Fuzzy Conventions

Marcin Pęski

University of Toronto

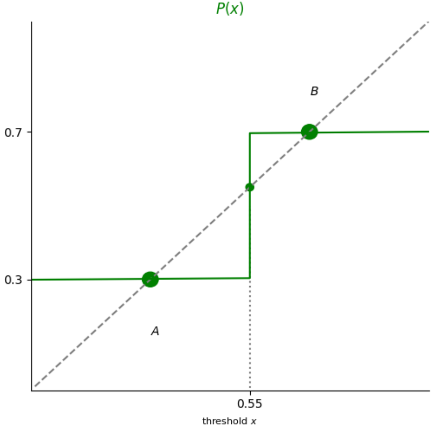
October 25, 2022

Introduction

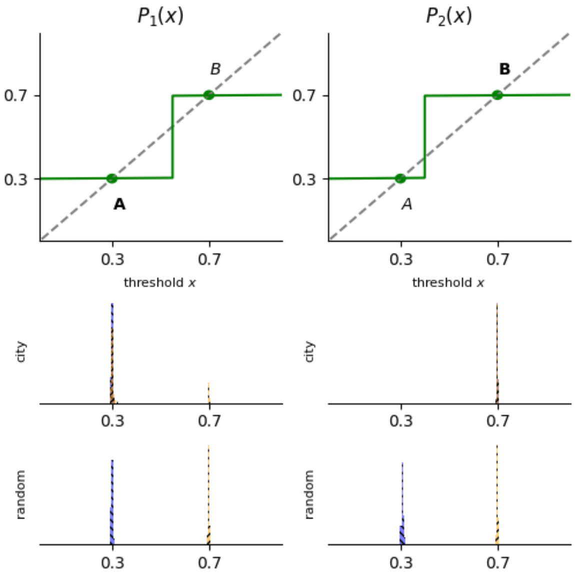
- Social interactions, positive externalities.
    - wearing a mask,
    - engaging in criminal activity,
    - technology adoption.
- A typical result: emergence of a (homogeneous) convention.
- But, in reality, conventions are often fuzzy:
    - some, but not all, wear masks,
    - married couples that use both IPhone and Android.

Introduction

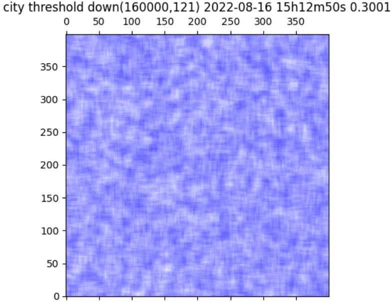
- Granovetter 78: People care not only about their neighbors, but they differ wrt. tastes, preferences.
- $P(x)$ - probability that you choose action 1 if at least fraction x of your neighbors chooses 1.

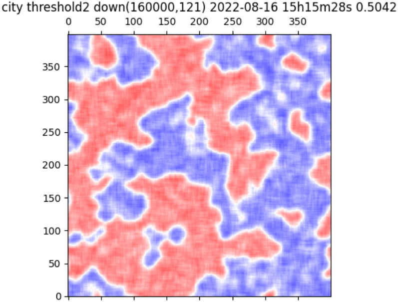


Introduction



Introduction





- City network with 160,000 agents, each agent has 120 neighbors.
- Blue - most of the neighbors play 0, red - most of the neighbors play 1.

- *Fuzzy convention* x: almost all agents observe $\sim x$ fraction of neighbors playing 1.
- *Random-utility dominant outcome*:

$$x^* \in \arg\max_x \int_0^x \left(y - P^{-1}(y)\right) dy,$$

  - risk-dominance (Harsanyi-Selten 88),

Equilibrium selection

- All sufficiently fine networks have an equilibrium that is fuzzy convention x^*.
- For some networks ("city"), fuzzy convention x^* is the only equilibrium.

Identification:

- Maximum range of average equilibrium behavior across all networks.

- Random utility models: matching (Dagsvik 00, Choo-Siow 06, Menzel 15, Peski 17, 22), games (Alvarez et al 22)
- Dynamic coordination models:
  - evolutionary approach: Kandori et al 93, Young 93, Ellison 93, Ellison 00,
  - contagion: Lee Valentyi 00, Morris 00,
  - here - static equilibrium.
- Bayesian equilibrium in network games: Jackson Yariv 07, Galeotti et al 10
  - here: complete information
- Large (but finite) degree networks.

1. Random utility dominant fuzzy convention on each network.
2. "Unique" selection on the city network.
3. Largest equilibrium set.

RU-dominant convention

- Agents i, j live on a network with weights $g_{ij} = g_{ji} \geq 0$,
  - $g_i = \sum_j g_{ij}$ is degree of agent i,
  - each node has one agent,
- I.i.d payoff shocks $\tau_i \sim P(.)$.
- Profile a is equilibrium if for each i

$$\tau_i \leq \beta_i^a \Longrightarrow a_i = 1,$$

  where $\beta_i^a := \frac{1}{g_i} \sum g_{ij} a_j$ is the average neighborhood behavior.
- Granovetter (78) is equivalent to a binary random-utility coordination game.

RU-dominant convention

Definition

Profile a is ε-*fuzzy convention* x if

$$\frac{1}{N}\left\{i : \left|\beta_i^a - x\right| \geq \varepsilon\right\} \leq \varepsilon.$$

RU-dominant convention

Random utility dominant outcome

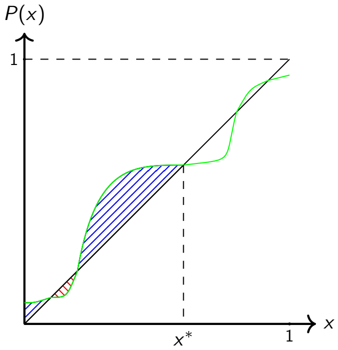
Definition

R*andom utility (RU-) dominant* outcome

$$x^* \in \arg\max_x \int_0^x \left(y - P^{-1}(y)\right) dy.$$

RU-dominant convention

Random utility dominant outcome

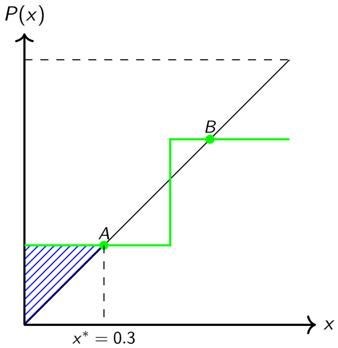


$$x^* \in \arg\max_x \int_0^x \left(y - P^{-1}(y)\right) dy.$$

- Generically, (a) unique and (b) strictly stable fixed point of P.

