Computer assisted proofs in differential equations
Lecture 2 - Dec. 11, 2019

Marcio Gameiro

Rutgers University

and

University of São Paulo at São Carlos - ICMC-USP

12th Americas Conference on Differential Equations and Nonlinear Analysis (Americas XII)
December 9-13, 2019, Mathematics Research Center (CIMAT)
Guanajuato, México

Example 2 - Lorenz

- Consider the Lorenz system

$$\begin{aligned}
\dot{x}_1 &= \sigma(x_2 - x_1) \\
\dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 \\
\dot{x}_3 &= -\beta x_3 + x_1 x_2
\end{aligned}$$

Example 2 - Lorenz

- Consider the Lorenz system

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2 \end{aligned}$$

- For $\beta > 0$ and $\rho > 1$ the equilibria are

$$\left\{(0,0,0), \left(\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1\right)\right\}.$$

Example 2 - Lorenz

- Consider the Lorenz system

$$
\begin{aligned}
\dot{x}_1 &= \sigma(x_2 - x_1) \\
\dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 \\
\dot{x}_3 &= -\beta x_3 + x_1 x_2
\end{aligned}
$$

- For $\beta > 0$ and $\rho > 1$ the equilibria are

$$\left\{(0,0,0), \left(\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1\right)\right\}.$$

- At the classical parameter values $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$, $\left(\sqrt{72}, \sqrt{72}, 27\right)$ is an equilibrium solution that is approximated by

$$\bar{x} = \begin{pmatrix} 8.4853 \\ 8.4853 \\ 27 \end{pmatrix}.$$

Example 2 - Lorenz

- We have
$$Df(\bar{x}) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - \bar{x}_3 & -1 & -\bar{x}_1 \\ \bar{x}_2 & \bar{x}_1 & -\beta \end{pmatrix}.$$

Example 2 - Lorenz

- We have

$$Df(\bar{x}) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - \bar{x}_3 & -1 & -\bar{x}_1 \\ \bar{x}_2 & \bar{x}_1 & -\beta \end{pmatrix}.$$

- Rather than looking for an explicit expression for $Df(\bar{x})^{-1}$ we compute a numerical inverse and set

$$A = \begin{pmatrix} -0.052 & -0.018 & 0.059 \\ 0.048 & -0.018 & 0.059 \\ -0.012 & -0.118 & 0 \end{pmatrix}.$$

Example 2 - Lorenz

- We have

$$Df(\bar{x}) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - \bar{x}_3 & -1 & -\bar{x}_1 \\ \bar{x}_2 & \bar{x}_1 & -\beta \end{pmatrix}.$$

- Rather than looking for an explicit expression for $Df(\bar{x})^{-1}$ we compute a numerical inverse and set

$$A = \begin{pmatrix} -0.052 & -0.018 & 0.059 \\ 0.048 & -0.018 & 0.059 \\ -0.012 & -0.118 & 0 \end{pmatrix}.$$

- As in the previous example, we take $Y_0 = \|Af(\bar{x})\|_\infty$ and $Z_0 = \|\mathrm{I} - ADf(\bar{x})\|_\infty$.

Example 2 - Lorenz

- To obtain Z_2 we use the standard change of variables, for $c \in \overline{B_r(\bar{x})}$ set $b = c - \bar{x} \in \overline{B_r(0)}$, and compute

$$
\begin{aligned}
& Df(c) - Df(\bar{x}) = Df(b + \bar{x}) - Df(\bar{x}) \\
&= \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - (\bar{x}_3 + b_3) & -1 & -(\bar{x}_1 + b_1) \\ \bar{x}_2 + b_2 & \bar{x}_1 + b_1 & -\beta \end{pmatrix} - \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - \bar{x}_3 & -1 & -\bar{x}_1 \\ \bar{x}_2 & \bar{x}_1 & -\beta \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ -b_3 & 0 & -b_1 \\ b_2 & b_1 & 0 \end{pmatrix}.
\end{aligned}
$$

