Fuzzy Conventions

Marcin Pęski

University of Toronto

November 3, 2021

Introduction

- Social interactions, positive externalities.
  - maintaining neat front yard,
  - engaging in criminal activity,
  - technology adoption.
- A typical result: emergence of a (homogeneous) convention.
- But, in reality, conventions are often fuzzy:
  - there are countries where multiple languages are used,
  - married couples that use both IPhone and Android.
- People care not only about their neighbors, but they differ wrt. tastes, preferences.

Introduction

- Model: binary coordination games on networks with random utility.
- (Statistical) heterogeneous preferences: i.i.d payoff shocks
- We are interested in
  - the set X of average (i.e., aggregate) behavior $x \in [0, 1]$ in static (complete information) equilibria,
  - when each agent number of connection is large.
- **Question**: What can we say about equilibrium sets? How do they depend on the network?

Introduction

- 4 theorems
- The largest set:
  - there exists a set $X = [0, 1]$ st. for each network, the equilbria belong to X,
  - X is the equilibrium set for some networks
  - partial identification theory.
- The smallest set
  - there exists $x^* \in X$ such that each network has an equilibrium in x^*,
  - x^* is its only equilibrium on some networks,
  - equilibrium selection theory.
- The largest and smallest sets are easy to determine from the distribution of payoff shocks.

- Emergence of conventions: evolutionary approach
  - risk-dominance [Harsanyi and Selten(1988)],
  - complete networks [Kandori *et al.*(1993)Kandori, Mailath and Rob], [Young(1993)],
  - line [Ellison(1993)] and some other networks [Ellison(2000)],
  - all networks: [Peski(2010)].
- Contagion [Morris(2000a)]:
  - some networks (lattices) admit contagion: a finite group of agents can spread risk-dominant behavior to the rest of the network,
  - contagion only works for risk-dominant actions.
- Here,
  - random utility instead of noise (or a perturbation),
  - static solution concept.

Model

Game $G(g, u, \varepsilon)$

- Network with weights $g_{ij} = g_{ji} \geq 0$.
- Agent i's payoffs:

$$\sum_{j \neq i} g_{ij} u_i (a_i, a_j, \varepsilon_i),$$

  - binary actions $a_i \in \{0, 1\}$,
  - i.i.d. payoff shocks $\varepsilon_i \sim F$,
  - positive externalities (given each payoff shock),

- Average behavior:
$$\mathrm{Av}(a) = \frac{1}{\sum_i g_i} \sum_i g_i a_i,$$
where $g_i = \sum_j g_{ij}$ is the "degree" of i.
  - average action "per interaction",
  - alternative $\mathrm{Av}_{\mathrm{alt}}(a) = \frac{1}{N} \sum_i a_i$,
  - if network is balanced ($g_i = g_j$, the two are the same),
  - results extend, with one exception (in which case, I don't know).
- Equilibrium set:
$$\mathrm{Eq}(g, \varepsilon) = \{\mathrm{Av}(a) : a \text{ is a Nash equilibrium in game } G(g, u, \varepsilon)\}.$$

- We are interested in asymptotics of $\mathrm{Eq}(g,.)$ as
  - $d(g) = \max_{i,j} \frac{g_{ij}}{g_i} \to 0$ - large degree,
  - $w(g) = \max_{i,j} \frac{g_i}{g_j} < w_{\max} < \infty$ is bounded - not too much inequality.

Model

Almost inclusion and almost equality

- Say that $A \subseteq_\eta B$ if for each $a \in A$, there is $b \in B$ st. $|a - b| \leq \eta$.
- Say that $A =_\eta B$ if $A \subseteq_\eta B$ and $B \subseteq_\eta A$.

- Define a profile of neighborhood fractions β^a: for each i

$$\beta_i^a = \frac{1}{g_i} \sum_{i \neq j} g_{ij} a_j.$$

- a is a Nash equilibrium given payoff shocks $\varepsilon = (\varepsilon_i)$ if for each i,

$$u_i\left(a_i, \beta_i^a, \varepsilon\right) \geq u_i\left(1 - a_i, \beta_i^a, \varepsilon\right).$$

Model

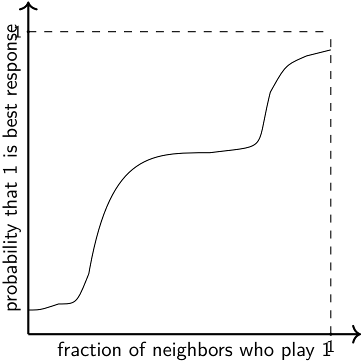
- Define the probability that 1 is a best response if x fraction of neighbors plays 1:

$$P(x) = F\{\varepsilon : u(1, x, \varepsilon) \geq u(0, x, \varepsilon)\}.$$

- P is increasing, $P(x) \in [0, 1]$

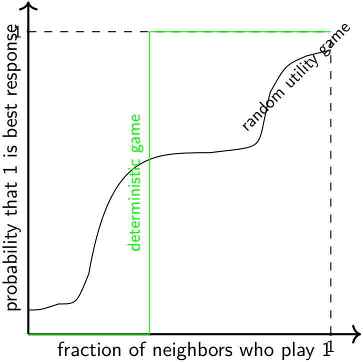
Model

- Let $P(x) = F\{\varepsilon : u(1, x, \varepsilon) \geq u(0, x, \varepsilon)\}$,
- P is increasing, $P(x) \in [0, 1]$.



