Non-distortionary belief elicitation

Marcin Pęski   Colin Stewart

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Introduction

We are interested in experiments that

- include belief elicitation:
    - testing belief-dependent models, cognitive uncertainty, self-confidence, information processing,
- elicitation is (or can be) incentivized:
    - incentives improve accuracy (Schlag et al. (2015) and many others),
    - example: binarized Becker-DeGroot-Marschak (BDM) scoring rule,
- cognitive load constraints make elicitation of "all" beliefs difficult, and
- the researcher is interested in *action-dependent* questions.

Introduction

Action-dependent questions:

- IQ test or math test (e.g. Möbius et al. (2022), Zimmermann (2020))
  - What is your rank? How likely are you in the top 50%?
  - How many questions you answered correctly?
- cognitive uncertainty (e.g. Enke and Graeber (2023), Hu (2023))
  - Is your answer within x% of the correct answer?
  - How much would you pay for the experimenter to choose the correct answer (expected regret)?
- field experiment: job training (e.g. Abebe et al. (2020)):
  - What is your expected wage?

Action-independent question:

- beliefs in auctions (e.g. Armantier and Treich (2009)):
  - What is the expected payoff from bid b (not necessarily the chosen bid)?

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A good scoring rule incentivizes reporting true belief *given the action*:

1. A subject is asked a question with answers (a) to (d) and receives a reward of 1 for a correct answer
2. is then asked the probability q that her answer is correct with a reward of $2 - 2(1-q)^2$ if the answer is indeed correct and $2 - 2q^2$ otherwise.

But it may distort incentives to choose the action:

- Suppose she assigns probabilities $(1/2, 1/4, 1/4, 0)$ to the correctness of answers (a), (b), (c), and (d), respectively,
- hence (a) is the payoff maximizing answer, but
- choosing (a) and reporting belief $1/2$ gives total expected payoff $7/4$
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Should we worry about distortions?

- problems with interpretation of experimental results,
- honest instructions (Danz et al. (2022)),
- ethical and legal issues,
- past literature:
  - "moral hazard" in (Chambers and Lambert, 2021),
  - "hedging" (Blanco et al. (2010)),
  - "contamination" (Healy (2024),
  - measurement in (Chassang et al. (2012),

Introduction

Research objective

How to incentivize belief elicitation without distortion?
When can it be done?

Answer

- Sufficient conditions: questions about expected payoffs or "affine" transformations thereof (e.g. expected regret) can be incentivized.
- A complete characterization in three classes of decision problems.

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Outline

1. Introduction
2. Model
3. Sufficient conditions
4. Necessary conditions
5. Representation in special cases
6. Comments and conclusions

Model

- Decision problem:

$$\max_a \sum_\theta p(\theta)u(a,\theta)$$

  - no redundant or dominated actions,
  - unknown state $\theta \in \Theta$,
  - privately known belief $p \in \Delta\Theta$,
  - static problem, no learning, actions do not affect states,
- Benchmark case: experimentalist knows $u(.)$ and observes a and θ.
- DM is asked to report $r = \mathbb{E}_p X(a,.)$ ("linear" belief).
  - (Action-dependent) question $X(a,\theta) \in \mathbb{R}$:

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- (Action-dependent) question $X(a, \theta) \in \mathbb{R}$.
- DM is asked to report $r = \mathbb{E}_p X(a, .)$

Example

1. expected payoffs: $X(a, \theta) = u(a, \theta)$
2. expected regret: $X(a, \theta) = \max_b u(b, \theta) - u(a, \theta)$
3. (*ex post*) correct choice: $X(a, \theta) = \begin{cases} 1 & \text{if } a \in \arg\max_{b \in A} u(b; \theta) \\ 0 & \text{otherwise.} \end{cases}$
4. probability of state θ_0: $X(a, \theta) = \mathbb{1}\{\theta = \theta_0\}$

- Experimenter designs a scoring rule:

$$\max_{a,r} V(a, r, \theta),$$

- For example, subject randomly rewarded either for the decision problem or belief elicitation,

$$V(a, r, \theta) = (1 - \alpha)u(a, \theta) + \alpha V_0(a, r, \theta).$$

- But, only *total payoff* $V(.)$ matters.

