Fuzzy Conventions

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- Social interactions, positive externalities.
 - wearing a mask,
 - engaging in criminal activity,
 - technology adoption.
- A typical result: emergence of a (homogeneous) convention.
- But, in reality, conventions are often fuzzy:
 - some, but not all, wear masks,
 - married couples that use both IPhone and Android.
- People care not only about their neighbors, but they differ wrt. tastes, preferences.

- Binary coordination games on networks with random utility,
- (Statistical) heterogeneous preferences: i.i.d payoff shocks,
- I am interested in the set of average (i.e., aggregate) behavior $x \in [0,1]$
 - in static,
 - complete information equilibria,
 - when each agent number of connection is large.
- Q: What can we say about equilibrium sets? How do they depend on the network?

- agents i, j live on a network with weights $g_{ij} = g_{ji} \ge 0$,
 - $g_i = \sum_j g_{ij}$ is degree of agent i,
- payoffs: $\sum_{j\neq i} g_{ij} u(a_i, a_j, \varepsilon_i)$,
 - each i chooses $a_i \in \{0, 1\}$,
 - i.i.d. payoff shocks $\varepsilon_i \sim F$,
 - positive externalities: $u(.,.,\varepsilon_i)$ has increasing differences for each ε ,
- average behavior Av (a) = $\frac{1}{\sum_i g_i} \sum_i g_i a_i$,
- equilibrium set

$$\mathsf{Eq}\left(g,\varepsilon\right)=\left\{\mathsf{Av}\left(a\right):a\text{ is a Nash equilibrium in game }G\left(g,\varepsilon\right)\right\},$$

Model

- Object of interest: $\lim Eq(g,.)$ as

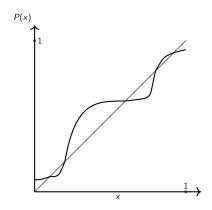
 - $d\left(g\right) = \max_{i,j} \frac{g_{ij}}{g_i} \rightarrow 0$ large degrees, $w\left(g\right) = \max_{i,j} \frac{g_i}{g_i} < w_{\max} < \infty$ is bounded not too much inequality.

Results

• 4 theorems that characterize the largest and the smallest possible limit of equilibrium sets across all networks.

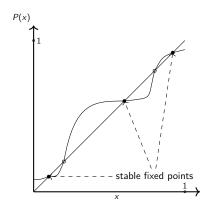
Results

- Let $P(x) = F\{\varepsilon : u(1, x, \varepsilon) \ge u(0, x, \varepsilon)\},\$
- continuum best response



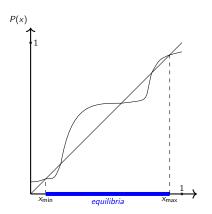
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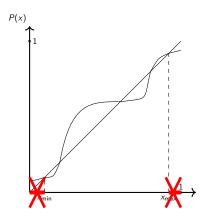
Results

• **Theorem 1**: There exists a sequence of networks such that the limit equilibrium set is $[x_{min}, x_{max}]$.



Results

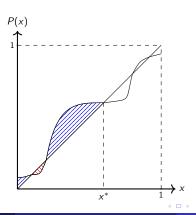
• **Theorem 2**: All limit equilibrium sets are contained in $[x_{min}, x_{max}]$.



Results

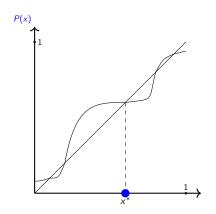
• Define random utility (RU-) dominant outcome

$$x^* \in \arg\max_{x} \int\limits_{0}^{x} \left(y - P^{-1}(y)\right) dy.$$



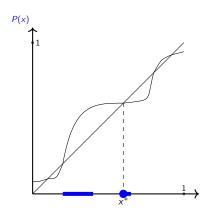
Results

• **Theorem 3**: There exists a sequence of networks such that the limit equilibrium set is $\{x^*\}$.



Results

• **Theorem 4**: All limit equilibrium sets contain x^* .