RU-dominant convention

Random utility dominant outcome

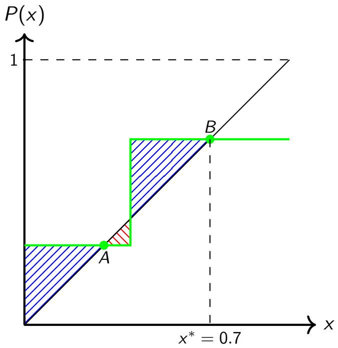


$$x^* \in \arg\max_x \int_0^x \left(y - P^{-1}(y)\right) dy.$$

- RU-dominance chooses A equilibrium in the first example from the introduction.

RU-dominant convention

Random utility dominant outcome

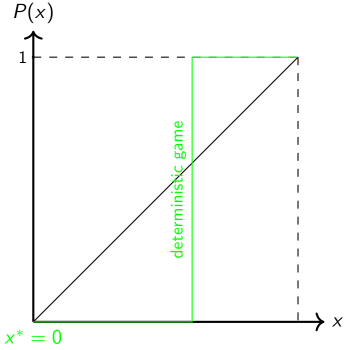


$$x^* \in \arg\max_x \int_0^x \left(y - P^{-1}(y)\right) dy.$$

- RU-dominance chooses B equilibrium in the second example from the introduction.

RU-dominant convention

Random utility dominant outcome

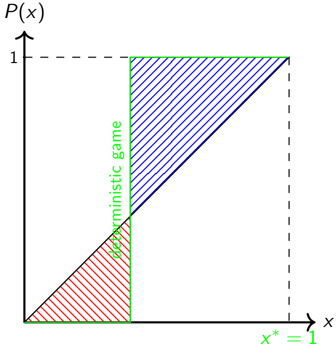


$$x^* \in \arg\max_x \int_0^x \left(y - P^{-1}(y)\right) dy.$$

- When game is deterministic, RU-dominance is equivalent to Harsanyi-Selten risk-dominance

RU-dominant convention

Random utility dominant outcome



$$x^* \in \arg\max_x \int_0^x \left(y - P^{-1}(y)\right) dy.$$

- When game is deterministic, RU-dominance is equivalent to Harsanyi-Selten risk-dominance.

RU-dominant convention

Random utility dominant outcome

- Formula

$$x^* \in \arg\max_x \int_0^x (P(y) - y)\, dy$$

appears in Morris and Shin (06).

  - continuum toy model,
  - observe that the coordination game has a potential,
  - the above outcome maximizes potential,
  - hence it is robust to incomplete information.

RU-dominant convention

- *Large degrees*: Let $d(g) = \max_{i,j} \frac{g_{ij}}{g_i} \to 0$.

RU-dominant convention

- *Large degrees*: Let $d(g) = \max_{i,j} \frac{g_{ij}}{g_i} \to 0$.
- *Limited inequality*: Let $w(g) = \max_{i,j} \frac{g_i}{g_j} < w^*$.

RU-dominant convention

Theorem

For each $\eta > 0$ and $w < \infty$,

$$\lim_{d(g)\to 0, w(g)\leq w} \mathit{Prob}\left(\exists a \text{ is equilibrium st. } a \text{ is } \eta\text{-fuzzy convention } x^*\right) = 1.$$

- Each sufficiently fine network, with a large probability, has an equilibrium that is a fuzzy convention x^*.

RU-dominant convention

- Granovetter's model is a potential game ([?]):

$$V(a;\tau) = \frac{1}{2}\sum_{i,j} g_{ij} a_i a_j - \sum g_i a_i \tau_i .$$

- The proof shows that, if the network is sufficiently large and fine, for almost all realizations of τ, any *global* maximizer of V is a fuzzy convention x^*.
- Hence, fuzzy convention x^* is also
  - robust to incomplete information (Ui 2001) and
  - stochastically stable under logistic dynamics (Blume 1993, 95).

RU-dominant convention
Proof

- Concentration inequality.
- Calculations on the potential function:

$$V(a;\tau) = \frac{1}{2}\sum_{i,j} g_{ij} a_i a_j - \sum g_i a_i \tau_i$$

RU-dominant convention

Proof: Concentration inequality

- *Law of Large Numbers*: for each function f,

$$\frac{1}{\sum g_i} \sum_i g_i f\left(\tau_i, \beta_i^a\right) \to \frac{1}{\sum g_i} \sum_i g_i \, \mathbb{E} \, f\left(., \beta_i^a\right) \text{ as } N \to \infty$$

RU-dominant convention

Proof: Concentration inequality

- *Law of Large Numbers*: for each bounded function f,

$$\frac{1}{\sum g_i} \sum_i g_i f\left(\tau_i, \beta_i^a\right) \to \frac{1}{\sum g_i} \sum_i g_i \, \mathbb{E} f\left(., \beta_i^a\right) \text{ as } N \to \infty$$

(if $w\left(g\right) = \max \frac{g_i}{g_j}$ remains bounded.)