Example 2 - Lorenz

- Thus, denoting $A = (a_{i,j})_{i,j=1}^{3}$, we have

$$
\begin{aligned}
&\|A[Df(c) - Df(\bar{x})]\|_\infty = \\
&\max_{i=1,2,3} \max_{\|b\|_\infty \le r} \left\{ |a_{i,2}||b_3| + |a_{i,3}||b_2| + |a_{i,3}||b_1| + |a_{i,2}||b_1| \right\} \\
&\le \max_{i=1,2,3} \left\{ 2(|a_{i,2}| + |a_{i,3}|) \right\} r.
\end{aligned}
$$

Example 2 - Lorenz

- Thus, denoting $A = (a_{i,j})_{i,j=1}^{3}$, we have

$$
\begin{aligned}
&\|A[Df(c) - Df(\bar{x})]\|_\infty = \\
&\max_{i=1,2,3} \max_{\|b\|_\infty \le r} \left\{ |a_{i,2}||b_3| + |a_{i,3}||b_2| + |a_{i,3}||b_1| + |a_{i,2}||b_1| \right\} \\
&\le \max_{i=1,2,3} \left\{ 2(|a_{i,2}| + |a_{i,3}|) \right\} r.
\end{aligned}
$$

- Hence we set

$$
Z_2 = \max_{i=1,2,3} \left\{ 2(|a_{i,2}| + |a_{i,3}|) \right\}.
$$

Example 3 - Fisher

- The following system is a finite dimensional approximation of Fisher's equation,

$$\dot{x}_k = f_k(x, \lambda) = (\lambda - k^2)x_k - \lambda \sum_{\substack{k_1 + k_2 = k \\ k_i = -N, \ldots, N}} x_{|k_1|} x_{|k_2|}, \qquad k = 0, \ldots, N,$$

where λ is a parameter.

Example 3 - Fisher

- The following system is a finite dimensional approximation of Fisher's equation,

$$\dot{x}_k = f_k(x, \lambda) = (\lambda - k^2)x_k - \lambda \sum_{\substack{k_1+k_2=k \\ k_i=-N,\dots,N}} x_{|k_1|}x_{|k_2|}, \qquad k = 0, \dots, N,$$

  where λ is a parameter.
- Denoting

$$(x * y)_n = \sum_{\substack{k_1+k_2=n \\ k_i=-N,\dots,N}} x_{|k_1|}y_{|k_2|}, \qquad n = 0, \dots, N$$

  where $x = (x_0, \dots, x_N), y = (y_0, \dots, y_N) \in \mathbb{R}^{N+1}$.

Example 3 - Fisher

- The following system is a finite dimensional approximation of Fisher's equation,

$$\dot{x}_k = f_k(x, \lambda) = (\lambda - k^2)x_k - \lambda \sum_{\substack{k_1+k_2=k \\ k_i=-N,\dots,N}} x_{|k_1|}x_{|k_2|}, \qquad k = 0, \dots, N,$$

  where λ is a parameter.
- Denoting

$$(x * y)_n = \sum_{\substack{k_1+k_2=n \\ k_i=-N,\dots,N}} x_{|k_1|}y_{|k_2|}, \qquad n = 0, \dots, N$$

  where $x = (x_0, \dots, x_N), y = (y_0, \dots, y_N) \in \mathbb{R}^{N+1}$.
- We have

$$\|x * y\|_1 \leq \|x\|_1 \|y\|_1.$$

Example 3 - Fisher

- For $N = 1$

$$\dot{x} = f(x, \lambda) = \begin{pmatrix} \lambda x_0 - \lambda(x_0^2 + 2x_1^2) \\ (\lambda - 1)x_1 - \lambda(2x_0 x_1) \end{pmatrix},$$

