Computer assisted proofs in differential equations
Lecture 4 - Dec. 13, 2019

Marcio Gameiro

Rutgers University

and

University of São Paulo at São Carlos - ICMC-USP

12th Americas Conference on Differential Equations and Nonlinear Analysis (Americas XII)
December 9-13, 2019, Mathematics Research Center (CIMAT)
Guanajuato, México

Systems of first order differential equations

- Consider the IVP

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^N,$$

where $f = (f_1, \ldots, f_N)$ is a polynomial nonlinearity.

Systems of first order differential equations

- Consider the IVP

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^N,$$

where $f = (f_1, \ldots, f_N)$ is a polynomial nonlinearity.

- We denote

$$f(x) = \sum_{|\alpha|=0}^{M} c_\alpha x^\alpha \quad \text{with } c_\alpha \in \mathbb{R}^N,$$

where $\alpha \in \mathbb{N}^N$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_N}$.

Systems of first order differential equations

- Consider the IVP

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^N,$$

where $f = (f_1, \ldots, f_N)$ is a polynomial nonlinearity.

- We denote

$$f(x) = \sum_{|\alpha|=0}^{M} c_\alpha x^\alpha \quad \text{with } c_\alpha \in \mathbb{R}^N,$$

where $\alpha \in \mathbb{N}^N$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_N}$.

- We assume that time rescaling, if needed, is absorbed implicitly in the coefficients c_α.

Systems of first order differential equations

- Consider the IVP

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^N,$$

  where $f = (f_1, \dots, f_N)$ is a polynomial nonlinearity.
- We denote

$$f(x) = \sum_{|\alpha|=0}^{M} c_\alpha x^\alpha \quad \text{with } c_\alpha \in \mathbb{R}^N,$$

  where $\alpha \in \mathbb{N}^N$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_N}$.
- We assume that time rescaling, if needed, is absorbed implicitly in the coefficients c_α.
- The goal is to obtain solutions $x(t) = (x_1(t), \dots, x_N(t))$ with power series expansion

$$x_n(t) = \sum_{k=0}^{\infty} (a_n)_k \, t^k = \sum_{k=0}^{\infty} a_{n,k} \, t^k, \quad n = 1, \dots, N.$$

Systems of first order differential equations

- Denote $a_n := \{(a_n)_k\}_{k\geq 0} = \{a_{n,k}\}_{k\geq 0}$ for $n = 1, \ldots, N$, and $a := (a_1, \ldots, a_N)$.

Systems of first order differential equations

- Denote $a_n := \{(a_n)_k\}_{k\geq 0} = \{a_{n,k}\}_{k\geq 0}$ for $n = 1,\dots,N$, and $a := (a_1,\dots,a_N)$.
- The n-th component of $f(x(t))$ is of the form

$$f_n(x(t)) = \sum_{k\geq 0} (\phi_n(a))_k t^k,$$

  where the k-th Taylor coefficient $(\phi_n(a))_k$ of $f_n(x(t))$ is a combination of Cauchy products involving the terms of a.
- To make $\phi_n(a)$ explicit, we use the notation

$$a^k := a * a^{k-1} = \underbrace{a * \cdots * a}_{k \text{ times}}, \quad \text{with} \quad a^1 = a.$$

Systems of first order differential equations

- Denote $a_n := \{(a_n)_k\}_{k\geq 0} = \{a_{n,k}\}_{k\geq 0}$ for $n = 1, \ldots, N$, and $a := (a_1, \ldots, a_N)$.
- The n-th component of $f(x(t))$ is of the form

$$f_n(x(t)) = \sum_{k\geq 0} (\phi_n(a))_k t^k,$$

  where the k-th Taylor coefficient $(\phi_n(a))_k$ of $f_n(x(t))$ is a combination of Cauchy products involving the terms of a.
- To make $\phi_n(a)$ explicit, we use the notation

$$a^k := a * a^{k-1} = \underbrace{a * \cdots * a}_{k \text{ times}}, \quad \text{with} \quad a^1 = a.$$

- It is also convenient to denote

$$(a^0)_k := \delta_{k0}.$$

  This is natural in view of $1 = \sum_{k=0}^{\infty} \delta_{k0} t^k$. Also, we now have $a = a * a^0$.

Systems of first order differential equations

- We then have

$$\phi(a) = \sum_{|\alpha|=0}^{M} c_\alpha a^\alpha = \sum_{|\alpha|=0}^{M} c_\alpha\, a_1^{\alpha_1} * \cdots * a_N^{\alpha_N} \quad \text{with } c_\alpha \in \mathbb{R}^N,$$

which is just the polynomial f with multiplications re-interpreted as Cauchy products.

Systems of first order differential equations

- We then have

$$\phi(a) = \sum_{|\alpha|=0}^{M} c_\alpha a^\alpha = \sum_{|\alpha|=0}^{M} c_\alpha \, a_1^{\alpha_1} * \cdots * a_N^{\alpha_N} \quad \text{with } c_\alpha \in \mathbb{R}^N,$$

  which is just the polynomial f with multiplications re-interpreted as Cauchy products.
- We need to solve

$$\begin{aligned} \dot{x}_n(t) - f_n(x(t)) &= \sum_{k\geq 1} k a_{n,k} t^{k-1} - \sum_{k\geq 0} \phi_n(a)_k t^k \\ &= \sum_{k\geq 1} \left(k a_{n,k} - \phi_n(a)_{k-1}\right) t^{k-1} = 0. \end{aligned}$$

Systems of first order differential equations

- We then have

$$\phi(a) = \sum_{|\alpha|=0}^{M} c_\alpha a^\alpha = \sum_{|\alpha|=0}^{M} c_\alpha \, a_1^{\alpha_1} * \cdots * a_N^{\alpha_N} \quad \text{with } c_\alpha \in \mathbb{R}^N,$$

  which is just the polynomial f with multiplications re-interpreted as Cauchy products.
- We need to solve

$$\begin{aligned} \dot{x}_n(t) - f_n(x(t)) &= \sum_{k\geq 1} k a_{n,k} t^{k-1} - \sum_{k \geq 0} \phi_n(a)_k t^k \\ &= \sum_{k\geq 1} \left(k a_{n,k} - \phi_n(a)_{k-1} \right) t^{k-1} = 0. \end{aligned}$$

