result with Sage:

```
sage: html("$\lim_{s=\infty} A(s) = %s$" % latex(A(infinity)))
```

Problem 2

Problem 2

A motorboat cruises with constant motor power along a river. First, the boat travels a distance of 10 km down the river before returning the same way up the river. The proper speed denotes the speed with which the boat would travel on water at rest.

In the following, it shall be assumed that the proper speed of the boat be constant and the water of the river flows at a constant speed of $5\frac{\mathrm{km}}{\mathrm{h}}$. The time needed for the transposition maneuver shall be neglected.

The total time (in hours) for the boat's roundtrip, within the model, is given by the term

$$t(x) = \frac{10}{x+5} + \frac{10}{x-5},$$

for $x > 5$. x denotes the proper speed of the boat in units of $\frac{\mathrm{km}}{\mathrm{h}}$.

1. Based on the model, determine the total time in minutes for a journey with the proper speeds $10\,\frac{\mathrm{km}}{\mathrm{h}}$ and $20\,\frac{\mathrm{km}}{\mathrm{h}}$.
2. Explain, why the first and second summand of the term $t(x)$ describe the time needed to travel down and up the river, respectively.
3. Explain, why the term $t(x)$ cannot be used to calculate the total time when $0 < x < 5$.
4. Demonstrate that the terms $f(x)$ and $t(x)$ are equivalent.

5. Describe how one can use the graph in order to get an approximate value for the proper speed of the boat based on the total time taken between 2 and 14 hours. Based on the model, calculate the proper speed of the boat if four hours are needed for the whole journey.

Solution of part 2a

In order to calculate the total time of travel, we simply have to insert the values $x = 10$ and $x = 20$ in $t(x)$ and multiply the result with 60 to obtain the time in minutes.

We find a total time of

$$60t(10) = 60\left(\frac{10}{15} + \frac{10}{5}\right) = 160\,\mathrm{min}$$

for a proper speed of $10\frac{\mathrm{km}}{\mathrm{h}}$ and

$$60t(20) = 60\left(\frac{10}{25} + \frac{10}{15}\right) = 64\,\mathrm{min}.$$

for a proper speed of $20\frac{\mathrm{km}}{\mathrm{h}}$. We can verify these results with Sage. Furthermore, we use Sage to plot the total time with respect to the proper speed in the interval between $10\frac{\mathrm{km}}{\mathrm{h}}$ and $20\frac{\mathrm{km}}{\mathrm{h}}$.

```
sage: t(x) = 10/(x+5)+10/(x-5)
sage: print "Total time for 10 km/h:", 60*t(10), "minutes"
sage: print "Total time for 20 km/h:", 60*t(20), "minutes"
sage: p1 = plot(60*t(x), x, (10, 20))
sage: show(p1, figsize=(4, 2.8))
```

Solution of part 2b

The time needed to travel a distance s with constant speed x can be expressed as $\frac{s}{x}$. However, the boat travels with its proper speed relative to the surrounding water. Depending on the boat's direction of travel, its speed relative to the shore is obtained by adding or subtracting the speed of the water to or from the boat's proper speed. For a distance $s = 10\,\mathrm{km}$ and a flow velocity of $5\frac{\mathrm{km}}{\mathrm{h}}$, the two summands of the term $t(x)$ correspond to the time needed to travel down and up the river.

Solution of part 2c

For $0 < x < 5$, we can use the first part of the term $t(x)$ to calculate the time needed for travelling down the river. However, the second term cannot be used to calculate the time needed for the return voyage because the water's flow velocity exceeds the speed of the boat. Therefore, the boat will not be able to reach the point of departure. $t(x)$ should therefore yield an infinitely large value.

However, the graph shows that the function yields negative values on the interval $0 < x < 5$, which obviously precludes an interpretation in terms of a total time.

```
sage: p13 = plot(60*t(x), x, (0, 30), exclude=(5,), ymax=800,
  ↪ ymin=-800)
sage: show(p13, figsize=(4, 2.8))
```

Solution of part 2d

Finding the common denominator, we add up the two terms and find

$$\begin{aligned} t(x) &= \frac{10}{x+5} + \frac{10}{x-5} \\ &= \frac{10(x-5)+10(x+5))}{(x+5)(x-5)} \\ &= \frac{20x}{x^2-25} \\ &= f(x) \end{aligned}$$

With Sage, we can verify this result by calculating the difference of the two functions. However, the method `rational_simplify()` is essential for the program to work.

```
sage: print t(x)-f(x)
sage: print (t(x)-f(x)).rational_simplify()
```

Solution of part 2e

The proper speed can be read off the graph by searching the point of the curve where the y-value matches the specified total time. The corresponding x-value represents the proper speed of the boat. For a total time of 4 hours one finds a proper speed of approximately $8\frac{\mathrm{km}}{\mathrm{h}}$. The exact value can be determined by means of the inverse function f^{*-1} defined in part 1d. We obtain

$$f^{*-1}(4) = 5\frac{1+\sqrt{5}}{2}.$$

Sage confirms this result:

```
sage: totaltime = 4
sage: print "Proper speed for a total time of {}h: {}km/h".format(
...          totaltime, f_inv(totaltime).n(10))
```

The graphic construction can be visualized with Sage as follows:

```
sage: p14 = plot(f(x), x, (5.1, 14))
sage: x4, y4 = f_inv(totaltime), totaltime
sage: l1 = line([(x4, y4), (0, y4)], color='red')
sage: l2 = line([(x4, y4), (x4, 0)], color='red')
sage: show(p14+l1+l2, aspect_ratio=1, xmin=0, ymin=0, ymax=14,
  ↪ figsize=4)
```

3.1.20 Nonlinear equation

Problem

Determine the solutions of $(4x-3)\cdot\ln\left(x^2-5x+7\right)=0$ for $x\in\mathbb{R}$.

Solution

The zeros of the function on the left-hand side are determined by the zeros of the two factors.

Solving the requirement $4x-3=0$ for x immediately yields the first zero $x_1=3/4$.

The second factor vanishes provided the argument of the logarithm equals one. One thus needs to find solutions of $x^2-5x+7=1$. The solutions of the resulting quadratic equation $x^2-5x+6=0$ are obtained by means of

$$x_{2,3}=\frac{5\pm\sqrt{25-24}}{2}.$$

We thus obtain two more zeros $x_2=2$ and $x_3=3$.

These results can easily be checked with Sage:

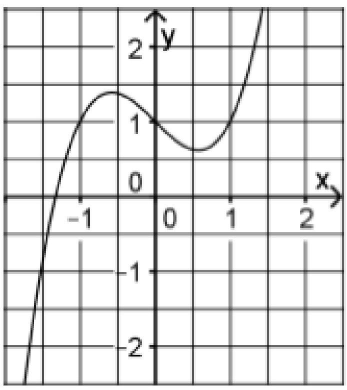
```
sage: solve((4*x-3)*ln(x^2-5*x+7) == 0, x)
```

3.1.21 Properties of graphs of functions

Problem

Given are the functions f, g and h defined on $\mathbb{R}$ by $f(x)=x^2-x+1$, $g(x)=x^3-x+1$, and $h(x)=x^4+x^2+1$.

1. The figure depicts the graph of one of the three functions. Indicate which functions is represented by the graph. Argue why the graph cannot represent the other two functions.



2. The first derivative of the function h is given by h'. Evalute $\int_0^1 h'(x)\mathrm{d}x$.

Solution of part a

The graph displays two extrema. Therefore, it cannot represent the function $f(x)$ because the derivative of a quadratic function possesses only one zero. Furthermore, the function displayes in the figure takes on negative values, excluding the function $h(x)$ aus. As a consequence the graph represents the function $g(x)$. We check our conjecture with the help of Sage:

```
sage: ranges = {'xmin': -2, 'xmax': 2.5, 'ymin': -2.5, 'ymax': 2.5}
sage: p = sum([plot(x^2-x+1, color='blue', **ranges),
...            plot(x^3-x+1, color='red', **ranges),
...            plot(x^4+x^2+1, color='green', **ranges)])
sage: p.show(figsize=(2.7, 3))
```

The graph of the function $g(x)$ shown in red indeed fits the graph in the original figure.

Solution of part b

The antiderivative of the derivative of a function is the function itself. It follows

$$\int_0^1 h'(x)\mathrm{d}x = h(1) - h(0) = 3 - 1 = 2.$$

In Sage, we start by differentiating the function $h(x)$ as a check and continue by evaluating the required definite integral:

```
sage: h(x) = x^4+x^2+1
sage: dh(x) = diff(h, x)
sage: print 'Derivative of h(x):', dh
sage: print 'Value of the definite integral:', integrate(dh(x), x,
  ↪ 0, 1)
```

Of course, according to our reasoning above, we obtain the same result by subtracting the function taken at the limits of integration:

```
sage: h(x) = x^4+x^2+1
sage: h(1)-h(0)
```

3.1.22 Properties of sin(2x)

Problem

We are given the function $f : x \mapsto \sin(2x)$ defined on $\mathbb{R}$. What are the amplitude, period, and its range.

Solution

The sine function has amplitude 1, period 2π and the range $[-1, 1]$. The given function f has the same amplitude and range. Due the factor of 2 in the argument, its period is only π.

We can check these statements by means of Sage:

```
sage: plot(sin(2*x), (0, 2*pi), figsize=(4, 2.5))
```

3.1.23 Construction of the function given its properties

Problem

For each of the following sets of requirements name a function satisfying them.

1. The function g has a maximum domain given by $]-\infty; 5]$.
2. The function k has a zero at $x = 2$ as well as pole at $x = -3$ without changing its sign. The graph of k has an asymptote the straight line given by $y = 1$ als Asymptote.

Solution of part a

A function with domain $[0; \infty[$ is given by $x \mapsto \sqrt{x}$. Therefore, $g(x) = \sqrt{5-x}$ is one of the functions with the given domain.

Solution of part b

The function $k(x)$ can be chosen as rational function. Beacuase of the zero at $x = 2$ the numerator must contain at least a factor $x - 2$. The pole at $x = -3$ without change of sign is obtained by means of a factor $(x+3)^2$ in the denominator. In order to obtain the desired asymptotic behavior for $|x| \to \infty$, the factor in the numerator must be squares. We thus arrive at

$$k(x) = \frac{(x-2)^2}{(x+3)^2}.$$

We demonstrate with the help of Sage that this function indeed has the required properties.

Zero at $x = 2$:

```
sage: k(x) = ((x-2)/(x+3))^2
sage: plot(k, xmin=0, xmax=4, ymin=-0.1, ymax=0.5, figsize=(4, 2.5))
```

Pole at $x = -3$ without change of sign:

```
sage: k(x) = ((x-2)/(x+3))^2
sage: plot(k, xmin=-6, xmax=0, ymin=0, ymax=1000, figsize=(4, 2.5))
```

Asymptotic approach to the straight line $y = 1$ for $|x| \to 1$:

```
sage: k(x) = ((x-2)/(x+3))^2
sage: xmax = 1000
sage: xmin = -xmax
sage: p = plot(k, xmin=xmin, xmax=xmax, ymin=0.5, ymax=1.5)
sage: p = p+line([(xmin, 1), (xmax, 1)], linestyle='dashed')
sage: p.show(figsize=(4, 2.5))
```

3.1.24 Determine parameter when the derivative vanishes

Problem

We are given the set of functions $f_a : x \mapsto x\mathrm{e}^{ax}$ defined on $\mathbb{R}$ with $a \in \mathbb{R}\backslash\{0\}$. Determine the value of a for which the derivative of f_a at $x = 2$ vanishes.

Solution

The derivative of the given function is obtained as

$$\frac{\mathrm{d}f_a}{\mathrm{d}x} = (1 + ax)\mathrm{e}^{ax}$$

so that

$$\left.\frac{\mathrm{d}f_a}{\mathrm{d}x}\right|_{x=2} = (1 + 2a)\mathrm{e}^{2a}.$$

As a consquence, the derivate vanishes provided $1 + 2a = 0$, i.e. for $a = -1/2$.

This calculation can be checked by means of Sage:

```
sage: f(x, a) = x*exp(a*x)
sage: df = f.derivative(x)
sage: print 'Derivative of f:', df
sage: solve(df(2, a) == 0, a)
```

3.1.25 Domain and values of the lograrithm

Problem

The function $f : x \mapsto \sqrt{1 - \ln x}$ with maximal domain $\mathbb{D}$ is given.

1. Determine $\mathbb{D}$.
2. Determine the value $x \in \mathbb{D}$ for which $f(x) = 2$.

Solution of part a

The logarithm is only defined for arguments $x > 0$, and the square root only for arguments $x \geq 0$. For values $x > e$, the logarithm of x yields values bigger than 1 and thus the argument of the square root would be smaller than 0. On the other hand, in the range $0 < x \leq e$, the logarithm yields values smaller or equal to 1 such that the argument of the square root becomes bigger or equal to zero. The maximal domain is hence given by

$$\mathbb{D} =]0; e]\,.$$

We plot the function with Sage.

```
sage: f(x) = sqrt(1-ln(x))
sage: plot(f(x), (0, e), figsize=(4, 2.8))
```

Solution of part b

To obtain the corresponding value for x, we solve the equation for x:

$$\sqrt{1-\ln x} = 2 \Rightarrow \ln(x) = -3 \Rightarrow x = e^{-3}$$

The result is confirmed by Sage:

```
sage: solve(f(x)==2, x)
```

3.1.26 Intergral of a symmetric function

Problem

Demonstrate that the graph of the function $g : x \mapsto x^2 \cdot \sin x$ defined over $\mathbb{R}$ is point symmetric with respect to the coordinate origin, and obtain the value of the integral

$$\int_{-\pi}^{\pi} x^2 \cdot \sin x \, \mathrm{d}x\,.$$

Solution

First we show that $f(x) = -f(-x)$ holds:

$$f(-x) = (-x)^2 \cdot \sin(-x) = x^2 \cdot (-\sin(x)) = -f(x)$$

This can also be checked with Sage

```
sage: f(x) = x^2*sin(x)
sage: if f(-x) == -f(x):
...       print "The function is odd."
sage: else:
...       print "The function is not odd."
```

and is confirmed by the shape of the graph of the function:

```
sage: plot(f(x), (-pi, pi), figsize=(4, 2.8))
```

If the boundaries of an integral over an odd function are symmetric with respect to zero, as is the case here, the integral vanishes.

Sage can confirm that:

```
sage: print "Value of the integral:", integral(f, x, -pi, pi)
```

3.1.27 Qubic function

Problem

A polynomial function of degree three defined over $\mathbb{R}$ is given. Its graph G_f has a maximum at $x = 1$ and a minimum at $x = 4$.

1. Establish that the graph of the derivative f' of f is a parabola which intersects the x-axis at the points (1|0) and (4|0) and opens to the top.
2. Establish that 2.5 is the x-coordinate of the point of inflection of G_f.

Solution of part a

In general, the derivative of a cubic function is a parabola, and extrema of the cubic function correspond to roots of the parabola. Since there is a maximum at $x = 1$, the second derivative of the function has to be negative at that point, while at the minimum at $x = 4$ the second derivative has to be positive. Thus, the graph of f' is a parabola opening to the top.

To illustrate a concrete example for the function f, we choose the derivative

$$f'(x) = (x - 1)(x - 4)$$

which evidently has the correct roots. We now plot the function f, as well as its first and second derivate, with Sage and can thereby check the properties of the function f just discussed.

```
sage: df(x) = (x-1)*(x-4)
sage: ddf(x) = derivative(df, x)
sage: f(x) = integral(df, x)
sage: p1 = plot(f(x), (-3, 6), color='red', legend_label="$f(x)$")
sage: p2 = plot(df(x), (-3, 6), color='green', legend_label="$f'(x)$")
sage: p3 = plot(ddf(x), (-3, 6), color='blue', legend_label="$f''(x)$")
sage: show(p1+p2+p3, figsize=(4, 2.8), ymin=-10, ymax=10)
```

Solution of part b

The point of inflection x_0 of a graph of the function f has to fulfill the condition:

$$f''(x_0) = 0\,.$$

The derivative of f hence has to have an extremum. For a parabola, this can only be the case at the vertex. For reasons of symmetry, this point is always in the center between two points with the same function value. The point of inflection of f hence has to be centered between the two roots:

$$x_0 = \frac{1+4}{2} = 2.5\,.$$

The point of inflection of f can be already determined from the plot of part a). However, we can also calculate it with the help of Sage:

```
sage: print "Point of inflection at:", solve(ddf(x) == 0, x)[0]
```

3.1.28 Properties of the function

Problem

The function $f : x \mapsto \frac{\ln x}{x^2}$ with maximal domain $\mathbb{D}$ is given.

1. Give $\mathbb{D}$ as well as the roots of f, and determine $\lim\limits_{x\to 0} f(x)$.
2. Determine the x-coordinate of the point in which the graph of f has a horizonal tangent line.

Solution of part a

The logarithm is defined for arguments $x > 0$ only. The denominator x^2 contributes a gap in the domain at $x = 0$. The maximal domain is hence given by

$$\mathbb{D} =]0; \infty[\,.$$

We obtain the roots from the roots of the numerator:

$$\ln(x) = 0 \Rightarrow x = 1 \, .$$

We plot the function with Sage.

```
sage: f(x) = ln(x)/x**2
sage: plot(f(x), (0, 4), ymin=-2, figsize=(4, 2.8))
```

We can also verify the root with Sage.

```
sage: solve(f(x)==0, x)
```

As the graph produced by Sage suggests, the function goes to $-\infty$ as $x \to 0$. This can also be established by the fact that on the one hand the enumerator goes to $-\infty$ and on the other hand the denominator of the function goes to 0^+.

Solution of part b

A horizontal tangent line corresponds to an extremum of the function. To identify such a point, we have to determine the derivative first and, subsequently, set it equal to 0:

$$f'(x) = \frac{1 - 2\ln(x)}{x^3} \stackrel{!}{=} 0 \Rightarrow \ln x = \frac{1}{2}$$

This yields a horizontal tangent line at

$$x = e^{\frac{1}{2}} = \sqrt{e}$$

which we add to the sketch of the function:

```
sage: df = derivative(f, x)
sage: x0 = solve(df(x) == 0, x)[0].right()
sage: print "Horizontal tangent line at", x0, "=", float(x0)
sage: p1 = plot(f(x), (0, 4), x, ymin=0,  exclude=[0])
sage: p2 = plot(f(x0), (0, 4), x, color='red')
sage: show(p1+p2, figsize=(4, 2.8))
```

3.1.29 The construction of functions given set of properties

Problem

Give the term and the domain of a function which fulfills the given property/properties.

1. The point (2|0) is a point of inflection of the graph of g.
2. The graph of the function h is strictly monotonically decreasing and concave.

Solution of part a

A function with the desired point of inflection can be obtained from the following requirements:

$$f(2) = 0$$
$$f''(2) = 0$$

Furthermore, we require that $f'''(2) \neq 0$ and choose in particular $f'''(2) = 1$. Taking these requirements into account, the integration yields

$$f(x) = \frac{1}{6}(x-2)^3 + c(x-2)\,.$$

A plot made by Sage can confirm the point of inflection:

```
sage: g(x, c) = (x-2)^3/6+c*(x-2)
sage: p1 = plot(g(x, 0), (0, 4), color='red',
    legend_label="$f'(2)=0$")
sage: p2 = plot(g(x, 1), (0, 4), color='green',
    legend_label="$f'(2)=1$")
sage: p3 = plot(g(x, -1), (0, 4), color='blue',
    legend_label="$f'(2)=-1$")
sage: show(p1+p2+p3, figsize=(4, 2.8))
```

Solution of part b

The simplest solution for a strictly monotonically increasing and convex function is the exponential function. With a negative sign, it is turned into a strictly monotonically decreasing, concave function as can be confirmed by calculation:

$$h(x) = -e^x$$
$$h'(x) = -e^x < 0$$
$$h''(x) = -e^x < 0$$

A drawing by Sage confirms this:

```
sage: h(x) = -exp(x)
sage: plot(h(x), (-2, 2), figsize=(4, 2.8))
```

3.1.30 Graphical integration

Problem

Figure 1 depicts the graph of a function f defined over $\mathbb{R}$.