Incentivizability

Question X is *incentivizable* if there exists a scoring rule V such that

$$\arg\max_{a,r} \mathbb{E}_p V(a, r, .) = \left\{ (a, \mathbb{E}_p X(a; .)) : a \in \arg\max_{b \in A} \mathbb{E}_p u(b; \cdot) \right\},$$

- strict incentives for reporting beliefs $\mathbb{E}_p X(a; .)$,
- without distorting the behavior in the original problem,
- one question only,
  - with $\min(|\Theta| - 1, |A|)$ questions, everything can be elicited (see also Chen et al. (2026))
- "linear" property of beliefs, $\mathbb{E}_p X(a; .)$ (practical interest, but see also Lambert et al. (2008) and Lambert (2019)).

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Sufficient conditions

Lemma

The following questions are incentivizable: for any $d \in \mathbb{R}^{\Theta}$

- $X(a, \theta) = d(\theta)$,
- $X(a, \theta) = u(a, \theta) + d(\theta)$.

Questions about payoffs plus an action-independent variable can be incentivized.

Sufficient conditions

Assume $X(a,\theta) = u(a,\theta) + d(\theta)$ and w.l.o.g. normalize $0 < X(a,\theta) < 1$.

Becker-DeGroot-Marschak (Becker et al. (1964)):

- subject reports $r \in [0,1]$,
- random number x is drawn uniformly from $[0,1]$,
- if $x \leq r$, the subject receives $X(a,\theta)$,
- otherwise, if $r \leq x$, the subject receives x.

Sufficient conditions

$X(a,\theta)$	◉	○	0
$X(a,\theta)$	◉	○	0.01
$\dots$			$\dots$
$X(a,\theta)$	◉	○	r
$X(a,\theta)$	○	$\nearrow$ ◉	$r + 0.01$
$\dots$			$\dots$
$X(a,\theta)$	○	◉	1

Sufficient conditions

Becker-DeGroot-Marschak:

$$V(r,a,\theta)=\int_0^r X(a;\theta)\,dx+\int_r^1 x dx = X(a;\theta)\,r-\frac{r^2}{2}+\frac{1}{2}$$

- is maximized by $r=\mathbb{E}_p X(a,.)$, and
- the expected optimal payoff

$$\max_r \mathbb{E}_p V(r,a,.)=\frac{1}{2}(\mathbb{E}_p X)^2+\frac{1}{2}=\frac{1}{2}(\mathbb{E}_p u(a,.)+\mathbb{E}_p d)^2+\frac{1}{2}$$

is maximized by $a\in\arg\max\mathbb{E}_p u(a,.)$.

Sufficient conditions

Equivalent questions

Question Y is *equivalent* to X on $B \subseteq A$ if and only if there exist $\gamma, \kappa : B \longrightarrow \mathbb{R}$ and $\gamma(a) \neq 0$ st.

$$Y(a,\theta) = \gamma(a)X(a,\theta) + \kappa(a)$$

Lemma

If X is incentivizable, and Y is equivalent to X, then Y is incentivizable.

- Proof: Use (observed) actions to the invert affine transformation to compute and incentivize X.

Sufficient conditions

Aligned representation

Question X is *aligned* with u on $B \subseteq A$ if and only if it is equivalent on B to

$$u(a,\theta) + d(\theta) \text{ or } \ d(\theta) \text{ for some } d \in \mathbb{R}^{\Theta}$$

- X is aligned (with payoffs u) if it is an "affine transformation" of $u + d$ or d,
- in the latter case, it is action-independent,
- aligned on a subset: the existence of affine transformations required to $B \subseteq A$.
- **Lemma**: Any X that is aligned on the full set of actions A is incentivizable.

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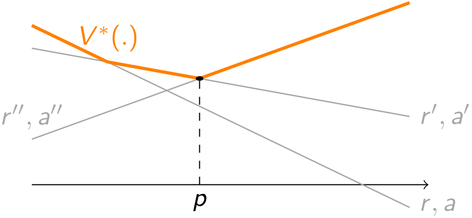
Necessary conditions

Example

- Value of information function:

$$V^*(p) = \max_{a,r} \mathbb{E}_p V(r, a, .)$$

- is convex and
- it is strictly convex at p whenever there are multiple optimal actions



Necessary conditions

Example

- DM chooses $a \in \{x_1, x_2, y, z\}$ to match the state $\theta \in \{x_1, x_2, y, z\}$:

$$u(a, \theta) = 1\{a = \theta\}.$$

- Question "What's the probability that the colors of the action and the state match?"

$$X(a, \theta) = \begin{cases} 1 & a, \theta \in \{x_1, x_2\} \text{ or } a = \theta, \\ 0 & \text{otherwise} \end{cases}$$

- X is not aligned. It is also not incentivizable.

