BARGAINING WITH MECHANISMS

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Abstract. Two players bargain over an allocation of a single indivisible good with transfers, with one-sided incomplete information about preferences. Both players can offer arbitrary mechanisms to determine the allocation. We show that there is a unique perfect Bayesian equilibrium outcome. In the equilibrium, one of the players proposes a menu that is optimal for the uninformed player among all menus such that each type of the informed player receives at least her payoff under complete information. The optimal menu can be implemented with three allocations. Under a natural assumption on the uninformed player beliefs, the optimal menu coincides with the Myerson's neutral solution to the bargaining problem in this environment.

1. Introduction

In a standard model of bargaining, one party proposes an allocation of the bargaining surplus and the other party either accepts or rejects it. However, offers made during real-world negotiations are often much more complex. Instead of a single allocation, parties may offer menus of allocations for the other party to choose from.[1] They

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[1]See Jackson et al. 2020 for real-world and experimental examples. I had an opportunity to observe bargaining over a pension plan reform that took place in 2016-18 between three Ontario universities and the representatives of faculty and staff. Among other issues, the parties negotiated the size of the spousal benefit, early retirement options, inflation indexation, etc. While the universities only cared about the total actuarial cost, the preferences of the labor side were uncertain, mostly due to its heterogeneity (for instance, the staff, but not the faculty, valued early retirement more than the

may offer to settle the dispute with an arbitrator.[2] They may also offer to alter the bargaining protocol, for example, by dividing the dispute into smaller areas and settling them separately, or by establishing deadlines.[3] We teach our students (and our children) that a fair cake division can be found through procedures like "I divide and you choose."[4] All such offers can be represented as a mechanism, the outcome of which determines the final allocation. The goal of this paper is to study the role of mechanisms as offers in a strategic model of bargaining by addressing the following questions: Does expanding the scope of offers to general mechanisms affect the way in which parties bargain? Which mechanisms are offered in equilibrium? Is the equilibrium efficient?

A natural setting for studying mechanisms as offers is when there is incomplete information about player preferences. To stay as close as possible to the existing literature, we consider a version of multi-round random proposer model (Okada (1996)). There are two players, Alice and Bob, who decide who should get a single indivisible good. Probabilistic allocation as well as transfers are allowed. Bob's value for the good is known known. Alice's value is her private information. In each period, a randomly chosen player is given an opportunity to offer a mechanism, which the other player accepts or rejects. A mechanism is defined as an arbitrary game, where players' choices determine the final allocations. When the offer is accepted, the mechanism is implemented and the bargaining ends. We study perfect Bayesian equilibria (PBE) with the only restriction that Bob's off-path beliefs about Alice's types do not change after his actions. By varying the probability with which the proposer is chosen, the

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spousal benefit). Ultimately, the universities proposed a menu of options, and the labor side chose an option from this menu.

[2]During the 2019-2020 dispute between the Ontario government and teacher unions, both parties called upon the other to accept mediation but could not agree on the same mediator (Rushowy (2020), Moodie (2019)).

[3]EU accession negotiations typically take the form of independent bargaining over 30-40 areas.

[4]An example of such mechanism is the Texas shoot-out clause used in the dissolution of a partnership: one partner names a price and the other partner is obliged to either sell her shares or buy the shares of the first partner at the price. I am grateful to T. Tröger for this example.

model can span a whole range of bargaining games, including those where all offers are made by either the informed, or the uninformed player.

Any strategic model of bargaining under incomplete information must deal with two types of problems. Due to a screening problem, a Bob's offer may be acceptable to some but not all types of the opponent. This may lead to a delay and a new offer for the remaining types, which may change the incentives to accept the original one. Due to a signaling problem, by making, or rejecting an offer, Alice reveals some private information, which may hurt her and benefit Bob future bargaining rounds. Due to a "belief punishment" problem, Alice may accept or make an unfavorable offer because off-path deviations are punished with beliefs that lead to a low continuation payoff. The signaling and belief-threat problems typically lead to multiplicity of equilibria that can sometimes be resolved through equilibrium refinements.

This paper shows that when players are allowed to offer arbitrary mechanisms, the problems have a satisfactory solution. The main result is that Bob's PBE payoff is unique and Alice's payoff is generically unique. In the equilibrium, one of the players proposes a screening menu that is optimal for Bob, subject to the constraint that each of Alice's types receives at least her complete information payoff. The menu has a natural interpretation: Bob gets a fraction of the probability of receiving the good equal to his bargaining power and either purchases the remaining fraction from Alice at the price equal to his valuation, or sells his own fraction at optimal (for him) price. The final outcome is not *ex-post* efficient (it is *interim* efficient). The solution has natural comparative statics with respect to information: Bob is better off when his information improves. When Bob's beliefs converge to certainty, the outcome converges to the complete information Nash solution.

The proof of the uniqueness parallels the argument for the uniqueness of the subgame perfect equilibrium payoffs in Rubinstein's alternating-offer game. We develop step-by-step bounds to contain the equilibrium payoffs and show that the lower and upper bounds converge to the same outcome. Two types of mechanisms play a role in the proof. On the one hand, Bob's ability to offer menus (of allocations, for Alice to choose) allows him to screen among Alice's types without them worrying about revealing information. On the other hand, Alice's ability to offer menus (for Bob to choose) of menus (of allocations, for Alice to choose) allows her to protect herself from "punishment with beliefs". To see a simple intuition for the latter point, suppose that Alice considers an off-path deviation to one of two mechanisms $m \in \{m_1, m_2\}$ with the property that, for each of Bob's beliefs, one of the two mechanisms would be acceptable to Bob, but none of them is acceptable across all his beliefs. She can be stopped from such a deviation if she is afraid that after off-path offer m, Bob's beliefs will change to those that find m unacceptable. Such punishment with beliefs would not be possible if she were able to offer a menu $\{m_1, m_2\}$ of mechanisms and let Bob choose whichever mechanism he prefers.

The main result is significant for multiple reasons. First, because both the informed and uninformed agents design mechanisms, our model is an example of a dynamic informed principal problem (Myerson (1983)). The uniqueness without any equilibrium refinement is a rare result in the informed principal literature where, typically, multiple equilibria can be supported by belief punishment threats (Mylovanov and Tröger (2012)).

The availability of sophisticated offers plays an important role for uniqueness. If players are only able to offer simple allocations, bargaining games with both the informed and informed players making offers typically have multiple equilibria (Ausubel and Deneckere 1989, Gul and Sonnenschein (1988)). The uniqueness can sometimes can be restored by equilibrium refinements (Grossman and Perry (1986b)).

Second, although assumptions explicitly disallow commitment across periods, the equilibrium outcome is the same as if Bob could commit himself to any mechanism subject to the constraint that each Alice type receives at least her complete information payoff. When the bargaining power of the two agents is equal, the latter is equal to the Nash solution. The constraint is clearly a consequence of the connection between the random-proposer bargaining model and the Nash solution.

To make this point starker, consider a special case of our model in which the uninformed party makes all the offers. In this case, the unique outcome is that Bob offers to sell the good to Alice at a price that maximizes his static monopoly profits, or his commitment payoff. The outcome is inefficient if Alice's valuation is between Bob's and the monopoly price. This result can be contrasted with the Coase conjecture, which predicts that the uninformed player sells at the price equal to the lowest possible value of the informed player, and the equilibrium is efficient. In the bargaining literature, the Coase conjecture has been typically associated with the "gap case" of the durable monopoly problem with offers made only by the uninformed party (Gul, Sonnenschein, and Wilson (1986)), but the Coasian forces play role also in alternating-offer models as well (Gul and Sonnenschein (1988)).

An important assumption of our model is that once the mechanism is offered and accepted, the players are committed to its implementation. Although our assumption is shared by the Coasian bargaining literature, and also the more recent literature on dynamic mechanism design with limited commitment (Skreta (2006), Doval and Skreta (2018), Liu et al. (2019), and others), we also allow for a wider range of mechanisms than this literature typically considers. For example, an agreement on negotiation protocol may force players to restrict their future options, set a deadline, or choose an ex-post inefficient outcome. In other words, we allow players to commit *jointly.* Our model is applicable in situations in which such a commitment is possible, either because the nature of dividing the surplus makes it impossible to divide it again, or because renegotiation is costly, or the agreement is enforced by an arbitrator or a court.

Section 6.3 discusses a version of the model where, if both players are willing, each agreement can be renegotiated. We show that, unexpectedly, renegotiation cannot decrease Bob's equilibrium payoffs.

Third, when bargaining powers of the two players are equal and Bob's beliefs satisfy a natural assumption, we show that the equilibrium outcome is equivalent to Myerson's neutral solution (Myerson (1984)). The neutral solution is defined as an incentive compatible revelation mechanism that satisfies probability invariance, extension, and random dictatorship axioms. Myerson's characterization considers a maximization problem to maximize the weighted welfare of privately known types, then uses it to derive virtual valuations, and finds weights such that the welfare maximizing outcome balances virtual valuations across players. The belief assumption is closely related to increasing virtual utility assumption well-known from the mechanism design literature.

This result contributes to the Nash program (originated in Nash (1953)), which studies strategic foundations of cooperative games. Apart from ours, the only other paper to implement Myerson's solution in a strategic game is Clippel, Fanning, and Rozen (2020). That paper constructs a bargaining game that implements neutral solution in a general class of environments but under the assumption that types are ex post verifiable.

Ours is not the first paper to use sophisticated offers in bargaining. Mechanisms as offers have been considered in axiomatic theories of bargaining in Harsanyi and Selten (1972), Myerson (1979), and Myerson (1984). Certain mechanisms, like menus, also appear in some work on strategic bargaining under one-sided incomplete information. With an exception of Jackson et al. 2020, all related papers that we are aware of work solely with two types. Sen (2000) (see also Inderst (2003)) studies a two-type alternating offer game, where players can offer menus but not general mechanisms and demonstrates the existence of a unique outcome in a refinement of PBE (perfect sequential equilibrium due to Grossman and Perry (1986a)). The equilibrium behavior depends on whether the high type prefers her own complete information Nash payoff, or the Nash allocation of the low type. In a similar bargaining environment, Wang (1998) studies the Coasian bargaining model with Bob making all the offers. He shows that, in the unique equilibrium, Bob separates Alice's two types with an optimal screening contract. In particular, the Coase conjecture fails as Bob retains all power subject to the incentive compatibility constraints. More recently, Strulovici (2017) assumes that, instead of ending the game, any accepted offer becomes the status quo for future bargaining. In that setting, in the unique equilibrium, the uninformed player is unable to offer an inefficient payoff to type u_1' in order to screen out the more extreme type u_1''.

Jackson et al. 2020 considers a general bargaining environment. Although the authors allow for incomplete information on both sides, they make a strong assumption that the total value of bargaining surplus is commonly known. This assumption implies that there are no incentive problems that stop agents from truthfully revealing their information. In the unique equilibrium, the agents use menus to implement information revelation in a single round of bargaining. The result is robust to small perturbations of the common knowledge assumption.

2. Model

2.1. **Environment.** There are two agents, Alice and Bob, who jointly decide on an allocation of a single indivisible good. Once the decision is reached, the good is immediately consumed and cannot be re-traded. An allocation is a pair (q, t), where q is the probability that Alice gets the good, and t is a transfer from Alice to Bob. Let $X = \{(q, t) : q \in [0, 1], t \in \mathbb{R}\}$be the space of allocations. Allowing for allocations where none of the agents gets the good with a positive probability would not change any of the results (such allocations would play no role in the equilibrium).

Alice's payoff from an allocation is equal to $qu - t$, where $u \geq 0$ is her preference parameter. Bob's payoff is equal to $(1 - q) v + t$, where $v > 0$. (All results extend to $v = 0$, but the proofs require minor modification to handle the possibility of 0 payoffs for Bob.) Bob's preference parameter v is commonly known. Alice's parameter u is privately known by her and Bob has beliefs $\mu \in \Delta U$, where $U = [u_{\min}, u_{\max}]$ for some

$\max(v, u_{\min}) < u_{\max}$. Note that this case incorporates both "gap" and "no-gap" cases from the literature on the Coasian bargaining.

2.2. **Bargaining game.** In each period, the proposer is chosen randomly: Alice with probability β and Bob with probability $1-\beta$. As usual, β is interpreted as a measure of Alice's bargaining power. The proposer chooses a mechanism m from the set of mechanisms $\mathcal{M}$ and the other player either accepts or rejects. If the offer is accepted, mechanism m is implemented in the same period, the allocation is determined in a continuation equilibrium of the mechanism, and the game ends with players receiving their respective payoffs from the allocation. The mechanisms and their equilibria are formally defined below. If the offer is rejected, the game moves onto the next period. All actions (mechanism choices and acceptance decisions) are perfectly observed. For the sake of the proof of the equilibrium existence, we also assume that, at each instance, the players observe independent public randomization device. The players discount with a common factor $\delta < 1$.

The solution concept is a *perfect Bayesian equilibrium* (or, simply, equilibrium), in which (a) the players best respond to the opponent's strategy, and, in Bob's case, given his beliefs, and (b) at each decision point, Bob updates his beliefs through Bayes's formula after almost all of Alice's decisions, where almost all is with respect to her strategy in the given period. The requirement that the beliefs are updated only after Alice's moves would be satisfied in a sequential equilibrium.[5] The precise definition of an equilibrium is postponed till the Technical Appendix B.6.

An equilibrium *payoff outcome* (y, y_B) is a (measurable) function $y : U \to \mathbb{R}$ and a payoff $y_B \in \mathbb{R}$, with the interpretation that $y(u)$ is the expected payoff of Alice's type u, and y_B is the expected payoff of Bob. Let $E(\delta, \mu)$ be the set of expected equilibrium payoff outcomes in a game where the discount factor is equal to δ, and Bob's beliefs are equal to μ.

[5]Because the space of actions is large (in fact, infinitely dimensional), it is not clear what the right notion of completely mixed strategies, hence sequential equilibrium is.

2.3. **Mechanisms.** A *mechanism* is any normal-form or extensive-form game such that the action choices determine the final allocation in X. Formally, a mechanism is a tuple $m = \left(\left(S_i^k\right)_{i=A,B}^{k\leq K}, \chi\right)$, where $K \leq \infty$ is the number of rounds in the mechanism, S_i^k is a of actions for player i in period t, and $\chi : \prod_i^K S_i^k \to X$ is an allocation function. Examples include:

- *simple offers:* players do not make any choices and receive a predetermined allocation;
- *menus for Alice*: Alice chooses an allocation $x \in Y$ from a compact set of allocations $Y \subseteq X$. Let $\mathcal{Y}$ be the space of all menus;
- *menus for Bob of menus for Alice*: Bob chooses one of menu for Alice $Y \in W$ from a (Hausdorff topology) compact set of menus $W \subseteq \mathcal{Y}$, followed by Alice who chooses an allocation from the menu;
- original bargaining game, or any alteration of the bargaining protocol of the original game.

The details of a mechanism are less important than equilibrium payoffs outcomes which can be attained in the mechanism. For instance, if $Y \subseteq X$ is a menu, then Alice type u's equilibrium payoffs are uniquely equal to

$$y\left(u;Y\right) = \max_{(q,t)\in Y} qu - t.$$

Bob's expected payoff is derived by integrating over Alice's optimal choices, with the caveat that a non-generic (but possibly, positive measure) type might be indifferent between two choices with different payoff consequences for Bob. More generally, the next result provides a partial characterization of equilibrium payoffs.

Lemma 1. *Fix an equilibrium payoff outcome (y, y_B) of a mechanism m with beliefs μ. Then, y is increasing and convex, with bounded subdifferentials $\partial y(u) \subseteq [0,1]$. Let*

$$\pi(u; y) = v - y(u) + (u - v) \cdot \begin{cases} \max \partial y(u) & u \geq v \\ \min \partial y(u) & u < v \end{cases} \quad \textit{and}$$

$$\Pi(\mu; y) = \int \pi(u; y)\, d\mu(y).$$

Then, $y_B \leq \Pi(\mu; y)$. Moreover, there is a menu $Y \subseteq X$ such that $y = y(.; Y)$.