1. Determine an approximate value of $\int\limits_3^5 f(x)\mathrm{d}x$ with the help of figure 1.

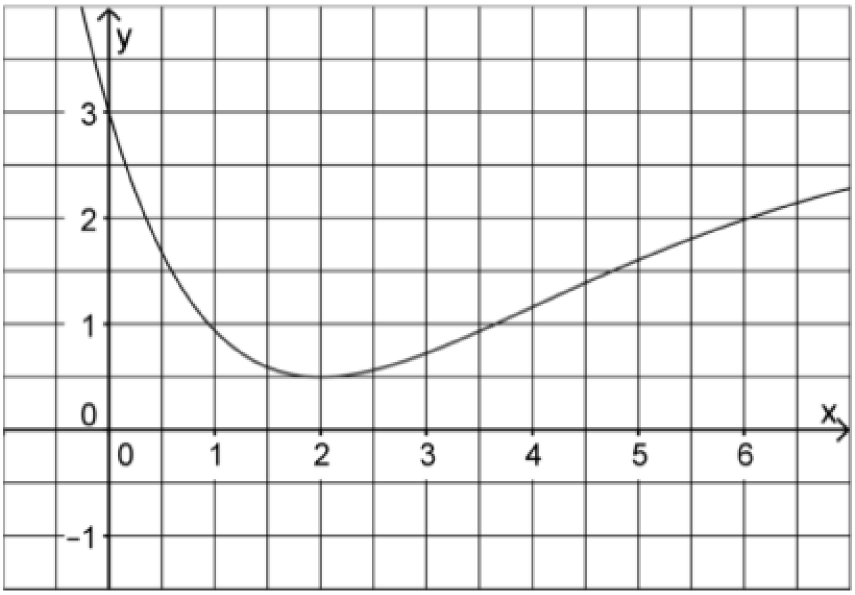


Fig. 3.1: Abb. 1

The function F is the antiderivative of f defined over $\mathbb{R}$ with $F(3) = 0$.

2. Give an approximate value of the derivative of F at $x = 2$ with the help of figure 1.

3. Show that $F(b) = \int_3^b f(x)\mathrm{d}x$ with $b \in \mathbb{R}$ holds.

Solution of part a

The integral corresponds to the area under the curve between the x-values 3 and 5. Thanks to the drawn grid, we can estimate the area to be about 9 squares. One square has the height and width of 0.5, respectively. The area of a square thus corresponds to 0.25, and the value of the integral is hence about 2.25.

Because the function of the graph is not explictly given, it is not easy to verify this result with Sage. However, we can try to approximately reproduce the graph with an interpolation. To this end, we choose a polynomial of degree 4 of the form

$$ax^4 + bx^3 + cx^2 + dx + e\,.$$

To fix the parameters a to e, we have to choose five points characterizing the graph. For this purpose, we choose the points $(0|3)$, $(1|1)$, $(2|0.5)$, $(4|1.2)$ and $(6|2)$.

With the help of Sage, we can solve the corresponding linear system:

```
sage: var('a, b, c, d, e')
sage: f(x) = a*x**4 + b*x**3 + c*x**2 + d*x + e
sage: equations = [f(0) == 3, f(1) == 1, f(2) == 0.5, f(4) == 1.2,
  ↪ f(6) == 2]
sage: solution = solve(equations, a, b, c, d, e,
  ↪ solution_dict=True)[0]
```

```
sage: f(x) = f(x).substitute(solution)
sage: print f(x)
```

In the interval $[0,6]$, the graph of the function agrees with the given graph quite well:

```
sage: p1 = plot(f(x), (0, 6), ymin=0, figsize=(4, 2.8))
sage: p1
```

The value of the integral can be approximately reproduced as well:

```
sage: print "Value of the integral:", float(integrate(f(x), x, 3,
   5))
```

Solution of part b

The derivative of the function F is the original function f. Hence, we just have to read off the value of f at $x = 2$ which we have already done in part a):

$$F'(2) = f(2) = 0.5$$

The corresponding tangent line is depicted as a green line in the following part.

Solution of part c

Since F is an antiderivative of f, we have

$$\int_3^b f(x)\mathrm{d}x = F(b) - F(3)\,.$$

The statement to be proven follows from the supposition $F(3) = 0$.

We add $F(x)$ to the plot of f:

```
sage: F(x) = integral(f(x), x)
sage: F_3(x) = F(x) - F(3)
sage: p2 = plot(F_3(x), (0, 6), ymin=-2, color = 'red')
sage: p3 = plot(F_3(2)+0.5*(x-2), (0, 6), color='green')
sage: show(p1+p2+p3, figsize=(4, 2.8))
```

3.2 Stochastics

3.2.1 Blood types statistics

Problem

The following table depicts the distribution of blood types and rhesus factors amongst the german population:

	0	A	B	AB
Rh+	35%	37%	9%	4%
Rh-	6%	6%	2%	1%

During a morning, 25 people donate blood in a hospital. In the following it shall be assumed that those 25 people represent a random sample of the population.

1. Determine the probability that exactly ten of the donors have blood type A.
2. Find the probability that more than half of the donors have blood type A and rhesus factor Rh+.

The following table shows which donor blood is suitable for the different recipients:

		Donor							
		0 Rh-	0 Rh+	A Rh-	A Rh+	B Rh-	B Rh+	AB Rh-	AB Rh+
Recipient	AB Rh+	✓	✓	✓	✓	✓	✓	✓	✓
	AB Rh-	✓		✓		✓		✓	
	B Rh+	✓	✓			✓	✓		
	B Rh-	✓				✓			
	A Rh+	✓	✓	✓	✓				
	A Rh-	✓		✓					
	0 Rh+	✓	✓						
	0 Rh-	✓							

3. A patient with blood type B and rhesus factor Rh- needs donor blood. Determine how many randomly selected people have to donate blood in order to obtain at least one suitable donor blood with a probability higher than 95 %.

Solution of part a

The probability to find exactly ten people with blood type A from 25 blood donors is given by the binomial distribution. The probability for each single donor to have blood type A is:

$$P(\mathrm{A}) = P(\mathrm{A}, \mathrm{Rh}+) + P(\mathrm{A}, \mathrm{Rh}-) = 43\% \,.$$

The desired probability is:

$$\begin{aligned} W_{0.43}^{25}(10) &= \binom{25}{10} \cdot 0.43^{10} \cdot (1 - 0.43)^{15} \\ &= \frac{25!}{10! \cdot 15!} 0.43^{10} \cdot 0.57^{15} \approx 15.4\% \,. \end{aligned}$$

This random experiment can be simulated with Sage:

```
sage: import numpy as np
sage: from numpy.random import random_sample
sage: repetitions = 100000
sage: people = 25
sage: people_a = 10
sage: p_a = 0.43
sage: hits = 0
sage: for _ in range(repetitions):
sage:     if np.sum(random_sample(people) < p_a) == people_a:
sage:         hits = hits+1
sage: print("Empirical probability to have 10 people with blood type A out of 25 people: {:5.2%}".format(
sage:         float(hits)/repetitions))
```

Solution of part b

The probability that more than half of the donors have blood type 0 and rhesus factor Rh+ can easily be determined with Sage by summation:

```
sage: def bernoulli(N, p, n):
sage:     return p^n*(1-p)^(N-n)*binomial(N, n)

sage: p_0_rhneg = 0.35
sage: sum = 0
sage: for hits in range((people+1)//2, people+1):
sage:     sum = sum+bernoulli(people, p_0_rhneg, hits)
sage: print("Probablity that more than half of the donors have blood type 0 Rh+: {:5.2%}".format(float(sum)))
```

Solution of part c

According to the table, people with blood type 0 Rh- as well as B Rh- can donate to a recipient with blood type B and rhesus factor Rh-. The probability to find a suitable donor thus equals

$$P(0, \mathrm{Rh-}) + P(\mathrm{B}, \mathrm{Rh-}) = 8\% \,.$$

The probability that a person is not a suitable donor then equals 92%. The probability that there is no suitable donor amongst n people thus amounts to 0.92^n. We look for the smallest number n for which

$$(0.92)^n \leq 0.05 \,.$$

Taking the logarithm and bearing in mind that $\ln(0.92)$ is negative, one finds

$$n \geq \frac{\ln(0.05)}{\ln(0.92)} \approx 35.9\,.$$

One hence needs at least 36 donors.

With the help of a random experiment one can approximately determine with Sage the probability that amongst 36 donors there is at least one suitable donor.

```
sage: repetitions = 100000
sage: n = 36
sage: p = 0.08
sage: hits = 0
sage: for _ in range(repetitions):
sage:     if np.sum(random_sample(n) < p):
sage:         hits = hist+1
sage: print("Probability that there is a suitable donor amongst {}
 ↪  people: {:5.2%}".format(n, float(hits)/repetitions))
```

3.2.2 Statistics of a medical test

Problem

0.074% of newborn children have a specific metabolic disorder. If this disorder is identified at an early stage, a future disease can be prevented by means of an appropriate treatment. For an early diagnosis, one can begin with a simple test. If the test result indicates a metabolic disorder, we call it positive.

If a newborn child has a metabolic disorder, the test result is positive with a probability of 99.5%. If a newborn child does not have a metabolic disorder, the probability that the test result is erroneously positive is 0.78%.

The test is conducted with a randomly selected newborn child. One considers the following results:

S: „There is a metabolic disorder.“

T: „The test is positive.“

1. Describe the event $\overline{S \cup T}$ in the present context.
2. Calculate the probabilities $P(T)$ and $P_T(S)$. Interpret the result for $P_T(S)$ in the present context.

   (for checking purposes: $P(T) \approx 0.85\%, P_T(S) < 0.1$)
3. During a screening, a huge number of randomly selected newborn children is tested. Determine the number of children per million tested newborn children expected on average to have a metabolic disorder while the test shows a negative result.

Solution of part a

First, we simplify the statement:

$$\overline{S \cup T} = \overline{S} \cap \overline{T}.$$

This formula thus describes the event that there is no metabolic disorder and the test result is negative.

Solution of part b

$P(T)$ describes the probability for a positive test result. It results from the probability of a positive test for a healthy newborn child as well as the probability of a positive test for an ill newborn child.

$$P(T) = (1 - 0.00074) \cdot 0.0078 + 0.00074 \cdot 0.995 \approx 0.00853\,.$$

$P_T(S)$ is computed as follows:

$$P_T(S) = \frac{P(S \cap T)}{P(T)} = \frac{0.00074 \cdot 0.995}{0.00853} \approx 0.0863\,.$$

This means that for a positive test, only in 8.63% of all cases a metabolic disorder is found.

Solution of part c

The probability that a randomly selected child has a metabolic disorder and is tested positively is:

$$P(S \cap \overline{T}) = 0.00074 \cdot (1 - 0.995) \approx 3.7 \cdot 10^{-6}$$

Thus, for one million tested children this event occurs for about four children.

With Sage, we can simulate the test and determine the number of all occuring events.

```
sage: import numpy as np
sage: from numpy.random import random_sample
sage: children = 1000000
sage: ps = 0.00074
sage: pst = 0.995
sage: pnst = 0.0078

sage: test_s = random_sample(children)
sage: ill_children = np.sum(random_sample(children) < ps)
sage: ill_children_pos = np.sum(random_sample(ill_children) < pst)
sage: ill_children_neg = ill_children-ill_children_pos
sage: healthy_children = children-ill_children
sage: healthy_children_pos = np.sum(random_sample(healthy_children)
  ↪ < pnst)
sage: healthy_children_neg = healthy_children-healthy_children_pos

sage: print("""{} children were tested as follows:
```

```
sage: {} children were ill and were tested positively.
sage: {} children were ill and were tested negatively.
sage: {} children were healthy and were tested positively.
sage: {} children were healthy and were tested negatively.""").format(
sage:     children, ill_children_pos, ill_children_neg, healthy_children_pos, healthy_children_neg)
```

3.2.3 Random Game

Problem

In order to earn money for the equipment of the playing area in the children's unit of the hospital a prize draw is offered. After the player paid two euros, three balls are drawn at random without replacement from a container in which there are three red, three green and three blue balls. If the three balls have the same colour, the player wins and receives a certain amount of money; otherwise he looses and obtains no money. Afterwards, the drawn balls are put back into the container.

1. Show that the probability for winning one game equals $\frac{1}{28}$.
2. Compute which amount of money, in case of winning, has to be paid out so that an average gain of 1.25 euros per game for the equipment of the playing area can be expected.

Solution of part a

The probability that three balls with the same colour are drawn can be computed as follows. First, there are 9 balls in the container. Now, one ball is drawn at random. Consequently, eight balls remain in the container. Two of these balls have the same colour as the already drawn ball. The probability of drawing one of these two balls in the next turn thus is $\frac{2}{8}$. Afterwards, the last ball with the same colour has to be drawn from the container with the remaining seven balls. The corresponding probability is $\frac{1}{7}$.

The total probability of winning hence is:

$$\frac{2}{8} \cdot \frac{1}{7} = \frac{1}{28} \approx 0.0357\,.$$

This prize draw can be simulated with Sage. For that purpose, we shuffle the balls and consider the colour of the first three balls.

```
sage: from random import choice

sage: def game():
sage:     urn = ['r', 'r', 'r', 'g', 'g', 'g', 'b', 'b', 'b']
sage:     shuffle(urn)
sage:     return urn[0] == urn[1] == urn[2]

```

```
sage: games = 100000
sage: winnings = 0

sage: for _ in range(games):
sage:     if game():
sage:         winnings = winnings+1

sage: print("In {} of {} cases, the three balls had the same
 ↪  colour.".format(winnings, games))
```

Solution of part b

The expected earnings E per game are computed for a stake of 2 euros per game and a prize money of x euro to be

$$E(x) = 2 - \frac{1}{28} \cdot x\,.$$

If the earnings per game are to be 1.25 euros, i.e. $E(x) = 1{,}25$, the result is $x = 21$. The prize money hence has to be 21 euros.

With Sage, we can test what impact a prize money of 21 euros has on the earnings per game.

```
sage: games = 100000
sage: stake = 2
sage: prize = 21
sage: earnings = 0

sage: for _ in range(games):
sage:     earnings = earnings+stake
sage:     if game():
sage:         earnings = earnings-prize

sage: print("{} games have been played and {} euros have been
 ↪  earned. "
           "This corresponds to {:.2f} euros per game.".format(
sage:                       games, earnings, float(earnings/games)))
```

3.2.4 Survey

Problem

In a city the election of the mayor is approaching. 12% of the eligible voters are young voters, i.e. persons aged between 18 and 24 years. Before the election campaign, a representative poll amongst the eligible voters is conducted. According to the poll, 44% of the polled eligible voters have already decided in favour of a candidate. One out of seven of the respondents, who have not yet decided in favour of a candidate, is a young voter.

The following outcomes are considered:

J: “A person randomly selected from the respondents is a young voter.”

K: “A person randomly selected from the respondents has decided already in favour of a candidate.”

1. Compile a completely filled fourfold table for the context described above.
2. Demonstrate that $P_J(\overline{K}) > P_{\overline{J}}(\overline{K})$. Justify that, in spite of validity of this inequality, it is not reasonable to concentrate predominantly on young voters in the election campaign.
3. On a specific day during his campaign, the candidate of party A speaks to 48 randomly selected eligible voters. Determine the probability that there are exactly six young voters amongst them.

Solution of part a

The problem yields the following fourfold table:

	K	$\overline{K}$	$\sum$
J		X	12%
$\overline{J}$		$6 \cdot X$	88%
$\sum$	44%	56%	100%

The statement “One out of seven respondents, who have not yet decided in favour of a candidate, is a young voter.” is considered in the column $\overline{K}$. The sum over the yet undecided eligible voters yields $X = 8\%$. The empty entries in the rows J and $\overline{J}$ can be completed by subtraction.

	K	$\overline{K}$	$\sum$
J	4%	8%	12%
$\overline{J}$	40%	48%	88%
$\sum$	44%	56%	100%

Solution of part b

To show the inequality, we compute $P_J(\overline{K})$ and $P_{\overline{J}}(\overline{K})$.

$$P_J(\overline{K}) = \frac{P(\overline{K} \cap J)}{P(J)} = \frac{8\%}{12\%} = 66.7\%$$
$$P_{\overline{J}}(\overline{K}) = \frac{P(\overline{K} \cap \overline{J})}{P(\overline{J})} = \frac{48\%}{88\%} \approx 54.5\%$$

The inequality $P_J(\overline{K}) > P_{\overline{J}}(\overline{K})$ is thus fulfilled. Nevertheless, it is not reasonable to concentrate on the young voters during the election campaign. For the outcome of the election, the total number of voters has to be considered. Even though by percentage, more older voters than young voters have already decided in favour of a candidate, this is not the case in absolute numbers. At 8% young voters and 48% older voters, who have not yet taken their decision, the election campaign should better focus on older voters.

Solution of part c

The probability for an eligible voter to be a young voter is 12%. The probability that there are exactly six young voters amongst 48 voters can be determined from the binomial distribution:

$$P_{0.12}^{48}(6) = \binom{48}{6} \cdot 0.12^6 \cdot (1-0.12)^{42} = 17.07\%$$

We can simulate this experiment with Sage.

```
sage: import numpy as np
sage: from numpy.random import random_sample
sage: iterations = 1000000
sage: people = 48
sage: young_voters = 6
sage: p = 0.12
sage: six_young_voters = np.sum(random_sample((people, iterations))
 ↪  < p, axis=0) == 6
sage: hits = np.sum(six_young_voters)
sage: print("The probability that amongst {} randomly selected
 ↪  people there are exactly {} young voters is: {:4.2%}".format(
sage:     people, young_voters, float(hits)/iterations))
```

3.2.5 Null hypothesis

Problem

According to the survey, the candidate of party A would have received about 50% of the votes if the election had taken place at the time of the survey. A success at the first ballot, for which more than 50% of all votes are required, is hence questionable. Thus, the election campaign consultant put in place by party A suggests an additional campaign in the final stage of the election battle. However, the treasurer of party A would prefer to avoid the high costs caused by an additional campaign, if possible.

1. In order to come to a decision on the realization of an additional campaign, the null hypothesis “The candidate of party A would currently receive at most 50% of all votes.” is to be tested by means of a sample of 200 eligible voters on a level of significance of 5%. Determine the associated decision rule.
2. Justify that the choice of the null hypothesis for the described test is in accordance with the concern of the election campaign consultant to achieve a success already at the first ballot.

Solution of part a

We want to disprove the null hypothesis. For that we assume that 50% of the voters vote for the candidate of party A. In a survey of 200 people, we have to determine the number k of people who vote for our candidate such that the level of significance is 5%. Thus, the equation

$$1 - P_{0.5}^{200}(X \leq k) \leq 0.05$$

has to be solved for k. From a mathematical table for the binomial distribution, we can determine $k \approx 112$. Alternatively, we can use Sage:

```
sage: from scipy.stats import binom
sage: total = 200
sage: p = 0.5
sage: for approving in (111, 112, 113):
sage:     print "Level of significance for {} approvals:
 ↪  {:4.2f}%".format(
sage:         approving, (1-binom.cdf(approving-1, total, p))*100)
```

Furthermore, we can simulate the survey and check in how many surveys at least 112 people would indicate to vote for candidate A, although the probability that a person votes for candidate A is 50%.

```
sage: import numpy as np
sage: from numpy.random import random_sample
sage: repetitions = 10000
sage: p = 0.5
sage: people = 200
sage: threshold = 112
sage: for_A = random_sample((people, repetitions)) < p
sage: above_threshold = np.sum(for_A, axis=0) >= threshold
sage: cases = np.sum(above_threshold)

sage: print(("The probability that at a survey of {} people at least
 ↪  "
             "{} people vote for candidate A\nif the probability to
              ↪  "
             "decide for candidate A is {:2.0%}, equals:  {:3.2%}
              ↪  ").format(
         people, threshold, float(p), float(cases)/repetitions))
```

Solution of part b

With the chosen null hypothesis one can relatively safely say that with at least 112 positive statements the candidate of party A will be elected. If the first survey is correct about the candidate receiving only about 50% of the votes, the null hypothesis will probably be disproven and the funds for an additional campaign get approved.

