Computer assisted proofs in differential equations
Lecture 3 - Dec. 12, 2019

Marcio Gameiro

Rutgers University

and

University of São Paulo at São Carlos - ICMC-USP

12th Americas Conference on Differential Equations and Nonlinear Analysis (Americas XII)

December 9-13, 2019, Mathematics Research Center (CIMAT)

Guanajuato, México

Sequence Spaces

- We denote
$$S_{\mathbb{I}}(\mathbb{C}) := \{a = \{a_n\} \mid a_n \in \mathbb{C}, n \in \mathbb{I}\}$$
where $\mathbb{I} = \mathbb{N}$ or $\mathbb{I} = \mathbb{Z}$. We also use $S_{\mathbb{I}}(\mathbb{R})$.

Sequence Spaces

- We denote
$$S_{\mathbb{I}}(\mathbb{C}) := \{a = \{a_n\} \mid a_n \in \mathbb{C}, n \in \mathbb{I}\}$$
where $\mathbb{I} = \mathbb{N}$ or $\mathbb{I} = \mathbb{Z}$. We also use $S_{\mathbb{I}}(\mathbb{R})$.
- For $\nu > 0$ we define
$$\|a\|_{1,\nu} := \sum_{n\in\mathbb{Z}} |a_n|\nu^{|n|}.$$

Sequence Spaces

- We denote
$$S_{\mathbb{I}}(\mathbb{C}) := \{a = \{a_n\} \mid a_n \in \mathbb{C}, n \in \mathbb{I}\}$$
where $\mathbb{I} = \mathbb{N}$ or $\mathbb{I} = \mathbb{Z}$. We also use $S_{\mathbb{I}}(\mathbb{R})$.
- For $\nu > 0$ we define
$$\|a\|_{1,\nu} := \sum_{n \in \mathbb{Z}} |a_n| \nu^{|n|}.$$
- We make use of the weighted ℓ^1 space (analogous for $\ell^1_{\nu,\mathbb{N}}$)
$$\ell^1_{\nu,\mathbb{Z}} := \left\{ a \in S_{\mathbb{Z}}(\mathbb{C}) \mid \|a\|_{1,\nu} < \infty \right\}.$$

Sequence Spaces

- We denote
$$S_{\mathbb{I}}(\mathbb{C}) := \{a = \{a_n\} \mid a_n \in \mathbb{C}, n \in \mathbb{I}\}$$
where $\mathbb{I} = \mathbb{N}$ or $\mathbb{I} = \mathbb{Z}$. We also use $S_{\mathbb{I}}(\mathbb{R})$.
- For $\nu > 0$ we define
$$\|a\|_{1,\nu} := \sum_{n \in \mathbb{Z}} |a_n| \nu^{|n|}.$$
- We make use of the weighted ℓ^1 space (analogous for $\ell^1_{\nu,\mathbb{N}}$)
$$\ell^1_{\nu,\mathbb{Z}} := \left\{a \in S_{\mathbb{Z}}(\mathbb{C}) \mid \|a\|_{1,\nu} < \infty\right\}.$$
- We often just denote ℓ^1_ν for $\ell^1_{\nu,\mathbb{N}}$ or $\ell^1_{\nu,\mathbb{Z}}$.

Sequence Spaces

- We denote
$$S_{\mathbb{I}}(\mathbb{C}) := \{a = \{a_n\} \mid a_n \in \mathbb{C}, n \in \mathbb{I}\}$$
where $\mathbb{I} = \mathbb{N}$ or $\mathbb{I} = \mathbb{Z}$. We also use $S_{\mathbb{I}}(\mathbb{R})$.
- For $\nu > 0$ we define
$$\|a\|_{1,\nu} := \sum_{n \in \mathbb{Z}} |a_n| \nu^{|n|}.$$
- We make use of the weighted ℓ^1 space (analogous for $\ell^1_{\nu,\mathbb{N}}$)
$$\ell^1_{\nu,\mathbb{Z}} := \left\{ a \in S_{\mathbb{Z}}(\mathbb{C}) \mid \|a\|_{1,\nu} < \infty \right\}.$$
- We often just denote ℓ^1_ν for $\ell^1_{\nu,\mathbb{N}}$ or $\ell^1_{\nu,\mathbb{Z}}$.
- For the convolution or Cauchy products we have
$$\|a * b\|_{1,\nu} \leq \|a\|_{1,\nu} \|b\|_{1,\nu}.$$

Taylor Methods for Initial Value Problems

- Denote $D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$ and let

$$C^\omega(D_r(z_0), \mathbb{C}) = \{f\colon D_r(z_0) \to \mathbb{C} \mid f \text{ is bounded and analytic}\}.$$

Taylor Methods for Initial Value Problems

- Denote $D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$ and let

$$C^\omega(D_r(z_0), \mathbb{C}) = \{f\colon D_r(z_0) \to \mathbb{C} \mid f \text{ is bounded and analytic}\}.$$

- We use the supremum norm

$$\|f\|_\infty := \sup\{|f(z)| \mid z \in D_r(z_0)\}.$$

Lemma

Let $f \in C^\omega(D_r(z_0), \mathbb{C})$ with Taylor series given by

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

If $0 < \nu < r$, then $a = \{a_k\}_{k=0}^{\infty} \in \ell_\nu^1$.

Lemma

Let $a = \{a_k\}_{k=0}^{\infty} \in \ell_\nu^1$ *and define*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Then $f \in C^\omega(D_\nu(z_0), \mathbb{C})$ *and*

$$\|f\|_\infty \leq \|a\|_{1,\nu}.$$

Lemma

Let $a = \{a_k\}_{k=0}^{\infty} \in \ell_\nu^1$ *and define*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Then $f \in C^\omega(D_\nu(z_0), \mathbb{C})$ *and*

$$\|f\|_\infty \leq \|a\|_{1,\nu}.$$

Proof.