Literature

- Emergence of conventions: evolutionary approach
 - risk-dominance (Harsanyi-Selten 88),
 - complete networks (Kandori, Mailath Rob 93), (Young 93), line and some other networks (Ellison 93, Ellison 00), all networks (Peski 10).
- Global games and robustness to incomplete information
- Contagion (Morris 00):
 - some networks (lattices) admit contagion: a finite group of agents can spread risk-dominant behavior to the rest of the network,
 - contagion only works towards risk-dominant action.
- Here,
 - random utility instead of noise (or a perturbation),
 - static solution concept,
 - no aggregate uncertainty.

Notation

• Define a profile of neighborhood fractions β^a : for each i

$$\beta_i^a = \frac{1}{g_i} \sum_{i \neq j} g_{ij} a_j,$$

• $A \subseteq_{\eta} B$ if for each $a \in A$, there is $b \in B$ st. $|a - b| \le \eta$, $A =_{\eta} B$ if $A \subseteq_{\eta} B$ and $B \subseteq_{\eta} A$.

• Let g_{complete}^n be the complete graph with n nodes

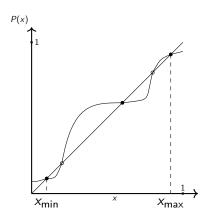


Theorem

If x is a stable fixed point of P, then, for each $\eta > 0$,

$$\lim_{n\to\infty} \operatorname{Prob}\left(\{x\}\subseteq_{\eta} \operatorname{Eq}\left(g_{\operatorname{complete}}^n,\varepsilon\right)\right) \geq 1-\eta.$$

very simple proof,



• Generically, x_{\min} and x_{\max} - the smallest and the largest fixed points - are stable.

Corollary

There exists a sequence of graphs g^n such that

$$\lim_{n\to\infty} Prob([x_{\min},x_{\max}] \subseteq_{\eta} Eq(g^n,\varepsilon)) \geq 1-\eta.$$









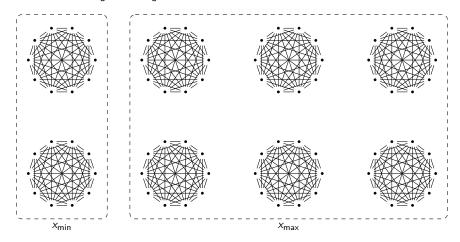








• Here, $x = \frac{2}{8}x_{\min} + \frac{6}{8}x_{\max}$.



 So far, we showed existence of networks g such that with a large probability,

$$[x_{\min}, x_{\max}] \subseteq_{\eta} \mathsf{Eq}(g, \varepsilon)$$
.

• Next, we show that, for any g st. $d(g) = \max_{i,j} \frac{g_{ij}}{g_i}$ is sufficiently small,

$$\mathsf{Eq}\left(g,\varepsilon\right)\subseteq_{\eta}\left[x_{\mathsf{min}},x_{\mathsf{max}}\right].$$

Theorem

For any $w_{\text{max}} < \infty$, any sequence of graphs g_n , if $d\left(g_n\right) \to 0$ and $w\left(g_n\right) \leq w_{\text{max}}$, then

$$\lim_{n\to\infty} Prob\left(Eq\left(g^n,\varepsilon\right) \subseteq_{\eta} \left[x_{\min},x_{\max}\right] \right) = 1.$$

- Proof: surprisingly complicated.
- W.l.o.g., we want to show that, with a large probability, there is no profile a st Av $(a) > x_{\text{max}}$ and a is an equilibrium.

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- Bound

Prob (there exists a st.Av(a) $\geq x$ and a is equilibrium) $\leq \# \{a : Av(a) > x\} \cdot Prob(a \text{ is equilibrium}).$

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- W.l.o.g., we want to show that, with a large probability, there is no profile a st Av $(a) > x_{\text{max}}$ and a is an equilibrium.
- It is easy to show that a is unlikely to be an equilibrium: there exists $\delta > 0$ st. for each a,

Prob (a is equilibrium)
$$\leq \exp(-\delta N)$$
.