3.2.6 Expected value of election process

Problem

After the election, party A may fill three seats in a committee. Out of the eight female and four male city councillors, who were interested in a seat in this comitee, three people are determined to be members of the committee by lot.

The random variable X describes the number of female members of the committee of party A. Figure 1 depicts the probability distribution of the random variable X with $P(X = 0) = \frac{1}{55}$ and $P(X = 3) = \frac{14}{55}$.

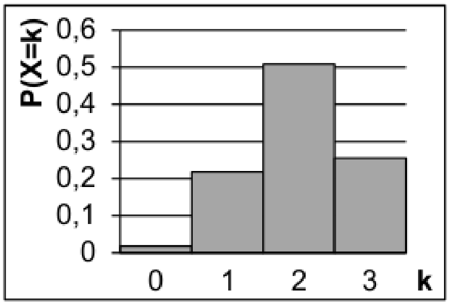


Abb. 1

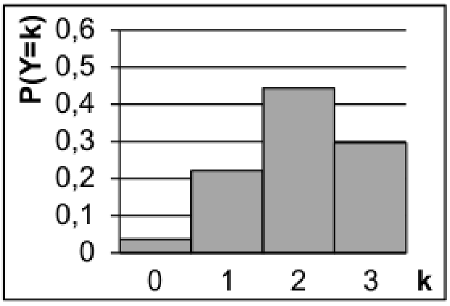


Abb. 2

1. Compute the probability $P(X = 1)$ and $P(X = 2)$.

   (Result: $P(X = 1) = \frac{12}{55}, P(X = 2) = \frac{28}{55}$*)*

2. Determine the expected value and the variance of the random variable X.

   (Result: $E(X) = 2, \mathrm{Var}(X) = \frac{6}{11}$*)*

3. Figure 2 shows the probability distribution of the binomially distributed random variable Y with the parameters $n = 3$ and $p = \frac{2}{3}$. Show by calculation that Y has the same expected value as the random variable X but has a bigger variance than X. Explain how one can see by comparison of figures 1 and 2 that $\mathrm{Var}(Y) > \mathrm{Var}(X)$ holds.

Solution of part a

The drawing of lots corresponds to drawing from an urn without replacement. This yields a hypergeometric distribution. With the number of female ($f = 8$) and male city councillors ($m = 4$) one obtains for $N = 3$ members of the committee

$$P(X) = \frac{\frac{f!}{X!(f-X)!}\frac{m!}{(N-X)!(m-N+X)!}}{\frac{(f+m)!}{N!(f+m-N)!}} .$$

For $X = 1$ one finds

$$P(1) = \frac{12}{55} .$$

Since the sum of probabilities has to equal one, it follows that

$$P(2) = 1 - P(0) - P(1) - P(3) = \frac{28}{55} .$$

The probabilities can also be easily determined with the help of Sage.

```
sage: def hypergeometric(M, N, n, k):
sage:     return binomial(M, k) * binomial(N - M, n - k) /
  ↪  binomial(N, n)

sage: f = 8
sage: m = 4
```

```
sage: N = 3
sage: for X in range(N+1):
sage:     print("P(X={}) = {}".format(X, hypergeometric(f, m+f, N,
 ↪  X)))
```

Solution of part b

Generally, the expected value of the distribution can be computed with the formula

$$E(X) = \sum_k k \cdot P(k).$$

The variance is then given by

$$\mathrm{Var}(X) = E(X^2) - \big(E(X)\big)^2$$

with

$$E(X^2) = \sum_k k^2 \cdot P(k)\,.$$

With the probabilities from the first part of the problem, one obtains

$$\begin{aligned} E(X) &= \frac{1}{55}(0 \cdot 1 + 1 \cdot 12 + 2 \cdot 28 + 3 \cdot 14) = \frac{110}{55} = 2 \\ E(X^2) &= \frac{1}{55}(0 \cdot 1 + 1 \cdot 12 + 4 \cdot 28 + 9 \cdot 14) = \frac{250}{55} = \frac{50}{11} \end{aligned}$$

and hence

$$\mathrm{Var}(X) = \frac{50}{11} - 4 = \frac{6}{11}\,.$$

With Sage, we can determine these results easily as well.

```
sage: E_X = sum(hypergeometric(f, m+f, N, k)*k for k in range(N+1))
sage: E_X2 = sum(hypergeometric(f, m+f, N, k)*k^2 for k in
 ↪  range(N+1))
sage: print(u"E(X) = {} \nVar(X) = {}".format(E_X, E_X2-E_X^2))
```

Solution of part c

For the given binomial probability distribution

$$P(Y = k) = \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{3-k} \binom{3}{k}\,,$$

we can determine the expected value as well as the variance with the help of Sage.

```
sage: def bernoulli(N, p, k):
sage:     return p^k*(1-p)^(N-k)*binomial(N, k)

sage: N = 3
sage: p = 2/3
sage: for k in range(N+1):
sage:     print("P(X={}) = {}".format(k, bernoulli(N, p, k)))

sage: E_Y = sum(bernoulli(N, p, k)*k for k in range(N+1))
sage: E_Y2 = sum(bernoulli(N, p, k)*k^2 for k in range(N+1))
sage: print(u"E(Y) = {} \nVar(Y) = {}".format(E_Y, E_Y2-E_Y^2))
```

Of cource, one can obtain these results by explicit calculations along the lines of the previous part of the problem if one determines the probabilities first.

Comparing the results with part b, one sees that the expected value is equal but the variance is bigger.

This can already be concluded by means of the figures because the probability for Y at $k = 2$ is smaller, while at $k = 0$ and $k = 3$ it is clearly bigger than for X. Hence, the probability distribution for Y is “broader” and has a bigger variance.

3.2.7 Two red and three white balls...

Problem

Urn A contains two red and three white balls. Urn B contains three red and two white balls. We will assume the following experiment:

> We choose a random ball from urn A and put it into urn B. Then, we choose a random ball from urn B and put it into urn A.

1. Determine all possible contents of urn A after the experiment.
2. Consider the following event E: “After the experiment, urn A contains the same amount of white balls as before the experiment.”

Solution of part a

The experiment consists of two steps. As there are two types of balls that can be chosen in each step, we have to consider four different sequences of the experiment. In the following, the ball movement as well as the final content of urn A are displayed for each of the four sequences.

1. $\bullet\bullet \circ \circ\circ \underset{\bullet}{\overset{\bullet}{\rightleftarrows}} \bullet\bullet\bullet \circ \circ \quad \Rightarrow \mathrm{A}: \bullet\bullet \circ \circ\circ$

2. $\bullet\bullet \circ \circ\circ \underset{\circ}{\overset{\bullet}{\rightleftarrows}} \bullet\bullet\bullet \circ \circ \quad \Rightarrow \mathrm{A}: \bullet \circ \circ \circ \circ$

3. $\bullet\bullet \circ \circ\circ \underset{\bullet}{\overset{\circ}{\rightleftarrows}} \bullet\bullet\bullet \circ \circ \quad \Rightarrow \mathrm{A}: \bullet\bullet\bullet \circ \circ$

4. $\bullet\bullet \circ \circ\circ \underset{\circ}{\overset{\circ}{\rightleftarrows}} \bullet\bullet\bullet \circ \circ \quad \Rightarrow \mathrm{A}: \bullet\bullet \circ \circ\circ$

The first and the fourth sequence provide the same final content of urn A. Therefore, the three possible final contents of urn A are: $\bullet \circ \circ \circ \circ$, $\bullet\bullet \circ \circ\circ$ and $\bullet\bullet\bullet \circ \circ$.

Solution of part b

The probability for urn A to finally contain three white balls is given by the sum of the probabilities of the first and the fourth sequence from part a.

1. The probability of drawing a red ball in the first step equals $\frac{2}{5}$, because two of the five balls in urn A are red. Urn B then contains four red balls and two white balls. Therefore, the probability of drawing one of the four red balls in the second step equals $\frac{4}{6} = \frac{2}{3}$. The probability for this sequence is thus obtained as $\frac{2}{5} \cdot \frac{2}{3} = \frac{4}{15}$.
2. Along the same lines, the probability of drawing a white ball in both steps can be calculated as $\frac{3}{5} \cdot \frac{1}{2} = \frac{3}{10}$.

The probability for event E is the sum of the probabilities for these two sequences:

$$p(\mathrm{E}) = \frac{4}{15} + \frac{3}{10} = \frac{17}{30} \approx 0.5667$$

The probability for the complementary event $\bar{E}$

$$p(\bar{\mathrm{E}}) = 1 - p(\mathrm{E}) = \frac{13}{30}$$

is smaller than the probability of the event E.

The probability for event E can be computed with Sage in two different ways. The first possibility is to simulate every possible sequence of the experiment with distinguishable balls. We therefore consider that every ball will be chosen with the same probability. This approach will lead to an exact result and will also prove our solution of part a.

```
sage: frequency_e = 0
sage: total = 0
sage: A0 = ['w', 'w', 'w', 'r', 'r']
sage: B0 = ['w', 'w', 'r', 'r', 'r']
sage: for a_ball in A0:
...       A1 = A0[::]
...       B1 = B0[::]
...       A1.remove(a_ball)
...       B1.append(a_ball)
...       for b_ball in B1:
...           print 'A->B:', a_ball,
...           A2 = A1[::]
...           A2.append(b_ball)
...           print '  B->A:', b_ball,
...           total = total+1
...           if A2.count('w') == 3:
```

```
...                  frequency_e = frequency_e+1
...                  print "    A: ", A2, "<==="
...              else:
...                  print "    A: ", A2
sage: print "p(E) = %s/%s" % (frequency_e, total)
```

A somewhat simpler solution consists in determining the probability for E by means of a simulation. To this end, we move a randomly chosen ball from urn A to urn B and another one in the opposite direction. The number of occurrences of event E is counted. This approach will give us only an approximation of the exact result. Furthermore, in order to reach a satisfying approximation, many repetitions are needed so that the runtime of this code exceeds the runtime of the first solution.

```
sage: import random
sage: def move_ball(urn1, urn2):
...         ball = random.choice(urn1)
...         urn1.remove(ball)
...         urn2.append(ball)
...         return urn1, urn2
sage: frequency_e = 0
sage: iterations = 100000
sage: for _ in range(iterations):
...         A = ['w', 'w', 'w', 'r', 'r']
...         B = ['w', 'w', 'r', 'r', 'r']
...         move_ball(A, B)
...         move_ball(B, A)
...         if A.count('w') == 3 :
...             frequency_e = frequency_e+1
sage: print "Approximation for the probability p(E) = ",
  ↪ float(frequency_e/iterations)
```

3.2.8 Bernoulli process

Problem

Consider a Bernoulli process with a probability of 0,9 and a length of 20. Specify an event with the probability given by

$$0{,}9^{20} + 20 \cdot 0{,}1 \cdot 0{,}9^{19}$$

Solution

The probability of having exactly n hits in a Bernoulli process with probability p and length N equals

$$P(n) = \binom{N}{n} \cdot p^n \cdot (1-p)^{N-n}.$$

The probability specified in this problem therefore corresponds to the sum of the probabilities of having exactly 20 hits

$$\binom{20}{20} \cdot 0{,}9^{20} \cdot 0{,}1^{0} = 0{,}9^{20}$$

and having exactly 19 hits

$$\binom{20}{19} \cdot 0{,}9^{19} \cdot 0{,}1^{1} = 20 \cdot 0{,}9^{19} \cdot 0{,}1.$$

In other words, the corresponding event consists in having at least 19 hits.

We will simulate the Bernoulli process by means of Sage and evaluate the probability of having at least 19 hits. Before doing so, we calculate the decimal value of the given probability.

```
sage: p = 0.9
sage: q = 0.1
sage: p_E = p^20 + 20*q*p^19
sage: print "Probability p(E) =", p_E
```

Now, we determine the probability for finding a given number of hits based on 50000 repetitions of the Bernoulli process. Finally, we compare the probability $p(E)$ of finding at least 19 hits with the exact result.

```
sage: import numpy as np
sage: threshold_value = 19
sage: frequency_e = np.zeros(21)
sage: iterations = 50000
sage: for _ in range(iterations):
...       hits = sum(np.random.random(20) < p)
...       frequency_e[hits] = frequency_e[hits]+1
sage: probabilities = frequency_e/iterations
sage: headline = ' Hit  probability'
sage: print headline
sage: print "-"*len(headline)
sage: for hits, p_of_e in enumerate(probabilities):
...       print "%6i       %g" % (hits, p_of_e)
sage: p_geq_19 = probabilities[19]+probabilities[20]
sage: print "Approximation of the probability p(E) =", p_geq_19
```

The result is close to the exact result. The difference arises from the finite number of realizations.

3.2.9 Random variable and expected value

Aufgabe

Consider a random variable X with the possible values $0, 1, 2$ and 3. The table shows the probability distribution of X with $p_1, p_2 \in [0; 1]$.

k	|	0	1	2	3
$P(X = k)$	|	p_1	$\frac{3}{10}$	$\frac{1}{5}$	p_2

Demonstrate that the expectation value of X cannot exceed 2.2.

Solution

The expectation value of a random variable X is given by the sum over the products of all possible values and their corresponding probabilities:

$$E(X) = \sum_k k \cdot p_k.$$

For the given probabilities, the expectation value becomes

$$E(X) = 0 \cdot p_1 + 1 \cdot \frac{3}{10} + 2 \cdot \frac{1}{5} + 3 \cdot p_2 = \frac{7}{10} + 3 \cdot p_2.$$

p_1 and p_2 cannot be chosen arbitrarily because all probabilities need to sum up to one:

$$p_1 + \frac{3}{10} + \frac{1}{5} + p_2 = 1$$
$$\Rightarrow \quad p_2 = \frac{1}{2} - p_1$$

Furthermore, the probabilities p_1 and p_2 may not be negative, so that p_2 takes on its maximum value for $p_1 = 0$. Therefore, the probability p_2 can take values from the interval $[0; \frac{1}{2}]$.

Taking the derivative of the expectation value with respect to the probability p_2

$$E'(p_2) = 3$$

one finds that the expectation value increases with increasing probability p_2. The maximum of the expectation value is therefore reached for the maximum value of p_2 and is found to agree with the expected value

$$E\left(\frac{1}{2}\right) = \frac{7}{10} + \frac{3}{2} = \frac{11}{5} = 2.2.$$

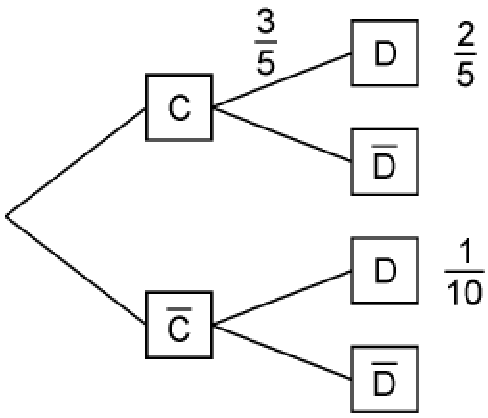
For the implementation in Sage, we will use k as an index for the corresponding probability. p_1 and p_2 defined in the problem will now be referred to as p_0 and p_3. In the Sage interface, it is possible to set the values for p_1, p_2, and p_3 from which p_0 is determined. At first, we will keep the values set to $p_1 = \frac{3}{10}$ and $p_2 = \frac{1}{5}$, as specified in the problem. By changing the value of p_3 in the allowed interval, we can determine the maximum of the expectation value. By varying the other probabilities, we can also explore how the maximum of the expectation value depends on the probabilities fixed in the problem. We use a parameter `eps` to cope with rounding errors which occur when adding probabilities. The necessity of this parameter becomes clear by setting it to 0.

```
sage: @interact
sage: def _(p1=slider(0., 1., 0.1),
...            p2=slider(0., 1., 0.1),
...            p3=slider(0., 1., 0.01), eps=3e-16):
sage:     p0 = 1-p1-p2-p3
sage:     if p0 >= -eps:
...           print 'p0 =', p0
...           print 'E =', p1+2*p2+3*p3
...       else:
...           print 'p0 =', p0, 'Negative values are forbidden.'
```

3.2.10 Tree diagram

Aufgabe

The following tree diagram is related to a random experiment with events C and D.



1. Calculate $P(\bar{D})$.
2. Prove that C and D are statistically dependent.
3. Modify the value $\frac{1}{10}$ in the tree diagram so that C and D are statistically independent.

Solution of part a

The probability $P(D)$ results from the given tree diagram as

$$P(D) = P(C \cap D) + P(\bar{C} \cap D) = \frac{2}{5} + \frac{1}{10} = \frac{1}{2}$$

The condition $P(D) + P(\bar{D}) = 1$ leads to $P(\bar{D}) = \frac{1}{2}$.

Solution of part b

Two events C and D are statistically dependent if the occurrence of event C has an influence on the probability of event D, i.e. $P(D|C) \neq P(D|\bar{C})$. From the tree diagram we read off $P(D|C) = \frac{3}{5}$.

In addition, we need

$$P(C) = \frac{P(C \cap D)}{P(D|C)} = \frac{2/5}{3/5} = \frac{2}{3},$$

which results in $P(\bar{C}) = 1 - P(C) = \frac{1}{3}$ and finally in

$$P(D|\bar{C}) = \frac{P(\bar{C} \cap D)}{P(\bar{C})} = \frac{1/10}{1/3} = \frac{3}{10}.$$

This proves $P(D|C) \neq P(D|\bar{C})$ so that C and D are indeed statistically dependent.

