

Fuzzy Conventions

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Introduction

- ▶ Social interactions, positive externalities.
 - ▶ maintaining neat front yard,
 - ▶ engaging in criminal activity,
 - ▶ technology adoption.
- ▶ A typical result: emergence of a (homogeneous) convention.
- ▶ But, in reality, conventions are often fuzzy:
 - ▶ there are countries where multiple languages are used,
 - ▶ married couples that use both iPhone and Android.
- ▶ People care not only about their neighbors, but they differ wrt. tastes, preferences.

Introduction

- ▶ Model: binary coordination games on networks with random utility.
- ▶ (Statistical) heterogeneous preferences: i.i.d payoff shocks
- ▶ We are interested in
 - ▶ average (i.e., aggregate) behavior $x \in [0, 1]$ in static (complete information) equilibria,
 - ▶ and each agent no. of connection grows.
- ▶ **Question:** What can we say about limit equilibria? How do they depend on the network?

Introduction

- ▶ 4 theorems
- ▶ The largest set:
 - ▶ there exists a set $X = [0, 1]$ st. for each network, the equilibria belong to X ,
 - ▶ there exists a network st. X is the equilibrium set,
 - ▶ partial identification theory.
- ▶ The smallest set
 - ▶ there exists $x^* \in X$ such that each network has an equilibrium in x^* ,
 - ▶ there exists a network st. x^* is its only equilibrium,
 - ▶ equilibrium selection theory.
- ▶ The largest and smallest sets are easy to determine from the distribution of payoff shocks.

Introduction

Literature

- ▶ Emergence of conventions: evolutionary approach
 - ▶ risk-dominance [Harsanyi and Selten(1988)],
 - ▶ complete networks [Kandori *et al.*(1993)Kandori, Mailath and Rob], [Young(1993)],
 - ▶ line [Ellison(1993)] and some other networks [Ellison(2000)],
 - ▶ all networks: [Peski(2010)].
- ▶ Contagion [Morris(2000a)]:
 - ▶ some networks (lattices) admit contagion: a finite group of agents can spread risk-dominant behavior to the rest of the network,
 - ▶ contagion only works for risk-dominant actions.
- ▶ Here,
 - ▶ random utility instead of noise (or a perturbation),
 - ▶ static solution concept.

Model

Game $G(g, u, \varepsilon)$

- ▶ Network $g_{ij} = g_{ji}$
 - ▶ $g_i = \sum_j g_{ij}$ - “degree” of i .
- ▶ Agent i 's payoffs: $\sum_{j \neq i} g_{ij} u_i(a_i, a_j, \varepsilon_i)$,
 - ▶ binary actions $a_i \in \{0, 1\}$,
 - ▶ i.i.d. payoff shocks $\varepsilon_i \sim F$,
 - ▶ positive externalities (given each payoff shock),
- ▶ Average behavior:

$$\text{Av}(a) = \frac{1}{\sum_i g_i} \sum_i g_i a_i.$$

- ▶ Equilibrium set:

$$\text{Eq}(g, \varepsilon) = \{\text{Av}(a) : a \text{ is a Nash equilibrium in game } G(g, u, \varepsilon)\}$$

.

Model

Average behavior

- ▶ Average behavior: $Av(a) = \frac{1}{\sum_i g_i} \sum_i g_i a_i$.
 - ▶ average action “per interaction”,
 - ▶ alternative $Av_{alt}(a) = \frac{1}{N} \sum_i a_i$,
 - ▶ if network is balanced ($g_i = g_j$, the two are the same),
 - ▶ results extend, with one exception (in which case, I don't know).

Model

Asymptotics

- ▶ We are interested in asymptotics of $\text{Eq}(g, \cdot)$ as
 - ▶ $d(g) = \max_{i,j} \frac{g_{ij}}{g_i} \rightarrow 0$ - large degree,
 - ▶ $w(g) = \max_{i,j} \frac{g_i}{g_j} < w_{\max} < \infty$ is bounded - not too much inequality.

Model

Almost inclusion and almost equality

- ▶ Say that $A \subseteq_{\eta} B$ if for each $a \in A$, there is $b \in B$ st.
 $|a - b| \leq \eta$.
- ▶ Say that $A =_{\eta} B$ if $A \subseteq_{\eta} B$ and $B \subseteq_{\eta} A$.