But, there are many profiles a:

$$\# \{a : Av(a) > x\} \sim exp((x \log x + (1-x) \log (1-x)) N).$$

- Proof: surprisingly complicated.
- W.l.o.g., we want to show that, with a large probability, there is no profile a st Av $(a) > x_{\text{max}}$ and a is an equilibrium.
- Problem: there are too many candidate profiles a.
- Observation I: the above proof treats events "a is equilibrium" for all as as disjoint, whereas they are often correlated.
- Observation II: events "a is equilibrium" and "a' is equilibrium" are correlated more if β^a and $\beta^{a'}$ are similar.
 - $\beta_i^a = \frac{1}{g_i} \sum_j g_{ij} a_j$.
- Idea: divide all profiles a into "groups" with similar β^a .

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• The correlation is stronger if $\beta^a \sim \beta^{a'}$, where β^a is a profile of "neighborhood fractions $\beta^a_i = \frac{1}{g_i} \sum_{j \neq i} g_{ij} a_j$), or

$$d\left(\beta_{i}^{a},\beta_{i}^{a'}\right) = \sqrt{\frac{1}{\sum g_{i}^{2}} \sum g_{i}^{2} \left(\beta_{i}^{a} - \beta_{i}^{a'}\right)^{2}} \text{ is small.}$$

• We show that for each a_0 st. Av $(a_0) > x$, if δ is sufficiently small and $d(g) \le \delta$, then

Prob $(\{a: d(\beta^a, \beta^{a_0}) \le \delta\}$ contains an equilibrium) $\le \exp(-\delta N)$.

Set of "neighborhood fraction" profiles

$$\mathcal{B} = \{\beta^a : a \text{ is a profile}\}.$$

- $\mathcal{N}(\mathcal{B}, \delta)$ is the smallest n such that there exists $b_1, ..., b_n \in \mathcal{B}$ st. \mathcal{B} can be covered with balls radius δ and centers at b_i (metric entropy).
- For some constant c > 0,

$$\mathcal{N}\left(\mathcal{B},\delta
ight)\leq\exp\left(crac{1}{\delta^{2}}d\left(g
ight)N
ight).$$

$$\begin{split} &\operatorname{\mathsf{Prob}}\left(\left\{a:d\left(\beta^{a},\beta\right)\leq\delta\right\} \text{ contains an equilibrium}\right)\\ &\leq &\mathcal{N}\left(\mathcal{B},\delta\right) \cdot \sup_{a_{0}:\operatorname{\mathsf{Av}}\left(a_{0}\right)>x}\operatorname{\mathsf{Prob}}\left(\left\{a:d\left(\beta^{a},\beta^{a_{0}}\right)\leq\delta\right\} \text{ contains an equilibrium}\right).\\ &\leq \exp\left(c\frac{1}{\delta^{2}}d\left(g\right)\mathcal{N}-\delta\mathcal{N}\right), \end{split}$$

which is small if d(g) is small enough.

Random utility dominant outcome

- So far, we characterized a tight upper bound on the equilibrium set.
- Next, we turn to a lower bound.

Random utility dominant outcome

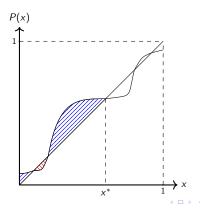
• Define random utility (RU-) dominant outcome

$$x^* \in \arg\max_{x} \int\limits_{0}^{x} \left(y - P^{-1}(y)\right) dy.$$

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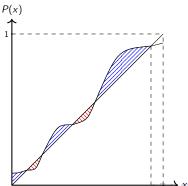


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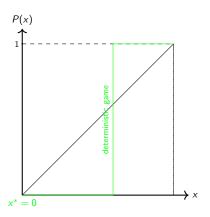
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• RU-outcome can be x_{min} or x_{max} .



