Non-distortionary belief elicitation

Marcin Pęski   Colin Stewart

October 31, 2025

Introduction

We are interested in experiments that

- include belief elicitation:
  - testing belief-dependent models, cognitive uncertainty, self-confidence, information processing,
- elicitation is (or can be) incentivized:
  - incentives improve accuracy (Schlag et al. (2015) and many others),
  - example: binarized Becker-DeGroot-Marschak (BDM) scoring rule (Hossain and Okui (2013)),
  - incentivization does not need to be explicit (Danz et al. (2022)),
- cognitive load constraints make elicitation of "all" beliefs difficult, and
- the researcher is interested in *action-dependent* questions.

Introduction

Action-dependent questions:

- field experiment: job training (e.g. Abebe et al. (2020)):
  - What is your expected wage?
- IQ test or math test (e.g. Möbius et al. (2022), Zimmermann (2020))
  - What is your rank? How likely are you in the top 50%?
  - How many questions you answered correctly?
- cognitive uncertainty (e.g. Enke and Graeber (2023), Hu (2023))
  - Is your answer within x% of the correct answer?
  - How much would you pay for the experimenter to choose the correct answer (expected regret)?

Action-independent question:

- beliefs in auctions (e.g. Armantier and Treich (2009)):
  - What is the expected payoff from bid b (not necessarily the chosen bid)?

Introduction

Action-dependent questions:

- field experiment: job training (e.g. Abebe et al. (2020)):
  - What is your expected wage?
- IQ test or math test (e.g. Möbius et al. (2022), Zimmermann (2020))
  - What is your rank? How likely are you in the top 50%?
  - How many questions you answered correctly?
- cognitive uncertainty (e.g. Enke and Graeber (2023), Hu (2023))
  - Is your answer within x% of the correct answer?
  - How much would you pay for the experimenter to choose the correct answer (expected regret)?

Action-independent question:

- beliefs in auctions (e.g. Armantier and Treich (2009)):
  - What is the expected payoff from bid b (not necessarily the chosen bid)?

Introduction

A good scoring rule incentivizes reporting true belief *given the action*. Consider a a subject who

1. is asked a multiple choice question with answers (a) to (d) and receives a reward of 1 for a correct answer
2. is then asked the probability q that her answer is correct with a reward of $2 - 2(1-q)^2$ if the answer is indeed correct and $2 - 2q^2$ otherwise.

But it may distort incentives to choose the action:

- Suppose she assigns probabilities $(1/2, 1/4, 1/4, 0)$ to the correctness of answers (a), (b), (c), and (d), respectively,
- hence (a) is the payoff maximizing answer, but
- choosing (a) and reporting belief $1/2$ gives total expected payoff $7/4$
- choosing (d) and reporting belief 0 gives total expected payoff 2.

Introduction

A good scoring rule incentivizes reporting true belief *given the action*. Consider a a subject who

1. is asked a multiple choice question with answers (a) to (d) and receives a reward of 1 for a correct answer
2. is then asked the probability q that her answer is correct with a reward of $2 - 2(1 - q)^2$ if the answer is indeed correct and $2 - 2q^2$ otherwise.

But it may distort incentives to choose the action:

- Suppose she assigns probabilities $(1/2, 1/4, 1/4, 0)$ to the correctness of answers (a), (b), (c), and (d), respectively,
- hence (a) is the payoff maximizing answer, but
- choosing (a) and reporting belief $1/2$ gives total expected payoff $7/4$
- choosing (d) and reporting belief 0 gives total expected payoff 2.

Introduction

Incentivizing elicitation of action-dependent beliefs may distort behavior:

- issues with interpretation, instructions, field experiments,
- "moral hazard" in (Chambers and Lambert, 2021) and less related "hedging" (Blanco et al. (2010)) or "contamination" (Healy (2024)

Questions

How to incentivize belief elicitation without distortion?
When can it be done?

Answer

Only questions about expected payoffs or "affine" transformations thereof (e.g. expected regret) can be incentivized.

Introduction

Questions

How to incentivize belief elicitation without distortion?
When can it be done?

Answer

Only questions about expected payoffs or "affine" transformations thereof (e.g. expected regret) can be incentivized.

Outline

Model

- Decision problem:

$$\max_a \sum_\theta p(\theta)u(a,\theta)$$

  - no redundant or dominated actions,
  - unknown state $\theta \in \Theta$,
  - privately known belief $p \in \Delta\Theta$.
- Benchmark case (but not limited too) : Experimentalist knows θ and $u(.)$.
- (Action-dependent) question $X(a,\theta) \in \mathbb{R}$:
  - DM is asked to report $r = \mathbb{E}_p X(a,.)$ ("linear" belief).