Solution of part c

In contrast to the previous task, the condition $P(D|C) = P(D|\bar{C})$ has to hold. $P(\bar{C})$ is still given as $\frac{1}{3}$ so that

$$P(\bar{C} \cap D) = P(D|\bar{C}) \cdot P(\bar{C}) = P(D|C) \cdot P(\bar{C}) = \frac{3}{5} \cdot \frac{1}{3} = \frac{1}{5}.$$

We will calculate all probabilities of the tree diagram with Sage by using the conditions

$$\begin{aligned}
P(C) + P(\bar{C}) &= 1 \\
P(D|C) + P(\bar{D}|C) &= 1 \\
P(D|\bar{C}) + P(\bar{D}|\bar{C}) &= 1 \\
P(D|C) \cdot P(C) &= P(C \cap D) \\
P(\bar{D}|C) \cdot P(C) &= P(C \cap \bar{D}) \\
P(D|\bar{C}) \cdot P(\bar{C}) &= P(\bar{C} \cap D) \\
P(\bar{D}|\bar{C}) \cdot P(\bar{C}) &= P(\bar{C} \cap \bar{D}).
\end{aligned}$$

The values of $P(D|C)$, $P(C \cap D)$, and $P(\bar{C} \cap D)$ can be modified in the list `probabilities`.

```
sage: var('p_c p_cb p_d_if_c p_db_if_c p_d_if_cb p_db_if_cb')
sage: var('p_c_and_d p_c_and_db p_cb_and_d p_cb_and_db')
sage: probabilities = [p_d_if_c==3/5,
sage:                              p_c_and_d==2/5,
sage:                              p_cb_and_d==1/10]
sage: equations = [p_c+p_cb==1,
sage:                      p_d_if_c+p_db_if_c==1,
sage:                      p_d_if_cb+p_db_if_cb==1,
sage:                      p_d_if_c*p_c==p_c_and_d,
sage:                      p_db_if_c*p_c==p_c_and_db,
sage:                      p_d_if_cb*p_cb==p_cb_and_d,
sage:                      p_db_if_cb*p_cb==p_cb_and_db]
sage: solution = solve(probabilities+equations,
...                       p_c, p_cb,
...                       p_d_if_c, p_db_if_c, p_d_if_cb, p_db_if_cb,
...                       p_c_and_d, p_c_and_db, p_cb_and_d,
    ↪ p_cb_and_db,
...                       solution_dict=True)[0]
sage: print 'P(C) =', solution[p_c]
```

```
sage: print '   P(D|C) =', solution[p_d_if_c],
sage: print '   P(DC) =', solution[p_c_and_d]
sage: print '   P(D|C) =', solution[p_db_if_c],
sage: print '   P(DC) =', solution[p_c_and_db]
sage: print 'P(C) =', solution[p_cb]
sage: print '   P(D|C) =', solution[p_d_if_cb],
sage: print '   P(DC) =', solution[p_cb_and_d]
sage: print '   P(D|C) =', solution[p_db_if_cb],
sage: print '   P(DC) =', solution[p_cb_and_db]
```

3.2.11 JIM - survey data analysis

Problem 1

In the course of the so-called JIM survey, in 2012 the use of information and media by adolescents of an age between 12 and 19 years was studied in Germany. The following table represents a subset of results for a representative sample of adolescents, among them 102 boys. For four kinds of devices, the number of girls and boys within the sample of 200 adolescents possessing the respective device is given.

	Girls	Boys
Smart phone	42	52
Computer	77	87
TV set	54	65
Stationary game console	37	62

1. Determine the probability that one person chosen at random out of the 200 adolescents is female and does not possess a TV set.
2. Out of the 200 adolescents, one person possessing a TV set was chosen at random. Find the probability that this person is female.
3. Justify that the events ”One person chosen at random out of 200 adolescents possesses a TV set“ and ”One person chosen at random out of 200 adolescents is a girl.“ are not independent.
4. According to the survey, 55% of the girls of an age between 12 and 19 years possess a TV set. Give the value of the sum

$$\sum_{i=0}^{12} B(25; 0.55; i)$$

in percent. Justify that this value in general does not represent the probability that among 25 girls of a class in 9th grade less than half possess a TV set.

Solution of part 1a

There is a total of 98 girls in the group, 54 of them owning a TV set. Accordingly, 44 girls do not own a TV set. The probability to find a girl not owning a TV set therefore is given by

$$\frac{44}{200} = 22\%.$$

We check the result by generating a list of 200 adolescents either being a girl with or without a TV set or a boy with or without a TV set. Then we draw at random out of this list and compile the number of persons in each category.

```
sage: import random
sage: girls = 98
sage: boys = 102
sage: girls_with_tv = 54
sage: boys_with_tv = 65
sage: adolescents = (["Girls with TV"]*girls_with_tv
...                  + ["Girls without TV"]*(girls-girls_with_tv)
...                  + ["Boys with TV"]*boys_with_tv
...                  + ["Boys without TV"]*(boys-boys_with_tv)
...                  )
sage: iterations = 60000
sage: frequencies = {"Girls with TV": 0,
...                  "Girls without TV": 0,
...                  "Boys with TV": 0,
...                  "Boys without TV": 0}
sage: for _ in range(iterations):
...         key = random.choice(adolescents)
...         frequencies[key] = frequencies[key]+1
sage: table(list(frequencies.items()))
```

Now we can determine the probability of finding a girl without a TV set.

```
sage: print "Probability for a girl without TV set: {:4.1%}".format(
...          (float(frequencies["Girls without TV"])/iterations))
```

Solution of part 1b

It is stated in the problem text that the randomly chosen person owns a TV set and thus is either one of the 65 boys owning a TV set or one of the 54 girls with a TV set. The total number of persons owning a TV set thus amounts to 119. The probability that this person is a girl then is found as

$$\frac{54}{119} \approx 45.4\%$$

We make use of the simulation of part a) in order to empirically check this result.

```
sage: with_tv = frequencies["Girls with TV"]+frequencies["Boys with
  ↪  TV"]
sage: print "Probability for a person with TV to be female:
  ↪  {:4.1%}".format(
...          (float(frequencies["Girls with TV"])/with_tv))
```

Solution of part 1c

The two events A ”One person chosen at random out of 200 adolescents possesses a TV set“ and B ”One person chosen at random out of 200 adolescents is a girl.“ were independent provided

$$P(B|A) = P(B|\bar{A}) = P(B)$$

holds.

In part b) we already evaluated the probability for a person owning a TV set to be a girl. This value corresponds to $P(B|A)$. It remains to determine the probability that a person chosen at random is a girl:

$$P(B) = \frac{98}{200} = 49\%.$$

It follows

$$P(B|A) = \frac{54}{119} \neq \frac{49}{100} = P(B)$$

and therefore the events A und B are not independent.

Solution of part 1d

We determine the sum by means of Sage and obtain approximately 30.6%.

```
sage: def bernoulli(N, p, n):
...       return p^n*(1-p)^(N-n)*binomial(N, n)

sage: p = 0.55
sage: ntot = 25
sage: nmax = 12
sage: probsum = 0
sage: for i in range(nmax+1):
...       probsum = probsum+bernoulli(ntot, p, i)
sage: print "The sum amounts to {:4.1%}".format(float(probsum))
```

Alternatively, the result can be obtained directly as:

```
sage: from scipy.special import bdtr
sage: print "The sum amounts to {:4.1%}".format(bdtr(nmax, ntot, p))
```

The survey was carried out with adolescents between the ages of 12 and 19. However, it is not known whether it is representative for the 9th grade (about 14 to 15 years of age). Therefore, it is not premissible to make use of the value of the sum as the probability that out of 25 girls of 9th grade less than half own a TV set.

If, however, we assume that indeed 55% of the girls in 9th grade possess a TV set, we can use Sage to empirically check that the sum represents the probability that less then half of 25 girls possess a TV set.

```
sage: import numpy as np
sage: threshold = 12
sage: p = 0.55
sage: frequency = 0
sage: iterations = 50000
sage: for _ in range(iterations):
...         girls_with = sum(np.random.random(25) < p)
...         if girls_with <= threshold:
...             frequency = frequency+1
sage: print("Probability that less than half of the girls possess "
...           "a TV: {:4.1%}".format(float(frequency)/iterations))
```

Problem 2

According to the JIM survey, considerably less than 90% of the adolescents own a computer. Therefore, the city council of a provincial town is approached to install a workspace with computers in the local youth centre. The city council is only willing to invest the requested funds if less than 90% of the adolescents in the provincial town own a computer.

1. The decision on the approval of the funds shall be based on an inquiry in the provincial town among 100 randomly chosen adolescents between 12 and 19 years of age. The probability that the funds are mistakenly approved, shall be at most 5%. Determine the corresponding rule for which at the same time the probability is minimal that the funds are mistakenly not approved.
2. Determine the probability that exactly 85 among the 100 adolescents interviewed own a computer, provided the percentage of adolescents owning a computer among the adolescents in the provincial town is as large as among the adolescents represented in the table.

Solution of part 2a

For the given hypothesis test, we need to check how many of the 100 adolescents interviewed may own a coomputer such that the probability that more than 90% of the adolescents own a computer is at most 5%.

We assume that the random variable X representing the number of adolescents owning a computer is binomially distributed. Assuming a threshold of 90% of adolescents owning a computer, we determine the maximum value C for a sample size of 100 for which the probability does not exceed 5%:

$$\sum_{i=0}^{C} B(100; 0.9; i) \leq 5\%$$

We determine the sum by means of Sage:

```
sage: p = 0.9
sage: adolescents = 100
```

```
sage: C = 0
sage: probsum = bernoulli(adolescents, p, C)
sage: while probsum < 0.05:
...         C = C+1
...         probsum = probsum+bernoulli(adolescents, p, C)
...     C = C-1
sage: print("The workspace should be approved if {} or fewer "
...           "adolescents own a computer.").format(C)
```

We can check the limiting value $C = 84$ by means of a simulation.

```
sage: frequency_C = 0
sage: frequency_Cp1 = 0
sage: iterations = 50000
sage: C = 84
sage: for _ in range(iterations):
...         adolescents_with = sum(np.random.random(100) < p)
...         if adolescents_with <= C:
...             frequency_C = frequency_C+1
...         if adolescents_with <= C + 1:
...             frequency_Cp1 = frequency_Cp1+1
sage: print("Empirical probability, that at 90% probability to own a
↪ computer "
...           "{} out of 100 adolescents or less own a computer:
↪ {:3.1%}".format(
...           C, float(frequency_C)/iterations))
sage: print("Empirical probability, that at 90% probability to own a
↪ computer "
...           "{} out of 100 adolescents or less own a computer:
↪ {:3.1%}".format(
...           C+1, float(frequency_Cp1)/iterations))
```

Solution of part 2b

The percentage recorded in the table of adolescents owning a computer is

$$\frac{77 + 87}{200} = 82\%.$$

At a probability of 82% for owning a computer, the probability that exactly 85 out 100 adolescents own a computer amounts to

$$P(X = 85) = B(100; 0.82; 85)$$

With Sage we find $P(X = 85) \approx 8.1\%$.

```
sage: print "Probability to find exactly 85 adolescents owning a
↪ computer: {:3.1%}".format(
...             float(bernoulli(100, 0.82, 85)))
```

Problem 3

It can be assumed that among the adolescents owning a smart phone, the percentage of those owning a stationary game console is larger than among those not owning a smart phone. Determine for the 200 adolescents recorded in the table, how big the number of persons owning, both, a smart phone and a stationary game console must be, so that the assumption is valid for the adolescents recorded in the table.

Solution of part 3

This problem is concerned with the dependence of events. For the following, we introduce the events A „One person chosen at random out of 200 adolescents owns a stationary game console.“ and B „One person chosen at random out of 200 adolescents owns a smart phone.“

We demand that the two events are statistically dependent in a way that

$$P(A|B) > P(A|\bar{B})$$

is fulfilled. From the table we obtain $P(A) = (37 + 62)/200 = 49.5\%$ and $P(B) = (42 + 52)/200 = 47\%$.

By means of

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

and

$$P(A \cap B) + P(A \cap \bar{B}) = P(A)$$

the above condition can be transformed into

$$\begin{aligned} \frac{P(A \cap B)}{P(B)} &> \frac{P(A) - P(A \cap B)}{P(\bar{B})} \\ P(A \cap B)P(\bar{B}) &> P(A)P(B) - P(A \cap B)P(B) \\ P(A \cap B)[P(\bar{B}) + P(B)] &> P(A)P(B) \\ P(A \cap B) &> P(A)P(B) \\ P(A \cap B) &> 0.495 \cdot 0.47 \end{aligned}$$

Out of 200 adolescents, at least 23.3% must own a smart phone and a stationary game console for the hypothesis formulated in the problem to hold. This threshold amounts to 47 adolescents.

The limit for $P(A \cap B)$ beyond which A and B depend on each other as requested, can also be determined with the help of Sage by solving a linear system of equations:

```
sage: var('p_aub p_aunb p_b p_nb p_a')
sage: probabilities = [p_a == 0.47, p_b == 0.495]
sage: equations = [p_b + p_nb == 1,
...                p_aub + p_aunb == p_a,
...                p_aub/p_b == p_aunb/p_nb]
```

```
sage: solution = solve(equations + probabilities, p_aub, p_aunb,
 ↪  p_b, p_nb, p_a,
...                  solution_dict=True)[0]
sage: print "Statistical independence requires P(AB) =
 ↪  {:4.1%}".format(
...          float(solution[p_aub]))
```

3.2.12 A scrapbook

Problem

Customers of a supermarket receive as a function of the value of their purchase a certain amount of packages containing animal pictures which can be collected in a scrapbook. Each package contains five pictures. The scrapbook contains places for a total of 200 different pictures. Large numbers of pictures are produced with equal probability and distributed randomly among the packages. The pictures in a given package do not necessarily differ.

1. Justify that the term

$$\frac{200 \cdot 199 \cdot 198 \cdot 197 \cdot 196}{200^5}$$

   gives the probability that all animal pictures in a given package differ.

2. In the scrapbook of a boy, 15 pictures are still missing. He goes shopping with his mother and finally receives two packages with animal pictures. Determine the probability that the two packages contain only pictures which the boy already has in his scrapbook.

Childrens' favorites are the 3D pictures where the animals appear as three-dimensional. 20 of the 200 pictures provided for the scrapbook are 3D pictures.

3. Determine how many packages a child needs at least to obtain a 3D picture with a probability of at least 99%.

Solution of part a

In order to obtain five different pictures, the first picture may be selected among all 200 pictures, the second picture may be selected out of 199 pictures etc. The respective probabilities are obtained by division by the number of different pictures, i.e. 200. The probability that all five pictures are different, is obtained as the product of the respective probabilities:

$$\frac{200 \cdot 199 \cdot 198 \cdot 197 \cdot 196}{200^5}$$

Evaluating this expression by means of Sage, one finds

```
sage: print "The probability to obtain five different pictures is
 ↪  given by {:4.1%}.".format(
...          float(200*199*198*197*196/200**5))
```

This value can be confirmed empirically through a simulation using Sage:

```
sage: from numpy.random import randint
sage: iterations = 100000
sage: different = 0
sage: for _ in range(iterations):
...       a = set(randint(200, size=5))
...       if len(a) == 5:
...           different +=1
sage: print "Empirical probability to obtain five different pictures: {:4.1%}".format(
...           float(different)/iterations)
```

Solution of part b

The probability that one of the pictures found by the boy in one of the two packages is already contained in his collection is given by

$$\frac{200-15}{200} = \frac{37}{40}.$$

The probability that all ten pictures found by the boy in the two packcages are already contained in his collection is again obtained by multiplication:

$$P = \left(\frac{37}{40}\right)^{10}$$

By means of Sage we can evaluate this expression

```
sage: print "Probability to obtain no new picture: {:4.1%}".format(float((37/40)**10))
```

and verify it by means of a simulation

```
sage: iterations = 100000
sage: property = set(range(185))
sage: no_new = 0
sage: for _ in range(iterations):
...       pictures = set(randint(200, size=10))
...       if pictures.issubset(property):
...           no_new = no_new+1
sage: print "Empirical probability for not obtaining a new picture: {:4.1%}".format(
...           float(no_new/iterations))
```

Solution of part c

The probability that a picture is not a 3D picture amounts to

$$\frac{200-20}{200} = \frac{9}{10}.$$

Correspondingly, the probability that among n pictures none is a 3D picture is given by

$$P(n) = \left(\frac{9}{10}\right)^n \,.$$

We now need to determine n such that the probability term is smaller than 1%. We thus solve

$$P(n) = 0.01$$

and obtain

$$n = \frac{\log(0.01)}{\log(0.9)} = 43.7\,.$$

As the pictures are only available in packages of five pictures, one needs nine packages in order to received a 3D picture with a probability of 99%. We again use Sage to check this result by a simulation.

```
sage: nr_packages = 9
sage: pictures_per_package = 5
sage: iterations = 100000
sage: threeD_pictures = set(range(20))
sage: threeD_found = 0
sage: for _ in range(iterations):
...       mypictures = set(randint(200,
 ↪  size=nr_packages*pictures_per_package))
...       if not mypictures.isdisjoint(threeD_pictures):
...           threeD_found = threeD_found+1
sage: print "Empirical probability to obtain at least one 3D
 ↪  picture: {:4.1%}".format(
...           float(threeD_found/iterations))
```

3.2.13 The wheel of fortune

Problem

A supermarket organizes a competition in order to raise money for the equipement of the local kindergarden. The five sectors of the wheel of fortune used for this purpose are numbered from 1 to 5. The size of the sectors is proportional to the value of the numbers, e.g., the sector with number 3 is three times as large as the sector with the number 1. After the player has paid six euros, the wheel of fortune will be turned once. Does the player obtain one of the numbers 1 to 4, he will receive the corresponding value in a corresponding amount of euros. If he obtains the number 5, he receives a ticket for a leisure park with a value of fifteen euros.

1. Determine the angle spanned by the sector with number 1 as well as the probability that the player will win the ticket in a single game.

   (Partial result: Size of angle: 24°)

2. Determine the expectation value for the payment per game if winning a ticket is equivalent to receiving a payment of fifteen euros. Interprete the result.
3. The supermarket needs to pay to the leisure park only ten euros per ticket. Therefore, as a result of the competition, one can expect a surplus to be donated to the local kindergarden. Determine the expected surplus provided that the game is played 6000 times.

Solution of part 1a

The angle spanned by sector 1 can be obtained as follows:

$$x + 2x + 3x + 4x + 5x = 360 \quad \Leftrightarrow \quad x = 24\,.$$

The probability to win a ticket is given by

$$\frac{5}{1+2+3+4+5} = \frac{5}{15} = \frac{1}{3}.$$

Solution of part 1b

The expectation value is determined by means of the formula

$$E = \sum_X P(X) \cdot X,$$

where X is the amount of the payment. Assuming that winning the ticket is equivalent to a payment of fifteen euros, we obtain

$$E = \frac{1}{15} + \frac{2}{15} \cdot 2 + \frac{3}{15} \cdot 3 + \frac{4}{15} \cdot 4 + \frac{5}{15} \cdot 15 = 7\,.$$

A player on average receives seven euros per round.

A corresponding simulation can be carried out with Sage.

```
sage: import random
sage: amounts = [1]*1+[2]*2+[3]*3+[4]*4+[15]*5
sage: games = 6000
sage: total_win = 0
sage: for _ in range(games):
...       total_win = total_win+random.choice(amounts)
sage: print "expectation value = {:4.2f}
 ↪  euros".format(float(total_win)/games)
```

Solution of part 1c

In contrast to part b, the payment now has to be replaced by the yield for the supermarket, i.e., 5, 4, 3, 2 and -4. We thus obtain for the expectation value

$$E = \frac{1}{15} \cdot 5 + \frac{2}{15} \cdot 4 + \frac{3}{15} \cdot 3 + \frac{4}{15} \cdot 2 + \frac{5}{15} \cdot (-4) = \frac{2}{3}.$$

The yield per game for the supermarket amounts to 67 cents. On the basis of 6000 games, a surplus of about

$$6000 \cdot \frac{2}{3} \, \text{euros} = 4000 \, \text{euros}$$

should result. For such a large number of repetitions, the actual result should not deviate by too much from the expectation value.

We simulate the game with Sage as seen from the supermarket or the kindergarden.

```
sage: amounts = [5]*1+[4]*2+[3]*3+[2]*4+[-4]*5
sage: games = 6000
sage: surplus = 0
sage: for _ in range(games):
...       surplus = surplus+random.choice(amounts)
sage: print "Surplus for the kindergarden: {} euros".format(surplus)
sage: print "Average surplus per game for the kindergarden: {:4.2f}
  ↪ euros".format(
...          float(surplus/games))
```

An impression of the bandwidth of the results can be obtained by means of a simulation. A series of 6000 games is repeated many times and the frequency of surplusses is displayed in a histogram.

```
sage: import matplotlib.pyplot as plt
sage: repetitions = 500
sage: games = 6000
sage: surplusses = []
sage: for repetition in range(repetitions):
...       surplus = 0
...       for _ in range(games):
...           surplus = surplus+random.choice(amounts)
...       surplusses.append(surplus)
sage: plt.hist(surplusses, bins=30)
sage: plt.show()
```

3.2.14 Biathlon

Problem

In the winter sport biathlon, during each shooting round, five targets have to be hit. In the course of an individual race, a biathlet executes a shooting round by shooting on each target once. This shooting round is modeled by a Bernoulli chain of length 5 with a probablity p to score a hit.

