Combinatorial Topological Framework for Nonlinear Dynamics

CRM Tutorial - Part 1

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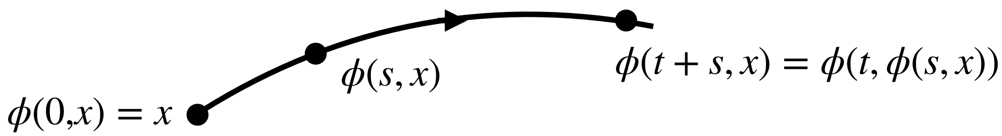
A **dynamical system** is a continuous function $\phi\colon \mathbb{T}^+ \times X \to X$ where $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$ satisfying $\phi(0,x) = x$ and $\phi(t+s,x) = \phi(t,\phi(s,x))$.



(Semi)-flow: $\mathbb{T} = \mathbb{R}$, typically generated from the solutions to an ODE $\dot{x} = F(x)$.

Map: $\mathbb{T} = \mathbb{Z}$, generated by iterating the continuous map $f(x) = \phi(1,x)$.
We do not assume injectivity nor surjectivity.

(Forward)-orbit: $\gamma_x^+(t) = \varphi(t,x)$ for $t \in \mathbb{T}^+$, which for maps is a sequence $\gamma_x^+(n) = f^n(x)$.

Complete orbit: $\gamma_x\colon \mathbb{T} \to X$ such that $\gamma_x(0) = x$ and $\phi(t,\gamma_x(s)) = \gamma_x(t+s)$.

Invariant set: $\phi(t,S) = S, \forall t \geq 0$ iff S is a union of complete orbits.

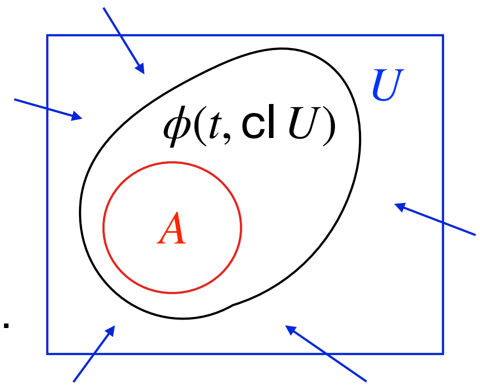
Maximal invariant set: $\text{Inv}(N)$ is the largest invariant set in $N \subset X$.

Isolated invariant set: $S = \text{Inv}(U) \subset \text{int}(U)$ for some **isolating** neighborhood $U \supset S$.

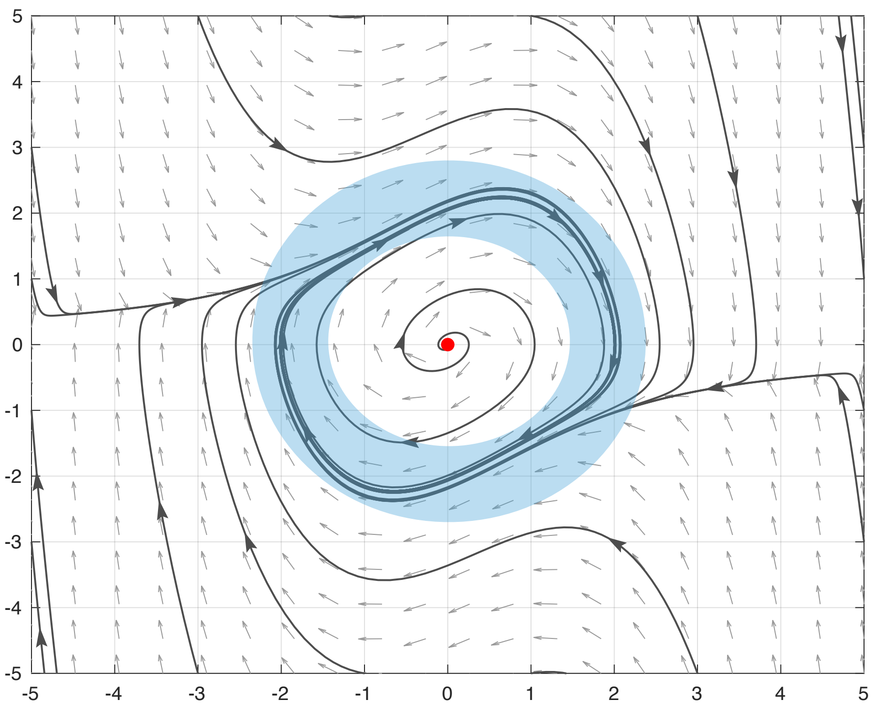
$U \subset X$ is an **attracting neighborhood** if $\phi(t, \mathrm{cl}\, U) \subset \mathrm{int}(U)\ \forall t \geq t_0 > 0$.

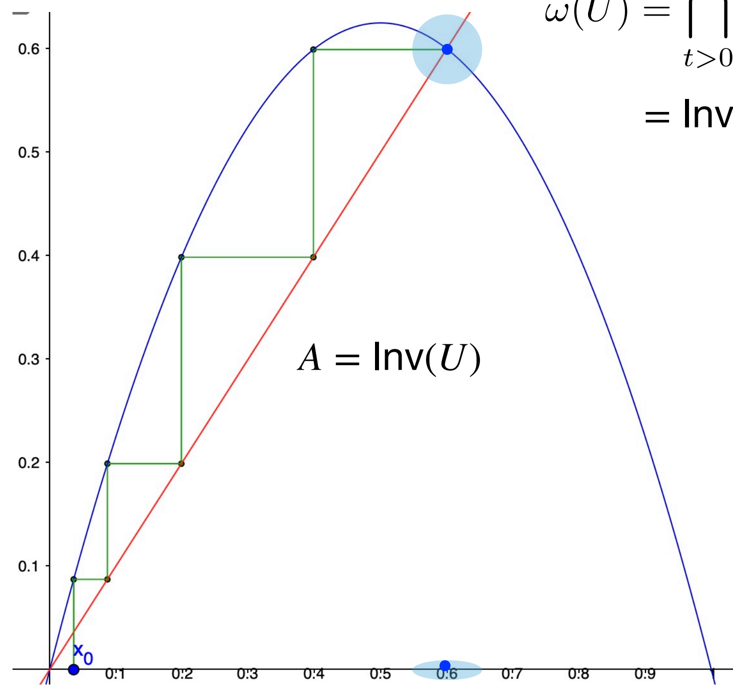
$U \subset X$ is an **attracting block** if $\phi(t, \mathrm{cl}\, U) \subset \mathrm{int}(U)\ \forall t > 0$.

$A \subset X$ is an **attractor** if $A = \mathrm{Inv}(U)$ for some attracting neighborhood.



$$\omega(U) = \bigcap_{t>0} \mathrm{cl}\,(\phi(U, [t, \infty)))$$
$$= \mathrm{Inv}(U)$$





Lattices

A *bounded, distributive lattice* is a set L with the binary operations $\vee, \wedge : \mathsf{L} \times \mathsf{L} \to \mathsf{L}$ satisfying the following axioms:

(i) (idempotent) $a \wedge a = a \vee a = a$ for all $a \in \mathsf{L}$,

(ii) (commutative) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ for all $a, b \in \mathsf{L}$,

(iii) (associative) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$ for all $a, b, c \in \mathsf{L}$,

(iv) (absorption) $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ for all $a, b \in \mathsf{L}$.

(v) (distributive) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in \mathsf{L}$.

(vi) (neutral elements) $\exists 0, 1 \in \mathsf{L}$ such that $0 \wedge a = 0$, $0 \vee a = a$, $1 \wedge a = a$, and $1 \vee a = 1$ for all $a \in \mathsf{L}$.

All sublattices contain 0 and 1, and all homomorphisms preserve 0 and 1.

Think sets! ie. $\vee = \cup$ and $\wedge = \cap$

A Computational Framework for Global Dynamics

Observable dynamics: attractors

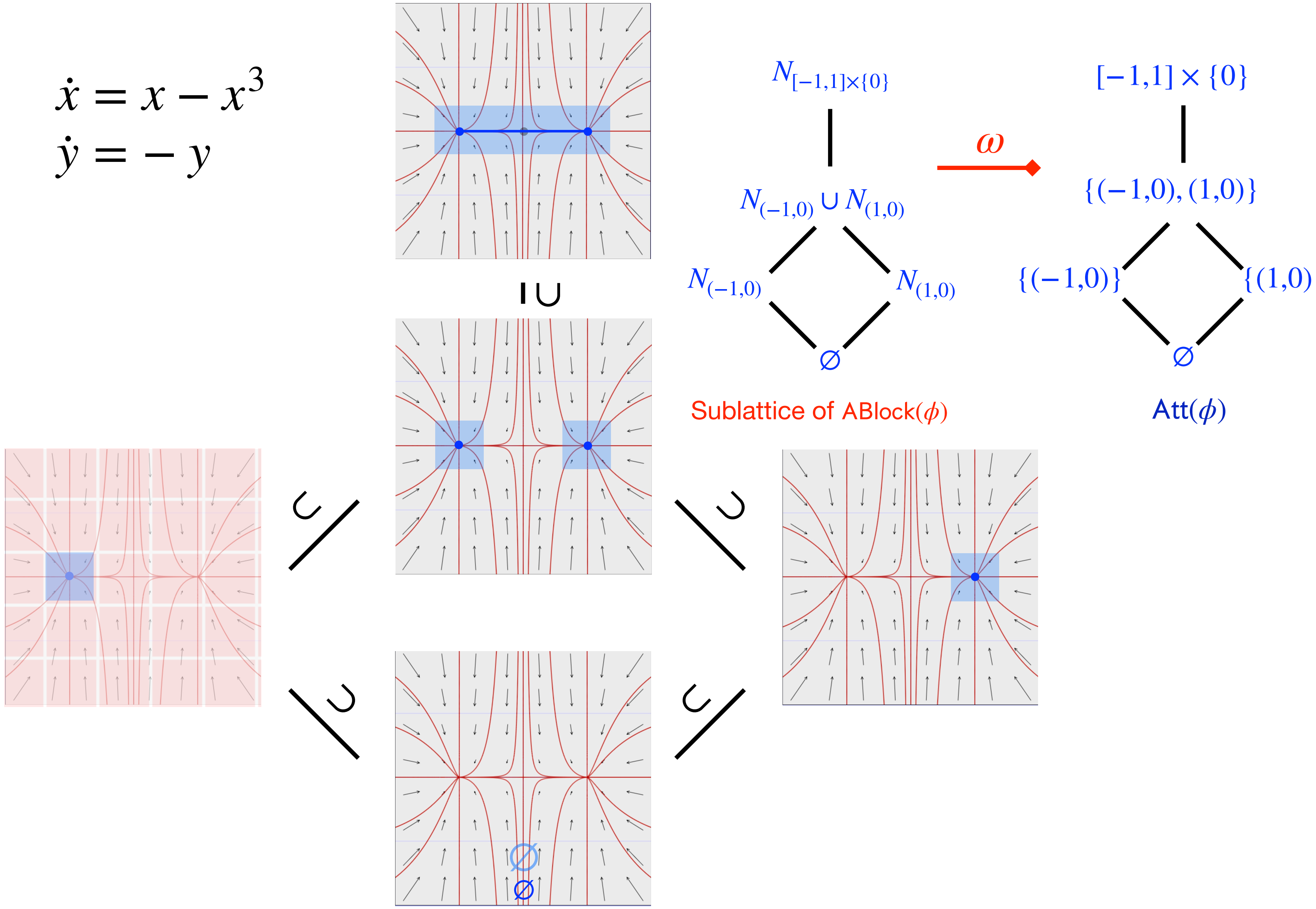
Computable dynamics: attracting blocks (neighborhoods)

$$\begin{array}{ll} \mathsf{ABlock}(\phi) & \vee = \cup,\ \wedge = \cap \\ \big\downarrow \omega & \text{surjective lattice homomorphism between distributive lattices} \\ \mathsf{Att}(\phi) & \vee = \cup,\ \wedge = \omega(\cdot \cap \cdot) \end{array}$$

Goal 1: combinatorialize for (finite) computations

Goal 2: dualize for concise descriptions and algorithms

$$\dot{x} = x - x^3$$
$$\dot{y} = -y$$



ABlock(ϕ) Att(ϕ)

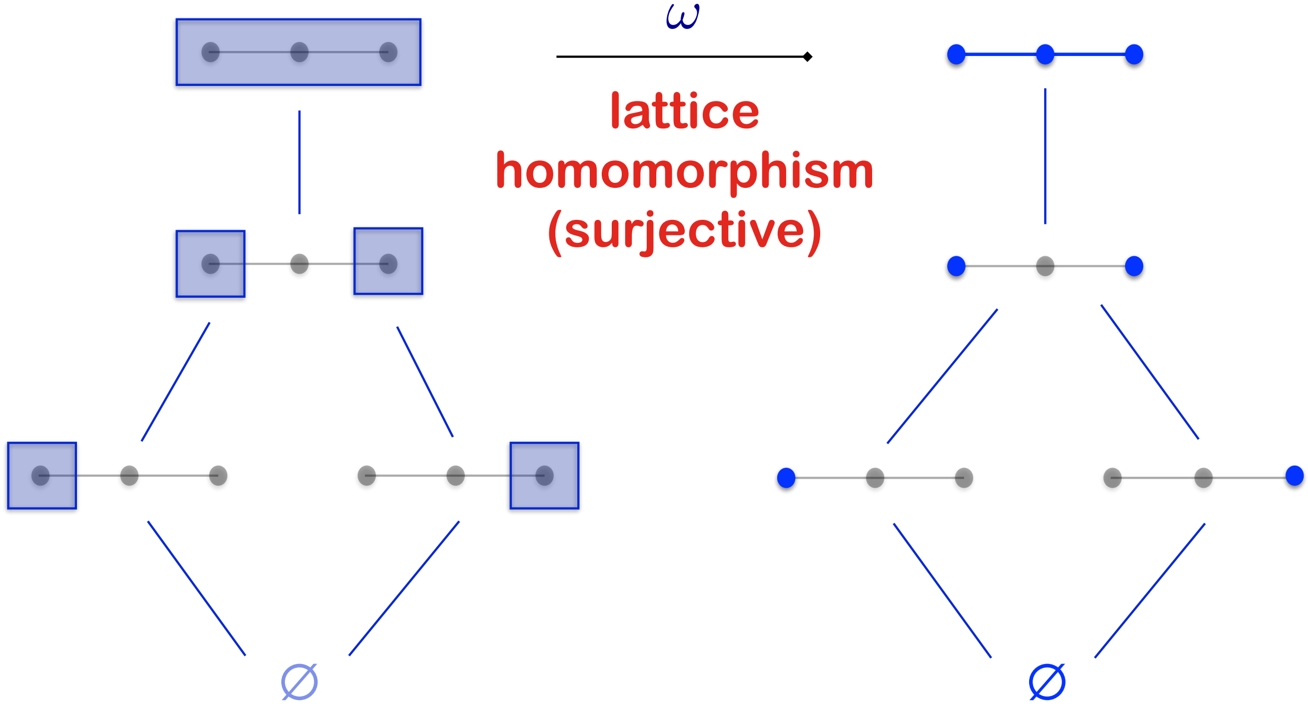
Built on a finite grid of regular closed sets.

$\vee = \cup$

$\wedge = \mathrm{cl}(\mathrm{int}(\cdot \cap \cdot))$

$\vee = \cup$

$\wedge = \omega \cap$



J - functor

A lattice L has a naturally induced partial order as follows.

Given $a, b \in \mathsf{L}$ define

$$a \leq b \quad \Leftrightarrow \quad a \wedge b = a.$$

Given a lattice L, an element $0 \neq c \in \mathsf{L}$ is *join-irreducible* if

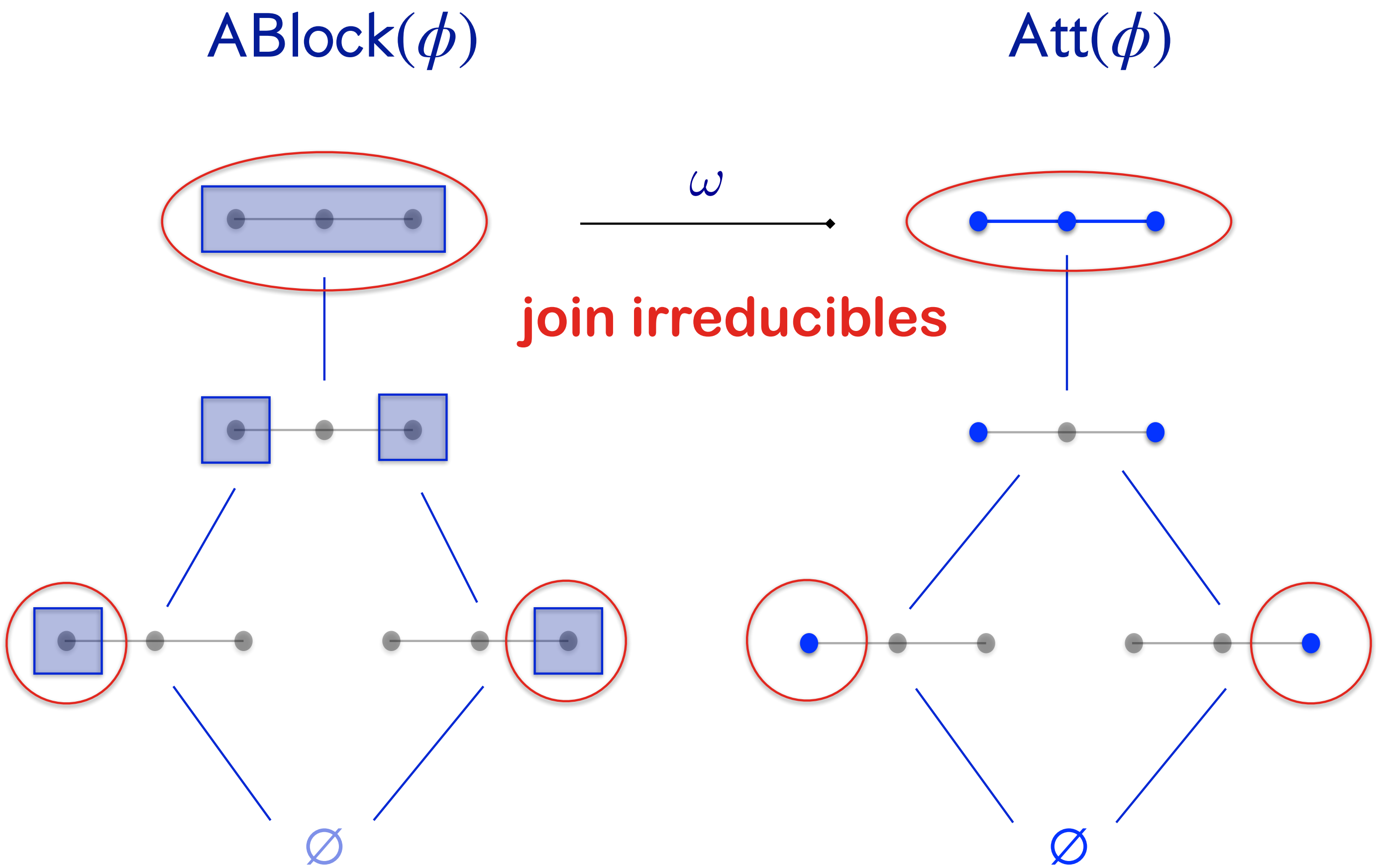
$c = a \vee b$ implies $c = a$ or $c = b$ for all $a, b \in \mathsf{L}$.