RU-dominant convention

Proof: Concentration inequality

- *Hoeffding*: for each bounded function f,

$$\text{Prob}\left(\left|\sum_i g_i f\left(\tau_i, \beta_i^a\right) - \sum_i g_i \, \mathbb{E}\, f\left(., \beta_i^a\right)\right| \geq \varepsilon \sum g_i\right) \leq B \exp\left(-c_\varepsilon N\right).$$

RU-dominant convention

Proof: Concentration inequality

- *Uniform concentration*:

$$\mathrm{Prob}\left(\sup_a\left|\sum_i g_i f\left(\tau_i,\beta_i^a\right)-\sum_i g_i\,\mathbb{E}\,f\left(.,\beta_i^a\right)\right|\geq\varepsilon\sum g_i\right)$$

- *Uniform concentration*: for each K-Lipschitz function f,

$$\begin{aligned}
&\text{Prob}\left(\sup_a \left|\sum_i g_i f\left(\tau_i, \beta_i^a\right) - \sum_i g_i \,\mathbb{E}\, f\left(., \beta_i^a\right)\right| \geq \varepsilon \sum g_i\right) \\
\leq & B\exp\left(-c_{\varepsilon,K,d(g)} N\right),
\end{aligned}$$

where $\lim_{d\to 0} c_{\varepsilon,K,d} = 0$.

RU-dominant convention

Proof: Concentration inequality

$$\begin{aligned}\text{Prob}\left(\sup_a F(\beta^a) \le \epsilon\right) &= \text{Prob}\left(\sup_{\beta\in\mathcal{B}} F(\beta) \le \epsilon\right)\\ &\le |\mathcal{B}| \sup_{\beta\in\mathcal{B}} \text{Prob}\left(F(\beta) \le \epsilon\right)\\ &= |\mathcal{B}| \sup_a \text{Prob}\left(F(\beta^a) \le \epsilon\right).\end{aligned}$$

where $\mathcal{B} = \{\beta^a : a \text{ is a profile}\}$

$$
\begin{aligned}
\text{Prob}\left(\sup_a F\left(\beta^a\right) \leq \epsilon\right) &= \text{Prob}\left(\sup_{\beta \in \mathcal{B}} F\left(\beta\right) \leq \epsilon\right) \\
&\leq |\mathcal{B}| \sup_{\beta \in \mathcal{B}} \text{Prob}\left(F\left(\beta\right) \leq \epsilon\right) \\
&= |\mathcal{B}| \sup_a \text{Prob}\left(F\left(\beta^a\right) \leq \epsilon\right).
\end{aligned}
$$

where $\mathcal{B} = \{\beta^a : a \text{ is a profile}\}$

$$
\begin{aligned}
\text{Prob}\left(\sup_a F\left(\beta^a\right) \le \epsilon\right) &= \text{Prob}\left(\sup_{\beta\in\mathcal{B}} F\left(\beta\right) \le \epsilon\right) \\
&\le \left|\mathcal{B}\right| \sup_{\beta\in\mathcal{B}} \text{Prob}\left(F\left(\beta\right) \le \epsilon\right) \\
&= \left|\mathcal{B}\right| \sup_a \text{Prob}\left(F\left(\beta^a\right) \le \epsilon\right).
\end{aligned}
$$

where $\mathcal{B} = \{\beta^a : a \text{ is a profile}\}$

RU-dominant convention

Proof: Concentration inequality

$$
\begin{aligned}
\text{Prob}\left(\sup_a F\left(\beta^a\right) \leq \epsilon\right) &= \text{Prob}\left(\sup_{\beta \in \mathcal{B}} F\left(\beta\right) \leq \epsilon\right) \\
&\leq \left|\mathcal{B}\right| \sup_{\beta \in \mathcal{B}} \text{Prob}\left(F\left(\beta\right) \leq \epsilon\right) \\
&= \left|\mathcal{B}\right| \sup_a \text{Prob}\left(F\left(\beta^a\right) \leq \epsilon\right).
\end{aligned}
$$

where $\mathcal{B} = \{\beta^a : a \text{ is a profile}\}$

$$\text{Prob}\left(\sup_a F\left(\beta^a\right) \leq \varepsilon\right) \leq |\mathcal{B}| \sup_a \text{Prob}\left(F\left(\beta^a\right) \leq \varepsilon\right).$$

RU-dominant convention

Proof: Concentration inequality

$$\text{Prob}\left(\sup_a F(\beta^a) \le \varepsilon\right) \le |\mathcal{B}| \sup_a \text{Prob}\left(F(\beta^a) \le \varepsilon\right).$$

- Unfortunately, counting measure is too large:

$$|\mathcal{B}| = |\{\beta^a : a \text{ is a profile}\}| = |\{a \text{ is a profile}\}| = 2^N.$$

RU-dominant convention

Proof: Concentration inequality

$$\text{Prob}\left(\sup_a F\left(\beta^a\right) \leq \varepsilon\right) \leq |\mathcal{B}| \sup_a \text{Prob}\left(F\left(\beta^a\right) \leq \varepsilon\right).$$

- Fortunately, metric entropy is small enough, if $d\left(g\right)$ is small

$$\mathcal{N}\left(\mathcal{B}, \delta\right) \leq \exp\left(\frac{1}{\delta^2} d\left(g\right) N\right).$$

RU-dominant convention

Proof: Concentration inequality

$$\text{Prob}\left(\sup_a F\left(\beta^a\right) \leq \varepsilon\right) \leq \mathcal{N}\left(\mathcal{B}, \delta\right) \sup_a \text{Prob}\left(\sup_{a' : \|a\prime - a\| \leq \delta} F\left(\beta^{a'}\right) \leq \varepsilon\right).$$

- Fortunately, metric entropy is small enough, if $d\left(g\right)$ is small

$$\mathcal{N}\left(\mathcal{B}, \delta\right) \leq \exp\left(\frac{1}{\delta^2} d\left(g\right) N\right).$$