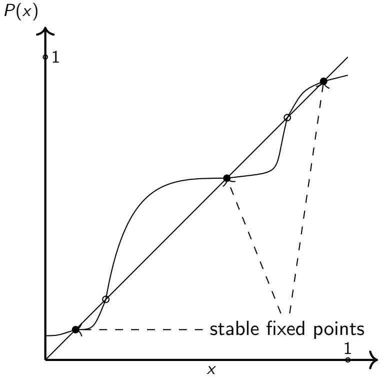
Model

- Let $P(x) = F\{\varepsilon : u(1, x, \varepsilon) \geq u(0, x, \varepsilon)\}$,
- P is increasing, $P(x) \in [0, 1]$.



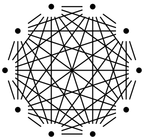
Model

- Let $P(x) = F\{\varepsilon : u(1, x, \varepsilon) \geq u(0, x, \varepsilon)\}$,
- Fixed points of P correspond to equilibria in the continuum model.



Complete graph

- Let g^n_{complete} be the complete graph with n nodes



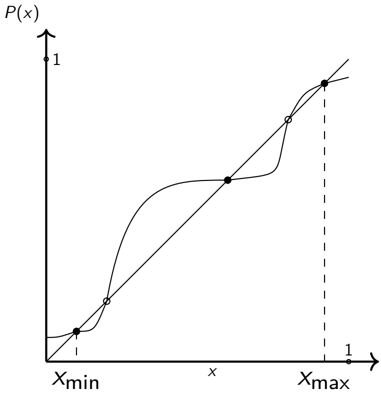
Theorem

If x is a stable fixed point of P, then, for each $\eta > 0$,

$$\lim_{n\to\infty} Prob\left(\{x\} \subseteq_\eta Eq\left(g^n_{complete}, \varepsilon\right)\right) \geq 1 - \eta.$$

- very simple proof.

Complete graph



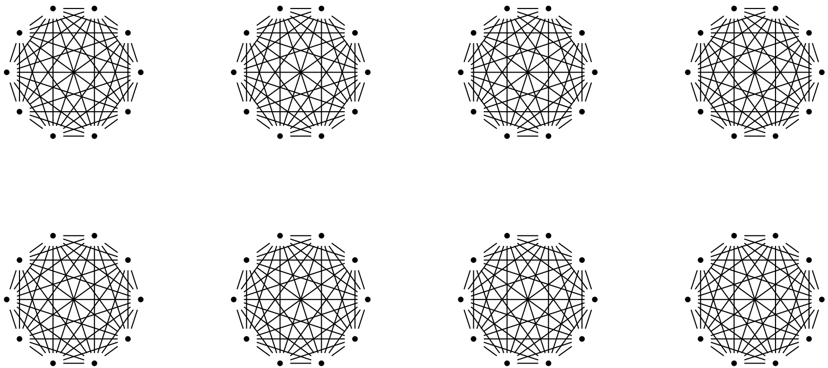
- Generically, x_{min} and x_{max} - the smallest and the largest fixed points - are stable.

Complete graph

Corollary

There exists a sequence of graphs g^n such that

$$\lim_{n\to\infty} Prob\left([x_{\min}, x_{\max}] \subseteq_\eta Eq\left(g^n, \varepsilon\right)\right) \geq 1-\eta.$$



Largest equilibrium set

- So far, we showed existence of networks g such that with a large probability,
$$[x_{\min}, x_{\max}] \subseteq_\eta \mathrm{Eq}\left(g, \varepsilon\right).$$
- Next, we show that, for any g st. $d\left(g\right) = \max_{i,j} \frac{g_{ij}}{g_i}$ is sufficiently small,
$$\mathrm{Eq}\left(g, \varepsilon\right) \subseteq_\eta [x_{\min}, x_{\max}].$$

Largest equilibrium set

Theorem

For any $w_{\max} < \infty$, *any sequence of graphs* g_n, *if* $d(g_n) \to 0$ *and* $w(g_n) \leq w_{\max}$, *then*

$$\lim_{n\to\infty} Prob\left(Eq\left(g^n, \varepsilon\right) \subseteq_\eta \left[x_{\min}, x_{\max}\right]\right) = 1.$$

Largest equilibrium set

- Proof: surprisingly complicated.
- W.l.o.g., we want to show that, with a large probability, there is no profile a st $\mathrm{Av}(a) > x_{\max}$ and a is an equilibrium.

Largest equilibrium set

- Proof: surprisingly complicated.
- W.l.o.g., we want to show that, with a large probability, there is no profile a st $\mathrm{Av}(a) > x_{\max}$ and a is an equilibrium.
- Bound

$$\begin{aligned} &\mathrm{Prob}\left(\text{there exists } a \text{ st.} \mathrm{Av}(a) \geq x \text{ and } a \text{ is equilibrium}\right) \\ \leq &\#\left\{a : \mathrm{Av}(a) > x\right\} \cdot \mathrm{Prob}\left(a \text{ is equilibrium}\right). \end{aligned}$$

Largest equilibrium set

- Proof: surprisingly complicated.
- W.l.o.g., we want to show that, with a large probability, there is no profile a st $\mathrm{Av}(a) > x_{\max}$ and a is an equilibrium.
- It is easy to show that a is unlikely to be an equilibrium: there exists $\delta > 0$ st. for each a,

$$\mathrm{Prob}(a \text{ is equilibrium}) \leq \exp(-\delta N).$$

- But, there are many profiles a:

$$\#\{a : \mathrm{Av}(a) > x\} \sim \exp((x \log x + (1-x)\log(1-x))N).$$

Largest equilibrium set

- Proof: surprisingly complicated.
- W.l.o.g., we want to show that, with a large probability, there is no profile a st $\mathrm{Av}(a) > x_{\max}$ and a is an equilibrium.
- Problem: there are too many candidate profiles a.
- Observation: the above proof treats events "a is equilibrium" for all as as disjoint, whereas they are often correlated.
- Observation II: events "a is equilibrium" and "a' is equilibrium" are correlated more if β^a and $\beta^{a'}$ are similar.
- Idea: divide all profiles a into "groups" with similar β^a.