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- Take beliefs $\left(p^\epsilon_{x_1}, p^\epsilon_{x_2}, p^\epsilon_y, p^\epsilon_z\right) = \left(\frac{1}{8} - \epsilon, \frac{3}{8} + \epsilon, \frac{3}{8} + \epsilon, \frac{1}{8} - \epsilon\right)$,
- DM is indifferent between x_2 and y

$$\mathbb{E}_{p_\epsilon} u(x_2, \theta) = \mathbb{E}_{p_\epsilon} u(y, \theta) = \frac{3}{8} + \epsilon$$

- but green prob. is constant and red prob. is changing:

$$r_{x_2} = \mathbb{E}_{p_\epsilon} X(x_2, \theta) = \frac{1}{2} \text{ and}$$
$$r_y = \mathbb{E}_{p_\epsilon} X(y, \theta) = \frac{3}{8} + \epsilon$$

- If V incentivizes X, then V^* must be, at the same time, affine and strictly convex along p_ϵ. Contradiction.

If a and b are best responses at the same belief, and there is no other optimal action, we say that a, b are *adjacent*.

Adjacency Lemma

If X is incentivizable, then X is aligned with u on each pair of adjacent actions $\{a, b\}$.

Observation

Question X is *aligned* with u on $\{a, b\}$

$$\mathrm{X}(a) = \gamma(a)(\mathrm{u}(a) + \mathrm{d}) + \kappa(a),$$
$$\mathrm{X}(b) = \gamma(b)(\mathrm{u}(b) + \mathrm{d}) + \kappa(b),$$

if and only if there is $x \neq 0$ and y such that

$$\bar{\mathrm{X}}(a) = x\bar{\mathrm{X}}(b) + y\Delta_a^b,$$

where

- $\bar{\mathrm{X}}(a, \theta) = \mathrm{X}(a, \theta) - \frac{1}{|\Theta|} \sum_{\theta'} \mathrm{X}(a, \theta')$.
- $\Delta_a^b(\theta) = \bar{\mathrm{u}}(b, \theta) - \bar{\mathrm{u}}(a, \theta)$.

Necessary conditions

Adjacency Lemma

- For any actions a, b such that DM is indifferent between a, b at beliefs p, p'

$$\mathbb{E}_p u(a,\theta) = \mathbb{E}_p u(b,\theta), \mathbb{E}_{p'} u(a,\theta) = \mathbb{E}_{p'} u(b,\theta), \text{ and}$$

the report after a is constant

$$\mathbb{E}_p X(a,\theta) = \mathbb{E}_{p'} X(a,\theta),$$

the report after b must be constant as well:

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- Hence,

$$dp \perp 1, \Delta_a^b, \mathrm{X}(a) \text{ implies } dp \perp \mathrm{X}(b),$$

- $\Rightarrow \bar{\mathrm{X}}(b) \in \mathrm{span}(\bar{\mathrm{X}}(a), \Delta_a^b)$,
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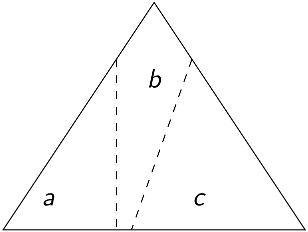
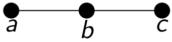
- Alignment on all actions is sufficient for incentivizability.
- Alignment on pairs of adjacent actions is necessary.
- To close the gap between sufficient and necessary conditions, we look at the adjacency graph.