- We define $F_n(a) = \{F_n(a)_k\}_{k\geq 0}$, for each $n = 1, \ldots, N$, component-wise by

$$(F_n(a))_k := \begin{cases} a_{n,0} - x_{0,n} & \text{if } k = 0, \\ k a_{n,k} - \phi_n(a)_{k-1} & \text{if } k \geq 1. \end{cases}$$

Systems of first order differential equations

- Using similar arguments as in the previous example we can derive the estimates for the radii polynomial in the space $X := (\ell^1_\nu)^N$.

Systems of first order differential equations

- Using similar arguments as in the previous example we can derive the estimates for the radii polynomial in the space $X := (\ell^1_\nu)^N$.
- In particular, a graphically intuitive representation of A is given by

$$A = \left[\begin{array}{cc|cc|cc|cc}
A^{(N)}_{1,1} & 0 & A^{(N)}_{1,2} & 0 & \cdots & \cdots & A^{(N)}_{1,n} & 0 \\
0 & \Lambda_N & 0 & 0 & \cdots & \cdots & 0 & 0 \\
\hline
A^{(N)}_{2,1} & 0 & A^{(N)}_{2,2} & 0 & & & A^{(N)}_{2,n} & 0 \\
0 & 0 & 0 & \Lambda_N & & & 0 & 0 \\
\hline
\vdots & \vdots & & & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & & & \ddots & \vdots & \vdots \\
\hline
A^{(N)}_{n,1} & 0 & A^{(N)}_{n,2} & 0 & \cdots & \cdots & A^{(N)}_{n,n} & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & \Lambda_N
\end{array}\right].$$

Systems of first order differential equations

- Using similar arguments as in the previous example we can derive the estimates for the radii polynomial in the space $X := (\ell^1_\nu)^N$.
- In particular, a graphically intuitive representation of A is given by

$$A = \left[\begin{array}{cc|cc|cc|cc}
A^{(N)}_{1,1} & 0 & A^{(N)}_{1,2} & 0 & \cdots & \cdots & A^{(N)}_{1,n} & 0 \\
0 & \Lambda_N & 0 & 0 & \cdots & \cdots & 0 & 0 \\
\hline
A^{(N)}_{2,1} & 0 & A^{(N)}_{2,2} & 0 & & & A^{(N)}_{2,n} & 0 \\
0 & 0 & 0 & \Lambda_N & & & 0 & 0 \\
\hline
\vdots & \vdots & & & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & & & \ddots & \vdots & \vdots \\
\hline
A^{(N)}_{n,1} & 0 & A^{(N)}_{n,2} & 0 & \cdots & \cdots & A^{(N)}_{n,n} & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & \Lambda_N
\end{array}\right].$$

- Where

$$(\Lambda_K a)_k = \begin{cases} 0 & \text{if } 0 \le k \le K, \\ \frac{1}{k} a_k & \text{if } k \ge K+1. \end{cases}$$

Example - Lorenz system

- Consider the vector field

$$f(x) = \begin{pmatrix} \sigma(x_2 - x_1) \\ \rho x_1 - x_2 - x_1 x_3 \\ -\beta x_3 + x_1 x_2 \end{pmatrix},$$

Example - Lorenz system

- Consider the vector field

$$f(x) = \begin{pmatrix} \sigma(x_2 - x_1) \\ \rho x_1 - x_2 - x_1 x_3 \\ -\beta x_3 + x_1 x_2 \end{pmatrix},$$

- Let $a = (a_1, a_2, a_3) \in (\ell^1_\nu)^3$, and

$$(\phi_j(a))_k = \begin{cases} \sigma((a_2)_k - (a_1)_k), & j = 1 \\ \rho(a_1)_k - (a_2)_k - (a_1 * a_3)_k, & j = 2 \\ -\beta(a_3)_k + (a_1 * a_2)_k, & j = 3 \end{cases}$$

Example - Lorenz system

- Consider the vector field

$$f(x) = \begin{pmatrix} \sigma(x_2 - x_1) \\ \rho x_1 - x_2 - x_1 x_3 \\ -\beta x_3 + x_1 x_2 \end{pmatrix},$$

- Let $a = (a_1, a_2, a_3) \in (\ell^1_\nu)^3$, and

$$(\phi_j(a))_k = \begin{cases} \sigma((a_2)_k - (a_1)_k), & j = 1 \\ \rho(a_1)_k - (a_2)_k - (a_1 * a_3)_k, & j = 2 \\ -\beta(a_3)_k + (a_1 * a_2)_k, & j = 3 \end{cases}$$

- After rescaling time by L we solve recursively

$$\begin{pmatrix} (a_1)_k \\ (a_2)_k \\ (a_3)_k \end{pmatrix} = \frac{L}{k} \begin{pmatrix} \sigma((a_2)_{k-1} - (a_1)_{k-1}) \\ \rho(a_1)_{k-1} - (a_2)_{k-1} - (a_1 * a_3)_{k-1} \\ -\beta(a_3)_{k-1} + (a_1 * a_2)_{k-1} \end{pmatrix}.$$