1. Give an expressions for the following events A and B which described the probability for the event as a function of p.

A: „The biathlete scores exactly four hits.“
B: „The biathlete scores a hit only for the first two shots.“

2. Explain by way of example why modeling a shooting round by means of a Bernoulli chain might disagree with reality.

Solution of part a

We start by considering the probability for event B. Since the probability of a hit is given by math:p, the probability for a miss equals $1-p$. Correspondingly, the probability for scoring a hit for exactly the first two shots is obtained as $p^2(1-p)^3$. We check this statement by means of simulation. However, we should not expect perfect agreement.

```
sage: p = 0.7
sage: rounds = 1000000
sage: goal = [True, True, False, False, False]
sage: successes = 0
sage: for round in range(rounds):
...       result = [random() < p for _ in range(5)]
...       if result == goal:
...           successes = successes+1
sage: print N(successes/rounds), p^2*(1-p)^3
```

Let us now consider event A. In analogy to the previous consideration, the probability for a given sequence of four hits and and one miss equals $p^4(1-p)$. However, the shot which misses is not fixed. The number of possibilites to distribute M events on N positions is given by the binomial coefficient

$$\binom{N}{M} = \frac{N!}{M!(N-M)!}.$$

In our case, the desired probability is obtained as

$$\binom{5}{4} p^4(1-p) = 5p^4(1-p).$$

After briefly verifying the binomial coefficient of which we make use here

```
sage: binomial(5, 4)
```

we once more check our result for the probability by means of a simulation:

```
sage: p = 0.7
sage: rounds = 1000000
sage: successes = 0
sage: for round in range(rounds):
...       result = [random() < p for _ in range(5)]
...       if sum(result) == 4:
...           successes = successes+1
sage: print N(successes/rounds), 5*p^4*(1-p)
```

Solution of part b

The Bernoulli chain assumes that the probability of a hit is the same for each shot. However, in reality the probability of a hit might for example decrease after a miss.

3.2.15 Talkshow

Problem

A talkshow host invites three politicians, a newswoman and two members of a citizens' action committee. During the discussion round, the participants will be sitting in a semi-circle with the host in the middle and each participant taken as an individual.

1. Give an expression which allows to determine the number of possible seating arrangements if no other constraints need to be taken into account.
2. The station has decided that the newswoman will take a seat next to the host and that to the other side of the host, a politician shall be seated. Determine the number of possible seating arrangements accounting for these constraints.

Solution of part a

If we want to generate all possible seating arrangements, we start with the first seat for which we the choice among six person. For the second seat, five person are left and so on. In total, we obtain

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

possibilities.

If we indicate the host by H, the politicans by 1, 2, and 3, the newswoman by N and the members of the citizens' action committee by C and c, we can list all seating arrangements:

```
sage: for n, a in enumerate(Arrangements(["1", "2", "3", "N", "C",
 ↪  "c"], 6)):
...       if not n % 8:
...           print "%3i" % (n/8+1),
...       print "%sH%s" % ("".join(a[:3]), "".join(a[3:])),
...       if not (n+1) % 8:
...           print
```

Our list indeed comprises $8 \cdot 90 = 720$ different seating arrangements.

Solution of part b

We can attribute the seats by proceeding as follows: The newswoman is placed on one of the two seats next to the host (2 possibilities) and one of the three politicians is seated on the other side of the host (3 possibilities). It remains to place four persons on four seats which, in analogy to our reasoning in part a, yields $4 \cdot 3 \cdot 2 \cdot 1 = 24$ possiblities. In total, we obtain $2 \cdot 3 \cdot 24$ different seating arrangements which we can list:

```
sage: persons = set(["1", "2", "3", "C", "c"])
sage: n = 0
...   for jleft in (True, False):
...       for pmiddle in ("1", "2", "3"):
...           for others in Arrangements(persons-set([pmiddle]), 4):
...               if jleft:
...                   a = "".join(others[:2])+"NH"+pmiddle+"".join(others[2:])
...               else:
...                   a = "".join(others[:2])+pmiddle+"HN"+"".join(others[2:])
...               if not n % 8:
...                   print "%3i" % (n/8+1),
...               print a,
...               if not (n+1) % 8:
...                   print
...               n = n+1
```

We obtain $18 \cdot 8 = 144$ seating arrangements as expected.

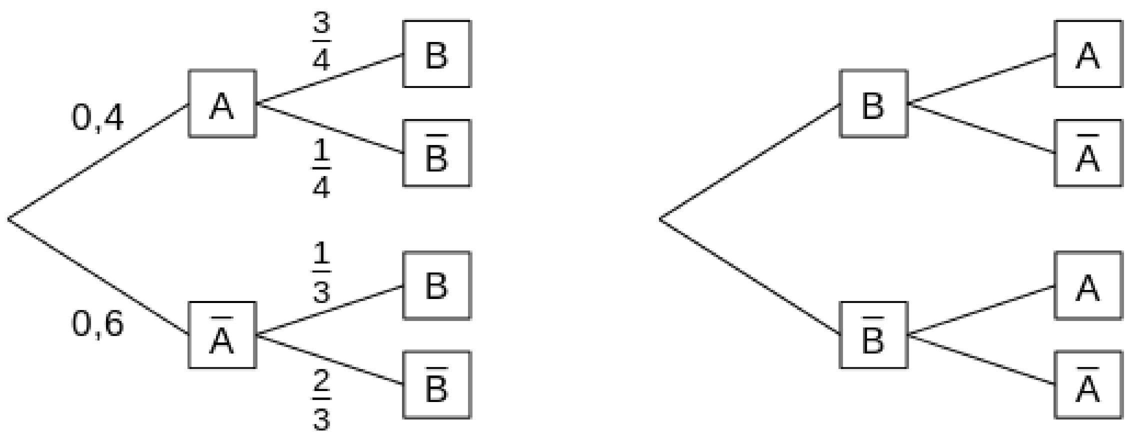
3.2.16 Tree diagrams

Problem

The two tree diagrams belong to the same random experiment with events A and B.

Compute the probability $P(B)$ and, subsequently, add the corresponding probabilities to each branch in the right tree diagram.

(Partial result: P(B)=0.5)



Solution

The probability $P(B)$ results from the information given in the left tree diagram as

$$P(B) = P(B \cap A) + P(B \cap \bar{A}) = P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A}) = \frac{3}{4} \cdot 0.4 + \frac{1}{3} \cdot 0.6 = 0.5$$

Then, the condition $P(B) + P(\bar{B}) = 1$ yields $P(\bar{B}) = 0.5$. The probabilities corresponding to the remaining branches can be determined by means of Bayes' theorem.

$$\begin{aligned}
P(A|B) &= \frac{P(B|A) \cdot P(A)}{P(B)} = \frac{\frac{3}{4} \cdot 0.4}{0.5} = \frac{3}{5} \\
P(\bar{A}|B) &= \frac{P(B|\bar{A}) \cdot P(\bar{A})}{P(B)} = \frac{\frac{1}{3} \cdot 0.6}{0.5} = \frac{2}{5} \\
P(A|\bar{B}) &= \frac{P(\bar{B}|A) \cdot P(A)}{P(\bar{B})} = \frac{\frac{1}{4} \cdot 0.4}{0.5} = \frac{1}{5} \\
P(\bar{A}|\bar{B}) &= \frac{P(\bar{B}|\bar{A}) \cdot P(\bar{A})}{P(\bar{B})} = \frac{\frac{2}{3} \cdot 0.6}{0.5} = \frac{4}{5}
\end{aligned}$$

We can use Sage to determine all probabilities of the right tree diagram on the basis of the given probabilities and the conditions

$$\begin{aligned}
P(B) + P(\bar{B}) &= 1\,, \\
P(A|B) + P(\bar{A}|B) &= 1\,, \\
P(A|\bar{B}) + P(\bar{A}|\bar{B}) &= 1\,, \\
P(A|B) \cdot P(B) &= P(B|A) \cdot P(A)\,, \\
P(A|\bar{B}) \cdot P(\bar{B}) &= P(\bar{B}|A) \cdot P(A)\,, \\
P(\bar{A}|B) \cdot P(B) &= P(B|\bar{A}) \cdot P(\bar{A})\,, \\
P(\bar{A}|\bar{B}) \cdot P(\bar{B}) &= P(\bar{B}|\bar{A}) \cdot P(\bar{A})\,.
\end{aligned}$$

The values of the left tree diagram can be set in the list `probabilities`.

```
sage: var('p_a p_ab p_b_if_a p_bb_if_a p_b_if_ab p_bb_if_ab')
sage: var('p_b p_bb p_a_if_b p_ab_if_b p_a_if_bb p_ab_if_bb')
sage: probabilities = [p_a == 0.4,
sage:                         p_ab == 0.6,
sage:                         p_b_if_a == 3/4,
sage:                         p_bb_if_a == 1/4,
sage:                         p_b_if_ab == 1/3,
sage:                         p_bb_if_ab == 2/3]
sage: equations = [p_b+p_bb == 1,
sage:               p_a_if_b+p_ab_if_b == 1,
sage:               p_a_if_bb+p_ab_if_bb == 1,
sage:               p_a_if_b*p_b == p_b_if_a*p_a,
sage:               p_ab_if_b*p_b == p_b_if_ab*p_ab,
sage:               p_a_if_bb*p_bb == p_bb_if_a*p_a,
sage:               p_ab_if_bb*p_bb == p_bb_if_ab*p_ab]
sage: solution = solve(probabilities+equations,
sage:                p_a, p_ab, p_b_if_a, p_bb_if_a, p_b_if_ab,
    ↪ p_bb_if_ab,
sage:                p_b, p_bb, p_a_if_b, p_ab_if_b, p_a_if_bb,
    ↪ p_ab_if_bb,
```

```
sage:                    solution_dict=True)[0]
sage: print 'P(B) =', solution[p_b]
sage: print '   P(A|B) =', solution[p_a_if_b],
sage: print '   P(A|B) =', solution[p_ab_if_b],
sage: print '\nP(B) =', solution[p_bb]
sage: print '   P(A|B) =', solution[p_a_if_bb],
sage: print '   P(A|B) =', solution[p_ab_if_bb]
```

3.2.17 Tossing a coin

Problem

For a random experiment an ideal coin is tossed until it shows heads (H) for a second time or tails (T) for a second time. The event space is set to be: $\{HH; TT; HTH; HTT; THH; THT\}$.

1. Argue that this random experiment is not a Laplace experiment.
2. The random variable X assigns to each event the number of coins tossed. Compute the expectation value of X.

Solution of part a

For a Laplace experiment each outcome has the same probability. Here, this would mean that each outcome has a probability of $\frac{1}{6}$. Because an ideal coin is used for the random experiment, we can determine the probabilities of each event as follows:

$$P(HH) = P(TT) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$
$$P(HTH) = P(HTT) = P(THH) = P(THT) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Because at least one of the probabilities now differs from $\frac{1}{6}$, this random experiment is not a Laplace experiment.

The different probabilities also become apparent if one simulates the random experiment with Sage.

```
sage: def toss():
sage:     return Set(["T", "H"]).random_element()

sage: def event():
sage:     event = toss()+toss()
sage:     if event[0] != event[1]:
sage:         event = event+toss()
sage:     return event

```

```
sage: eventspace = {"HH": 0, "TT": 0,
sage:                   "HTH": 0, "HTT": 0, "THH": 0, "THT": 0}
sage: iterations = 10000
sage: for n in range(iterations):
sage:     e = event()
sage:     eventspace[e] = eventspace[e]+1
sage: for k, v in eventspace.items():
sage:     print "Frequency of the event %3s: %5i" % (k, v)
```

Solution of part b

The expectation value of X can be determined with the probabilities obtained in part a:

$$E(X) = 2 \cdot P(HH) + 2 \cdot P(TT) + 3 \cdot P(HTH) + 3 \cdot P(HTT) + 3 \cdot P(THH) + 3 \cdot P(THT) = 2.5.$$

A simulation with Sage yields:

```
sage: iterations = 10000
sage: x = 0
sage: for n in range(iterations):
sage:     x = x+len(event())
sage: print "The expectation value of X is about: ", "%4.2f" %
  ↪  float(x/iterations)
```

3.2.18 Seminar problem

Problem

Eight girls and six boys, among them Anna and Tobias, participate in a seminar. For a presentation, a team of four persons is formed by drawing from the participants at random.

1. For each of the following events, give an expression allowing to compute the respective probability.

   A: "Anna and Tobias are in the team."

   B: "The team consists of the same number of boys and girls."

2. Describe an event in this context which has the probability represented by the following expression:

$$\frac{\binom{14}{4} - \binom{6}{4}}{\binom{14}{4}}$$

Solution of part a

The combinatorial problem of forming a team of four persons from 14 participants, corresponds to drawing 4 balls from 14 without replacement and disregarding the order. Accordingly, there are

$$\binom{14}{4} = 1001$$

possibilities to form a team.

Because in event A, the team members besides Anna and Tobias are arbitrary, there are $\binom{12}{2} = 66$ possibilities to realize that event.

Accordingly, the probability for event A is

$$P(A) = \frac{66}{1001} \approx 6.6\%\,.$$

We can check this value by means of a simulation with Sage. In doing so, 4 elements from the numbers 1 to 14 are drawn and Anna and Tobias are assigned to the values 1 and 2, respectively.

```
sage: iterations = 30000
sage: frequency = 0
sage: for _ in range(iterations):
sage:     team = Subsets(14, 4).random_element()
sage:     if 1 in team and 2 in team:
sage:         frequency = frequency+1
sage: print 'Approximation for the probability P(A) = %4.1f%%' %
   ↪  float(100*frequency/iterations)
```

When realizing event B, there are $\binom{8}{2} = 28$ different possibilities to choose two girls and $\binom{6}{2} = 15$ for the boys.

Together there are thus $28 \cdot 15 = 420$ possibilities to form a team of two girls and two boys. The corresponding probability is

$$P(B) = \frac{420}{1001} \approx 42.0\%\,.$$

As before, this result can be checked by means of a simulation. The girls are assigned to the numbers smaller or equal to 8 and the numbers above 8 correspond to boys.

```
sage: iterations = 30000
sage: frequency = 0
sage: for _ in range(iterations):
sage:     team = Subsets(14, 4).random_element()
sage:     number_girls = 0
sage:     number_boys = 0
sage:     for member in team:
sage:         if member <= 8:
sage:             number_girls = number_girls+1
sage:         else:
sage:             number_boys = number_boys+1
```

```
sage:     if number_girls == number_boys:
sage:         frequency = frequency+1
sage: print 'Approximation for the probability P(B) = %4.1f%%' % float(100*frequency/iterations)
```

Solution of part b

The given probability can be simplified to

$$1 - \frac{\binom{6}{4}}{\binom{14}{4}} .$$

The corresponding event is hence complementary to an event with the probability

$$\frac{\binom{6}{4}}{\binom{14}{4}} .$$

The latter corresponds for example to the event “The team contains only boys.” because the number of possibilities to choose 4 boys equals $\binom{6}{4}$. The complementary event to this then is “The team has at least one girl.”

The second simulation from part a can be easily ajusted to check our interpretation.

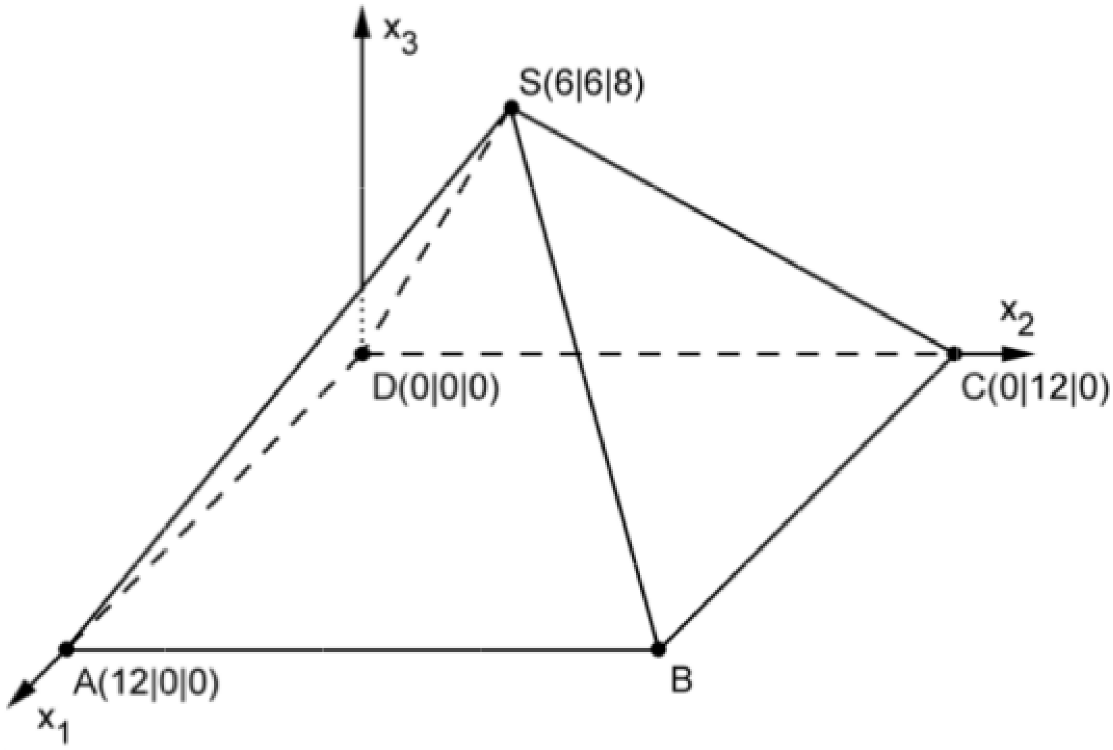
```
sage: iterations = 30000
sage: frequency = 0
sage: for _ in range(iterations):
sage:     team = Subsets(14, 4).random_element()
sage:     for member in team:
sage:         if member <= 8:
sage:             frequency = frequency+1
sage:             break
sage: p = (binomial(14, 4)-binomial(6, 4))/binomial(14, 4)
sage: print 'given probability = %6.3f' % float(p)
sage: print 'simulated probability = %6.3f' % float(frequency/iterations)
```

3.3 Geometry

3.3.1 Pyramid

Problem

The figure depicts a model of an exhibition pavilion which has the shape of a regular four-sided pyramid with a square base and is placed on a horizontal surface. In this model, the triangle BCS describes the southern exterior wall of the pavilion. In the coordinate system, the length unit corresponds to 1 m, i.e. the base of the pavilion has a lateral length of 12 m.



1. Give the coordinates of the point B and determine the volume of the pavilion.
2. In the model, the southern exterior wall of the pavilion lies in the plane E. Determine the equation of E in its point-normal form.

   (possible result: $E : 4x_2 + 3x_3 - 48 = 0$*)*
3. The interior work of the pavilion requires a thin and as short as possible bar between the center point of the base and the southern exterior wall. Determine at which height above the base the bar has to be attached to the exterior wall.

Solar panels are mounted flush on some part of the southern exterior wall. In the model, the solar panels cover a triangular surface the vertices of which are the top S, as well as the centers of the edges $[SB]$ and $[SC]$.

4. Find the area of the surface covered by the solar panels.
5. The electric power delivered by the solar panels depends amongst other things on the

magnitude of the inclination angle with respect to the horizontal. The table gives the percentage of the delivered power compared to the maximally possible power as a function of the inclination angle. Estimate this percentage for the solar panels of the pavilion - after calculation of the inclination angle - by making use of the table.

inclination angle	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
percentage of maximal power	87%	93%	97%	100%	100%	98%	94%	88%	80%	69%

Solution of part a

The coordinates of point B can be computed by adding the vectors of D to A and C, respectively, yielding $B(12|12|0)$.