Example 3 - Fisher

- For $N = 1$

$$\dot{x} = f(x, \lambda) = \begin{pmatrix} \lambda x_0 - \lambda(x_0^2 + 2x_1^2) \\ (\lambda - 1)x_1 - \lambda(2x_0 x_1) \end{pmatrix},$$

- For $N = 2$

$$\dot{x} = f(x, \lambda) = \begin{pmatrix} \lambda x_0 - \lambda(x_0^2 + 2x_1^2 + 2x_2^2) \\ (\lambda - 1)x_1 - \lambda(2x_0 x_1 + 2x_1 x_2) \\ (\lambda - 4)x_2 - \lambda(2x_0 x_2 + x_1^2) \end{pmatrix},$$

Example 3 - Fisher

- For $N = 1$

$$\dot{x} = f(x, \lambda) = \begin{pmatrix} \lambda x_0 - \lambda(x_0^2 + 2x_1^2) \\ (\lambda - 1)x_1 - \lambda(2x_0 x_1) \end{pmatrix},$$

- For $N = 2$

$$\dot{x} = f(x, \lambda) = \begin{pmatrix} \lambda x_0 - \lambda(x_0^2 + 2x_1^2 + 2x_2^2) \\ (\lambda - 1)x_1 - \lambda(2x_0 x_1 + 2x_1 x_2) \\ (\lambda - 4)x_2 - \lambda(2x_0 x_2 + x_1^2) \end{pmatrix},$$

- For $N = 3$

$$\dot{x} = f(x, \lambda) = \begin{pmatrix} \lambda x_0 - \lambda(x_0^2 + 2x_1^2 + 2x_2^2 + 2x_3^2) \\ (\lambda - 1)x_1 - \lambda(2x_0 x_1 + 2x_1 x_2 + 2x_2 x_3) \\ (\lambda - 4)x_2 - \lambda(2x_0 x_2 + x_1^2 + 2x_1 x_3) \\ (\lambda - 9)x_3 - \lambda(2x_0 x_3 + 2x_1 x_2) \end{pmatrix}.$$

Example 3 - Fisher

- Returning to the general case

$$\frac{\partial f_k}{\partial x_0}(x) = \lambda - k^2 - 2\lambda x_k$$

$$\frac{\partial f_k}{\partial x_l}(x) = \lambda - k^2 - 2\lambda \begin{cases} x_{|k-l|} + x_{k+l} & \text{if } k + l \leq N \\ x_{|k-l|} & \text{otherwise.} \end{cases}$$

Example 3 - Fisher

- Returning to the general case

$$\frac{\partial f_k}{\partial x_0}(x) = \lambda - k^2 - 2\lambda x_k$$

$$\frac{\partial f_k}{\partial x_l}(x) = \lambda - k^2 - 2\lambda \begin{cases} x_{|k-l|} + x_{k+l} & \text{if } k + l \leq N \\ x_{|k-l|} & \text{otherwise.} \end{cases}$$

- As usual, we take

$$Y_0 = \|Af(\bar{x})\|_1 \quad \text{and} \quad Z_0 = \|I - ADf(\bar{x})\|_1 .$$

Example 3 - Fisher

- Returning to the general case

$$\frac{\partial f_k}{\partial x_0}(x) = \lambda - k^2 - 2\lambda x_k$$

$$\frac{\partial f_k}{\partial x_l}(x) = \lambda - k^2 - 2\lambda \begin{cases} x_{|k-l|} + x_{k+l} & \text{if } k + l \leq N \\ x_{|k-l|} & \text{otherwise.} \end{cases}$$

- As usual, we take

$$Y_0 = \|Af(\bar{x})\|_1 \quad \text{and} \quad Z_0 = \|I - ADf(\bar{x})\|_1 .$$