The set of join-irreducible elements in L is denoted by $\mathsf{J}(\mathsf{L})$.

c is join-irreducible iff there exists a unique element $a \in \mathsf{L}$ such that $a < c$ and there is no z such that $a < z < c$.

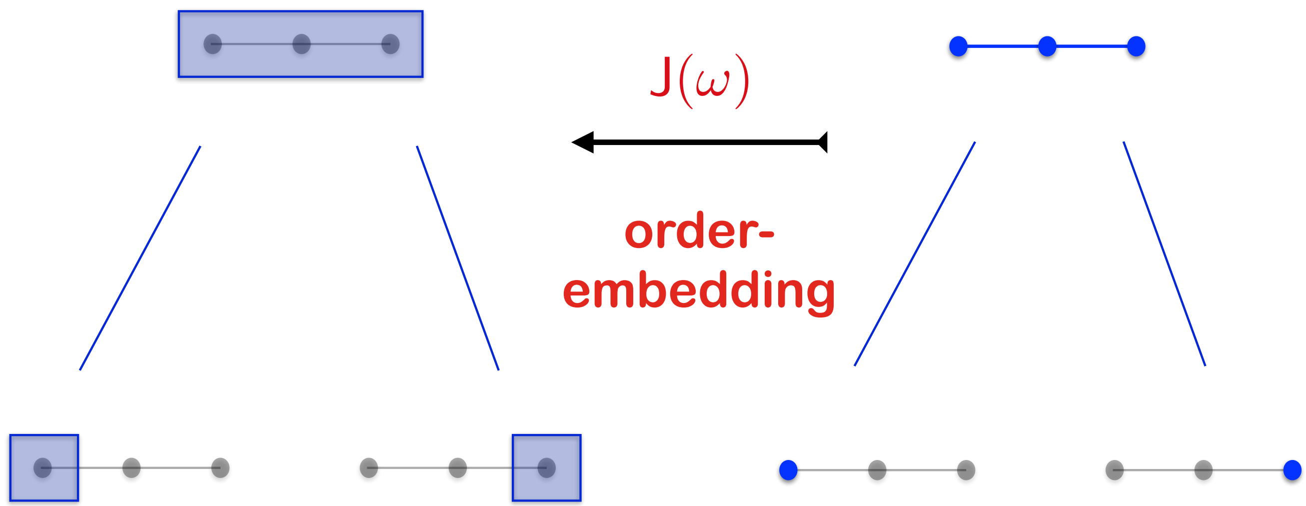
J is a contravariant functor from finite distributive lattices to finite posets.

In all lattices we consider, the order $\leq$ is induced by inclusion.



ABlock(ϕ)
Att(ϕ)
ω
join irreducibles
∅
∅

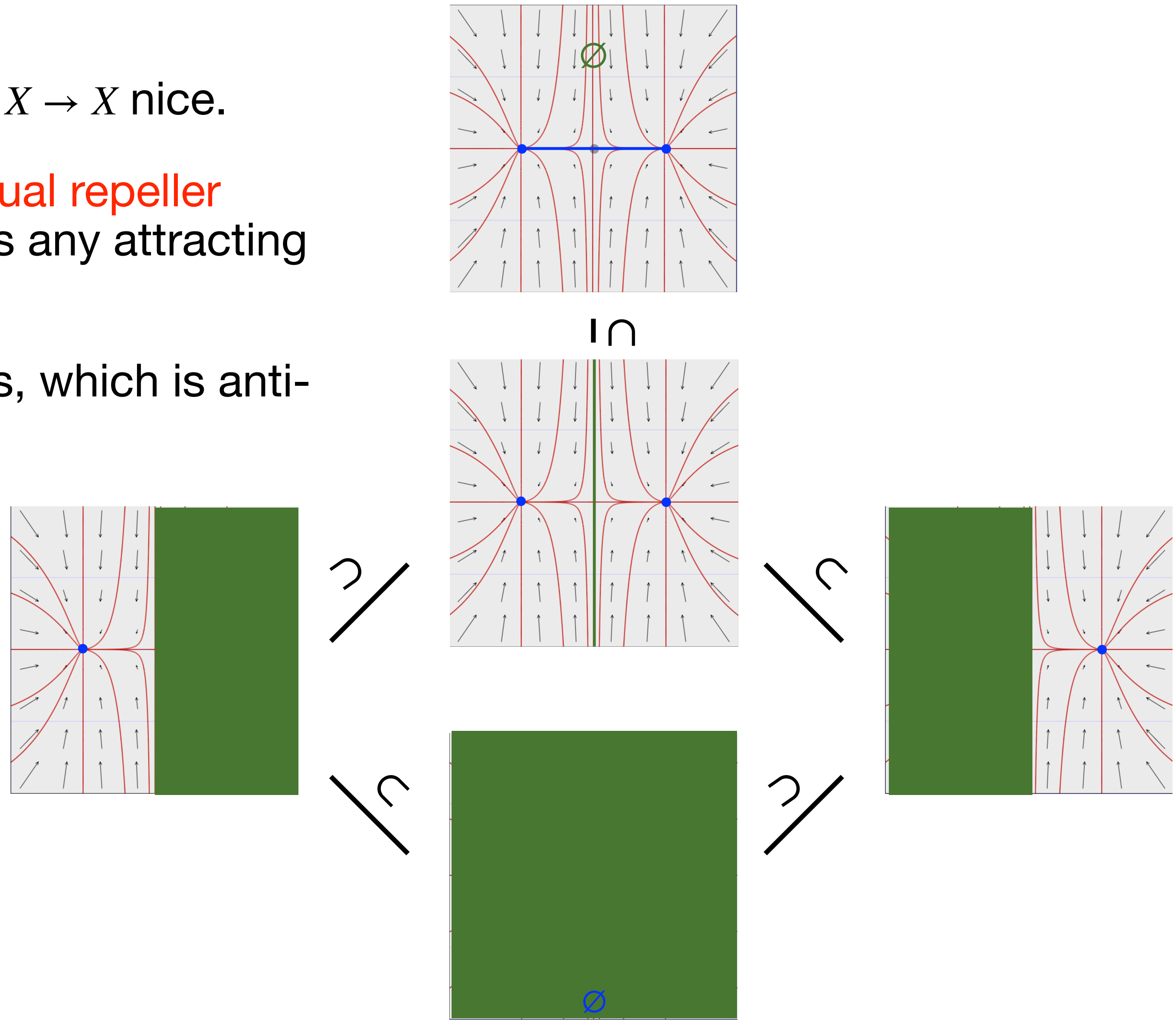
J is a contravariant functor



Let X be compact and $\phi\colon \mathbb{T}^+ \times X \to X$ nice.

Then each attractor A has a dual repeller $A^* := \mathrm{Inv}^+(U^c) = \alpha(U^c)$ where U is any attracting neighborhood of A.

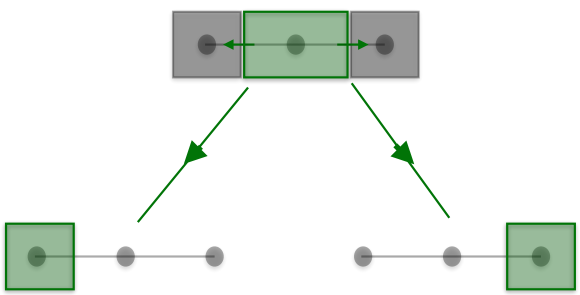
$\mathrm{Rep}(\phi)$ is the lattice of repellers, which is anti-isomorphic to $\mathrm{Att}(\phi)$.



Conley form: generalization of set difference to distributive lattices.

$$N_A \wedge (\overleftarrow{N_A})^{\#} = \mathrm{cl}(N_A \cap (\overleftarrow{N_A})^c) \qquad A \wedge (\overleftarrow{A})^* = A \cap (\overleftarrow{A})^*$$

poset of isolating neighborhoods

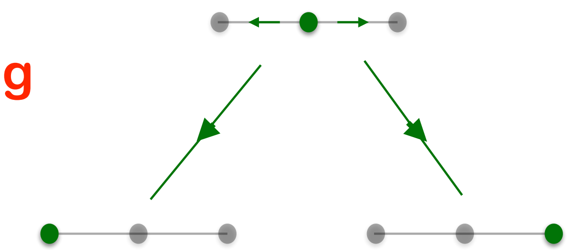


T(N)

π

order-embedding

poset of invariant sets

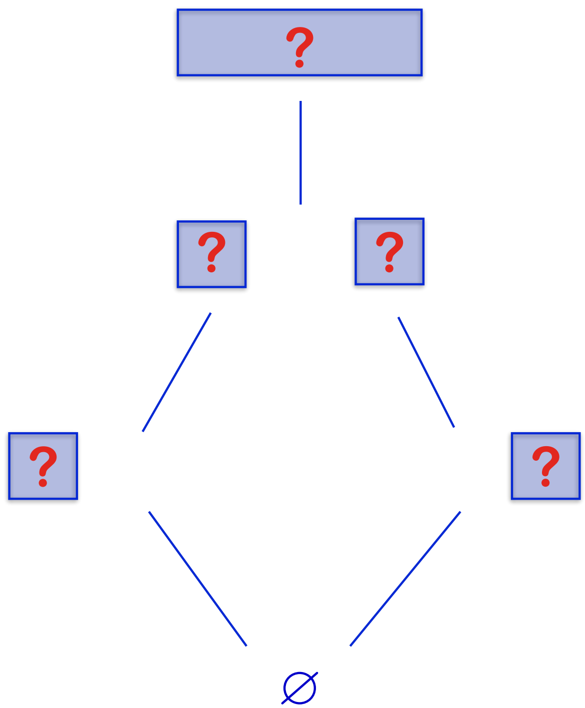


M(A)

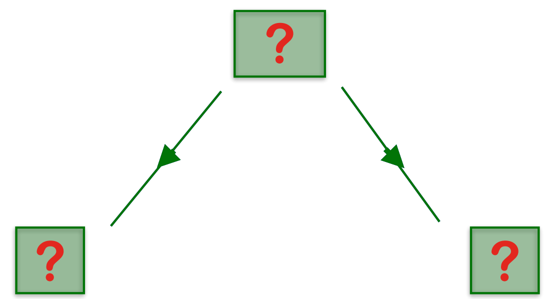
(tesselated) Morse decomposition

$$\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$$

attracting block lattice



poset of isolating neighborhoods



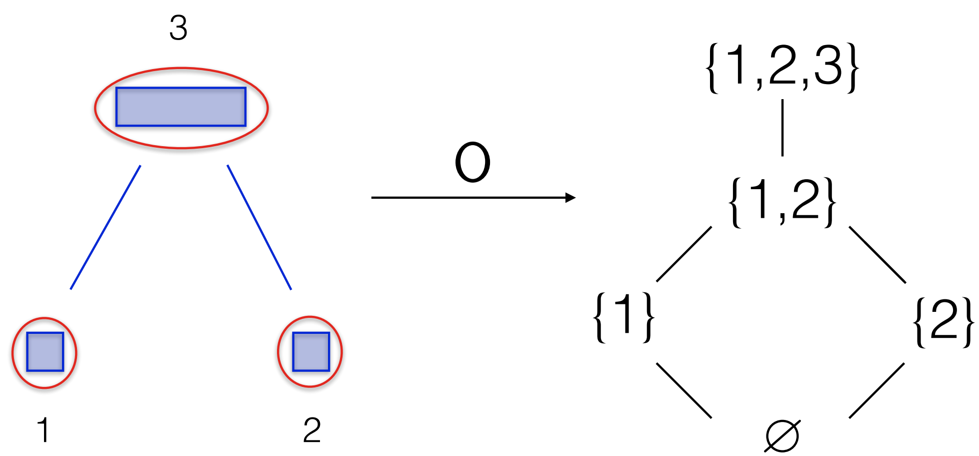
These structures are robust!

O - functor

$S \subset \mathsf{P}$ is a *down set of* P if $q \leq p$ and $p \in S$ implies $q \in S$.

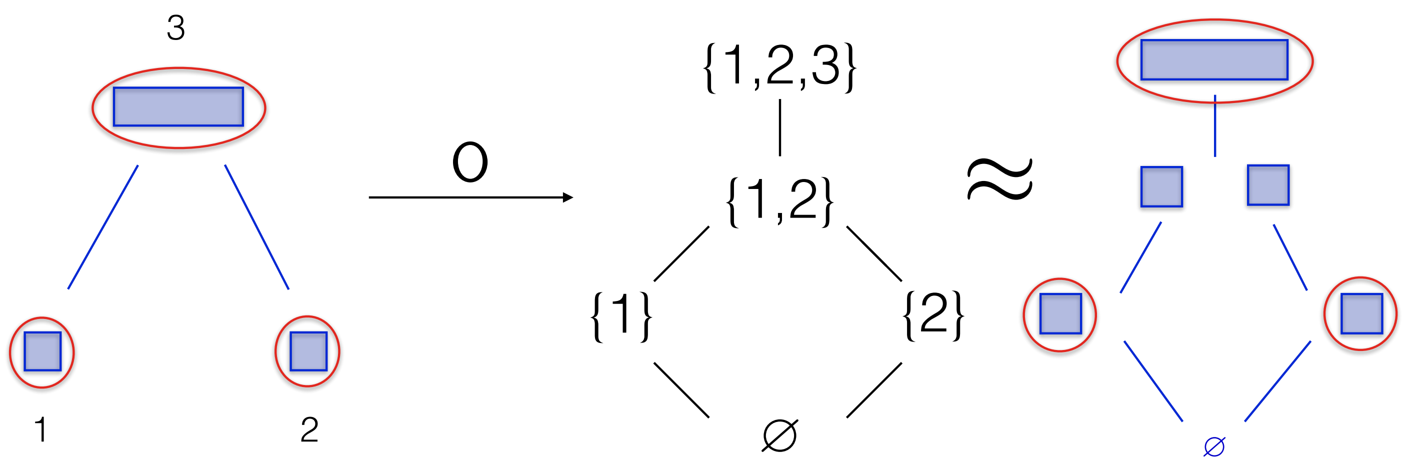
The collection of down-sets of a finite poset P generates a finite distributive lattice denoted by $\mathsf{O}(\mathsf{P})$ with respect to $\vee = \cup$ and $\wedge = \cap$.

O is a contravariant functor from finite posets to finite distributive lattices.



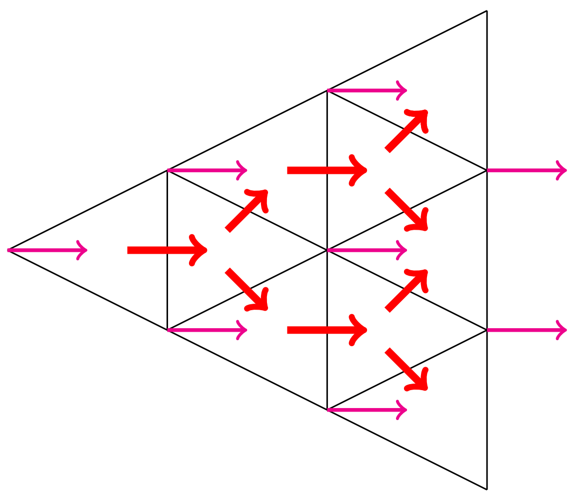
Birkhoff's Representation Theorem: **J induces a duality between finite distributive lattices and finite posets.**

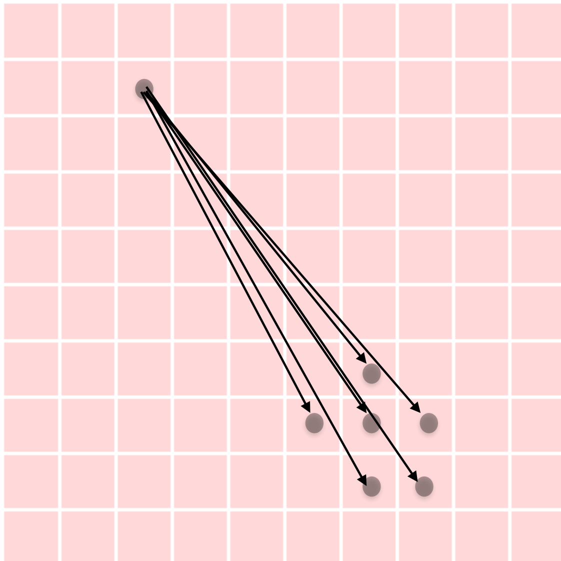
$$O(J(L)) \approx L \quad \textbf{and} \quad J(O(P)) \approx P$$



Combinatorialize and Dualize

Combinatorial multivalued map (CMVM) ...