The volume V of the pyramid can be determined by means of the formula

$$V = \frac{1}{3}Ah = \frac{1}{3}144 \cdot 8 = 384$$

with base area A and height h. In general, the formula

$$V = \frac{1}{3}(\vec{A} \times \vec{C}) \cdot \vec{S},$$

holds, which can easily be evaluated with Sage.

```
sage: a = vector([12, 0, 0])
sage: d = vector([0, 0, 0])
sage: c = vector([0, 12, 0])
sage: s = vector([6, 6, 8])

sage: b = a + c
sage: print("B = {}".format(b))

sage: v = 1/3 * a.cross_product(b) * s
sage: print("The volume of the pyramid is {}m³.".format(v))
```

Solution of part b

The plane E which is given by the points B, C and S shall be computed. The equation is to be given in normal-point form. One obtains the normal vector by evaluating the cross-product of two vectors spanning the plane from the point vectors.

$$\vec{n} = (\vec{C} - \vec{B}) \times (\vec{S} - \vec{B}) = \begin{pmatrix} -12 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -6 \\ -6 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 96 \\ 72 \end{pmatrix}.$$

With the vector $\vec{B}$ to point B on the plane, the equation of the plane in point-normal form thus becomes

$$E : \vec{n} \cdot \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \vec{B} \right) = 96(x_2 - 12) + 72x_3 = 0$$

or

$$E : 96x_2 + 72x_3 - 1152 = 0\,.$$

With Sage, one obtains this result as follows:

```
sage: var("x_1, x_2, x_3")
sage: n = (c-b).cross_product(s-b)
sage: print("Normal vector: {}".format(n))
sage: E = n.dot_product(vector([x_1, x_2, x_3])-b) == 0
sage: print("E : {}".format(E))
```

The result corresponds up to a factor of 48 to the result given in the problem.

Solution of part c

The bar from the center of the base $S'(6|6|0)$ to the southern exterior wall shall be as short as possible. Thus, the normal to the plane E through the point S' is sought after. Once the normal is determined, we can compute the intersection with the plane and hence read off the height from its x_3-coordinate.

The normal vector $\vec{n}$ of the plane was already calculated in part b. We define the auxiliary line

$$\vec{h}(t) = \vec{x}_{S'} + t\vec{n} = \begin{pmatrix} 6 \\ 6 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 96 \\ 72 \end{pmatrix}$$

and seek its intersection with the plane E. Plugging the coordinates $x_2(t)$ und $x_3(t)$ into the equation of the plane and solving for t yields $t = \frac{1}{25}$. The desired height then equals 2.88 m.

This result can be confirmed by Sage by following the method just described and making use of the abstractly formulated equation of the plane derived in part b.

```
sage: var("t")
sage: h = vector([6, 6, 0]) + n * t
sage: intersection_equation = n.dot_product(h-b) == 0
sage: print(intersection_equation)

sage: result = solve(intersection_equation, t)
sage: t0 = result[0]
sage: print(t0)

sage: p = h.subs(t0)
sage: print("Height of attachment: {} m = {} m".format(p[2],
  ↪ float(p[2])))
```

Solution of part d

The area can be computed with the help of the cross-product:

$$F = \frac{1}{2}\left|\frac{1}{2}(\vec{S} - \vec{B}) \times \frac{1}{2}(\vec{S} - \vec{C})\right|\,.$$

We leave the calculations to Sage and obtain

```
sage: sb2 = (s-b)/2
sage: sc2 = (s-c)/2
sage: F = abs(sb2.cross_product(sc2))/2
sage: print("F = {}m²".format(F))
```

Solution of part e

The inclination angle of the plane E with respect to the x_1-x_2-plane equals the angle between the two normals of the planes. For the x_1-x_2-plane, the normal vector corresponds to the unit vector along the x_3-direction. The normal vector of the plane E is the vector $\vec{n}$ which was determined in part b. The desired angle can be computed by means of the following formula:

$$\phi = \cos^{-1}\left(\frac{\vec{n}\cdot\vec{e}_3}{|\vec{n}|\,|\vec{e}_3|}\right).$$

```
sage: x_3 = vector([0,0,1])
sage: print("Inclination angle: {}°".format((arccos(n*x_3/n.norm())
    * 180/pi).n(digits=3)))
```

For an angle of 53,1°, one can estimate the delivered power with the help of the table to be 96 to 97% of the maximal power.

3.3.2 Straight lines in 3d

Problem

The straight lines $g : \vec{X} = \begin{pmatrix}8\\1\\7\end{pmatrix} + \lambda\cdot\begin{pmatrix}3\\1\\2\end{pmatrix}$, $\lambda \in \mathbb{R}$, and $h : \vec{X} = \begin{pmatrix}-1\\5\\-9\end{pmatrix} + \mu\cdot\begin{pmatrix}1\\-2\\4\end{pmatrix}$, $\mu \in \mathbb{R}$, are given in a cartesian coordinate system. The lines g and h intersect at the point T.

1. Determine the coordinates of T.

   (result: $T(2|-1|3)$)

2. Give the coordinates of two points P and Q which lie on g and are at equal distance from T.

3. Two points U and V of the line h together with the points P and Q form the rectangle $PUQV$. Describe a method to determine the coordinates of U and V.

Solution of part a

One obtains the intersection of the lines g and h by equating the equations of the two lines. The equations for the three coordinates yield the system of linear equations

$$\begin{aligned} 3\lambda - \mu &= -9 \\ \lambda + 2\mu &= 4 \\ 2\lambda - 4\mu &= -16 \end{aligned}$$

with the solution $\lambda = -2$, $\mu = 3$. Plugging this into the equation of the lines yields the given coordinates of the point T.

The solution can be determined with the help of Sage:

```
sage: var("lamb, mu")
sage: g = vector([8, 1, 7]) + lamb * vector([3, 1, 2])
sage: h = vector([-1, 5, -9]) + mu * vector([1, -2, 4])
sage: result = solve([g[0] == h[0],
sage:                 g[1] == h[1],
sage:                 g[2] == h[2]], mu, lamb)
sage: print("Values at the intersection: {}".format(result[0]))
sage: t = h.subs(result[0][0])
sage: print("Intersection at: T = {}".format(t))
```

To illustrate the result one can draw the lines in a three-dimensional coordinate system with Sage.

```
sage: labeloffset = vector([0, 0, 2])
sage: pg = line([g(lamb = -4), g(lamb = 0)], color = 'blue')
sage: tg = text3d("g", g(lamb = 0) + labeloffset, color='blue',
   ↪ horizontal_alignment='left')
sage: ph = line([h(mu = 0), h(mu = 4)], color = 'purple')
sage: th = text3d("h", h(mu = 0) + labeloffset, color='purple',
   ↪ horizontal_alignment='left')
sage: pt = point(t, size=10, color='red')
sage: tt = text3d("T", t + labeloffset, color='red',
   ↪ horizontal_alignment='left')
sage: p1 = pg + tg + ph + th + pt + tt
sage: show(p1, aspect_ratio=1)
```

Solution of part b

The intersection T can be obtained by plugging the value of λ determined in part a into the equation of line g. Now, two points P and Q are to be determined which lie on g at equal distance from T. To this end, one adds a chosen value to λ and subtracts the same value from λ.

In part a, we found $\lambda = -2$. The two points P and Q can be obtained, for example, by plugging the values $\lambda = -1$ and $\lambda = -3$ into the equations of the line. These points are added to the coordinate system.

```
sage: p = g(lamb = result[0][1].right() + 1)
sage: print("P {}".format(p))
sage: pp = point(p, size=10, color='green')
sage: tp = text3d("P", p + labeloffset, color='green',
 ↪ horizontal_alignment='left')
sage: q = g(lamb = result[0][1].right() - 1)
sage: print("Q {}".format(q))
sage: pq = point(q, size=10, color='green')
sage: tq = text3d("Q", q + labeloffset, color='green',
 ↪ horizontal_alignment='left')
sage: p2 = p1 + pp + tp + pq + tq
sage: show(p2, aspect_ratio=1)
```

Solution of part c

In the following, a method is described for finding two points U and V which lie on line h such that $PUQV$ forms a rectangle.

The points P and Q are opposite corners of the rectangle. It follows that the line g between these points is a diagonal of the rectangle. Since T is exactly the center between those two points, it has to be the center of the rectangle as well.

The two other points shall lie on the line h. The same considerations as before imply that the line h between the points U and V is the second diagonal of the rectangle.

To obtain a rectangle from a quadrangle with known diagonals, the diagonals have to have the same length and the center of the diagonals has to be the intersection of these. Thus, the two points U and V have to be at equal distance from T, just as the points P and Q.

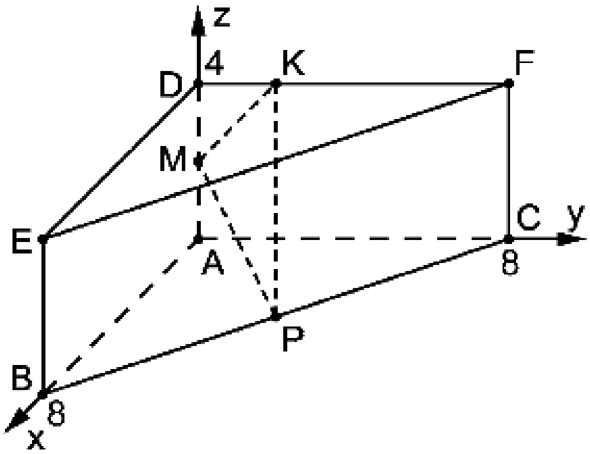
With Sage, we first compute the distance between the points T and Q. Subsequently, the value μ is determined for which the corresponding point on the line h is at equal distance from T as the point Q. Then, the points U and V can be obtained by plugging $\pm\mu$ into the equation of line h.

```
sage: from sage.plot.polygon import Polygon

sage: disance = (t-q).norm()
sage: print("Distance between T and Q: {}".format(disance))
sage: result = solve(mu*vector([1, -2, 4]).norm() == disance, mu)
sage: print(result[0])
sage: mu_1 = result[0].right()
sage: mu_2 = -mu_1

sage: rectangle = line3d([q, t+mu_1*vector([1, -2, 4]), p,
 ↪ t+mu_2*vector([1, -2, 4]), q], color='orange', thickness=5)
sage: show(p2 + rectangle, aspect_ratio=1)
```

3.3.3 Prism

Problem

The image shows a right prism $ABCDEF$ with the points $A(0|0|0)$, $B(8|0|0)$, $C(0|8|0)$ and $D(0|0|4)$.



1. Determine the distance between the two vertices B and F.
2. The points M and P correspond to the midpoints of the edges $[AD]$ and $[BC]$. The point $K(0|y_K|4)$ is located on the edge $[DF]$. Determine y_K so that the triangle KMP is orthogonal in M.

Solution of part a

First, we need to calculate the coordinates of the vertex F located above the point C in z-direction, at the same height as the point D. F therefore has the coordinates $F(0|8|4)$.

The distance between B and F can be calculated as

$$\overline{BF} = |\vec{B} - \vec{F}| = \sqrt{8^2 + (-8)^2 + (-4)^2} = 12.$$

We can verify the result with the help of Sage by constructing the point F and determining its distance to the point B.

```
sage: a = vector([0, 0, 0])
sage: b = vector([8, 0, 0])
sage: c = vector([0, 8, 0])
sage: d = vector([0, 0, 4])
sage: f = c + d - a
sage: print 'distance B-F:', norm(b-f)
```

Solution of part b

The midpoint M of the edge between the points A and D can be determined by

$$\vec{M} = \vec{A} + 1/2 \cdot (\vec{D} - \vec{A}).$$

Correspondingly, we obtain the coordinates of the midpoints $M(0|0|2)$ and $P(4|4|0)$.

```
sage: m = a + 1/2 * (d - a)
sage: p = b + 1/2 * (c - b)
sage: print "m:", m, ", p:", p
```

In order to determine the y-value of the point K, we will use the condition that the triangle KMP has to be orthogonal in M. This implies that the inner product of the vectors $\vec{k}$ and $\vec{p}$ connecting the point M to the points K and P, respectively, must vanish. The vectors can be determined as

$$\vec{k} = \begin{pmatrix} 0 \\ y_K \\ 2 \end{pmatrix} \qquad \vec{p} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$$

with the inner product

$$\vec{k} \cdot \vec{p} = 4y_k - 4 \overset{!}{=} 0,$$

Therefore, we find $y_K = 1$.

Sage offers a fast way to verify the solution:

```
sage: y = var('y')
sage: k = vector([0, y, 4])
sage: solve((m-k).dot_product(m-p) == 0, y)
```

3.3.4 The plane

Problem

Given the plane $E : 3x_2 + 4x_3 = 5$,

1. explain the special location of the plane E in the coordinate system.
2. verify by calculation whether a sphere with center $Z(1|6|3)$ and radius 7 intersects the plane E.

Solution of part a

The equation for the plane is independent of the x_1-component. Therefore, the plane E is aligned parallel to the x_1-axis.

We can use Sage to depict the plane in three dimensions by solving the plane's equation for x_3.

```
sage: var('x1')
sage: var('x2')
sage: var('x3')
sage: solution = solve(3*x2+4*x3 == 5, x3, solution_dict=1)[0]
sage: p1 = plot3d(solution[x3], (x1, -10, 10), (x2, -10, 10))
sage: p1
```

Solution of part b

First, we will use Sage to get an overview of the problem.

```
sage: p2 = sphere(center=(1, 6, 3), size=7, color='red', opacity=1)
sage: show(p1 + p2, aspect_ratio=1)
```

It is easy to see that the sphere and the plane intersect.

In order to analytically verify this observation, we will determine the distance between the plane and the center of the sphere. The normal vector of the plane can be derived from its Hesse normal form:

$$\vec{n} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix},$$

or in normalized form:

$$\vec{n}_0 = \begin{pmatrix} 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}.$$

In order to calculate the distance between the plane and the center of the sphere, we define points along the line aligned parallel to the normal $\vec{n}_0$ and going through the center of the sphere as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} + t \cdot \begin{pmatrix} 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}.$$

The value of $|t|$ corresponds to the distance of the current point from the center of the sphere. The value t of the intersection point of the line and the plane can be determined by inserting the coordinates of the line into the plane's equation. We find

$$\begin{aligned} &3 \cdot \left(6 + \frac{3}{5}t\right) + 4 \cdot \left(3 + \frac{4}{5}t\right) - 5 = 0 \quad \Leftrightarrow \\ &t = -5 \quad \Leftrightarrow \\ &|t| = 5. \end{aligned}$$

Thus, the distance between the plane and the center of the sphere equals 5 and is smaller than the radius 7 of the sphere. Therefore, the sphere intersects the plane. The calculation can be verified by Sage:

```
sage: t = var('t')
sage: n = vector([0, 3, 4])
sage: n0 = n/norm(n)
sage: z = vector([1, 6, 3])
sage: radius = 7
sage: line = z+t*n0
sage: solution = solve(3*line[1]+4*line[2] == 5, t,
 ↪  solution_dict=True)[0]
```

```
sage: distance = abs(solution[t])
sage: print "Distance: center of the sphere - plane:", distance
sage: if distance < radius:
...        print('plane intersects sphere')
sage: else:
...        print('plane does not intersect sphere')
```

3.3.5 The cuboid

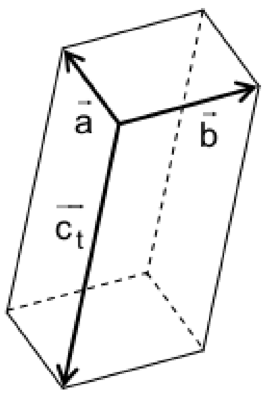
Problem

The vectors

$$\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{c_t} = \begin{pmatrix} 4t \\ 2t \\ -5t \end{pmatrix},$$

generate a geometric body for every t with $t \in \mathbb{R}\backslash\{0\}$. The figure below illustrates this for an example of t.

1. Prove that the generated body is always a cuboid.
2. Determine the values of t leading to a cuboid with a volume of 15.



Solution of part a

The generated body is a cuboid if and only if the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually orthogonal. Therefore, we calculate all possible inner products of two vectors and verify that all of them vanish.

$$\vec{a} \cdot \vec{b} = 2 \cdot (-1) + 1 \cdot 2 + 2 \cdot 0 = 0$$
$$\vec{a} \cdot \vec{c} = 2 \cdot 4t + 1 \cdot 2t + 2 \cdot (-5t) = 0$$
$$\vec{b} \cdot \vec{c} = (-1) \cdot 4t + 2 \cdot 2t + 0 \cdot (-5t) = 0$$

The fact that every inner product vanishes proves that the generated body is a cuboid for any value of t. The inner products can also be calculated with Sage:

```
sage: t = var('t')
sage: a = vector([2, 1, 2])
sage: b = vector([-1, 2, 0])
sage: c = vector([4*t, 2*t, -5*t])
sage: print u"a\u00b7b =", a.dot_product(b)
sage: print u"a\u00b7c =", a.dot_product(c)
sage: print u"b\u00b7c =", b.dot_product(c)
```

Solution of part b

The volume of a cuboid can be calculated with the scalar triple product:

$$\begin{aligned} V &= \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right| \\ &= \left|\begin{pmatrix}2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}2\cdot(-5t)-0\cdot 2t\\ 0\cdot 4t-(-1)\cdot(-5t)\\ (-1)\cdot 2t-2\cdot 4t\end{pmatrix}\right| \\ &= \left|2\cdot(-10t)+1\cdot(-5t)+2\cdot(-10t)\right| \\ &= 45\,|t|\,. \end{aligned}$$

Together with the condition that the cuboid's volume should be equal to 15, we obtain

$$V = 45\,|t| \overset{!}{=} 15$$

which can be rewritten as

$$|t| = \frac{1}{3} \Leftrightarrow t = \pm\frac{1}{3}.$$

We obtain two possible solutions for t. In one case, the vector c points towards the upper half-space while in the other case it points towards the lower half-space.

This result can be verified with the help of Sage. In particular, the evaluation of the triple product is significantly simplified.

```
sage: V = abs(a.dot_product(b.cross_product(c)))
sage: print "Volume =", V
sage: solve(V == 15, t)
```

3.3.6 The sphere

Problem

A sphere is centered at $M(-3|2|7)$. The point $P(3|4|4)$ is located on the surface of this sphere.

1. The point Q is also located on the surface of the sphere. The line segment $[PQ]$ contains the center M. Determine the coordinates of the point Q.
2. Prove that the sphere touches the x_1x_2-plane.

Solution of part a

Since both points, P and Q, are located on the surface of the sphere and the line segment connecting these points contains the center of the sphere, Q can be represented as:

$$\vec{Q} = \vec{M} + \left(\vec{M} - \vec{P}\right).$$

As a result, we obtain

$$Q(-9|0|10),$$

which can be verified with Sage:

```
sage: M = vector([-3, 2, 7])
sage: P = vector([3, 4, 4])
sage: print "Q:", M + M - P
```

Solution of part b

In order to verify whether the sphere touches the x_1x_2-plane, we only need to calculate the distance between the plane and the center of the sphere and compare it to the sphere's radius. This distance is equal to the absolute value of the center's x_3-coordinate, which equals 7.

The radius of the sphere can be determined from the distance between the point P on the sphere and the sphere's center:

$$r = \left|\vec{M} - \vec{P}\right| = \sqrt{(-3-3)^2 + (2-4)^2 + (7-4)^2} = 7$$

Thus, the distance between the plane and the center of the sphere equals the radius of the sphere, so that the sphere touches the plane in one point. The coordinates of this point can be obtained by projecting the sphere's center to the x_1x_2-plane, yielding $S(-3|2|0)$.