Largest equilibrium set

- Proof: surprisingly complicated.
- W.l.o.g., we want to show that, with a large probability, there is no profile a st $\mathrm{Av}(a) > x_{\max}$ and a is an equilibrium.
- Problem: there are too many candidate profiles a.
- Observation: the above proof treats events "a is equilibrium" for all as as disjoint, whereas they are often correlated.
- Observation II: events "a is equilibrium" and "a' is equilibrium" are correlated more if β^{a} and $\beta^{a'}$ are similar.
- Idea: divide all profiles a into "groups" with similar β^{a}.

Largest equilibrium set

- Proof: surprisingly complicated.
- W.l.o.g., we want to show that, with a large probability, there is no profile a st $\mathrm{Av}\,(a) > x_{\max}$ and a is an equilibrium.
- Problem: there are too many candidate profiles a.
- Observation: the above proof treats events "a is equilibrium" for all as as disjoint, whereas they are often correlated.
- Observation II: events "a is equilibrium" and "a' is equilibrium" are correlated more if β^a and $\beta^{a'}$ are similar.
- Idea: divide all profiles a into "groups" with similar β^a.

Largest equilibrium set

- Proof: surprisingly complicated.
- W.l.o.g., we want to show that, with a large probability, there is no profile a st $\mathrm{Av}(a) > x_{\max}$ and a is an equilibrium.
- Problem: there are too many candidate profiles a.
- Observation: the above proof treats events "a is equilibrium" for all as as disjoint, whereas they are often correlated.
- Observation II: events "a is equilibrium" and "a' is equilibrium" are correlated more if β^a and $\beta^{a'}$ are similar.
- Idea: divide all profiles a into "groups" with similar β^a.

Largest equilibrium set

- The correlation is stronger if $\beta^a \sim \beta^{a'}$, where β^a is a profile of "neighborhood fractions $\beta_i^a = \frac{1}{g_i}\sum_{j\neq i} g_{ij} a_j$), or

$$d\left(\beta_i^a, \beta_i^{a'}\right) = \sqrt{\frac{1}{\sum g_i^2}\sum g_i^2\left(\beta_i^a - \beta_i^{a'}\right)^2} \text{ is small.}$$

- We show that for each a_0 st. $\mathrm{Av}(a_0) > x$, if δ is sufficiently small and $d(g) \leq \delta$, then

$$\mathrm{Prob}\left(\{a : d(\beta^a, \beta^{a_0}) \leq \delta\} \text{ contains an equilibrium}\right) \leq \exp(-\delta N).$$

Largest equilibrium set

- Set of "neighborhood fraction" profiles

$$\mathcal{B} = \{\beta^a : a \text{ is a profile}\}.$$

- Metric entropy: $\mathcal{N}(\mathcal{B}, \delta)$ is the smallest n such that there exists $b_1, ..., b_n \in \mathcal{B}$ st. $\mathcal{B}$ can be covered with balls radius δ and centers at b_i.
- For some constant $c > 0$,

$$\mathcal{N}(\mathcal{B}, \delta) \leq \exp\left(c\frac{1}{\delta^2}d(g)N\right).$$

Largest equilibrium set

$$
\begin{aligned}
&\operatorname{Prob}\left(\left\{a : d\left(\beta^{a}, \beta\right) \le \delta\right\} \text{ contains an equilibrium}\right) \\
\le &\mathcal{N}\left(\mathcal{B}, \delta\right) \cdot \sup_{a_0 : \operatorname{Av}(a_0) > x} \operatorname{Prob}\left(\left\{a : d\left(\beta^{a}, \beta^{a_0}\right) \le \delta\right\} \text{ contains an equilibrium}\right). \\
\le &\exp\left(c \frac{1}{\delta^2} d\left(g\right) N - \delta N\right),
\end{aligned}
$$

which is small if $d\left(g\right)$ is small enough.

RU-dominance

Random utility dominant outcome

- So far, we characterized a tight upper bound on the equilibrium set.
- Next, we turn to a lower bound.

RU-dominance

Random utility dominant outcome

- Define *random utility (RU-) dominant* outcome

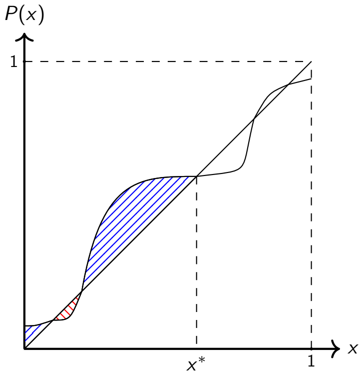
$$x^* \in \arg\max_x \int_0^x \left(y - P^{-1}(y)\right) dy.$$

RU-dominance

Random utility dominant outcome

- Define *random utility (RU-) dominant* outcome

$$x^* \in \arg\max_x \int_0^x \left(y - P^{-1}(y)\right) dy.$$



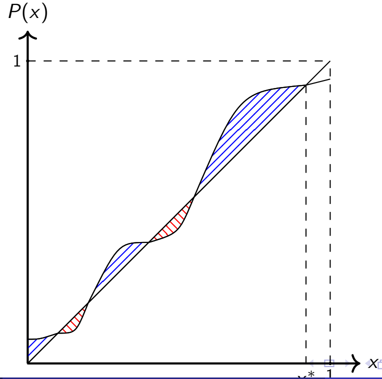
RU-dominance

Random utility dominant outcome

- Define *random utility (RU-) dominant* outcome

$$x^* \in \arg\max_x \int_0^x \left(y - P^{-1}(y)\right) dy.$$

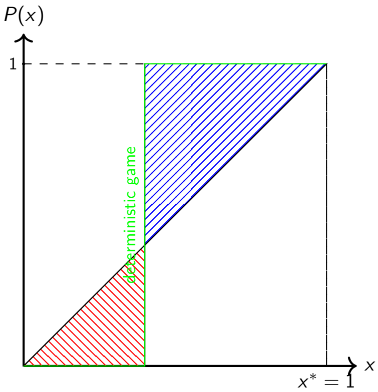
- RU-outcome can be $x_{\min}$ or $x_{\max}$.