belief symplex $\Delta\Theta$

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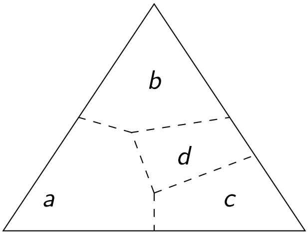
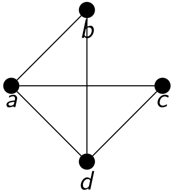
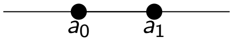




belief symplex $\Delta\Theta$ adjacency graph

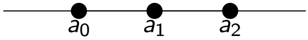
Necessary conditions

Adjacency paths



- For adjacent a_0, a_1, there exist $x_1 \neq 0, y_1$ st.

$$\bar{X}(a_0) = x_1 \bar{X}(a_1) + y_1 \Delta_{\mathbf{a_0}}^{\mathbf{a_1}}$$

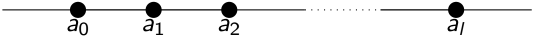


- These conditions carry over through adjacency paths ...

$$\begin{aligned}\bar{X}(a_0) &= x_1\bar{X}(a_1) + y_1\Delta_{a_0}^{a_1} \\ &= x_1x_2\bar{X}(a_2) + x_1y_2\Delta_{a_1}^{a_2} + y_1\Delta_{a_0}^{a_1}\end{aligned}$$

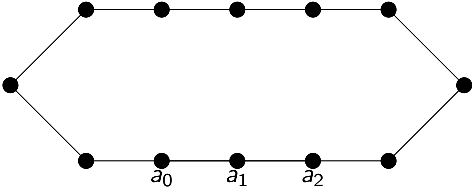
Necessary conditions

Adjacency paths



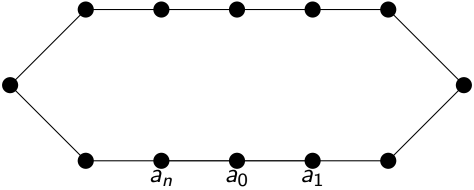
- These conditions carry over through adjacency paths ...

$$\bar{\mathrm{X}}(a_0) = x_1 ... x_l \bar{\mathrm{X}}(a_l) + \sum_{0<i\leq l} x_1 ... x_{i-1} y_i \Delta^{a_i}_{a_{i-1}}$$



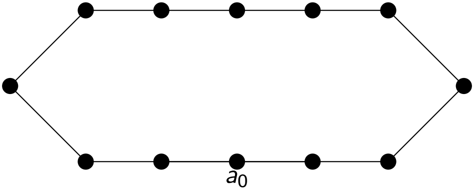
- ... and come back through cycles for $x = x_1 ... x_n$:

$$\bar{X}(a_0) = x_1 ... x_{n+1} \bar{X}(a_0) + \sum_{i=1,..,n+1} x_1 ... x_{i-1} y_i \Delta^{a_i}_{a_{i-1}}$$



Adjacency on Cycles Lemma

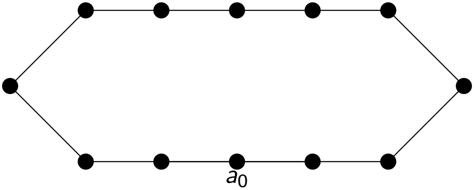
Suppose $C = (a_0, ..., a_n)$ is a cycle such that vectors $\Delta_{a_0}^{a_1}, ..., \Delta_{a_0}^{a_n}$ are linearly independent.
Then, if X is incentivizable, then it is either aligned on C, or $\bar{X}(a_0) \in \text{span}\{\Delta_{a_0}^{a_1}, ..., \Delta_{a_0}^{a_n}\}$.



- Because $\Delta_{a_0}^{a_1} + ... + \Delta_{a_n}^{a_0} = 0$, we have

$$(1 - x_1 ... x_{n+1})\bar{X}(a_0) = \sum_{i>0} (x_1 ... x_i y_{i+1} - y_1)\Delta_{a_i}^{a_{i+1}}$$

- If $\bar{X}(a_0) \in \text{span}\{\Delta_{a_0}^{a_1}, ..., \Delta_{a_0}^{a_n}\}$, all the bracketed terms are 0 due to the linear independence.