RU-dominant convention

Proof: Potential calculations

- For each profile a,

$$V(a;\tau) = \frac{1}{2}\sum_{i,j} g_{ij} a_i a_j - \sum g_i a_i \tau_i.$$

RU-dominant convention

Proof: Potential calculations

- For each equilibrium profile a,

$$V(a;\tau) = \frac{1}{2}\sum_{i,j} g_{ij} a_i a_j - \sum g_i a_i \tau_i.$$

where

$$a_i = \mathbf{1}\left\{\tau_i \leq \beta_i^a\right\}.$$

RU-dominant convention

Proof: Potential calculations

- For each equilibrium profile a,

$$V(a;\tau) = \frac{1}{2}\sum_{i,j} g_{ij}\mathbf{1}\{\tau_i \leq \beta_i^a\}\mathbf{1}\left\{\tau_j \leq \beta_j^a\right\} - \sum g_i \mathbf{1}\{\tau_i \leq \beta_i^a\}\tau_i.$$

- For each equilibrium profile a,

$$V(a;\tau)=\frac{1}{2}\sum_{i,j}g_{ij}\mathbf{1}\{\tau_i\le\beta_i^a\}\mathbf{1}\left\{\tau_j\le\beta_j^a\right\}-\sum g_i\mathbf{1}\{\tau_i\le\beta_i^a\}\tau_i.$$

- Due to concentration inequalities

$$\mathbb{E}\,\mathbf{1}\{\tau_i\le\beta_i^a\}=P(\beta_i^a),$$
$$\mathbb{E}\,\mathbf{1}\{\tau_i\le\beta_i^a\}\tau_i=\int_0^{\beta_i^a}ydP(y).$$

RU-dominant convention

Proof: Potential calculations

- For each equilibrium profile a,

$$V\left(a;\tau\right)=\frac{1}{2}\sum_{i,j}g_{ij}P\left(\beta_i^a\right)P\left(\beta_j^a\right)-\sum g_i\int_0^{\beta_i^a}ydP\left(y\right).$$

RU-dominant convention

Proof: Potential calculations

- Due to $2P\left(\beta_i^a\right)P\left(\beta_j^a\right) \leq P\left(\beta_i^a\right)^2 + P\left(\beta_j^a\right)^2$,

$$\frac{1}{2}\sum_{i,j} g_{ij} P\left(\beta_i^a\right) P\left(\beta_j^a\right) \leq \frac{1}{2}\sum_i g_i \left(P\left(\beta_i^a\right)\right)^2$$

- Hence, for each equilibrium profile a,

$$V\left(a;\tau\right) \leq \sum_i g_i \left[\frac{1}{2}\left(P\left(\beta_i^a\right)\right)^2 - \int_0^{\beta_i^a} y dP\left(y\right)\right].$$

- For each equilibrium profile a,

$$V(a;\tau) \leq \sum_i g_i \left[\frac{1}{2} \left(P\left(\beta_i^a\right) \right)^2 - \int_0^{\beta_i^a} y dP(y) \right].$$

- The RHS is maximized by $\beta_i^a = x^*$.

RU-dominant selection

- So far: fuzzy convention x^* is an equilibrium on each sufficiently fine network.
- Next: on some networks, there are no other equilibria.

RU-dominant selection

Theorem

Suppose that $0 < P(0) < P(1) < 1$.
For each $\eta > 0$, *there is a network g such that with probability* $1 - \eta$, *each equilibrium is* η*-fuzzy convention* x^*.

- The assumption ensures that, for each action, there is a positive probability that the action is dominant.

RU-dominant selection
Proof

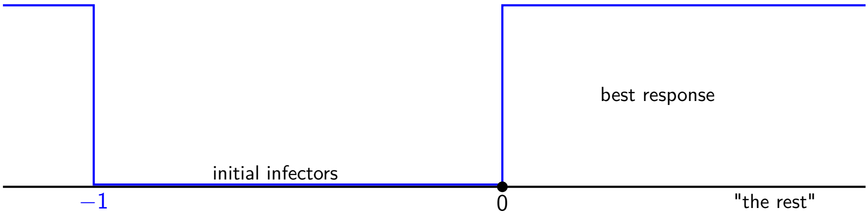
- 2+-dimensional lattices (city network)
  - 1-dimensional lattice (line) is not enough
- A result about static equilibrium:
  - but proof based on best response dynamics.
  - review of contagion arguments (Ellison 93, Blume 95, Lee and Valentinyi 00, Morris 00),
  - contagion wave on "toy" line,
  - why line is not enough and why 2-dimensional lattice is.

RU-dominant selection

Proof: Review of contagion arguments

- Start with deterministic case, but with small group of initial infectors.
- Assume 0 is risk-dominant.
- We want to show that 0 is the only equilibrium.
- -$>$ contagion.

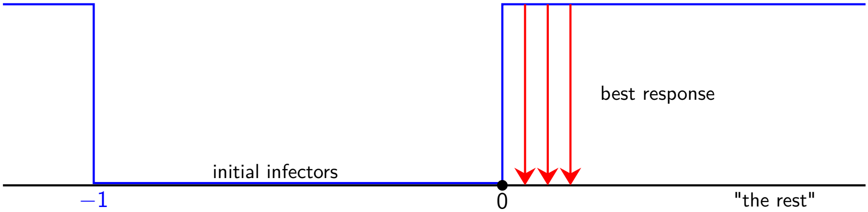
- Ellison 93: suppose that action 0 is risk-dominant,
- initial infectors $-1 \leq i \leq 0$ play 0; the rests play 1,



RU-dominant selection

Proof: Review of contagion arguments

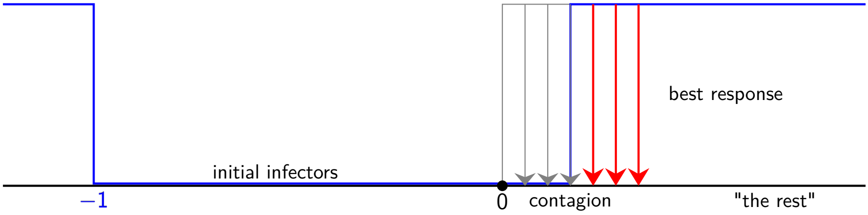
- Ellison 93: suppose that action 0 is risk-dominant,
- initial infectors $-1 \leq i \leq 0$ play 0; the rests play 1,
- best response dynamics