We can use Sage to illustrate this in three-dimensional space. The mouse can be used to rotate the view.

```
sage: var('x')
sage: var('y')
sage: z = 0
sage: radius = norm(M-P)
sage: print 'sphere radius          :', radius
sage: print 'distance center - plane:', M[2]
sage: p1 = plot3d(z, (x, -15, 15), (y, -15, 15), opacity=0.7)
sage: p2 = sphere(center=(-3, 2, 7), size=radius, color='red',
  ↪ opacity=0.7)
sage: show(p1 + p2, aspect_ratio=1)
```

3.3.7 The triangle

Problem

In a cartesian coordinate system, a triangle ABC located in the plane $E : x_1 + x_2 + x_3 = 4$ is defined by the points $A(4|0|0)$, $B(0|4|0)$, and $C(0|0|4)$.

1. Determine the area of the triangle ABC.

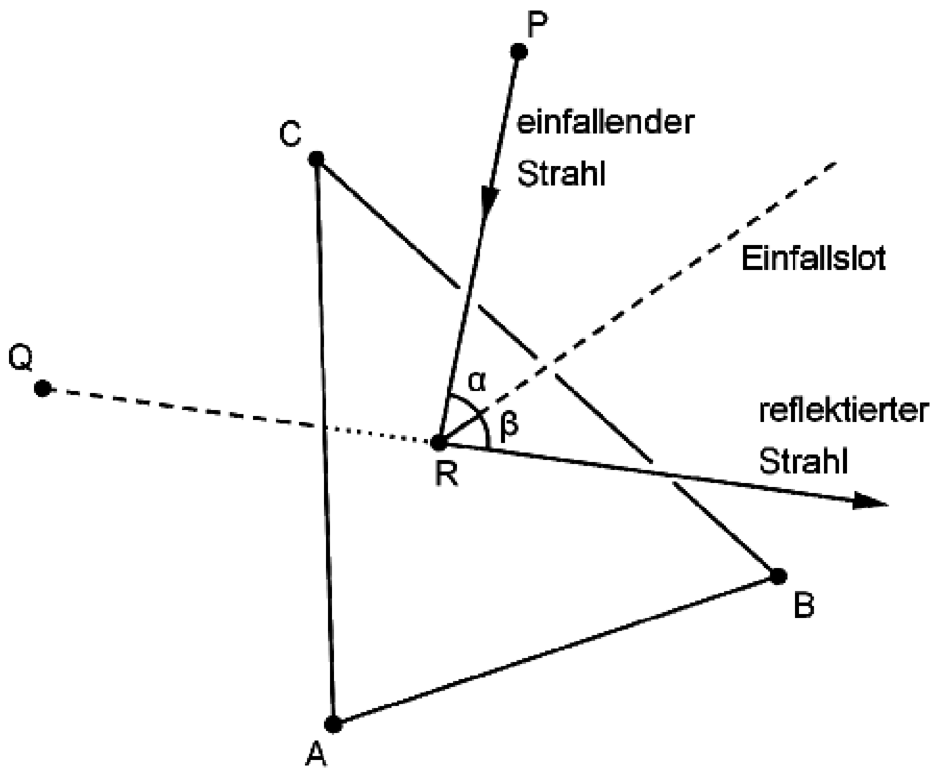
The triangle ABC is exemplary for a mirror. Within the model, the point $P(2|2|3)$ indicates the position of a light source emitting a beam of light, and the direction of this light beam is described by the vector

$$\vec{v} = \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix} .$$

2. Specify the equation of a straight line g along which the light beam propagates in the model. Determine the coordinates of the point R in which g intersects the plane E and establish that the light beam impinges on the triangular mirror.

   (check: $R(1{,}5|1{,}5|1)$ *)*

The incoming light beam (*einfallender Strahl*) is reflected in the point of the mirror represented within the model by point R. For an observer, the reflected light beam (*reflektierter Strahl*) seems to originate from a position described within the model by point $Q(0|0|1)$ (cf. figure).



3. Demonstrate that the points P and Q are symmetric with respect to the plane E.

The normal on plane E in point R is referred to as axis of indicence (*Einfallslot*).

4. The two straight lines along which the incoming and the reflected light beams propagate within the model are located in the plane F. Determine an equation for F in Hessian normal form. Demonstrate that the axis of incidence is also situated in the plane F.

   (possible partial result: $F : x_1 - x_2 = 0$ *)*

5. Demonstrate that the magnitudes of the angles β between the reflected light beam and the axis of incidence and α between the incoming light beam and the axis of incidence agree.

Solution of part a

In order to determine the area of the triangle, we first need to determine the connecting vectors $\vec{AB}$ and $\vec{AC}$:

$$\vec{AB} = \vec{B} - \vec{A} = \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix},$$
$$\vec{AC} = \vec{C} - \vec{A} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix}.$$

Now, we can determine the area of the triangle by means of the cross product:

$$A = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| = \frac{1}{2}\left|\begin{pmatrix} 16 \\ 16 \\ 16 \end{pmatrix}\right| = 8\sqrt{3}.$$

We check the result by means of Sage:

```
sage: a = vector([4, 0, 0])
sage: b = vector([0, 4, 0])
sage: c = vector([0, 0, 4])
sage: ab = b - a
sage: ac = c - a
sage: A = 1/2 * abs(ab.cross_product(ac))
sage: print "Area of the triangle:", A
```

Furthermore, we graphically represent the triangle by means of Sage:

```
sage: from sage.plot.polygon import Polygon
sage: labeloffset = vector([0, 0, 0.3])
sage: p1 = polygon([a, b, c])
sage: for p, label in ((a, 'A'), (b, 'B'), (c, 'C')):
sage:     p1 = p1+point(p, size=10)
sage:     p1 = p1+text3d(label, p+labeloffset, color='black',
   ↪ horizontal_alignment='left')
sage: show(p1)
```

Solution of part b

The straight line must go through point P and run along the vector $\vec{v}$. The representation of the straight line is then obtained as

$$\vec{g} = \vec{P} + \lambda\vec{v} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix} .$$

In order to obtain the point of intersection R, we insert the coordinates of the straight line into the equation of the plane:

$$\begin{aligned} 2 - \lambda + 2 - \lambda + 3 - 4\lambda = 7 - 6\lambda &\stackrel{!}{=} 4 \\ \rightarrow \lambda &= \frac{1}{2}. \end{aligned}$$

The point of intersection is then obtained as

$$\vec{R} = \vec{P} + \frac{1}{2}\vec{v} = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{pmatrix} .$$

We check this point of intersection by means of Sage

```
sage: p = vector(QQ, (2, 2, 3))
sage: v = vector(QQ, (-1, -1, -4))
sage: plane = Polyhedron(eqns=[(-4, 1, 1, 1)])
sage: straight_line = Polyhedron(vertices=[p], rays=[-v, v])
sage: r = straight_line.intersection(plane).vertices()[0].vector()
sage: print 'point of intersection R', r
```

and insert the straight line into the graphics of part a). As we can see, the straight line hits the triangular mirror. This can also be deduced from the fact that all coordinates of the point of intersection are positive.

```
sage: p2 = line([p, r], color='red', thickness=2)
sage: for pt, label in ((p, 'P'), (r, 'R')):
sage:     p2 = p2+point(pt, size=10)
sage:     p2 = p2+text3d(label, pt+labeloffset, color='black',
  ↪ horizontal_alignment='left')
sage: show(p2 + p1)
```

Solution of part c

We first consider the connecting vector $\vec{PQ}$:

$$\vec{PQ} = \vec{Q} - \vec{P} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix} .$$

From the equation of the plane E, it is straightforward to deduce that a non-normalized normal vector of E is given by

$$\vec{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} .$$

This vector obviously is parallel to the connecting vector $\vec{PQ}$. The points P and Q thus lie on the same normal of E. For the two points to be symmetric with respect to E, they need to have the same distance to the plane. We verify this by inserting the midpoint of the connecting line into the equation of the plane:

$$\vec{M} = \vec{P} + \frac{1}{2}\vec{PQ} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$E: \quad 1 + 1 + 2 = 4$$

The points are thus symmetric with respect to the plane.

We add the connection between the points P and Q as well as the reflected light beam to our sketch:

```
sage: q = vector([0, 0, 1])
sage: Q = point(q, size=10)
sage: Qt = text3d("Q", q + labeloffset, color='black',
  ↪  horizontal_alignment='left')
sage: pq = line([p, q], color='black', thickness=2)
sage: g_refl = line([r, r + r-q], color='red', thickness=2)
sage: g_refl_q = line([q, r], color='red', thickness=1,
  ↪  linestyle='--')
sage: p3 = Q + Qt + pq + g_refl + g_refl_q
sage: show(p1 + p2 + p3)
```

Solution of part d

We consider the two straight lines

$$\vec{g}_1 = \vec{R} + \lambda\vec{v} = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix}$$

and

$$\vec{g}_2 = \vec{R} + \mu(\vec{R} - \vec{Q}) = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{pmatrix} + \mu \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix} .$$

Together with the common point of intersection R, the equation of the plane is easily obtained as

$$F: \quad \vec{X} = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix} .$$

This equation of the plane can be expressed in terms of the coordinates x_1 and x_2:

$$E: \quad x_1 - x_2 = 0\,.$$

The axis of incidence normal to the plane E and going through point R can be represented as

$$\vec{e} = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Obviously, the coordinates x_1 and x_2 of e satisfy the equation of the plane F so that e lies in F.

We add the plane F and the axis of incidence e to our drawing:

```
sage: axis_of_incidence = vector([1,1,1])
sage: F = polygon((vector([0,0,0]), vector([4,4,0]), vector([4,4,4]), vector([0,0,4])), color='green')
sage: e = line([r, r + axis_of_incidence], color='black', thickness=2)
sage: p4 = e + F
sage: show(p1 + p2 + p3 + p4)
```

Solution of part e

The angles α and β are easily calculated by means of a scalar product:

$$\cos(\alpha) = \frac{-\vec{v}\cdot\vec{n}}{|\vec{v}|\,|\vec{n}|} = -\frac{1}{\sqrt{18}} \begin{pmatrix} -1 \\ -1 \\ -4 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \sqrt{\frac{2}{3}}$$

and analogously:

$$\cos(\beta) = \frac{\vec{QR}\cdot\vec{n}}{\left|\vec{QR}\right|\,|\vec{n}|} = \frac{2}{\sqrt{18}} \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \sqrt{\frac{2}{3}}$$

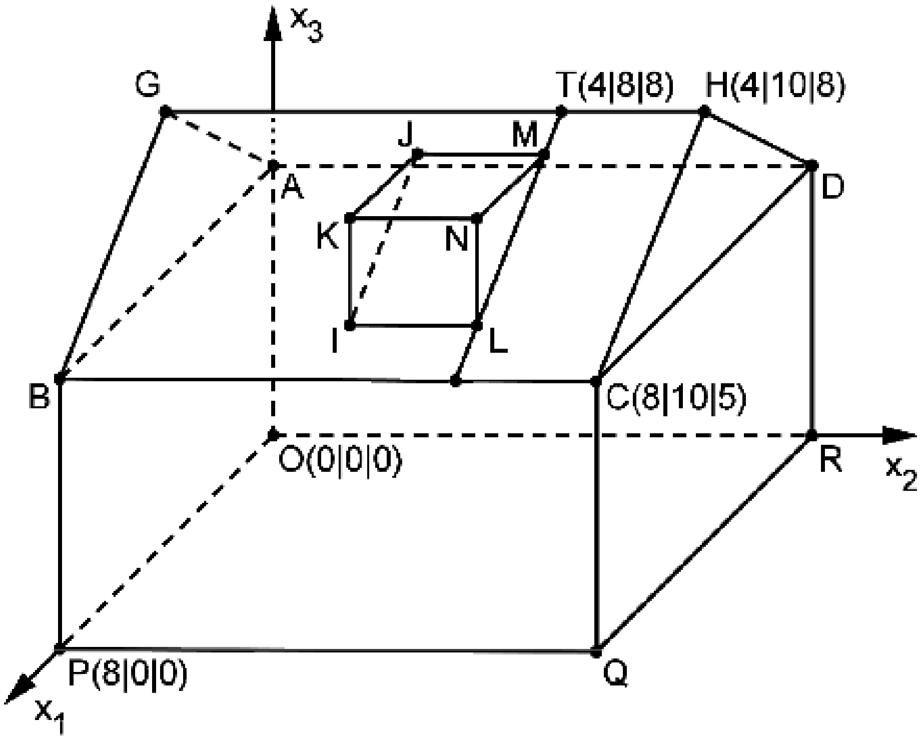
The angles α and β thus have the same magnitude which is also confirmed by our sketch and by explicit evaluation with the help of Sage.

```
sage: n = axis_of_incidence.normalized()
sage: cosa = -n.dot_product(v.normalized())
sage: cosb = n.dot_product((r-q).normalized())
sage: print simplify(cosa-cosb)
```

3.3.8 A house

Problem

The figure displays exemplarily a family home erected on a horizontal surface. On one of the rectangular roof surfaces, a dormer shall be erected. The points A, B, C, D, O, P, Q and R are vertices of a cuboid. The straight triangular prism $LMNIJK$ represents the dormer while the straight line $[GH]$ represents the roof ridge, i.e. the upper horizontal edge of the roof. A length unit in the coordinate system corresponds to 1 m, i.e., the house has a length of 10 m.



1. Calculate the area of the roof surface represented in the model by the rectangle $BCHG$.
2. In the town where the family home is located, a charter for the erection of dormers exists which has to be abided by every constructor. This charter allows the erection of a dormer provided the inclination of the roof surface of the respective roof against the horizonal is at least 35°. Demonstrate by means of a calculation that for the family home under consideration, the erection of a dormer is permissible.

Within the model, the roof on which the dormer is to be erected lies in the plane

$$E : 3x_1 + 4x_3 - 44 = 0.$$

The dormer shall be erected in such a way that its distance from the lateral edge of the roof surface represented in the model by the line HC equals 2 m while its distance from the roof ridge is 1 m. In order to obtain the coordinates of point M, the straight line

$$t : \vec{X} = \begin{pmatrix} 4 \\ 8 \\ 8 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

going through point $T(4|8|8)$ is considered.

3. Justify that t lies in the plane E and is situated at a distance 2 from the straight line HC.
4. Now, a point M is determined on the straight line t such that the distance between dormer and roof ridge is 1 m. Determine the coordinates of M.

(Result: $M(4.8|8|7.4)$)

The points M and N lie on the straight line

$$m : \vec{X} = \begin{pmatrix} 4.8 \\ 8 \\ 7.4 \end{pmatrix} + \mu \cdot \begin{pmatrix} 6 \\ 0 \\ -1 \end{pmatrix}, \quad \mu \in \mathbb{R},$$

which in the model determines the inclination of the roof surface of the dormer. The line $[NL]$ parallel to the x_3 axis within the model represents the height of the dormer which shall be 1.4 m. In order to determine the coordinates of N und L, the plane F obtained by shifting E by 1.4 in the positive x_3 direction is considered.

5. Justify that $3x_1 + 4x_3 - 49.6 = 0$ is an equation describing F.
6. Determine the coordinates of N and L.

   (Partial result: $N(7.2|8|7)$)

Solution of part a

The area of a rectangle is obtained as a product of length and width:

$$A = |BC|\,|BG|$$

From the figure, we determine the points $B(8|0|5)$ and $G(4|0|8)$. Together with $C(8|10|5)$ we obtain

$$A = \left|\begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix}\right| \left|\begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}\right| = 50$$

The area thus amounts to 50m^2.

In Sage, we first define all points and then check the result.

```
sage: o = vector([0,0,0])
sage: p = vector([8,0,0])
sage: c = vector([8, 10, 5])
sage: width, length, height = c
sage: h = vector([4, 10, 8])
sage: t = vector([4, 8, 8])
sage: a = o+vector([0, 0, height])
sage: b = p+vector([0, 0, height])
sage: g = h-vector([0, length, 0])
sage: print "Area: %sm²" % float(norm(b-g)*norm(c-b))
```

Solution of part b

The angle between two vectors $\vec{a}$ and $\vec{b}$ can be calculated by means of the cosine rule:

$$\cos\alpha = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

By means of this formula, Sage yields a value of about $37°$. The house thus is in accordance with the charter.

```
sage: ba = (a-b).normalized()
sage: bg = (g-b).normalized()
sage: print "Inclination of roof: %4.1f°" %
  ↪  float(arccos(ba.dot_product(bg))*180/pi)
```

Solution of part c

We insert the straight line t into the equation for the plane:

$$3(4 + 4\lambda) + 4(8 - 3\lambda) - 44 = 0.$$

t thus lies in the plane. The straight line

$$HC = H + \lambda\,(H - C) = \begin{pmatrix} 4 \\ 10 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

obviously is parallel to t. The distance between t and HC equals the distance between the points T and H:

$$\left|\vec{H} - \vec{T}\right| = \left|\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right| = 2$$

This result is confirmed by Sage:

```
sage: norm(h-t)
```

Solution of part d

The distance of a point given by λ from the point T ist determined by

$$\left|\lambda \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}\right| = 25\lambda^2 .$$

For $\lambda = \pm\frac{1}{5}$ the distance equals 1. This result is confirmed by Sage.

```
sage: lamb = solve(abs(x*(c-h)) == 1, x)
sage: print "The solutions for  are:", lamb
```

However, only the positive solution for λ makes sense, because the other point is situated above the roof ridge. Thus we obtain the point $M(4.8|8|7.4)$.

```
sage: m = t + lamb[1].right() * (c-h)
sage: print "point M:", m
```

Solution of part e

Shifting E by 1.4 in the positive x_3 direction, the equation of the plane is rewritten as:

$$F: 3x_1 + 4(x_3 - 1.4) - 44 = 0 \Leftrightarrow$$
$$F: 3x_1 + 4x_3 - 49.6 = 0$$

Solution of part f

Inserting the equation for the straight line m into the equation for the plane F yields

$$3(4.8 + 6\mu) + 4(7.4 - \mu) - 49.6 = 14\mu - 5.6 = 0$$
$$\Rightarrow \ \mu = 0.4$$

By insertion into the equation for the straight line m, one obtains the point $N(7.2|8|7)$, and by shifting by -1.4 in the x_3 direction the point $L(7.2|8|5.6)$ is found. These results are obtained by means of Sage as follows:

```
sage: mu = solve(3*(4.8+6*x) + 4*(7.4-x) - 49.6 == 0, x)[0].right()
sage: n = m + mu*vector([6, 0, -1])
sage: l = n + vector([0, 0, -7/5])
sage: print "Coordinates of N: ", n, ", L:", l
```

3.3.9 Parallelogram

Problem

A straight line g goes through the points A(0|1|2) and B(2|5|6).

1. Demonstrate that the distance between points A and B is 6. The points C and D lie on g and have each the distance 12 from A. Determine the coordinates of C and D.
2. The points A, B and E(1|2|5) together with one more point shall form the vertices of a parallelogram. There exist several possibilities for the position of the fourth vertex. State the coordinates of two of the possible fourth vertices.

Solution of part a

The vector connecting points A and B has the coordinates (2, 4, 4). Its length is therefore given by $\sqrt{2^2 + 4^2 + 4^2} = \sqrt{36} = 6$. The points C and D can be obtained by adding or subtracting twice the vector from A to B to the position vector of A. We thus obtain the points C (4|9|10) and D(-4|-7|-6).

We now implement this reasoning in Sage. First we calculate the distance between points A and B. then we determine the coordinates of points C and D. Finally, we verify that the distance between points C and D on the one hand and the point A on the other hand equals indeed 12.

```
sage: a = vector([0, 1, 2])
sage: b = vector([2, 5, 6])
sage: print 'Length of vector from A to B:', norm(b-a)
sage: c = a+2*(b-a)
sage: d = a-2*(b-a)
sage: print 'Coordinates of C:', c
sage: print 'Coordinates of D:', d
sage: print 'Distance of points A and C:', norm(c-a)
sage: print 'Distance of points A and D:', norm(d-a)
```

Solution of part b

Choosing two of three possible vectors between the given points, one adds one vector to the end of the other one to obtain the fourth point.