The estimate

$$\begin{aligned}\|a\|_{1,\nu} &= \sum_{k=0}^{\infty} |a_k| \nu^k \geq \sup_{|z-z_0|<\nu} \sum_{k=0}^{\infty} |a_k| |z - z_0|^k \\ &\geq \sup_{|z-z_0|<\nu} \left| \sum_{k=0}^{\infty} a_k (z - z_0)^k \right| = \sup_{|z-z_0|<\nu} |f(z)| = \|f\|_\infty\end{aligned}$$

shows that $f \in C^\omega(D_\nu(z_0), \mathbb{C})$. □

Example

- Let $z_0 = 0 \in \mathbb{C}$ and consider $a \in \ell_1^1$ given by

$$a_0 = 0 \quad \text{and} \quad a_k = \frac{1}{k^2} \quad k \geq 1.$$

Example

- Let $z_0 = 0 \in \mathbb{C}$ and consider $a \in \ell_1^1$ given by

$$a_0 = 0 \quad \text{and} \quad a_k = \frac{1}{k^2} \quad k \geq 1.$$

- The function

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{k^2} z^k,$$

is analytic and bounded on $D_1(0) \subset \mathbb{C}$ as $\|f\|_\infty \leq \frac{\pi^2}{6}$.

Taylor Methods for Initial Value Problems

Example

- Let $z_0 = 0 \in \mathbb{C}$ and consider $a \in \ell_1^1$ given by

$$a_0 = 0 \quad \text{and} \quad a_k = \frac{1}{k^2} \quad k \geq 1.$$

- The function

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{k^2} z^k,$$

  is analytic and bounded on $D_1(0) \subset \mathbb{C}$ as $\|f\|_\infty \leq \frac{\pi^2}{6}$.

- The function f' has power series

$$f'(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^{k-1}$$

  which is analytic on $D_1(0)$ but is no longer bounded. To see, observe that $f'(z) = -\frac{\ln(1-z)}{z}$ for $|z| < 1$.

Taylor Methods for Initial Value Problems

Example

- In terms of sequence spaces f' corresponds to

$$\left\{ \frac{1}{k} \right\}_{k=1}^{\infty},$$

which is not in ℓ_1^1.

Example

- In terms of sequence spaces f' corresponds to

$$\left\{\frac{1}{k}\right\}_{k=1}^{\infty},$$

which is not in ℓ_1^1.

- This discussion shows that differentiation is not a bounded linear operator on $C^\omega(D_1(0), \mathbb{C})$ and its induced action on ℓ_1^1 is not bounded either.

Example

- In terms of sequence spaces f' corresponds to

$$\left\{ \frac{1}{k} \right\}_{k=1}^{\infty},$$

which is not in ℓ_1^1.

- This discussion shows that differentiation is not a bounded linear operator on $C^\omega(D_1(0), \mathbb{C})$ and its induced action on ℓ_1^1 is not bounded either.

- The following Lemma shows that we can recover derivatives by "giving up" some portion of the domain of analyticity. Or that differentiation induces a bounded linear map from ℓ_ν^1 into $\ell_{\nu'}^1$, for any $0 < \nu' < \nu$.

Taylor Methods for Initial Value Problems

Lemma (Cauchy Bounds for Analytic Functions)

Let $\nu > 0$, $z_0 \in \mathbb{C}$ *and* $a = \{a_k\}_{k \in \mathbb{N}} \in \ell^1_\nu$. *Define* $f \colon D_\nu(z_0) \to \mathbb{C}$ *by*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Let $0 < \sigma \leq 1$ *and* $\nu' = \nu e^{-\sigma} < \nu$. *Then*

$$\sup_{z \in D_{\nu'}(z_0)} |f'(z)| \leq \frac{1}{\nu\sigma} \|a\|_{1,\nu}.$$

Proof.

- Since f is analytic in $D_\nu(z_0)$

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1},$$

for all $z \in D_\nu(z_0)$. Choose $0 < \sigma \leq 1$ and define $\nu' = \nu e^{-\sigma}$. We can check that $k e^{-\sigma(k-1)} < 1/\sigma$ for all $k \geq 0$.

$\square$

Taylor Methods for Initial Value Problems

Proof.

- Since f is analytic in $D_\nu(z_0)$

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1},$$

  for all $z \in D_\nu(z_0)$. Choose $0 < \sigma \leq 1$ and define $\nu' = \nu e^{-\sigma}$. We can check that $ke^{-\sigma(k-1)} < 1/\sigma$ for all $k \geq 0$.

- Thus, for any $z \in D_{\nu'}(z_0)$ we have that

$$\begin{aligned} |f'(z)| &\leq \sum_{k=1}^{\infty} k|a_k||z|^{k-1} \leq \sum_{k=1}^{\infty} k|a_k|(\nu e^{-\sigma})^{k-1} \\ &\leq \sum_{k=1}^{\infty} \frac{1}{\nu} k e^{-\sigma(k-1)} |a_k| \nu^k \leq \frac{1}{\nu} \sum_{k=0}^{\infty} \frac{1}{\sigma} |a_k| \nu^k = \frac{1}{\nu\sigma} \|a\|_{1,\nu}. \end{aligned}$$

$\square$

Example - a first order scalar IVP

- Consider
$$\dot{x} = x(x-1), \quad x(t_0) = x_0 \in \mathbb{R}.$$

Example - a first order scalar IVP

- Consider
$$\dot{x} = x(x-1), \quad x(t_0) = x_0 \in \mathbb{R}.$$
- Rescaling of time $t \to \frac{t+t_0}{L}$ we get
$$\dot{x} - f(x) = \dot{x} - Lx(x-1) = 0, \quad x(0) = x_0 \in \mathbb{R}.$$