- For Z_2, for $c \in \overline{B_r(\bar{x})}$ and $h \in \overline{B_1(0)}$, we have

$$\begin{aligned} ([Df(c) - Df(\bar{x})]h)_k &= \left(-k^2 + \lambda\right) h_k - 2\lambda \left(\sum_{l=1}^{N} c_{|k-l|} h_l + \sum_{l=0}^{N-k} c_{k+l} h_l \right) \\ &\quad - \left(-k^2 + \lambda\right) h_k + 2\lambda \left(\sum_{l=1}^{N} \bar{x}_{|k-l|} h_l + \sum_{l=0}^{N-k} \bar{x}_{k+l} h_l \right) \\ &= -2\lambda \left((c * h - \bar{x} * h \right)_k = -2\lambda \left((c - \bar{x}) * h \right)_k . \end{aligned}$$

Example 3 - Fisher

- It follows that

$$
\begin{aligned}
&\|A[Df(c) - Df(\bar{x})]\|_1 = \sup_{\|h\|=1} \|A[Df(c) - Df(\bar{x})]h\|_1 \\
&\leq 2|\lambda| \sup_{\|h\|=1} \|A(c - \bar{x}) * h\|_1 \leq 2|\lambda| \sup_{\|h\|=1} \|A\|_1 \|(c - \bar{x}) * h\|_1 \\
&\leq 2|\lambda| \sup_{\|h\|=1} \|A\|_1 \|c - \bar{x}\| \|h\|_1 \leq 2|\lambda| \|A\|_1 r
\end{aligned}
$$

- Therefore, we take

$$Z_2 = 2|\lambda| \|A\|_1.$$

Localized Radii Polynomials in Finite Dimensions

Theorem (Localized radii polynomials in finite dimensions)

Consider a C^1 *map* $f : U \to \mathbb{R}^n$ *where* $U \subset \mathbb{R}^n$ *is open. Let* $\bar{x} \in U$ *and* $A \in M_n(\mathbb{R})$. *Let* $r_* > 0$ *be such that* $\overline{B_{r_*}(\bar{x})} \subset U$. *Let* Y_0, Z_0, *and* $Z_2 : (0, r_*] \to [0, \infty)$ *be non-negative constants satisfying*

$$
\begin{aligned}
\|Af(\bar{x})\| &\leq Y_0 \\
\|\mathrm{I} - ADf(\bar{x})\| &\leq Z_0 \\
\|A[Df(c) - Df(\bar{x})]\| &\leq Z_2(r)r, \quad \text{for all } c \in \overline{B_r(\bar{x})} \text{ and all } r \in (0, r_*].
\end{aligned}
$$

Define the radii polynomial

$$p(r) = Z_2(r)r^2 - (1 - Z_0)r + Y_0.$$

If there exists $r_0 \in (0, r_*]$ *such that* $p(r_0) < 0$, *then the matrix* A *is invertible and there exists a unique* $\tilde{x} \in \overline{B_{r_0}(\bar{x})}$ *satisfying* $f(\tilde{x}) = 0$.

Effectiveness of Radii Polynomials

Theorem

Let $U \subset \mathbb{R}^N$ be an open set and let $f \in C^2(U, \mathbb{R}^N)$. If $\tilde{x} \in U$ is a nondegenerate zero of f, then there exists $\epsilon > 0$ such that if $\bar{x} \in \overline{B_\epsilon(\tilde{x})}$ then the following holds. There exists $A(\bar{x}) \in M_N(\mathbb{R})$, non-negative constants $r_*(\bar{x})$, $Y_0(\bar{x})$, $Z_0(\bar{x})$, and $Z_2(\bar{x})$ and positive constants $r_0(\bar{x}) < r_*(\bar{x})$ such that

$$p(r_0(\bar{x})) < 0$$

where

$$p(r) = Z_2(\bar{x})r^2 - (1 - Z_0(\bar{x}))r + Y_0(\bar{x})$$

is the associated radii polynomial and $\overline{B_{r_*(\bar{x})}(\bar{x})} \subset U$.