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- If $\bar{X}(a_0) \notin \text{span}\{\Delta_{a_0}^{a_1}, ..., \Delta_{a_0}^{a_n}\}$,

$$x_1 x_2 ... x_{n+1} = 1 \text{ and } x_1 ... x_{i-1} y_i = y_1 \text{ for each } i,$$

- Substitution yields

$$\begin{aligned}
\bar{X}(a_0) &= x_1 ... x_l \bar{X}(a_l) + \sum_{0<i\leq l} x_1 ... x_{i-1} y_i \Delta_{a_{i-1}}^{a_i} \\
&= x_1 ... x_l \bar{X}(a_l) + y_1 \sum_{0<i\leq l} \Delta_{a_{i-1}}^{a_i} \\
&= x_1 ... x_l \bar{X}(a_l) + y_1 [\bar{u}(a_l) - \bar{u}(a_0)]
\end{aligned}$$

or, after some algebra,

$$\bar{X}(a_l) = -y_{l+1} \left(\bar{u}(a_l) - \left[\frac{1}{y_1} \bar{X}(a_l) + \bar{u}(a_0) \right] \right)$$

which is the aligned representation.

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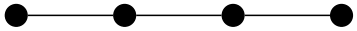
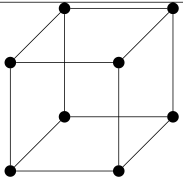
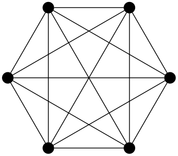
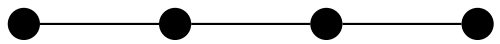
Necessary conditions

	Tree	**Product problem**	**Complete graph**
Adjacency graph			
Examples	monotone problems, cognitive uncertainty (Enke and Graeber, 2023),	random problem selection (Azrieli et al., 2018), test with ≥ 3 questions	multiple choice question, prediction problems
Necessary and sufficient conditions	aligned on each pair of adjacent actions	product-aligned	aligned

Special cases: Tree



Theorem: Incentivizability on tree-like problems

Suppose that the adjacency graph is a tree.

Then, X is incentivizable if and only if it satisfies the Adjacency Lemma for each adjacent pair.

- Proof: pasting together scoring rules over two disjoint set connected by a single adjacent pair.
- Example: monotone decision problems

Special cases: Tree

Enke and Graeber (2023))

- DM chooses an estimate of some objective probability $A = \Theta = \{0, 1/n, ..., 1\}$ incentivized with

$$u(a,\theta) = -(a-\theta)^2.$$

- What is the probability that the *ex post* correct CE is within ϵ of the chosen CE:

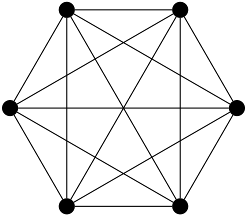
$$X(a,\theta) = \begin{cases} 1 & |a-\theta| < \epsilon \\ 0 & \text{otherwise.} \end{cases}$$

  - a measure of cognitive confidence, not incentivized
- Adjacency Lemma => X is not incentivizable for sufficiently large n.
- But, we could incentivize a question about expected payoffs/regret instead:

$$X'(a,\theta) = 1 - u(a,\theta).$$

Special cases: Complete graph

Decision problems with complete graph:



Example: multiple-choice question.

Special cases: Complete graph

Theorem: Incentivizability on complete graphs

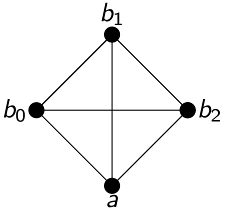
Suppose that $|A| \geq 4$, the adjacency graph is a complete, and for all actions a, b_0, b_1, b_2, vectors $\Delta_a^{b_0}, \Delta_a^{b_1}, \Delta_a^{b_2}$ are linearly independent.

Then, X is incentivizable if and only if it has aligned representation.

- Complete graphs have lots of cycles. Proof

Complete adjacency graph

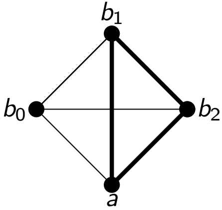
Proof



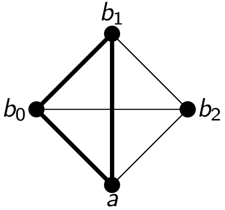
- Fix a st. $\bar{X}(a) \neq 0$ and consider 3-cycles.
- Suppose vectors $\Delta_a^{b_0}, \Delta_a^{b_1}, \Delta_a^{b_2}$ are linearly independent.

Complete adjacency graph

Proof



- Consider 3-cycles that contain action a.