... directed graph or binary relation.

Extension of Birkhoff's Representation Theorem

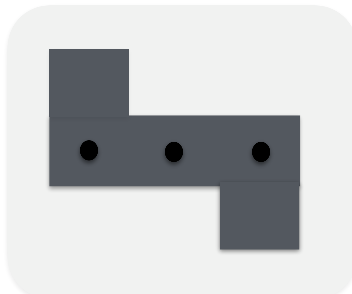
(K.,Kasti, VanderVorst)

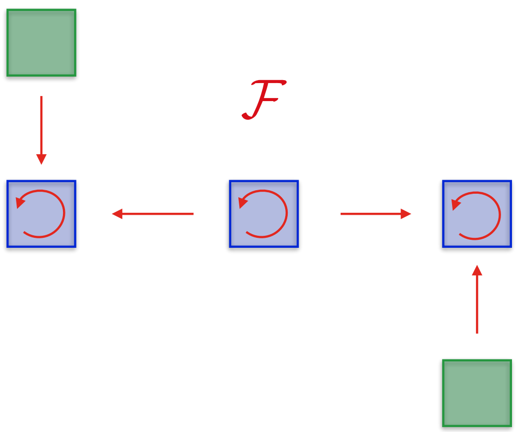
$$\begin{tikzcd}
\mathsf{K} \arrow[r, leftrightarrow] \arrow[d, "h"] & \mathsf{Invset}^+(\mathcal{F}) = \{\mathcal{U} \mid \mathcal{F}(\mathcal{U}) \subset \mathcal{U}\} \quad \vee = \cup,\ \wedge = \cap \arrow[d, "\omega"] \\
\mathsf{L} \arrow[r, leftrightarrow] & \mathsf{Att}(\mathcal{F}) = \{\mathcal{A} \mid \mathcal{F}(\mathcal{A}) = \mathcal{A}\} \quad \vee = \cup,\ \wedge = \omega(\cdot \cap \cdot)
\end{tikzcd}$$

$$\omega(U) = \bigcap_{n\geq 1} \bigcup_{m\geq n} \mathscr{F}^m(U)$$

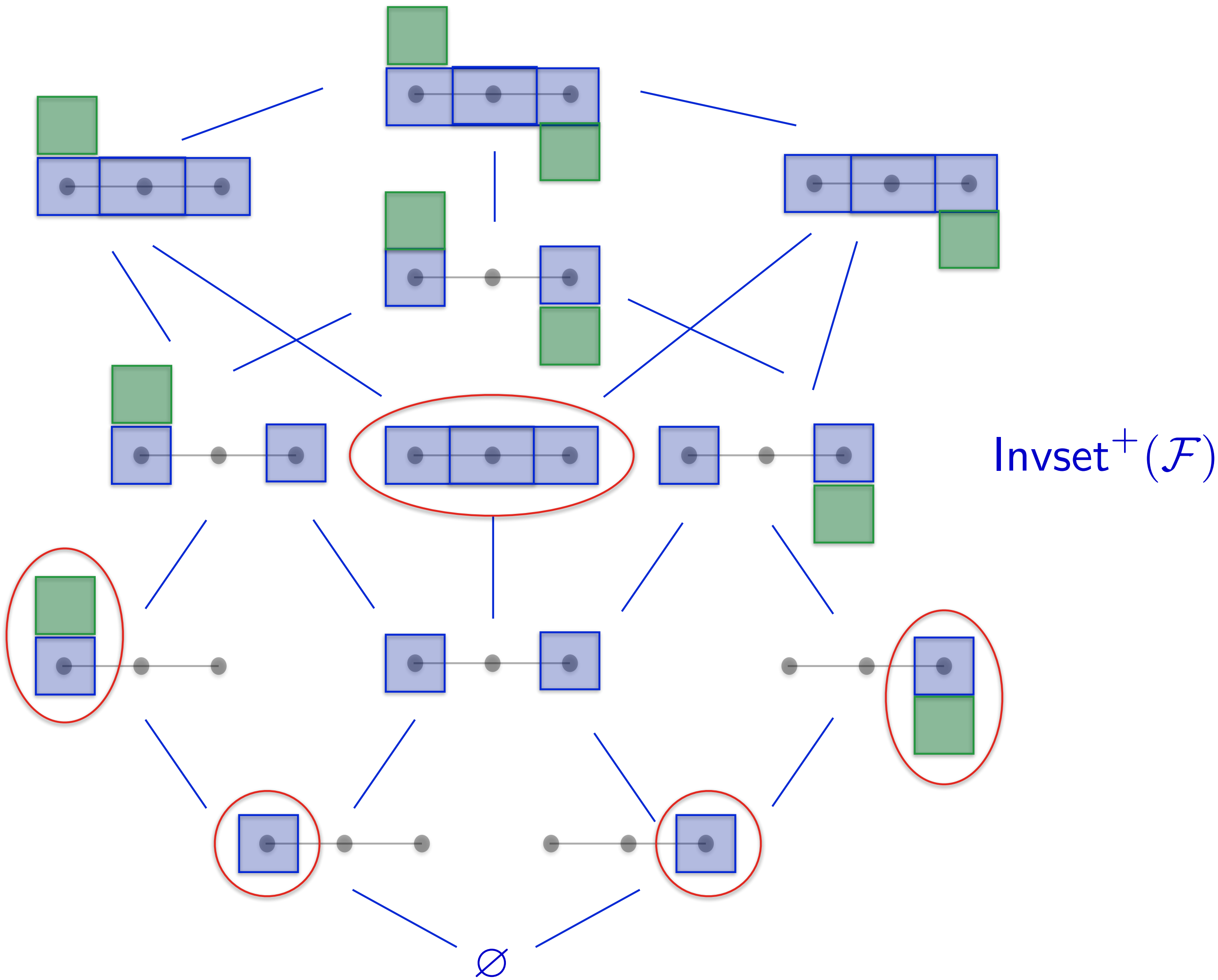
There is a duality between surjective lattice homomorphisms on finite distributive lattices and finite binary relations / directed graphs / combinatorial multivalued maps— up to condensation, transitivity, and isomorphism.

Computationally: represent $\mathcal{F}$ as a state transition graph on a grid of regular closed subsets.

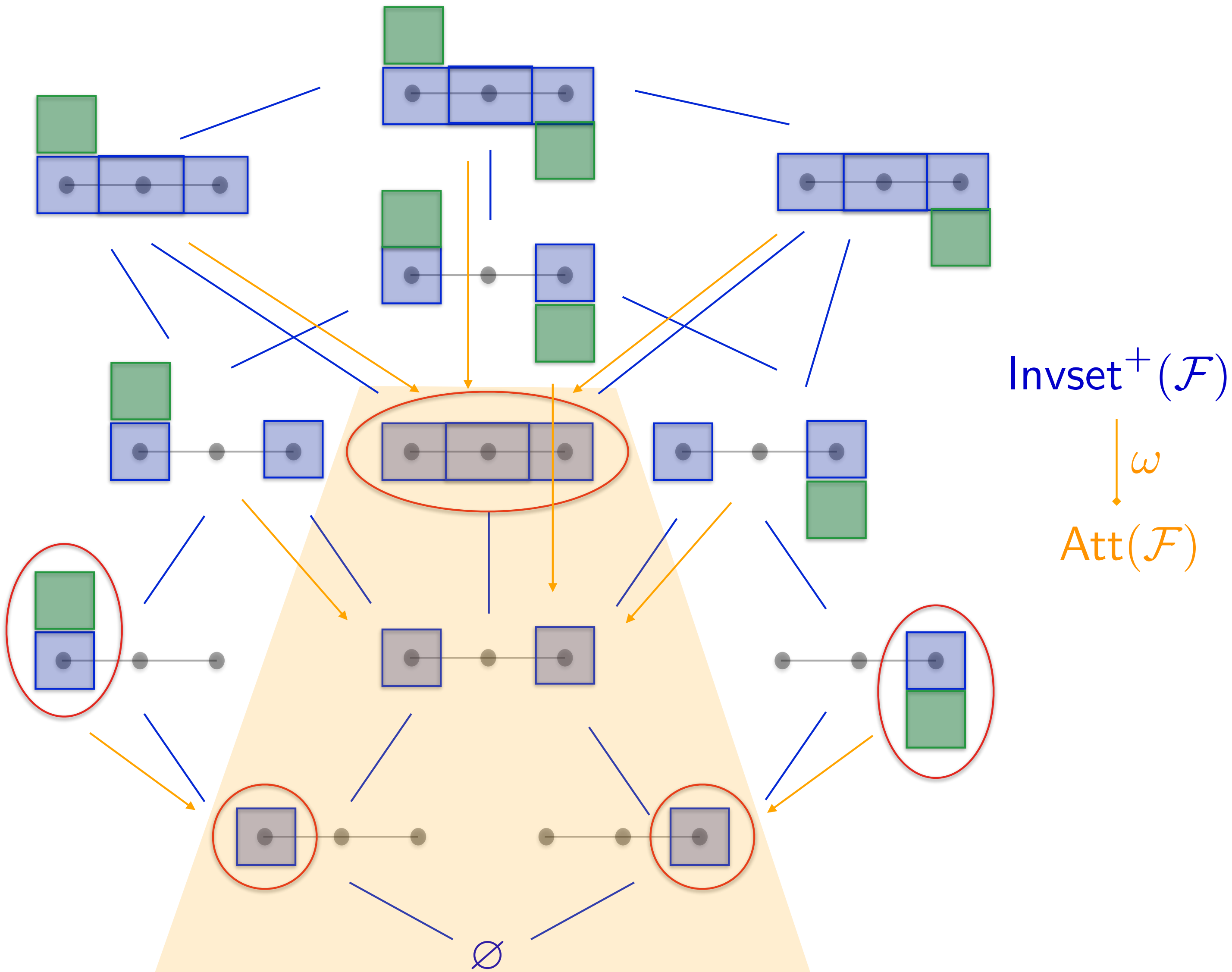




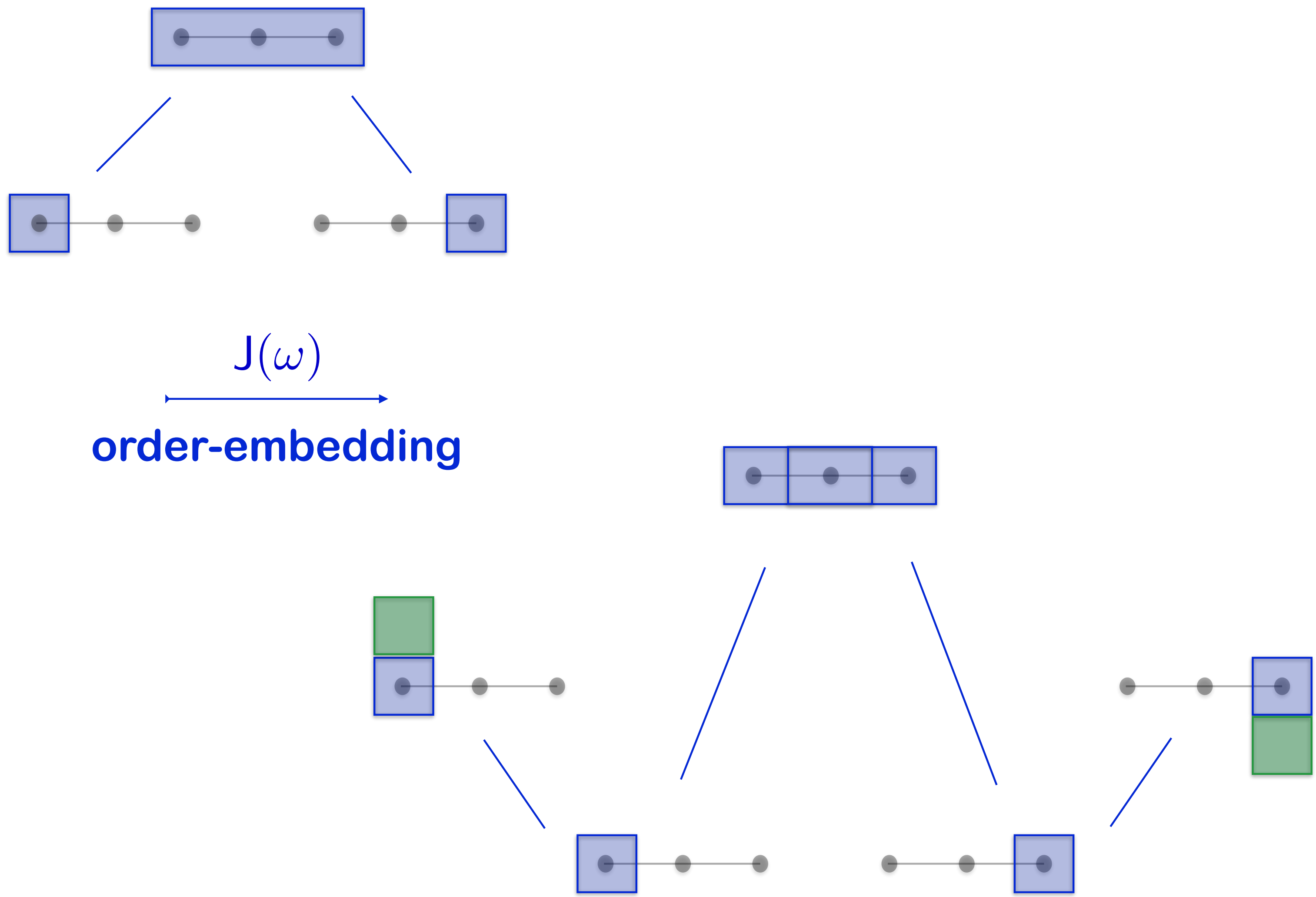
$$
\begin{tikzcd}
\mathrm{Invset}^{+}(\mathcal{F}) \arrow[r, leftrightarrow, "?"] \arrow[d, "\omega"] & \mathsf{N} \hookrightarrow \mathsf{ABlock}_{\mathsf{R}}(\phi) \arrow[d, "\omega"] \\
\mathrm{Att}(\mathcal{F}) \arrow[r, leftrightarrow, "?"] & \mathsf{A} \hookrightarrow \mathsf{Att}(\phi)
\end{tikzcd}
$$



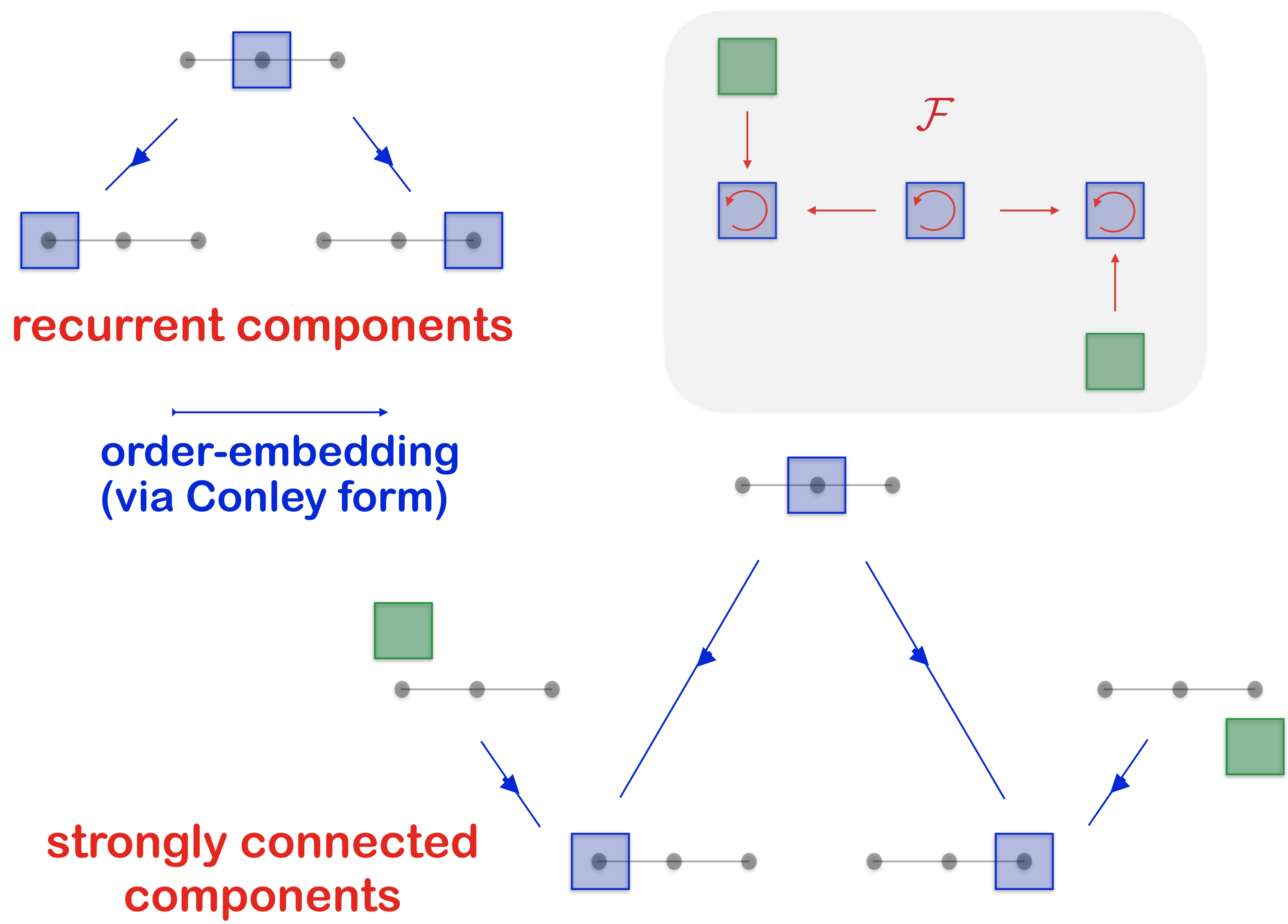
$\mathsf{Invset}^+(\mathcal{F})$
$\varnothing$