RU-dominance

Random utility dominant outcome

- When game is deterministic, RU-dominance is equivalent to Harsanyi-Selten risk-dominance



RU-dominance

Random utility dominant selection

Theorem

Suppose that $0 < P(0) < P(1) < 1$. *There exists a sequence of networks* g^n *st. for each* $\eta > 0$,

$$\lim_{n\to\infty} Prob\left(Eq\left(g^n, \varepsilon\right) =_\eta \{x^*\}\right) \geq 1 - \eta.$$

- For some networks, x^* is the unique average equilibrium behavior.
- The assumption ensures that, for each action, there is a positive probability that the action is dominant.

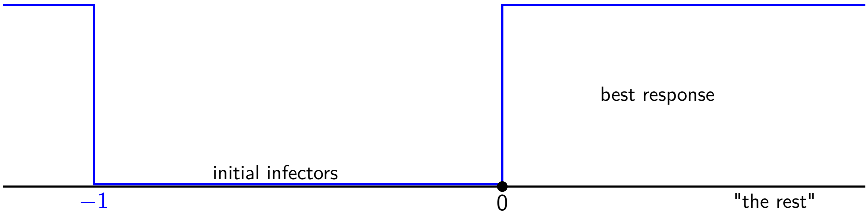
- Network sequence: 2-dimensional lattices
  - not necessarily 1-dimensional lattice (line)
- Combination of ideas from
  - [Ellison(1993)] - deterministic contagion wave on line, and
  - [Morris(2000b)] - deterministic contagion on lattices, plus
  - new ideas (RU-dominance, random utility vs deterministic games, initial infectors vs obstacles).

- Start with deterministic case, but with small group of initial infectors.
- Assume 0 is risk-dominant.
- We want to show that 0 is the only equilibrium.
- -$>$ contagion.

RU-dominance

Proof: Contagion wave on line

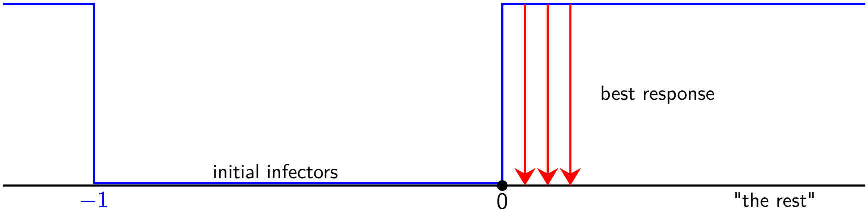
- [Ellison(1993)]: suppose that action 0 is risk-dominant,
- initial infectors $-1 \leq i \leq 0$ play 0; the rests play 1,



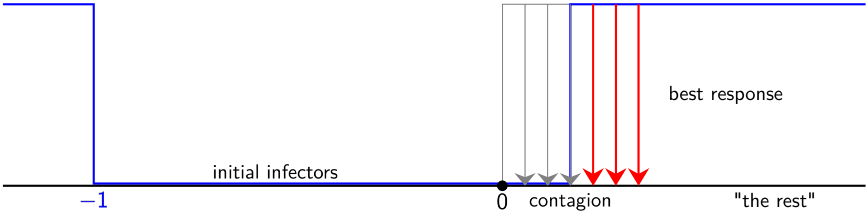
RU-dominance

Proof: Contagion wave on line

- [Ellison(1993)]: suppose that action 0 is risk-dominant,
- initial infectors $-1 \leq i \leq 0$ play 0; the rests play 1,
- best response dynamics



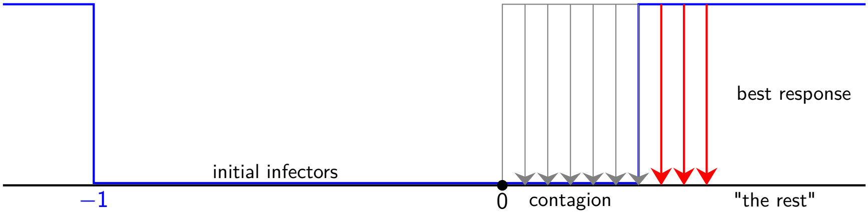
- [Ellison(1993)]: suppose that action 0 is risk-dominant,
- initial infectors $-1 \leq i \leq 0$ play 0; the rests play 1,
- best response dynamics -> contagion



RU-dominance

Proof: Contagion wave on line

- [Ellison(1993)]: suppose that action 0 is risk-dominant,
- initial infectors $-1 \leq i \leq 0$ play 0; the rests play 1,
- best response dynamics -> contagion

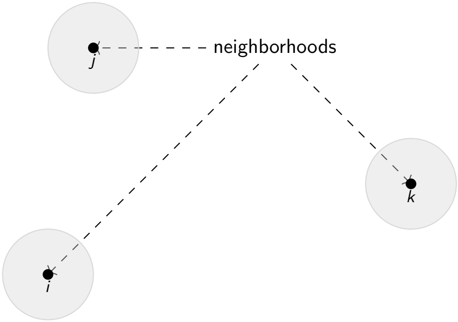


- [Morris(2000b)] "Contagion" shows that the same mechanics works on other networks, like 2 (or higher)-dimensional lattices.
- Key step: around $1/2$ of neighbors of "threshold agents" are already infected.