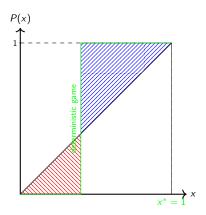
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 When game is deterministic, RU-dominance is equivalent to Harsanyi-Selten risk-dominance



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Random utility dominant outcome

Formula

$$x^* \in \arg\max_{x} \int_{0}^{x} \left(y - P^{-1}(y) \right) dy$$

appears in Morris and Shin (06).

- continuum toy model,
 - observe that the coordination game has a potential,
 - the above outcome maximizes potential,
 - hence it is robust to incomplete information.

Assume 0 < P(0) < P(1) < 1. There exists a sequence of networks g^n st. for each $\eta > 0$,

$$\lim_{n\to\infty} Prob(Eq(g^n,\varepsilon)) =_{\eta} \{x^*\}) \ge 1 - \eta.$$

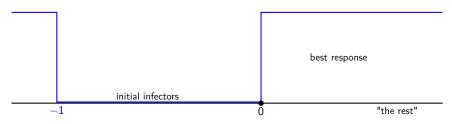
- For some networks, x^* is the unique average equilibrium behavior.
- The assumption ensures that, for each action, there is a positive probability that the action is dominant.

Proof

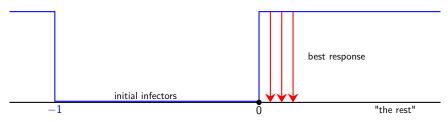
- Networks: 2-dimensional lattices
 - line (1-dimensional lattice) is not enough
- Static result, but proof based on best response dynamics.
 - review of contagion arguments (Ellison 93, Blume 93, Morris 00),
 - contagion wave on "toy" line,
 - why line is not enough and why 2-dimensional lattice is.

- Start with deterministic case, but with small group of initial infectors.
- Assume 0 is risk-dominant.
- We want to show that 0 is the only equilibrium.
- -> contagion.

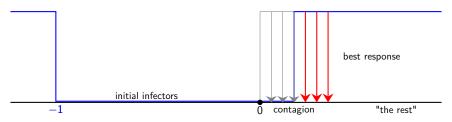
- Ellison 93: suppose that action 0 is risk-dominant,
- initial infectors $-1 \le i \le 0$ play 0; the rests play 1,



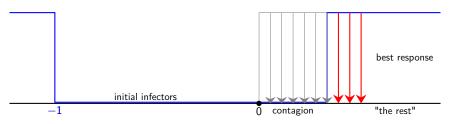
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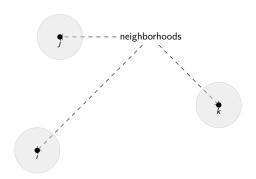


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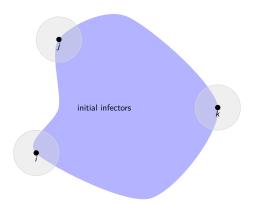


- Blume 93, Morris 00 the same mechanics works on other networks, like 2 (or higher)-dimensional lattices.
- Key step: half of neighbors of "threshold agents" must be infected to spread contagion.

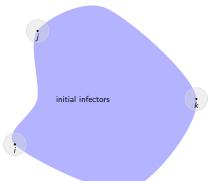
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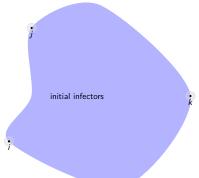
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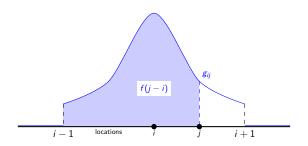


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Proof: Contagion wave on toy line

- Random utility payoffs (so, not deterministic)
- Toy line: Continuum of agents in each location.



- ullet Toy line: agents in location i are connected with agents in location j
 - connection density $g_{ij} = g_{ji} = g_{i+1,j+1}$ for any I,
 - $g_{ij} = 0$ for j > i + 1,
 - $f(j-i) = \frac{1}{g_i} \int_{i-1}^{j} g_{il} dl$,
 - f(x) + f(1-x) = 1.