Model

Definitions

- ▶ Define a profile of neighborhood fractions β^a : for each i

$$\beta_i^a = \frac{1}{g_i} \sum_{j \neq i} g_{ij} a_j.$$

- ▶ a is a Nash equilibrium given payoff shocks $\varepsilon = (\varepsilon_i)$ if for each i ,

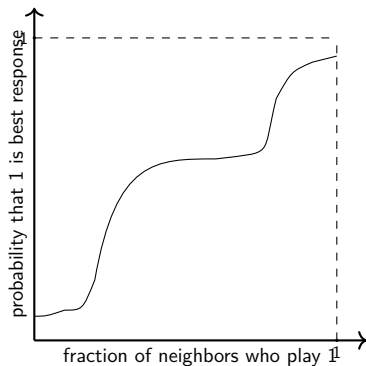
$$u_i(a_i, \beta_i^a, \varepsilon) \geq u_i(1 - a_i, \beta_i^a, \varepsilon).$$

Model

- ▶ Let $P(x) = F\{\varepsilon : u(1, x, \varepsilon) \geq u(0, x, \varepsilon)\}$,
 - ▶ probability that 1 is a best response if x fraction of neighbors plays 1.
- ▶ P is increasing, $P(x) \in [0, 1]$

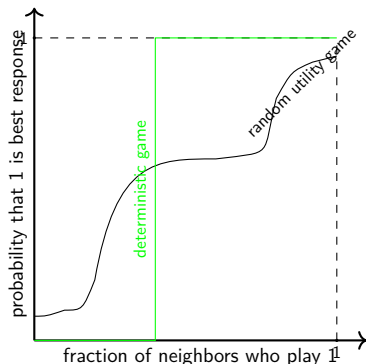
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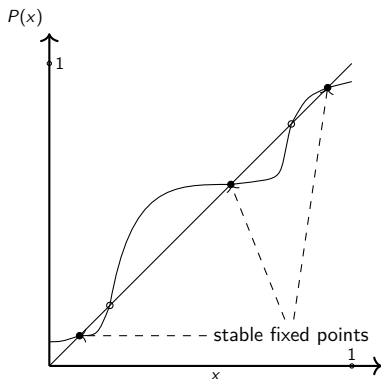
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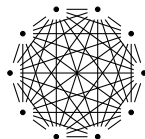
Model

- ▶ Let $P(x) = F\{\varepsilon : u(1, x, \varepsilon) \geq u(0, x, \varepsilon)\}$,
- ▶ Fixed points of P correspond to equilibria in the continuum model.



Complete graph

- ▶ Let g_{complete}^n be the complete graph with n nodes



Theorem

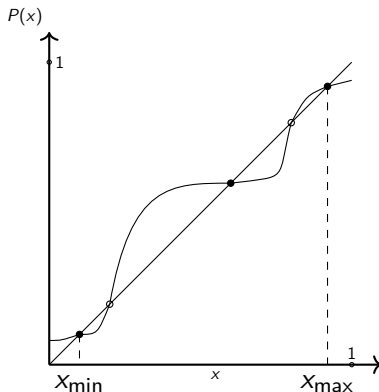
If x is a stable fixed point of P , then, for each $\eta > 0$,

$$\lim_{n \rightarrow \infty} \text{Prob} \left(\{x\} \subseteq_{\eta} \text{Eq} \left(g_{\text{complete}}^n, \varepsilon \right) \right) \geq 1 - \eta.$$

- ▶ *very simple proof.*

Complete graph

- Generically, x_{\min} and x_{\max} - the smallest and the largest fixed points - are stable.



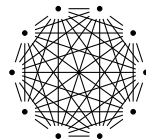
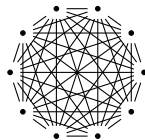
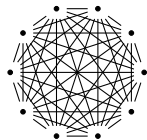
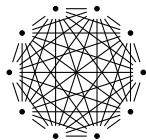
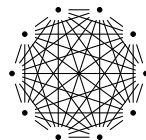
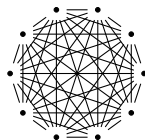
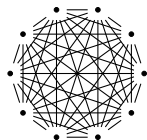
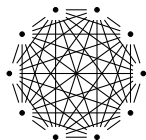
- For large n , both x_{\min} and x_{\max} are close to some equilibrium behaviors.

Complete graph

Corollary

There exists a sequence of graphs g^n such that

$$\lim_{n \rightarrow \infty} \text{Prob}([x_{\min}, x_{\max}] \subseteq_{\eta} \text{Eq}(g^n, \varepsilon)) \geq 1 - \eta.$$



Largest equilibrium set

- ▶ So far, we showed existence of networks g such that $[x_{\min}, x_{\max}] \subseteq_{\eta} \text{Eq}(g, \varepsilon)$ with a large probability.
- ▶ Next, we show that $\text{Eq}(g, \varepsilon) \subseteq_{\eta} [x_{\min}, x_{\max}]$ for any g st. $d(g) = \max_{i,j} \frac{g_{ij}}{g_i}$ is sufficiently small.