RU-dominance

Proof: Contagion wave on lattices

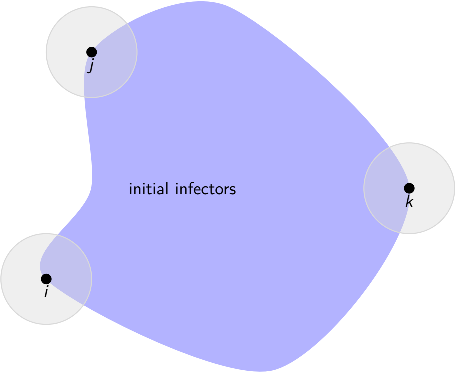
- [Morris(2000b)] "Contagion" shows that the same mechanics works on other networks, like 2 (or higher)-dimensional lattices.
- Key step: around 1/2 of neighbors of "threshold agents" are already infected.



RU-dominance

Proof: Contagion wave on lattices

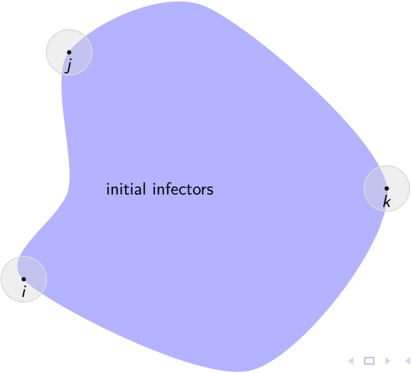
- [Morris(2000b)] "Contagion" shows that the same mechanics works on other networks, like 2 (or higher)-dimensional lattices.
- Key step: around 1/2 of neighbors of "threshold agents" are already infected.



RU-dominance

Proof: Contagion wave on lattices

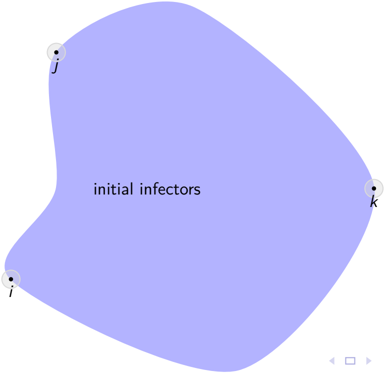
- [Morris(2000b)] "Contagion" shows that the same mechanics works on other networks, like 2 (or higher)-dimensional lattices.
- Key step: around $1/2$ of neighbors of "threshold agents" are already infected - $>$ initial infectors must be large enough relative to neighborhoods.



RU-dominance

Proof: Contagion wave on lattices

- [Morris(2000b)] "Contagion" shows that the same mechanics works on other networks, like 2 (or higher)-dimensional lattices.
- Key step: around $1/2$ of neighbors of "threshold agents" are already infected - $>$ initial infectors must be large enough relative to neighborhoods.



RU-dominance

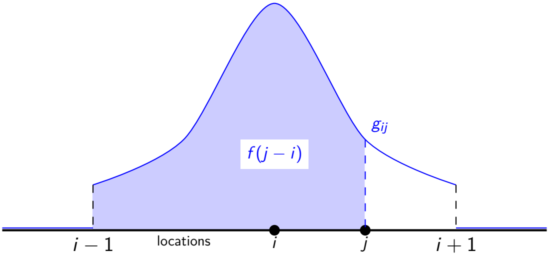
Proof: Contagion wave on line, RU case

- Intermediate step: Random utility, but with continuum of agents in each location.
  - except for initial infectors.

RU-dominance

Proof: Contagion wave on line, RU case

- "Real line" network: agent in location i is connected with agents in location j
- connection density $g_{ij} = g_{ji} = g_{i+l,j+l}$ for any l,
  - $g_{ij} = 0$ for $j > i + 1$,
  - $f\,(j-i) = \frac{1}{g_i} \int_{i-1}^{j} g_{il} dl$.



RU-dominance

Proof: Contagion wave on line, RU case

- "Real line" network: agent in location i is connected with agents in location j
- connection density $g_{ij} = g_{ji} = g_{i+l,j+l}$ for any l,
  - $g_{ij} = 0$ for $j > i + 1$,
  - $f(j-i) = \frac{1}{g_i} \int_{i-1}^{j} g_{il} dl$,
- $f(x) + f(-x) = 1$.

RU-dominance

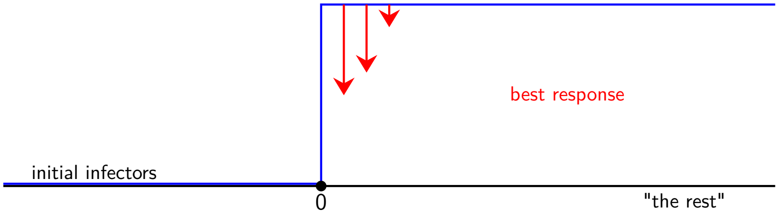
Proof: Contagion wave on line, RU case

- Assume that $x^* = 0$ is *RU*-dominant, i.e.

$$\int_0^x \left(y - P^{-1}(y)\right) dy < 0 \text{ for each } x > 0.$$

- With a positive probability, there are "initial infectors" for whom 0 is dominant.
- Will contagion spread to the whole (or almost whole) network?
- If yes, then each equilibrium must have average behavior close to 0.