$\mathsf{Invset}^+(\mathcal{F})$
ω
$\mathsf{Att}(\mathcal{F})$
$\varnothing$



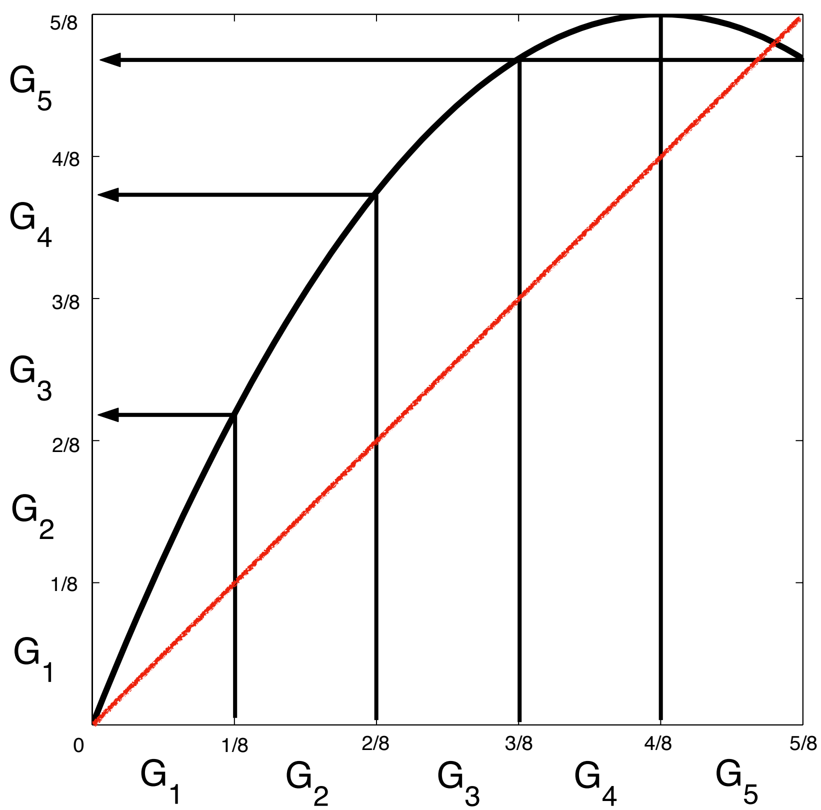
$\mathsf{J}(\omega)$
order-embedding


$\mathcal{F}$
recurrent components
order-embedding
(via Conley form)
strongly connected
components

Combinatorial Dynamics

$X = [0, 5/8] \qquad f\colon X \to X$

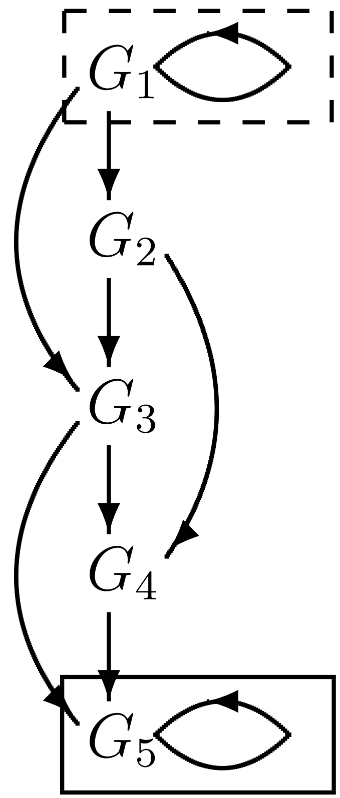
$f(x) = 2.5x(1-x)$



$\mathcal{X} = \{G_1, G_2, G_3, G_4, G_5\} \qquad \mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$

Multivalued Map $\Leftrightarrow$ Directed Graph

$\Leftrightarrow$ Binary Relation



$|G_1| = [0, 1/8]$

$|\{G_1, G_2, G_3\}| = [0, 3/8]$

$f([0, 1/8]) = [0, 35/128]$

$\mathcal{F}(G_1) = \{G_1, G_2, G_3\}$

$f(|G_1|) \subset \mathrm{int}|\mathcal{F}(G_1)|$

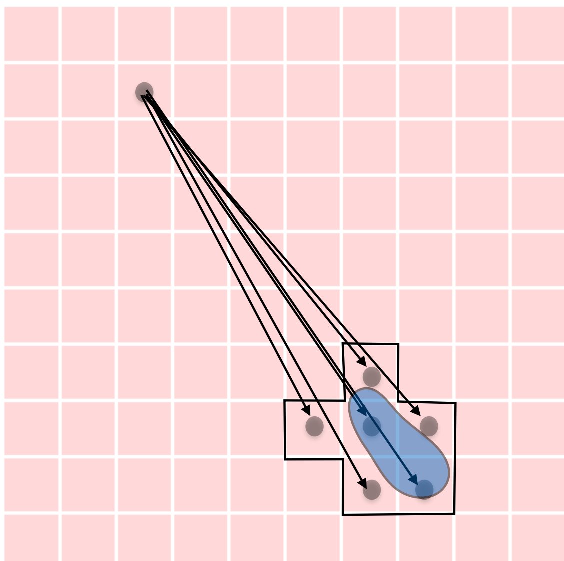
Outer Approximation

$\mathcal{X}$ is a labeling set for some finite *"grid"* on a compact metric space X

$\mathcal{F}\colon \mathcal{X} \rightrightarrows \mathcal{X}$ is an *outer approximation* of $f\colon X \to X$ if

$f(|\mathcal{U}|) \subset \mathrm{int}|\mathcal{F}(\mathcal{U})|$ for all $\mathcal{U} \subset \mathcal{X}$

For explicit maps, OA's can be rigorously computed using interval arithmetic with outward rounding.



Proposition: If $\mathcal{F}$ is an outer approximation of f, and $\mathcal{U}$ is *forward invariant*, i.e. $\mathcal{F}(\mathcal{U}) \subset \mathcal{U}$, then $|\mathcal{U}|$ is an attracting block, which must contain a nonempty attractor for f.

Proposition: $\{f : \mathcal{F}$ is an outer approximation of $f\}$ is open in the space of dynamical systems on X.

Proposition: If x_n for $n \in \mathbb{Z}$ is an orbit of f, i.e. $x_{n+1} = f(x_n)$, then any sequence $G_n \in \mathcal{X}$ with $x_n \in G_n$ is a walk through the graph of $\mathcal{F}$, i.e. $G_{n+1} \in \mathcal{F}(G_n)$.

Interval arithmetic

$$
\begin{aligned}
[a,b]+[c,d] &= [a+c,b+d] \\
[a,b]-[c,d] &= [a-d,b-c] \\
[a,b]\times[c,d] &= [\min\{ac,ad,bc,bd\},\max\{ac,ad,bc,bd\}] \\
[a,b]\ /\ [c,d] &= [\min\{a/c,a/d,b/c,b/d\},\max\{a/c,a/d,b/c,b/d\}]
\end{aligned}
$$

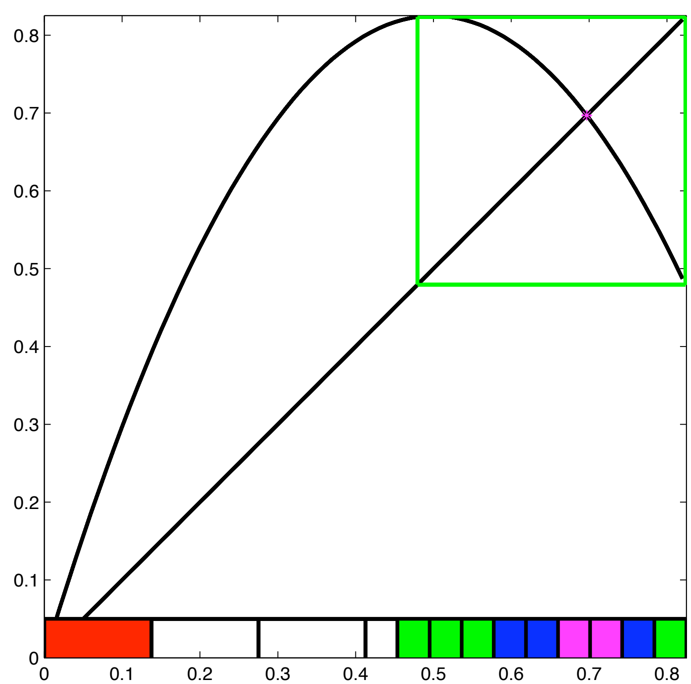
Provides enclosure of arithmetic operations--
if $x \in [a,b]$ **and** $y \in [c,d]$**, then** $x \cdot y \in [a,b]\cdot[c,d]$**,**
at the cost of at most 4 times the original cost.

We can extend to nonarithmetic functions.

Evaluating explicit maps using this arithmetic without outward rounding of floating point numbers provides rigorous outer approximations on a computer.

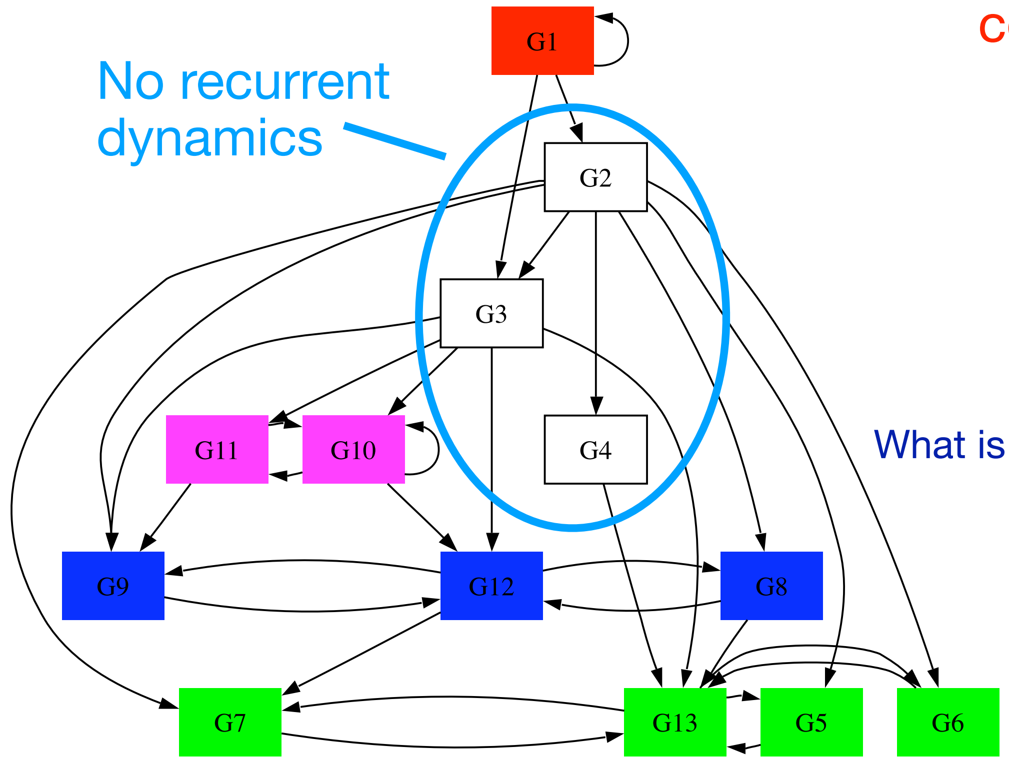
Recurrence

$$f(x) = 3.3x(1-x)$$

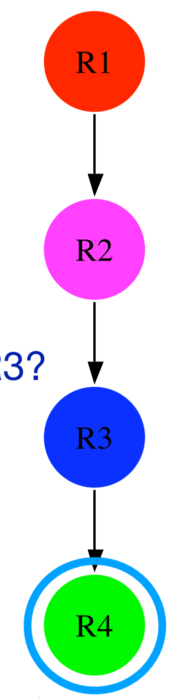


Strongly connected components ⟵ Morse graph of recurrent components

No recurrent dynamics



What is R3?

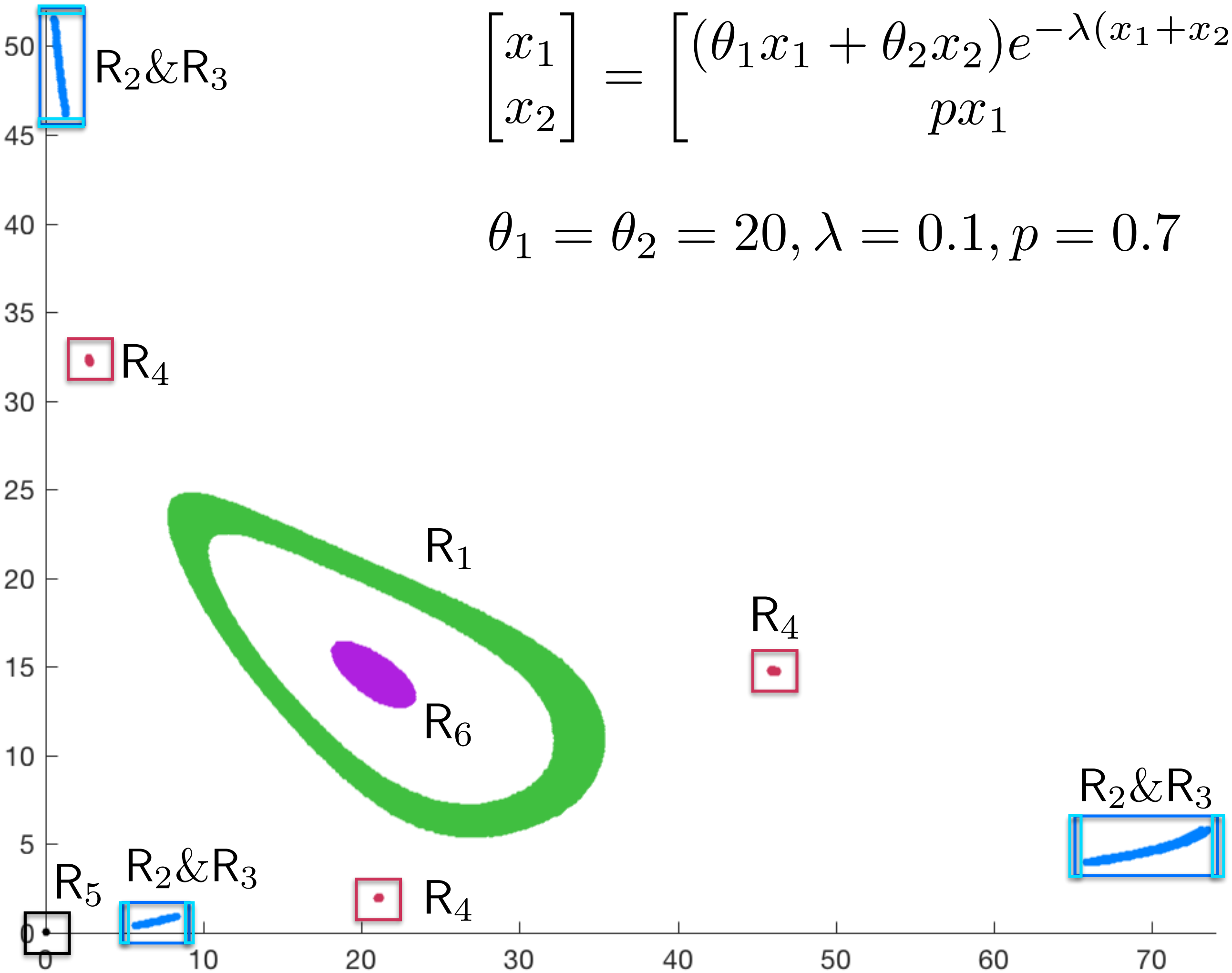


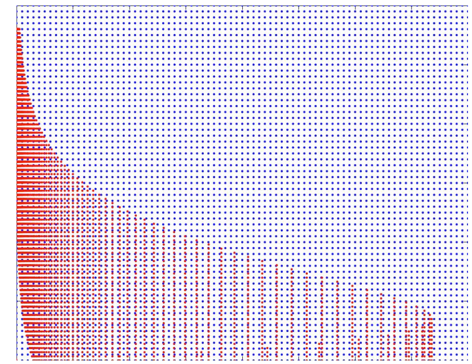
There exist fast graph algorithms to compute the Morse graph from the multivalued map.