Continuation of Equilibria

- For $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, consider the ODE

$$\dot{x} = f(x, \lambda).$$

Continuation of Equilibria

- For $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, consider the ODE

$$\dot{x} = f(x, \lambda).$$

- For the case $m = 1$, that is, $f : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ with $\Lambda \subset \mathbb{R}$ we want to compute branches of solutions in the solution set

$$\mathcal{S} = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid f(x, \lambda) = 0\} \subset \mathbb{R}^n \times \mathbb{R}.$$

Continuation of Equilibria

- For $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, consider the ODE

$$\dot{x} = f(x, \lambda).$$

- For the case $m = 1$, that is, $f\colon \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ with $\Lambda \subset \mathbb{R}$ we want to compute branches of solutions in the solution set

$$\mathcal{S} = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid f(x, \lambda) = 0\} \subset \mathbb{R}^n \times \mathbb{R}.$$

- For this is use a rigorous version of the Parameter Continuation and the Pseudo-Arclength Continuation methods.

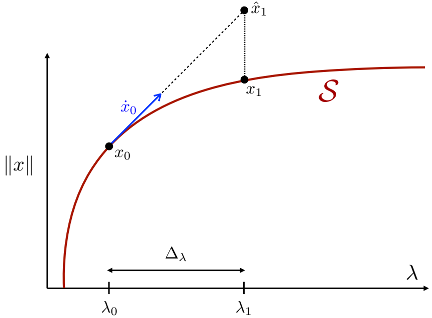
Parameter Continuation

- Start with (x_0, λ_0) and compute

$$\dot{x}_0 = -D_x f(x_0, \lambda_0)^{-1} \frac{\partial f}{\partial \lambda}(x_0, \lambda_0) \quad \text{and} \quad \hat{x}_1 = x_0 + \Delta_\lambda \dot{x}_0.$$

Parameter Continuation

- Start with (x_0, λ_0) and compute
$$\dot{x}_0 = -D_x f(x_0, \lambda_0)^{-1} \frac{\partial f}{\partial \lambda}(x_0, \lambda_0) \quad \text{and} \quad \hat{x}_1 = x_0 + \Delta_\lambda \dot{x}_0.$$
- Fixing $\lambda_1 = \lambda_0 + \Delta_\lambda$, apply Newton's method $x_1^{(0)} = \hat{x}_1$
$$x_1^{(n+1)} = x_1^{(n)} - \left(D_x f(x_1^{(n)}, \lambda_1) \right)^{-1} f(x_1^{(n)}, \lambda_1), \quad n \geq 0,$$
to obtain the solution (x_1, λ_1). We repeat this procedure.



Pseudo-Arclength Continuation

- We make $X = (x, \lambda)$ variable and consider the problem $f(X) = 0$. Starting with $X_0 = (x_0, \lambda_0)$ we compute $\dot{X}_0$ by solving

$$D_X f(X_0)\dot{X}_0 = \left[D_x f(x_0, \lambda_0) \quad \frac{\partial f}{\partial \lambda}(x_0, \lambda_0) \right] \dot{X}_0 = 0 \in \mathbb{R}^n.$$

Pseudo-Arclength Continuation

- We make $X = (x, \lambda)$ variable and consider the problem $f(X) = 0$. Starting with $X_0 = (x_0, \lambda_0)$ we compute $\dot{X}_0$ by solving

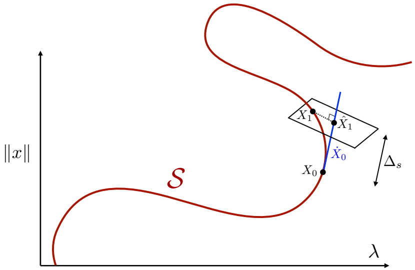
$$D_X f(X_0)\dot{X}_0 = \left[D_x f(x_0, \lambda_0) \quad \frac{\partial f}{\partial \lambda}(x_0, \lambda_0) \right] \dot{X}_0 = 0 \in \mathbb{R}^n.$$