Model

- Decision problem:

$$\max_a \sum_\theta p(\theta)u(a,\theta)$$

  - no redundant or dominated actions,
  - unknown state $\theta \in \Theta$,
  - privately known belief $p \in \Delta\Theta$.
- Benchmark case (but not limited too) : Experimentalist knows θ and $u(.)$.
- (Action-dependent) question $X(a,\theta) \in \mathbb{R}$:
  - DM is asked to report $r = \mathbb{E}_p X(a,.)$ ("linear" belief).

Example

1. expected payoffs: $X(a,\theta) = u(a,\theta)$
2. expected regret: $X(a,\theta) = \max_b u(b,\theta) - u(a,\theta)$
3. (*ex post*) correct choice: $X(a,\theta) = \begin{cases} 1 & \text{if } a \in \arg\max_{b \in A} u(b;\theta) \\ 0 & \text{otherwise.} \end{cases}$
4. probability of state θ_0: $X(a,\theta) = \mathbb{1}\{\theta = \theta_0\}$

- Incentivization through scoring rule:

$$\max_{a,r} V(a, r, \theta),$$

- where, for example, subject randomly rewarded either for the decision problem or belief elicitation,

$$V(a, r, \theta) = (1 - \alpha)u(a, \theta) + \alpha V_0(a, r, \theta).$$

- But, only *total payoff* $V(.)$ matters.

Incentivizability

Question X is *incentivizable* if there exists a scoring rule V such that

$$\arg\max_{a,r} \mathbb{E}_p V(a, r, .) = \left\{ (a, \mathbb{E}_p X(a; .)) : a \in \arg\max_{b \in A} \mathbb{E}_p u(b; \cdot) \right\},$$

- strict incentives for reporting beliefs $\mathbb{E}_p X(a; .)$,
- without distorting the behavior in the original problem,
- one question only,
- "linear" property of beliefs, $\mathbb{E}_p X(a; .)$ (practical interest, but see also Lambert et al. (2008) and Lambert (2019)).

Outline

Sufficient conditions

Lemma

The following questions are incentivizable:

- $X(a, \theta) = d(\theta)$ *for any* $d \in \mathbb{R}$,
- $X(a, \theta) = u(a, \theta) + d(\theta)$ *for any* $d \in \mathbb{R}$,
- ...

Questions about payoffs plus an action-independent variable can be incentivized.

Sufficient conditions

Assume $X(a,\theta) = u(a,\theta) + d(\theta)$ and w.l.o.g. normalize $0 < X(a,\theta) < 1$.

Becker-DeGroot-Marschak (Becker et al. (1964)):

- subject reports $r \in [0,1]$,
- random number x is drawn uniformly from $[0,1]$,
- if $x \leq r$, the subject receives $X(a,\theta)$,
- otherwise, if $r \leq x$, the subject receives x.

Sufficient conditions

$X(a,\theta)$	◉	○	0
$X(a,\theta)$	◉	○	0.01
...			...
$X(a,\theta)$	◉	○	r
$X(a,\theta)$	○	◉	$r+0.01$
...			...
$X(a,\theta)$	○	◉	1

Sufficient conditions

Becker-DeGroot-Marschak:

$$V(r,a,\theta) = \int_0^r X(a;\theta)\,dx + \int_r^1 x dx = X(a;\theta)\,r - \frac{r^2}{2} + \frac{1}{2}$$

- is maximized by $r = \mathbb{E}_p X(a,.)$, and
- the expected optimal payoff

$$\max_r \mathbb{E}_p V(r,a,.) = \frac{1}{2}(\mathbb{E}_p X)^2 + \frac{1}{2} = \frac{1}{2}(\mathbb{E}_p u(a,.) + \mathbb{E}_p d)^2 + \frac{1}{2}$$

is maximized by $a \in \arg\max \mathbb{E}_p u(a,.)$.

Sufficient conditions

Lemma

For any question X, any $\gamma, \kappa : A \longrightarrow \mathbb{R}$, let $Y(a,\theta) = \gamma(a)X(a,\theta) + \kappa(a)$. If X is incentivizable, then Y is incentivizable.

Affine transformations of incentivizable questions can be incentivized.

Proof.

Take $V_Y(a, r, \theta) = V_X(a, \frac{1}{\gamma(a)}(r - \kappa(a)), \theta)$. $\square$

Sufficient conditions

Aligned representation

Question X is *aligned* with u on $B \subseteq A$ if and only if there are $\gamma, \kappa : B \longrightarrow \mathbb{R}$, and $d \in \mathbb{R}^{\Theta}$ such that for each $a \in B$

$$X(a,\theta) = \gamma(a)\left(u(a,\theta) + d(\theta)\right) + \kappa(a), \text{ or}$$
$$X(a,\theta) = \gamma(a) d(\theta) + \kappa(a)$$

- X is aligned (with payoffs u) if it is an "affine transformation" of u, with "degrees of freedom" γ, κ, and d,
- in the second case, we say that X is *trivial*,
- aligned on a subset $B \subseteq A$.