Largest equilibrium set

Theorem

For any $w_{\max} < \infty$, any sequence of graphs g_n , if $d(g_n) \rightarrow 0$ and $w(g_n) \leq w_{\max}$, then

$$\lim_{n \rightarrow \infty} \text{Prob}(Eq(g^n, \varepsilon) \subseteq_{\eta} [x_{\min}, x_{\max}]) = 1.$$

Largest equilibrium set

- ▶ Proof: surprisingly complicated.
- ▶ W.l.o.g., we want to show that, with a large probability, there is no profile a st $Av(a) > x_{\max}$ and a is an equilibrium.

Largest equilibrium set

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- ▶ W.l.o.g., we want to show that, with a large probability, there is no profile a st $Av(a) > x_{\max}$ and a is an equilibrium.
- ▶ Bound

$$\begin{aligned} & \text{Prob}(\text{there exists } a \text{ st. } Av(a) \geq x \text{ and } a \text{ is equilibrium}) \\ & \leq \# \{a : Av(a) > x\} \cdot \text{Prob}(a \text{ is equilibrium}). \end{aligned}$$

Largest equilibrium set

- ▶ Proof: surprisingly complicated.
- ▶ W.l.o.g., we want to show that, with a large probability, there is no profile a st $Av(a) > x_{\max}$ and a is an equilibrium.
- ▶ it is easy to show that a is unlikely to be an equilibrium: there exists $\delta > 0$ st. for each a ,

$$\text{Prob}(a \text{ is equilibrium}) \leq \exp(-\delta N).$$

- ▶ But, there are many profiles a :

$$\#\{a : Av(a) > x\} \sim \exp((x \log x + (1-x) \log(1-x)) N).$$

Largest equilibrium set

- ▶ Proof: surprisingly complicated.
- ▶ W.l.o.g., we want to show that, with a large probability, there is no profile a st $Av(a) > x_{\max}$ and a is an equilibrium.
- ▶ Problem: there are too many candidate profiles a . /pause
- ▶ Observation: the above proof treats events “ a is equilibrium” for all a s as disjoint, whereas they are often correlated. /pause
- ▶ Observation II: events “ a is equilibrium” and “ a' is equilibrium” are correlated more if β^a and $\beta^{a'}$ are similar. /pause
- ▶ Idea: divide all profiles a into “groups” with similar β^a .

Largest equilibrium set

- ▶ The correlation is stronger if $\beta^a \sim \beta^{a'}$, where β^a is a profile of “neighborhood fractions $\beta_i^a = \frac{1}{g_i} \sum_{j \neq i} g_{ij} a_j$), or

$$d(\beta_i^a, \beta_i^{a'}) = \sqrt{\frac{1}{\sum g_i^2} \sum g_i^2 (\beta_i^a - \beta_i^{a'})^2} \text{ is small.}$$

- ▶ We show that for each a_0 st. $\text{Av}(a_0) > x$, if δ is sufficiently small and $d(g) \leq \delta$, then

$$\text{Prob}(\{a : d(\beta^a, \beta^{a_0}) \leq \delta\} \text{ contains an equilibrium}) \leq \exp(-\delta N).$$

Largest equilibrium set

- ▶ Set of “neighborhood fraction” profiles

$$\mathcal{B} = \{\beta^a : a \text{ is a profile}\}.$$

- ▶ Metric entropy: $\mathcal{N}(\mathcal{B}, \delta)$ is the smallest n such that there exists $b_1, \dots, b_n \in \mathcal{B}$ st. \mathcal{B} can be covered with balls radius δ and centers at b_i .
- ▶ Bound on metric entropy: for some constant $c > 0$,

$$\mathcal{N}(\mathcal{B}, \delta) \leq \exp\left(c \frac{1}{\delta^2} d(g) N\right).$$

Largest equilibrium set

$$\begin{aligned} & \text{Prob}(\{a : d(\beta^a, \beta) \leq \delta\} \text{ contains an equilibrium}) \\ & \leq \mathcal{N}(\mathcal{B}, \delta) \cdot \sup_{a_0: \text{Av}(a_0) > x} \text{Prob}(\{a : d(\beta^a, \beta^{a_0}) \leq \delta\} \text{ contains an equilibrium}) \\ & \leq \exp\left(c \frac{1}{\delta^2} d(g) N - \delta N\right), \end{aligned}$$

which is small if $d(g)$ is small enough.

RU-dominance

Random utility dominant outcome

- ▶ So far, we characterized a tight upper bound on the equilibrium set.
- ▶ Next, we turn to a lower bound.