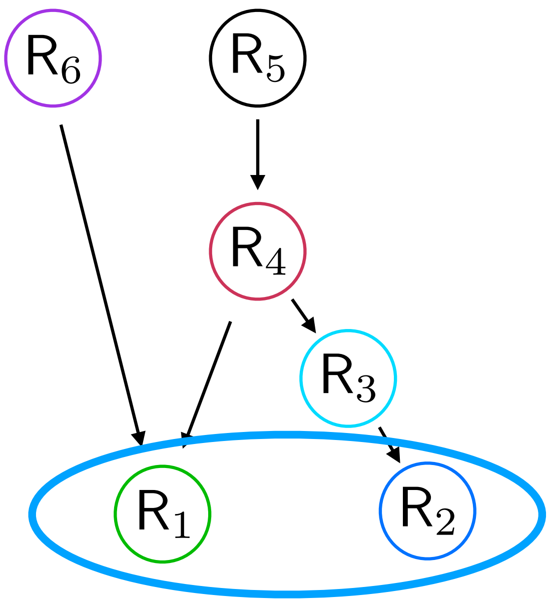
R4 contains a nonempty attractor

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\theta_1 x_1 + \theta_2 x_2) e^{-\lambda(x_1 + x_2)} \\ p x_1 \end{bmatrix}$$

$$\theta_1 = \theta_2 = 20, \lambda = 0.1, p = 0.7$$







Conley Index and Reconstruction of Dynamics

Algebraic topological invariant for isolated invariant sets.

Homological index is computable from the information contained in an outer approximation with nice images. There are fast algorithms in the cubical setting.

Theorems guarantee certain types of dynamics within an isolating neighborhood:

- Nontrivial Conley index implies nonempty invariant set.
- Lefschetz fixed point theorem for existence of fixed points.
- Extension of Lefschetz theorem to existence of periodic orbits.
- Extension of Lefschetz theorem to existence of chaotic dynamics.
- Connection matrix theory for existence of connecting orbits between Morse sets.

Conley Index

Given a pair of compact sets (N, L) with $L \subset N$ define the index map $f_{(N,L)} \colon \mathbb{N} \times (N/L, [L]) \to (N/L, [L])$ by

$$f_{(N,L)}([x]) = \begin{cases} [f(x)] & \text{if } x, f(x) \subset N \backslash L, \\ [L] & \text{otherwise.} \end{cases}$$

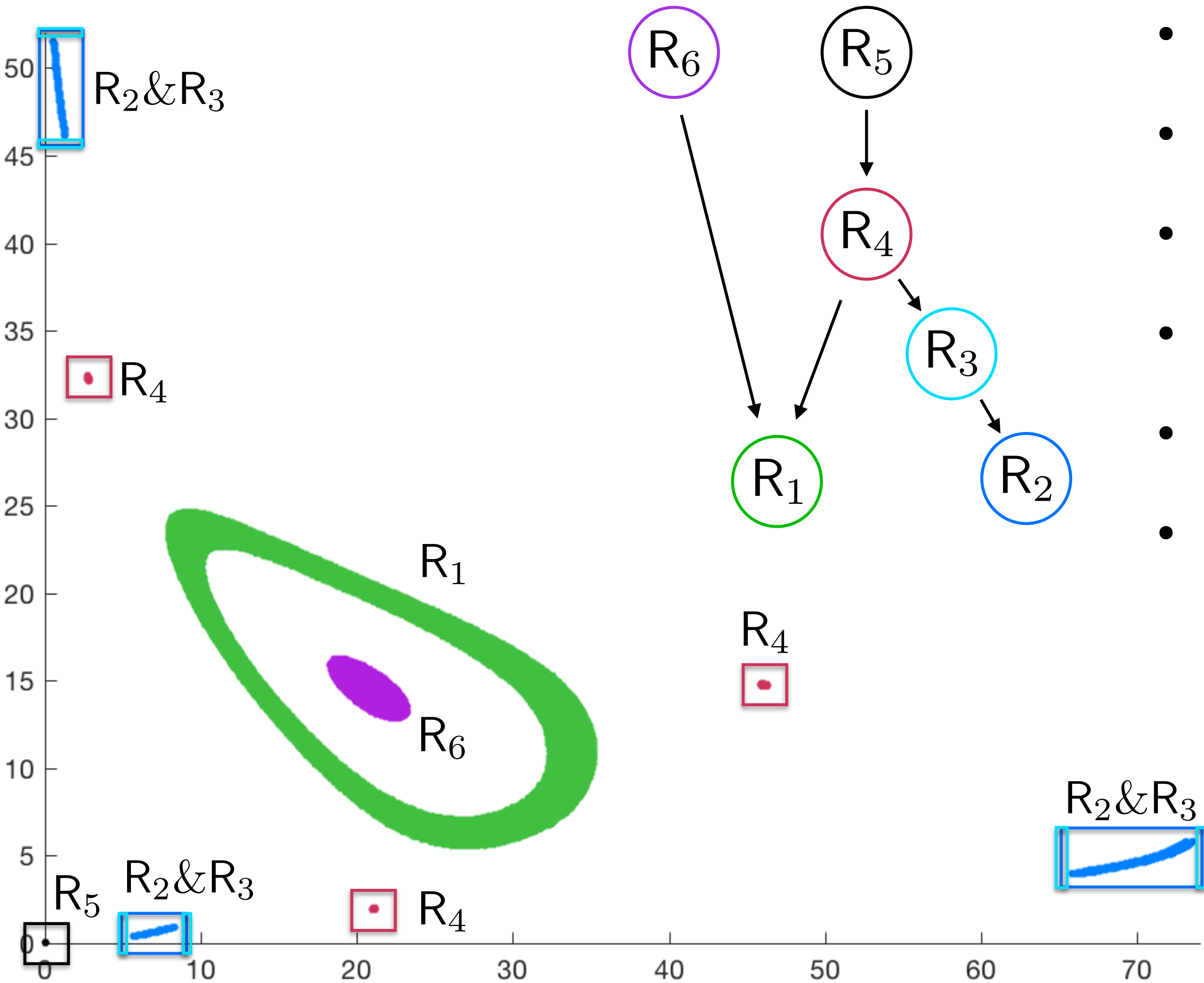
(N, L) is an *index pair* if $f_{(N,L)}$ is well-defined, continuous, and $\mathrm{cl}(N \backslash L)$ isolates $\mathrm{Inv}(N)$.

Definition: If (N, L) is an *index pair*, then the *(homological) Conley index* for $S = \mathrm{Inv}(N)$ is

$$\mathrm{Con}_*(S) = [H_*(N/L, [L]), (f_{(N.L)})_*]_s$$

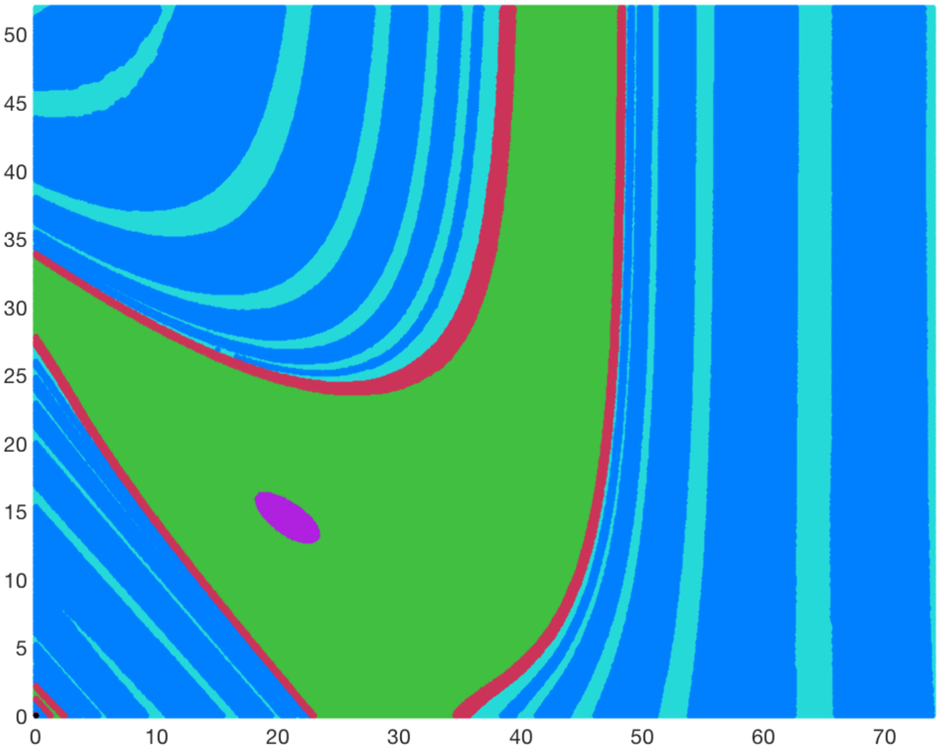
where $[\cdot, \cdot]_s$ denotes shift equivalence class.

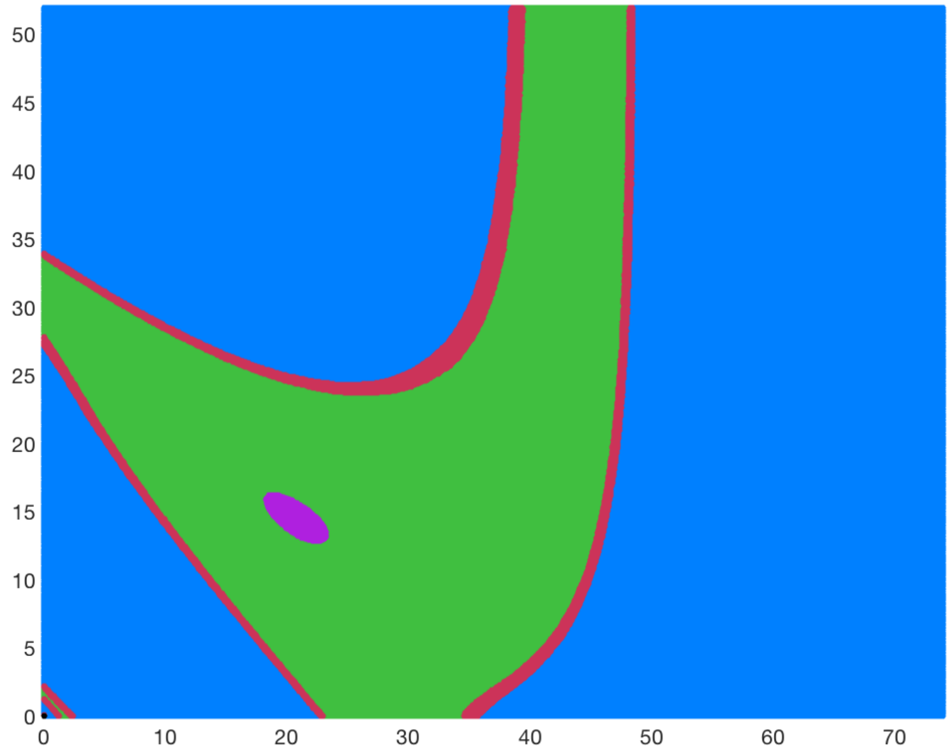
Theorem: If $\mathrm{Con}_*(S)$ is nontrivial, i.e. $(f_{(N,L)})_*$ is not nilpotent, then $S \neq \varnothing$.



- R_6 contains a fixed point
- R_5 has trivial index*
- R_4 contains a period-3 orbit
- R_3 has trivial index
- R_2 contains chaotic dynamics
- Multistability

Basins of Attraction via order theory

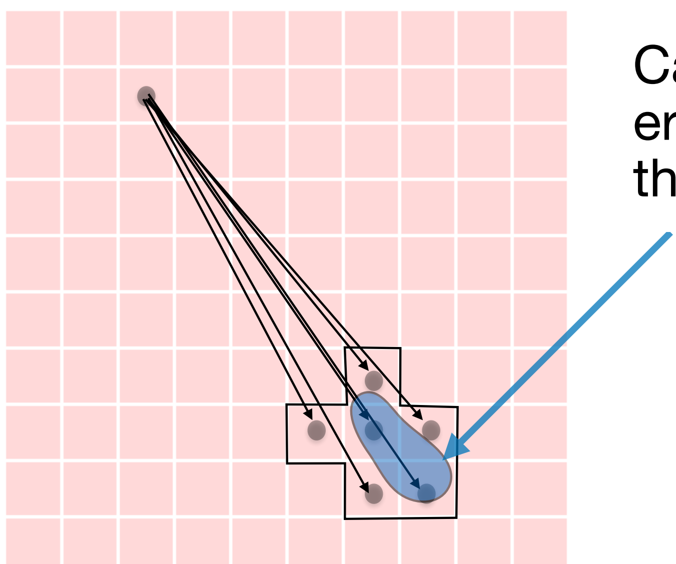




An algorithmic approach to lattices and order in dynamics (Kalies, Kasti, and VanderVorst)

Computer-assisted proofs of dynamics

- Dynamics extracted from a combinatorial model are rigorously valid for an open class of systems, ie. all continuous selectors of the multivalued map.
- Methods do not require a close approximation / fine resolution; they are accurate but not necessarily precise, ie. the extracted dynamics is correct but may not be interesting if the model is very coarse.

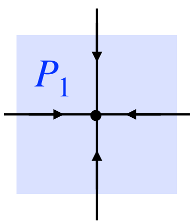


Can incorporate experimental error, modeling error, statistical measures of confidence into the images of the multivalued map.

Conley Index for Flows

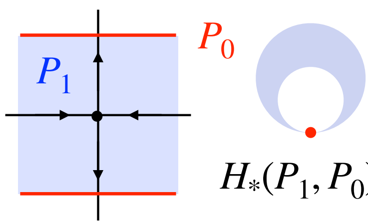
The index map is the identity for a flow, which implies that the Conley index is just the relative homology $H_*(P_1, P_0)$, which over a field can be represented by a list of Betti numbers.

(1) $\mathrm{cl}(P_1 \backslash P_0)$ isolates $\mathrm{Inv}(P_1)$, (2) P_0 is forward-invariant within P_1, (3) P_0 is an exit set for P_1.



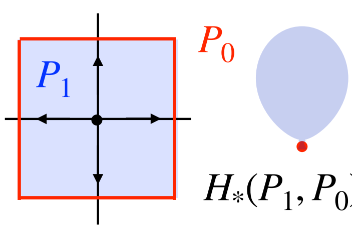
$P_0 = \varnothing$

$H_*(P_1, P_0) \to (1,0,\ldots)$



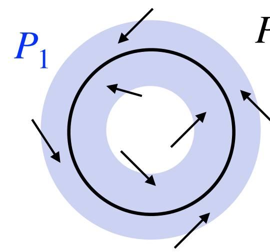
$P_1/P_0 \approx S^1$

$H_*(P_1, P_0) \to (0,1,0,\ldots)$



$P_1/P_0 \approx S^2$

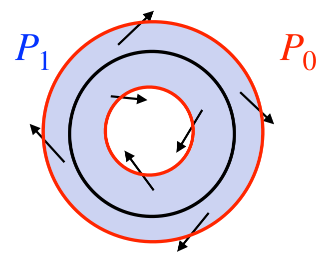
$H_*(P_1, P_0) \to (0,0,1,0,\ldots)$



$P_0 = \varnothing$

$P_1/P_0 \approx S^0 \vee S^1$

$H_*(P_1, P_0) \to (1,1,0,\ldots)$



$P_1/P_0 \approx S^1 \vee S^2$

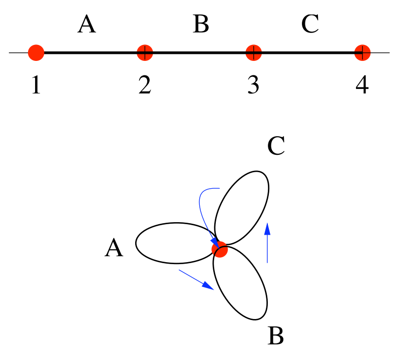
$H_*(P_1, P_0) \to (0,1,1,0,\ldots)$

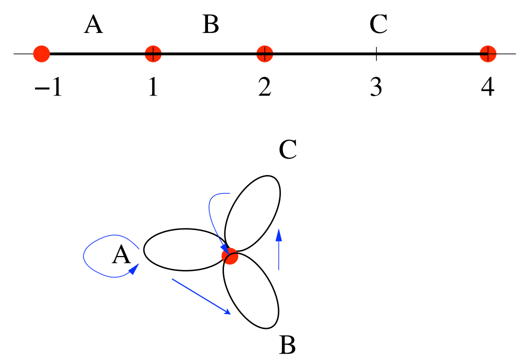
Converses?

Conley index for Maps: Shift Equivalence

$$f(x) = x + 1 \qquad\qquad g(x) = 2x$$

$$(P_1, P_0) = ([1,4], \{1,2,3,4\}) \qquad\qquad (P_1, P_0) = ([-1,4], \{-1,1,2,4\})$$





$$(f_P)_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad\qquad (g_P)_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Theorem: If $\sum_{n=0}^{\infty}(-1)^n \mathrm{tr}((f_P)_n) \neq 0$ **then** $P_1 \backslash P_0$ **contains a fixed point. [Lefschetz]**

Let $\mathscr{F} : \mathscr{X} \to \mathscr{X}$ be an outer approximation and $\mathscr{M}$ a Morse set.