Proof: Contagion wave on line, RU case

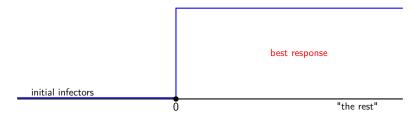
• For simplicity, assume that $x^* = 0$ is RU-dominant, i.e.

$$\int_{0}^{x} \left(y - P^{-1} \left(y \right) \right) dy < 0 \text{ for each } x > 0.$$

• Starting from arbitrary profile with a group of initial infectors playing x^* , best response dynamics will spread x^* to the whole line.

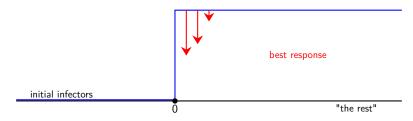
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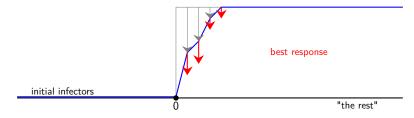
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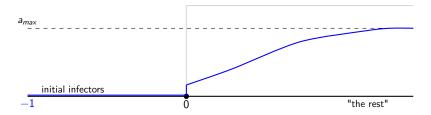
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Proof: Contagion wave on line, RU case

• Suppose that stops before spreading everywhere.



Proof: Contagion wave on line, RU case

• If the contagion stops, then at each location i > 0,

$$a_{i}\leq P\left(\int a_{i+k}df\left(k\right)\right).$$

Taking inverse and integrating by parts

$$P^{-1}\left(a_{i}
ight)\leq\int a_{i+k}df\left(k
ight)=\int_{0}^{a_{\max}}f\left(i-j
ight)da_{j}.$$

• Integrate over $a_i \in [0, a_{max}]$,

$$\int_0^{a_{\max}} P^{-1}\left(a_i\right) da_i \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f\left(i-j\right) da_j da_i.$$

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$$P^{-1}(a_i) \leq \int a_{i+k} df(k) = \int_0^{a_{\max}} f(i-j) da_j.$$

• Integrate over $a_i \in [0, a_{\text{max}}]$,

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Proof: Contagion wave on line, RU case

• Integrate over $a_i \in [0, a_{\text{max}}]$,

$$\int_{0}^{a_{\text{max}}} P^{-1}(a_{i}) da_{i}$$

$$\leq \int_{0}^{a_{\text{max}}} \int_{0}^{a_{\text{max}}} f(i-j) da_{j} da_{i}$$

$$= \frac{1}{2} \int_{0}^{a_{\text{max}}} f(i-j) da_{j} da_{i} + \frac{1}{2} \int_{0}^{a_{\text{max}}} \int_{0}^{a_{\text{max}}} f(j-i) da_{j} da_{i}$$

• Integrate over $a_i \in [0, a_{\text{max}}]$,

$$\begin{split} & \int_{0}^{a_{\text{max}}} P^{-1}\left(a_{i}\right) da_{i} \\ & \leq \int_{0}^{a_{\text{max}}} \int_{0}^{a_{\text{max}}} f\left(i-j\right) da_{j} da_{i} \\ & = \frac{1}{2} \int_{0}^{a_{\text{max}}} f\left(i-j\right) da_{j} da_{i} + \frac{1}{2} \int_{0}^{a_{\text{max}}} \int_{0}^{a_{\text{max}}} f\left(j-i\right) da_{j} da_{i} \\ & = \frac{1}{2} \int_{0}^{a_{\text{max}}} \int_{0}^{a_{\text{max}}} \left[f\left(i-j\right) + f\left(j-i\right) \right] da_{j} da_{i} \end{split}$$

• Integrate over $a_i \in [0, a_{\text{max}}]$,

$$\int_{0}^{a_{\text{max}}} P^{-1}(a_{i}) da_{i}
\leq \int_{0}^{a_{\text{max}}} \int_{0}^{a_{\text{max}}} f(i-j) da_{j} da_{i}
= \frac{1}{2} \int_{0}^{a_{\text{max}}} f(i-j) da_{j} da_{i} + \frac{1}{2} \int_{0}^{a_{\text{max}}} \int_{0}^{a_{\text{max}}} f(j-i) da_{j} da_{i}
= \frac{1}{2} \int_{0}^{a_{\text{max}}} \int_{0}^{a_{\text{max}}} [f(i-j) + f(j-i)] da_{j} da_{i}$$

• Recall that f(i - j) + f(j - i) = 1.