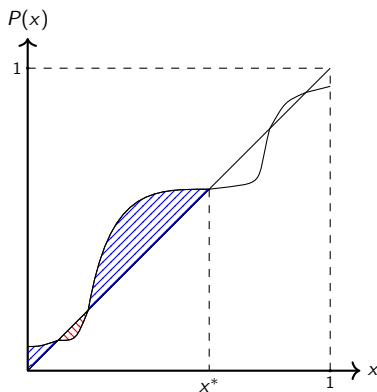
RU-dominance

Random utility dominant outcome

- Define *random utility (RU-) dominant* outcome

$$x^* \in \arg \max_x \int_0^x \left(y - P^{-1}(y) \right) dy.$$

- RU-outcome is generically a strictly stable fixed point of P .



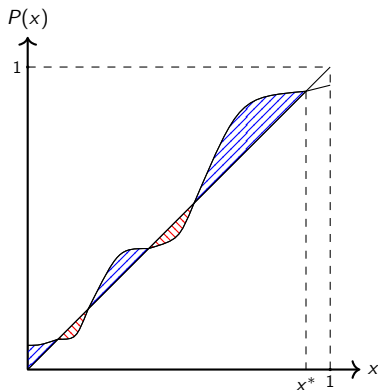
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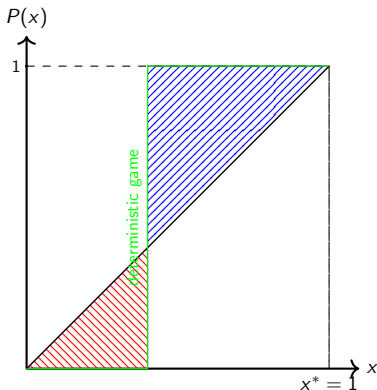
- RU-outcome can be x_{\min} or x_{\max} .



RU-dominance

Random utility dominant outcome

- ▶ When game is deterministic, RU-dominance is equivalent to Harsanyi-Selten risk-dominance



RU-dominance

Random utility dominant selection

Theorem

Suppose that $0 < P(0) < P(1) < 1$. There exists a sequence of networks g^n st. for each $\eta > 0$,

$$\lim_{n \rightarrow \infty} \text{Prob}(Eq(g^n, \varepsilon) =_{\eta} \{x^*\}) \geq 1 - \eta.$$

- ▶ For some networks, x^* is the unique average equilibrium behavior.
- ▶ The assumption ensures that, for each action, there is a positive probability that the action is dominant.

RU-dominance

Proof

- ▶ Network sequence: 2-dimensional lattices
 - ▶ not necessarily 1-dimensional lattice (line)
- ▶ Combination of ideas from
 - ▶ [Ellison(1993)] - deterministic contagion wave on line, and
 - ▶ [Morris(2000b)] - deterministic contagion on lattices, plus
 - ▶ new ideas (RU-dominance, random utility vs deterministic games, initial infectors vs obstacles).

RU-dominance

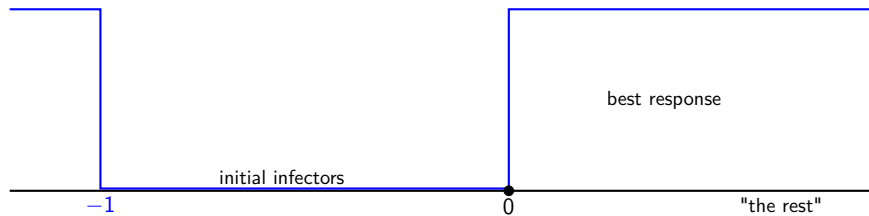
Proof: Contagion wave on line

- ▶ Start with deterministic case, but with small group of initial infectors.
- ▶ Assume 0 is risk-dominant.
- ▶ We want to show that 0 is the only equilibrium.
- ▶ \rightarrow contagion.

RU-dominance

Proof: Contagion wave on line

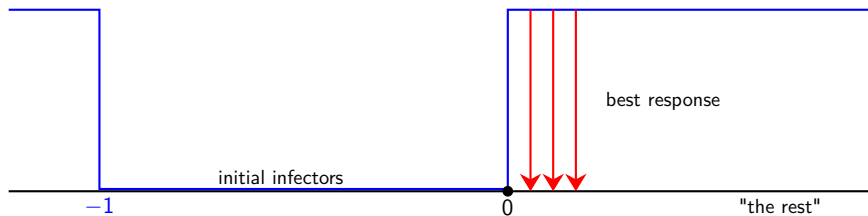
- ▶ [Ellison(1993)]: suppose that action 0 is risk-dominant,
- ▶ initial infectors $-1 \leq i \leq 0$ play 0; the rests play 1,