RU-dominant selection

Proof: Review of contagion arguments

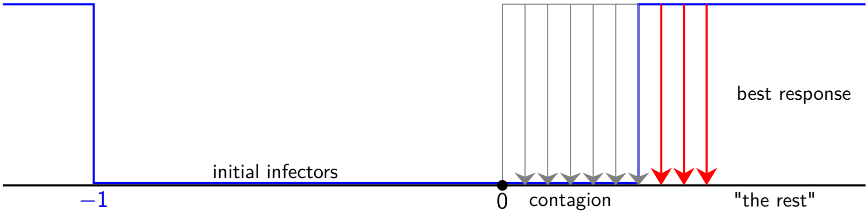
- Ellison 93: suppose that action 0 is risk-dominant,
- initial infectors $-1 \leq i \leq 0$ play 0; the rests play 1,
- best response dynamics -> contagion



RU-dominant selection

Proof: Review of contagion arguments

- Ellison 93: suppose that action 0 is risk-dominant,
- initial infectors $-1 \leq i \leq 0$ play 0; the rests play 1,
- best response dynamics -> contagion

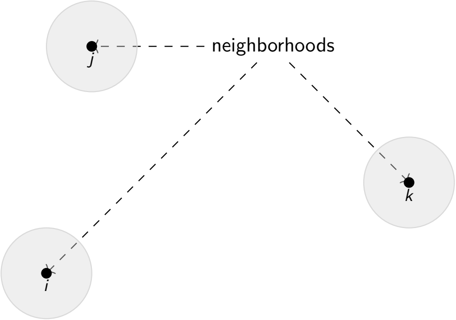


- Blume 95- the same mechanics works on other networks, like 2 (or higher)-dimensional lattices.
- Key step: half of neighbors of "threshold agents" must be infected to spread contagion.

RU-dominant selection

Proof: Review of contagion arguments

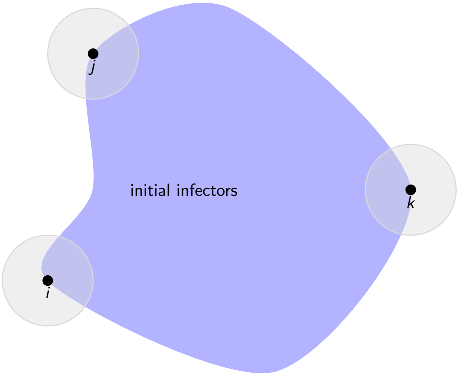
- Blume 93, Morris 00 - the same mechanics works on other networks, like 2 (or higher)-dimensional lattices.
- Key step: half of neighbors of "threshold agents" must be infected to spread contagion.



RU-dominant selection

Proof: Review of contagion arguments

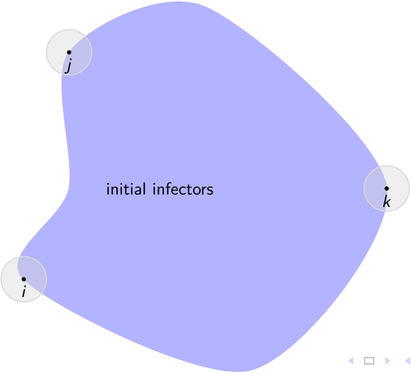
- Blume 93, Morris 00 - the same mechanics works on other networks, like 2 (or higher)-dimensional lattices.
- Key step: half of neighbors of "threshold agents" must be infected to spread contagion.



RU-dominant selection

Proof: Review of contagion arguments

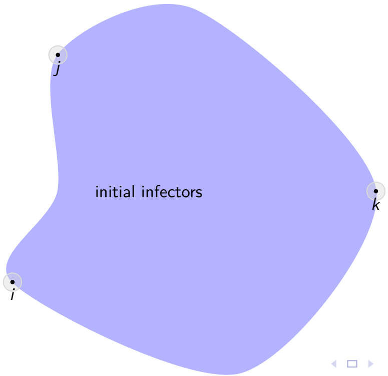
- Blume 93, Morris 00 - the same mechanics works on other networks, like 2 (or higher)-dimensional lattices.
- Key step: half of neighbors of "threshold agents" must be infected to spread contagion
- - $>$ initial infectors must be large enough relative to neighborhoods.



RU-dominant selection

Proof: Review of contagion arguments

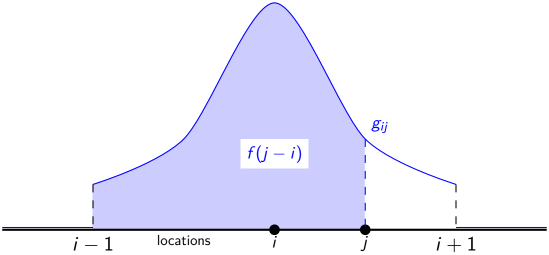
- Blume 93, Morris 00 - the same mechanics works on other networks, like 2 (or higher)-dimensional lattices.
- Key step: half of neighbors of "threshold agents" must be infected to spread contagion
- - $>$ initial infectors must be large enough relative to neighborhoods.



- Random utility payoffs (so, not deterministic)
- Toy line: Continuum of agents in each location.

RU-dominant selection

Proof: Contagion wave on line, RU case



- Toy line: agents in location i are connected with agents in location j
  - connection density $g_{ij} = g_{ji} = g_{i+l,j+l}$ for any l,
  - $g_{ij} = 0$ for $j > i+1$,
  - $f(j-i) = \frac{1}{g_i} \int_{i-1}^{j} g_{il} dl$,
  - $f(x) + f(1-x) = 1$.