Example - Lorenz system

- We have

$$Y_0 := \max\{Y_0^{(1)}, Y_0^{(2)}, Y_0^{(3)}\},$$

Example - Lorenz system

- We have
$$Y_0 := \max\{Y_0^{(1)}, Y_0^{(2)}, Y_0^{(3)}\},$$
- where
$$Y_0^{(1)} = \frac{L}{N+1}|\sigma((\bar{a}_2)_N - (\bar{a}_1)_N)|\nu^{N+1},$$
$$Y_0^{(2)} = \nu \sum_{k=N}^{2N} \frac{L}{k+1}\,|\rho(\bar{a}_1)_k - (\bar{a}_2)_k - (\bar{a}_1 * \bar{a}_3)_k|\,\nu^k,$$
$$Y_0^{(3)} = \nu \sum_{k=N}^{2N} \frac{L}{k+1}\,|-\beta(\bar{a}_3)_k + (\bar{a}_1 * \bar{a}_2)_k|\,\nu^k,$$

Example - Lorenz system

- We also define

$$Z(r) := \frac{\nu}{N+1}\left(\max\left\{z_0^{(1)}, z_0^{(2)}, z_0^{(3)}\right\} + \max\left\{z_1^{(2)}, z_1^{(3)}\right\} r\right),$$

Example - Lorenz system

- We also define

$$Z(r) := \frac{\nu}{N+1}\left(\max\left\{z_0^{(1)}, z_0^{(2)}, z_0^{(3)}\right\} + \max\left\{z_1^{(2)}, z_1^{(3)}\right\} r\right),$$

- where

$$\begin{aligned}
z_0^{(1)} &= 2L|\sigma| \\
z_0^{(2)} &= L(|\rho| + 1 + \|\bar{a}_1\|_{1,\nu} + \|\bar{a}_3\|_{1,\nu}) \\
z_1^{(2)} &= 2L \\
z_0^{(3)} &= L(|\beta| + \|\bar{a}_1\|_{1,\nu} + \|\bar{a}_2\|_{1,\nu}) \\
z_1^{(3)} &= 2L.
\end{aligned}$$

Example - Lorenz system

- It then follows that

$$\|T(\bar{a}) - \bar{a}\| \le Y_0$$

and

$$\sup_{u \in \overline{B_r(0)}} \|DT(u)\| \le Z(r).$$

Example - Lorenz system

- It then follows that
$$\|T(\bar{a}) - \bar{a}\| \leq Y_0$$
and
$$\sup_{u \in \overline{B_r(0)}} \|DT(u)\| \leq Z(r).$$

- Hence we have the radii polynomial
$$p(r) = Z(r)r - r + Y_0$$
and we can apply the contraction mapping radii polynomial theorem.

Example - Lorenz system

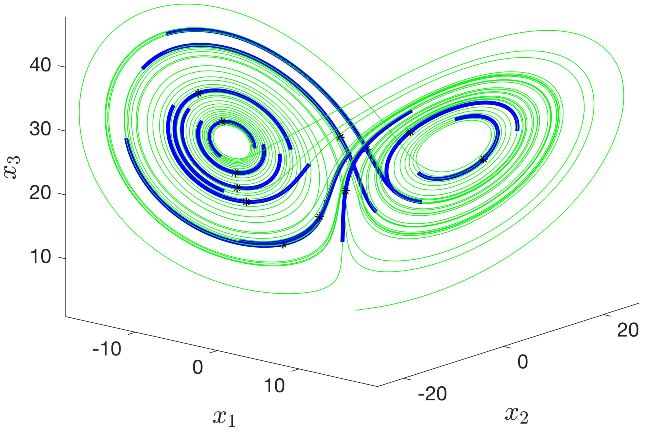


Figure: The green curve is a numerical integration of a single initial condition for 50 times units. The blue curves represent one step of Taylor integration with order $N = 200$ for different initial conditions, plotted for times $t \in (-0.2, 0.2)$.

Periodic solutions of vector fields

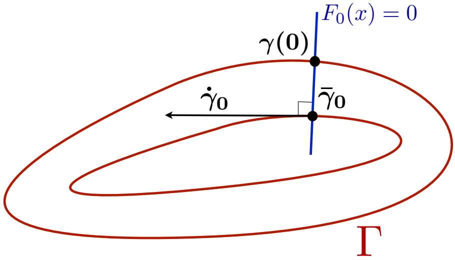
- Given a periodic solution $\gamma(t)$ of an ODE $\dot{x} = f(x)$, then $\gamma_\theta(t) := \gamma(t+\theta)$ is also a periodic solution for all for all $\theta \in \mathbb{R}$.

Periodic solutions of vector fields

- Given a periodic solution $\gamma(t)$ of an ODE $\dot{x} = f(x)$, then $\gamma_\theta(t) := \gamma(t+\theta)$ is also a periodic solution for all for all $\theta \in \mathbb{R}$.
- We make use of the so-called Poincaré phase condition in order to isolate $\gamma(t)$ in function space

$$\dot{\gamma}_0 \cdot (\bar{\gamma}_0 - \gamma(0)) = 0,$$

for fixed $\bar{\gamma}_0 = (\bar{\gamma}_{0,1}, \ldots, \bar{\gamma}_{0,n}) \in \mathbb{R}^n$ and $\dot{\gamma}_0 = (\dot{\gamma}_{0,1}, \ldots, \dot{\gamma}_{0,n}) \in \mathbb{R}^n$.



Periodic solutions of vector fields

- Let $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ be a $\frac{2\pi}{\omega}$-periodic solution to $\dot{x} = f(x)$, with $f\colon \mathbb{R}^n \to \mathbb{R}^n$.

Periodic solutions of vector fields

- Let $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ be a $\frac{2\pi}{\omega}$-periodic solution to $\dot{x} = f(x)$, with $f\colon \mathbb{R}^n \to \mathbb{R}^n$.
- For each $j = 1, \ldots, n$, denote by $a_j := \{(a_j)_k\}_{k\in\mathbb{Z}}$ the sequence of Fourier coefficients of γ_j, that is

$$\gamma_j(t) = \sum_{k\in\mathbb{Z}} (a_j)_k e^{ik\omega t}.$$

Periodic solutions of vector fields

- Let $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ be a $\frac{2\pi}{\omega}$-periodic solution to $\dot{x} = f(x)$, with $f\colon \mathbb{R}^n \to \mathbb{R}^n$.
- For each $j = 1, \dots, n$, denote by $a_j := \{(a_j)_k\}_{k \in \mathbb{Z}}$ the sequence of Fourier coefficients of γ_j, that is

$$\gamma_j(t) = \sum_{k \in \mathbb{Z}} (a_j)_k e^{ik\omega t}.$$

- Then there is a $\nu > 1$ such that $a_j \in \ell_\nu^1$ for each $j = 1, \dots, n$. Denote $a = (a_1, a_2, \dots, a_n) \in \ell_\nu^1 \times \ell_\nu^1 \times \cdots \times \ell_\nu^1 = \left(\ell_\nu^1\right)^n$.