RU-dominance

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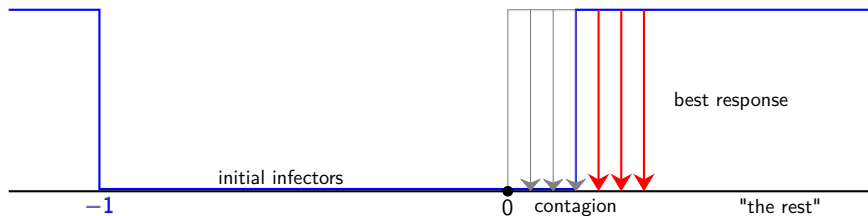
- ▶ [Ellison(1993)]: suppose that action 0 is risk-dominant,
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- ▶ best response dynamics



RU-dominance

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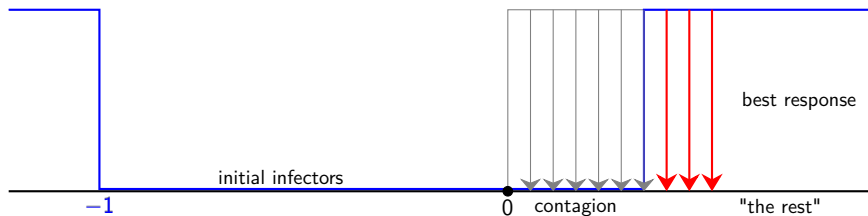
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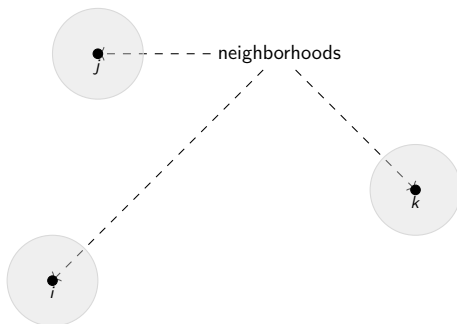
Proof: Contagion wave on lattices

- ▶ [Morris(2000b)] “Contagion” shows that the same mechanics works on other networks, like 2 (or higher)-dimensional lattices.
- ▶ Key step: around $1/2$ of neighbors of “threshold agents” are already infected.

RU-dominance

Proof: Contagion wave on lattices

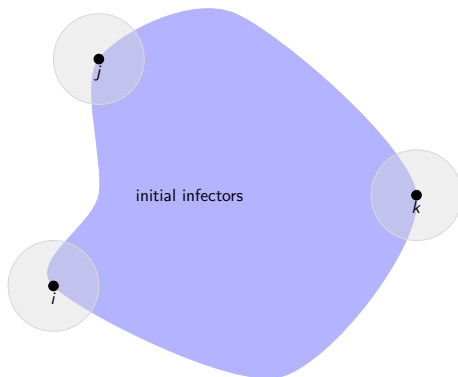
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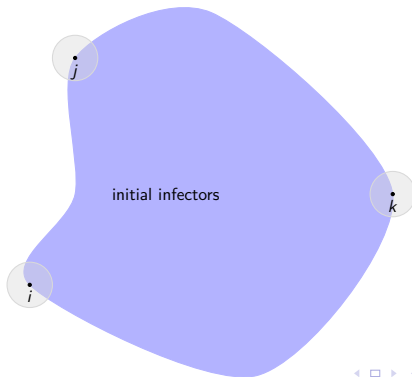
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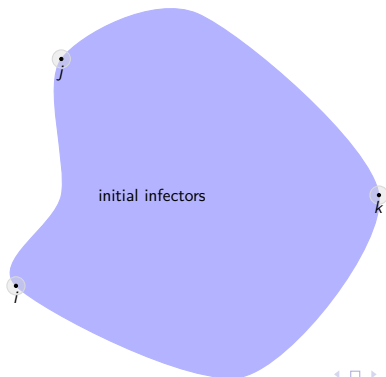
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RU-dominance

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RU-dominance

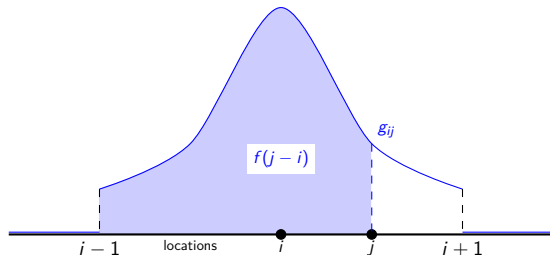
Proof: Contagion wave on line, RU case

- ▶ Intermediate step: Random utility, but with continuum of agents in each location.
 - ▶ except for initial infectors.