The Lemma is standard. Its first part is essentially the envelope theorem. The second part shows that all equilibrium payoffs can be attained in some menu. It is a version of the revelation principle for our environment.[6] The proof constructs menu Y as the set of expected discounted allocations that each type receives in equilibrium.

It must be emphasized that, as in informed principal models, the revelation principle does not capture the full role of a mechanism. The reason is that Bob's beliefs may change whenever mechanism is offered or accepted by Alice. This is especially important for off-path choices, where, in equilibrium, a "belief-punishment" threat might stop Alice from proposing a mechanism. A lower bound on Alice's payoffs will depend on her ability to design mechanisms that forestall such threats.

In order to fully describe the relevant aspects of a mechanism, its equilibrium correspondence is needed. For each mechanism m, let $E(\mu; m)$ be the set of perfect Bayesian equilibrium payoff outcomes (y, y_B) in mechanism m given Bob's beliefs μ.

Assume that the space of beliefs is equipped with weak* topology and the space of payoff outcomes $\mathbb{R}^U \times \mathbb{R}$ has a topology of uniform convergence. A mechanism is *Kakutani* if the correspondence $E(.; m) : \Delta U \rightrightarrows \mathbb{R}^U \times \mathbb{R}$ is u.h.c., compact-, convex- and non-empty valued. The Technical Appendix B shows that each simple offer, menu, and menu of menus is Kakutani (Corollary 1).

[6]The representation of bargaining outcomes as an incentive-compatible mechanism goes back to Myerson (1979) and Ausubel and Deneckere (1989).

Let $\mathcal{M}$ be the space of mechanisms available to players. We assume that $\mathcal{M}$ contains all menus and menus of menus and it only consists of Kakutani mechanisms. The statement of the main result refers only to menus. The proof relies heavily on the availability of menus of menus of a particular kind. Whether $\mathcal{M}$ contains any other mechanisms is irrelevant for the results and proofs. The restriction to Kakutani mechanisms plays a role in the proof of the existence of equilibrium in Section 4.

2.4. **Complete information.** Under complete information, Alice's parameter is known to be u. In this case, the ability to offer general mechanisms plays no role and the model becomes analogous to classic models of bargaining (Okada (1996), Rubinstein (1982)) with surplus equal to $\max(u, v)$. It is well-known that the equilibrium is unique and the payoffs for Alice and Bob are $(\beta \max(u, v), (1-\beta)\max(u, v))$. When $\beta = 1/2$, complete information payoffs are equal to the Nash solution (Nash Jr (1950)).

3. Main result

3.1. **Optimal screening menus.** For each Bob's belief μ, define the optimal screening price

$$P(\mu) = \arg\max_p \mu\{u : u \geq p\}(p - v).$$

The optimal screening price is generically unique, and if it is not, let $p^*(\mu) = \max P(\mu)$ be the largest solution to the maximization problem. If $u_{\min} \geq v$, selling at price(s) $P(\mu)$ would maximize Bob's payoffs if Bob was allowed to unilaterally choose mechanism (Bulow and Roberts (1989)).

For each $\alpha \in [0, 1]$, define a three-allocation menu for Alice:

$$Y_{\alpha,p} = \{(0, -\alpha v), (\alpha, 0), (1, (1-\alpha)p)\}.$$

Under this menu, Alice gets the good with probability α, and she may either sell her probability share at (per-unit) price v to Bob, do nothing or purchase the remaining

$1-\alpha$ probability from Bob at price p. This mechanism has a flavor of Texas shootout clause described in Footnote 4.

Th e outcome of mechanism $Y_{\alpha,p}$ is not efficient for types u such that $v<u<p$. In such a case, the mechanism allocates the good to Alice with probability α, whereas it is efficient to give it to Alice with probability 1.

Note for future reference that, although menus $Y_{\alpha,p}$ for $p\in P^*(\mu)$ lead to the same expected payoff for Bob, Alice (weakly) prefers the lowest price. All the types above the lowest price have the strict preference for such an outcome.

A straightforward extension of Bulow and Roberts (1989) shows that menu $Y_{\alpha,p}$ for each $p\in P^*(\mu)$ is a solution to the Bob's optimal mechanism problem under the constraint that Alice's utility is at least equal to $\alpha\max(u,v)$. To simplify the notation, write $y_{\alpha,p}$ instead of $y(.,Y_{\alpha,p})$.

Lemma 2. *For each α, μ, and $p\in P^*(\mu)$,*

$$\Pi(\mu;y_{\alpha,p})=(1-\alpha)\Pi(\mu;y_{0,p})=\max_{Y:\forall_u y(u;Y)\geq\alpha\max(u,v)}\Pi(\mu;y(u;Y))=:\Pi^*_\alpha(\mu). \tag{1}$$

The last equality defines the value of the optimization problem subject to α-constraint. If $\alpha=\beta$, then the constraint in the optimization problem (1) ensures that each type of Alice receives her complete information payoff (see Section 2.4). The assumption that $v>0$ ensures that $\Pi^*_0(\mu)=\Pi(\mu;y_{0,\mu})>0$ for any belief μ.

3.2. **Main result.** We are ready to state the main result of this paper:

Theorem 1. *Bob's perfect Bayesian equilibrium payoffs in the bargaining game with beliefs μ are unique and equal to $\Pi^*_\beta(\mu)$. For each $p\in P(\mu)$, there is an equilibrium with Alice's payoffs $y_{\beta,p}$.*

Bob's equilibrium payoff is unique and it is equal to the expected payoff from his optimal screening menu among all menus that ensure each of Alice's types receive her complete information payoff. The same payoff would be obtained if Bob were able to commit to an optimal mechanism subject to the complete information constraint. If the optimal screening menu is unique, the payoff of each of Alice's types is also unique.

The optimal payoff is convex in μ. An implication is that it has a natural comparative statics with respect to information: Bob is better off if his information improves in the sense of Blackwell's ordering. In particular, Bob is worse off due to his incomplete information about Alice's preferences. Each of Alice's types is either the same or better off under incomplete information.

Bob's optimal payoff is continuous in his beliefs. In particular, when $\mu \to \delta_u$ for some Alice type u, Bob's payoff converges to $(1-\beta)\,u$, i.e., his complete information payoff against type u. This stands in contrast to the Coase conjecture literature, where the durable monopolist payoff in the limit $\delta \to 1$ typically depends on the support of its beliefs, and may change discontinuously with beliefs (see also Section 6.2 for further discussion o the relation to the Coasian conjecture.)

3.3. **Proof intuition.** The proof of the Theorem is divided into two parts. Section 4 presents a partial construction of the equilibrium. In the equilibrium, if player i is chosen to be the proposer, the player offers menu $Y_{\alpha^i,p}$, where $\alpha^A = 1-\delta+\delta\beta$, $\alpha^B = \delta\beta$ and p is one of the optimal prices given Bob's beliefs. Such offer is immediately accepted. The incentives for Bob come from the fact that, because Alice expects to receive at least $\delta\beta \max(u,v) = \alpha^B \max(u,v)$ in the continuation bargaining game, Bob's payoff given this constraint cannot be improved relative to menu $Y_{\alpha^B,\mu}$ due to Lemma 2. The incentives for Alice are provided by a mixture of a similar argument as well as a belief threat that ensures that she gets no more than $\alpha^A \max(u,v)$ in the continuation game after off-path offer (see Lemma 3 below). The equilibrium construction is only partial because in two subgames, the beliefs and behavior are obtained as a solution to some fixed-point problem.

The only mechanisms used in the equilibrium (other than off-path deviations) are menus $Y_{\alpha^i,p'}$ for $i = A, B$ and any price $p' \in U$. In particular, the equilibrium remains an equilibrium if no other mechanisms are available. The menus play two roles. First, they address the screening problem described in the introduction: players are able to attain equilibrium payoffs for all types of Alice without a costly delay. They also help with the signaling problem, as they make it possible for Alice to reveal her private information only when it is too late for Bob to benefit from it.

Section 5 shows that there cannot be any other equilibrium payoff outcome. The basic idea is to modify arguments from the complete information bargaining literature: if other payoffs could have been attained, one of the players would have a profitable deviation.

More specifically, if Bob's payoffs are lower than $\Pi^*_\beta(\mu)$ for some belief, we first identify his "worst possible payoff" by finding the largest $\alpha_{\max}$ such that his payoffs are equal to $\Pi^*_{\alpha_{\max}}(\mu)$ for some belief μ. If $\alpha_{\max} > \beta$, we consider an equilibrium that implements such payoffs. There is a possible deviation for Bob, where, whenever he is chosen as the proposer, he offers menu $Y_{\delta\alpha_{\max},p}$, where $p \in P^*(\mu)$. Due to $\alpha_{\max} > \beta$ (and the complete information logic of the game), if accepted with probability 1, such an offer would increase Bob's expected payoffs strictly above the purported equilibrium payoffs, which would lead to a contradiction with the original payoff being derived in equilibrium.

To show that such a menu will indeed be accepted, notice first there must be Alice's types in the support of Bob's beliefs who receive at most $\alpha_{\max}(u,v)$ in the continuation game (this is a consequence of the choice of $\alpha_{\max}$ and Lemma 2). Due to the discounting, such types should accept offer of $y_{\delta\alpha_{\max},p}(u) \geq \delta y_{\alpha_{\max},p}(u)$ today. The other types will accept as well due to an unraveling logic - if Bob's continuation beliefs assign probability 1 only to the types that rejected his offer, the same argument applies and some of those types will receive $\alpha_{\max}\max(u,v)$ in the continuation game. But then, due to the discounting, they should have accepted the offer in the first place. The details can be found in Section 5.1.

Similarly, we show that each Alice's type must receive a payoff at least $\beta \max(u, v)$. If not, we first identify the lowest possible $\alpha_{\min}$ such that there is an equilibrium and Alice's type u who receive payoff at least $\alpha_{\min} \max(u, v)$. If μ are Bob's beliefs in the equilibrium that implements such payoffs, Bob's payoffs cannot be higher than $\Pi^*_{\alpha_{\min}}(\mu)$ due to Lemma 2. Consider Alice's deviation, where, whenever she is a proposer, she offers menu $Y_{1-\delta(1-\alpha_{\min}),p}$. If Bob does not change his beliefs upon seeing such an offer, the expected payoff from such a menu is equal to $\delta\Pi^*_{\alpha_{\min}}(\mu)$, which is the same as Bob can expect from rejecting Alice's offer. Hence Bob should accept it, and the complete information logic of $\alpha_{\min} < \beta$ implies that such a deviation would increase Alice's payoffs.

Of course, Bob can update his beliefs following Alice's offer to $\psi \in \Delta U$ for which $p \notin P^*(\mu)$. Then, menu $Y_{1-\delta(1-\alpha_{\min}),p}$ does not guarantee possible continuation payoff $\delta\Pi^*_{\alpha_{\min}}(\psi)$. If so, Bob would reject Alice's offer, which would stop Alice from making it. To deal with such belief threats, Alice can instead offer a menu $\left\{Y_{1-\delta(1-\alpha_{\min}),p} : p \in U\right\}$ of menus. If the menu of menu is accepted, Bob, with beliefs ψ, can choose a menu from the menu of menus to maximize his expected payoff. Alice's offer is constructed in such a way that Bob can choose menu $Y_{1-\delta(1-\alpha_{\min}),p^*(\psi)}$ to guarantee himself payoff $\delta\Pi^*_{\alpha_{\min}}(\psi)$. The menu of menus protects Alice from "belief punishment" threat, as whatever are Bob's post-offer beliefs, Bob is able to choose a menu that is both satisfactory for him and for Alice.

The proof requires that all menus and menus of menus are available. The results of (Ausubel and Deneckere 1989) for alternating-offer bargaining imply that, when players are only able to propose single offers, the bargaining game has multiple equilibria. We do not know if the theorem also holds if $\mathcal{M}$ contains only menus but no menus of menus. However, in such a case, our proof shows that $\Pi^*_\beta(\mu)$ is a lower bound on Bob's equilibrium payoffs.

When $\beta = 0$, Bob makes all the offers, and the outcome of the bargaining game is equivalent to Bob's optimal screening menu (without any constraints). This result extends Wang (1998) from two types to a continuum. Wang (1998) assumes that Bob can offer menus, but not menus of menus nor any other mechanism. As we remark above, the availability of menus is sufficient to show that $\Pi^*_\beta(\mu)$ is the lower bound on Bob's payoffs.

4. Equilibrium

We are going to show that for each beliefs μ, and each price $p \in P^*(\mu)$, there is an equilibrium with payoffs $\left(y_{\beta,p}, \Pi^*_\beta(\mu)\right)$. Further equilibrium payoffs (for Alice) can be obtained by using a public randomization device.

This equilibrium beliefs, actions, and continuation payoffs are illustrated in Figure 1. The thick solid lines describe the equilibrium behavior. The thick dashed lines mean that the equilibrium actions depend on the state of the game.

Let $\alpha^A = 1 - \delta + \delta\beta$ and $\alpha^B = \delta\beta$. The constants are chosen so that (a) β is the expected value of α^i, where i is the random proposer, (b) Alice prefers to receive $\alpha^B = \delta\beta$ instead of waiting one period for β, and (c) Bob prefers to receive $1 - \alpha^A = \delta(1-\beta)$ instead of waiting one period for $1-\beta$. In the equilibrium, the proposing player j offers menu $Y_{\alpha^j,p}$ for some (any) $p \in P^*(\mu)$. The offer is accepted. Because of (a), the expected equilibrium payoffs before the proposer is chosen are equal to $\Pi^*_\beta(\mu)$ for Bob and $y_{\beta,p}$ for Alice. If the equilibrium offer is rejected, the game moves to the next period. Additionally, if Alice is the one rejecting the offer (i.e., $j = B$), Bob updates his beliefs so that they are concentrated on type $u_{\max}$. Under such beliefs, the optimal price is $u_{\max}$ and Alice's expected continuation payoffs are equal to $\alpha^B \max(u,v) = \delta\beta \max(u,v)$. Because of (b) and (c), if $j = A$, then Bob is indifferent between accepting or rejecting the offer (note that $1 - \alpha^A = \delta(1-\beta)$), and, if $j = B$, then Alice is either indifferent (types $u \leq p$) or prefers to accept the offer (types $u > p$).

If Alice offers mechanism $m \neq Y_{\alpha^j,p}$, Bob's beliefs, his best response decision, and the continuation payoffs depends on mechanism m.