RU-dominant selection

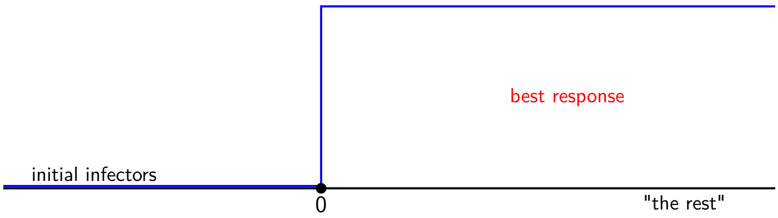
Proof: Contagion wave on line, RU case

- For simplicity, assume that $x^* = 0$ is *RU*-dominant, i.e.

$$\int_0^x \left(y - P^{-1}(y)\right) dy < 0 \text{ for each } x > 0.$$

- Starting from arbitrary profile with a group of initial infectors playing x^*, best response dynamics will spread x^* to the whole line.

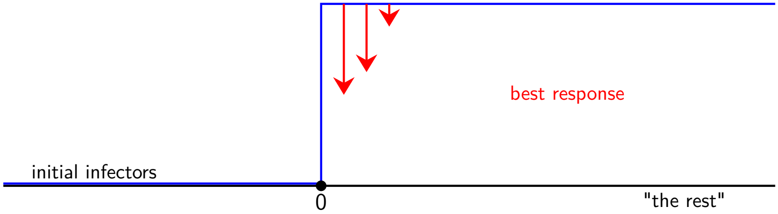
- Initial infectors play $x^* = 0$.



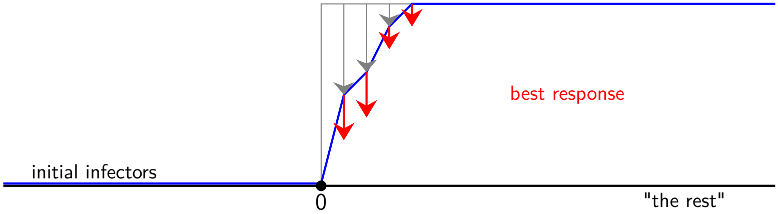
RU-dominant selection

Proof: Contagion wave on line, RU case

- Initial infectors play $x^* = 0$.



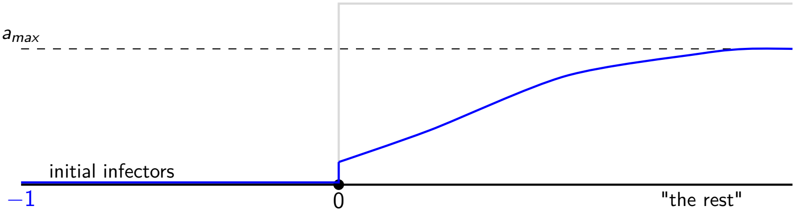
- Initial infectors play $x^* = 0$.



RU-dominant selection

Proof: Contagion wave on line, RU case

- Suppose that stops before spreading everywhere.



RU-dominant selection

Proof: Contagion wave on line, RU case

- If the contagion stops, then

$$a_i \leq P\left(\int a_{i+k} df(k)\right) \text{ for each } i.$$

- We are going to show that the above implies

$$\int_0^{a_{\max}} \left(a - P^{-1}(a)\right) da \geq 0$$

which will violate 0 being RU-dominant.

RU-dominant selection

Proof: Contagion wave on line, RU case

- If the contagion stops, then

$$a_i \leq P\left(\int a_{i+k} df\left(k\right)\right) \text{ for each } i.$$

- We are going to show that the above implies

$$\int_0^{a_{\max}} \left(a - P^{-1}\left(a\right)\right) da \geq 0$$

which will violate 0 being RU-dominant.

RU-dominant selection

Proof: Contagion wave on line, RU case

- If the contagion stops, then at each location $i > 0$,

$$a_i \leq P\left(\int a_{i+k} df(k)\right).$$

- Taking inverse and integrating by parts

$$P^{-1}(a_i) \leq \int a_{i+k} df(k) = \int_0^{a_{\max}} f(i-j)\, da_j.$$

- Integrate over $a_i \in [0, a_{\max}]$,

$$\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i.$$

RU-dominant selection

Proof: Contagion wave on line, RU case

- If the contagion stops, then at each location $i > 0$,

$$a_i \leq P\left(\int a_{i+k} df\left(k\right)\right).$$

- Taking inverse and integrating by parts

$$P^{-1}\left(a_i\right) \leq \int a_{i+k} df\left(k\right) = \int_0^{a_{\max}} f\left(i-j\right) da_j.$$

- Integrate over $a_i \in [0, a_{\max}]$,

$$\int_0^{a_{\max}} P^{-1}\left(a_i\right) da_i \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f\left(i-j\right) da_j da_i.$$

- If the contagion stops, then at each location $i > 0$,

$$a_i \leq P\left(\int a_{i+k} df(k)\right).$$

- Taking inverse and integrating by parts

$$P^{-1}(a_i) \leq \int a_{i+k} df(k) = \int_0^{a_{\max}} f(i-j)\, da_j.$$

- Integrate over $a_i \in [0, a_{\max}]$,

$$\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i.$$

RU-dominant selection

Proof: Contagion wave on line, RU case

- Integrate over $a_i \in [0, a_{\max}]$,

$$\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i$$

RU-dominant selection

Proof: Contagion wave on line, RU case

- Integrate over $a_i \in [0, a_{\max}]$,

$$\begin{aligned}
&\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \\
\leq &\int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i \\
= &\frac{1}{2}\int_0^{a_{\max}} f(i-j)\, da_j da_i + \frac{1}{2}\int_0^{a_{\max}} \int_0^{a_{\max}} f(j-i)\, da_j da_i
\end{aligned}$$

RU-dominant selection

Proof: Contagion wave on line, RU case

- Integrate over $a_i \in [0, a_{\max}]$,

$$
\begin{aligned}
&\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \\
\leq &\int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i \\
= &\frac{1}{2} \int_0^{a_{\max}} f(i-j)\, da_j da_i + \frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} f(j-i)\, da_j da_i \\
= &\frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} \left[f(i-j) + f(j-i) \right] da_j da_i
\end{aligned}
$$

- Integrate over $a_i \in [0, a_{\max}]$,

$$\begin{aligned}
&\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \\
\leq &\int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i \\
= &\frac{1}{2} \int_0^{a_{\max}} f(i-j)\, da_j da_i + \frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} f(j-i)\, da_j da_i \\
= &\frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} [f(i-j) + f(j-i)]\, da_j da_i
\end{aligned}$$

- Recall that $f(i-j) + f(j-i) = 1$.