Periodic solutions of vector fields

- Let $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ be a $\frac{2\pi}{\omega}$-periodic solution to $\dot{x} = f(x)$, with $f\colon \mathbb{R}^n \to \mathbb{R}^n$.
- For each $j = 1, \dots, n$, denote by $a_j := \{(a_j)_k\}_{k \in \mathbb{Z}}$ the sequence of Fourier coefficients of γ_j, that is

$$\gamma_j(t) = \sum_{k \in \mathbb{Z}} (a_j)_k e^{ik\omega t}.$$

- Then there is a $\nu > 1$ such that $a_j \in \ell^1_\nu$ for each $j = 1, \dots, n$. Denote $a = (a_1, a_2, \dots, a_n) \in \ell^1_\nu \times \ell^1_\nu \times \cdots \times \ell^1_\nu = \left(\ell^1_\nu\right)^n$.
- Assuming that f is a polynomial vector field, then the j-th component of $f(\gamma(t))$ is of the form

$$f_j(\gamma(t)) = \sum_{k \in \mathbb{Z}} (\phi_j(a))_k e^{ik\omega t},$$

  where the k-th Fourier coefficient $(\phi_j(a))_k$ of $f_j(\gamma(t))$ is a combination of discrete convolutions involving the terms of a.

Periodic solutions of vector fields

- Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a $\frac{2\pi}{\omega}$-periodic solution to $\dot{x} = f(x)$, with $f : \mathbb{R}^n \to \mathbb{R}^n$.
- For each $j = 1, \dots, n$, denote by $a_j := \{(a_j)_k\}_{k \in \mathbb{Z}}$ the sequence of Fourier coefficients of γ_j, that is

$$\gamma_j(t) = \sum_{k \in \mathbb{Z}} (a_j)_k e^{ik\omega t}.$$

- Then there is a $\nu > 1$ such that $a_j \in \ell^1_\nu$ for each $j = 1, \dots, n$. Denote $a = (a_1, a_2, \dots, a_n) \in \ell^1_\nu \times \ell^1_\nu \times \cdots \times \ell^1_\nu = \left(\ell^1_\nu\right)^n$.
- Assuming that f is a polynomial vector field, then the j-th component of $f(\gamma(t))$ is of the form

$$f_j(\gamma(t)) = \sum_{k \in \mathbb{Z}} (\phi_j(a))_k e^{ik\omega t},$$

  where the k-th Fourier coefficient $(\phi_j(a))_k$ of $f_j(\gamma(t))$ is a combination of discrete convolutions involving the terms of a.
- For each $j = 1, \dots, n$, denote $\phi_j(a) := \left\{(\phi_j(a))_k\right\}_{k \in \mathbb{Z}}$ and $\phi(a) = (\phi_1(a), \phi_2(a), \dots, \phi_n(a))$.

Periodic solutions of vector fields

- The assumption that γ is a periodic solution to the ODE is equivalent to
$$\dot{\gamma}_j(t) - f_j(\gamma(t)) = \sum_{k\in\mathbb{Z}} (ik\omega(a_j)_k - (\phi_j(a))_k)e^{ik\omega t} = 0,$$
for each $j = 1, \ldots, n$.

Periodic solutions of vector fields

- The assumption that γ is a periodic solution to the ODE is equivalent to
$$\dot{\gamma}_j(t) - f_j(\gamma(t)) = \sum_{k\in\mathbb{Z}} (ik\omega(a_j)_k - (\phi_j(a))_k) e^{ik\omega t} = 0,$$
for each $j = 1, \dots, n$.
- Letting $x = (\omega, a)$, this can be written as
$$(F_j(x))_k := ik\omega(a_j)_k - (\phi_j(a))_k = 0,$$
for all $k \in \mathbb{Z}$ and $j = 1, \dots, n$.

Periodic solutions of vector fields

- The assumption that γ is a periodic solution to the ODE is equivalent to
$$\dot{\gamma}_j(t) - f_j(\gamma(t)) = \sum_{k\in\mathbb{Z}} (ik\omega(a_j)_k - (\phi_j(a))_k) e^{ik\omega t} = 0,$$
for each $j = 1, \ldots, n$.
- Letting $x = (\omega, a)$, this can be written as
$$(F_j(x))_k := ik\omega(a_j)_k - (\phi_j(a))_k = 0,$$
for all $k \in \mathbb{Z}$ and $j = 1, \ldots, n$.
- In terms of the Fourier coefficients, the Poincaré phase condition is equivalent to
$$\bar{\gamma}_0 \cdot f(\bar{\gamma}_0) - \sum_{n=1}^{N} f_n(\bar{\gamma}_0) \sum_{k\in\mathbb{Z}} a_{n,k} = 0.$$

Periodic solutions of vector fields

- However this condition can be relaxed, by fixing $k_0 \in \mathbb{N}$, to the following

$$F_0(x) := \sum_{j=1}^{n} \bar{\gamma}_{0,j}\dot{\gamma}_{0,j} - \sum_{j=1}^{n}\sum_{|k|\le k_0} \dot{\gamma}_{0,j}(a_j)_k = 0.$$