Example - a first order scalar IVP

- Consider

$$\dot{x} = x(x-1), \quad x(t_0) = x_0 \in \mathbb{R}.$$

- Rescaling of time $t \to \frac{t+t_0}{L}$ we get

$$\dot{x} - f(x) = \dot{x} - Lx(x-1) = 0, \quad x(0) = x_0 \in \mathbb{R}.$$

- Using a Taylor series expansion

$$x(t) = \sum_{k=0}^{\infty} a_k t^k.$$

Example - a first order scalar IVP

- Consider
$$\dot{x} = x(x-1), \quad x(t_0) = x_0 \in \mathbb{R}.$$
- Rescaling of time $t \to \frac{t+t_0}{L}$ we get
$$\dot{x} - f(x) = \dot{x} - Lx(x-1) = 0, \quad x(0) = x_0 \in \mathbb{R}.$$
- Using a Taylor series expansion
$$x(t) = \sum_{k=0}^{\infty} a_k t^k.$$
- We get $x(0) = a_0 = x_0$ and
$$\sum_{k=0}^{\infty}(k+1)a_{k+1}t^k + L\sum_{k=0}^{\infty} a_k t^k - L\sum_{k=0}^{\infty}(a*a)_k t^k = 0,$$
where $(a*a)_k = \sum_{j=0}^{k} a_{k-j}a_j$ is the Cauchy product.

Example - a first order scalar IVP

- Consider
$$\dot{x} = x(x-1), \quad x(t_0) = x_0 \in \mathbb{R}.$$
- Rescaling of time $t \to \frac{t+t_0}{L}$ we get
$$\dot{x} - f(x) = \dot{x} - Lx(x-1) = 0, \quad x(0) = x_0 \in \mathbb{R}.$$
- Using a Taylor series expansion
$$x(t) = \sum_{k=0}^{\infty} a_k t^k.$$
- We get $x(0) = a_0 = x_0$ and
$$\sum_{k=0}^{\infty}(k+1)a_{k+1}t^k + L\sum_{k=0}^{\infty} a_k t^k - L\sum_{k=0}^{\infty}(a*a)_k t^k = 0,$$
where $(a*a)_k = \sum_{j=0}^{k} a_{k-j}a_j$ is the Cauchy product.
- Matching like powers of t gives the infinite sequence of equations
$$(k+1)a_{k+1} + La_k - L(a*a)_k = 0, \quad k \geq 0.$$

Example - Fixed point formulation

- These equations gives the recursion relation

$$a_k = \begin{cases} x_0 & \text{for } k = 0 \\ \frac{L}{k}\left(-a_{k-1} + (a * a)_{k-1}\right) & \text{for } k \geq 1, \end{cases}$$

which can be solved iteratively.

Example - Fixed point formulation

- These equations gives the recursion relation

$$a_k = \begin{cases} x_0 & \text{for } k = 0 \\ \frac{L}{k}\left(-a_{k-1} + (a * a)_{k-1}\right) & \text{for } k \geq 1, \end{cases}$$

  which can be solved iteratively.

- Denote by $\tilde{a} = \{\tilde{a}_k\}_{k=0}^{\infty}$ the sequence defined iteratively above. If we prove that $\tilde{a} \in \ell_\nu^1$, then we have that $\tilde{x}\colon(-\nu,\nu) \to \mathbb{R}$, defined by

$$\tilde{x}(t) := \sum_{k=0}^{\infty} \tilde{a}_k t^k,$$

  is the unique solution and it is analytic over the interval $(-\nu,\nu)$.

Example - Fixed point formulation

- For a fixed choice of N, we obtain an approximate solution

$$\bar{x}^{(N)}(t) := \sum_{k=0}^{N} \tilde{a}_k t^k.$$

Example - Fixed point formulation

- For a fixed choice of N, we obtain an approximate solution

$$\bar{x}^{(N)}(t) := \sum_{k=0}^{N} \tilde{a}_k t^k.$$

- To show that $\tilde{a} \in \ell_\nu^1$ it is sufficient to show that $\sum_{k \geq N+1} |\tilde{a}_k| \nu^k < \infty$.

Example - Fixed point formulation

- For a fixed choice of N, we obtain an approximate solution

$$\bar{x}^{(N)}(t) := \sum_{k=0}^{N} \tilde{a}_k t^k.$$

- To show that $\tilde{a} \in \ell_\nu^1$ it is sufficient to show that $\sum_{k \geq N+1} |\tilde{a}_k| \nu^k < \infty$.
- Furthermore, observe that

$$\sup_{t \in (-\nu,\nu)} \left| \tilde{x}(t) - \bar{x}^{(N)}(t) \right| = \sup_{t \in (-\nu,\nu)} \left| \sum_{k=N+1}^{\infty} \tilde{a}_k t^k \right| \leq \sum_{k \geq N+1} |\tilde{a}_k| \nu^k,$$

which provides an explicit bound on the error between the true and approximate solution.

Example - Fixed point formulation

- For a fixed choice of N, we obtain an approximate solution

$$\bar{x}^{(N)}(t) := \sum_{k=0}^{N} \tilde{a}_k t^k.$$

- To show that $\tilde{a} \in \ell_\nu^1$ it is sufficient to show that $\sum_{k\geq N+1} |\tilde{a}_k|\nu^k < \infty$.
- Furthermore, observe that

$$\sup_{t\in(-\nu,\nu)} \left|\tilde{x}(t) - \bar{x}^{(N)}(t)\right| = \sup_{t\in(-\nu,\nu)} \left|\sum_{k=N+1}^{\infty} \tilde{a}_k t^k\right| \leq \sum_{k\geq N+1} |\tilde{a}_k|\nu^k,$$

  which provides an explicit bound on the error between the true and approximate solution.
- Fix N as above and denote

$$\bar{a} = (\bar{a}_0, \bar{a}_1, \dots, \bar{a}_{N-1}, \bar{a}_N, 0, 0, 0, \dots) := (\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_{N-1}, \tilde{a}_N, 0, 0, 0, \dots).$$

Example - Fixed point formulation

- We prove the existence in the restricted sequence space

$$X = \left\{ u \in \ell_\nu^1 \mid u_k = 0 \text{ for } 0 \le k \le N \right\}.$$

