TA ASSIGNMENT PROBLEM

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Abstract. The goal is to formulate a TA assignment problem and discuss some problems with tryig to solve it using stability-like notions.

1. Model

Let S be the set of students (i.e., candidate TAs). Let C be the set of courses. A (TA) assignment is a mapping $\phi : C \to \mathbb{R}_+^S$ We refer to $\phi(c)$ as the assignment to course c. If ϕ is an assignment, then for each s, define $\phi(s) \in \mathbb{R}_+^C$ by $\phi(s, c) = \phi(c, s)$. We refer to $\phi(s)$ as the assignment of student s. Let Φ be the space of the assignments.

Let $A_S = \mathbb{R}_+^S$ be the space of possible assignments to courses. Analogously, let $A_C = \mathbb{R}_+^C$ be the space of possible assignments for students.

Each course c has a subset of assignments $F_c \subseteq A_S$ that are feasible for it. Examples of feasibility constraints include:

- budget: Each course has a budget of h_c TA hours that need to be allocated and assignment $\phi(c) \in A_S$ is feasible for c if and only if

$$\sum_s \phi(c, s) = h_c,$$

- minimum assignment hours: there is a constraint τ_c such that an assignment is feasible only if for each s, either $\phi(c, s) = 0$ or $\phi(c, s) \geq \tau_c$.

Similarly, each student s has a subset $A_s \subseteq A_C$ of assignments that are feasible for them. Examples of student feasibility constraints:

- maximum assignment: each student s cannot work for more than κ_s hours:

$$\sum_c \phi(c, s) \leq \kappa_s,$$

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- minimum assignment: student s must work for at least l_s hours,
- balance across semesters: There is a set M of "semesters" and each course c is associated with a subset $M_c \subseteq M$ of "semesters" in which this course is taught. The student s assigned hours in semester m are equal to

  $$h_s(m;\phi) = \sum_{c:m\in M_c} \frac{1}{|M_c|}\phi(c,s).$$

  It is required that the student assignment across semesters is balanced: for some constant $\rho > 1$, for each m, m',

  $$\frac{h_s(m,\phi)}{h_s(m',\phi)} \leq \rho.$$

An assignment ϕ is *feasible* if $\phi(i) \in A_i$ for each $i \in S \cup C$.

Each student s has preferences over possible assignments represented by utility function $u_s : A_C \to \mathbb{R}$. Similarly, each course c has preferences over assignments represented by $u_c : A_S \to \mathbb{R}$.

An assignment is individually rational (for students) if none of the students is better off by dropping some courses from its assignment (the logic is that students cannot be made to work for a course, and a typical TA contract is written separately for each course. If TAs sign up for a bundle, i.e., for each of their assignment, the individual rationality would require that each student prefers their assignment to $\mathbf{0}$ assignment). Formally, for each s, each $D \subseteq C$, $u(\phi(s)) \geq u(\phi_D(s))$, where ϕ_D is an assignment obtained from ϕ by zeroing all courses not in D $\phi_D(c,s) = \begin{cases} \phi(c,s) & c \in D \\ 0 & \text{otherwise} \end{cases}$.

Let Φ^* be the set of feasible and individually rational assignments.

A social planner has preferences over Φ^*.

2. Core and stability

Suppose that the social planner would like its assignment to be stable, in some sense. It is not clear what exactly stable assignment means - as there are many options available. Perhaps the most straightforward would be t consider individually rational assignments that are not blocked by 2-agent (student and course) coalitions. But, with many-to-many assignments, 2-agent coalitions may not be sufficient and one may want to think about some sort of core-type concept.

Instead of focusing on a stability notion, I want to illustrate the problem with few examples why the any stability-like notion can be problematic.

2.1. **No externalities.** Suppose that student preferences are additive: for each s, there exist constants $\nu_s(c)$ such that for each $a \in A_C$,

$$u_s(a) = \sum_c a_c \nu_s(c).$$

Additionally, suppose that all assignments of student s are subject to maximum constraint κ_s.

Similarly, suppose that course preferences are additive: for each c, there exist constants $\nu_c(s)$ such that for each $a \in A_S$,

$$u_c(a) = \sum_s a_s \nu_c(s).$$

Additionally, suppose that all assignments of course c are subject to maximum constraint h_s.