- Fix a pseudo-arclength parameter $\Delta_s > 0$ and compute the predictor

$$\hat{X}_1 = \bar{X}_0 + \Delta_s \dot{X}_0 \in \mathbb{R}^{n+1}.$$

Pseudo-Arclength Continuation

- We make $X = (x, \lambda)$ variable and consider the problem $f(X) = 0$. Starting with $X_0 = (x_0, \lambda_0)$ we compute $\dot{X}_0$ by solving
$$D_X f(X_0)\dot{X}_0 = \left[D_x f(x_0, \lambda_0) \quad \frac{\partial f}{\partial \lambda}(x_0, \lambda_0) \right] \dot{X}_0 = 0 \in \mathbb{R}^n.$$
- Fix a pseudo-arclength parameter $\Delta_s > 0$ and compute the predictor
$$\hat{X}_1 = \bar{X}_0 + \Delta_s \dot{X}_0 \in \mathbb{R}^{n+1}.$$
- Now correct towards $\mathcal{S}$ on the hyperplane perpendicular to the tangent vector $\dot{X}_0$ which contains the predictor $\hat{X}_1$.



Pseudo-Arclength Continuation

- The equation of the hyperplane is

$$E(X) = (X - \hat{X}_1) \cdot \dot{X}_0 = 0.$$

Pseudo-Arclength Continuation

- The equation of the hyperplane is

$$E(X) = (X - \hat{X}_1) \cdot \dot{X}_0 = 0.$$

- Apply Newton's method to

$$F(X) = \begin{pmatrix} E(X) \\ f(X) \end{pmatrix} = 0,$$

with the initial condition $\hat{X}_1$ to obtain a new solution X_1

Pseudo-Arclength Continuation

- The equation of the hyperplane is

$$E(X) = (X - \hat{X}_1) \cdot \dot{X}_0 = 0.$$

- Apply Newton's method to

$$F(X) = \begin{pmatrix} E(X) \\ f(X) \end{pmatrix} = 0,$$

  with the initial condition $\hat{X}_1$ to obtain a new solution X_1
- Repeat the procedure to numerically obtain a branch of solutions parameterized by the pseudo-arc length parameter s.

Rigorous Parameter Continuation for Zeros of Functions

- Approximate solutions

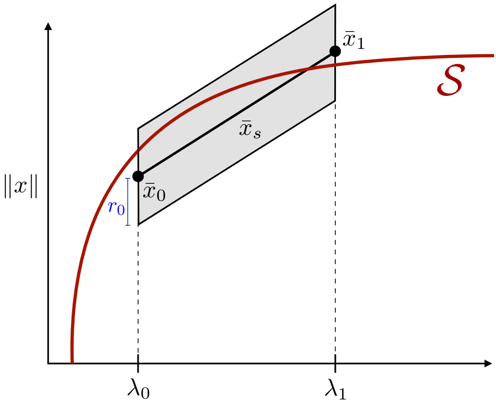
$$\lambda_s = (1-s)\lambda_0 + s\lambda_1 \quad \text{and} \quad \bar{x}_s = (1-s)\bar{x}_0 + s\bar{x}_1, \quad s \in [0,1].$$