RU-dominance

Proof: Contagion wave on line, RU case

- ▶ “Real line” network: agent in location i is connected with agents in location j
- ▶ connection density $g_{ij} = g_{ji} = g_{i+l,j+l}$ for any l ,
 - ▶ $g_{ij} = 0$ for $j > i + 1$,
 - ▶ $f(j - i) = \frac{1}{g_i} \int_{i-1}^j g_{il} dl$.



RU-dominance

Proof: Contagion wave on line, RU case

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 - ▶ $g_{ij} = 0$ for $j > i + 1$,
 - ▶ $f(j - i) = \frac{1}{g_i} \int_{i-1}^j g_{il} dl$,
- ▶ $f(x) + f(-x) = 1$.

RU-dominance

Proof: Contagion wave on line, RU case

- ▶ Assume that $x^* = 0$ is *RU*-dominant, i.e.

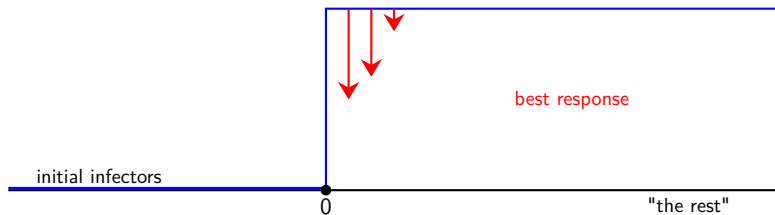
$$\int_0^x (y - P^{-1}(y)) dy < 0 \text{ for each } x > 0.$$

- ▶ With a positive probability, there are “initial infectors” for whom 0 is dominant.
- ▶ Will contagion spread to the whole (or almost whole) network?
- ▶ If yes, then each equilibrium must have average behavior close to 0.

RU-dominance

Proof: Contagion wave on line, RU case

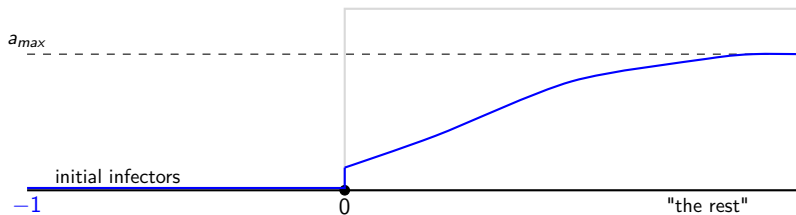
- ▶ Start with a profile like in [Ellison(1993)] on line network.
- ▶ Apart from initial infectors (who all have 0 as dominant action), each other location has “continuum” of agents.



RU-dominance

Proof: Contagion wave on line, RU case

- ▶ If the contagion spreads to the whole line - we are done.
- ▶ Suppose that it stops.



RU-dominance

Proof: Contagion wave on line, RU case

- ▶ If the contagion stops, then at each location $i > 0$,

$$a_i \leq P \left(\int a_{i+k} df(k) \right).$$

- ▶ Taking inverse and integrating by parts

$$P^{-1}(a_i) \leq \int a_{i+k} df(k) = \int_0^{a_{\max}} f(i-j) da_j.$$

- ▶ Integrate over $a_i \in [0, a_{\max}]$,

$$\int_0^{a_{\max}} P^{-1}(a_i) da_i \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j) da_j da_i.$$

RU-dominance

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RU-dominance

Proof: Contagion wave on line, RU case

- Integrate over $a_i \in [0, a_{\max}]$,

$$\begin{aligned} & \int_0^{a_{\max}} P^{-1}(a_i) da_i \\ & \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j) da_j da_i \end{aligned}$$

RU-dominance

Proof: Contagion wave on line, RU case

- Integrate over $a_i \in [0, a_{\max}]$,

$$\begin{aligned} & \int_0^{a_{\max}} P^{-1}(a_i) da_i \\ & \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j) da_j da_i \\ & = \frac{1}{2} \int_0^{a_{\max}} f(i-j) da_j da_i + \frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} f(j-i) da_j da_i \end{aligned}$$

RU-dominance

Proof: Contagion wave on line, RU case

- Integrate over $a_i \in [0, a_{\max}]$,

$$\begin{aligned} & \int_0^{a_{\max}} P^{-1}(a_i) da_i \\ & \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j) da_j da_i \\ & = \frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j) da_j da_i + \frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} f(j-i) da_j da_i \\ & = \frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} [f(i-j) + f(j-i)] da_j da_i \end{aligned}$$

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- Recall that $f(i-j) + f(j-i) = 1$.

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- We get contradiction with $\int_0^{a_{\max}} (y - P^{-1}(y)) dy < 0$.