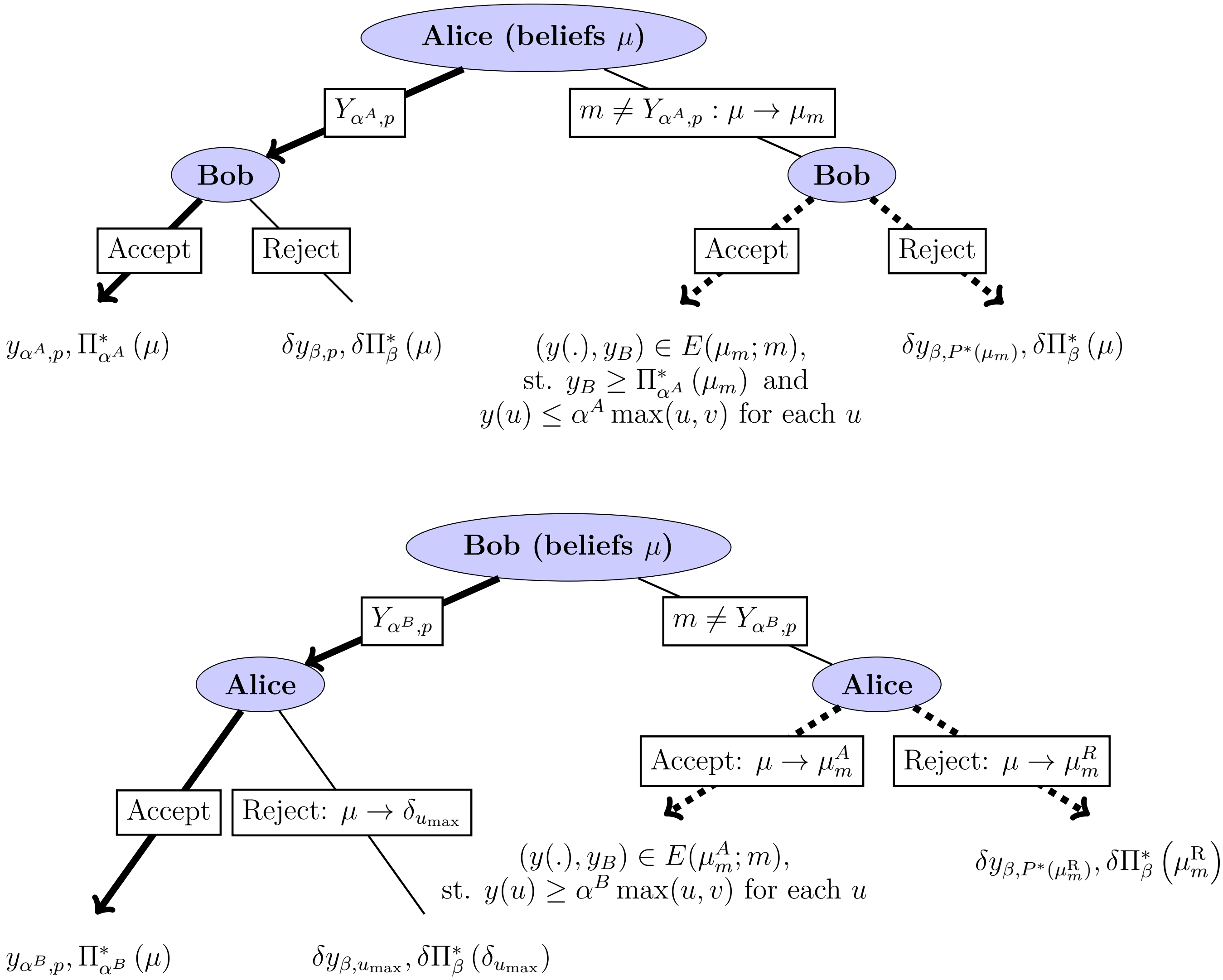

FIGURE 1. Equilibrium.

Lemma 3. *For each α and each m, if m is a Kakutani mechanism, then there exists $\mu_m \in \Delta U$ and $(y, y_B) \in E(\mu_m; m)$ st. either $y(u) \leq \alpha \max(u,v)$ for each u, or $y_B \leq \Pi^*_\alpha(\mu_m)$.*

Lemma 3 is a counterpart to Lemma 2, in the sense that it shows that either there are a belief and continuation payoffs that are worse for Bob than the optimal screening menu subject to α-constraints, or there are a belief and continuation payoffs such that *each* Alice's type fails the α-constraints. The identified belief has a two-element support. The proof of the Lemma is essentially a fixed point result. Because the relevant space of beliefs is one-dimensional, the proof is elementary.

After Alice offer $m \neq Y_{\alpha^A,p}$, Bob updates his beliefs to μ_m, where μ_m, y, and y_B are as in the Lemma for $\alpha = \alpha^A$. If $y_B > \Pi^*_{\alpha^A}(\mu_m) = \delta\Pi^*_\beta(\mu_m)$ (hence $y(u) \leq \alpha \max(u,v)$ for each u), the offer is accepted. Otherwise, the offer is rejected. Because Bob accepts the offer only if $y(u) \leq \alpha^A \max(u,v)$ for each u, none of Alice's types strictly prefers to offer a mechanism that is going to be accepted. Because the rejection leads to strictly lower payoffs than the equilibrium payoffs, Alice does not want to offer a mechanism that is rejected. Hence each type of Alice (weakly) prefers to offer $Y_{\alpha^A,p}$.

If Bob offers a mechanism $m \neq Y_{\alpha^B,p}$, Alice chooses her behavior optimally by comparing the payoffs y in a continuation equilibrium of the accepted mechanism $\left(y^{\mathrm{A}}, y^{\mathrm{A}}_B\right) \in E\left(\mu^{\mathrm{A}}_m; m\right)$ with continuation payoffs from the bargaining game after rejection $\delta y_{\mu^{\mathrm{R}}_m, P^*(\mu^{\mathrm{R}}_m)}$. The probability of acceptance, the updated beliefs after acceptance, μ^{A}_m, and after rejection, μ^{R}_m, as well as the continuation payoffs in the accepted mechanism or, if rejected, the bargaining game are determined in an equilibrium. The existence of these objects is proven in the Technical Appendix (section C.4). Importantly, their exact values do not matter from the point of view of providing Bob with incentives not to deviate to m. Notice that Alice's continuation payoffs after rejection are not smaller than $\left(\delta\beta\alpha^A + \delta(1-\beta)\alpha^B\right)\max(u,v) = \delta\beta\max(u,v) = \alpha^B \max(u,v)$ for each type u. In equilibrium, this must be also a lower bound on her payoff after accepting the mechanism, as well as her expected payoffs after Bob proposes m. Treating the subgame following Bob's proposal as a mechanism, Lemma 2 together with the second half of Lemma 1 shows that Bob's expected payoffs from m cannot be larger than $\Pi^*_{\alpha^B}(\mu)$. Hence Bob (weakly) prefers to offer menu $Y_{\alpha^B,p}$.

5. Payoff bounds

This sections shows that no other equilibrium has different payoffs for Bob. The first part shows that Bob's payoffs cannot be lower than $\Pi^*_\beta(\mu)$. The second part shows that for each Alice's type, her expected payoffs cannot be lower than $\beta \max(u, v)$. Lemma 2 implies that Bob's payoff cannot be higher than $\Pi^*_\beta(\mu)$. This shall conclude the proof of Theorem 1.

In both cases, we show that if the respective player's lowest equilibrium payoffs are lower than they should be, the player has a profitable deviation.

5.1. **Lower bound on Bob's equilibrium payoffs.** This subsection shows that for all beliefs μ and all payoff outcomes $(y, y_B) \in E(\delta, \mu)$, Bob's payoffs are not smaller than $y_B \leq \Pi^*_\beta(\mu)$. To restate the claim in a more convenient way, define

$$\alpha_{\max} = \sup\left\{\alpha : \exists (\mu, y, y_B) \in E(\delta) \text{ st. } y_B \leq \Pi^*_\alpha(\mu)\right\}.$$

The goal is to show that $\alpha_{\max} \leq \beta$.

Two simple preliminary results are needed. The first one shows that for each equilibrium payoff outcome, there must be some types with payoffs close to $\alpha_{\max} \max(u, v)$.

Lemma 4. *For each belief μ and payoff outcome $(y, y_B) \in E(\delta, \mu)$, and for each $\eta > 0$, there exists type $u \in \operatorname{supp}\mu$ such that $y(u) < (\alpha_{\max} + \eta)\max(u, v)$.*

Proof. On the contrary, suppose that there exists $\eta > 0$, belief μ, and payoff outcome $(y, y_B) \in E(\delta, \mu)$ such that $y(u) \geq (\alpha_{\max} + \eta)\max(u, v)$ for each $u \in \operatorname{supp}\mu$.

Consider payoff function $y'(u) = \max(y(u), (\alpha_{\max} + \eta)\max(u, v))$. We are going to show that $\pi(u; y') \geq \pi(u; y)$ for each $u \in \operatorname{supp}\mu$ (see the definition of π in Lemma 1). Note that for any such u, either $y(u) > (\alpha_{\max} + \eta)\max(u, v)$ or $y(u) = (\alpha_{\max} + \eta)\max(u, v)$. In the former case, $\partial y(u) = \partial y'(u)$ and $\pi(u; y) = \pi(u; y')$. In the latter case, if $u \leq v$, then $\min \partial y'(u) = 0 \leq \min \partial y(u)$, and if $u \geq v$, $\max \partial y'(u) = \max(\partial y(u) \cup \{\alpha_{\max}\}) \geq \max(\partial y(u))$, which implies that $\max \partial y(u) \leq \max \partial y'(u)$. In both cases, $\pi(u; y) \leq \pi(u; y')$.

Hence $\Pi(\mu; y) \leq \Pi(\mu; y')$. Lemma 2 implies that $\Pi(\mu; y') \leq \Pi^*_{\alpha_{\max}+\eta}(\mu)$. But this contradicts the definition of $\alpha_{\max}$. □

The second result shows that, if $\alpha > \beta$, then Bob has a proposal that is better for him than $\Pi^*_\alpha(\mu)$ and that Alice would rather choose prefer to wait for $\alpha \max(u, v)$.

Lemma 5. *If $\alpha_0 > \beta$, then there is $\varepsilon > 0$ such that for each $\alpha \geq \alpha_0$, there is menu Y' such that*

$$y(u, Y') \geq \delta\alpha \max(u, v) + \varepsilon \max(u, v) \text{ and} \tag{2}$$
$$\beta\delta\Pi^*_\alpha(\mu) + (1-\beta)\Pi(\mu; y(u, Y')) \geq (1+\varepsilon)\Pi^*_\alpha(\mu).$$

Proof. Let $\varepsilon \leq \frac{(\alpha_0-\beta)(1-\delta)}{2-\beta-\alpha_0}$. Let $Y' = Y_{\delta\alpha+\varepsilon,p}$ for any $p \in P^*(\mu)$ and notice that

$$y(u, Y_{\delta\alpha+\varepsilon,p}) = y_{\delta\alpha+\varepsilon,p} \geq \delta\alpha \max(u, v) + \varepsilon \max(u, v).$$

Take $\alpha \geq \alpha_0 > \beta$. Due to Lemma 2,

$$\begin{aligned}
&\beta\delta\Pi^*_\alpha(\mu) + (1-\beta)\Pi(\mu; y_{\delta\alpha+\varepsilon,p}) - (1+\varepsilon)\Pi^*_\alpha(\mu) \\
=&\Pi^*_0(\mu)\left[\beta\delta(1-\alpha) + (1-\beta)(1-\delta\alpha-\varepsilon) - (1+\varepsilon)(1-\alpha)\right] \\
=&\Pi^*_0(\mu)\left[(\alpha-\beta)(1-\delta) - \varepsilon(2-\beta-\alpha)\right] \geq 0,
\end{aligned}$$

where the last inequality follows from the choice of ε. □

Suppose, by contradiction, that $\alpha_{\max} > \beta$. Fix α_0 so that $\beta < \alpha_0 < \alpha_{\max}$ and choose ε as in Lemma 5. By the definition of $\alpha_{\max}$, for each $\eta > 0$, one can find equilibrium beliefs μ and payoff outcome $(y, y_B) \in E(\delta, \mu)$ such that $\Pi(\mu; y) < \Pi_{\alpha_{\max}-\eta}(\mu)$. From now on, assume that η is small enough so that $\eta < \alpha_{\max} - \alpha_0$, $2\eta \leq \varepsilon$ and $\varepsilon(1-(\alpha_{\max}-\eta)) > \beta\delta\eta$. By the choice of ε, there exists menu Y' such that (2) holds for $\alpha = \alpha_{\max} - \eta$.

Consider an equilibrium that implements (μ, y, y_B). If Bob is chosen as the proposer, consider Bob's strategy (possibly, a deviation), in which he offers Y'. We claim that almost all Alice's types in the support of Bob's beliefs must accept such an offer.

Indeed, if not, let ψ be the beliefs and (z, z_B) be the payoff outcome associated with the continuation equilibrium after Alice rejects Y'. It must be that ψ is absolutely continuous wrt μ, hence supp$\psi \subseteq$ suppμ. By Lemma 4, there are types $u \in \operatorname{supp}\psi$, for whom the expected payoff from rejection, $\delta z(u)$, is strictly smaller than

$$\delta\left(\alpha_{\max}+\eta\right)\max\left(u,v\right) \leq \delta\left(\alpha_{\max}-\eta\right)\max\left(u,v\right)+2\eta\max\left(u,v\right) \leq y\left(u,Y'\right),$$

or payoffs from accepting Y' (the last inequality comes from the choice of η and menu Y'). Because the payoffs are continuous in types, there must be a strictly positive ψ-mass of types who have strictly higher payoffs Y', which leads to a contradiction with rejection of Y' being a best response for almost all rejecting types.

Compute the expected payoff from such a strategy before the proposer is chosen. If Alice is chosen to be the proposer, Bob's expected payoff is not lower than

$$\begin{aligned}\delta\Pi^*_{\alpha_{\max}}\left(\mu\right) &= \delta\left(1-\alpha_{\max}\right)\Pi^*_0\left(\mu\right) = \delta\left(1-\left(\alpha_{\max}-\eta\right)\right)\Pi^*_0\left(\mu\right) - \delta\eta\Pi^*_0\left(\mu\right)\\ &= \delta\Pi^*_{\alpha_{\max}-\eta}\left(\mu\right) - \delta\eta\Pi^*_0\left(\mu\right)\end{aligned}$$

due to the choice of $\alpha_{\max}$. If Bob is chosen, his expected payoff from (accepted with probability 1) offer Y' is $\Pi\left(\mu; y\left(.;Y'\right)\right)$. Hence the expected payoff is at least

$$\begin{aligned}&\beta\delta\Pi^*_{\alpha_{\max}-\eta}\left(\mu\right) - \beta\delta\eta\Pi^*_0\left(\mu\right) + \left(1-\beta\right)\Pi\left(\mu; Y'\right)\\ \geq&\left(1+\varepsilon\right)\Pi^*_{\alpha_{\max}-\eta}\left(\mu\right) - \beta\delta\eta\Pi^*_0\left(\mu\right) = \left(\left(1+\varepsilon\right)\left(1-\left(\alpha_{\max}-\eta\right)\right) - \beta\delta\eta\right)\Pi^*_0\left(\mu\right)\\ \geq&\left(1-\left(\alpha_{\max}-\eta\right)\right)\Pi^*_0\left(\mu\right) + \left(\varepsilon\left(1-\left(\alpha_{\max}-\eta\right)\right) - \eta\beta\delta\right)\Pi^*_0\left(\mu\right)\\ >&\Pi^*_{\alpha_{\max}-\eta}\left(\mu\right) > \Pi\left(\mu; y\right),\end{aligned}$$

where the last inequality follows from the choice of equilibrium beliefs μ and payoff outcome (y, y_B). The inequality contradicts (y, y_B) being an equilibrium payoff outcome for the game with beliefs μ. The contradiction shows that $\alpha_{\max} = \beta$.

5.2. **Lower bound on Alice's payoffs.** This subsection shows that Alice type u's equilibrium payoffs are not smaller than $\beta \max(u, v)$. To restate the claim in a more convenient way, define

$$\alpha_{\min} = \inf \{\alpha : \exists \mu, (y, y_B) \in E(\delta, \mu), \text{ and } u \text{ st. } y(u) \leq \alpha \max(u, v)\}.$$

The goal is to show that $\alpha_{\min} \geq \beta$.

Let

$$W_\alpha = \{Y_{\alpha,p} : p \in U\}$$

be a menu of menus $Y_{\alpha,p}$ across all possible prices p.