Rigorous Parameter Continuation for Zeros of Functions

- Approximate solutions

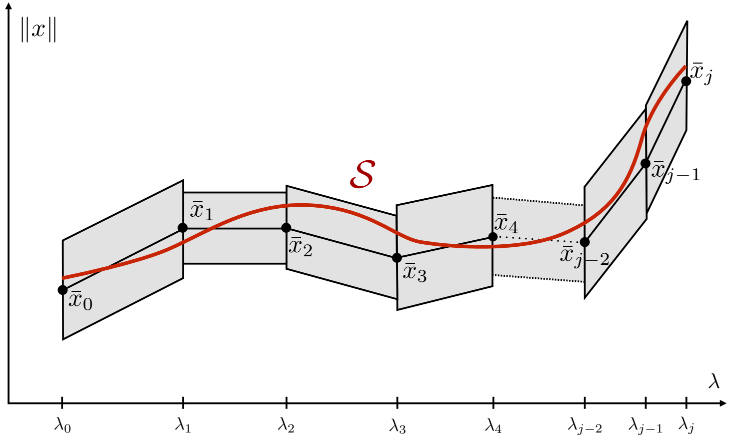
$$\lambda_s = (1-s)\lambda_0 + s\lambda_1 \quad \text{and} \quad \bar{x}_s = (1-s)\bar{x}_0 + s\bar{x}_1, \quad s \in [0,1].$$

- Apply uniform contraction theorem



Global Smooth Branches of Solutions

- Glue the local branches to get a smooth global branch.



Example

- Consider again the Fisher's example

$$\dot{x}_k = f_k(x, \lambda) = (\lambda - k^2)x_k - \lambda(x^2)_k, \qquad k = 0, \ldots, N,$$

where λ is a parameter and

$$(x^2)_k = (x * x)_k = \sum_{\substack{k_1 + k_2 = k \\ k_i = -N, \ldots, N}} x_{|k_1|} x_{|k_2|}, \qquad k = 0, \ldots, N.$$

Example

- Consider again the Fisher's example

$$\dot{x}_k = f_k(x, \lambda) = (\lambda - k^2)x_k - \lambda(x^2)_k, \qquad k = 0, \dots, N,$$

where λ is a parameter and

$$(x^2)_k = (x * x)_k = \sum_{\substack{k_1 + k_2 = k \\ k_i = -N, \dots, N}} x_{|k_1|} x_{|k_2|}, \qquad k = 0, \dots, N.$$

- We take an initial solution and compute branches $\mathcal{B}_N$ starting at this same initial solution, but changing the projection dimension N.

Example

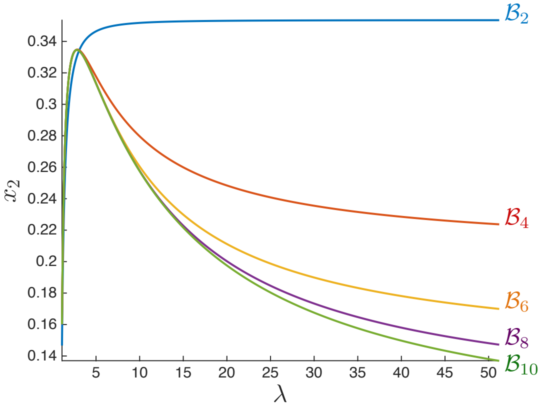


Figure: Branches of solutions $\mathcal{B}_N$ for $N \in \{2, 4, 6, 8, 10\}$.

Radii Polynomials on Banach spaces

Theorem (Radii polynomial for the contraction mapping theorem)

Suppose that X is a Banach space, that $T : X \to X$ is a Fréchet differentiable mapping, and that $\bar{x} \in X$. Let $Y_0 \geq 0$ and $Z : (0, \infty) \to [0, \infty)$ a non-negative function satisfying that

$$\| T(\bar{x}) - \bar{x} \|_X \leq Y_0,$$

and

$$\sup_{x \in \overline{B_r(\bar{x})}} \| DT(x) \|_{B(X)} \leq Z(r).$$

Define the radii polynomial

$$p(r) := Z(r) r - r + Y_0.$$

If there exists $r_0 > 0$ such that $p(r_0) < 0$, then there exists a unique $\tilde{x} \in \overline{B_{r_0}(\bar{x})}$ so that $T(\tilde{x}) = \tilde{x}$.

Radii Polynomials on Banach spaces

- Consider a differentiable map $F\colon X \to X$, where X is Banach space.