Periodic solutions of vector fields

- However this condition can be relaxed, by fixing $k_0 \in \mathbb{N}$, to the following
$$F_0(x) := \sum_{j=1}^{n} \bar{\gamma}_{0,j}\dot{\gamma}_{0,j} - \sum_{j=1}^{n}\sum_{|k|\le k_0} \dot{\gamma}_{0,j}(a_j)_k = 0.$$
- Considering $X := \mathbb{C} \times \left(\ell^1_\nu\right)^n$ with norm
$$\|x\|_X = \|(\omega, a)\|_X := \max\left\{|\omega|, \|a_1\|_{1,\nu}, \|a_2\|_{1,\nu}, \dots, \|a_n\|_{1,\nu}\right\}.$$

Periodic solutions of vector fields

- However this condition can be relaxed, by fixing $k_0 \in \mathbb{N}$, to the following

$$F_0(x) := \sum_{j=1}^{n} \bar{\gamma}_{0,j} \dot{\gamma}_{0,j} - \sum_{j=1}^{n} \sum_{|k| \le k_0} \dot{\gamma}_{0,j} (a_j)_k = 0.$$

- Considering $X := \mathbb{C} \times \left(\ell_\nu^1\right)^n$ with norm

$$\|x\|_X = \|(\omega, a)\|_X := \max\left\{|\omega|, \|a_1\|_{1,\nu}, \|a_2\|_{1,\nu}, \dots, \|a_n\|_{1,\nu}\right\}.$$

- For each $j = 1, \dots, n$, let

$$F_j(x) = \left\{(F_j(x))_k\right\}_{k \in \mathbb{Z}}.$$

- We need to solve

$$F(x) = \begin{pmatrix} F_0(x) \\ F_1(x) \\ \vdots \\ F_n(x) \end{pmatrix} = 0$$

in the Banach space $X = \mathbb{C} \times \left(\ell_\nu^1\right)^n$.

Example - Lorenz system

- Consider again the Lorenz system is given by

$$\begin{cases} \dot{x}_1 = \sigma(x_2 - x_1) \\ \dot{x}_2 = \rho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 = -\beta x_3 + x_1 x_2, \end{cases}$$

Example - Lorenz system

- Consider again the Lorenz system is given by

$$\begin{cases} \dot{x}_1 = \sigma(x_2 - x_1) \\ \dot{x}_2 = \rho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 = -\beta x_3 + x_1 x_2, \end{cases}$$

- The functional F takes the form

$$F_j(x) = \begin{cases} \sum_{j=1}^{3} \bar{\gamma}_{0,j}\dot{\gamma}_{0,j} - \sum_{j=1}^{3} \dot{\gamma}_{0,j} \sum_{|k| \le k_0} (a_j)_k, & j = 0 \\ \{ik\omega(a_1)_k - (\phi_1(a))_k\}_{k\in\mathbb{Z}}, & j = 1 \\ \{ik\omega(a_2)_k - (\phi_2(a))_k\}_{k\in\mathbb{Z}}, & j = 2 \\ \{ik\omega(a_3)_k - (\phi_2(a))_k\}_{k\in\mathbb{Z}}, & j = 3, \end{cases}$$

where

$$(\phi_j(a))_k = \begin{cases} \sigma\left((a_2)_k - (a_1)_k\right), & j = 1 \\ \rho(a_1)_k - (a_2)_k - (a_1 * a_3)_k, & j = 2 \\ -\beta(a_3)_k + (a_1 * a_2)_k, & j = 3. \end{cases}$$

Example - Lorenz system

For the classical parameter values $\rho = 28$, $\beta = -8/3$ and $\sigma = 10$ we prove the existence of the periodic orbits below (with $\nu = 1.01$, except for the orbit AAAAAABBABBBAAAA where $\nu = 1.009$).

orbit itinerary	period	N	$\mathcal{I}$
AB	1.5587	80	$[3.0112 \times 10^{-14} \quad 0.0078691]$
AAB	2.3059	110	$[6.0866 \times 10^{-13} \quad 0.0041978]$
ABB	2.3059	110	$[5.1273 \times 10^{-13} \quad 0.0031017]$
AAAB	3.0236	140	$[2.1023 \times 10^{-12} \quad 0.0018607]$
AABB	3.0843	140	$[2.6987 \times 10^{-13} \quad 0.0015571]$
ABBB	3.0236	140	$[2.016 \times 10^{-12} \quad 0.0014146]$
AAAAB	3.7256	170	$[1.6132 \times 10^{-12} \quad 0.00089004]$
AAABB	3.8203	170	$[2.6086 \times 10^{-12} \quad 0.00059818]$
AABAB	3.8695	170	$[3.5391 \times 10^{-12} \quad 0.00064646]$
AABBB	3.8203	170	$[2.1882 \times 10^{-12} \quad 0.00055587]$
AAAABABB	6.0824	300	$[2.7194 \times 10^{-11} \quad 6.8867 \times 10^{-5}]$
AABBAABB	8.957	560	$[2.6555 \times 10^{-11} \quad 6.4514 \times 10^{-5}]$
AAAAAABBABBBAAAA	11.9973	870	$[1.2693 \times 10^{-10} \quad 3.0684 \times 10^{-5}]$

Example - Lorenz system

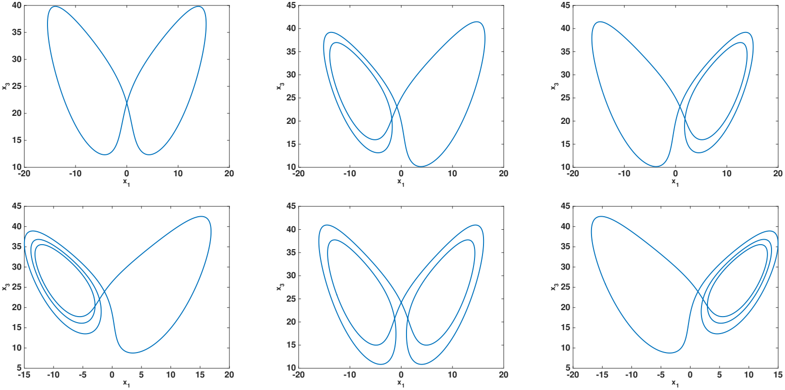


Figure: **Top left:** AB, **Top center:** AAB, **Top right:** ABB, **Bottom left:** AAAB, **Bottom center:** AABB, **Bottom right:** ABBB.