If W_α is offered by Alice and accepted by Bob, Bob's expected continuation equilibrium payoff from such a menu is unique and equal to $\Pi^*_\alpha\left(\mu^W\right)$, where μ^W is Bob's belief after being offered menu W_α. To see that, notice first that Bob cannot receive a higher payoff due to Lemma 2. The payoff is attained in the equilibrium, in which Bob chooses menu $Y_{\alpha,p}$ for some $p \in P^*\left(\mu^W\right)$ and whatever is Bob's choice, whenever indifferent, Alice always picks the allocation that is more favorable to Bob. Finally, there are no other equilibria. To see why, notice that the potential multiplicity is due to atomic Alice's types $u = p$ who are indifferent between the two allocations in menu $Y_{\alpha,p}$. If such a type plans to make a choice that is not favorable to Bob, Bob can always offer a menu $Y_{\alpha,p-d}$ for some small $d > 0$. Then, the atom $u = p$ has a strict preference to choose the better allocation, and the lowest possible expected payoff in such a menu converges to the highest payoff in menu $Y_{\alpha,p}$ as $d \to 0$.

On the contrary to our claim, suppose that $\alpha_{\min} < \beta$. Let η be such that $0 < \eta \leq \frac{1}{1+\beta}(\beta - \alpha_{\min})(1 - \delta)$. By the definition of $\alpha_{\min}$, for each $\eta > 0$, one can find a belief μ, an equilibrium payoff outcome $(y, y_B) \in E(\delta, \mu)$, and a type and u such that $y(u) < (\alpha_{\min} + \eta)\max(u,v)$. Consider an equilibrium that implements (y, y_B). If Alice is chosen as the proposer, consider Alice's strategy (possibly, a deviation), in which she offers menu of menus $W_{1-\delta(1-\alpha_{\min})-\eta}$. Then, Bob will accept such a menu of menus with probability 1. Indeed, let μ^W be Bob's belief after Alice's proposal and let y^W be her continuation payoff. As we check above, Bob's payoff from accepting is equal to

$$\Pi^*_{1-\delta(1-\alpha_{\min})-\eta}\left(\mu^W\right) = \delta\left(1-\alpha_{\min}\right)\Pi^*_0\left(\mu^W\right) + \eta\Pi^*_0\left(\mu^W\right) > \delta\Pi_{\alpha_{\min}}\left(\mu^W\right),$$

which, due to the definition of $\alpha_{\min}$ and Lemma 2, is the lower bound on Bob's equilibrium continuation payoffs after rejecting Alice's offer.

Anticipating Bob's acceptance, the payoff from such a strategy to Alice's type $u > v$, if she is the proposer, is equal to $(1-\delta(1-\alpha_{\min})-\eta)\max(u,v)$. If Bob is the proposer, her payoff cannot be smaller than $\delta\alpha_{\min}\max(u,v)$. Hence her expected payoff at the beginning of the period is not smaller than

$$\begin{aligned}
&\beta\left(\left(1-\delta\left(1-\alpha_{\min}\right)-\eta\right)\max(u,v)\right) + (1-\beta)\,\delta\alpha_{\min}\max(u,v)\\
&=\left(\beta-\beta\delta+\delta\alpha_{\min}-\beta\eta\right)\max(u,v)\\
&=\left(\alpha_{\min}+\eta\right)\max(u,v) + \left(\left(\beta-\alpha_{\min}\right)(1-\delta)-(1+\beta)\,\eta\right)\max(u,v)\\
&>\left(\alpha_{\min}+\eta\right)\max(u,v) > y(u),
\end{aligned}$$

where the inequalities are due to the choice of η. But leads to the contradiction with y being equilibrium payoff.

6. Comments

6.1. **Neutral solution.** The neutral solution is an axiomatic solution concept for bargaining problems with incomplete information proposed in Myerson (1984). It is defined as the minimal set of incentive compatible outcomes that satisfies three axioms: (a) probability invariance axiom that ensures that solution is robust to a change in the problem parameters that does not affect its decision-theoretic structure, (b) extension axiom than connects solutions to related bargaining problems, and (c) random-dictatorship axiom that defines a fair and natural mechanism in simple division games. Myerson's characterization shows that that the neutral solution is an allocation that equalizes virtual valuations of the two players, where the valuations are derived from some welfare optimization problem.

We are going to show that, under a natural assumption on Bob's beliefs, the optimal screening menu $Y_{1/2,p^*(\mu)}$, belongs to the neutral solution.[7] Hence, Theorem 1 applied to the case of equal bargaining powers $\beta = \frac{1}{2}$ is a contribution to the Nash program. In order to explain the result without unnecessary technicalities, assume that Bob's beliefs have finite support and that $u_{\min} \geq v$. Let $f_u = \mu(u)$ be the probability of type u. Let $q^*_u = \begin{cases} 1 & u \geq p^*(\mu) \\ 1/2 & u < p^*(\mu) \end{cases}$ and $t^*_u = \begin{cases} p^*(^m) & u \geq p^*(\mu) \\ 0 & u < p^*(\mu) \end{cases}$ be the allocation mappings from mechanism $Y_{1/2,p^*(\mu)}$. We are going to show that $(q^*_\cdot, t^*_\cdot)$ satisfies the sufficient and necessary conditions for the neutral solution from Myerson (1984). Consider an optimization problem, which maximizes the sum of Bob's utility and weighted utilities of each of Alice's types:

$$\max_{q_\cdot, t_\cdot} \sum_{u \in \operatorname{supp}\mu} \lambda_u (q_u u - t)\, du + \sum_{u \in \operatorname{supp}\mu} ((1 - q_u) v + t_u) f_u$$
$$\text{s.t.} q_u u - t_u \geq q_{u'} u - t_{u'} \text{ for each } u, u'.$$

Coefficients $\lambda_u \geq 0$ are weights assigned to type u. The constraints ensure that the allocation is incentive compatible.

Standard arguments show that it is w.l.o.g. to consider only immediate downward incentive constraints, i.e., constraints for u and $u' = u_-$, where $u_- = \max\{u' \in \operatorname{supp}\mu : u' < u\}$.

[7]This result has been known to Roger Myerson since 80ies (private communication), but, as far as I know, it has not been published anywhere.

(Similarly, define $u_{+} = \min\{u' \in \text{supp}\mu : u' > u\}$.) To obtain the Lagrangian, one multiplies the incentive constraints for u and u_{-} by factor $\alpha_u \geq 0$ and adds them up to the objective function:

$$\sum_{u \in \text{supp}\mu} \sum_{i=A,B} V_i \left(q_u, t_u, u, \lambda, \alpha\right), \tag{3}$$

where α_u is the Lagrange multiplier associated with constraint and V_i are *virtual evaluations*:

$$V_A \left(q, t, u, \lambda, \alpha\right) = \left(\lambda_u + \alpha_u\right)\left(qu - t\right) - \alpha_{u+}\left(qu_{+} - t\right),$$
$$V_B \left(q, t, u, \lambda, \alpha\right) = \left(\left(1 - q\right) v + t\right) f_u.$$

If $(q_{\cdot}, t_{\cdot})$ is a neutral solution, then, informally, it is a (a) solution to the optimization problem (3) for some weights λ, where (b) the weights are chosen so that the virtual utilities of the two players are equal (at least for the types with a strictly positive weight λ_u).

To see why it is the case for $(q^*_{\cdot}, t^*_{\cdot})$, notice first that the first-order conditions for transfers t require that for each u,

$$-\lambda_u - \alpha_u + \alpha_{u+} + f_u = 0.$$

Because $q^*_u = \frac{1}{2}$ for $u < p^*(\mu)$, the first-order conditions for allocation probabilities for such types hold with equality and imply that

$$\left(\lambda_u + \alpha_u\right) u - \alpha_{u+} u_{+} - v f_u = 0.$$

The two equations imply that $\lambda_u + \alpha_u = f_u + \alpha_{u+}$, and $\alpha_{u+}(u_+ - u) = f_u(u - v)$. Hence

$$\begin{aligned} V_A(q^*, t^*, u, \lambda, \alpha) &= (f_u + \alpha_{u+})\frac{1}{2}u - \alpha_{u+}\frac{1}{2}u_+ \\ &= \frac{1}{2}f_u u - \frac{1}{2}\alpha_{u+}(u_+ - u) \\ &= \frac{1}{2}f_u v = V_B(q^*, t^*, u, \lambda, \alpha). \end{aligned}$$

The above calculations verify only (some) necessary conditions. The sufficient conditions are established under an additional assumption on Bob's beliefs. The assumption on the beliefs is closely related to the well-known requirement that that a virtual value is increasing.[8]

Proposition 1. *Suppose that the support of μ is finite, $p^*(\mu)$ is the unique solution to the screening problem (1), and that $(u - v)\frac{f_u}{(u_+ - u)} - \sum_{u' \geq u_+} f_{u'}$ is strictly increasing in $u \in \operatorname{supp}\mu$. Then, $(q^*_., t^*_.)$ is a neutral bargaining solution.*

It is not clear at this moment whether the relation between the neutral solution and the optimal constrained screening menu is just a pure coincidence or it extends to other bargaining environments. Note that the existence of transfers and the associated first-order condition are crucial for the above argument. An earlier version of the paper studied an environment without transfers in which the optimal constrained screening mechanism differs from the neutral solution.

6.2. **Comparison to the Coasian bargaining.** When $\beta = 0$, or all the offers are made by the uninformed agent, our model becomes very similar to seller-only bargaining models studied intensively in the durable-good monopolist literature. A famous result from this literature is the Coasian conjecture: in the gap case, the monopolist must price the good at the lowest possible value of the buyer (Gul, Sonnenschein, and

[8]In our setting, the virtual value from the mechanism design literature is equal to $u - v - \frac{\sum_{u' \geq u} f_{u'}}{f_u/(u_+ - u)}$.

Wilson (1986)). This solution exhibits three features: (a) it is ex-post efficient, (b) the uninformed agent's payoff is as if he faces an informed player type that is worst for him, and (c) each type of the informed agent is able to mimic the behavior of the type that would maximize her payoffs.

In contrast, in our bargaining model, when $\beta = 0$, Bob offers the optimal screening menu $Y_{0,\mu}$, which is accepted. Bob's payoff is much higher as if he sold the good at the lowest Alice's value. The outcome is also not efficient.

In order to explain why the Coase conjecture fails in our paper, recall the basic logic of the Coasian bargaining literature. First, the uninformed player is not able to commit to not offering a trade to a low type in the future. He may want to postpone the transaction with the low type in order to reach a better deal with a higher type first. Because such a deal would be unacceptable to the low type, a rejection would convince the uninformed player that he is facing the low type, rendering him more inclined to offer a trade that is acceptable to such a type in the next period. Because any offer that is acceptable to the low type is highly attractive to the high type, if the delay between offers is not costly enough, the high type has an incentive to imitate the low type, reject the initial offer, which in turn destroys the equilibrium.

In our setting, by choosing an appropriate menu, Bob can simultaneously make an offer that is acceptable to high and low types. Because both types are expected to accept it, its rejection does not generate any information, and, in particular, it does not have to be interpreted as evidence that Bob is facing the "low" type. [9]

A companion paper, Peski (2019), studies war-of-attrition bargaining in a similar environment, except that players have additional ability to commit to their offers due to reputational types. Interestingly, more commitment leads to a Coasian-type result: in the unique (rational and patient limit) equilibrium, Bob proposes a menu $\mathcal{C}$ of all

[9]A similar mechanism is at play in Board and Pycia (2014) which considers a Coasian bargaining model, but with the informed player having an access to an outside option. In equilibrium, the low types prefer to exit the market, and the rejection of on offer is not meaningful in itself unless reinforced by exit.

allocations that give him at least his worst possible complete information payoff. Bob is typically strictly worse off than under the optimal menu $Y_{\alpha,P^*(\mu)}$; Alice types are better off, some of them strictly so. The disparity between standard and reputational versions of the model is striking to a reader familiar with Abreu and Gul (2000).

6.3. **Renegotiation.** So far, we have assumed that allocations determined by an accepted mechanism are final and cannot be renegotiated. At first glance, it may seem that the ability to commit jointly is responsible for Bob's high constrained-commitment payoff, and that allowing for renegotiation might introduce forces that would reduce Bob's payoff.

In order to examine the effect of renegotiation, we consider the following modification of the basic model. Suppose that after a mechanism is implemented and an allocation is chosen, one of the players can request renegotiation and the other player either accepts or rejects. (Who requests the renegotiation is not relevant, but the decision to renegotiate must be made jointly.) If the renegotiation request is rejected, the game ends, and the original allocation prevails. If the request is accepted, the previous agreement is forgotten, and the players restart the bargaining game (with a possibility for future renegotiation(s)) in the next period.

We claim that Bob's equilibrium payoff under renegotiation cannot be lower than the payoff without renegotiation $\Pi^*_\beta(\mu)$. The argument described in Section 5.1 remains valid under the following modification:

A potential complication due to renegotiation is that, if Bob offers a menu Y' and it is accepted, the payoffs of the agents depend not only on the payoffs in Y', but possibly also on the continuation game in which renegotiation occurs. In particular, Alice may choose a sub-optimal allocation because she anticipates it to be renegotiated. However, we claim that the problem is not relevant here, and, with probability 1, Alice accepts Y', chooses an allocation optimal for her type, and refuses any further renegotiation (if requested). On the contrary, suppose that Alice accepts the menu, one of the players requests renegotiation, which is, in turn, accepted.

Let ψ and (z, z_B) be the beliefs and the payoff outcome associated with the continuation equilibrium after Alice rejects Y'. It must be that ψ is absolutely continuous wrt μ, hence $\text{supp}\psi \subseteq \text{supp}\mu$. By Lemma 4, there are types $u \in \text{supp}\,\psi$, for whom the expected payoff from rejection, $\delta z(u)$, is strictly smaller than

$$\delta(\alpha_{\max} + \eta)\max(u, v) \leq \delta y_{\alpha_{\max},\mu}(u) + 2\eta\max(u, v) \leq y(u, Y'),$$

or payoffs from accepting Y' (the last inequality comes from the choice of menu Y'). Because the payoffs are continuous in types, there must be a strictly positive ψ-mass of types who have strictly higher payoffs Y', which leads to a contradiction with a rejection of Y' being a best response for almost all rejecting types.

Hence, Y' is accepted, Alice behaves as if it is final, she chooses optimally, and the outcome is not renegotiated. The rest of the argument from Section 5.1 remains unchanged.

Although we do not know the upper bound on Bob's payoff, the argument in Section 5.2 is not valid under renegotiation due to the problem described above. In particular, if Bob accepts Alice's counter-offer, Alice's behavior in the menu of menus may be sub-optimal, and lead to subsequent renegotiation. If the payoff from the continuation game is sufficiently low, Bob will reject Alice's counter-offer in equilibrium, which may lead Alice to accept Bob's offer in the previous period.

The fact that a reduction in commitment abilities does not reduce the uninformed party's bargaining power is surprising. At the same time, we note that there are alternative ways of modeling renegotiation, in which Coasian-type forces may dominate and reduce Bob's payoff. We leave these investigations for future research.

Appendix A. Remaining proofs

A.1. **Proof of Lemma 1.** The properties of function y follows from standard arguments based on the envelope theorem. For each mechanism and an equilibrium that implements payoff outcome (y, y_B), let $q(u)$ be the equilibrium probability that Alice gets the good if she is type u and let $t(u)$ be the expected transfer of type u. The standard arguments imply that $q(u)$ is increasing and that $\partial y(u) \in [\lim_{u' \nearrow u} q(u'), \lim_{u' \searrow u} q(u')]$. Because $y(u) = q(u)u - t(u)$, Bob's payoff from interaction with type u is equal to

$$v(1 - q(u)) + t(u) = -y(u) + q(u)(u - v) + v \leq \pi(u; y),$$

where the last inequality follows from the fact that $q(u) \geq 0$ for each u. For the last claim, construct menu $Y = \text{cl}\{(q(u), t(u)) : u \in U\}$ (taking closure does not change the incentive compatibility of allocation mapping q, t).