Example - Fixed point formulation

- We prove the existence in the restricted sequence space

$$X = \left\{ u \in \ell^1_\nu \mid u_k = 0 \text{ for } 0 \le k \le N \right\}.$$

- We do this by using the following map

$$T(u)_k := \begin{cases} 0 & \text{if } k = 0, \dots, N, \\ \dfrac{L}{k}\left(-\bar{a}_{k-1} - u_{k-1} + \left((\bar{a}+u) * (\bar{a}+u)\right)_{k-1}\right) & \text{if } k \ge N+1. \end{cases}$$

Example - Fixed point formulation

- We prove the existence in the restricted sequence space

$$X = \left\{ u \in \ell^1_\nu \mid u_k = 0 \text{ for } 0 \leq k \leq N \right\}.$$

- We do this by using the following map

$$T(u)_k := \begin{cases} 0 & \text{if } k = 0, \ldots, N, \\ \dfrac{L}{k}\left(-\bar{a}_{k-1} - u_{k-1} + ((\bar{a}+u) * (\bar{a}+u))_{k-1}\right) & \text{if } k \geq N+1. \end{cases}$$

- If we prove that T is a contraction, then substituting $a = \bar{a} + u$ into the recursion formula gives as the fixed point for T

$$\tilde{a}_{\text{tail}} = (0, 0, \ldots, 0, 0, \tilde{a}_{N+1}, \tilde{a}_{N+2}, \tilde{a}_{N+3}, \ldots).$$

Example - Fixed point formulation

- We prove the existence in the restricted sequence space

$$X = \left\{ u \in \ell^1_\nu \mid u_k = 0 \text{ for } 0 \le k \le N \right\}.$$

- We do this by using the following map

$$T(u)_k := \begin{cases} 0 & \text{if } k = 0, \dots, N, \\ \dfrac{L}{k}\left(-\bar{a}_{k-1} - u_{k-1} + ((\bar{a} + u) * (\bar{a} + u))_{k-1}\right) & \text{if } k \ge N + 1. \end{cases}$$

- If we prove that T is a contraction, then substituting $a = \bar{a} + u$ into the recursion formula gives as the fixed point for T

$$\tilde{a}_{\text{tail}} = (0, 0, \dots, 0, 0, \tilde{a}_{N+1}, \tilde{a}_{N+2}, \tilde{a}_{N+3}, \dots).$$

- Therefore, we prove the existence of a fixed point of $T\colon X \to X$

$$T(u)_k = \frac{L}{k}(-\bar{a} - u + (\bar{a} + u) * (\bar{a} + u))_{k-1} \quad \text{any } k \ge N + 1.$$

Example - Fixed point formulation

Lemma

Choose $\nu > 0$. *If*

$$Y_0 := L\nu \sum_{k=N}^{2N} \frac{1}{k+1} \left|(-\bar{a} + \bar{a} * \bar{a})_k\right| \nu^k,$$

and

$$Z(r) := \frac{L\nu}{N+1} \left(1 + 2\|\bar{a}\|_{1,\nu} + 2r\right),$$

then

$$\|T(0)\|_{1,\nu} \leq Y_0,$$

and for any $r > 0$

$$\sup_{\|u\|_{1,\nu} \leq r} \|DT(u)\|_{B(\ell^1_\nu, \ell^1_\nu)} \leq Z(r).$$

Example - Fixed point formulation

Theorem

Fix L, N, and ν. Consider the radii polynomial

$$p(r) := Z(r)r - r + Y_0 = \frac{2L\nu}{N+1} r^2 - \left(1 - \frac{(1 + 2\|\bar{a}\|_{1,\nu})L\nu}{N+1}\right) r + Y_0.$$

If there exists $r_0 > 0$ such that $p(r_0) < 0$, then there exists a unique

$$\tilde{u} \in \overline{B_{r_0}(0)} \subset X = \left\{ u \in \ell^1_\nu \mid u_k = 0 \text{ for } 0 \le k \le N \right\}$$

so that $\tilde{a} := \bar{a} + u \colon (-\nu, \nu) \to \mathbb{R}$ is an analytic solution to the initial value problem.

Example - Fixed point formulation

Proof of Lemma.

- Since $\bar{a}_k = 0$ for $k \geq N + 1$, the product $(\bar{a} * \bar{a})_k = \sum_{j=0}^{k} \bar{a}_{k-j}\bar{a}_j$ vanishes for $k \geq 2N + 1$.

$\square$

Example - Fixed point formulation

Proof of Lemma.

- Since $\bar{a}_k = 0$ for $k \geq N+1$, the product $(\bar{a} * \bar{a})_k = \sum_{j=0}^{k} \bar{a}_{k-j}\bar{a}_j$ vanishes for $k \geq 2N+1$.
- Hence we can compute

$$\|T(0)\|_{1,\nu} = \sum_{k=N+1}^{\infty} |T(0)_k| \nu^k = L \sum_{k=N+1}^{\infty} \left| \frac{1}{k} (-\bar{a} + \bar{a} * \bar{a})_{k-1} \right| \nu^k$$

$$= L\nu \sum_{k=N}^{\infty} \frac{1}{k+1} |(-\bar{a} + \bar{a} * \bar{a})_k| \nu^k = L\nu \sum_{k=N}^{2N} \frac{1}{k+1} |(-\bar{a} + \bar{a} * \bar{a})_k| \nu^k = Y_0.$$

□

Example - Fixed point formulation

Proof of Lemma.

