Online Appendix to "Bargaining over Heterogeneous Pie with Mechanisms and Incomplete Information."

This is the Online Appendix to "Bargaining over Heterogeneous Pie with Mechanisms and Incomplete Information." It contains the proofs of results from Section 6.

Proof of Proposition 1

Let $\left(x^{j,\delta}\right)$ be the Rubinstein's allocation for type u^*, i.e., the outcome of the complete information game of Bob and Alice's type u where player j begins and players can make any simple offer, $\Gamma^j\left(\delta, \mathcal{S}, \delta_{u^*}\right)$. Let $A = \left\{x : v\left(x\right) \geq \delta v\left(x^{B,\delta}\right)\right\}$ be the set of allocations x that are acceptable to Bob who expects allocation $x^{B,\delta}$ in the continuation game if he rejects the current offer x.

We construct an equilibrium such that

- in Alice offer subgame,
  - Alice type u always offers her best allocation from set A, $x\left(u\right) \in \arg\max_{x\in A} u\left(x\right)$. (Notice that Alice is indifferent between two allocations only when $u = v$, in which case, Bob is also indifferent among her optimal choices.)
  - Bob accepts any offer $x \in A$.
  - If Alice offers $x \notin A$, Bob rejects, changes his beliefs to $\mu^* = \delta_{u^*}$, and expects from now on to follow the complete information equilibrium with payoff $v\left(x^{B,\delta}\right)$ in the next period.
- Bob's offer subgame,
  - Bob (with beliefs μ) chooses an offer so to maximize his expected payoff
    $$\int_{u:u(x)\geq\delta u(x(u))} v\left(u\left(x\right)\right) d\mu\left(u\right) + \delta v\left(x^{B,\delta}\right) \int^2_{u:u(x)\geq\delta u(x(u))} d\mu\left(u\right),$$
    subject to the above strategy of Alice.
  - Alice type u accepts any offer x such that $u\left(x\right) \geq \delta u\left(x\left(u\right)\right)$ and rejects otherwise.

Clearly, the strategies are best responses and the equilibrium payoffs are as required.

Proof of Theorem 3

The lower bound on Alice's payoff is a consequence of the proof of the second part of Lemma 3 - that proof does not depend on N.

We show the bound on Bob's payoff. For each Alice's type u, and each $v \geq 0$, define $u^*(v) = \max_{x:v(u)\geq v} u(x)$ as the largest payoff of type u that is consistent with Bob receiving at least v. For each $v \in [0,1]$, let $Y(v)$ be a menu $Y(v) = \{x : v(x) \geq v\}$. Let

$$e_B^{A,\delta} = \inf\left\{e_B : (e_A, e_B) \in E^A(\delta, \mu; \mathcal{M}) \text{ for any } \mu \in \Delta\mathcal{U}\right\}$$

be the lowest Bob's equilibrium payoff across all possible beliefs in the game in which Alice makes the first offer.

We will show that in the subgame where Bob makes the first offer, Alice will accept any menu that contains menu $Y\left(\delta e_B^{A,\delta} + 1 - \delta\right)$ in its interior. In any equilibrium of the game where Alice makes the first offer, Bob's expected payoff is not lower than $e_B^{A,\delta}$. Hence, it is impossible for every type u of Alice to expect more than $u^*\left(e_B^{A,\delta}\right)$. It follows that, in the game where Bob makes the first offer, a positive-measure fraction of Alice's types must accept any offer that is strictly larger than $\delta u^*\left(e_B^{A,\delta}\right)$. An argument similar to the one used in part "Equilibrium cannot be too high" part of Section 5.1 shows that all Alice types u must accept any menu with payoffs $y(u) > \delta u^*\left(e_B^{A,\delta}\right)$. (If some types reject, then a positive fraction of them would receive tomorrow's payoffs that are lower than $u^*\left(e_B^{A,\delta}\right)$. But then, a rejection would not be a best response.) Due to linearity of u and v,