A.2. **Proof of Lemma 2.** For the first equality, notice that

$$\pi(u; y_{\alpha,p}) = \begin{cases} v - \alpha v & \text{if } u \leq v \\ \alpha(u - v) + v - \alpha u & \text{if } u \in (v, p) \\ (u - v) + v - (u - (1 - \alpha)p) & \text{if } u \geq p \end{cases} = (1 - \alpha)\pi(u; y_{0,p}).$$

We show the second inequality. Assume (w.l.o.g.) that $u_{\min} \leq v$. By Lemma 1, the value of the optimization problem $\Pi^*_\alpha(\mu)$ is equal to

$$\max_{y: y \text{ satisfies } \alpha\text{-constraints}} \Pi(\mu; y),$$

where α-constraints mean that y is increasing, convex, and $\partial y(u) \in [0, 1]$ and $y(u) \geq \alpha \max(u, v)$ for each u. We are going to show that (a) $\Pi^*_\alpha \leq (1 - \alpha)\Pi^*_0$ and that (b) $\Pi^*_\alpha \geq (1 - \alpha)\Pi^*_0$.

For (a), take any y st. $y(u) \geq \alpha \max(u,v)$. Define

$$y'(u) = \begin{cases} \alpha v & \text{if } u \leq v \\ y(u) - (y(v) - \alpha v) & \text{if } u \geq v \end{cases}.$$

Then, $y'(u)$ satisfies the α-constraints. Moreover, for each $u \leq v$, $\min \partial y(u) \geq 0 = \partial y'(u)$ and $y(u) \geq \alpha v = y'(u)$. Hence, $\pi(u;y) \leq \pi(u;y')$ for each $u \leq v$. For each $u \geq v$, $\partial y(u) = \partial y'(u)$ and $y(u) \geq y'(u)$, hence $\pi(u;y) \leq \pi(u;y')$. It follows that $\Pi(\mu;y) \leq \Pi(\mu;y')$.

From now on, we assume that $y(u) = \alpha v$ for each $u \leq v$. Then,

$$\Pi(\mu;y) = \int_{u_{\min}}^{u_{\max}} \pi(u;y)\, d\mu(u) = \int_{v}^{u_{\max}} \pi(u;y)\, d\mu(u).$$

Because $y(u) \geq \alpha u$ for each $u \geq v$, it must be that $\alpha \in \partial y(v)$ and $\partial y(u) \geq \alpha$ for each $u > v$. Let $y'(u) = \frac{1}{1-\alpha}(y(u) - \alpha \max(u,v))$. Then, y' satisfies 0-constraints. Moreover,

$$\begin{aligned}
\Pi(\mu;y) &= \int_{v}^{u_{\max}} \left[(\max \partial y(u))(u-v) + v - y(u)\right] d\mu(u) \\
&= \int_{v}^{u_{\max}} \left[((1-\alpha)(\max \partial y'(u)) + \alpha)(u-v) + v - y(u)\right] d\mu(u) \\
&= \int_{v}^{u_{\max}} \left[(1-\alpha)(\max \partial y'(u))(u-v) + (v - \alpha v) - (y(u) - \alpha u)\right] d\mu(u) \\
&= (1-\alpha) \int_{v}^{u_{\max}} \left[(\max \partial y'(u))(u-v) + v - y'(u)\right] d\mu(u) = (1-\alpha)\,\Pi(\mu;y').
\end{aligned}$$

This shows that $\Pi^*_\alpha \leq (1-\alpha)\,\Pi^*_0$.

For (b), take any y that satisfies 0-constraints. By the above argument, we can assume that $y(0) = 0$. Define y' so that for each u, $y'(u) = \alpha \max(u,v) + (1-\alpha)\, y(u)$.

Then similar calculations show that $\Pi(\mu;y') = (1-\alpha)\Pi(\mu;y)$, which shows that $\Pi^*_\alpha \geq (1-\alpha)\Pi^*_0$.

Finally, Bulow and Roberts (1989)shows that for each $p \in P^*(\mu)$,

$$\max_{y:y \text{ satisfies 0-constraints}} \Pi(\mu;y) = \Pi(\mu;y_{0,p}).$$

A.3. **Proof of Lemma 3.** The identified belief has a two-element support: $\mu_m = \mu_\rho = \rho\delta_{u_{\max}} + (1-\rho)\delta_{u^*_{\min}}$, where δ is the Dirac's delta, $u^*_{\min} = \max(v, u_{\min})$, and the weight that the belief puts on $u_{\max}$, ρ, is yet to be determined. We restrict the possible choices of ρ to $\rho \geq \rho^* = \frac{u^*_{\min}-v}{u_{\max}-v}$. Then $\rho(u_{\max}-v) \geq (u^*_{\min}-v)$, which implies that $P^*(\mu_\rho) = \{u_{\max}\}$. It follows that Alice's payoffs in the optimal screening menu subject to α-constraint are exactly equal to the constraint, i.e., $y_{\alpha,u_{\max}} = \alpha\min(u,v)$ for each u. Moreover, $\Pi^*_\alpha(\mu_\rho) = (1-\alpha)\Pi^*_0(\mu_\rho) = (1-\alpha)\rho(u_{\max}-v)$.

In the course of the proof, we are going to provide an upper bound on Bob's payoffs y_B as a function of Alice's payoffs y (see Lemma 1):

$$\begin{aligned} y_B \leq\ & (1-\rho)\left[\max \partial y(u^*_{\min})(u^*_{\min}-v) + v - y(u^*_{\min})\right] \\ & + \rho\left[\max \partial y(u_{\max})(u_{\max}-v) + v - y(u_{\max})\right]. \end{aligned}$$

Due to the convexity of y, $\max \partial y(u^*_{\min}) \leq \frac{y(u_{\max})-y\left(u^*_{\min}\right)}{u_{\max}-u^*_{\min}} = q$ and $\max \partial y(u_{\max}) \leq 1$. Hence the above is not larger than

$$\leq (1-\rho)\left[q(u^*_{\min}-v) - y(u^*_{\min})\right] + \rho\left[u_{\max}-v-y(u_{\max})\right] + v.$$

Define sets

$$\begin{aligned} Y_0 &= \left\{y \in \mathbb{R}^U : y(u^*_{\min}) \geq \alpha u^*_{\min} \text{ and } y(u_{\max}) \leq \alpha u_{\max}\right\}, \\ Y_1 &= \left\{y \in \mathbb{R}^U : \text{either } y(u^*_{\min}) \leq \alpha u^*_{\min} \text{ or } y(u_{\max}) \geq \alpha u_{\max}\right\}, \end{aligned}$$

and, for each $i = 0, 1$,

$$B_i = \{\rho \in [\rho^*, 1] : \text{there is } (y, y_B) \in E(\mu_\rho; m) \text{ st. } y \in Y_i\}$$

Then, $[\rho^*, 1] \subseteq B_0 \cup B_1$. Because $E(.; m)$ is u.h.c. (as m is Kakutani), sets B_i are closed. The following three cases are exhaustive. In each of the cases, we find ρ and $(y, y_B) \in E(\delta, \mu_\rho)$ such that either $y(u) \le \alpha \max(u, v)$ for each u, or $y_B \le \Pi^*_\alpha(\mu_\rho)$.

- $\rho^* \in B_0$: Let $\rho = \rho^* = \frac{u^*_{\min} - v}{u_{\max} - v}$ and take any $(y, y_B) \in E(\mu_\rho; m)$ be such that $y(u^*_{\min}) \ge \alpha u^*_{\min}$ or $y(u_{\max}) \le \alpha u_{\max}$. In this case, $q = \frac{y(u_{\max}) - y\left(u^*_{\min}\right)}{u_{\max} - u^*_{\min}} \le \alpha$. Because $y(u_{\max}) \le y(u^*_{\min}) + q(u_{\max} - u^*_{\min})$, Bob's expected payoffs are not higher than

$$\begin{aligned}
y_B \le& (1-\rho)\left[q\left(u^*_{\min} - v\right) - y\left(u^*_{\min}\right)\right] + \rho\left[\left(u_{\max} - v\right) - y\left(u_{\max}\right)\right] + v \\
\le& (1-\rho) q\left(u^*_{\min} - v\right) + \rho\left[\left(u_{\max} - v\right) - q\left(\left(u_{\max} - v\right) - \left(u^*_{\min} - v\right)\right)\right] + v - y\left(u^*_{\min}\right) \\
=& q\left(u^*_{\min} - v\right) + \rho(1-q)\left(u_{\max} - v\right) + v - y\left(u^*_{\min}\right) \\
=& q\left(u^*_{\min} - v\right) + (1-q)\left(u^*_{\min} - v\right) + v - y\left(u^*_{\min}\right) \\
=& u^*_{\min} - y\left(u^*_{\min}\right) \le (1-\alpha) u^*_{\min} = (1-\alpha)\rho u_{\max} = \Pi^*_\alpha(\mu_\rho).
\end{aligned}$$

- $1 \in B_1$: Take $\rho = 1$ and let $(y, y_B) \in E(\mu_\rho; m)$ be such that $y(u^*_{\min}) \le \alpha u^*_{\min}$ or $y(u_{\max}) \ge \alpha u_{\max}$. There are two subcases:
  - If $y(u_{\max}) \le \alpha u_{\max}$, then $y(u^*_{\min}) \le \alpha u^*_{\min}$ and, due to monotonicity and convexity, $y(u) \le \alpha \max(u, v)$ for each $u \in U$.
  - If $y(u_{\max}) \ge \alpha u_{\max}$, then Bob's expected payoff is not higher than

$$y_B \le (1-\rho)\left[q\left(u^*_{\min} - v\right) - y\left(u^*_{\min}\right)\right] + \rho\left[\left(u_{\max} - v\right) - y\left(u_{\max}\right)\right] + v \le (1-\alpha) u_{\max} = \Pi^*_\alpha(\mu_1).$$

- There is $\rho \in B_0 \cap B_1$. Let $(y^i, y^i_B) \in E(\mu_\rho; m)$ be such that $y^i \in Y^i$ for each $i = 0, 1$. Because sets Y^i cover the entire space of payoff functions, and because

$E(\mu_\rho; m)$ is convex, there exists a convex combination $(y, y_B) = \gamma(y^1, y_B^1) + (1-\gamma)(y^0, y_B^0)$ such that $(y, y_B) \in \mathrm{bd}Y^0 \cap \mathrm{bd}Y^1$. There are two subcases:

- If $y(u^*_{\min}) = \alpha u^*_{\min}$ and $y(u_{\max}) \leq \alpha u_{\max}$, then due to monotonicity and convexity, $y(u) \leq \alpha \max(u, v)$ for each $u \in U$.
- If $y(u^*_{\min}) \geq \alpha u^*_{\min}$ and $y(u_{\max}) = \alpha u_{\max}$, then $q \leq \alpha$ and Bob's expected payoffs are not higher than

$$\begin{aligned} y_B \leq & (1-\rho)\left[q(u^*_{\min} - v) - y(u^*_{\min})\right] + \rho\left[(u_{\max} - v) - y(u_{\max})\right] + v \\ \leq & \rho(1-\alpha) u_{\max} = \Pi(\mu_\rho; y_{\alpha, u_{\max}}). \end{aligned}$$

A.4. **Proof of Proposition 1.** The proof verifies the necessary conditions of for neutral equilibria from Myerson (1984). For convenience, we reproduce the relevant result (restated for the problem at hand) below:

Theorem. *(Theorem 4, Myerson (1984))$(q_., t_.)$ is a neutral bargaining solution if and only if $(q_u, t_u)_{u\in U}$ is incentive compatible and there exist sequences $(\lambda_u^\varepsilon)_{u\in U}$, $(\alpha_u^\varepsilon)_{u\in U}$, $(\omega_u^\varepsilon)_{u\in U}$,and (ω_B^ε) for $\varepsilon \to 0$ such that*

- *(8.1) $\lambda_u^\varepsilon > 0$, $\alpha_u^\varepsilon \geq 0$ for each ε and each u,*
- *(8.2) for each ε,*

$$\left((\lambda_u^\varepsilon + \alpha_u^\varepsilon)\omega_u^\varepsilon - \alpha_{u+}\omega_{u+}^\varepsilon\right) = \frac{1}{2}\max_{q,t}\left(V_A(q, t, u, \lambda^\varepsilon, \alpha^\varepsilon) + V_B(q, t, u, \lambda^\varepsilon, \alpha^\varepsilon)\right) \text{ for each } u, \text{ and}$$

$$\omega_B^\varepsilon = \frac{1}{2}\sum_u \max_{q,t}\left(V_A(q, t, u, \lambda^\varepsilon, \alpha^\varepsilon) + V_B(q, t, u, \lambda^\varepsilon, \alpha^\varepsilon)\right),$$

- *(8.3)* $\limsup_{\varepsilon\to 0}\omega_u^\varepsilon \leq q_u u - t_u$ *for each u and* $\limsup_{\varepsilon\to 0}\omega_B^\varepsilon \leq \sum_u f_u\left[v(1-q_u) + t_u\right]$.

The difficulty is to make sure that all λs are strictly positive. (Myerson (1984) describes "almost equivalent" necessary conditions that do not require strict positivity for all λs).

For each $u \in \text{supp}\mu$, let

$$R_u = \sum_{u' \geq u} f_{u'} (u - v)$$

be the expected revenue from fixed price u. (Here and below, the summation takes place over elements of the support $\text{supp}\mu$.) Then, by definition, R_u is maximized at $u = p^* (\mu)$. let

$$r_u = -\frac{1}{u_+ - u} \left(R_{u_+} - R_u \right) = \sum_{u' \geq u_+} f_{u'} - \frac{1}{(u_+ - u)} (u - v) f_u.$$

An interpretation of r_u is the marginal increase in Bob's payoff at price u. The belief assumption implies that r_u is decreasing. Due to the definition and the uniqueness of $p^* (\mu)$ as the highest optimal price, we have $r_{p^*(u)} < 0$, and $r_{p^*(\mu)_-} > 0$. Moreover, because r_u is decreasing, $r_u < 0$ for each $u \geq p^* (\mu)$.

Take any $\varepsilon > 0$ such that $\varepsilon < \frac{1}{|\text{supp}\mu|} r_{p^*(\mu)_-}$ and, for each u, $\varepsilon < \frac{1}{|\text{supp}\mu|} f_u$. Define $\lambda_B^\varepsilon = 1$ and

$$\lambda_u^\varepsilon = \begin{cases} \varepsilon & \text{if } u > p^* (\mu) \\ r_{p^*(\mu)_-} - \varepsilon \left| \{ u \in \text{supp}\mu : u > p^* (\mu) \} \right| & \text{if } u = p^* (u) \\ r_{u_-} - r_u & \text{if } u < p^* (u) . \end{cases}$$

Also, let for each u,

$$\alpha_u^\varepsilon = \sum_{u' \geq u} f_u - \sum_{u' \geq u} \lambda_u^\varepsilon \text{ for each } u.$$

The properties of r and the choice of ε imply that $\lambda_u^\varepsilon > 0$ for each u. The choice of ε ensures that $\alpha_u^\varepsilon \geq 0$ for each $u > p^* (\mu)$. For $u \leq p^* (\mu)$,

$$\alpha_u^\varepsilon = \sum_{u' \geq u} f_u - r_{u_-} = \frac{1}{(u - u_-)} (u_- - v) f_{u_-} \geq 0.$$

It follows that conditions (8.1) are satisfied.

Let

$$\omega_u^\varepsilon = \frac{1}{2}u \leq q_u^* u - t_u^* \text{ where the inequality is strict only for } u \geq p^*(\mu),$$
$$\omega_B^\varepsilon = \frac{1}{2}v + \frac{1}{2}\sum_{u>p^*(\mu)} \left(f_u\left(p^*(u) - v\right)\right) + \frac{1}{2}\varepsilon \sum_{u\geq p^*(\mu)} \left(u - p^*(\mu)\right).$$

Conditions (8.3) follow from the fact that $\lim \omega_B^\varepsilon = \frac{1}{2}v + \frac{1}{2}\sum_{u>p^*(\mu)} \left(f_u\left(p^*(u) - v\right)\right)$, which is Bob's expected payoff in the mechanism $(q_\cdot^*, t_\cdot^*)$.