Example - Lorenz system

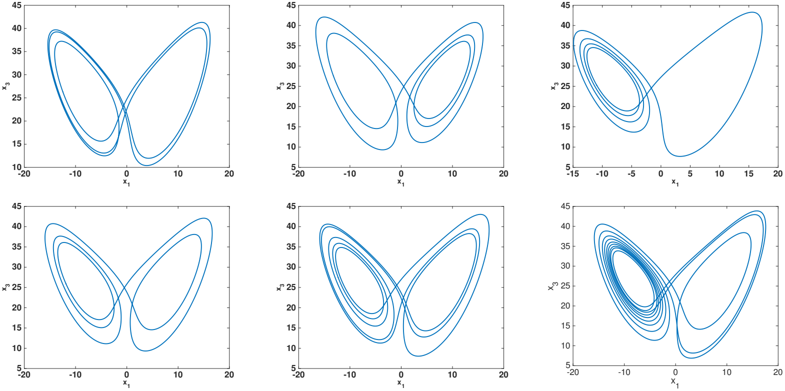


Figure: **Top left:** AABAB, **Top center:** AABBB, **Top right:** AAAAB, **Bottom left:** AAABB, **Bottom center:** AAAABABB, **Bottom right:** AAAAABAAAABB.

Example - Lorenz system

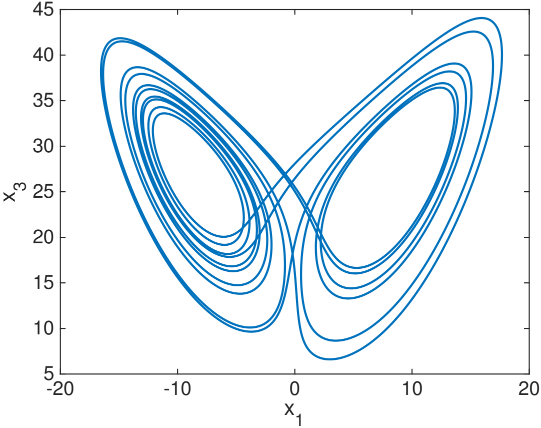


Figure: A periodic orbit AAAAAABBABBBAAAA with period $p \approx 11.9973$. The C^0-error between the true solution and the numerical approximation is $r = 1.2693 \times 10^{-10}$. The proof was obtained with $N = 870$ and $\nu = 1.009$.

The Parameterization Method for (Un)Stable Manifolds

Theorem ((Un)Stable Manifold Theorem)

Consider the ODE

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n$$

with $f \in C^k(\mathbb{R}^n)$, $k \in \{1, 2, 3, \ldots, \infty\} \cup \{\omega\}$ *for which* $f(0) = 0$. *Assume*

$$Df(0) = \begin{bmatrix} A^u & 0 \\ 0 & A^s \end{bmatrix}$$

is hyperbolic where $A^u \in M_d(\mathbb{R})$, $A^s \in M_{n-d}(\mathbb{R})$, *and the real parts of the all the eigenvalues of* A^u *and* A^s *are positive and negative, respectively. Then, there exists a neighborhood* $U = U^u \times U^s \subset \mathbb{R}^d \times \mathbb{R}^{n-d}$ *of the origin and* C^k *functions* $P^u \colon U^u \to \mathbb{R}^n$ *and* $P^s \colon U^s \to \mathbb{R}^n$ *that are tangent to* $\mathbb{R}^d \times 0$ *and* $0 \times \mathbb{R}^{n-d}$ *at the point* 0, *respectively, such that*

$$W^u_{loc}(0) = P^u(U^u) \quad \text{and} \quad W^s_{loc}(0) = P^s(U^s).$$

The Parameterization Method

- Denote the diagonal matrix of stable eigenvalues by

$$\Lambda = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_m \end{bmatrix}$$

The Parameterization Method

- Denote the diagonal matrix of stable eigenvalues by

$$\Lambda = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_m \end{bmatrix}$$

- and the matrix whose columns are the associated associated eigenvectors by

$$A_0 = [\xi_1 | \dots | \xi_m].$$

The Parameterization Method

- Denote the diagonal matrix of stable eigenvalues by

$$\Lambda = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_m \end{bmatrix}$$

- and the matrix whose columns are the associated associated eigenvectors by

$$A_0 = [\xi_1 | \dots | \xi_m].$$

- In the basis defined by the stable eigenvectors, the linearized equation restricted to the stable subspace is

$$\dot{y} = \Lambda y, \quad y \in \mathbb{R}^m.$$

The Parameterization Method

- Denote the diagonal matrix of stable eigenvalues by

$$\Lambda = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_m \end{bmatrix}$$

- and the matrix whose columns are the associated associated eigenvectors by

$$A_0 = [\xi_1 | \dots | \xi_m].$$

- In the basis defined by the stable eigenvectors, the linearized equation restricted to the stable subspace is

$$\dot{y} = \Lambda y, \quad y \in \mathbb{R}^m.$$

- The associated flow is given by $\psi(t, x) = e^{\Lambda t} x$.

The Parameterization Method

- Denote the diagonal matrix of stable eigenvalues by

$$\Lambda = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_m \end{bmatrix}$$

- and the matrix whose columns are the associated associated eigenvectors by

$$A_0 = [\xi_1 | \dots | \xi_m].$$

- In the basis defined by the stable eigenvectors, the linearized equation restricted to the stable subspace is

$$\dot{y} = \Lambda y, \quad y \in \mathbb{R}^m.$$

- The associated flow is given by $\psi(t, x) = e^{\Lambda t} x$.
- Our goal is to construct an analytic function $P \colon B^m \to \mathbb{R}^n$ such that $P(B^m) = W^s_{loc}(\tilde{x})$.