Corollary

Any X that is aligned on A is incentivizable.

Aligned representation

Question X is *aligned* with u on $B \subseteq A$ if and only if there are $\gamma, \kappa : B \longrightarrow \mathbb{R}$, and $d \in \mathbb{R}^{\Theta}$ such that for each $a \in B$

$$\begin{aligned} X(a,\theta) =& \gamma(a)\left(u(a,\theta) + d(\theta)\right) + \kappa(a), \text{ or} \\ X(a,\theta) =& \gamma(a)d(\theta) + \kappa(a) \end{aligned}$$

Example

1. ✓ expected payoffs: $X(a,\theta) = u(a,\theta)$
2. ✓ expected regret: $X(a,\theta) = u(a,\theta) - \max_b u(b,\theta)$
3. ✗ (*ex post*) correct choice: $X(a,\theta) = \begin{cases} 1 & \text{if } a \in \arg\max_{b\in A} u(b;\theta) \\ 0 & \text{otherwise.} \end{cases}$
4. ✓ probability of state θ_0: $X(a,\theta) = \mathbb{1}\{\theta = \theta_0\}$

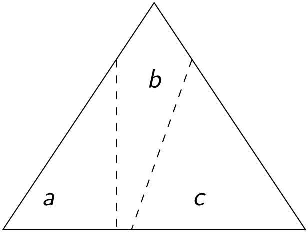
Outline

Representation

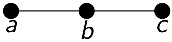
- Alignment on all actions is sufficient for incentivizability.
- We will show that alignment on pairs of adjacent actions is necessary.
  - $a, b \in A$ are *adjacent* if there is a belief $p \in \Delta(\Theta)$ at which a and b are both optimal and there is no other optimal action.
- How to close the gap between sufficient and necessary conditions depends on the adjacency graph:
  - $a, b \in A$ are *adjacent* if there is a belief $p \in \Delta(\Theta)$ at which a and b are both optimal and there is no other optimal action.
- Three classes of decision problems.

Representation

Adjacency graph



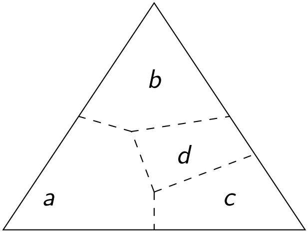
belief symplex $\Delta\Theta$



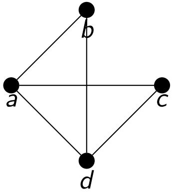
adjacency graph

Representation

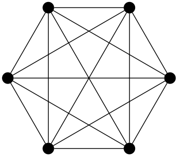
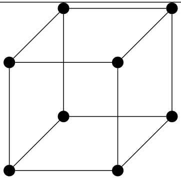
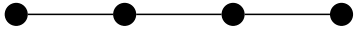
Adjacency graph



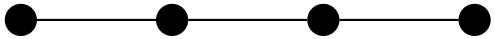
belief symplex $\Delta\Theta$



adjacency graph

	Complete graph	**Product problem**	**Tree**
Adjacency graph			
Examples	multiple choice question, prediction problems	random problem selection (Azrieli et al., 2018), test with ≥ 3 questions	monotone problems, cognitive uncertainty (Enke and Graeber, 2023),
Necessary and sufficient conditions	aligned	product-aligned	aligned on each pair of adjacent actions

Special cases: Tree



Theorem: Incentivizability on tree-like problems

Suppose that the adjacency graph is a tree.

Then, X is incentivizable if and only if it satisfies the Adjacency Lemma for each adjacent pair.

- Proof: scoring rules paste scoring rules over two disjoint set connected by a single adjacent pair.
- Example: monotone decision problems

Special cases: Tree

(inspired by) Enke and Graeber (2023))

- DM chooses certainty equivalent $a^*(p; q, y)$ of a lottery $O^{1-q}1^q$.
- The utility of the lottery is subject to cognitive uncertainty θ.
- Choice is BDM incentivized:

$$\begin{aligned} a^*(p; q, y) =& \arg\max_a \mathbb{E}_p \left[\frac{a}{y} q u_0(y, \theta) + \int_a^y p u_0(z, \theta) dz \right] \\ =& \arg\max_a \mathbb{E}_p u(a, \theta) \end{aligned}$$

- What is the probability that the *ex post* correct CE is within ϵ of the chosen CE:

$$X(a, \theta) = \begin{cases} 1 & |a - a^*(\delta_\theta)| < \epsilon \\ 0 & \text{otherwise.} \end{cases}$$

- Adjacency Lemma => X is not incetivizable for generic u_0.