• Integrate over $a_i \in [0, a_{max}]$,

$$\begin{split} & \int_{0}^{a_{\text{max}}} P^{-1}\left(a_{i}\right) da_{i} \leq \int_{0}^{a_{\text{max}}} \int_{0}^{a_{\text{max}}} f\left(i-j\right) da_{j} da_{i} \\ = & \frac{1}{2} \int_{0}^{a_{\text{max}}} f\left(i-j\right) da_{j} da_{i} + \frac{1}{2} \int_{0}^{a_{\text{max}}} \int_{0}^{a_{\text{max}}} f\left(j-i\right) da_{j} da_{i} \\ = & \frac{1}{2} \int_{0}^{a_{\text{max}}} \int_{0}^{a_{\text{max}}} \left[f\left(i-j\right) + f\left(j-i\right) \right] da_{j} da_{i} \\ = & \frac{1}{2} \int_{0}^{a_{\text{max}}} \int_{0}^{a_{\text{max}}} da_{j} da_{i} = \int_{0}^{a_{\text{max}}} a da. \end{split}$$

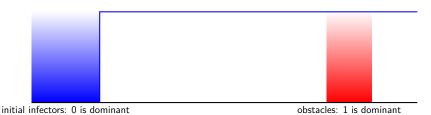
• We get contradiction with $\int_0^{a_{\max}} (y - P^{-1}(y)) dy < 0$.

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- But! so far we assumed that locations contain continuum.
- Contagion can be also stopped by unusual payoff shocks, like those that make 1 dominant.



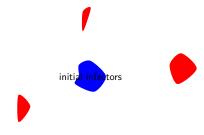
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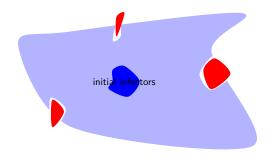
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- So far, we showed that there are networks g such that Eq $(g, \varepsilon) \subseteq_n \{x^*\}$ with a large probability.
- Next, we show that if $d(g) = \max_{i,j} \frac{g_{ij}}{g_i}$ is sufficiently small, than $\{x^*\} \subseteq_n \text{Eq}(g,\varepsilon)$.

Theorem

For any sequence of graphs g_n , if $d\left(g_n\right) \to 0$, then

$$\lim_{n} Prob(\{x^*\} \subseteq_{\eta} Eq(g_n)) = 1.$$

- Hence $\{x^*\}$ is the smallest equilibrium set.
- Equilibrium selection theory: no matter what network, there is an equilibrium with aggregate behavior,
 - the proof tries to make this idea more precise.
- Analog of a result from Morris "Contagion": if all but finitely many agents play risk-dominant action, the best response dynamics won't move towards risk-dominated action.

Proof: Morris "Contagion"

- Morris: "Contagion":
- Initial profile a^0 : all but fintely many play risk-dominant action 0
- Consider a best response dynamics $a^0 < a^1 < a^2 < ...$
 - each "round" only one agent changes action
- For each profile a, define capacity to infect:

$$\mathcal{F}_{0}\left(a
ight)=\sum_{i,j:a_{i}=1,a_{j}=0}g_{ij}.$$

Proof: Morris "Contagion"

- Capacity must go down "significantly" at each round:
 - if *i* changes action from 0 to 1 as a best response, the capacity changes by by

$$\sum_{j:a_j=0}g_{ij}-\sum_{j:a_j=1}g_{ij}.$$

but, if 1 is a best response and 0 is risk-dominant, then

$$\sum_{j:a_j=0} g_{ij} < \frac{1}{2} < \sum_{j:a_j=1} g_{ij}!$$

- Because the capacity cannot be negative, contagion has to stop.
- If the initial profile was close to 0, the capacity was small and the contagion will stop very soon, with most agents not changing their actions.