The Parameterization Method

- To obtain the constraints needed to solve for P, we required that P be a conjugacy between the original flow φ restricted to $W^s_{\text{loc}}(\tilde{x})$ and the flow ψ of the linear equation.

The Parameterization Method

- To obtain the constraints needed to solve for P, we required that P be a conjugacy between the original flow φ restricted to $W^s_{\text{loc}}(\tilde{x})$ and the flow ψ of the linear equation.
- The most obvious restriction is that P must map fixed points to fixed points and hence

$$P(0) = \tilde{x}.$$

The Parameterization Method

- To obtain the constraints needed to solve for P, we required that P be a conjugacy between the original flow φ restricted to $W^s_{\text{loc}}(\tilde{x})$ and the flow ψ of the linear equation.
- The most obvious restriction is that P must map fixed points to fixed points and hence
$$P(0) = \tilde{x}.$$
- To obtain the conjugacy we assume that
$$DP(0) = A_0$$

The Parameterization Method

- To obtain the constraints needed to solve for P, we required that P be a conjugacy between the original flow φ restricted to $W^s_{\text{loc}}(\tilde{x})$ and the flow ψ of the linear equation.
- The most obvious restriction is that P must map fixed points to fixed points and hence
$$P(0) = \tilde{x}.$$
- To obtain the conjugacy we assume that
$$DP(0) = A_0$$
- and
$$\varphi\left(t, P(\sigma)\right) = P\left(e^{\Lambda t}\sigma\right),$$
for all $\sigma \in B^m$.

The Parameterization Method

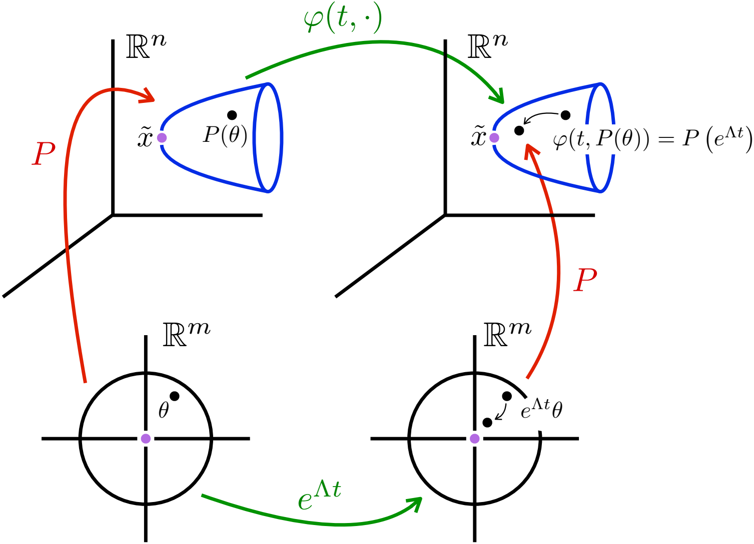


Figure: **Schematic of the Parameterization Method**

The Parameterization Method

Lemma (Parameterization Lemma)

Let $P \colon B^m \subset \mathbb{R}^m \to \mathbb{R}^n$ be a smooth function with

$$P(0) = \tilde{x} \qquad \textit{and} \qquad DP(0) = A_0.$$

Then $P(\sigma)$ satisfies the conjugacy relationship above if and only if P is a solution of the partial differential equation

$$\lambda_1 \sigma_1 \frac{\partial}{\partial \sigma_1} P(\sigma_1, \ldots, \sigma_m) + \ldots + \lambda_m \sigma_m \frac{\partial}{\partial \sigma_m} P(\sigma_1, \ldots, \sigma_m) = f(P(\sigma_1, \ldots, \sigma_m))$$

for all $\sigma = (\sigma_1, \ldots, \sigma_m) \in B^m$.

The invariance equation above can be written more concisely as

$$DP(\sigma)\Lambda\sigma = f(P(\sigma)), \quad \text{for} \quad \sigma \in B^m.$$

Example - Lorenz system

- Consider the Lorenz system having one equilibrium $\tilde{x}$ with only one stable eigenvalue ξ.

Example - Lorenz system

- Consider the Lorenz system having one equilibrium $\tilde{x}$ with only one stable eigenvalue ξ.
- For manifolds of dimension $m = 1$ the invariance equation becomes

$$\lambda s \frac{\partial}{\partial s} P(s) = f[P(s)].$$

Example - Lorenz system

- Consider the Lorenz system having one equilibrium $\tilde{x}$ with only one stable eigenvalue ξ.
- For manifolds of dimension $m = 1$ the invariance equation becomes

$$\lambda s \frac{\partial}{\partial s} P(s) = f[P(s)].$$

- We look for a power series solution

$$P(s) = \sum_{n=0}^{\infty} p_n s^n = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} s^n.$$

Example - Lorenz system

- Consider the Lorenz system having one equilibrium $\tilde{x}$ with only one stable eigenvalue ξ.
- For manifolds of dimension $m = 1$ the invariance equation becomes

$$\lambda s \frac{\partial}{\partial s} P(s) = f[P(s)].$$

- We look for a power series solution

$$P(s) = \sum_{n=0}^{\infty} p_n s^n = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} s^n.$$

- Imposing the first order constraints leads to

$$p_0 = \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \tilde{x}, \qquad \text{and} \qquad p_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \xi.$$

Example - Lorenz system

- To understand the terms with $n \geq 2$ first note that, by differentiating, we get that

$$\lambda s \frac{\partial}{\partial s} P(s) = \sum_{n=0}^{\infty} n\lambda \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} s^n.$$

Example - Lorenz system

- To understand the terms with $n \geq 2$ first note that, by differentiating, we get that

$$\lambda s \frac{\partial}{\partial s} P(s) = \sum_{n=0}^{\infty} n\lambda \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} s^n.$$