Special cases: product problems

- $\Theta = \times_i \Theta_i$, $A = \times_i A_i$, where $(\Theta_i, A_i, u_i(.,.))$ is a collection of tasks, and

$$u(a, \theta) = \sum_i u_i(a_i, \theta_i),$$

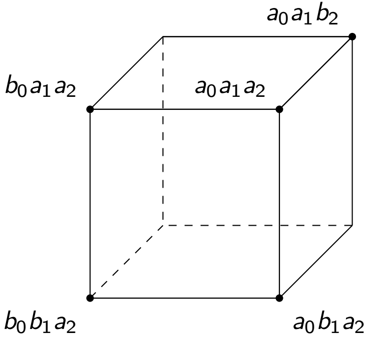
- Example: Random problem selection, true-false test
- Two actions $a, b \in A$ are adjacent if they differ in exactly one task: $a_{-i} = b_{-i}$ for some i

Special cases: product problems

- $\Theta = \times_i \Theta_i$, $A = \times_i A_i$, where $(\Theta_i, A_i, u_i(.,.))$ is a collection of tasks, and

$$u(a,\theta) = \sum_i u_i(a_i, \theta_i),$$

- Example: Random problem selection, true-false test
- Two actions $a, b \in A$ are adjacent if they differ in exactly one task: $a_{-i} = b_{-i}$ for some i



Special cases: product problems

Product-aligned representation

Question X is product aligned if there are parameters $\gamma(a), y_i, \kappa(a) \in \mathbb{R}$ and $d \in \mathbb{R}^\Theta$ such that for each a

$$X(a,\theta) = \qquad \Big(\sum_i \quad u_i(a_i,\theta) \qquad \Big)$$

Special cases: product problems

Product-aligned representation

Question X is product aligned if there are parameters $\gamma(a), y_i, \kappa(a) \in \mathbb{R}$ and $d \in \mathbb{R}^\Theta$ such that for each a

$$X(a,\theta) = \quad \left(\sum_i y_i u_i(a_i,\theta) \qquad \right)$$

- weaker condition than *aligned*, as y_is do not have to be the same or all positive.

Special cases: product problems

Product-aligned representation

Question X is product aligned if there are parameters $\gamma(a), y_i, \kappa(a) \in \mathbb{R}$ and $d \in \mathbb{R}^{\Theta}$ such that for each a

$$X(a,\theta) = \quad \left(\sum_i y_i u_i(a_i,\theta) + d(\theta)\right)$$

- weaker condition than *aligned*, as y_is do not have to be the same or all positive.

Special cases: product problems

Product-aligned representation

Question X is product aligned if there are parameters $\gamma(a), y_i, \kappa(a) \in \mathbb{R}$ and $d \in \mathbb{R}^{\Theta}$ such that for each a

$$X(a, \theta) = \gamma(a)\Big(\sum_i y_i u_i(a_i, \theta) + d(\theta)\Big) + \kappa(a)$$

- weaker condition than *aligned*, as y_is do not have to be the same or all positive.

Special cases: product problems

- X depends on task i *trivially* if, for each a_{-i}, the vectors $\{\bar{X}(a_i a_{-i}) : a_i \in A_i\}$ are collinear.

Theorem: Incentivizability on in product games

Suppose that

1. each task i is either
   - binary ($|A_i| = 2$), or
   - it has complete graph and vectors $\{\Delta_{a_i}^{b_i}, \Delta_{a_i}^{c_i}\}$ are linearly independent for all $a_i, b_i, c_i \in A_i$.
2. X depends non-trivially on at least 3 problems

Then, X is incentivizable iff it is product-aligned.

Special cases: product problems

Möbius et al. (2022)

- A subject writes IQ test with multiple-choice questions.
- The payoff is proportional to score: number of correct answers.

- "What is the difference between the two parts of the test?" corresponds to

$$X(a,\theta) = \sum_{i \leq \frac{1}{2}N} \mathbb{1}\{a_i = \theta_i\} - \sum_{i > \frac{1}{2}N} \mathbb{1}\{a_i = \theta_i\}$$

- This question is incentivizable.

Special cases: product problems

Möbius et al. (2022)

- A subject writes IQ test with multiple-choice questions.
- The payoff is proportional to score: number of correct answers.

- "How likely your score is above 50%?" corresponds to

$$x(a,\theta) = \begin{cases} 1 & \sum_i \mathbb{1}\{a_i = \theta_i\} \geq \frac{1}{2}N \\ 0 & \text{otherwise} \end{cases}.$$

- This question is NOT incentivizable.