$$\begin{aligned}
\delta u^*\left(e_B^{A,\delta}\right) &= \delta \max_{x:v(x)\geq e_B^{A,\delta}} u(x) = \max_{x:v(x)\geq e_B^{A,\delta}} u\left(\delta x + (1-\delta)\mathbf{0}_A\right) \\
&= \max_{x:v(\delta x + (1-\delta)\mathbf{0}_A)\geq \delta e_B^{A,\delta} + 1 - \delta} u\left(\delta x + (1-\delta)\mathbf{0}_A\right) \\
&\leq \max_{x:v(x)\geq \delta e_B^{A,\delta} + 1 - \delta} u(x) = u^*\left(\delta e_B^{A,\delta} + 1 - \delta\right).
\end{aligned}$$

(The inequality comes from the fact that the set of allocations X is convex, and, specifically, for each $x \in X$, $\delta x + (1-\delta)\mathbf{0}_A \in X$.) We conclude that Alice accepts any menu that contains menu $Y\left(\delta e_B^{A,\delta} + 1 - \delta\right)$ in its interior.

We are going to show that $e_B^{A,\delta} \geq \frac{\delta}{1+\delta}$. On the contrary, suppose that $e_B^{A,\delta} < \frac{\delta}{1+\delta}$. Then, there exists an equilibrium of the game where Alice makes the first offer with Bob's expected payoffs $e_B \leq e_B^{A,\delta}$. Consider a deviation where Bob rejects any offer from Alice and, instead, proposes menu $Y\left(\delta e_B^{A,\delta} + 1 - \delta\right)$. The above paragraph implies that such menu is accepted for sure. Bob's deviation is strictly profitable:

notice that, in the previous period,

$$\delta\left(\delta e_B^{A,\delta}+1-\delta\right)=e_B^{A,\delta}-\left(1-\delta^2\right)e_B^{A,\delta}+\delta\left(1-\delta\right)=e_B^{A,\delta}+\left(1-\delta\right)\left(\delta-\left(1+\delta\right)e_B^{A,\delta}\right)$$
$$>e_B^{A,\delta}+\left(1-\delta\right)\left(\delta-\left(1+\delta\right)\frac{\delta}{1+\delta}\right)=e_B^{A,\delta}.$$

It follows that $\lim_{\delta\to 1}\inf e_B^{A,\delta}\geq\lim_{\delta\to 1}\frac{\delta}{1+\delta}=\frac{1}{2}$.

Proof of Proposition 2

Fix $\delta<1$. For each player j and each $l=1,2$, define non-wasteful allocations x_l^j so that x_l^j is the equilibrium allocation in the complete information bargaining between Alice type τ_l and Bob in the subgame where player l makes the first offer. Let

$$A^j=\left\{x:\tau_l\left(x\right)\leq\tau_l\left(x_l^j\right)\text{ for each }l\right\}$$

be the set of allocations such that each type τ_l (weakly) prefers x_l^j to x^j. Let $x^j\in\arg\max_{x\in A}\tau\left(x\right)$ be the best non-wasteful allocation for Alice type τ among all allocations in A. It is easy to check that for each j,

$$\tau_1\left(x_1^j\right)=\tau\left(x^j\right)=\tau_2\left(x_2^j\right)=:u^j\text{ and }\delta u^A=u^B.$$

(Note that $\tau_l\left(x_l^j\right)=\tau_l\left(x^j\right)$ by the choice of x^j and symmetry. Moreover, $\tau\left(x^j\right)=\frac{1}{2}\tau_1\left(x^j\right)+\frac{1}{2}\tau_2\left(x^j\right)$.) The symmetry of two extreme types τ^l implies that

$$v\left(x_1^j\right)=v\left(x_2^j\right)=v^j\text{ and }v^A=\delta v^B.$$

Also, let $v_0^j=v\left(x^j\right)$. For each j, define a menu

$$Y^j=\left\{x_1^j,x^j,x_2^j\right\}\text{ for each }j.$$

Recall the definition of equilibrium correspondence E from Footnote 7. Let$\mathcal{M}^A$ be a class of mechanisms m such that for each belief $\mu\in\Delta\left\{\tau_1,\tau_2\right\}$, and each continuation equilibrium $\left(e_A,e_B\right)\in E\left(m,\mu\right)$, we have $e_A\left(t\right)>u^A$ for some $t\in\left\{\tau_1,t,\tau_2\right\}$. We have the following Lemma:

Lemma 6. *For each $m\in\mathcal{M}^A$, there exists a belief $\mu^m\in\Delta\left\{\tau_1,\tau_2\right\}$ and continuation equilibrium payoffs $e^m\in E\left(m,\mu\right)$ such that $e_B^m<v^A$.*

Proof. Fix $m\in\mathcal{M}_A$. The choice of allocation x^A as the τ-best allocation that satisfies incentive-compatibility conditions for types τ_l implies that for any allocation x, if

$\tau(x) > \tau\left(x^A\right) = u^A$, then $\tau_l(x) > u^A$ for some l. Hence, for any belief $\mu \in \Delta\{\tau_1, \tau_2\}$ and any mechanism equilibrium payoffs $(e_A, e_B) \in E(m, \mu)$, there is $l = 1, 2$ such that $e_A(\tau_l) > u^A$.

The compactness assumption (Footnote 7) implies that

$$E(m) = \{(e, \mu) : \mu \in \Delta\{\tau_1, \tau_2\}, e \in E(m, \mu)\}$$

is a compact and connected subset of compact Polish space $E^* \times \Delta\{\tau_1, \tau_2\}$. Let

$$W_l = \left\{(e, \mu) \in E(m) : e(\tau_l) > u^A\right\}.$$

Sets W_l are open, they cover $E(m)$, hence, either

- there is l_0 such that $E(m) \subseteq W_{l_0}$. In such a case, take $\mu^m = \delta(\tau_{l_0})$ be the measure assigning full probability to type τ_{l_0}, and take any $e^m \in E(m, \mu^m)$. Then, the choice of $x^A_{l_0}$ implies that $e^m_B < v^A$. Or,
- $W_1 \cap W_2 \neq \emptyset$, in which case take any $(e^m, \mu^m) \in W_1 \cap W_2$. The choice of x^A_l for each l implies that $e_B < v^A$.

$\square$

Construct an equilibrium:

- Alice's offer subgame: Alice offers menu Y^A and each of her type expects payoff u^A.
  - If Alice offers menu Y^A, Bob accepts it. If not, his continuation payoff is $\delta v^B = v^A$. Hence, accepting is a best response, and Alice's payoff is u^A.
  - If Alice offers mechanism $m \in \mathcal{M}^A$, Bob updates his beliefs to μ^m, and he expects that, if m is accepted, equilibrium with payoffs e^m is played, which leads to a payoff $e^m_B < v^A$ for Bob. Given that, Bob rejects the mechanism and expects discounted continuation payoff $\delta v^B = v^A$. In such a case, each Alice's type t expected payoff is not higher than $\delta u^B < u^A$.
  - If Alice offers mechanism $m \notin \mathcal{M}^A$, then, arbitrary continuation equilibrium is played. By the definition of class $\mathcal{M}^A$, each Alice's type receives a payoff not larger than u^A.

  Because no deviation leads to a higher payoff for any type, offer of menu Y^A is a best response.

- Bob's offer subgame: Bob offers menu Y^B and expects payoff $\mu\{\tau_1,\tau_2\} v^B + (1-\mu\{\tau_1,\tau_2\}) v_0^B$.
  - If Bob offers menu Y^B, Alice accepts it and each of her types receives payoffs u^B. If she were to reject, each of her types expects continuation payoffs $\delta u^A = u^B$. Hence, accepting is a best response.
  - If Bob offers any other mechanism $m \neq Y^B$, and Alice rejects it, each of her types expects continuation payoffs $\delta u^A = u^B$. If any of her types accepts it, it must be that she receives payoff of at least u^B. In any case, whatever is the continuation equilibrium payoff in the subgame after Bob offers m, each Alice's types receives at least u^B as her continuation payoff. The choice of allocations x_1^B, x^B, x_2^B implies that Bob's payoff cannot be larger than v^B.

  Hence, offering Y^B is a best response.