- Since $\bar{a}_k = 0$ for $k \geq N+1$, the product $(\bar{a} * \bar{a})_k = \sum_{j=0}^{k} \bar{a}_{k-j}\bar{a}_j$ vanishes for $k \geq 2N+1$.
- Hence we can compute

$$\|T(0)\|_{1,\nu} = \sum_{k=N+1}^{\infty} |T(0)_k|\,\nu^k = L \sum_{k=N+1}^{\infty} \left| \frac{1}{k} \left(-\bar{a} + \bar{a} * \bar{a}\right)_{k-1} \right| \nu^k$$

$$= L\nu \sum_{k=N}^{\infty} \frac{1}{k+1} \left|(-\bar{a} + \bar{a} * \bar{a})_k\right| \nu^k = L\nu \sum_{k=N}^{2N} \frac{1}{k+1} \left|(-\bar{a} + \bar{a} * \bar{a})_k\right| \nu^k = Y_0.$$

- For any $u, h \in X$ we have that

$$(DT(u)h)_k = \frac{L}{k}(-h + 2\bar{a} * h + 2u * h)_{k-1} \quad \text{for }, k \geq N+1.$$

$\square$

Proof of Lemma.

- Hence

$$\begin{aligned}
\|DT(u)h\|_{1,\nu} &= \sum_{k=N+1}^{\infty} \frac{L}{k}\left|(-h+2\bar{a}*h+2u*h)_{k-1}\right|\nu^k \\
&= L\nu \sum_{k=N}^{\infty} \frac{1}{k+1}\left|(-h+2\bar{a}*h+2u*h)_{k}\right|\nu^k \\
&\leq \frac{L\nu}{N+1} \sum_{k=N}^{\infty} \left|(-h+2\bar{a}*h+2u*h)_{k}\right|\nu^n \\
&\leq \frac{L\nu}{N+1} \left\|(-h+2\bar{a}*h+2u*h)\right\|_{1,\nu} \\
&\leq \frac{L\nu}{N+1} \left(\|h\|_{1,\nu} + 2\|\bar{a}\|_{1,\nu}\|h\|_{1,\nu} + 2\|u\|_{1,\nu}\|h\|_{1,\nu}\right).
\end{aligned}$$

$\square$

Proof of Lemma.

- Hence

$$
\begin{aligned}
\|DT(u)h\|_{1,\nu} &= \sum_{k=N+1}^{\infty} \frac{L}{k} \left|(-h + 2\bar{a} * h + 2u * h)_{k-1}\right| \nu^k \\
&= L\nu \sum_{k=N}^{\infty} \frac{1}{k+1} \left|(-h + 2\bar{a} * h + 2u * h)_k\right| \nu^k \\
&\leq \frac{L\nu}{N+1} \sum_{k=N}^{\infty} \left|(-h + 2\bar{a} * h + 2u * h)_k\right| \nu^n \\
&\leq \frac{L\nu}{N+1} \left\|(-h + 2\bar{a} * h + 2u * h)\right\|_{1,\nu} \\
&\leq \frac{L\nu}{N+1} \left(\|h\|_{1,\nu} + 2\|\bar{a}\|_{1,\nu}\|h\|_{1,\nu} + 2\|u\|_{1,\nu}\|h\|_{1,\nu}\right).
\end{aligned}
$$

- Taking the supremum over all $h \in X$ with $\|h\|_{1,\nu} = 1$ gives

$$
\|DT(u)\|_{B(\ell^1_\nu, \ell^1_\nu)} \leq \frac{L\nu}{N+1} \left(1 + 2\|\bar{a}\|_{1,\nu} + 2\|u\|_{1,\nu}\right).
$$

□

Proof of Lemma.

- Finally, for $r > 0$ and $u \in \overline{B_r(0)} \subset X$ we have the desired bound

$$\sup_{u \in \overline{B_r(0)}} \|DT(u)\|_{B(\ell^1_\nu, \ell^1_\nu)} \leq \frac{L\nu}{N+1}\Big(1 + 2\|\bar{a}\|_{1,\nu} + 2r\Big) = Z(r).$$

$\square$

Example - Fixed point formulation

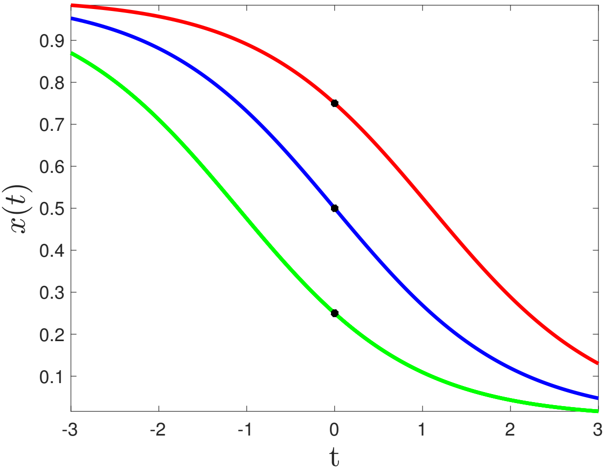

Figure: Solutions with $L = 3$, $\nu = 1$, and $N = 300$. For the green curve $x(0) = 1/4$, for the blue curve $x(0) = 1/2$, and the red curve $x(0) = 3/4$.

Example - Fixed point formulation based on Newton

- We begin by setting up a zero finding problem. Define $F\colon \ell^1_\nu \to S_{\mathbb{I}}(\mathbb{R})$ by
$$F_k(a) := \begin{cases} a_0 - x_0 & \text{for } k = 0 \\ ka_k + L\left(a_{k-1} - (a * a)_{k-1}\right) & \text{for } k \geq 1. \end{cases}$$
If $F(a) = 0$, then a is an analytic solution to the initial value problem.