Outline

Necessary conditions

- Example with failure of incentivizability
- Adjacency Lemma
- Adjacency on cycles
- Necessary (and sufficient) conditions on complete graph
- Comments

Example

- DM chooses $a \in \{x_1, x_2, y, z\}$ to match the state $\theta \in \{x_1, x_2, y, z\}$:

$$u(a, \theta) = 1\{a = \theta\}.$$

- Question "What's the probability that the colors of the action and the state match?"

$$X(a, \theta) = \begin{cases} 1 & a, \theta \in \{x_1, x_2\} \text{ or } a = \theta, \\ 0 & \text{otherwise} \end{cases}$$

- X is not aligned. It is also not incentivizable.

Example

- DM chooses $a \in \{x_1, x_2, y, z\}$ to match the state $\theta \in \{x_1, x_2, y, z\}$:

$$u(a, \theta) = 1\{a = \theta\}.$$

- Question "What's the probability that the colors of the action and the state match?"

$$X(a, \theta) = \begin{cases} 1 & a, \theta \in \{x_1, x_2\} \text{ or } a = \theta, \\ 0 & \text{otherwise} \end{cases}$$

- X is not aligned. It is also not incentivizable.

Example

- DM chooses $a \in \{x_1, x_2, y, z\}$ to match the state $\theta \in \{x_1, x_2, y, z\}$:

$$u(a, \theta) = 1\{a = \theta\}.$$

- Question "What's the probability that the colors of the action and the state match?"

$$X(a, \theta) = \begin{cases} 1 & a, \theta \in \{x_1, x_2\} \text{ or } a = \theta, \\ 0 & \text{otherwise} \end{cases}$$

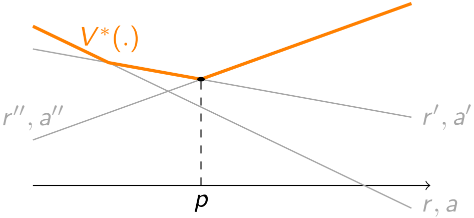
- X is not aligned. It is also not incentivizable.

Example

- Value of information function:

$$V^*(p) = \max_{a,r} \mathbb{E}_p V(r, a, .)$$

- is convex and
- it is strictly convex at p whenever there are multiple optimal actions



Example

- Take beliefs $\left(p^{\epsilon}_{x_1}, p^{\epsilon}_{x_2}, p^{\epsilon}_{y}, p^{\epsilon}_{z}\right) = \left(\frac{1}{8} - \epsilon, \frac{3}{8} + \epsilon, \frac{3}{8} + \epsilon, \frac{1}{8} - \epsilon\right)$,
- DM is indifferent between x_2 and y

$$\mathbb{E}_{p_\epsilon} u(x_2, \theta) = \mathbb{E}_{p_\epsilon} u(y, \theta) = \frac{3}{8} + \epsilon$$

- but green prob. is constant and red prob. is changing:

$$r_{x_2} = \mathbb{E}_{p_\epsilon} X(x_2, \theta) = \frac{1}{2} \text{ and}$$
$$r_y = \mathbb{E}_{p_\epsilon} X(y, \theta) = \frac{3}{8} + \epsilon$$

- If V incentivizes X, then V^* must be, at the same time, affine and strictly convex along p_ϵ. Contradiction.

Necessary conditions

If a and b are best responses at the same belief, and there is no other optimal action, we say that a, b are *adjacent*.

Adjacency Lemma

If X is incentivizable, then X is aligned with u on each pair of adjacent actions $\{a, b\}$.

Question X is *aligned* with u on $\{a, b\}$ if and only if there are $\gamma_a, \gamma_b, \kappa_a, \kappa_b \in \mathbb{R}$, and $d \in \mathbb{R}^{\Theta}$ such that

$$\begin{aligned} X(a, .) =& \gamma_a \left(u(a, .) + d(.)\right) + \kappa_a, \text{ and} \\ X(b, .) =& \gamma_b \left(u(a, .) + d(.)\right) + \kappa_b. \end{aligned}$$

- Let $\bar{\mathrm{X}}(a, \theta) = \mathrm{X}(a, \theta) - \frac{1}{|\Theta|} \sum_{\theta'} \mathrm{X}(a, \theta')$.
- Let $\Delta_a^b(\theta) = \bar{\mathrm{u}}(b, \theta) - \bar{\mathrm{u}}(a, \theta)$.
- X is aligned on $\{a, b\}$ iff for all $a, b \in B$, there is $x \neq 0$ and y such that

$$\bar{\mathrm{X}}(a) = x\bar{\mathrm{X}}(b) + y\Delta_a^b,$$

Necessary conditions

Adjacency Lemma

- For any actions a, b such that DM is indifferent between a, b at beliefs p, p'

$$\mathbb{E}_p u(a,\theta) = \mathbb{E}_p u(b,\theta), \mathbb{E}_{p'} u(a,\theta) = \mathbb{E}_{p'} u(b,\theta), \text{ and}$$

report after a is constant

$$\mathbb{E}_p X(a,\theta) = \mathbb{E}_{p'} X(a,\theta),$$

the report after b must be constant as well:

$$\mathbb{E}_p X(b,\theta) = \mathbb{E}_{p'} X(a,\theta).$$

- Hence,

$$dp \perp 1, \Delta_a^b, \mathrm{X}(a) \text{ implies } dp \perp \mathrm{X}(b),$$