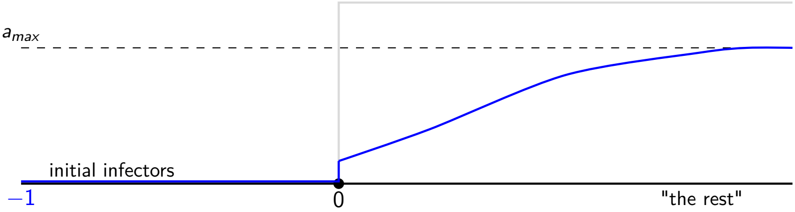
- Start with a profile like in [Ellison(1993)] on line network.
- Apart from initial infectors (who all have 0 as dominant action), each other location has “continuum” of agents.



RU-dominance

Proof: Contagion wave on line, RU case

- If the contagion spreads to the whole line - we are done.
- Suppose that it stops.



RU-dominance

Proof: Contagion wave on line, RU case

- If the contagion stops, then at each location $i > 0$,

$$a_i \leq P\left(\int a_{i+k} df(k)\right).$$

- Taking inverse and integrating by parts

$$P^{-1}(a_i) \leq \int a_{i+k} df(k) = \int_0^{a_{\max}} f(i-j)\, da_j.$$

- Integrate over $a_i \in [0, a_{\max}]$,

$$\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i.$$

RU-dominance

Proof: Contagion wave on line, RU case

- If the contagion stops, then at each location $i > 0$,

$$a_i \leq P\left(\int a_{i+k} df(k)\right).$$

- Taking inverse and integrating by parts

$$P^{-1}(a_i) \leq \int a_{i+k} df(k) = \int_0^{a_{\max}} f(i-j)\, da_j.$$

- Integrate over $a_i \in [0, a_{\max}]$,

$$\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i.$$

RU-dominance

Proof: Contagion wave on line, RU case

- If the contagion stops, then at each location $i > 0$,

$$a_i \leq P\left(\int a_{i+k} df(k)\right).$$

- Taking inverse and integrating by parts

$$P^{-1}(a_i) \leq \int a_{i+k} df(k) = \int_0^{a_{\max}} f(i-j)\, da_j.$$

- Integrate over $a_i \in [0, a_{\max}]$,

$$\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i.$$

- Integrate over $a_i \in [0, a_{\max}]$,

$$\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i$$

- Integrate over $a_i \in [0, a_{\max}]$,

$$\begin{aligned}
&\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \\
\leq &\int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i \\
= &\frac{1}{2}\int_0^{a_{\max}} f(i-j)\, da_j da_i + \frac{1}{2}\int_0^{a_{\max}} \int_0^{a_{\max}} f(j-i)\, da_j da_i
\end{aligned}$$

- Integrate over $a_i \in [0, a_{\max}]$,

$$
\begin{aligned}
&\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \\
\leq &\int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i \\
= &\frac{1}{2}\int_0^{a_{\max}} f(i-j)\, da_j da_i + \frac{1}{2}\int_0^{a_{\max}} \int_0^{a_{\max}} f(j-i)\, da_j da_i \\
= &\frac{1}{2}\int_0^{a_{\max}} \int_0^{a_{\max}} \left[f(i-j) + f(j-i)\right] da_j da_i
\end{aligned}
$$

- Integrate over $a_i \in [0, a_{\max}]$,

$$
\begin{aligned}
&\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \\
\leq &\int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i \\
= &\frac{1}{2}\int_0^{a_{\max}} f(i-j)\, da_j da_i + \frac{1}{2}\int_0^{a_{\max}} \int_0^{a_{\max}} f(j-i)\, da_j da_i \\
= &\frac{1}{2}\int_0^{a_{\max}} \int_0^{a_{\max}} [f(i-j) + f(j-i)]\, da_j da_i
\end{aligned}
$$

- Recall that $f(i-j) + f(j-i) = 1$.

- Integrate over $a_i \in [0, a_{\max}]$,

$$\begin{aligned}\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \leq& \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i \\ =&\frac{1}{2}\int_0^{a_{\max}} f(i-j)\, da_j da_i + \frac{1}{2}\int_0^{a_{\max}} \int_0^{a_{\max}} f(j-i)\, da_j da_i \\ =&\frac{1}{2}\int_0^{a_{\max}} \int_0^{a_{\max}} \left[f(i-j) + f(j-i)\right] da_j da_i \\ =&\frac{1}{2}\int_0^{a_{\max}} \int_0^{a_{\max}} da_j da_i = \int_0^{a_{\max}} a da.\end{aligned}$$

- We get contradiction with $\int_0^{a_{\max}} \left(y - P^{-1}(y)\right) dy < 0$.

- Integrate over $a_i \in [0, a_{\max}]$,

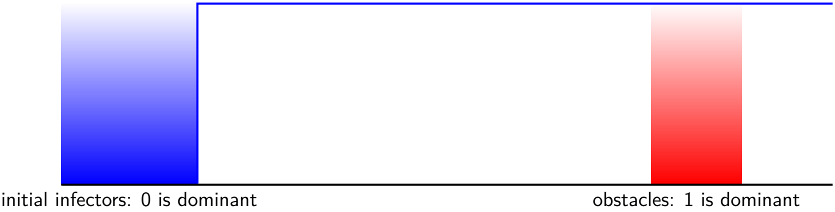
$$\int_0^{a_{\max}} P^{-1}(a_i)\, da_i \leq \int_0^{a_{\max}} \int_0^{a_{\max}} f(i-j)\, da_j da_i$$
$$= \frac{1}{2} \int_0^{a_{\max}} f(i-j)\, da_j da_i + \frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} f(j-i)\, da_j da_i$$
$$= \frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} [f(i-j) + f(j-i)]\, da_j da_i$$
$$= \frac{1}{2} \int_0^{a_{\max}} \int_0^{a_{\max}} da_j da_i = \int_0^{a_{\max}} a da.$$

- We get contradiction with $\int_0^{a_{\max}} \left(y - P^{-1}(y)\right) dy < 0$.