In this environment, one can show that the core is non-empty. Moreover, core allocations can be found by a version of Gale-Shapley algorithm. (Should I add this?)

2.2. **Courses that span across semesters.** It is well-known that any type of externality (or, more precisely, complementarities) in many-to-many matching problems leads to problems with the existence of a stable outcome (or core).

One source of externality that reduces the use of stability as a satisfactory solution concept is given by courses that span across multiple semesters.

Example 1. There are three students x, y, z, and four courses aF, aS, bY, cY. Courses bY and cY are offered across two semesters, the other two courses are offered only in one semester. Each course needs exactly 1 TA, and each student wants at most 1 course per semester. All allocations must be either 1 or 0. (In this example, partial allocations would not help anyway.) Student preferences over bundles of courses are

$$\begin{aligned} x &: \{aF, aS\} > bY > aF > \emptyset, \\ y &: aF > cY > \emptyset, \\ z &: cY > aS > \emptyset, \end{aligned}$$

Essentially, student x really likes course aF, but also needs money and would rather TA for aF and aS than for the whole-year course Y, but is willing to TA for aF only if that's the only option. Course preferences are

$$\begin{aligned} aF &: x > y > \emptyset \\ aS &: z > x > \emptyset, \\ bY &: x > \emptyset, \\ cY &: y > z > \emptyset. \end{aligned}$$

So, x and aF are top matches for e\each other, but otherwise, there are no obvious matches.

We show that there is no core. If y is assigned to aF, then z must be assigned to cY (as that is the top assignment for z and the second best assignment for cY after y is taken). But then, given that the best remaining for aS is x, x and pair $\{aF, aS\}$ are a blocking coalition (as the student and all courses prefer to rematch with each other).

If y is assigned to cY, then z must be assigned with aS (as z is the top choice for aS and aS is the top choice for z aparat from cY which prefers y. But then, since aS prefers z, bY is the best option for x. As a consequence aF is available for y who prefers it to cY.

If y is not assigned to anybody, then z gets cY, but then y and and cY is a blocking pair.

2.3. **Preferences over the number of assignments.** Another source of problematic externality is when either students (or, resp., courses) have preferences over the total number of assigned courses (resp. students). In fact, a reinterpretation of Example 1 illustrates a difficulty:

Example 2. There are three students x, y, z, and four courses a, a', b, c. Each student has up to 2h of work available. Courses b and c require 2h of work and they only want to hire the same TA. Courses a, a' require 1h of work each. Otherwise, the preferences are similar to Example 1: for students

$$\begin{aligned} x &: \{a, a'\} > b > a > \emptyset, \\ y &: a > c > \emptyset, \\ z &: c > a' > \emptyset, \end{aligned}$$

and for courses:

$$\begin{aligned} a &: x > y > \emptyset \\ a' &: z > x > \emptyset, \\ b &: x > \emptyset, \\ c &: y > z > \emptyset. \end{aligned}$$

There is no stable matching.

2.4. **Minimum assignment.** There are $n \geq 1$ students and $n + 1$ courses. Each student has $n + 1$ hours and each course needs n hours. The student s utility from assignment is linear in assignments (like in Section 2.1), with coefficients equal to

$$v_s(c) = \begin{cases} 1 + \varepsilon & c = s \\ 1 & c \neq s. \end{cases}.$$

Course c utility from the assignment $a \in A_S$ depends on the number of non-zero assignments and it equal to

$$u_c(a) = \varepsilon 1_{s=c} + \begin{cases} 1 & |\{s : a_s > 0\}| = 1 \\ 1 - \varepsilon & |\{s : a_s > 0\}| = 2 \\ 1 - 10 |\{s : a_s > 0\}| & |\{s : a_s > 0\}| > 2. \end{cases}$$

The first term is a small boost of the utility when the student has the same index as he course.

Then, the unique core allocation (or stable matching under any reasonable deifnition) is when each student is matched for n hours with the course with the same index and for remaining 1 hours is assigned to course $n+1$.

Of course, the above assignment is Pareto-optimal, but wildly inefficient.