Example - Fixed point formulation based on Newton

- We begin by setting up a zero finding problem. Define $F\colon \ell^1_\nu \to S_{\mathbb{I}}(\mathbb{R})$ by

$$F_k(a) := \begin{cases} a_0 - x_0 & \text{for } k = 0 \\ ka_k + L\left(a_{k-1} - (a * a)_{k-1}\right) & \text{for } k \geq 1. \end{cases}$$

  If $F(a) = 0$, then a is an analytic solution to the initial value problem.
- We define $F^{(N)}\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ by

$$F^{(N)}_k(a) := \begin{cases} a_0 - x_0 & \text{for } k = 0 \\ ka_k + L\left(a_{k-1} - (a * a)_{k-1}\right) & \text{for } 1 \leq k \leq N. \end{cases}$$

Example - Fixed point formulation based on Newton

- We begin by setting up a zero finding problem. Define $F\colon \ell^1_\nu \to S_{\mathbb{I}}(\mathbb{R})$ by

$$F_k(a) := \begin{cases} a_0 - x_0 & \text{for } k = 0 \\ ka_k + L\left(a_{k-1} - (a * a)_{k-1}\right) & \text{for } k \geq 1. \end{cases}$$

  If $F(a) = 0$, then a is an analytic solution to the initial value problem.
- We define $F^{(N)}\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ by

$$F^{(N)}_k(a) := \begin{cases} a_0 - x_0 & \text{for } k = 0 \\ ka_k + L\left(a_{k-1} - (a * a)_{k-1}\right) & \text{for } 1 \leq k \leq N. \end{cases}$$

- And solve numerically the truncated problem

$$F^{(N)}_k(a_0, \dots, a_N) = 0, \quad k = 0, \dots, N.$$

Example - Fixed point formulation based on Newton

- We begin by setting up a zero finding problem. Define $F\colon \ell^1_\nu \to S_{\mathbb{I}}(\mathbb{R})$ by

$$F_k(a) := \begin{cases} a_0 - x_0 & \text{for } k = 0 \\ ka_k + L\left(a_{k-1} - (a * a)_{k-1}\right) & \text{for } k \geq 1. \end{cases}$$

  If $F(a) = 0$, then a is an analytic solution to the initial value problem.
- We define $F^{(N)}\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ by

$$F^{(N)}_k(a) := \begin{cases} a_0 - x_0 & \text{for } k = 0 \\ ka_k + L\left(a_{k-1} - (a * a)_{k-1}\right) & \text{for } 1 \leq k \leq N. \end{cases}$$

- And solve numerically the truncated problem

$$F^{(N)}_k(a_0, \dots, a_N) = 0, \quad k = 0, \dots, N.$$

- If $\bar{a}^{(N)} = (\bar{a}_0, \bar{a}_1, \dots, \bar{a}_N)$ is an approximate zero, then we have an approximate solution

$$\bar{x}^{(N)}(t) := \sum_{k=0}^{N} \bar{a}_k t^k.$$

Example - Fixed point formulation based on Newton

- Define
$$\bar{a} := (\bar{a}_0, \bar{a}_1, \dots, \bar{a}_{N-1}, \bar{a}_N, 0, 0, 0, \dots) \in \ell^1_\nu.$$

Example - Fixed point formulation based on Newton

- Define
$$\bar{a} := (\bar{a}_0, \bar{a}_1, \dots, \bar{a}_{N-1}, \bar{a}_N, 0, 0, 0, \dots) \in \ell_\nu^1.$$
- We prove that there is a unique solution $\tilde{a}$ of $F(a) = 0$ in the ball of radius r_0 centered at $\bar{a}$ in ℓ_ν^1. This gives the analytic solution
$$\tilde{x}(t) = \sum_{k=0}^{\infty} \tilde{a}_k t^k \quad \text{for } t \in (-\nu, \nu).$$

Example - Fixed point formulation based on Newton

- Define

$$\bar{a} := (\bar{a}_0, \bar{a}_1, \dots, \bar{a}_{N-1}, \bar{a}_N, 0, 0, 0, \dots) \in \ell^1_\nu.$$

- We prove that there is a unique solution $\tilde{a}$ of $F(a) = 0$ in the ball of radius r_0 centered at $\bar{a}$ in ℓ^1_ν. This gives the analytic solution

$$\tilde{x}(t) = \sum_{k=0}^{\infty} \tilde{a}_k t^k \quad \text{for } t \in (-\nu, \nu).$$

- Comparing the true solution $\tilde{x}$ to the numerical approximation $\bar{x}^{(N)}$ we obtain the error bound

$$\|\tilde{x}(t) - \bar{x}^{(N)}(t)\|_\infty = \sup_{t \in (-\nu,\nu)} \left| \sum_{k=0}^{\infty} (\tilde{a}_k - \bar{a}_k) t^k \right| \le \sum_{k=0}^{\infty} |\tilde{a}_k - \bar{a}_k| \nu^k \le r_0.$$

Example - Fixed point formulation based on Newton

- The ka_k term in $F_k(a)$ for $k \geq 1$ implies that for some values of $a \in \ell^1_\nu$, $F(a) \notin \ell^1_\nu$.

Example - Fixed point formulation based on Newton

- The ka_k term in $F_k(a)$ for $k \geq 1$ implies that for some values of $a \in \ell^1_\nu$, $F(a) \notin \ell^1_\nu$.
- Hence it is not true that $F\colon \ell^1_\nu \to \ell^1_\nu$.

Example - Fixed point formulation based on Newton

- The ka_k term in $F_k(a)$ for $k \geq 1$ implies that for some values of $a \in \ell^1_\nu$, $F(a) \notin \ell^1_\nu$.
- Hence it is not true that $F\colon \ell^1_\nu \to \ell^1_\nu$.
- However, our approach only requires that $T\colon \ell^1_\nu \to \ell^1_\nu$. We carefully choose A to make sure the operator $T(a) := a - AF(a)$ satisfies this condition.

Example - Fixed point formulation based on Newton

- Using the fact that asymptotically $DF(\bar{a})$ is diagonal dominant, we approximate it by $A^{\dagger}$ whose action on a vector $h = \{h_k\}_{k\in\mathbb{N}} \in \ell_\nu^1$ is

$$(A^{\dagger}h)_k := \begin{cases} [DF(\bar{a})h^{(N)}]_k, & 0 \le k \le N \\ kh_k, & k \ge N+1. \end{cases}$$

  Here $h^{(N)} = (h_0, h_1, \ldots, h_N) \in \mathbb{R}^{N+1}$ is obtained by projecting h onto its first $(N+1)$ coordinates.