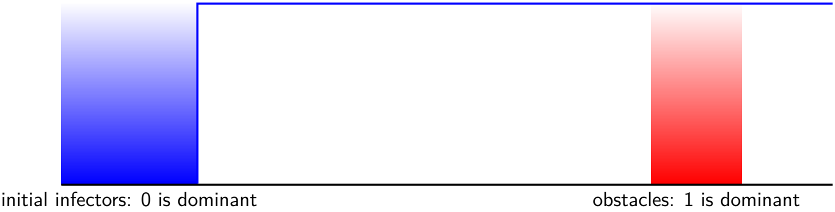
- Integrate over $a_i \in [0, a_{\max}]$,

$$
\begin{aligned}
&\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \\
\leq &\int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i \\
= &\frac{1}{2} \int_0^{a_{\max}} f(i-j)\, da_j da_i + \frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} f(j-i)\, da_j da_i \\
= &\frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} [f(i-j) + f(j-i)]\, da_j da_i \\
= &\frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} da_j da_i = \int_0^{a_{\max}} a da.
\end{aligned}
$$

RU-dominant selection

Proof: Contagion wave on line, RU case

- Hence the contagion must spread to the entire line.
- But! - so far we assumed that locations contain continuum.
- Contagion can be also stopped by unusual payoff shocks, like those that make 1 dominant.



RU-dominant selection

Proof: Contagion wave on line, RU case

- Hence the contagion has to spread to the entire line.
- But! - so far we assumed that locations contain continuum.
- Contagion can be also stopped by unusual payoff shocks, like those that make 1 dominant.

RU-dominant selection

Proof: Contagion wave on line, RU case

- We can compare the relative likelihood of infectors vs obstacles.
- On line, the latter can be more frequent.
- But not on 2-dimensional lattices.

RU-dominant selection

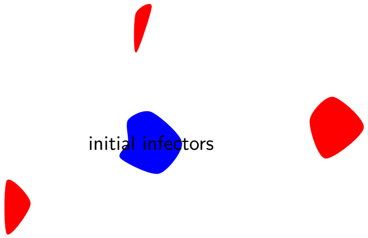
Proof: Contagion wave on line, RU case

- We can compare the relative likelihood of infectors vs obstacles.
- On line, the latter can be more frequent.
- But not on 2-dimensional lattices.

RU-dominant selection

Proof: Contagion wave on line, RU case

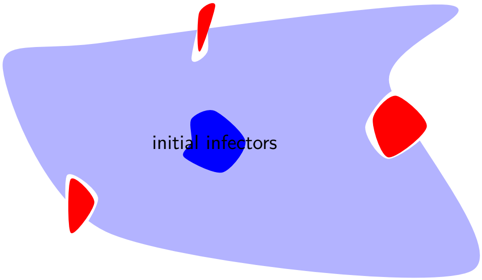
- We can compare the relative likelihood of infectors vs obstacles.
- On line, the latter can be more frequent.
- But not on 2-dimensional lattices.



RU-dominant selection

Proof: Contagion wave on line, RU case

- We can compare the relative likelihood of infectors vs obstacles.
- On line, the latter can be more frequent.
- But not on 2-dimensional lattices.



Robust equilibria

- So far,
  - each network has a fuzzy convention x^* equilibrium,
  - some networks only have such equilibrium.

Robust equilibria

- Let
$$a^*(\tau_i) = \mathbf{1}\{\tau_i \leq x^*\}.$$
- It is, with a large probability a fuzzy convention x^*:
$$\mathbb{E}\, a^*(\tau_i) = P(x^*) = x^*.$$

Robust equilibria

- The proofs show that
  - for each sufficiently fine network, with a large probability,
  - there exists an equilibrium that is close to a^*.
- Among all behaviors $a(\tau_i)$, a^* is the only one with such a property.
- Equilibrium selection.

Largest equilibrium set

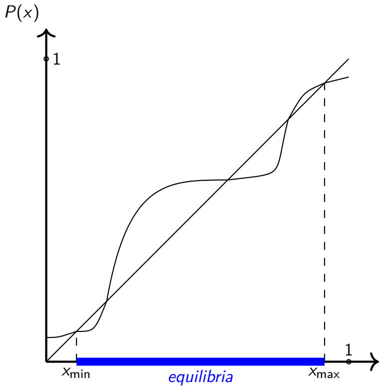
- So far, we showed that $\{x^*\}$ is the smallest set among all equilibrium sets of average behaviors across all networks.
- Next: What is the largest?
- Average equilibrium behavior

$$\mathrm{Av}\,(a) = \frac{1}{N} \sum a_i.$$

Largest equilibrium set

Theorem

There exists a sequence of networks g_n such that the sets of equilibrium average behavior converge to $[x_{\min}, x_{\max}]$.