- $\Rightarrow \bar{\mathrm{X}}(b) \in \mathrm{span}(\bar{\mathrm{X}}(a), \Delta_a^b)$,
- $\Rightarrow X$ is aligned on $\{a, b\}$.

Necessary conditions

Adjacency Lemma

- For any actions a, b such that DM is indifferent between a, b at beliefs p, p'

  $$\mathbb{E}_p u(a,\theta) = \mathbb{E}_p u(b,\theta), \mathbb{E}_{p'} u(a,\theta) = \mathbb{E}_{p'} u(b,\theta), \text{ and}$$

  report after a is constant

  $$\mathbb{E}_p X(a,\theta) = \mathbb{E}_{p'} X(a,\theta),$$

  the report after b must be constant as well:

  $$\mathbb{E}_p X(b,\theta) = \mathbb{E}_{p'} X(a,\theta).$$

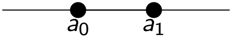
- Hence,

  $$dp \perp 1, \Delta_a^b, \mathrm{X}(a) \text{ implies } dp \perp \mathrm{X}(b),$$

- $\Rightarrow \bar{\mathrm{X}}(b) \in \mathrm{span}(\bar{\mathrm{X}}(a), \Delta_a^b)$,
- $\Rightarrow X$ is aligned on $\{a, b\}$.

Necessary conditions

Adjacency paths

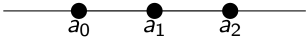


- For adjacent a_0, a_1, there exist $x_1 \neq 0, y_1$ st.

$$\bar{X}(a_0) = x_1 \bar{X}(a_1) + y_1 \Delta_{\mathbf{a_0}}^{\mathbf{a_1}}$$

Necessary conditions

Adjacency paths

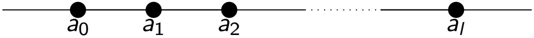


- These conditions carry over through adjacency paths ...

$$\begin{aligned}\bar{X}(a_0) &= x_1\bar{X}(a_1) + y_1\Delta^{a_1}_{a_0} \\ &= x_1x_2\bar{X}(a_2) + x_1y_2\Delta^{a_2}_{a_1} + y_1\Delta^{a_1}_{a_0}\end{aligned}$$

Necessary conditions

Adjacency paths

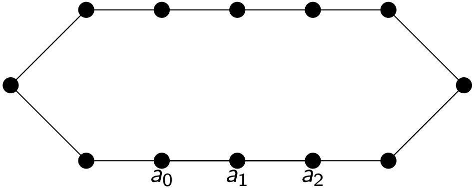


- These conditions carry over through adjacency paths ...

$$\bar{X}(a_0) = x_1 \ldots x_l \bar{X}(a_l) + \sum_{0<i\leq l} x_1 \ldots x_{i-1} y_i \Delta^{a_i}_{a_{i-1}}$$

Necessary conditions

Adjacency cycles

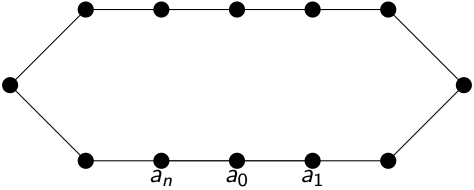


- ... and come back through cycles for $x = x_1 ... x_n$:

$$\bar{X}(a_0) = x_1 ... x_{n+1} \bar{X}(a_0) + \sum_{i=1,..,n+1} x_1 ... x_{i-1} y_i \Delta_{a_{i-1}}^{a_i}$$

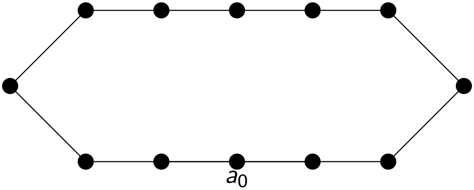
Necessary conditions

Adjacency on cycles



Adjacency on Cycles Lemma

Suppose $C = (a_0, ..., a_n)$ is a cycle such that vectors $\Delta_{\mathbf{a_0}}^{\mathbf{a_1}}, ..., \Delta_{\mathbf{a_0}}^{a_n}$ are linearly independent.
Then, if X is incentivizable, then it is either aligned on C, or $\bar{\mathrm{X}}(a_0) \in \mathrm{span}\{\Delta_{\mathbf{a_0}}^{\mathbf{a_1}}, ..., \Delta_{\mathbf{a_0}}^{a_n}\}$.