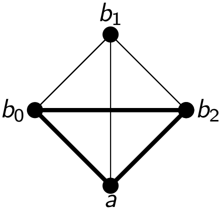
- The intersection

$$\operatorname{span}\{\Delta_a^{b_0}, \Delta_a^{b_1}\} \cap \operatorname{span}\{\Delta_a^{b_0}, \Delta_a^{b_2}\} \cap \operatorname{span}\{\Delta_a^{b_1}, \Delta_a^{b_2}\}$$

is equal to $\{0\}$ due to the linear independence assumption.

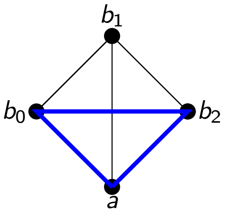
Complete adjacency graph

Proof



- $\bar{\mathrm{X}}(a)$ cannot belong to all of them.

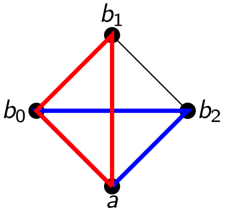
Complete adjacency graph
Proof



- So, there must be a cycle that contains a and that has aligned representation.

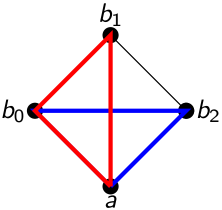
Complete adjacency graph

Proof



- We can apply the same argument to any other action, including b_1.
- But, the two "alignments" do not have to be the same.

Complete adjacency graph
Proof



- We can apply the same argument to any other action, including b_1.
- But, the two "alignments" do not have to be the same.

Lemma 2 (merging representations)

Suppose X is aligned on B and C and $a, b \in B$, $a \neq b$ are such that $\bar{\mathsf{X}}(a)$ and $\bar{\mathsf{X}}(b)$ are not collinear.
Then, X is aligned on $B \cup C$.

Special cases: product problems

- $\Theta = \times_i \Theta_i$, $A = \times_i A_i$, where $(\Theta_i, A_i, u_i(.,.))$ is a collection of tasks, and
$$u(a,\theta) = \sum_i u_i(a_i, \theta_i),$$
- Example: Random problem selection, true-false test
- Two actions $a, b \in A$ are adjacent if they differ in exactly one task: $a_{-i} = b_{-i}$ for some i

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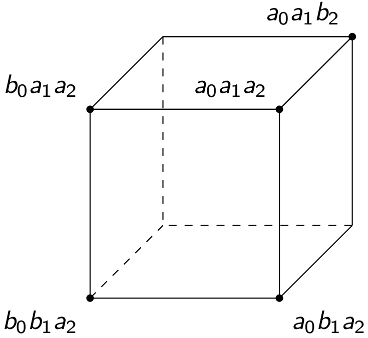

Special cases: Product problems

Product-aligned representation

Question X is product aligned if there are parameters $\gamma(a), y_i, \kappa(a) \in \mathbb{R}$ and $d \in \mathbb{R}^{\Theta}$ such that for each a

$$X(a, \theta) = \quad \left(\sum_i \quad u_i(a_i, \theta) \qquad \right)$$

Special cases: Product problems

Product-aligned representation

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- weaker condition than *aligned*, as y_is do not have to be the same or all positive.

Special cases: Product problems

Product-aligned representation

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$$X(a,\theta) = \quad \left(\sum_i y_i u_i(a_i,\theta) + d(\theta)\right)$$

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Special cases: Product problems

Product-aligned representation

Question X is product aligned if there are parameters $\gamma(a), y_i, \kappa(a) \in \mathbb{R}$ and $d \in \mathbb{R}^\Theta$ such that for each a

$$X(a,\theta) = \gamma(a)\Big(\sum_i y_i u_i(a_i,\theta) + d(\theta)\Big) + \kappa(a)$$

- weaker condition than *aligned*, as y_is do not have to be the same or all positive.

Special cases: Product problems

- X depends on task i *trivially* if, for each a_{-i}, the vectors $\{\bar{X}(a_i a_{-i}) : a_i \in A_i\}$ are collinear.