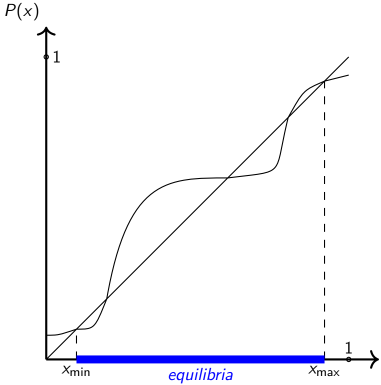


Largest equilibrium set

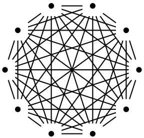
Theorem

There exists a sequence of networks g_n such that for each $\varepsilon > 0$

$$\lim_n Prob\left(\forall_{x\in[x_{\min},x_{\max}]}\exists_{a\text{ is equilibrium}}\ st.\ \ |Av(a)-x|<\varepsilon\right)=1.$$



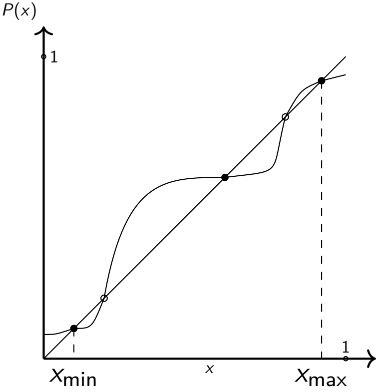
Largest equilibrium set



- Let g^n_{complete} be the complete graph with n nodes.
- If x is a stable fixed point of P, then, for each $\eta > 0$,

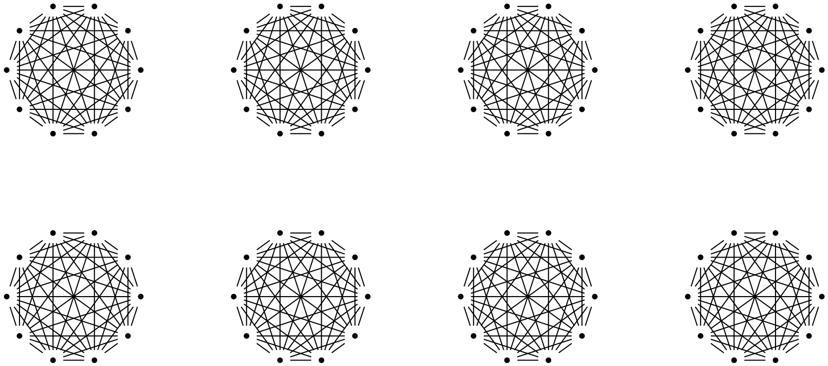
$$\lim_{n\to\infty} \text{Prob}\left(\{x\} \subseteq_\eta \text{Eq}\left(g^n_{\text{complete}}, \varepsilon\right)\right) \geq 1 - \eta.$$

Largest equilibrium set



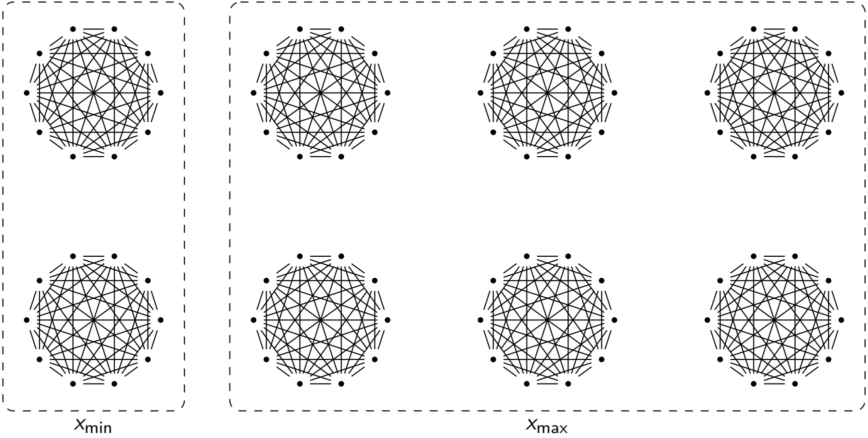
- Generically, $x_{\min}$ and $x_{\max}$ - the smallest and the largest fixed points - are stable.

Largest equilibrium set



- Idea: mix and match copies of complete networks.

Largest equilibrium set

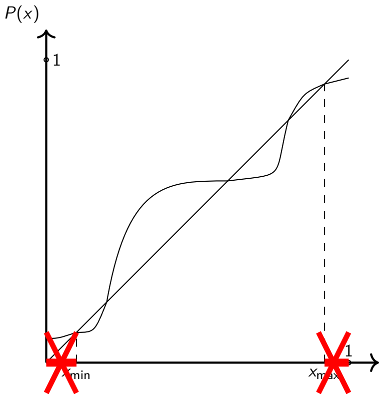


- Here, $x = \frac{2}{8}x_{min} + \frac{6}{8}x_{max}$.

Largest equilibrium set

Theorem

All limit equilibrium sets are contained in $[x_{\min}, x_{\max}]$.

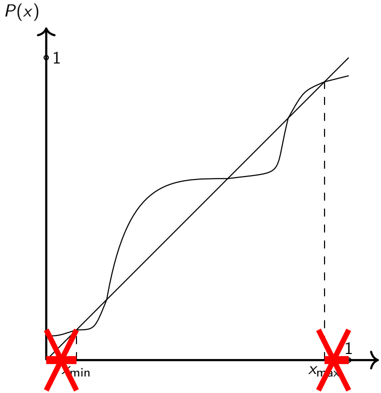


Largest equilibrium set

Theorem

For each $\eta > 0$ and $w < \infty$,

$$\lim_{d(g)\to 0, w(g)\leq w} Prob\left(a \text{ is equilibrium and } Av(a) \notin [x_{\min} - \eta, x_{\max} + \eta]\right) = 0.$$



Largest equilibrium set

- In fact, no equilibrium is larger than fuzzy convention $x^*_{\max}$ and smaller than fuzzy convention $x^*_{\min}$.
- The largest equilibrium set is $[x_{\min}, x_{\max}]$.
- Unique equilibrium when $x_{\min} = x_{\max}$.
- (Very partial) identification.

Largest equilibrium set

- Proof: similar to the proof of the first theorem.

Conclusion

- Heterogeneous payoffs in coordination games on network.
- We characterized the largest and the smallest possible set of equilibrium average behaviors across all networks.
- Results:
    - The largest set achieved on a collection of complete graphs,
    - partial identification theory,
    - The smallest set achieved on 2-dimensional (but not necessarily 1-dimensional) lattice,
    - equilibrium selection theory.
- Main assumptions:
    - independent payoff shocks,
    - large degree,
    - both assumptions are important.