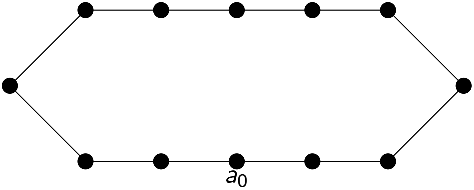
- Because $\Delta^{a_1}_{a_0} + ... + \Delta^{a_0}_{a_n} = 0$, we have

$$(1 - x_1 ... x_{n+1})\bar{\mathrm{X}}(a_0) = \sum_{i>0} (x_1 ... x_i y_{i+1} - y_1)\Delta^{a_{i+1}}_{a_i}$$

- If $\bar{\mathrm{X}}(a_0) \in \mathrm{span}\{\Delta^{a_1}_{a_0}, ..., \Delta^{a_n}_{a_0}\}$, all the bracketed terms are 0 due to the linear independence.

Necessary conditions

Adjacency cycles



- Because $\Delta^{a_1}_{a_0} + ... + \Delta^{a_0}_{a_n} = 0$, we have

$$(1 - x_1 ... x_{n+1})\bar{X}(a_0) = \sum_{i>0} (x_1 ... x_i y_{i+1} - y_1)\Delta^{a_{i+1}}_{a_i}$$

- If $\bar{X}(a_0) \in \text{span}\{\Delta^{a_1}_{a_0}, ..., \Delta^{a_n}_{a_0}\}$, all the bracketed terms are 0 due to the linear independence.

- If $\bar{\mathrm{X}}(a_0) \in \operatorname{span}\{\Delta_{\boldsymbol{a_0}}^{\boldsymbol{a_1}}, ..., \Delta_{\boldsymbol{a_0}}^{a_n}\}$,

$$x_1 x_2 ... x_{n+1} = 1 \text{ and } x_1 ... x_{i-1} y_i = y_1 \text{ for each } i,$$

- Substitution yields

$$\begin{aligned}
\bar{\mathrm{X}}(a_0) &= x_1 ... x_l \bar{\mathrm{X}}(a_l) + \sum_{0<i\leq l} x_1 ... x_{i-1} y_i \Delta_{a_{i-\mathbf{1}}}^{a_i} \\
&= x_1 ... x_l \bar{\mathrm{X}}(a_l) + y_1 \sum_{0<i\leq l} \Delta_{a_{i-\mathbf{1}}}^{a_i} \\
&= x_1 ... x_l \bar{\mathrm{X}}(a_l) + y_1 [\mathrm{u}(\bar{\mathrm{a}}_\mathrm{l}) - \mathrm{u}(\bar{\mathrm{a}}_0)]
\end{aligned}$$

or, after some algebra,

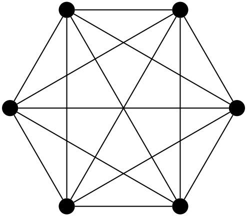
$$\bar{\mathrm{X}}(a_l) = -y_{l+1} \left(\bar{\mathrm{u}}(a_l) - \left[\frac{1}{y_1} \bar{\mathrm{X}}(a_l) + \bar{\mathrm{u}}(a_0) \right] \right)$$

which is the aligned representation.

Outline

Special cases: Complete graph

Decision problems with complete graph:



Example: multiple-choice question.

Special cases: Complete graph

Theorem: Incentivizability on complete graphs

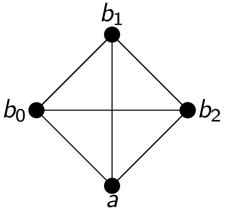
Suppose that $|A| \geq 4$, the adjacency graph is a complete, and for all actions a, b_0, b_1, b_2, vectors $\Delta_a^{b_0}, \Delta_a^{b_1}, \Delta_a^{b_2}$ are linearly independent.

Then, X is incentivizable if and only if it has aligned representation.

- Complete graphs have lots of cycles. Proof

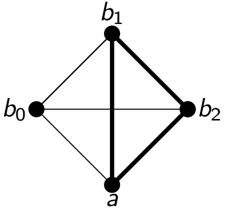
Complete adjacency graph

Proof



- Fix a st. $\bar{X}(a) \neq 0$ and consider 3-cycles.
- Suppose vectors $\Delta_a^{b_0}, \Delta_a^{b_1}, \Delta_a^{b_2}$ are linearly independent.