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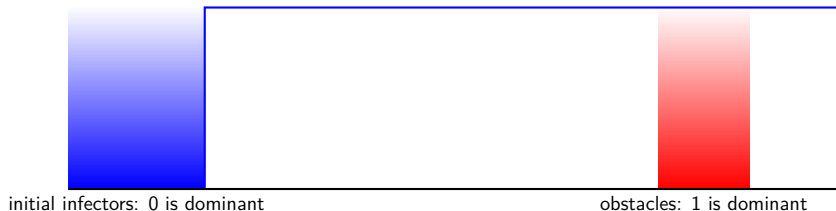
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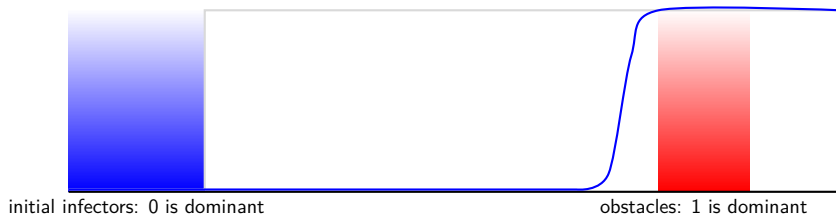
- ▶ Hence the contagion has to spread to the entire line.
- ▶ But! - so far we assumed that locations contain continuum.
- ▶ Contagion can be also stopped by unusual payoff shocks, like those that make 1 dominant.



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Proof: Contagion wave on line, RU case

- ▶ We can compare the relative likelihood of infectors vs obstacles.
- ▶ On line, the latter can be more frequent.
- ▶ But not on 2-dimensional lattices.

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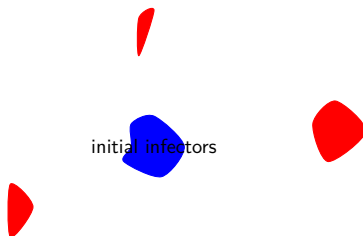
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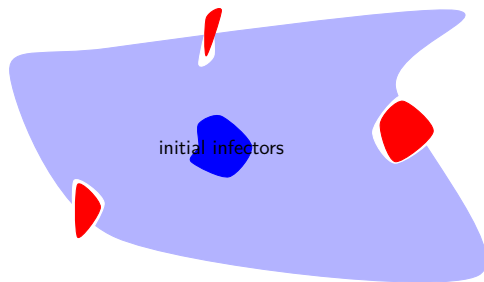
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Smallest equilibrium set

- ▶ So far, we showed that there are networks g such that $\text{Eq}(g, \varepsilon) \subseteq_{\eta} \{x^*\}$ with a large probability.
- ▶ Next, we show that if $d(g) = \max_{i,j} \frac{g_{ij}}{g_i}$ is sufficiently small, then $\{x^*\} \subseteq_n \text{Eq}(g, \varepsilon)$.

Smallest equilibrium set

Theorem

For any sequence of graphs g_n , if $d(g_n) \rightarrow 0$, then

$$\lim_n \text{Prob}(\{x^*\} \subseteq_{\eta} \text{Eq}(g_n)) = 1.$$

Smallest equilibrium set

- ▶ Hence $\{x^*\}$ is the smallest equilibrium set.
- ▶ Equilibrium selection theory: no matter what network, there is an equilibrium with aggregate behavior.

Smallest equilibrium set

- ▶ The proof makes this idea more precise.
- ▶ Suppose that

$$a_i^0 \in \arg \max_a u_i(a, x^*, \varepsilon_i)$$

is pl. i 's best response as if x^* of her neighbors play 1.

- ▶ The proof shows that the best response dynamics starting at a^0 ends (with a large prob.) in an equilibrium, in which very few agents switch their actions.
- ▶ Hence $a_i^0 \in \arg \max_a u_i(a, x^*, \varepsilon_i)$ is a pretty safe action to take, whatever is the true network.

Smallest equilibrium set

- ▶ Start with deterministic case.
- ▶ Suppose that $x^* = 0$ is risk-dominant.
- ▶ Let a^0 be a profile such that almost all, except for initial infectors, play 0.
- ▶ Can a small group of infectors initiate contagion?

Smallest equilibrium set

- ▶ No.
- ▶ Remarkably elegant argument from [Morris(2000a)].
- ▶ For each profile a , let

$$\mathcal{F}_0(a) = \sum_{i,j: a_i=1, a_j=0} g_{ij}$$

be the sum of links between infected and uninfected.

- ▶ Refer to $\mathcal{F}(a)$ as the capacity to infect.