We start by using the vector from A to B and from A to E:

```
sage: a = vector([0, 1, 2])
sage: b = vector([2, 5, 6])
sage: e = vector([1, 2, 5])
sage: a_to_b = b-a
sage: a_to_e = e-a
sage: f1 = b+a_to_e
sage: f2 = e+a_to_b
sage: f1, f2
```

The two ways to obtain the fourth vertex F yield the same result as it should be. One possible fourth vertex therefore is given by F(3|6|9).

Another parallelogram is obtained, if point B as being diagonally opposite to the new point.

```
sage: a = vector([0, 1, 2])
sage: b = vector([2, 5, 6])
sage: e = vector([1, 2, 5])
sage: b_to_a = a-b
sage: b_to_e = e-b
sage: f1 = a+b_to_e
sage: f2 = e+b_to_a
sage: f1, f2
```

For the sake of completeness we also determine the third possible vertex.

```
sage: a = vector([0, 1, 2])
sage: b = vector([2, 5, 6])
sage: e = vector([1, 2, 5])
sage: e_to_a = a-e
sage: e_to_b = b-e
sage: f1 = a+e_to_b
```

```
sage: f2 = b+e_to_a
sage: f1, f2
```

3.3.10 Pyramid and vectors

Problem

We consider the pyramid ABCDS with A(0|0|0), B(4|4|2), C(8|0|2), D(4|-4|0), and S(1|1|-4). Its base is a parallelogram.

1. Prove that the parallelogram ABCD is a rectangle.
2. The edge [AS] is normal to the base ABCD. The area of the base is $24\sqrt{2}$. Determine the volume of the pyramid.

Solution of part a

ABCD forms a rectangle if starting from one of the vertices the angle between the shortest vectors to the othe vertices is a right angle.

```
sage: a = vector([0, 0, 0])
sage: b = vector([4, 4, 2])
sage: c = vector([8, 0, 2])
sage: d = vector([4, -4, 0])
sage: print ' Distance A-B:', N(norm(b-a))
sage: print ' Distance A-C:', N(norm(c-a))
sage: print ' Distance A-D:', N(norm(d-a))
sage: (b-a).dot_product(d-a)
```

It follows that the vectors from A to B and from A to D are orthogonal to each other. The point C lies diagonally opposite of A. Therfore, the parallelogram is indeed a rectangle. Since this solution depends on the information that ABCD is a parallelogram, we check alos the other three inner angles.

```
sage: a = vector([0, 0, 0])
sage: b = vector([4, 4, 2])
sage: c = vector([8, 0, 2])
sage: d = vector([4, -4, 0])
sage: (c-b).dot_product(a-b), (d-c).dot_product(b-c),
  ↪  (a-d).dot_product(c-d)
```

Solution of part b

Since the vector from A to S is normal to the base, its length h equals the height of the pyramid. The area of the base is given as $A = 24\sqrt{2}$. We first briefly check the latter result.

```
sage: a = vector([0, 0, 0])
sage: b = vector([4, 4, 2])
sage: d = vector([4, -4, 0])
sage: norm(a-b)*norm(a-d)
```

The height of the pyramid is obtained as

```
sage: a = vector([0, 0, 0])
sage: s = vector([1, 1, -4])
sage: norm(s-a)
```

Then, volume takes on the value $V = \frac{h}{3}A = 48$. This result can be confirmed directly with the help of Sage.

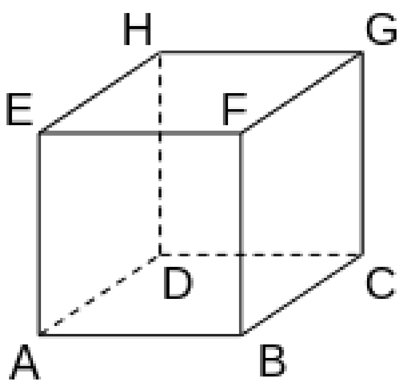
```
sage: a = vector([0, 0, 0])
sage: b = vector([4, 4, 2])
sage: c = vector([8, 0, 2])
sage: d = vector([4, -4, 0])
sage: s = vector([1, 1, -4])
sage: Polyhedron(vertices=[a, b, c, d, s]).volume()
```

3.3.11 The cube

Problem

Consider the cube $ABCDEFGH$ depicted in the figure.

The vertices D, E, F and H of this cube have the following coordinates in a cartesian coordinate system: $D(0|0|-2)$, $E(2|0|0)$, $F(2|2|0)$ and $H(0|0|0)$.

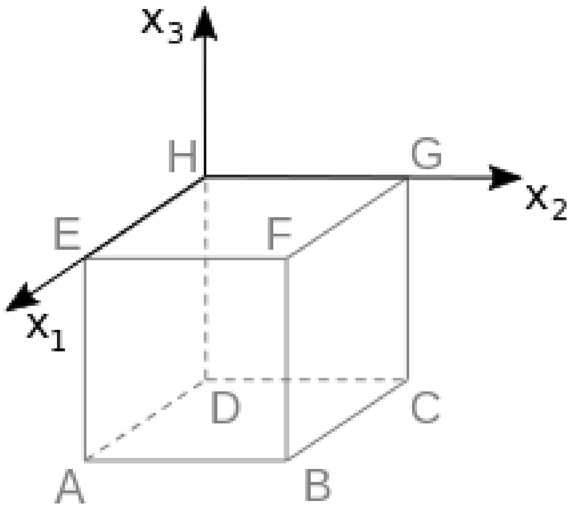


1. Add coordinate axes to the figure and label them accordingly. Give the coordinates of point A.

2. The point P lies on the edge $[FB]$ of the cube and has a distance of 3 from point H. Compute the coordinates of point P.

Solution of part a

The point H lies in the origin of the coordinate system. Moreover, point E lies on the positive x_1-axis and point D on the negative x_3-axis. Since the x_2-coordinate of F is positive, G lies on the positive x_2-axis. The coordinate system can be drawn as shown in the following figure.



The coordinates of A thus follow as $(2|0|-2)$.

Based on the vectors from H to D and E, respectively, one can obtain this result also with the help of Sage.

```
sage: H = vector([0, 0, 0])
sage: E = vector([2, 0, 0])
sage: D = vector([0, 0, -2])
sage: EH = E-H
sage: DH = D-H
sage: print "Point A:", EH+DH
```

Solution of part b

Point P can be determined as the intersection of the edge $[FB]$ and a sphere centered on H with radius 3. The edge is parametrized by the equation

$$[FB] : \vec{X} = \vec{F} + \lambda \cdot \vec{FB} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2\lambda \end{pmatrix}, \quad \lambda \in [0,1] .$$

Moreover, the sphere fulfills the equation

$$\left(\vec{X} - \vec{H}\right)^2 = 3^2 \quad \Leftrightarrow \quad {x_1}^2 + {x_2}^2 + {x_3}^2 = 9 .$$

By plugging the equation for the edge into the equation for the sphere, we obtain

$$2^2 + 2^2 + (-2\lambda)^2 = 9,$$

which can be solved for λ^2:

$$\lambda^2 = \frac{1}{4}$$

Formally, this equation has the two solutions $\lambda_1 = +\frac{1}{2}$ and $\lambda_2 = -\frac{1}{2}$. Because on the edge, the parameter can only take values between 0 and 1, only the solution $\lambda = +\frac{1}{2}$ is admissible. The coordinates of P are obtained by plugging this value into the equation describing the edge:

$$\vec{P} = \begin{pmatrix} 2 \\ 2 \\ -2 \cdot \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

The point P hence has the coordinates $(2|2|-1)$.

Alternatively we can determine the intersection point with Sage:

```
sage: var("Lambda")
sage: H = vector([0,0,0])
sage: F = vector([2,2,0])
sage: B = vector([2,2,-2])
sage: BF = B-F
sage: edge = F+Lambda*BF
sage: solutions = solve(edge*edge == 3^2, Lambda,
 ↪  solution_dict=True)
sage: if 0 <= solutions[0][Lambda] <= 1:
sage:     solution = solutions[0]
sage: else:
sage:     solution = solutions[1]
sage: print "Point P:", edge.substitute(solution)
```

3.3.12 Lines and points in 3d

Problem

The points $A(-2|1|4)$ and $B(-4|0|6)$ are given.

1. Determine the coordinates of the point C such that $\vec{CA} = 2 \cdot \vec{AB}$.
2. The straight line g passes through points A and B. Consider straight lines fulfilling the conditions I and II:
   (a) Each of the lines intersects the line g orthogonally.
   (b) The distance of each of these lines from point A is 3.

   Determine an equation for one of those lines.

Solution of part a

We start by plugging $\vec{CA} = \vec{A} - \vec{C}$ and $\vec{AB} = \vec{B} - \vec{A}$ into the equation

$$\vec{A} - \vec{C} = 2 \cdot \left(\vec{B} - \vec{A}\right) ,$$

and, subsequently, solve for $\vec{C}$:

$$\vec{C} = \vec{A} - 2 \cdot \left(\vec{B} - \vec{A}\right) = 3 \cdot \vec{A} - 2 \cdot \vec{B} = 3 \cdot \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} - 2 \cdot \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

This calculation can quickly be checked with Sage:

```
sage: A = vector([-2, 1, 4])
sage: B = vector([-4, 0, 6])
sage: C = A - 2*(B-A)
sage: print C
```

The coordinates of C are thus $(2|3|0)$.

Solution of part b

First, we observe that the distance from B to A is 3 because the line $[AB]$ has the length

$$|\vec{AB}| = \left| \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \right| = \sqrt{(-2)^2 + (-1)^2 + 2^2} = 3 \,.$$

Sage can also confirm this:

```
sage: A = vector([-2, 1, 4])
sage: B = vector([-4, 0, 6])
sage: BA = B-A
sage: print norm(BA)
```

Therefore, we choose B as the point on the line which has the required closest distance to A. We then are left with determining the direction vector $\vec{G}$ of the line. It has to be perpendicular to $\vec{AB}$ and thus has to fulfill the condition

$$\vec{AB} \cdot \vec{G} = 0 \,.$$

One can then simply guess a solution, for example

$$\vec{G} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} .$$

The equation of the line G thus is:

$$g : \vec{X} = \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} , \quad \lambda \in \mathbb{R} \,.$$

We can determine the direction vector with Sage choosing for example the x_1- and x_2-coordinates at discretion. To check the above result, we set them to 1 and 0, respectively.

```
sage: var('G_3')
sage: G_1 = 1
sage: G_2 = 0
sage: G = vector([G_1, G_2, G_3])
sage: solution = solve(BA*G == 0, G_3, solution_dict=True)[0]
sage: print "Direction vector G:", G.substitute(solution)
```

3.3.13 Planes and vectors

Problem

The plane $E : 2x_1 + x_2 + 2x_3 = 6$ as well as the points $P(1|0|2)$ and $Q(5|2|6)$ are given.

1. Show that the line passing through the points P and Q is perpendicular to the plane E.
2. The points P and Q are symmetric about the plane F. Determine an equation for F.

Solution of part a

The normal vector $\vec{n}$ of the plane E can be read off the plane's equation as

$$\vec{n} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

We choose the direction vector

$$\vec{PQ} = \vec{Q} - \vec{P} = \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$$

for the line PQ. It can be easily verfied that $\vec{PQ} = 2\vec{n}$ is true. The vectors $\vec{PQ}$ and $\vec{n}$ are thus collinear and therefore the line PQ is perpendicular to the plane E.

We can check this graphically with Sage:

```
sage: def E(x1, x2, x3):
sage:     return 2*x1 + x2 + 2*x3 - 6

sage: P = vector([1, 0, 2])
sage: Q = vector([5, 2, 6])

sage: p1 = implicit_plot3d(E, (-1, 7), (-4, 4), (-1, 7),
 ↪  color="blue")
sage: p2 = line([P, Q], color="red", thickness=2)
sage: labeloffset = vector([0, 0, 0.3])
```

```
sage: p3 = point(P, size=10)+text3d("P", P+labeloffset, color="black", horizontal_alignment="right")
sage: p4 = point(Q, size=10)+text3d("Q", P+labeloffset, color="black", horizontal_alignment="right")

sage: show(p1+p2+p3+p4, aspect_ratio=1)
```

Solution of part b

Since the points P and Q are supposed to be symmetric about the plane F, the line PQ is perpendicular to this plane. Furthermore, we have seen in part a that PQ is also perpendicular to plane E. Thus, E and F are parallel and have the same normal vector $\vec{n}$.

We choose the midpoint of the line $[PQ]$ as our reference point

$$\vec{A} = \frac{1}{2} \cdot \left(\vec{P} + \vec{Q}\right) = \frac{1}{2} \cdot \left(\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix} \right) = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

on the plane F. Its equation

$$\left(\vec{X} - \vec{A}\right) \cdot \vec{n} = 0$$

thus reads

$$2x_1 + x_2 + 2x_3 - 15 = 0\,.$$

This can again be verified by means of a 3D graphic:

```
sage: def F(x1, x2, x3):
sage:     return 2*x1 + x2 + 2*x3 - 15

sage: p5 = implicit_plot3d(F, (-1, 7), (-4, 4), (-1, 7), color="green")
sage: show(p1+p2+p3+p4+p5, aspect_ratio=1)
```

CHAPTER 4

Conclusion and outlook

4.1 How to successfully implement the methodology

Implementation of the presented project under Polish education circumstances, consultation with the academic community as well as the conducted evaluation allowed us to draw conclusions about the implementation conditions elaborated in the methodology. In order to identify strengths and weaknesses as well as opportunities and threats regarding the possibility of implementing the project in school practice, a SWOT analysis of the success of the project was carried out. It was formulated on the basis of the joint experience of the project participants, the analysis of the conditions of the local school environment and the external educational environment. We have identified a number of factors that influence the potential success of the project, understood as the implementation of the methodology developed within the project in such a way that it becomes a permanent element in the classroom. Although SageMath is a very useful tool in high school, it is very poorly popularized. It is clear from the evaluation questionnaires that pupils had not heard of him before. They have not heard of SageMath as well. Our actions under this project are designed to change this situation.

It will not be hard for students to be interested in students because they are embedded in the world of computer and information technology. They do not have any worries about it, they are familiar with new applications. These included interviews, evaluation questionnaires and the experiences of project teachers.

According to an evaluation study, 80% of students rated SageMath very well. Students found the SageMath tools very useful, emphasized SageMath on-line availability, intuitive commands, lots of online resources, help with home-work, advanced learning, and interest development.

However, in order for the student to be able to use the SageMath tools, the teacher should be familiar with the student's curiosity and motivation to use SageMath. In turn, for a teacher to have the ability to use the tools SageMath, should also be curious, motivated and convinced. Based on the experience of the teachers who took part in the project and who had not heard of SageMath before, we know that they have rated the highly usefulness of SageMath to teach their subject. The interviews highlighted the benefits of SageMath as free software, available online, with the ability to work on their own server, where you can save work, publish materials and evaluate student work. They have recognized the potential of SageMath to use in school and beyond school education, including the ability to: quickly perform multiple-parameter calculations, graphically present results, facilitate understanding of the topic, deepen problem analysis, explore content beyond the curriculum, tools for learner self-education, preparation for the subject competitions. Teachers also considered SageMath a very good tool for formulating generalizations based on a rapid study of multiple input problems, problem solving for many cases, quick validation of solutions, classroom preparation and classroom exercises, workshops and competitions.

But to say so about SageMath, teachers first had to go through the way of learning new software. It is known that “beginnings are difficult” - teachers frequently repeat it to the students. Ultimately, the time needed for SageMath was estimated by teachers to be around 10 hours and they did not find the software very difficult to learn. These statements, reviewed during the evaluation, suggest that the teachers who met SageMath expressed very positive opinions including the belief that SageMath would apply to both their needs and their extra-curricular and extra-curricular activities. And what about the teachers who did not touch the software themselves and only heard about it (for example, directly at the interview)? The case does not look so positive because the unknown does not inspire confidence. Although the attitude of curiosity - about 60% and skepticism - was about 20%, but also a large group of opponents - about 20%. The last ones do not want to spend time on new technologies, they are the supporters of calculations on the paper, they believe that the student will lose interest in the problem, since immediately can have a result. Among curious teachers were people interested in innovation, career advancement, active teaching methods, people worried about losing hours in connection with reform, people wishing to change schools to higher and above all younger ones. So this is the target group of teachers we are trying to reach in our publications, conferences, and dissemination of the results of the project.

An extremely important group from the point of view of the implementation conditions of the project results is also the external educational environment of the teacher. So, firstly, other teachers working in the same school. Among them are science-related teachers, IT teachers, or teachers who, through their participation in training, conferences or self-education, can encourage their colleagues to apply new technologies.

Another circle of the environment is the school’s management, which has a huge impact on motivating teachers for professional development, such as developing their curriculum, developing innovation, participating in training, applying active and innovative teaching methods - using new technology, subject competition organization or individual work of gifted student.

The next very important element are the methodological consultants. As part of their mission, they organize methodological meetings, workshops, consultations, or information carriers, and will certainly be interested in using new technologies from which SageMath is a great example. We took care to invite consultants to the conferences organized under the project. Going forward, we announced our teachers’ training centers. SageMath is definitely worth the price of training and we will be looking forward to it. Many teachers will be interested in such training. Our evaluation studies have shown this. Teachers who heard of SageMath were interested in this tool, asked whether there are courses in this field and admitted that they prefer to complete the course rather than independently learn new software.

A good time to implement our project is the time of reform, which is beieng implemented now in Poland, when teachers are no longer able to sleep peacefully, some will go to another school, others will face new employment conditions, and everyone will face the new program. So you have to be competitive. Good opportunity to dust off old habits, for example in the form of introducing new teaching methods, introducing innovation, creating a circle for "capable" or other activity. So teachers will be interested, there will be a lot of training for computer science teachers, promised by the Ministry of Education, will also regulate increasing the hours for the strict subjects (for computer science in particular), both in primary school and high school. It is a very good perspective to implement the results of this project.

By answering the question, "What conditions must be met to succeed," we have followed the path that information needs to be communicated about how the road can reach the teacher, and what factors will influence the success of the implementation of the results of our project. We asked the following questions:

Teacher -> Where do you find out information about SageMath? -> popularization of methodology in the media, conferences, other teacher, the offer of the training center, subject methodology, how will you find it worth? -> Show SageMath opportunities, workshops, teacher motivation for activity, openness, encouragement from another teacher or methodology, school management, reform and related change of work, reform and related curriculum changes. How to learn? self-education, demonstration eg computer science teacher, course, conference, publication Where will the classes take place? - if he has access to a computer lab, he / she may use the student's smartphones? What will he use? -> will choose from the curriculum content, which co-operation with teachers of the subject, innovation, projects, competitions, circle, work with a gifted student, preparing students for the subject competitions.

Headmaster -> Encourage,, Send to the course? Methodology/Subject Consultants-> will they come to the conference, will they popularize? Teacher Training Centers -> Does anybody know about SageMath, will they accept the conference invitation, or will they arrange a course? Ministry of Education -> Will the SageMath deployment be adequate in the network? - yes, the teacher has to implement a compulsory core curriculum, and the extension is planned by himself (possibly in agreement with the subject group) and has hours, because the program is general, the teacher himself develops the content and topics.

Will the curriculum be flexible? Under Polish education circumstances the syllabus is selected by teacher himself (possibly in the agreement with the school subject team). He chooses (for a given subject) a set of ready-made programs approved by the Ministry of Education or he / she develops himself as an author program (only for the approval of the Pedagogical Board at the school). So it all depends on the teacher !!