Radii Polynomials on Banach spaces

- Consider a differentiable map $F\colon X \to X$, where X is Banach space.
- Assume that we have $\bar{a} \in X$ such that $\|F(\bar{a})\|_X \approx 0$.

Radii Polynomials on Banach spaces

- Consider a differentiable map $F\colon X \to X$, where X is Banach space.
- Assume that we have $\bar{a} \in X$ such that $\|F(\bar{a})\|_X \approx 0$.
- As before, we want to define $T(a) = a - AF(a)$, where A is an approximated inverse of $DF(\bar{a})$.

Radii Polynomials on Banach spaces

- Consider a differentiable map $F\colon X \to X$, where X is Banach space.
- Assume that we have $\bar{a} \in X$ such that $\|F(\bar{a})\|_X \approx 0$.
- As before, we want to define $T(a) = a - AF(a)$, where A is an approximated inverse of $DF(\bar{a})$.
- If X is infinite dimensional, computing an an approximated inverse of $DF(\bar{a})$ is more challenging.

Radii Polynomials on Banach spaces

- Consider a differentiable map $F\colon X \to X$, where X is Banach space.
- Assume that we have $\bar{a} \in X$ such that $\|F(\bar{a})\|_X \approx 0$.
- As before, we want to define $T(a) = a - AF(a)$, where A is an approximated inverse of $DF(\bar{a})$.
- If X is infinite dimensional, computing an an approximated inverse of $DF(\bar{a})$ is more challenging.
- So we first approximate $DF(\bar{a})$ by an operator $A^{\dagger} \in B(X)$ chosen to be easier to work with. We then choose $A \in B(X)$ to be an approximate inverse of $A^{\dagger}$ and define the Newton-like operator

$$T(a) = a - AF(a).$$

Radii Polynomials on Banach spaces

- Consider a differentiable map $F: X \to X$, where X is Banach space.
- Assume that we have $\bar{a} \in X$ such that $\|F(\bar{a})\|_X \approx 0$.
- As before, we want to define $T(a) = a - AF(a)$, where A is an approximated inverse of $DF(\bar{a})$.
- If X is infinite dimensional, computing an an approximated inverse of $DF(\bar{a})$ is more challenging.
- So we first approximate $DF(\bar{a})$ by an operator $A^\dagger \in B(X)$ chosen to be easier to work with. We then choose $A \in B(X)$ to be an approximate inverse of $A^\dagger$ and define the Newton-like operator

$$T(a) = a - AF(a).$$

- The following theorem provides conditions for the existence of a fixed point of T and hence a zero of F. Note that $F: X \to Y$.

Radii Polynomials on Banach spaces

Theorem (Radii polynomial approach in infinite dimensions)

Let X and Y be Banach spaces and $F\colon X \to Y$ be a Fréchet differentiable mapping. Suppose that $\bar{x} \in X$, $A^{\dagger} \in B(X, Y)$, and $A \in B(Y, X)$. Moreover assume that A is injective. Let Y_0, Z_0, and Z_1 be positive constants and $Z_2\colon (0, \infty) \to [0, \infty)$ be a non-negative function satisfying

$$\begin{aligned}
&\|AF(\bar{x})\|_X \le Y_0, \\
&\|\mathrm{I}_X - AA^{\dagger}\|_{B(X)} \le Z_0, \\
&\|A[DF(\bar{x}) - A^{\dagger}]\|_{B(X)} \le Z_1, \\
&\|A[DF(c) - DF(\bar{x})]\|_{B(X)} \le Z_2(r)r, \quad \text{for all } c \in \overline{B_r(\bar{x})} \text{ and all } r > 0.
\end{aligned}$$

Define

$$p(r) := Z_2(r)r^2 - (1 - Z_0 - Z_1)r + Y_0.$$

If there exists $r_0 > 0$ such that $p(r_0) < 0$, then there exists a unique $\tilde{x} \in B_{r_0}(\bar{x})$ satisfying $F(\tilde{x}) = 0$.