Theorem: Incentivizability on in product games

Suppose that

1. each task i is either
   - binary ($|A_i| = 2$), or
   - it has complete graph and vectors $\{\Delta_{a_i}^{b_i}, \Delta_{a_i}^{c_i}\}$ are linearly independent for all $a_i, b_i, c_i \in A_i$.
2. X depends non-trivially on at least 3 problems

Then, X is incentivizable iff it is product-aligned.

Special cases: product problems

Möbius et al. (2022)

- A subject writes IQ test with multiple-choice questions.
- The payoff is proportional to score: number of correct answers.

- "How likely your score is above 50%?" corresponds to

$$X(a,\theta) = \begin{cases} 1 & \sum_i \mathbb{1}\{a_i = \theta_i\} \geq \frac{1}{2}N \\ 0 & \text{otherwise} \end{cases}.$$

  - a measure of self-confidence.
- This question is NOT incentivizable.
  - Suppose that the test has $N = 2$ questions,
  - both True with probability 51%,
  - - negatively correlated: different answers with prob. 98%.

Special cases: product problems

Möbius et al. (2022)

- A subject writes IQ test with multiple-choice questions.
- The payoff is proportional to score: number of correct answers.

- But, to elicit self-confidence, one can always ask about expected payoffs.
- Also, "What is the difference between the two parts of the test?" corresponds to

$$X(a,\theta) = \sum_{i \leq \frac{1}{2}N} \mathbb{1}\{a_i = \theta_i\} - \sum_{i > \frac{1}{2}N} \mathbb{1}\{a_i = \theta_i\}$$

- This question is incentivizable.

Outline

1. Introduction
2. Model
3. Sufficient conditions
4. Necessary conditions
5. Representation in special cases
6. Comments and conclusions

Joint incentivizability

Questions $\mathrm{X}, \mathrm{Y} : A \longrightarrow \mathbb{R}^{\Theta}$ are *jointly incentivizable* if there exists $V : \mathbb{R}^2 \times A \times \Theta \longrightarrow [0,1]$ st. for every $p \in \Delta(\Theta)$,

$$\begin{aligned} &\arg\max_{a,r,s} \mathbb{E}_p V\left(r, s, a, \theta\right) \\ = &\left\{\left(a, \mathbb{E}_p X\left(a;\theta\right), \mathbb{E}_p Y\left(a;\theta\right)\right) : a \in \arg\max_{b \in A} \mathbb{E}_p u\left(b;\cdot\right)\right\}. \end{aligned}$$

Comments

Multiple questions

Adjacency Lemma for 2 questions

Suppose that X and Y are jointly incentivizable. If actions a and b are adjacent, then there are ρ_X, ρ_Y and σ_x^y for $x, y = X, Y$, not all equal to 0, such that

$$\begin{aligned} \bar{\mathrm{X}}(b) &= \rho_X\left(\bar{\mathrm{u}}(b) - \bar{\mathrm{u}}(a)\right) + \sigma_X^X \bar{\mathrm{X}}(a) + \sigma_X^Y \bar{\mathrm{Y}}(a) \\ \text{and} \qquad \bar{\mathrm{Y}}(b) &= \rho_Y\left(\bar{\mathrm{u}}(b) - \bar{\mathrm{u}}(a)\right) + \sigma_Y^X \bar{\mathrm{X}}(a) + \sigma_Y^Y \bar{\mathrm{Y}}(a). \end{aligned}$$

- For any k, questions $X^1, ..., X^k$ are jointly incentivizable, if .... .

Lemma

All systems of $|\Theta| - 1$ questions are jointly incentivizable.

- With $|\Theta| - 1$, we can ask about all beliefs.

- Our techniques only apply to linear questions.
- Lambert (2019) studies elicitation of “properties” of beliefs, where a property corresponds to a discrete or continuum partition of the simplex
- A simple necessary condition: elicitable property must have "convex inverse images".
- Example: variance is (action-independent) non-incentivizable.

Conclusions

- Sufficient conditions: Aligned questions (i.e., questions about affine transformations of payoffs) are incentivizable.
- Necessary conditions: Adjacency Lemma.
- **"Theorem"** In three classes of decision problems, question X is incentivizable if and only if it satisfies the Adjacency Lemma.
- Special representations when the adjacency graph is complete, it's a tree, or in product problems.
- Other questions:
  - dynamic elicitation, learning,
  - action-dependent states ("moral hazard"),
  - "robust" elicitation.