Notice that, for each ε and each u, $\lambda_u^\varepsilon + \alpha_u^\varepsilon = f_u + \alpha_{u+}^\varepsilon$ and that

- for each $u < p^*(\mu)$, by definition,

$$\begin{aligned} & f_u(u - v) - \alpha_{u+}^\varepsilon (u_+ - u) \\ =& f_u(u - v) - \left(f_{u+} - \lambda_{u+}^\varepsilon + \alpha_{u++}^\varepsilon\right)(u_+ - u) \\ =& f_u(u - v) - \left(\frac{f_u}{u_+ - u}(u - v)\right)(u_+ - u) = 0, \text{ and} \end{aligned}$$

- for each $u \geq p^*(\mu)$,

$$f_u(u - v) - \alpha_{u+}^\varepsilon (u_+ - u) \geq f_u(u - v) - \sum_{u' \geq u_+} f_{u'}(u_+ - u) > 0.$$

Then, for each q and t,

$$\begin{aligned} & V_A(q, t, u, \lambda^\varepsilon, \alpha^\varepsilon) + V_B(q, t, u, \lambda^\varepsilon, \alpha^\varepsilon) \\ =& \left(\lambda_u^\varepsilon + \alpha_u^\varepsilon\right)(qu - t) - \alpha_{u+}^\varepsilon (qu_+ - t) + ((1 - q)v + t) f_u \\ =& \left(f_u + \alpha_{u+}^\varepsilon\right)(qu - t) - \alpha_{u+}^\varepsilon (qu_+ - t) + ((1 - q)v + t) f_u \\ =& q\left[f_u(u - v) - \alpha_{u+}^\varepsilon (u_+ - u)\right] + v f_u. \end{aligned}$$

The last expression is maximized by t_u^* and q_u^* for each u due to the above equalities and inequalities.

Equation (8.2) for Alice's type u is given by:

$$\begin{aligned}
&2\left(\left(\lambda_u^\varepsilon+\alpha_u^\varepsilon\right)\omega_u^\varepsilon-\alpha_{u+}^\varepsilon\omega_{u+}^\varepsilon\right)=\left(f_u+\alpha_{u+}^\varepsilon\right)u-\alpha_{u+}^\varepsilon u_+\\
=&vf_u+\left[f_u\left(u-v\right)-\alpha_{u+}^\varepsilon\left(u_+-u\right)\right]\\
=&\max_{q,t}V_A\left(q,t,u,\lambda^\varepsilon,\alpha^\varepsilon\right)+V_B\left(q,t,u,\lambda^\varepsilon,\alpha^\varepsilon\right)
\end{aligned}$$

Equation (8.2) for Bob comes from

$$\begin{aligned}
&\frac{1}{2}\sum_u V_A\left(q,t,u,\lambda^\varepsilon,\alpha^\varepsilon\right)+V_B\left(q,t,u,\lambda^\varepsilon,\alpha^\varepsilon\right)\\
=&\frac{1}{2}v+\frac{1}{2}\sum_{u\geq p^*(\mu)}\left(f_u\left(u-v\right)-\alpha_{u+}^\varepsilon\left(u_+-u\right)\right)\\
=&\frac{1}{2}v+\frac{1}{2}\sum_{u>p^*(\mu)}\left(f_u\left(p^*\left(u\right)-v\right)\right)+\frac{1}{2}\sum_{u\geq p^*(\mu)}\left(f_u\left(u-p^*\left(u\right)\right)\right)-\frac{1}{2}\sum_{u\geq p^*(\mu)}\left(u-p^*\left(\mu\right)\right)\left(\alpha_u^\varepsilon-\alpha_{u+}^\varepsilon\right)\\
=&\frac{1}{2}v+\frac{1}{2}\sum_{u>p^*(\mu)}\left(f_u\left(p^*\left(u\right)-v\right)\right)+\frac{1}{2}\varepsilon\sum_{u\geq p^*(\mu)}\left(u-p^*\left(\mu\right)\right)=\omega_B^\varepsilon.
\end{aligned}$$

where the last equality comes from the fact that $\alpha_u-\alpha_{u+}=f_u-\varepsilon$.

Appendix B. Equilibrium and Existence

B.1. **Incentive compatible and individually rational payoff outcomes.** Recall that a payoff outcome is a pair (y, y_B) of a function $y : U \to \mathbb{R}$ and $y_B \in \mathbb{R}$. Let $Y_0 = \mathbb{R}^U \times \mathbb{R}$ be the space of payoff outcomes equipped with the topology of uniform convergence.

An allocation mapping is a pair of measurable mappings $q, t : U \to X$. An allocation is *incentive compatible* if for each u, u'

$$q\left(u\right)u-t\left(u\right)\geq q\left(u'\right)u-t\left(u'\right).$$

For each allocation mapping q, t, let

$$y(u|q,t) = q(u)\,u - t(u) \text{ for each } u,$$
$$y_B(\mu|q,t) = \int ((1-q(u))\,v + t(u))\,d\mu(u)$$

be the expected payoffs and

$$IC(\mu) = \{(y(q,t), y(\mu|q,t)) \in Y_0 : \chi \text{ is incentive compatible}\},$$

Clearly, $IC(p)$ are non-empty (as it contains the payoffs from constant allocations) and convex.

Let $Y \subseteq Y_0$ be the subspace space of payoff outcomes (y, y_B) such that y is (weakly) increasing, convex, and Lipschitz, with the Lipschitz constant not larger than 1.

Lemma 6. *Y is a Polish space under the uniform convergence topology. For each p, $IC(p) \subseteq Y$, and the correspondence $IC : \Delta U \to Y$ is u.h.c., convex-, and non-empty-valued.*

B.2. **Payoff correspondences.** A *payoff correspondence* is a correspondence $E : \Delta U \rightrightarrows Y$ from beliefs into payoff vectors.

Payoff correspondence is *Kakutani* if it is u.h.c.[10] and for each μ, $E(\mu)$ is non-empty, convex, and compact. It follows that the set $E = \{(\mu, y) : y \in E(\mu)\} \subseteq \Delta U \times IC$ is compact.

For each payoff correspondence E and $x \in \mathbb{R}$, define xE to be payoff correspondence st. $E(\mu) = \{(xy, xy_B) : (y, y_B) \in Y\}$ for each μ. Similarly, we can define convex combination $\alpha E_1 + (1-\alpha) E_0$ of payoff correspondences E_0, E_1 for $\alpha \in (0,1)$.

B.3. **Mechanisms.** The set of payoff outcomes in a perfect Bayesian equilibrium in mechanism m is denoted as $E(\mu; m) \subseteq Y$. The equilibrium conditions ensure that the

[10]Recall that, for any topological spaces A, B, the correspondence $F : A \rightrightarrows B$ is u.h.c. if for each a, each open neighborhood B' of $F(a)$, there exists an open neighborhood A' of a such that for each $a' \in A'$, $F(a') \subseteq B'$.

equilibrium allocation must be incentive compatible, hence $E(\mu;m) \subseteq IC$. Because and players have access to public randomization, w.l.o.g. we assume that $E(\mu;m)$ is convex for each μ.

Remark 1. Payoff correspondence captures all important properties of a mechanism. In fact, we can refer to a payoff correspondence as an *abstract mechanism.* All the subsequent definitions work in exactly the same way of mechanisms are understood in this more general way.

A mechanism m is Kakutani if its payoff correspondence is Kakutani. In particular, any Kakutani mechanism has an equilibrium for each μ.

Let M be a set of mechanisms, equipped with some topology. We refer to M as a family of mechanisms.

Definition 1. Family M is *Kakutani* if M is compact Polish (as a set), each $m \in M$ is Kakutani, and the correspondence $E : M \times \Delta U \rightrightarrows Y$ induced by the equilibrium payoff correspondence is u.h.c.

B.4. **Menus.** An important class of mechanisms for bargaining games are menus. Menus are defined in Section 2.3 as a compact set of allocations for Alice to choose from. We generalize this definition to allow each player (Bob or Alice) to choose a continuation mechanism (rather than a single allocation) to be played later with the other player. Suppose that M is a Kakutani family of mechanisms. A menu of mechanisms for player i, denoted as M_i^m, is a multi-stage game, in which, in order,

- players observe a public randomization device,
- player i chooses $m \in M$ and, if $i = A$, Alice makes a cheap talk announcement and Bob updates his beliefs (following the choice of mechanism and the announcement),
- m is implemented.

The announcement serves to provide information about Alice. We assume that the space of announcements A to be sufficiently rich: it has a from $A = \Delta U \times A_0$ for some compact Polish set A_0 (the latter set can be a singleton).

B.5. **Equilibrium in menu of mechanisms.** Next, we define a notion of perfect Bayesian equilibrium in a menu of mechanisms M. We start with a menu for Alice.

Definition 2. We say that a tuple $(y, y_B, \mu) \in Y \times \Delta U$ is an equilibrium tuple in menu of mechanisms M_A^{m} if there exists a measurable strategy $\sigma : U \to \Delta (M \times A)$, measurable continuation payoffs $v : M \times A \to Y$, and, if $i = A$, measurable belief function $q : M \times A \to \Delta U$, such that the following conditions hold:

- payoff consistency: type $u_i \in U_i$, each player j

$$y (u) = \int v_A (u|m, a) \sigma (d (m, a) |u) \text{, for each } u \in U,$$
$$y_B = \int v_B (u|m, a) \sigma (d (m, a) |u) d\mu (u)$$

- best response: for each m, a, each $u \in U$

$$v_A (u|m, a) \leq y (u) ,$$

- belief consistency: for each continuous function $f : U \times M \times A \to \mathbb{R}$, we have

$$\int f (s, m, a) q (ds|m, a) \sigma (d (m, a) |u) d\mu (u) = \int f (u, m, a) \sigma (d (m, a) |u) d\mu (u) ,$$

- continuation payoffs: for each m, a, we have

$$v (m, a) \in E (m) (q (m, a)) .$$

We refer to the tuple (σ, q, v) as a (perfect Bayesian) equilibrium of menu of mechanisms M_A^{m}.

The definition for Bob's menu is analogous (and it can be obtained from above by replacing U by a single element set and dropping the belief consistency condition).

We point out non-standard features of the definition. Although the equilibrium describes the behavior of player i in the menu of mechanisms, it is silent about the behavior once the mechanism is selected. Instead, the definition points to a continuation payoff, taking as granted that the payoff can be implemented. The approach is modular: to focus on the behavior in the game at hand and leave the continuation behavior for some other definition. One consequence is that such a definition assumes that the one-shot-deviation principle always holds. Another consequence is that the definition does not require that the continuation behavior in the mechanism is measurable with respect to the history in the game at hand, as long as the continuation payoffs and beliefs at the beginning of the mechanism are measurable.

Second, the best response condition, together with the payoff consistency condition ensure that μ_i-almost all types best respond and receive payoffs as in y. The remaining 0-mass of types may either receive a lower payoff, or have no well-defined best response. This feature is without loss of generality, as we can always modify the equilibrium object to ensure the maximization for all types.

Definition 3. A tuple of payoffs $(y, y_B) \in Y$ is an equilibrium payoff outcome with correlating device (e.o.r.d.) in menu of mechanisms M_i^{m} with initial beliefs μ if there is a probability distribution $\gamma \in \Delta Y$ such that (y', μ) is an equilibrium tuple in menu of mechanisms M_i^{m} for γ-all y' and $y = \int y' d\gamma\left(y', y'_B\right)$ and $y_B = \int y'_B d\gamma\left(y', y'_B\right)$.

For each $\mu \in \Delta U$, let

$$
\begin{aligned}
E\left(M_i^{\mathrm{m}}\right)(\mu) &= \left\{(y, y_B) : y \text{ is e.o.r.d. in } M_i^{\mathrm{m}} \text{ with initial beliefs } \mu\right\} \\
&= \operatorname{con}\left\{(y, y_B) : (y, y_B, \mu) \text{ is equilibrium tuple in } M_i^{\mathrm{m}}\right\}.
\end{aligned}
$$

The equality in the second line is due to the Choquet Theorem. Hence $E\left(M_i^{\mathrm{m}}\right)$is the equilibrium correspondence in mechanism M_i^{m}.

Proposition 2. *If family of mechanisms M is Kakutani, then the family of menus of mechanisms $\{K_i^m : K \subseteq M, K \text{ is compact}\}$ is Kakutani as well.*

Recall that, by definition, all Kakutani mechanisms have an equilibrium. Thus, the proposition implies that each of the menu of mechanisms K_i^{m} for a compact subset K of a Kakutani family of mechanisms has an equilibrium.

Corollary 1. *Single offers, menus, and menus of menus are Kakutani mechanisms.*

Proof. The claim for single-offers is trivial. The claim for menus (where Alice is choosing from a compact set of single offers) follows from the Proposition 2. Proposition 2 implies that a compact family of menus (i.e., derived from a compact set of compact sets of X is Kakutani. Applied once again, the Proposition 2 implies that each menus of menus (where Bob is choosing from a compact set of Alice's menus) is Kakutani. □

B.6. **Equilibrium of the bargaining game.** The definition of equilibrium in the bargaining game builds upon the definition of equilibrium payoff outcome with correlating device in a menu of menus.

Definition 4. A tuple (y, y_B) is an equilibrium payoff outcome in the bargaining game with beliefs μ if there are Kakutani payoff correspondences E, E^A, E^B and $E^{A,m}, E^{B,m}$ for $m \in \mathcal{M}$ such that $(y, y_B) \in E(\mu)$ and

(1) $E = \beta E^A + (1-\beta) E^B$,
(2) for each player i, family of (abstract) mechanisms $\{E^{i,m} : m \in \mathcal{M}\}$ is Kakutani and E^i is a payoff correspondence in the menu of (abstract) mechanisms $\{E^{i,m} : m \in \mathcal{M}\}_i^{\mathrm{m}}$,
(3) for each player i, for each $m \in \mathcal{M}$,$E^{i,m}$ is a payoff correspondence in the menu of mechanisms $\{m, \delta E\}_{-i}^{\mathrm{m}}$.

Condition 1 ensures that the payoffs in the bargaining game are expectation over the choice of the proposer. Condition 2 describes the payoffs in the game in which a proposer chooses a mechanism. Condition 3 describes the payoffs in the subgame, in which the other player decides whether to accept or reject. In case of rejection, the continuation payoff is discounted.

Appendix C. Existence proofs

C.1. **Proof of Lemma 6.** The fact that Y_0 is Polish is standard. (Separability is a consequence of the fact that any function in Y_0 is continuous, hence determined by its values on a dense subset of U. The completeness follows from the Arzela-Ascoli Theorem and the fact that elements of Y_0 are equicontinuous.) The standard envelope theorem implies that for each $(y, y_B) \in IC(p)$, y is increasing in u, convex, and its subdifferential is bounded, $\partial y(u) \subseteq [0,1]$. Hence $IC(p) \subseteq Y_0$.

Let

$$\begin{aligned} \pi_{\min}(u; y) &= v - y + (u - v) \cdot \begin{cases} \max \partial y(u) & u \geq v \\ \min \partial y(u) & u < v \end{cases}, \\ \Pi_{\min}(\mu; y) &= \int \pi_{\min}(u; y) \, d\mu(y) . \end{aligned}$$

We are going to show that

$$IC(\mu) = \{(y, y_B) \in Y : y_A \in Y, \Pi_{\min}(\mu; y_A) \leq y_B \leq \Pi(\mu; y_A)\}$$

Indeed, if q, t is the physical allocation mapping implementing payoff outcome (y, y_B), then $q(u) \in \partial y(u)$ due to the envelope theorem, and $t(u) = q(u)u - y(u)$. Alternatively, for each $\Pi_{\min}(\mu; y) \leq y_B \leq \Pi(\mu; y)$, find α such that $y_B = \alpha \Pi_{\min}(\mu; y) + (1 - \alpha)\Pi(\mu; y)$, and consider an allocation

$$q(u) = \alpha \min \partial y(u) + (1 - \alpha) \max \partial y(u) \text{ and } t(u) = q(u)u - y(u) .$$

Then, Bob's payoff from this allocation is equal to y_B.