Smallest equilibrium set

- ▶ Whenever i changes action from 0 to 1 as a best response, the capacity increases by

$$\mathcal{F}_0(a') - \mathcal{F}_0(a) = \sum_{j:a_j=0} g_{ij} - \sum_{j:a_j=1} g_{ij}.$$

- ▶ But, if 1 is a best response and 0 is risk-dominant, then it's better be that

$$\sum_{j:a_j=0} g_{ij} < \frac{1}{2} < \sum_{j:a_j=1} g_{ij}.$$

- ▶ So, the capacity goes down every single infection!
- ▶ Because the capacity cannot be negative, contagion has to stop.
- ▶ If the initial profile was close to x^* , the capacity was small and the contagion will stop very soon, close to x^* .

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- ▶ If the initial profile was close to x^* , the capacity was small and the contagion will stop very soon, close to x^* .

Smallest equilibrium set

- ▶ Key feature of a good definition of capacity
 - ▶ it goes down with best responses,
 - ▶ it is small,
 - ▶ cannot be negative.
- ▶ The number of stages until the dynamics stops is related to the initial capacity.

Smallest equilibrium set

- ▶ RU case.
- ▶ Assume that RU -dominant outcome $x^* = P(x^*) = 0$.
- ▶ Let $a_i^0 = \arg \max u_i(a_i, x^*, \varepsilon_i)$ be the initial profile.
- ▶ Definition of capacity: Notice that

$$\sum_{i,j: a_i=1, a_j=0} g_{ij}$$

is not small (there are many players who play 1).



Smallest equilibrium set

- ▶ RU case.
- ▶ Assume that RU -dominant outcome $x^* = P(x^*) = 0$.
- ▶ Definition of capacity: Notice also that

$$\sum_{i,j: a_i=1, a_j=0} g_{ij} = \frac{1}{2} \sum_{i,j} g_{ij} (a_i - a_j)^2.$$

Smallest equilibrium set

- ▶ RU case.
- ▶ Assume that RU -dominant outcome $x^* = P(x^*) = 0$.
- ▶ Definition of capacity: Instead of

$$\frac{1}{2} \sum_{i,j} g_{ij} (a_i - a_j)^2,$$

we take

$$\mathcal{F}(a) = \frac{1}{2} \sum_{i,j} g_{ij} \left(P(\beta_i^a) - P(\beta_j^a) \right)^2.$$

- ▶ recall that β_i^a is the neighborhood fraction, and
- ▶ $P(\beta_i^a)$ is the “expected” best response of agent i .







Smallest equilibrium set

- ▶ Turns out that this is a good definition
 - ▶ capacity is small at a^0 as with a large probability $\beta_i^a \sim \beta_j^a$,
 - ▶ and it is a sum of a martingale and a decreasing process.
Ignoring (probabilistically) small terms, we get, for each T

$$\mathcal{F}(P(\beta^0)) \geq 2 \sum_i g_i \left[\int_{x^*}^{P(\beta_i^T)} (P^{-1}(y) - y) dy \right].$$

Conclusion

- ▶ Heterogeneous payoffs in coordination games on network.
- ▶ We characterized the largest and the smallest possible set of equilibrium average behaviors across all networks.
- ▶ Results:
 - ▶ The largest set achieved on a collection of complete graphs,
 - ▶ partial identification theory,
 - ▶ The smallest set achieved on 2-dimensional (but not necessarily 1-dimensional) lattice,
 - ▶ equilibrium selection theory.
- ▶ Main assumptions:
 - ▶ independent payoff shocks,
 - ▶ large degree,
 - ▶ both assumptions are important.

-  ELLISON, G. (1993). Learning, local interaction, and coordination. *Econometrica*, **61** (5), 1047.
-  — (2000). Basins of attraction, long-run stochastic stability, and the speed of step-by-step evolution. *Review of Economic Studies*, **67** (230), 17.
-  HARSANYI, J. C. and SELTEN, R. (1988). *A general theory of equilibrium selection in games*. Cambridge, Mass. and London: MIT Press.
-  KANDORI, M., MAILATH, G. J. and ROB, R. (1993). Learning, mutation, and long run equilibria in games. *Econometrica*, **61** (1), 29.
-  MORRIS, S. (2000a). Contagion. *Review of Economic Studies*, **67** (1), 57–78.
-  — (2000b). Contagion. *Review of Economic Studies*, **67** (1), 57–78.



PESKI, M. (2010). Generalized risk-dominance and asymmetric dynamics. *Journal of Economic Theory*, **145** (1), 216–248, publisher: Elsevier.



YOUNG, H. P. (1993). The evolution of conventions. *Econometrica*, **61** (1), 57–84.