The realization of the pair $(\mathscr{P}_1, \mathscr{P}_0) = (\mathscr{F}(\mathscr{M}), \mathscr{F}(\mathscr{M}) \backslash \mathscr{M})$ is an index pair.

An index map $(f_P)_*$ over a finite field can be computed from this pair.

Conley index information from the index map is represented by a collection of polynomials as follows.

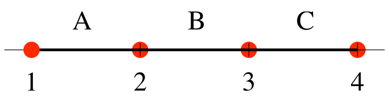
First $(f_P)_*$ is put into rational canonical form, a block diagonal matrix whose diagonal blocks correspond to polynomial factors of the characteristic polynomial. Two matrices are similar iff the sets of polynomial factors (including multiplicity) are equal.

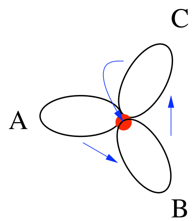
To accommodate shift equivalence, the set of polynomial factors is modified by dividing out the largest possible factor of x from each polynomial and then removing constant polynomials from the set. The resulting set determines the shift equivalence class of $(f_P)_*$.

Shift equivalence

$$f(x) = x + 1$$

$$(P_1, P_0) = ([1,4], \{1,2,3,4\})$$



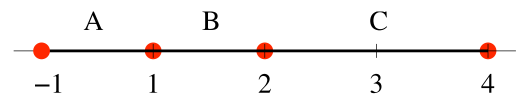


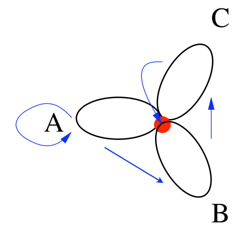
$$(f_P)_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\left(x^3 \to \{x \text{ mult } 3\} \to \varnothing\right) \text{ mod } 2$$

$$g(x) = 2x$$

$$(P_1, P_0) = ([-1,4], \{-1,1,2,4\})$$

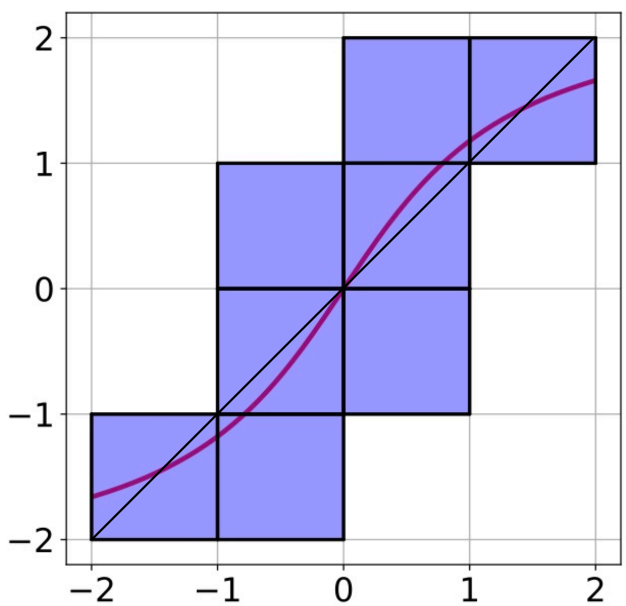


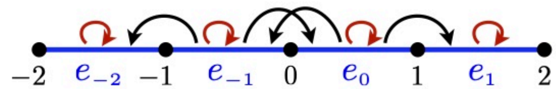


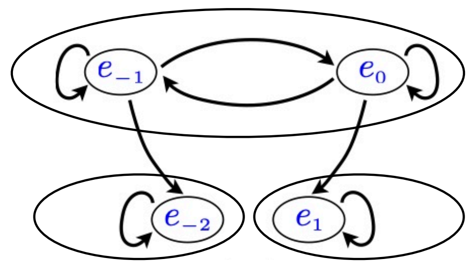
$$(g_P)_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

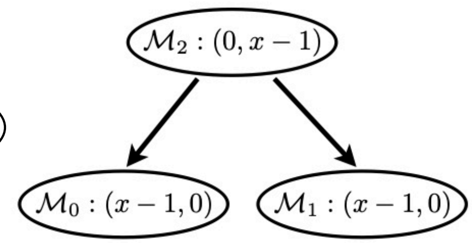
$$\left(x^3 + x^2 \to \{x \text{ mult } 2, x+1\} \to \{x+1\}\right) \text{ mod } 2$$

Conley Morse Graphs



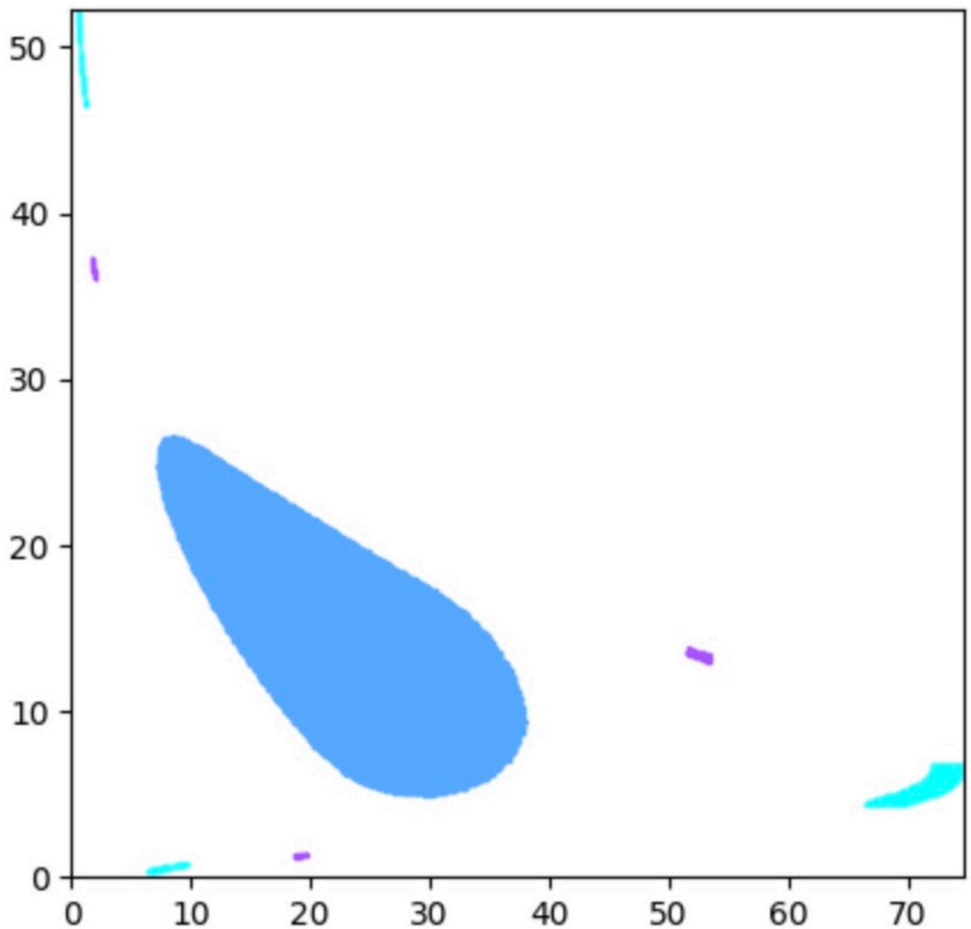




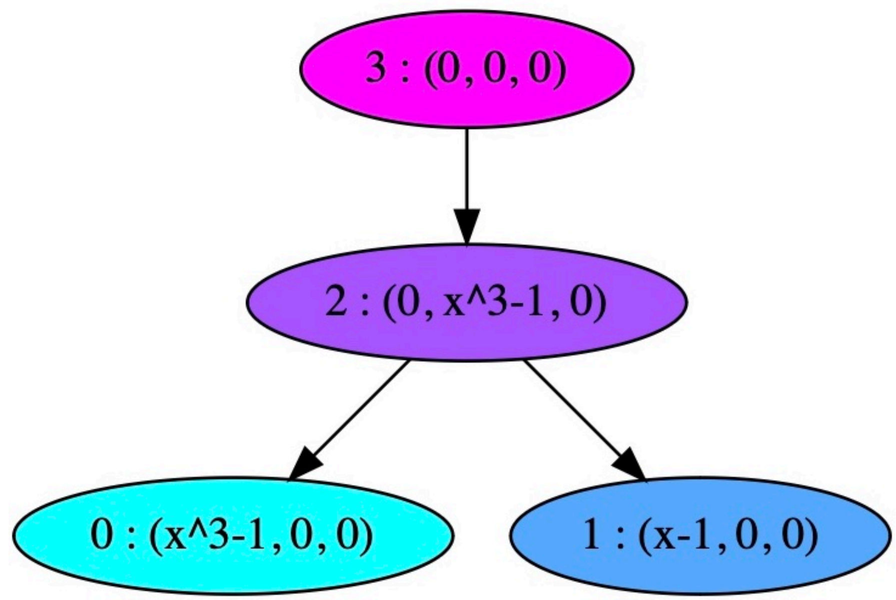


If the Conley index has $x \pm 1$ in one level of homology, then the corresponding Morse set contains a fixed point.

If the Conley index has $x^p \pm 1$ in one level of homology, then the corresponding Morse set contains a periodic orbit of period-p.



0
10
20
30
40
50
60
70



3 : (0, 0, 0)
2 : (0, x^3-1, 0)
0 : (x^3-1, 0, 0)
1 : (x-1, 0, 0)

Connection Matrices for Flows

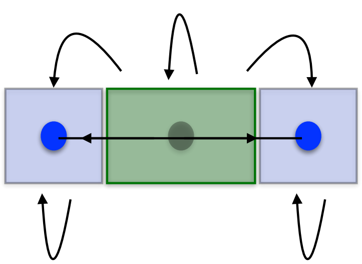
- The Morse graph for the combinatorial model represents a Morse decomposition.
- For each node in the Morse graph, the computed homology group is the homological Conley index for the associated Morse set.
- Flows: a connection matrix for the Morse decomposition is the boundary operator for a Conley complex for the combinatorial model.

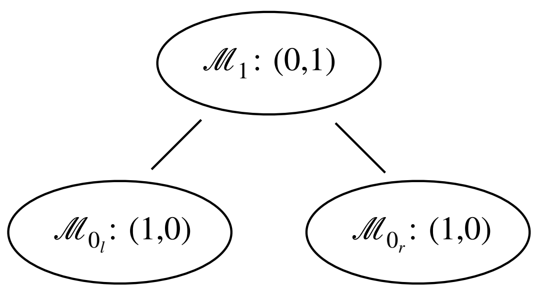
The information in a Conley-Morse graph can be organized in a chain complex.

A connection matrix is a boundary operator ($\triangle \circ \triangle = 0$)

$$\triangle : \bigoplus_{\mathscr{M} \in \mathrm{MG}} CH_*(\mathscr{M}) \to \bigoplus_{\mathscr{M} \in \mathrm{MG}} CH_*(\mathscr{M})$$

Connection matrices need not be unique, but they provide information about Conley indices of convex sets in the partial order as well as connecting orbit information.





$$\triangle_1 : CH_1(\mathscr{M}_1) \to CH_0(\mathscr{M}_{0_l}) \oplus CH_0(\mathscr{M}_{0_r})$$

$$\triangle_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

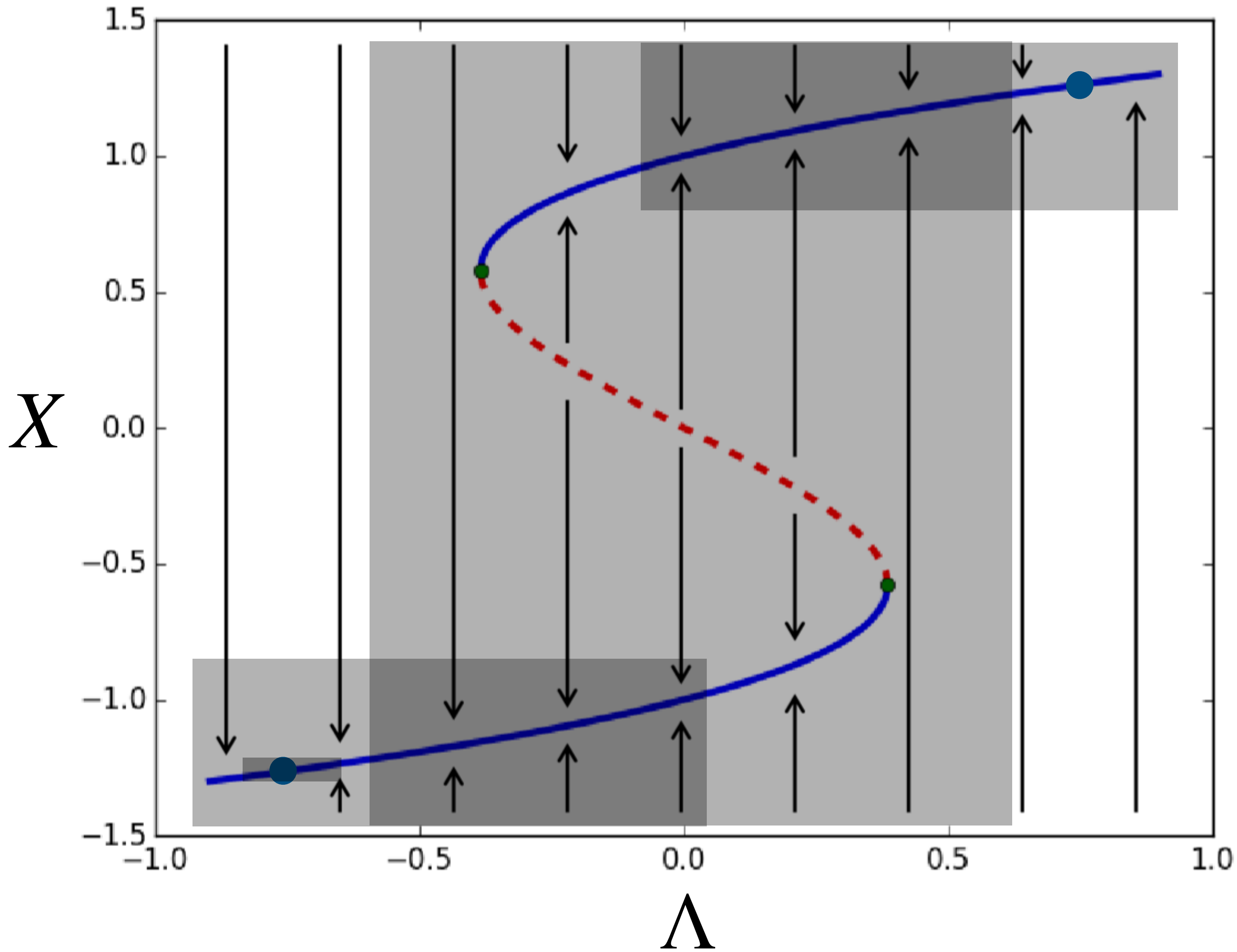
implies heteroclinic connections from $\mathscr{M}_1$ to both $\mathscr{M}_{0_l}$ and $\mathscr{M}_{0_r}$.

Continuation of the Conley index

Since isolating blocks exist, are robust, and provide index pairs, the Conley index is also robust with respect to changes in parameter, i.e. there is a continuation property (like the topological degree).

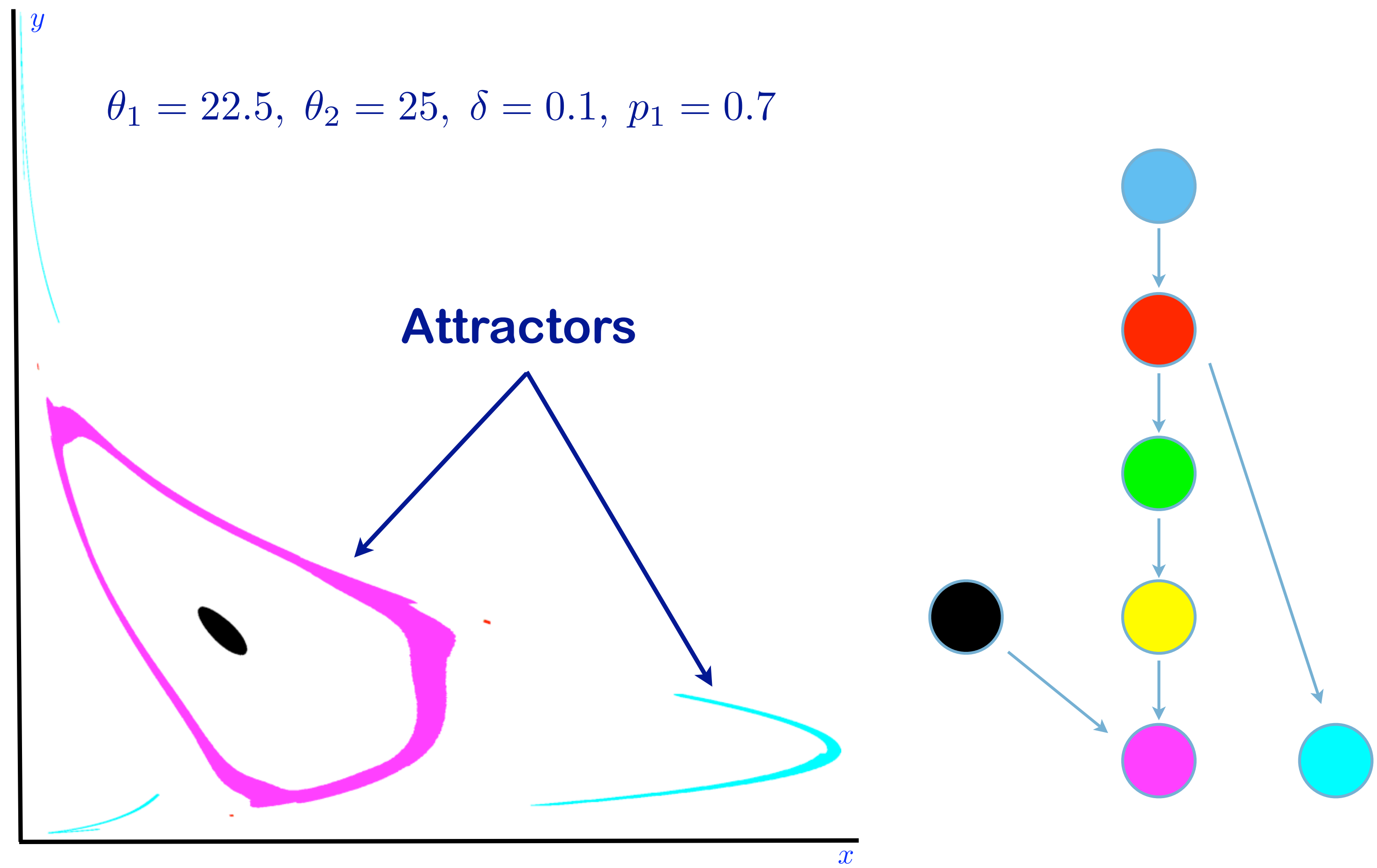
Local continuation: if S is an isolated invariant set for φ_{λ_0}, then there exists a neighborhood Q of λ_0 such that $\mathrm{Con}(S_\lambda) = \mathrm{Con}(S_{\lambda_0})$ for all $\lambda \in Q$.