RU-dominance

Proof: Contagion wave on line, RU case

- Hence the contagion has to spread to the entire line.
- But! - so far we assumed that locations contain continuum.
- Contagion can be also stopped by unusual payoff shocks, like those that make 1 dominant.



RU-dominance

Proof: Contagion wave on line, RU case

- Hence the contagion has to spread to the entire line.
- But! - so far we assumed that locations contain continuum.
- Contagion can be also stopped by unusual payoff shocks, like those that make 1 dominant.

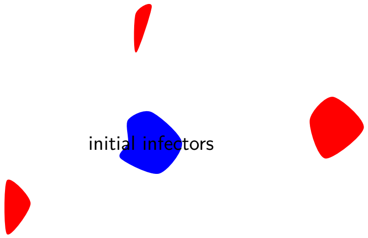
- We can compare the relative likelihood of infectors vs obstacles.
- On line, the latter can be more frequent.
- But not on 2-dimensional lattices.

RU-dominance

Proof: Contagion wave on line, RU case

- We can compare the relative likelihood of infectors vs obstacles.
- On line, the latter can be more frequent.
- But not on 2-dimensional lattices.

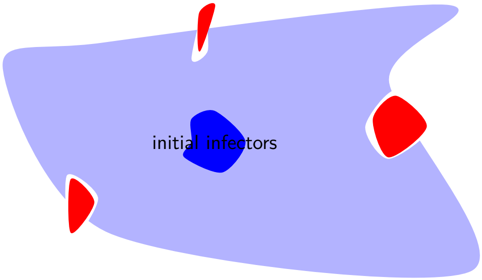
- We can compare the relative likelihood of infectors vs obstacles.
- On line, the latter can be more frequent.
- But not on 2-dimensional lattices.



RU-dominance

Proof: Contagion wave on line, RU case

- We can compare the relative likelihood of infectors vs obstacles.
- On line, the latter can be more frequent.
- But not on 2-dimensional lattices.



Smallest equilibrium set

- So far, we showed that there are networks g such that $\text{Eq}(g, \varepsilon) \subseteq_\eta \{x^*\}$ with a large probability.
- Next, we show that if $d(g) = \max_{i,j} \frac{g_{ij}}{g_i}$ is sufficiently small, than $\{x^*\} \subseteq_n \text{Eq}(g, \varepsilon)$.

Smallest equilibrium set

Theorem

For any sequence of graphs g_n, if $d(g_n) \to 0$, then

$$\lim_n Prob(\{x^*\} \subseteq_\eta Eq(g_n)) = 1.$$

Smallest equilibrium set

- Hence $\{x^*\}$ is the smallest equilibrium set.
- Equilibrium selection theory: no matter what network, there is an equilibrium with aggregate behavior.

Smallest equilibrium set

- The proof makes this idea more precise.
- Suppose that

$$a_i^0 \in \arg\max_a u_i\left(a, x^*, \varepsilon_i\right)$$

  is pl. i's best response as if x^* of her neighbors play 1.
- The proof shows that the best response dynamics starting at a^0 ends (with a large prob.) in an equilibrium, in which very few agents switch their actions.
- Hence $a_i^0 \in \arg\max_a u_i\left(a, x^*, \varepsilon_i\right)$ is a pretty safe action to take, whatever is the true network.

Smallest equilibrium set

- Start with deterministic case.
- Suppose that $x^* = 0$ is risk-dominant.
- Let a^0be a profile such that almost all, except for initial infectors, play 0.
- Can a small group of infectors initiate contagion?

Smallest equilibrium set

- No.
- Remarkably elegant argument from [Morris(2000a)].
- For each profile a, let

$$\mathcal{F}_0(a) = \sum_{i,j:a_i=1,a_j=0} g_{ij}$$

be the sum of links between infected and uninfected.

- Refer to $\mathcal{F}(a)$ as the capacity to infect.

Smallest equilibrium set

- Whenever i changes action from 0 to 1 as a best response, the capacity increases by

$$\mathcal{F}_0\left(a'\right) - \mathcal{F}_0\left(a\right) = \sum_{j:a_j=0} g_{ij} - \sum_{j:a_j=1} g_{ij}.$$

- But, if 1 is a best response and 0 is risk-dominant, then it's better be that

$$\sum_{j:a_j=0} g_{ij} < \frac{1}{2} < \sum_{j:a_j=1} g_{ij}.$$

- So, the capacity goes down every single infection!
- Because the capacity cannot be negative, contagion has to stop.
- If the initial profile was close to x^*, the capacity was small and the contagion will stop very soon, close to x^*.

Smallest equilibrium set

- Whenever i changes action from 0 to 1 as a best response, the capacity increases by

$$\mathcal{F}_0\left(a'\right) - \mathcal{F}_0\left(a\right) = \sum_{j:a_j=0} g_{ij} - \sum_{j:a_j=1} g_{ij}.$$

- But, if 1 is a best response and 0 is risk-dominant, then it's better be that

$$\sum_{j:a_j=0} g_{ij} < \frac{1}{2} < \sum_{j:a_j=1} g_{ij}.$$

- So, the capacity goes down every single infection!
- Because the capacity cannot be negative, contagion has to stop.
- If the initial profile was close to x^*, the capacity was small and the contagion will stop very soon, close to x^*.