Example - Fixed point formulation based on Newton

- Using the fact that asymptotically $DF(\bar{a})$ is diagonal dominant, we approximate it by $A^\dagger$ whose action on a vector $h = \{h_k\}_{k\in\mathbb{N}} \in \ell^1_\nu$ is

$$(A^\dagger h)_k := \begin{cases} [DF(\bar{a})h^{(N)}]_k, & 0 \le k \le N \\ kh_k, & k \ge N+1. \end{cases}$$

  Here $h^{(N)} = (h_0, h_1, \ldots, h_N) \in \mathbb{R}^{N+1}$ is obtained by projecting h onto its first $(N+1)$ coordinates.

- To define the approximate inverse A of $A^\dagger$, we first consider $(A^\dagger)^{(N)} \in M_{N+1}(\mathbb{R})$ given by

$$[(A^\dagger)^{(N)}]_{k,j} = [DF(\bar{a})]_{k,j} \quad \text{for } 0 \le k, j \le N.$$

Example - Fixed point formulation based on Newton

- This matrix is invertible, since it is lower triangular with positive elements on the diagonal. We numerically compute an approximate inverse $A^{(N)}$ of $(A^{\dagger})^{(N)}$ and use this matrix to define A by

$$(Ah)_k := \begin{cases} [A^{(N)} h^{(N)}]_k & 0 \leq k \leq N \\ \frac{1}{k} h_k & k \geq N+1. \end{cases}$$

Example - Fixed point formulation based on Newton

- This matrix is invertible, since it is lower triangular with positive elements on the diagonal. We numerically compute an approximate inverse $A^{(N)}$ of $(A^{\dagger})^{(N)}$ and use this matrix to define A by

$$(Ah)_k := \begin{cases} [A^{(N)}h^{(N)}]_k & 0 \leq k \leq N \\ \frac{1}{k}h_k & k \geq N+1. \end{cases}$$

- The radii polynomial automatically verifies that $A^{(N)}$ is invertible and hence that A is injective.

Example - Fixed point formulation based on Newton

Theorem

Let $\bar{a}^{(N)} = (\bar{a}_0, \ldots, \bar{a}_N)$ *be an approximate solution to* $F^{(N)}(a) = 0$ *and let* $\bar{a}$ *be as above. Let* $A^{(N)}$ *be an approximate inverse of* $(A^{\dagger})^{(N)}$*. Set*

$$
\begin{aligned}
Y_0 &:= \sum_{k=0}^{N} \left| \left(A^{(N)} F^{(N)}(\bar{a}) \right)_k \right| \nu^k + L\nu \sum_{k=N}^{2N} \frac{1}{k+1} \left| (\bar{a} - \bar{a} * \bar{a})_k \right| \nu^k, \\
Z_0 &:= \| I_{\mathbb{R}^{N+1}} - A^{(N)} DF^{(N)}(\bar{a}^{(N)}) \|_{1,\nu}, \\
Z_1 &:= \frac{L\nu}{N+1} \|\bar{g}\|_{1,\nu}, \quad Z_2 := 2L\nu \|A\|_{B(\ell^1_\nu, \ell^1_\nu)},
\end{aligned}
$$

where $\bar{g}_k := \delta_{k,0} - 2\bar{a}_k$*. Consider the radii polynomial*

$$
p(r) := Z_2 r^2 - (1 - Z_0 - Z_1) r + Y_0.
$$

Example - Fixed point formulation based on Newton

Theorem

If there exists $r_0 > 0$ such that $p(r_0) < 0$, then there exists a unique

$$\tilde{a} \in B_{r_0}(\bar{a}) \subset \ell_\nu^1$$

such that $\tilde{x} \colon (-\nu, \nu) \to \mathbb{R}$ defined by

$$\tilde{x}(t) := \sum_{k=0}^{\infty} \tilde{a}_k t^k$$

is an analytic solution to the initial value problem. Furthermore,

$$\left\| \tilde{x} - \sum_{k=0}^{N} \bar{a}_k t^k \right\|_\infty < r_0.$$

The proof follows from the radii polynomial theorem and the following lemmas.

Lemma

Under the conditions of the previous theorem

$$\|AF(\bar{a})\|_{1,\nu} \le Y_0.$$

Proof.

Observe that

$$
\begin{aligned}
&\|AF(\bar{a})\|_{1,\nu} = \\
&\sum_{k=0}^{N} \left| \left(A^{(N)} F^{(N)}(\bar{a}) \right)_k \right| \nu^k + \sum_{k>N} \left| \frac{1}{k} \left(k\bar{a}_k - L(\bar{a}_{k-1} - (\bar{a} * \bar{a})_{k-1}) \right) \right| \nu^k = \\
&\sum_{k=0}^{N} \left| \left(A^{(N)} F^{(N)}(\bar{a}) \right)_k \right| \nu^k + L \sum_{k=N+1}^{2N+1} \frac{1}{k} \left| \bar{a}_{k-1} - (\bar{a} * \bar{a})_{k-1} \right| \nu^k = \\
&\sum_{k=0}^{N} \left| \left(A^{(N)} F^{(N)}(\bar{a}) \right)_k \right| \nu^k + L\nu \sum_{k=N}^{2N} \frac{1}{k+1} \left| (\bar{a} - \bar{a} * \bar{a})_k \right| \nu^k.
\end{aligned}
$$

$\square$

Lemma

Under the conditions of the previous theorem

$$\|I - AA^{\dagger}\|_{B(\ell^1_\nu,\ell^1_\nu)} \leq Z_0.$$

Proof.