To show that correspondence IC is u.h.c. consider the following approximations to Π and $\Pi_{\min}$: for each n, let

$$\Pi^n_{\min}(\mu; y) = \int \left[n \min \left\{ \begin{array}{c} \left(y(u) - y\left(u - \frac{1}{n}\right)\right)(u - v), \\ \left(y\left(u + \frac{1}{n}\right) - y(u)\right)(u - v) \end{array} \right\} + v - y \right] d\mu(y)$$
$$\leq \Pi_{\min}(\mu; y) = \lim_{n\to\infty} \Pi^n_{\min}(\mu; y).$$
$$\Pi^n(\mu; y) = \int \left[n \max \left\{ \begin{array}{c} \left(y(u) - y\left(u - \frac{1}{n}\right)\right)(u - v), \\ \left(y\left(u + \frac{1}{n}\right) - y(u)\right)(u - v) \end{array} \right\} + v - y \right] d\mu(y)$$
$$\geq \Pi(\mu; y) = \lim_{n\to\infty} \Pi^n(\mu; y).$$

Here, we take $y(u) = y(u_{\min})$ for $u < u_{\min}$ and $y(u) = y(u_{\max}) + (u - u_{\max})$ for $u > u_{\min}$.

Take a sequence $(y_m, y_{B,m}) \in E(\mu_m)$ that converges to $\lim (y_m, y_{B,m}) = (y, y_B)$ and $\lim \mu_m = \mu$. Notice that $(y, y_B) \in Y$ as Y is Polish. Then, because $\Pi^n(\mu; y)$ is continuous in weak* topology on ΔU and the uniform topology on Y, we have

$$\Pi^n_{\min}(\mu; y) = \lim_m \Pi^n(\mu_m; y_m) \leq y_B = \lim y_{B,m} \leq \lim_m \Pi^m(\mu_m; y_m) \leq \Pi^n(\mu; y).$$

Because the above holds for each n, we get $\Pi_{\min}(\mu; y) \leq y_B \leq \Pi(\mu; y)$,which implies that $(y, y_B) \in E(\mu)$.

C.2. **Distributional equilibrium.** In this subsection, we provide an alternative and equivalent definition of equilibrium for menus of mechanisms and discuss its properties. The notion is a version of the equilibrium in distributional strategies from Milgrom and Weber (1985) but adapted to menu-of-mechanisms game.

From now on, assume that M is a compact family of mechanism.

A distributional strategy in Alice's menu of mechanism M_A^{m} with prior beliefs μ is a probability distribution $\alpha \in \Delta(U \times M \times A \times Y)$ such that $\mathrm{marg}_U \alpha = \mu$. In other words, a distributional strategy is a joint distribution over types, actions (mechanism and announcement), as well as continuation payoffs. Recall that $A = \Delta U \times A_0$, hence, the first part of the announcement $a = (q, a_0)$ can be interpreted as as "posterior beliefs" induced by the chosen mechanism and the announcement.

Definition 5. We say that a pair $(y, y_B, \mu) \in Y \times \Delta U$ is a tuple of distributional equilibrium payoffs in menu of mechanisms M_A^{m} with initial beliefs μ if there exists a distributional strategy $\alpha \in \Delta(U \times M \times A \times Y)$ such that the following conditions hold:

- payoff consistency: for any continuous function $f : U \to \mathbb{R}$,
$$\int f(u)\, y(u)\, d\mu(u) = \int f(u)\, v(u)\, \alpha\left(d\left(u, m, a, v, v_B\right)\right),$$
$$y_B = \int v_B \alpha\left(d\left(u, m, a, v, v_B\right)\right),$$
- best response:
$$\alpha\left\{v : v(u) \leq y(u) \text{ for each } u\right\} = 1,$$
$$\left(\bigcup_{q \in \Delta U} E(m)(q)\right) \cap \left\{v : v(u) \leq y(u) \text{ for each } u\right\} \neq \emptyset \text{ for each } m,$$
The first condition ensures that there are no on-path deviation. The second condition ensures that for each mechanism m, there is a belief q and continuation payoff that is worse than the equilibrium payoff for Alice: $v(u) \leq y(u)$ for each u. This takes care of off-path deviations),
- belief consistency: recall that $A = \Delta U \times A_0$. For each continuous function $f : U \times M \times \Delta U \times A_0 \to \mathbb{R}$, we have
$$\int \left(\int f(s, m, q, a_0)\, q(ds)\right) \alpha\left(d\left(u, m, q, a_0, v, v_B\right)\right)$$
$$= \int f(u, m, q, a_0)\, \alpha\left(d\left(u, m, q, a_0, v, v_B\right)\right).$$

The condition ensures that the announcement of continuation belief is correct in the Bayes updating sense,

- continuation payoffs: for each $m, a = (q, a_0)$, we have

$$\alpha \left\{ (u, m, q, a_0, v, v_B) : v \in E(m)(q) \right\} = 1.$$

We refer to α as a distributional equilibrium of menu of mechanisms M_i^{m}.

Lemma 7. *Tuple (y, μ) is an equilibrium tuple if and only if it is an distributional equilibrium tuple.*

Proof. Part 1. Suppose that $(y, y_B, \mu) \in Y \times \Delta U$ is an equilibrium tuple and let (σ, q, v) be the supporting strategy, belief function, and continuation payoffs. Define measure $\alpha \in \Delta(U \times M \times \Delta U \times A_0 \times Y)$ so that for any continuous function $f : U \times M \times \Delta U \times A_0 \times Y \to \mathbb{R}$, we have

$$\begin{aligned}
&\int \left(\int f(u, m, q(m, q, a_0), a_0, v(m, q, a_0))\, \sigma(d(m, q, a_0)\,|u) \right) d\mu(u) \\
&= \int f(u, m, q, a_0, v)\, \alpha(d(u, m, q, a_0, v, v_B)).
\end{aligned}$$

Then, the payoff consistency, best response, and continuation payoff conditions of Definition 5 are satisfied immediately. For the belief consistency condition, take any continuous $f : U \times M \times \Delta U \times A_0 \to \mathbb{R}$, and notice that

$$\begin{aligned}
&\int \left(\int f(s, m, q, a_0)\, q(ds) \right) \alpha(d(u, m, q, a_0, v, v_B)) \\
&= \int \left(\int \left(\int f(s, m, q(m, q, a_0), a_0)\, q(ds|m, q, a) \right) \sigma(d(m, q, a_0)\,|u) \right) d\mu(u) \\
&= \int \left(\int f(u, m, q(m, q, a_0), a_0)\, \sigma(d(m, q, a_0)\,|u) \right) d\mu(u) \\
&= \int f(u, m, q, a_0)\, \alpha(d(u, m, q, a_0, v, v_B)),
\end{aligned}$$

where the first and the third equality come from the definition of α and the second from the belief-consistency condition of Definition 2.

Part 2. Suppose that (y, y_B, μ) is a tuple of distributional equilibrium payoffs, and let α be a corresponding distributional equilibrium. Fix versions of conditional distributions $\alpha(.|u)$ and $\alpha(.|m,a)$ for each $u \in U$, and $m \in M, a \in A$. Define a measurable strategy $\sigma : U \to \Delta(M \times A)$, measurable belief function $\tilde{q} : M \times A \to \Delta U$, and measurable continuation payoffs $\tilde{v} : M \times A \to Y$:

$$\begin{aligned}
\sigma(u) &= \mathrm{marg}_{M\times A}\alpha(.|u), \\
\tilde{q}(m,a) &= \mathrm{marg}_U \alpha(.|m,a), \\
\tilde{v}(u|m,a) &= \int v(u)\,\alpha(d(v,v_B)|m,a,u), \\
\tilde{v}_B(m,a) &= \int v_B \alpha(d(u,v,v_B)|m,a)
\end{aligned}$$

The definitions of $\tilde{q}$ and $\tilde{\sigma}$ imply that for each continuous function $f : U \times M \times \Delta U \times A_0 \to \mathbb{R}$, we have

$$\begin{aligned}
&\int f(u,m,a)\,\sigma(d(m,a)|u)\,\mu(du) \\
=&\int f(u,m,a)\,\alpha(d(u_i,m,a)) \\
=&\int f(u,m,q,a')\,\alpha(d(u,m,q,a')), \\
=&\int\left(\int f(u,m,a)\left(\mathrm{marg}_U\alpha(du|m,a)\right)\right)\alpha(d(m,a)) \\
=&\int\left(\int f(u,m,a)\left(\mathrm{marg}_U\alpha(du|m,a)\right)\right)\alpha(d(m,a)) \\
=&\int\left(\int f(s,m,q,a')\,\tilde{q}(ds|m,q,a')\right)\alpha(d(m,a)).
\end{aligned}$$

In particular, $\tilde{q}$ satisfies the belief-consistency condition of Definition 2. By the belief consistency condition of Definition 5, we have

$$\int \left(\int f\left(s, m, q, a'\right) \left[q\left(ds\right) - \tilde{q}\left(ds|m, q, a'\right)\right] \right) \alpha\left(d\left(u, m, q, a', v\right)\right) = 0.$$

Because the claim holds for any continuous f, we obtain that $\tilde{q}\left(m, a\right) = q$, α_i-almost surely. Hence, by the continuation payoffs condition of Definition 5, we have $\left(\tilde{v}\left(m, a\right), \tilde{v}_B\left(m, a\right)\right) \in E\left(m, a\right)\left(\tilde{q}\left(m, a\right)\right)$, α-almost surely. Let W_1 be the set of pairs $\left(m, a\right)$ for which the relation does not hold.

By the best response condition, $\tilde{v}\left(m, a\right) \leq y$ α-almost surely. Let W_2 be the set of pairs $\left(m, a\right)$ for which the relation does not hold.

For each m, pick in a measurable way $\left(q^m, v^m\right)$ so that $v^m \in E\left(m\right)\left(q^m\right)$ and $v^m \leq y$. Modify $\tilde{v}$ to v for all pairs $\left(m, a\right) \in W_1 \cup W_2$ so that $v\left(m, a\right) = v^m$ and for all m, a. Because the modification is on α-null set, the payoff consistency and belief consistency of Definition 2 are satisfied. The construction ensures that the best response and the continuation payoffs conditions are satisfied as well. □

C.3. **Proof of Proposition 2.** The argument in Bob's case is relatively straightforward and omit it. From now on, assume that Alice chooses from the menu of mechanisms, i.e., $i = A$.

Let M be a Kakutani family of mechanisms. Let $\mathcal{M}$ be a collection of all compact subsets of M. Consider the equilibrium payoff correspondence $E^{\mathrm{m}} : \mathcal{M} \times \Delta U \rightrightarrows E_0$ defined so that for each $K \in \mathcal{M}$,

$$E^{\mathrm{m}}\left(K\right)\left(\mu\right) = E\left(K_i^{\mathrm{m}}\right)\left(\mu\right).$$

By definition, correspondence E^{m} is convex-valued. We want to show that correspondence $E^{\mathrm{m}}$is u.h.c. and that it is non-empty-valued.

The proof of u.h.c. is relatively straightforward and relies on the equivalence between equilibria and distributional equilibria established in Lemma 7. The proof of the existence is preceded by a general observation about continuous selectors approximating Kakutani correspondences.

C.3.1. *Upper hemicontinuity.*

Lemma 8. *Set* $E_0 = con\bigcup_{m\in M,\mu\in\Delta U,} E(m)(u) \subseteq Y$ *is compact.*

Proof. Take any sequence $y_n \in E$ and find m_n, μ_n such that $y_n \in E(m_n)(\mu_n)$. By taking subsequences, and using the fact that family M is Kakutani, we can assume that $m_n \to m \in M$ and $\mu_n \to \mu \in \Delta U$. For each $\varepsilon > 0$, let Y_ε be a finite set of elements of $E(m)(\mu)$ such that $E(m)(\mu) \subseteq \bigcup_{y\in Y_\varepsilon} B(y,\varepsilon)$, where $B(y,\varepsilon)$ is an open ball. Because $E : M \times \Delta U \rightrightarrows Y$ is u.h.c., for sufficiently high n, $y_n \in \bigcup_{y\in Y_\varepsilon} B(y,\varepsilon)$. By taking subsequences, there is $y_\varepsilon \in E(m)(\mu)$ such that $y_n \in B(y_\varepsilon,\varepsilon)$ for infinitely many n. Because the claim holds for all $\varepsilon > 0$, we can construct a Cauchy, hence convergent subsequence $y_n \to y \in E(m)(\mu)$. □

Lemma 9. *Correspondence* E^m*is u.h.c.*

Proof. It is enough to show that $E_0^{\mathrm{m}} : \mathcal{M} \times \Delta U \rightrightarrows E_0$ is u.h.c., where

$$E_0^{\mathrm{m}}(K)(\mu) = \{y : (y,\mu) \text{ is equilibrium pair in } K^{\mathrm{m}}\} .$$

Because E_0 is compact, we can rely on the characterization of upper hemicontinuity through sequences. Let $(y^n, \mu^n, K_n) \to (y, \mu, K)$ be a convergent sequence such that for each n, $y^n \in E_0^{\mathrm{m}}(K)(\mu^n)$. Let $\alpha^n \in \Delta(U \times K_n \times A \times E_0)$ be the sequence of associated equilibrium distributions. By taking a subsequence, we can assume that α^n converges to $\alpha \in \Delta(U \times K \times A \times E_0)$. Because all equilibrium conditions in the Definition 5 are preserved under weak limits, α is a distributional equilibrium in a menu of menus K_A^{m}. Moreover, y is the associated payoff vector and $\mu = \mathrm{marg}_U \alpha$ are the beliefs. □

C.3.2. *Continuous approximations.* In the next part of the proof, we use the following observation.

Suppose that A is compact Polish and B is either a compact subset of a Banach space or $B = E_0$. For each correspondence $F : A \rightrightarrows B$, and each $\varepsilon > 0$, define correspondence $U_\varepsilon F : A \rightrightarrows B$ so that for each $a \in A$,

$$U_\varepsilon F(a) = \left\{ \mathbb{E}_\mu b : \mu \in \Delta F \text{ and } \forall \alpha \geq 0,\ \mu\{(a', b') : d_A(a, a') \geq \alpha\varepsilon\} \leq \mathrm{e}^{-\alpha} \right\}.$$

Here, if B is a subset of a Banach space, $\mathbb{E}_\mu b$ is the barycenter of measure $\operatorname{marg}_B \mu$, and if $B = Y$, we take $\mathbb{E}_\mu b = (y, y_B) \in Y$ such that

$$y(u) = \int y'(u)\, d\mu(a, y', y'_B),$$
$$y_B = \int y'_B d\mu(a, y', y'_B).$$

Importantly, in both cases, mapping $\mu \to \mathbb{E}_\mu b$ is continuous in weak* topology on $\Delta(A \times B)$.

Lemma 10. *If F is u.h.c., convex- and non-empty-valued, then*

- *$U_\varepsilon F$ is continuous (as a correspondence),*
- *$U_\varepsilon F$ admits a continuous selector: a function $\phi_\varepsilon : A \to B$ such that for each $a \in A$, $\phi_\varepsilon(b) \in U_\varepsilon F$.*
- *$\lim_{\varepsilon \to 0} U_\varepsilon F \to F$ in the sense of the Hausdorff distance on subsets $A \times B$,*

Intuitively, $U_\varepsilon F$ is a continuous approximation of F.