Smallest equilibrium set

- Whenever i changes action from 0 to 1 as a best response, the capacity increases by

$$\mathcal{F}_0\left(a'\right) - \mathcal{F}_0\left(a\right) = \sum_{j:a_j=0} g_{ij} - \sum_{j:a_j=1} g_{ij}.$$

- But, if 1 is a best response and 0 is risk-dominant, then it's better be that

$$\sum_{j:a_j=0} g_{ij} < \frac{1}{2} < \sum_{j:a_j=1} g_{ij}.$$

- So, the capacity goes down every single infection!
- Because the capacity cannot be negative, contagion has to stop.
- If the initial profile was close to x^*, the capacity was small and the contagion will stop very soon, close to x^*.

Smallest equilibrium set

- Whenever i changes action from 0 to 1 as a best response, the capacity increases by

$$\mathcal{F}_0\left(a'\right) - \mathcal{F}_0\left(a\right) = \sum_{j:a_j=0} g_{ij} - \sum_{j:a_j=1} g_{ij}.$$

- But, if 1 is a best response and 0 is risk-dominant, then it's better be that

$$\sum_{j:a_j=0} g_{ij} < \frac{1}{2} < \sum_{j:a_j=1} g_{ij}.$$

- So, the capacity goes down every single infection!
- Because the capacity cannot be negative, contagion has to stop.
- If the initial profile was close to x^*, the capacity was small and the contagion will stop very soon, close to x^*.

Smallest equilibrium set

- Whenever i changes action from 0 to 1 as a best response, the capacity increases by

$$\mathcal{F}_0\left(a'\right) - \mathcal{F}_0\left(a\right) = \sum_{j:a_j=0} g_{ij} - \sum_{j:a_j=1} g_{ij}.$$

- But, if 1 is a best response and 0 is risk-dominant, then it's better be that

$$\sum_{j:a_j=0} g_{ij} < \frac{1}{2} < \sum_{j:a_j=1} g_{ij}.$$

- So, the capacity goes down every single infection!
- Because the capacity cannot be negative, contagion has to stop.
- If the initial profile was close to x^*, the capacity was small and the contagion will stop very soon, close to x^*.

Smallest equilibrium set

- Key feature of a good definition of capacity
  - it goes down with best responses,
  - it is small,
  - cannot be negative.
- The number of stages until the dynamics stops is related to the initial capacity.

Smallest equilibrium set

- RU case.
- Assume that *RU*-dominant outcome $x^* = P(x^*) = 0$.
- Let $a_i^0 = \arg\max u_i(a_i, x^*, \varepsilon_i)$ be the initial profile.
- Definition of capacity: Notice that

$$\sum_{i,j:a_i=1,a_j=0} g_{ij}$$

is not small (there are many players who play 1). .

Smallest equilibrium set

- RU case.
- Assume that *RU*-dominant outcome $x^* = P(x^*) = 0$.
- Definition of capacity: Notice also that

$$\sum_{i,j:a_i=1,a_j=0} g_{ij} = \frac{1}{2}\sum_{i,j} g_{ij}(a_i - a_j)^2.$$

Smallest equilibrium set

- RU case.
- Assume that RU-dominant outcome $x^* = P(x^*) = 0$.
- Definition of capacity: Instead of

$$\frac{1}{2}\sum_{i,j} g_{ij}\left(a_i - a_j\right)^2,$$

we take

$$\mathcal{F}(a) = \frac{1}{2}\sum_{i,j} g_{ij}\left(P\left(\beta_i^a\right) - P\left(\beta_j^a\right)\right)^2.$$

  - recall that β_i^a is the neighborhood fraction, and
  - $P\left(\beta_i^a\right)$ is the "expected" best response of agent i.

- Turns out that this is a good definition
  - capacity is small at a^0 as with a large probability $\beta_i^a \sim \beta_j^a$,
  - and it is a sum of a martingale and a decreasing process. Ignoring (probabilistically) small terms, we get, for each T

$$\mathcal{F}\left(P\left(\beta^0\right)\right) \geq 2\sum_i g_i \left[\int_{x^*}^{P\left(\beta_i^T\right)} \left(P^{-1}(y) - y\right) dy\right].$$

Conclusion

- Heterogeneous payoffs in coordination games on network.
- We characterized the largest and the smallest possible set of equilibrium average behaviors across all networks.
- Results:
    - The largest set achieved on a collection of complete graphs,
    - partial identification theory,
    - The smallest set achieved on 2-dimensional (but not necessarily 1-dimensional) lattice,
    - equilibrium selection theory.
- Main assumptions:
    - independent payoff shocks,
    - large degree,
    - both assumptions are important.

Ellison, G. (1993). Learning, local interaction, and coordination. *Econometrica*, **61** (5), 1047.

— (2000). Basins of attraction, long-run stochastic stability, and the speed of step-by-step evolution. *Review of Economic Studies*, **67** (230), 17.

Harsanyi, J. C. and Selten, R. (1988). *A general theory of equilibrium selection in games*. Cambridge, Mass. and London: MIT Press.

Kandori, M., Mailath, G. J. and Rob, R. (1993). Learning, mutation, and long run equilibria in games. *Econometrica*, **61** (1), 29.

Morris, S. (2000a). Contagion. *Review of Economic Studies*, **67** (1), 57–78.

— (2000b). Contagion. *Review of Economic Studies*, **67** (1), 57–78.

Peski, M. (2010). Generalized risk-dominance and asymmetric dynamics. *Journal of Economic Theory*, **145** (1), 216–248, publisher: Elsevier.

YOUNG, H. P. (1993). The evolution of conventions. *Econometrica*, **61** (1), 57–84.