Proof: Morris "Contagion"

- Key feature of a good definition of capacity
 - it decreases along best response dynamics,
 - it is small,
 - cannot be negative.
- The number of stages until the dynamics stops is related to the initial capacity.

- Our proof follows a similar idea.
- Let x*be RU-dominant outcome.
- Construct initial profile a^0 st. for each i,

$$a_i^0 \in \arg\max_{a} u_i(a, x^*, \varepsilon_i)$$

- many people play 0 and many play 1
- Consider best response dynamics $a^0 < a^1 < \dots < a^T$.
- We show that $\frac{T}{N} \sim O(d(g))$.
- Hence $a_i^0 \in \arg\max_a u_i(a, x^*, \varepsilon_i)$ is a pretty safe action to take, whatever is the true network.

Definition of capacity: original

$$\sum_{i,j:a_i=1,a_j=0}g_{ij}.$$

• The problem is that is is too big for the initial profile a^0 .

• Definition of capacity: notice that

$$\sum_{i,j:a_i=1,a_j=0} g_{ij} = \frac{1}{2} \sum_{i,j} g_{ij} (a_i - a_j)^2.$$

• Definition of capacity:

$$\frac{1}{2}\sum_{i,j}g_{ij}\left(a_i-a_j\right)^2.$$

• Definition of capacity:

$$\mathcal{F}(a) = \frac{1}{2} \sum_{i,j} g_{ij} \left(P(\beta_i^a) - P(\beta_j^a) \right)^2.$$

• replace a_i by the "continuum best response" to the neighborhood profile β^a .

• Definition of capacity:

$$\mathcal{F}(a) = \frac{1}{2} \sum_{i,j} g_{ij} \left(P(\beta_i^a) - P(\beta_j^a) \right)^2.$$

- Capacity is small at the initial profile a^0 (probabilistically),
 - $\beta_i^{a^0} \sim x^*$ for most i because $x^* = P(x^*)$ and $d(g) \sim 0$, and
 - $P\left(\beta_i^{a^0}\right) \sim P\left(\beta_j^{a^0}\right)$ for most i and j.

- Turns out that this is a good definition
 - capacity is small (probabilistically)
 - it is a sum of a martingale and a decreasing process,
 - ullet ignoring (probabilistically) small terms, we show that, for each T

$$\sum_{i} g_{i} \left[\int_{x^{*}}^{P\left(\beta_{i}^{a^{T}}\right)} \left(P^{-1}\left(y\right) - y\right) dy \right] \leq \mathcal{F}\left(a^{0}\right)$$

Comments

- Potential game
- Evolutionary literature
- Small degree

Comments

Potential

Binary coordination games have potential

$$V(a;\varepsilon) = \frac{1}{2} \sum_{i,j} g_{ij} a_i a_j - \sum_i \epsilon_i a_i$$
$$= a^T G a - a^T \varepsilon.$$

- Potential maximizers are
 - equilibria,
 - selected by evolutionary logistic dynamics (Blume)
 - robust to incomplete information.

Comments

Potential

Binary coordination games have potential

$$V(a; \varepsilon) = \frac{1}{2} \sum_{i,j} g_{ij} a_i a_j - \sum_i \epsilon_i a_i$$
$$= a^T G a - a^T \varepsilon.$$

- .
- One way to prove Theorem 4 is to show that V always has a local (or global) maximum close $a^{\rm O}$
 - quadratic <u>binary</u> form with a random linear term,
- But how?

Conclusion

- Heterogeneous payoffs in coordination games on network.
- We characterized the largest and the smallest possible set of equilibrium average behaviors across all networks.
- Results:
 - The largest set achieved on a collection of complete graphs,
 - partial identification theory,
 - The smallest set achieved on 2-dimensional (but not necessarily 1-dimensional) lattice,
 - equilibrium selection theory.
- Main assumptions:
 - independent payoff shocks,
 - · large degree,
 - both assumptions are important.