Global continuation: if N is an isolating neighborhood for φ_λ for all $\lambda \in Q$, then $\mathrm{Con}(S_{\lambda_1}) = \mathrm{Con}(S_{\lambda_2})$ for all $\lambda_1, \lambda_2 \in Q$

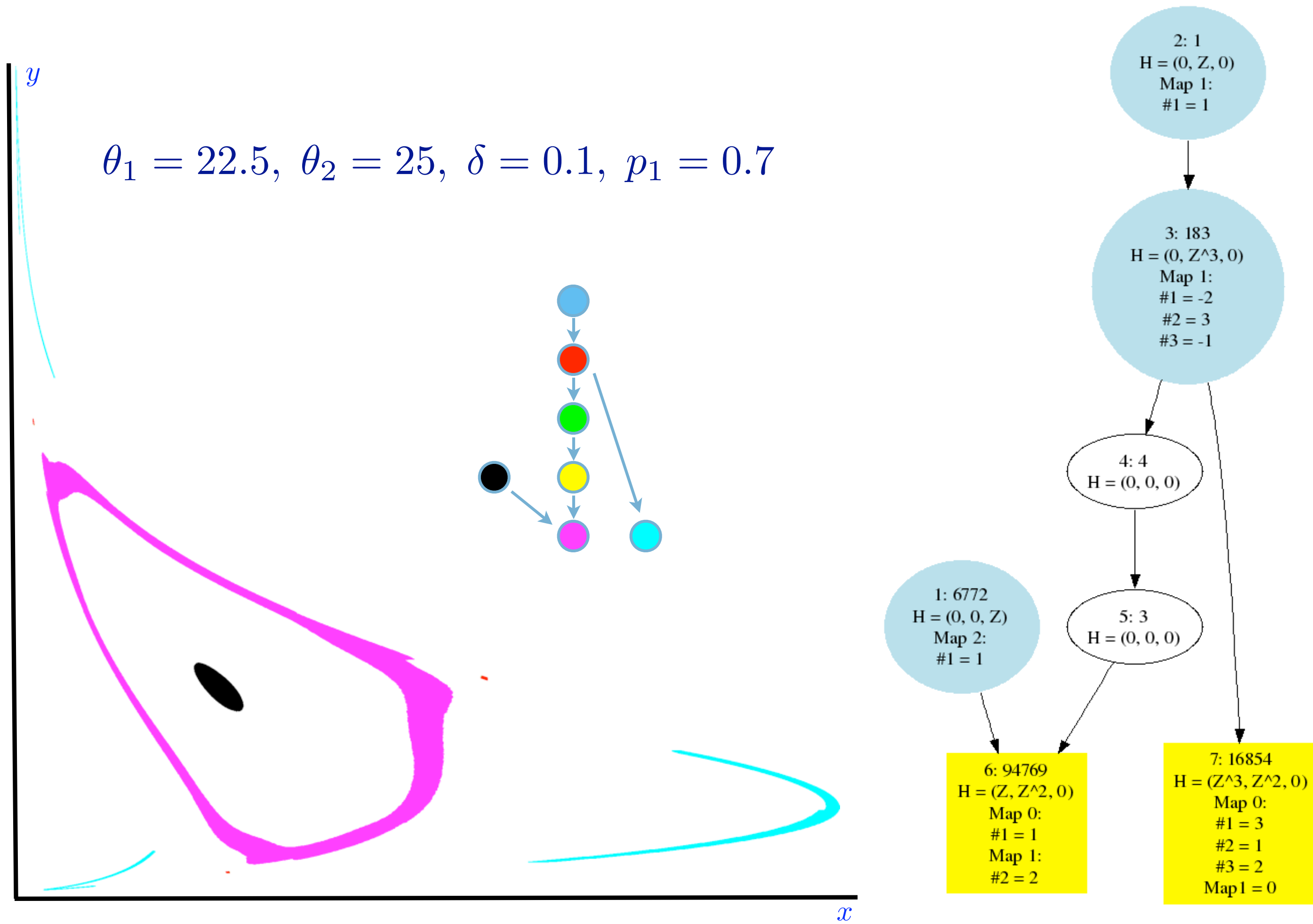


1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
X
−1.0
−0.5
0.0
0.5
1.0
Λ

Morse graph



Conley-Morse graph



Continuation Classes

Suppose $\mathcal{Q}$ is a grid on Λ with the simple intersection property, ie. empty or contractible.

Suppose two Morse decompositions $\{\mathcal{M}(p) \mid p \in \mathcal{P}_{Q_0}\}$ and $\{\mathcal{M}(p) \mid p \in \mathcal{P}_{Q_1}\}$ have been computed for $Q_0, Q_1 \in \mathcal{Q}$ with $Q_0 \cap Q_1 \neq \varnothing$.

If for each $p_0 \in \mathcal{P}_{Q_0}$ there is a unique $p_1 \in \mathcal{P}_{Q_1}$ such that $|\mathcal{M}_{Q_0}(p_0)| \cap |\mathcal{M}_{Q_1}(p_1)| \neq \varnothing$, and the induced mapping $\iota_{Q_0,Q_1} : \mathcal{P}_{Q_0} \to \mathcal{P}_{Q_1}$ is a directed graph isomorphism, then we say the corresponding Conley-Morse graphs $\mathrm{CMG}(Q_0)$ and $\mathrm{CMG}(Q_1)$ are equivalent. The equivalence classes of CMG's over grid elements are called continuation classes.

A nonlinear Leslie population model

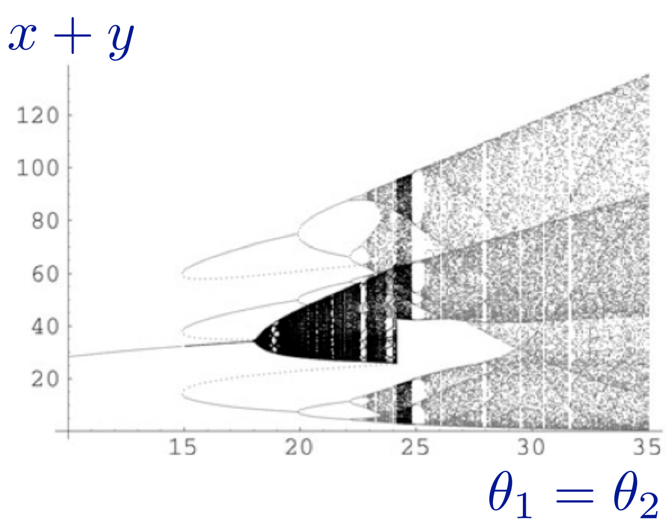
$$f(x, y; \theta_1, \theta_2, \delta, p_1) = (\theta_1 x + \theta_2 y)e^{-\delta(x+y)}, p_1 x)$$

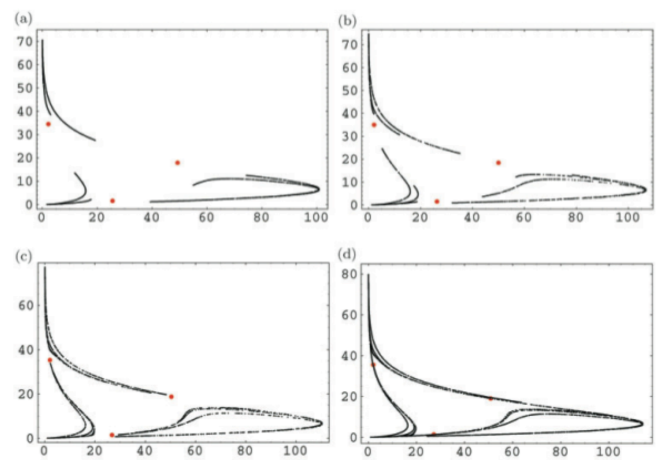
x, y **represent two age classes of a population**

θ_1, θ_2 **are fertility rates of each class**

δ **exponential decay in fertility rate [Ricker]**

p_1 **probability of surviving to next age class**

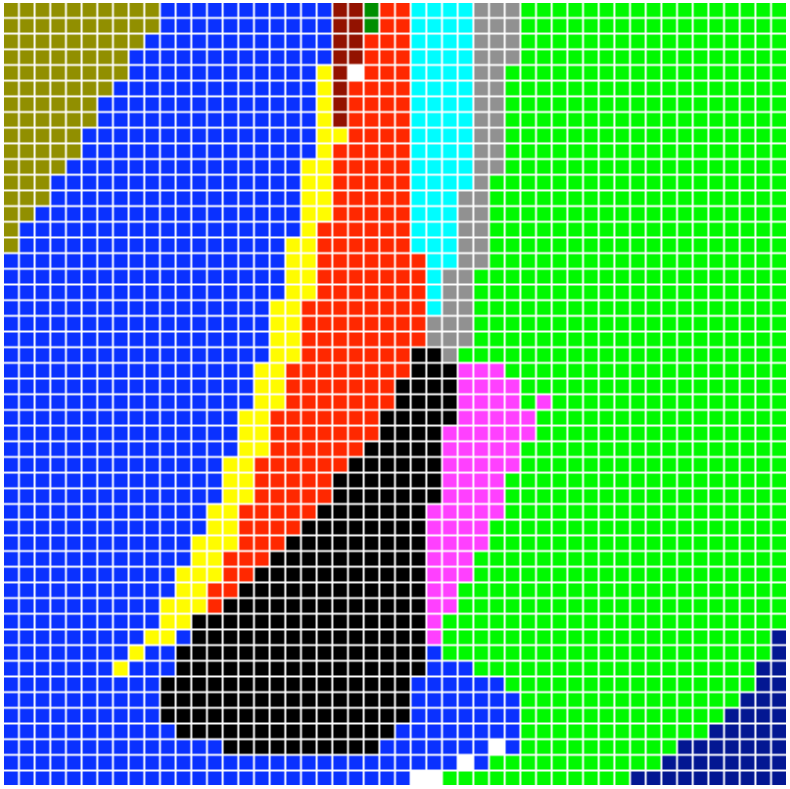




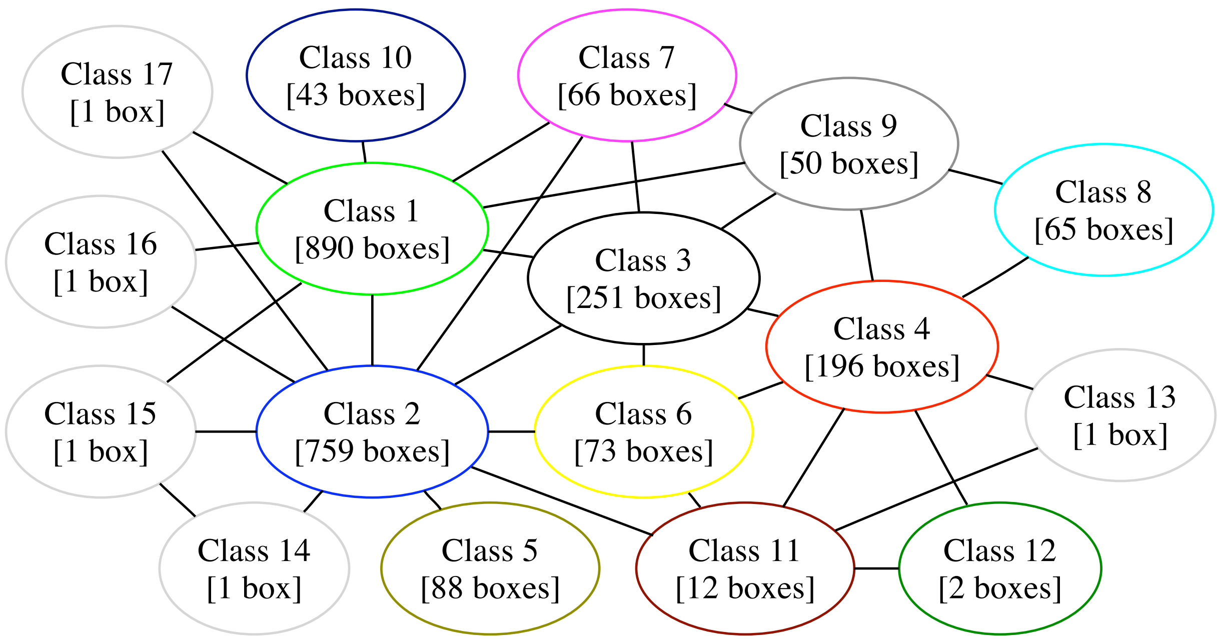
Continuation diagram for Leslie model in 2-d

$$f(x, y; \theta_1, \theta_2, \delta, p_1) = ((\theta_1 x + \theta_2 y)e^{-\delta(x+y)}, p_1 x)$$

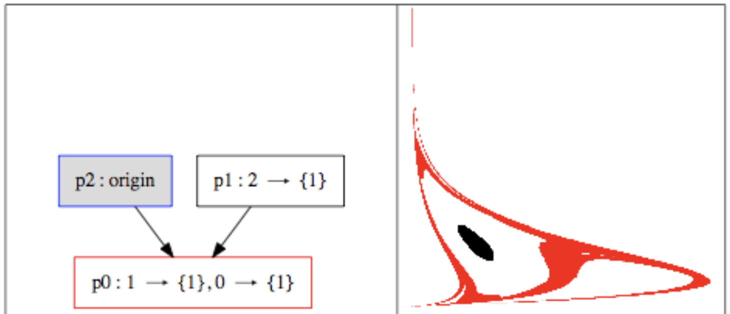
$$\delta = 0.1, p_1 = 0.7, \Lambda = \{(\theta_1, \theta_2) \in [8, 37] \times [3, 50]\}$$



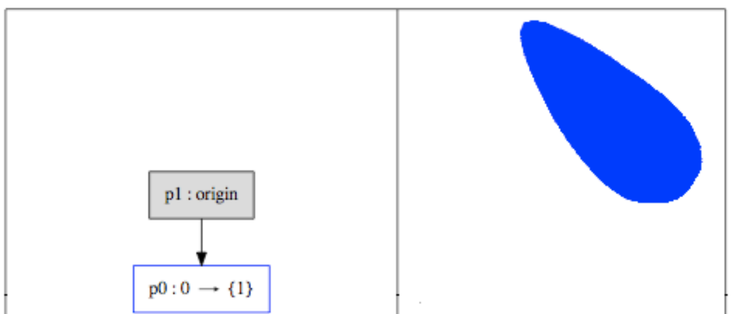
Continuation Graph



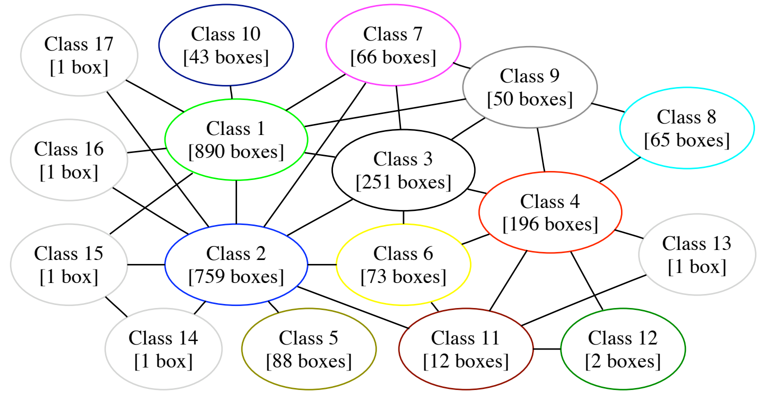
Arai, Kalies, Kokubu, Mischaikow, Oka, Pilarczyk

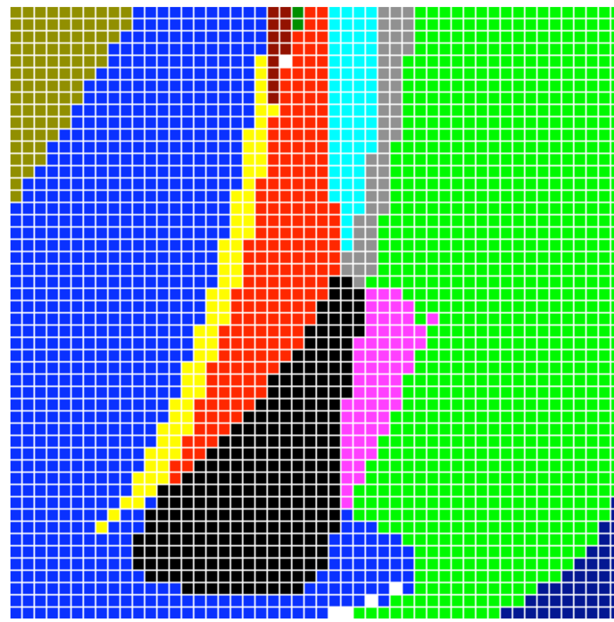


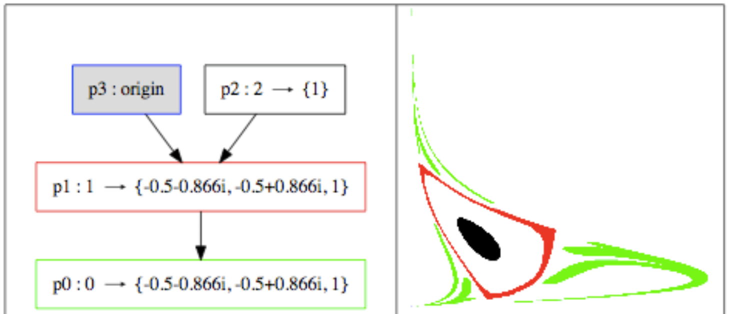
1. The Conley-Morse graph CMG(1) and the sets $\mathcal{M}(p)$ at the box $(35, 32)$.