Define $B = I - AA^{\dagger}$. Observe that if $k > N$ or $j > N$, then

$$B_{k,j} = \left(I - AA^{\dagger}\right)_{k,j} = \delta_{k,j} - \frac{1}{k} k\delta_{k,j} = 0.$$

The infinite dimensional operator B can thus be represented by the matrix $B^{(N)} = I_{\mathbb{R}^{N+1}} - A^{(N)}(A^{\dagger})^{(N)}$. Its matrix norm, induced by ℓ^1_ν, is given by

$$\left\|B^{(N)}\right\|_{\nu,N} := \max_{0\leq j\leq N} \frac{1}{\nu^j} \sum_{0\leq i\leq N} \left|B^{(N)}_{i,j}\right| \nu^i$$

Hence we have

$$\|I - AA^{\dagger}\|_{B(\ell^1_\nu,\ell^1_\nu)} \leq \|I_{\mathbb{R}^{N+1}} - A^{(N)} DF^{(N)}(\bar{a}^{(N)})\|_{1,\nu}$$

$\square$

Lemma

Under the conditions of the previous theorem

$$\|A[DF(\bar{x}) - A^{\dagger}]\|_{B(X)} \leq Z_1.$$

Proof.

Given $h \in \ell^1_\nu$ with $\|h\|_{1,\nu} \leq 1$, set

$$z := [DF(\bar{a}) - A^{\dagger}]h.$$

Note that

$$z_k = \begin{cases} 0 & k = 0, \ldots, N \\ -L\left(h_{k-1} - 2(\bar{a} * h)_{k-1}\right) & k > N, \end{cases}$$

and therefore, $z^{(N)} = (z_0, z_1, \ldots, z_N) = 0 \in \mathbb{R}^{N+1}$. Denote

$$\bar{g}_k := \delta_{k,0} - 2\bar{a}_k.$$

□

Example - Fixed point formulation based on Newton

Proof.

With this notation we have

$$h_{k-1} - 2(\bar{a} * h)_{k-1} = (\bar{g} * h)_{k-1}.$$

Hence,

$$\|Az\|_{1,\nu} = \sum_{k=0}^{N} \left|(A^{(N)} z^{(N)})_k\right| \nu^k + \sum_{k\geq N+1} \frac{1}{k} |z_k| \nu^k \leq$$

$$\frac{L\nu}{N+1} \sum_{k\geq N+1} |(\bar{g} * h)_{k-1}| \nu^{k-1} \leq \frac{L\nu}{N+1} \|\bar{g} * h\|_{1,\nu} \leq \frac{L\nu}{N+1} \|\bar{g}\|_{1,\nu}.$$

Thus we obtain the desired inequality for

$$Z_1 := \frac{L\nu}{N+1} \|\bar{g}\|_{1,\nu}.$$

□

Lemma

Under the conditions of the previous theorem

$$\|A[DF(b) - DF(\bar{a})]\|_{B(\ell^1_\nu, \ell^1_\nu)} \le Z_2 r, \quad \textit{for all } b \in \overline{B_r(\bar{a})}.$$

Proof.

Let $h \in \ell^1_\nu$ with $\|h\|_{1,\nu} \le 1$ and let $b \in \overline{B_r(\bar{a})}$. Then, for $k \ge 1$

$$([DF(b) - DF(\bar{a})]h)_k = kh_k + Lh_{k-1} - 2L(b * h)_{k-1}$$
$$-\left(kh_k + Lh_{k-1} - 2L(\bar{a} * h)_{k-1}\right) = -2L((b - \bar{a}) * h)_{k-1}$$

and therefore

$$\|[DF(b) - DF(\bar{a})]h\|_{1,\nu} = 2L \sum_{k \ge 1} |((b - \bar{a}) * h)_{k-1}| \nu^k =$$
$$2L\nu \sum_{k \ge 1} |((b - \bar{a}) * h)_{k-1}| \nu^{k-1} = 2L\nu \|(b - \bar{a}) * h\|_{1,\nu} \le 2L\nu r.$$

□

Example - Fixed point formulation based on Newton

Proof.

We conclude that for all $b \in \overline{B_r(\bar{a})}$,

$$\|A[DF(b) - DF(\bar{a})]\|_{B(\ell^1_\nu,\ell^1_\nu)} = \sup_{\|h\|_{1,\nu} \le 1} \|A[DF(b) - DF(\bar{a})]h\|_{1,\nu} \le$$

$$\|A\|_{B(\ell^1_\nu,\ell^1_\nu)} \|[DF(b) - DF(\bar{a})]h\|_{1,\nu} \le \|A\|_{B(\ell^1_\nu,\ell^1_\nu)} 2L\nu r.$$

where

$$\|A\|_{B(\ell^1_\nu,\ell^1_\nu)} = \max\left\{ \left\| A^{(N)} \right\|_{1,\nu}, \frac{1}{N+1} \right\}$$

$\square$

Example - Fixed point formulation based on Newton

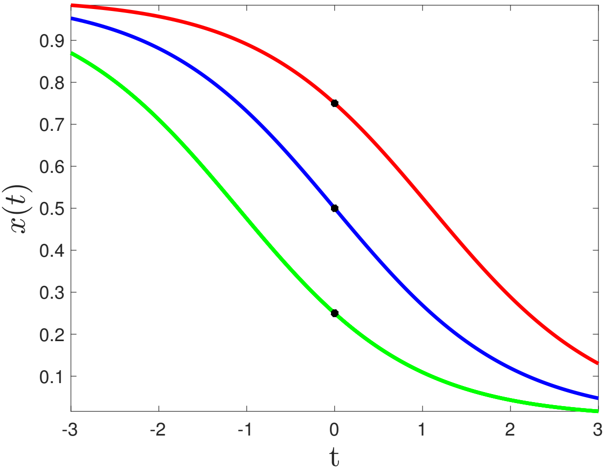

Figure: Solutions with $L = 3$, $\nu = 1$, and $N = 300$. For the green curve $x(0) = 1/4$, for the blue curve $x(0) = 1/2$, and the red curve $x(0) = 3/4$.