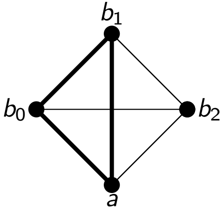
Complete adjacency graph
Proof



- Consider 3-cycles that contain action a.

Complete adjacency graph

Proof



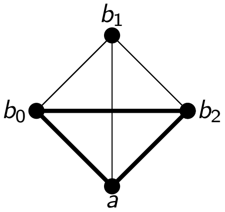
- The intersection

$$\text{span}\{\Delta_a^{b_0}, \Delta_a^{b_1}\} \cap \text{span}\{\Delta_a^{b_0}, \Delta_a^{b_2}\} \cap \text{span}\{\Delta_a^{b_1}, \Delta_a^{b_2}\}$$

is equal to $\{0\}$ due to the linear independence assumption.

Complete adjacency graph

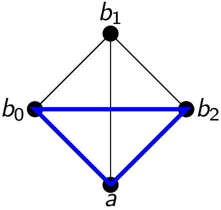
Proof



- $\bar{X}(a)$ cannot belong to all of them.

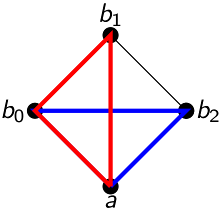
Complete adjacency graph

Proof



- So, there must be a cycle that contains a and that has aligned representation.

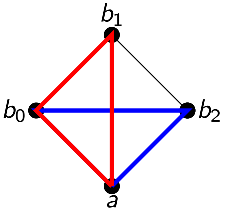
Complete adjacency graph
Proof



- We can apply the same argument to any other action, including b_1.
- But, the two "alignments" do not have to be the same.

Complete adjacency graph

Proof



- We can apply the same argument to any other action, including b_1.
- But, the two "alignments" do not have to be the same.

Lemma 2 (merging representations)

Suppose X is aligned on B and C and $a, b \in B$, $a \neq b$ are such that $\bar{\mathrm{X}}(a)$ and $\bar{\mathrm{X}}(b)$ are not collinear.
Then, X is aligned on $B \cup C$.

Outline

Joint incentivizability

Questions $X, Y : A \longrightarrow \mathbb{R}^{\Theta}$ are *jointly incentivizable* if there exists $V : \mathbb{R}^2 \times A \times \Theta \longrightarrow [0,1]$ st. for every $p \in \Delta(\Theta)$,

$$\begin{aligned} & \arg\max_{a,r,s} \mathbb{E}_p V\left(r, s, a, \theta\right) \\ = & \left\{ \left(a, \mathbb{E}_p X\left(a;\theta\right), \mathbb{E}_p Y\left(a;\theta\right)\right) : a \in \arg\max_{b \in A} \mathbb{E}_p u\left(b;\cdot\right) \right\}. \end{aligned}$$

Adjacency Lemma for 2 questions

Suppose that X and Y are jointly incentivizable. If actions a and b are adjacent, then there are ρ_X, ρ_Y and σ_x^y for $x, y = X, Y$, not all equal to 0, such that

$$\begin{aligned} \bar{\mathrm{X}}(b) &= \rho_X \left(\bar{\mathrm{u}}(b) - \bar{\mathrm{u}}(a)\right) + \sigma_X^X \bar{\mathrm{X}}(a) + \sigma_X^Y \bar{\mathrm{Y}}(a) \\ \text{and} \qquad \bar{\mathrm{Y}}(b) &= \rho_Y \left(\bar{\mathrm{u}}(b) - \bar{\mathrm{u}}(a)\right) + \sigma_Y^X \bar{\mathrm{X}}(a) + \sigma_Y^Y \bar{\mathrm{Y}}(a). \end{aligned}$$

Comments

Multiple questions

- For any k, questions $X^1, ..., X^k$ are jointly incentivizable, if .... .

Lemma

All systems of $|\Theta| - 1$ questions are jointly incentivizable.

- With $|\Theta| - 1$, we can ask about all beliefs.

Comments

Non-linear questions

- Our techniques only apply to linear questions.
- Lambert (2019) studies elicitation of "properties" of beliefs, where a property corresponds to a discrete or continuum partition of the simplex
- A simple necessary condition: elicitable property must have "convex inverse images".
- Example: variance is (action-independent) non-incentivizable.

Conclusions

- Sufficient conditions: Aligned questions (i.e., questions about affine transformations of payoffs) are incentivizable.
- Necessary conditions: Adjacency Lemma.
- **"Informal Theorem"** In three classes of decision problems, question X is incentivizable if and only if it satisfies the Adjacency Lemma.
- Special representations when the adjacency graph is complete, it's a tree, or in product problems.
- Other questions:
  - dynamic elicitation (signals?)
  - "robust" elicitation.