Proof. $U_\varepsilon F$ is clearly convex- and non-empty-valued. The upper hemicontinuity is obvious. To see the lower hemicontinuity, consider a sequence $a_n \to a$ and $b_n \in U_\varepsilon F(a_n)$ such that $b_n \to b$. Let μ_n be such that $b_n = \mathbb{E}_{\mu_n} b$ and $\mu_n\{(a', b') : d_A(a_n, a') \geq \alpha\varepsilon\} \leq$

e^{α}. Take $b_0 \in F(a)$, and let $\mu_n' = x_n \delta_a + (1 - x_n)\mu$, where $x_n = 1 - \mathrm{e}^{-\frac{d(a_n,a)}{\varepsilon}}$. Then,

$$\begin{aligned}
\mu_n'\{(a',b') : d_A(a,a') \geq \alpha\varepsilon\} &\leq (1 - x_n)\mu_n\{(a',b') : d_A(a,a') \geq \alpha\varepsilon\} \\
&\leq (1 - x_n)\mu_n\{(a',b') : d_A(a_n,a') \geq \alpha\varepsilon - d(a,a_n)\} \\
&\leq \mathrm{e}^{-\alpha}(1 - x_n)\mathrm{e}^{\frac{d(a_n,a)}{\varepsilon}} = \mathrm{e}^{-\alpha}.
\end{aligned}$$

In particular, $x_n b_0 + (1 - x_n) b_n \in U_\varepsilon F(a)$. The claim follows form the fact that $x_n \to 0$. The Michael Selection Theorem says that $U_\varepsilon F$ admits a continuous selector: a function $\phi_\varepsilon : A \to B$ such that for each $a \in A$, $\phi_\varepsilon(b) \in U_\varepsilon F$.

For the second claim, notice first that $F \subseteq U_\varepsilon F$ for each ε. Take any sequence $\varepsilon_n \to 0$, $a_n \to a$ and $b_n \in U_{\varepsilon_n}(a_n)$. Let μ_n be the associated distributions st. $b_n = \mathbb{E}_{\mu_n} b$. By taking subsequences, assume that $\mu_n \to \mu$ and $b_n \to b$. Then, $\mu \in \Delta F$, and for each $\xi > 0$,

$$\begin{aligned}
\mu\{(a',b') : d_A(a,a') \geq \xi\} &= \lim_n \mu_n\{(a',b') : d_A(a,a') \geq \xi\} \\
&\leq \lim_n \mu_n\left\{(a',b') : d_A(a_n,a') \geq \frac{1}{2}\xi\right\} = 0,
\end{aligned}$$

where the inequality comes from the fact that $a = \lim a_n$. Because the above is true for any $\xi > 0$, $\mu(\{a\} \times F(a)) = 1$ and $b = \lim b_n = \lim \mathbb{E}_{\mu_n} b = \lim \mathbb{E}_{\mu_n} b = \mathbb{E}_\mu b \in F(a)$. $\square$

C.3.3. *Existence of equilibrium.* We will show that the equilibrium payoff correspondence $E(M_A^{\mathrm{m}}) : \Delta U \rightrightarrows E_0$ is non-empty valued. Because finite subsets $K \subseteq M$ are dense in M, Lemma 9 implies that it is enough to show that $E(M_A^{\mathrm{m}})$ is for each finite $K \subseteq M$.

- For each $\varepsilon > 0$, each $k \in K$, let $U_\varepsilon E(k)$ be the ε-neighborhood around $E(k)$. Let $\phi_{\varepsilon,k} : \Delta U \to Y$ be a continuous selector from $U_\varepsilon E(k)$.

- Let $\Sigma = \{\omega \in \Delta (U \times K \times A \times E_0) : \text{marg}_U \omega = \mu\}$ be the set of distributional strategies. For each strategy ω, each $k \in K$, let $P_k : \Sigma \rightrightarrows \Delta U$ be correspondence of posterior beliefs after mechanism k that are consistent with the Bayes formula:
$$P_k(\omega) = \begin{cases} \frac{\text{marg}_U \omega(.,k)}{\omega(k)} & \text{if } \omega(k) > 0, \\ \Delta U_i & \text{if } \omega(k) = 0. \end{cases}$$
Clearly, P_k is u.h.c., non-empty-valued, and convex valued. Define $P : \Sigma \rightrightarrows (\Delta U)^K$ as $P = \times_{k\in K} P_k$. Let ϕ_ε^P be a continuous selector from $U_\varepsilon P$.
- Define correspondence $\sigma_\varepsilon : (\Delta U)^K \rightrightarrows \Sigma$ so that $\omega \in \sigma(\mu)$ if and only if for each k, all the types who choose mechanism k maximize their payoff $\phi_{\varepsilon,k}(\mu_k)(u_i)$:
$$\omega \left\{ (u,k) : k \in \arg\max_m (\phi_{\varepsilon,m}(\mu_m))(u) \right\} = \omega(k).$$
Clearly, σ_ε is u.h.c., non-empty-valued, and convex valued. Let ϕ_ε^σ be a continuous selector from $U_\varepsilon \sigma$.

The Tychonoff Fixed Point Theorem implies the existence of fixed point $\omega_\varepsilon = \phi_\varepsilon^\sigma \left(\phi_\varepsilon^P (\omega_\varepsilon) \right)$. Take a convergent subsequence of such fixed points so that

$$\omega_\varepsilon \to \omega, \ \phi_\varepsilon^P(\omega_\varepsilon) \to \mu, \text{ and } \phi_{\varepsilon,k}\left(\phi_\varepsilon^P(\omega_\varepsilon)\right) \to (y_k, y_{B,k}) \text{ as } \varepsilon \to 0$$

for some ω, μ, and y_k for each k. Then, for each k,

$$\omega \left\{ (u,k) : k \in \arg\max_r y_r(u) \right\} = \omega(k),$$
$$\mu \in P(\omega),$$
$$(y_k, y_{B,k}) \in E(k)(\mu_k) \text{ for each } k.$$

We construct a distributional equilibrium α: it is uniquely defined by (a) $\text{marg}_{U\times K}\omega = \text{marg}_{U\times K}\alpha$ and (b) $\alpha\{(u, k, \mu_k, a^0, y_k, y_{B,k})\} = 1$. The above properties of ω, μ and y_k imply that α is an equilibrium distribution that supports payoffs y such that for each type u, $y(u) = \max_k y_k(u)$.

C.4. **Existence part of the proof of Theorem 1.** Let m be a Kakutani mechanism. Define "abstract" mechanism m_0, where

$$E_0(m_0)(\mu) = \left\{\left(\delta y_{\beta,p}, \delta\Pi^*_\beta(\mu)\right) : p \in P^*(\mu)\right\}.$$
$$E(m_0)(\mu) = \mathrm{con}E_0(m_0)(\mu).$$

Correspondence $E(m_0) : \Delta U \rightrightarrows Y$ is clearly non-empty-valued and convex. Below, we are going to check that $E_0(m_0)$, hence $E(m_0)$, is u.h.c.. Hence, the "abstract" mechanism m_0 is Kakutani.

Consider a game, where Alice either accepts mechanism m (and an equilibrium from this mechanism is implemented) or rejects it, which leads to continuation payoffs in the bargaining game that are captured as equilibrium payoff outcomes of mechanism m^0. Because both $m$and m^0 are Kakutani, Proposition 2 implies that menu of mechanisms $\{m, m_0\}^{\mathrm{m}}_A$ is a Kakutani mechanism. Therefore, there exists a measurable strategy $\sigma : Y \to \Delta\{m, m_0\}$, a pair of beliefs $\mu^{\mathrm{A}}_m = q(m)$ and $\mu^{\mathrm{R}}_m = q(m_0)$, and continuation payoffs $\left(y^{\mathrm{A}}, y^{\mathrm{A}}_B\right) = \upsilon(m) \in E\left(\mu^{\mathrm{A}}_m; m\right)$ if the mechanism is accepted and $\left(\delta y^{\mathrm{R}}, \delta y^{\mathrm{R}}\right) = \upsilon(m_0) \in E(m_0)(\mu)$ if the mechanism is rejected such that (σ, q, υ) is perfect Bayesian equilibrium of the menu of mechanism $\{m, m_0\}^{\mathrm{m}}_A$.

We check that $E_0(m_0)$ is u.h.c. Take any sequence $(y_n, y_{B,n}) \in E(m_0)(\mu_n)$. By taking subsequences, we can assume that $\mu_n \to \mu$. Note that $y_{B,n} = \delta\Pi^*_\beta(\mu_n)$. Because Π^*_β is continuous, it must be that $y_{B,n} \to \Pi^*_\beta(\mu)$. For each n, there exists $p_n \in U$ such that $y_n = \delta y_{\beta,p_n}$. By taking subsequences, assume that $p_n \to p$. Clearly, $y_n \to \delta y_{\beta,p}$ uniformly over u. Finally, because $\Pi^*_\beta(\mu_n) = \Pi(\mu_n; y_{\beta,p_n})$, it must be that $\Pi^*_\beta(\mu) = \Pi(\mu; y_{\beta,p})$.

References

Abreu, Dilip and Faruk Gul (2000). “Bargaining and Reputation”. In: *Econometrica* 68.1, pp. 85–117. ISSN: 0012-9682. URL: https://www.jstor.org/stable/2999476 (visited on 02/12/2019).

Ausubel, Lawrence M. and Raymond J. Deneckere (1989). “A direct mechanism characterization of sequential bargaining with one-sided incomplete information”. In: *Journal of Economic Theory* 48.1. Publisher: Elsevier, pp. 18–46.

Board, Simon and Marek Pycia (2014). “Outside options and the failure of the Coase conjecture”. In: *American Economic Review* 104.2, pp. 656–71.

Bulow, Jeremy and John Roberts (1989). “The simple economics of optimal auctions”. In: *Journal of political economy* 97.5, pp. 1060–1090.

Clippel, Geoffroy de, Jack Fanning, and Kareen Rozen (2020). “Bargaining over Contingent Contracts Under Incomplete Information”. URL: https://ideas.repec.org/p/bro/econwp/2020-13.html.

Doval, Laura and Vasiliki Skreta (Nov. 2018). *Mechanism Design with Limited Commitment*. en. SSRN Scholarly Paper ID 3281132. Rochester, NY: Social Science Research Network. URL: https://papers.ssrn.com/abstract=3281132 (visited on 07/23/2019).

Grossman, Sanford J and Motty Perry (June 1986a). “Perfect sequential equilibrium”. In: *Journal of Economic Theory* 39.1, pp. 97–119. ISSN: 0022-0531. DOI: 10.1016/0022-0531(86)90022-0. URL: http://www.sciencedirect.com/science/article/pii/0022053186900220 (visited on 08/12/2019).

– (June 1986b). “Sequential bargaining under asymmetric information”. In: *Journal of Economic Theory* 39.1, pp. 120–154. ISSN: 0022-0531. DOI: 10.1016/0022-0531(86)90023-2. URL: http://www.sciencedirect.com/science/article/pii/0022053186900232.

Gul, Faruk and Hugo Sonnenschein (1988). “On delay in bargaining with one-sided uncertainty”. In: *Econometrica: Journal of the Econometric Society*, pp. 601–611.

Gul, Faruk, Hugo Sonnenschein, and Robert Wilson (June 1986). “Foundations of dynamic monopoly and the coase conjecture”. In: *Journal of Economic Theory* 39.1, pp. 155–190. ISSN: 0022-0531. DOI: 10.1016/0022-0531(86)90024-4. URL: http://www.sciencedirect.com/science/article/pii/0022053186900244.

Harsanyi, John C. and Reinhard Selten (1972). “A Generalized Nash Solution for Two-Person Bargaining Games with Incomplete Information”. In: *Management Science* 18.5(2), pp. 80–106. URL: https://econpapers.repec.org/article/inmormnsc/v_3a18_3ay_3a1972_3ai_3a5-part-2_3ap_3a80-106.htm (visited on 02/18/2019).

Inderst, Roman (Sept. 2003). “Alternating-offer bargaining over menus under incomplete information”. en. In: *Economic Theory* 22.2, pp. 419–429. ISSN: 1432-0479. DOI: 10.1007/s00199-002-0290-y. URL: https://doi.org/10.1007/s00199-002-0290-y.

Jackson, Matthew O. et al. (2020). *The Efficiency of Negotiations with Uncertainty and Multi-Dimensional Deals*. en. Tech. rep. Rochester, NY: Social Science Research Network.

Liu, Qingmin et al. (2019). “Auctions with limited commitment”. In: *American Economic Review* 109.3, pp. 876–910.

Milgrom, Paul R. and Robert J. Weber (1985). “Distributional strategies for games with incomplete information”. In: *Mathematics of operations research* 10.4, pp. 619–632.

Moodie, Jim (Dec. 2019). “No talks planned as teachers ratchet up the pressure in Sudbury and Ontario”. In: *Sudbury Star*.

Myerson, Roger B. (1979). “Incentive compatibility and the bargaining problem”. In: *Econometrica: journal of the Econometric Society*. Publisher: JSTOR, pp. 61–73.

Myerson, Roger B. (1983). “Mechanism design by an informed principal”. In: *Econometrica: Journal of the Econometric Society*, pp. 1767–1797.

– (1984). “Two-Person Bargaining Problems with Incomplete Information”. In: *Econometrica* 52.2, pp. 461–487. ISSN: 0012-9682. DOI: 10.2307/1911499. URL: https://www.jstor.org/stable/1911499 (visited on 02/12/2019).

Mylovanov, Tymofiy and Thomas Tröger (2012). “Informed-principal problems in environments with generalized private values”. en. In: *Theoretical Economics* 7.3, pp. 465–488. ISSN: 1555-7561. DOI: 10.3982/TE787. URL: https://onlinelibrary.wiley.com/doi/abs/10.3982/TE787 (visited on 08/23/2019).

Nash Jr, John F. (1950). “The bargaining problem”. In: *Econometrica: Journal of the Econometric Society*, pp. 155–162.

Nash, John (1953). “Two-person cooperative games”. In: *Econometrica: Journal of the Econometric Society*, pp. 128–140.

Okada, Akira (1996). “A noncooperative coalitional bargaining game with random proposers”. In: *Games and Economic Behavior* 16.1. Publisher: Elsevier, pp. 97–108.

Peski, Marcin (2019). “Bargaining over Heterogeneous Good with Structural Uncertainty”. Toronto. URL: http://individual.utoronto.ca/mpeski/papers/bargaining.pdf.

Rubinstein, Ariel (1982). “Perfect equilibrium in a bargaining model”. In: *Econometrica* 50.1.

Rushowy, Kristin (Nov. 2020). “Ontario government offers mediation amid faltering talks with teacher unions”. In: *Toronto Star*.

Sen, Arijit (2000). “Multidimensional Bargaining Under Asymmetric Information”. en. In: *International Economic Review* 41.2, pp. 425–450. ISSN: 1468-2354. DOI: 10.1111/1468-2354.00070. URL: https://onlinelibrary.wiley.com/doi/abs/10.1111/1468-2354.00070 (visited on 02/18/2019).

Skreta, Vasiliki (2006). “Sequentially Optimal Mechanisms”. In: *Review of Economic Studies* 73.4, pp. 1085–1111.

Strulovici, B. (2017). “Contract Negotiation and the Coase Conjecture: A Strategic Foundation for Renegotiation-Proof Contracts”. In: *Econometrica* 85.2, pp. 585–616. DOI: `10.3982/ECTA13637`.

Wang, Gyu Ho (1998). “Bargaining over a Menu of Wage Contracts”. In: *The Review of Economic Studies* 65.2, pp. 295–305. ISSN: 0034-6527. URL: `https://www.jstor.org/stable/2566974` (visited on 02/18/2019).