- For the Lorenz system, we see that

$$f(P(s)) = \sum_{n=0}^{\infty} \begin{pmatrix} \sigma(b_n - a_n) \\ \rho\, a_n - b_n - \sum_{k=0}^{n} a_{n-k}\, c_k \\ -\beta\, c_n + \sum_{k=0}^{n} a_{n-k}\, b_k \end{pmatrix} s^n.$$

Example - Lorenz system

- To understand the terms with $n \geq 2$ first note that, by differentiating, we get that

$$\lambda s \frac{\partial}{\partial s} P(s) = \sum_{n=0}^{\infty} n\lambda \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} s^n.$$

- For the Lorenz system, we see that

$$f(P(s)) = \sum_{n=0}^{\infty} \begin{pmatrix} \sigma(b_n - a_n) \\ \rho\, a_n - b_n - \sum_{k=0}^{n} a_{n-k}\, c_k \\ -\beta\, c_n + \sum_{k=0}^{n} a_{n-k}\, b_k \end{pmatrix} s^n.$$

- Upon matching like powers we have

$$n\lambda \begin{pmatrix} a_n \\ b_n \\ d_n \end{pmatrix} = \begin{pmatrix} \sigma(b_n - a_n) \\ \rho\, a_n - b_n - \sum_{k=0}^{n} a_{n-k}\, c_k \\ -\beta\, c_n + \sum_{k=0}^{n} a_{n-k}\, b_k \end{pmatrix},$$

for all $n \geq 2$.

Example - Lorenz system

- The sums contain "hidden" terms of order n, and we write

$$\sum_{k=0}^{n} a_{n-k} c_k = a_0 c_n + c_0 a_n + \sum_{k=1}^{n-1} a_{n-k} c_k,$$

  and

$$\sum_{k=0}^{n} a_{n-k} b_k = a_0 b_n + b_0 a_n + \sum_{k=1}^{n-1} a_{n-k} b_k,$$

  where the sums now depend only on terms of order strictly less than n.
- Isolating terms of order n on the left hand side gives

$$\begin{pmatrix} \sigma(b_n - a_n) - n\lambda a_n \\ \rho\, a_n - b_n - a_0 c_n - c_0 a_n - n\lambda b_n \\ -\beta\, c_n + a_0 b_n + b_0 a_n - n\lambda c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \sum_{k=1}^{n-1} a_{n-k}\, c_k \\ -\sum_{k=1}^{n-1} a_{n-k}\, b_k \end{pmatrix}.$$

Example - Lorenz system

- Noticing that we can rewrite the left hand side as

$$\begin{pmatrix} \sigma(b_n - a_n) - n\lambda a_n \\ \rho\, a_n - b_n - a_0 c_n - c_0 a_n - n\lambda b_n \\ -\beta\, c_n + a_0 b_n + b_0 a_n - n\lambda c_n \end{pmatrix} =$$

$$\begin{bmatrix} -\sigma - n\lambda & \sigma & 0 \\ \rho - c_0 & -1 - n\lambda & -a_0 \\ b_0 & a_0 & -\beta - n\lambda \end{bmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = [Df(\tilde{x}) - n\lambda \mathrm{I}]\, p_n .$$

- We can express the system above in matrix form as

$$[Df(\tilde{x}) - n\lambda \mathrm{I}]\, p_n = R_n,$$

where

$$R_n := \begin{pmatrix} 0 \\ \sum_{k=1}^{n-1} a_{n-k}\, c_k \\ -\sum_{k=1}^{n-1} a_{n-k}\, b_k \end{pmatrix}.$$

Example - Lorenz system

- Noticing that $n\lambda < \lambda < 0$ is never an eigenvalue for $n \geq 2$, since λ is the only stable eigenvalue, we have the recursive formula

$$p_n = \left[Df(\tilde{x}) - n\lambda \mathrm{I} \right]^{-1} R_n, \quad n \geq 2.$$

Example - Lorenz system

- Noticing that $n\lambda < \lambda < 0$ is never an eigenvalue for $n \geq 2$, since λ is the only stable eigenvalue, we have the recursive formula

$$p_n = \left[Df(\tilde{x}) - n\lambda \mathrm{I}\right]^{-1} R_n, \quad n \geq 2.$$

- Hence we have the N-th order approximation for the stable manifold

$$P^N(s) = \sum_{n=0}^{N} p_n s^n.$$

Example - Lorenz system

- For 2-dimensional manifolds the formulas becomes

$$\left[Df(\tilde{x}) - (m\lambda_1 + n\lambda_2)\mathrm{I}\right] p_{mn} = R_{mn},$$

where

$$R_{mn} = \begin{pmatrix} 0 \\ \sum_{k=0}^{m} \sum_{l=0}^{n} \hat{\delta}_{kl}^{mn} a_{(m-k)(n-l)} c_{kl} \\ -\sum_{k=0}^{m} \sum_{l=0}^{n} \hat{\delta}_{kl}^{mn} a_{(m-k)(n-l)} b_{kl} \end{pmatrix}.$$

Example - Lorenz system

- For 2-dimensional manifolds the formulas becomes

$$\left[Df(\tilde{x}) - (m\lambda_1 + n\lambda_2)\mathrm{I}\right] p_{mn} = R_{mn},$$

where

$$R_{mn} = \begin{pmatrix} 0 \\ \sum_{k=0}^{m}\sum_{l=0}^{n}\hat{\delta}_{kl}^{mn} a_{(m-k)(n-l)} c_{kl} \\ -\sum_{k=0}^{m}\sum_{l=0}^{n}\hat{\delta}_{kl}^{mn} a_{(m-k)(n-l)} b_{kl} \end{pmatrix}.$$

- The N-th order numerical approximation is

$$P^N(s_1, s_2) = \sum_{n=0}^{N}\sum_{m=0}^{n} p_{n-m,m} s_1^{n-m} s_2^{m}.$$