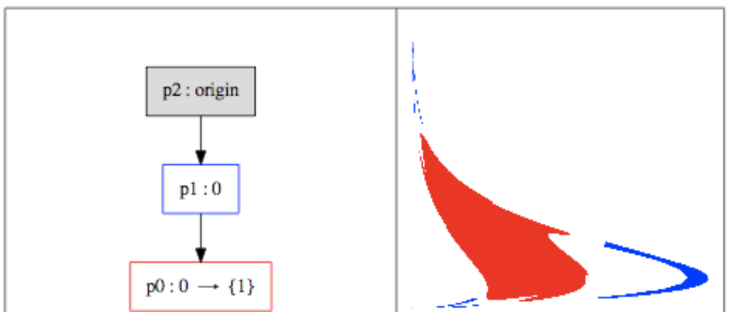
2. The Conley-Morse graph CMG(2) and the sets $\mathcal{M}(p)$ at the box $(10, 13)$.



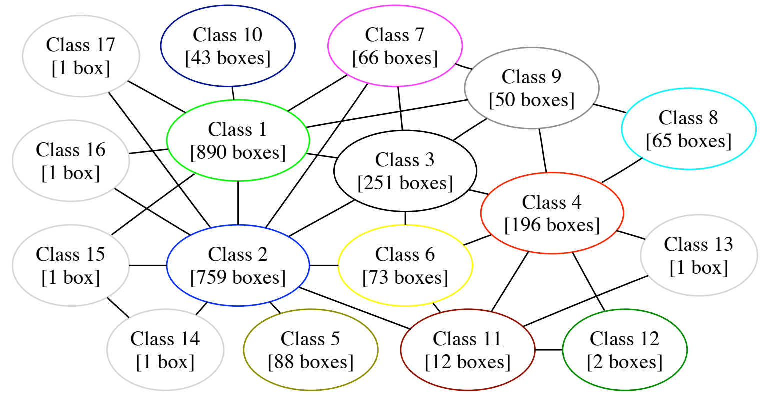


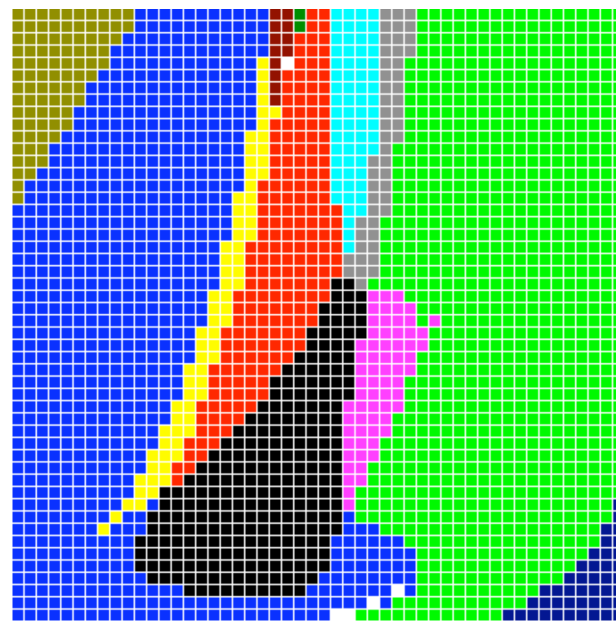


7. The Conley-Morse graph CMG(7) and the sets $\mathcal{M}(p)$ at the box $(31, 22)$.



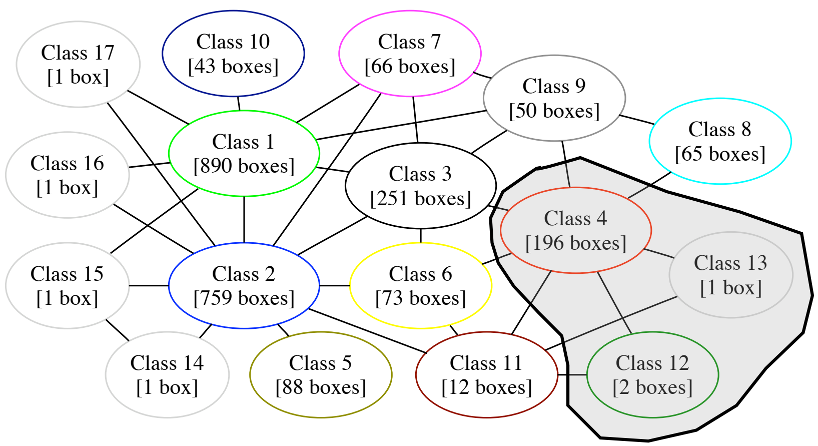
8. The Conley-Morse graph CMG(8) and the sets $\mathcal{M}(p)$ at the box $(27, 39)$.

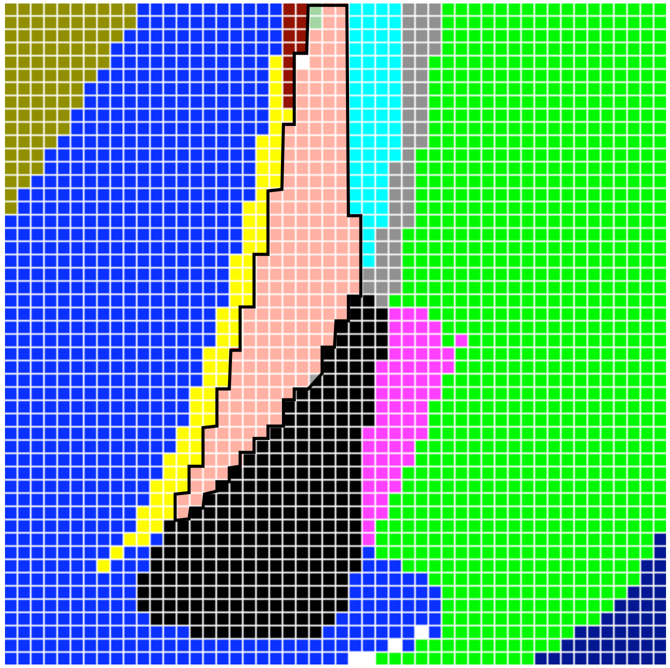




Multiple basins of attraction

Query: Which continuation classes have a CMG with more than one minimal element?





Resolution

All computations are rigorous, so any query result is a "theorem".

A resolution in both parameter space and phase space has been chosen a priori, and the dynamics computed is "effective" to this resolution -- what structures are detected depends on their size and hyperbolicity.

Two parameters can be in the same continuation class but have non-conjugate dynamics.

Two parameters can have conjugate dynamics and be in different continuation classes.

Advantage: queriable and can be extended to finer resolution in areas of interest.

Lifting lattice homomorphisms

QUESTION: Given a finite sublattice of attractors A , does there exist a finite sublattice of attracting neighborhoods such that $\mathrm{Inv} = \omega$ produces a lattice isomorphism? **Yes**

Franzosa 1986 (flows)
Franzosa and Mischaikow 1988 (semiflows)

$$
\begin{tikzcd}
 & \mathsf{ANbhd} \arrow[d, two heads, "\mathrm{Inv}"] \\
\mathsf{A} \arrow[ur, dotted, tail, "k"] \arrow[r, tail, "i"] & \mathsf{Att}
\end{tikzcd}
$$

EQUIVALENTLY: Given A, does there exist a lattice monomorphism k such that the diagram commutes?

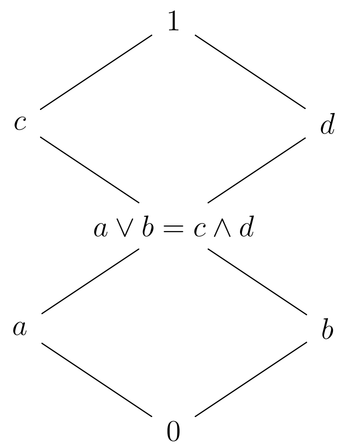
One could recast this as a general statement about lattices.

QUESTION: Given a finite lattice H and a lattice monomorphism ℓ and epimorphism h , does there exists a lattice monomorphism k such that the diagram commutes? **Not always**

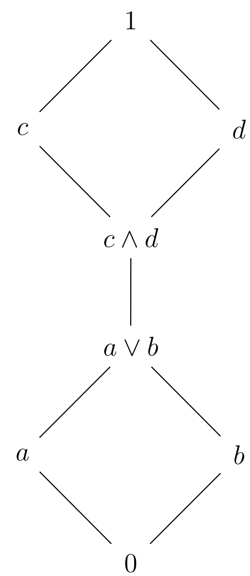
$$
\begin{tikzcd}
 & \mathsf{K} \arrow[d, two heads, "h"] \\
\mathsf{H} \arrow[ur, dotted, tail, "k"] \arrow[r, tail, "\ell"] & \mathsf{L}
\end{tikzcd}
$$

k is a *lift* of ℓ through h

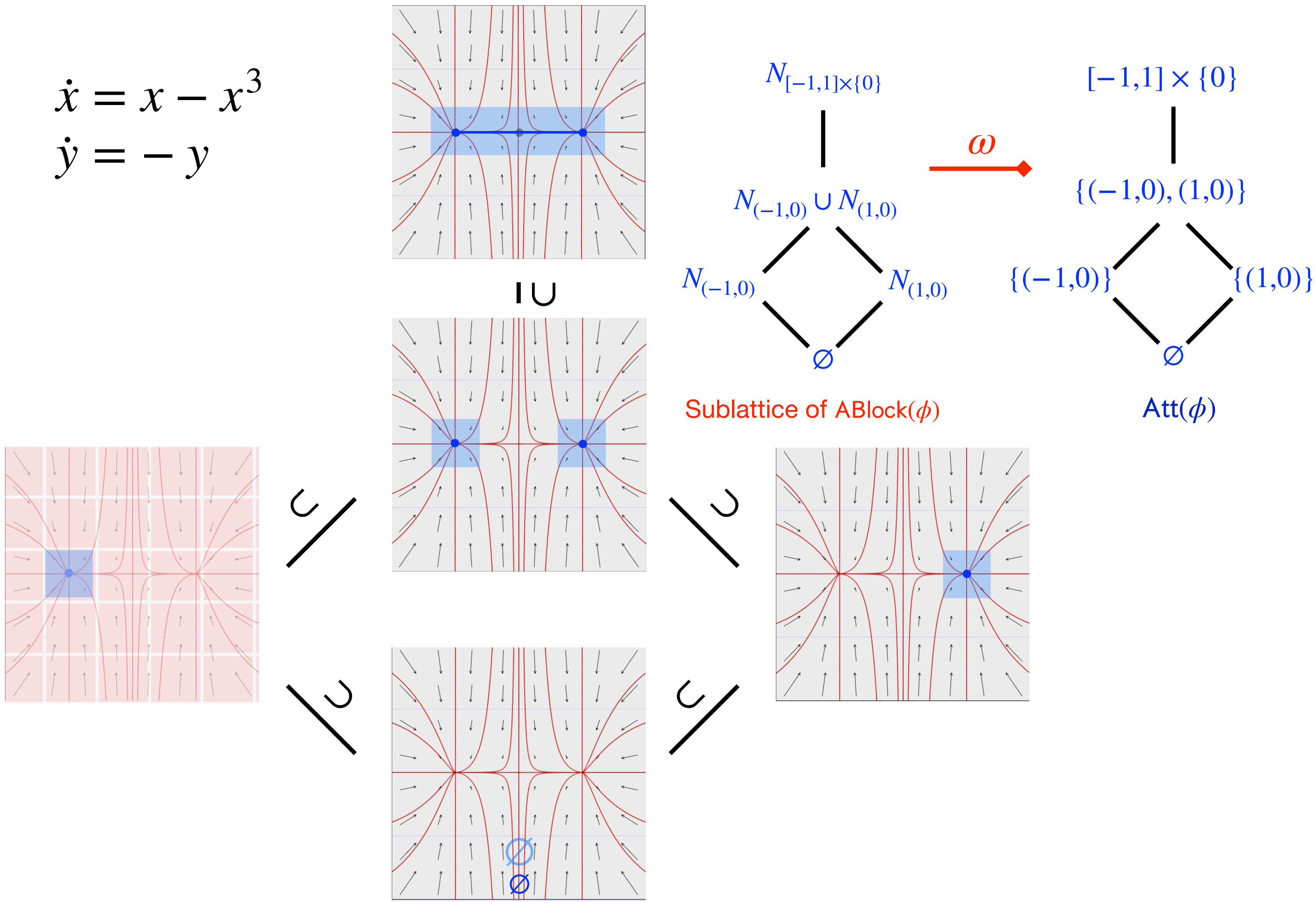
General lifting -- a counterexample



H=L



K



$\dot{x} = x - x^3$
$\dot{y} = -y$
$N_{[-1,1]\times\{0\}}$
$N_{(-1,0)} \cup N_{(1,0)}$
$N_{(-1,0)}$
$N_{(1,0)}$
$\varnothing$
ω
$[-1,1] \times \{0\}$
$\{(-1,0), (1,0)\}$
$\{(-1,0)\}$
$\{(1,0)\}$
$\varnothing$
Sublattice of ABlock(ϕ)
Att(ϕ)
$\varnothing$

Convergent outer approximations

$$\mathcal{F}_\rho(\xi) := \{\eta \in \mathcal{X} \mid B_\rho\big(f(|\xi|)\big) \cap |\eta| \neq \varnothing\}$$

Definition: $\mathcal{F}_o$ is the *minimal multivalued map* ($\rho = 0$), and $\mathcal{F}_\rho$ is the *ρ-minimal multivalued map*.

Let $\mathcal{F}_n \colon \mathcal{X}_n \rightrightarrows \mathcal{X}_n$ be a sequence of outer approximations for $f\colon X \to X$. Then $\mathcal{F}_n$ *converges* if $\operatorname{diam}(\mathcal{X}_n) \to 0$ and there exist ρ_n-minimal maps $\mathcal{F}_{\rho_n}$ with $\rho_n \to 0$ such that

$$\mathcal{F}_o \leq \mathcal{F}_n \leq \mathcal{F}_{\rho_n} \text{ on } \mathcal{X}_n.$$

Theorem: Let $f\colon X \to X$ be a continuous mapping on a compact metric space X. Let $\mathcal{F}_n \colon \mathcal{X}_n \rightrightarrows \mathcal{X}_n$ be a convergent sequence of outer approximations. Then for every finite sublattice $\mathsf{A} \subset \mathsf{Att}(X, f)$ there exists an n_A such that for all $n \geq n_\mathsf{A}$ there are lattice monomophisms $\mathsf{A} \to \mathsf{ASet}(\mathcal{X}_n, \mathcal{F}_n)$ and commuting diagrams

$$\begin{tikzcd} & \mathsf{ASet}(\mathcal{X}_n, \mathcal{F}_n) \arrow[d, "\omega(|\cdot|)"] \\ \mathsf{A} \arrow[r, tail, "i"] \arrow[ur, dotted, tail] & \mathsf{Att}(X, f). \end{tikzcd}$$

Can be replaced by Invset^+ if grids are a sequence of refinements.

The homomorphisms $\mathsf{A} \to \mathsf{ASet}(\mathcal{X}_n, \mathcal{F}_n)$ are called the lifts of A into $\mathsf{ASet}(\mathcal{X}_n, \mathcal{F}_n)$. There exist lifts such that the realization of the image of $\mathsf{A} \to \mathsf{Invset}^+(\mathcal{X}_n, \mathcal{F}_n)$, denoted by N, is a sublattice of $\mathsf{ANbhd}(X, f)$ and $\omega\colon \mathsf{N} \to \mathsf{A}$ is an lattice isomorphism. The same statement holds for finite sublattices of repellers $\mathsf{R} \subset \mathsf{RSet}(X, f)$.

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Thank you!