Linear Regression: Inference

EC 339

Marcio Santetti
Fall 2023

Motivation

A critique

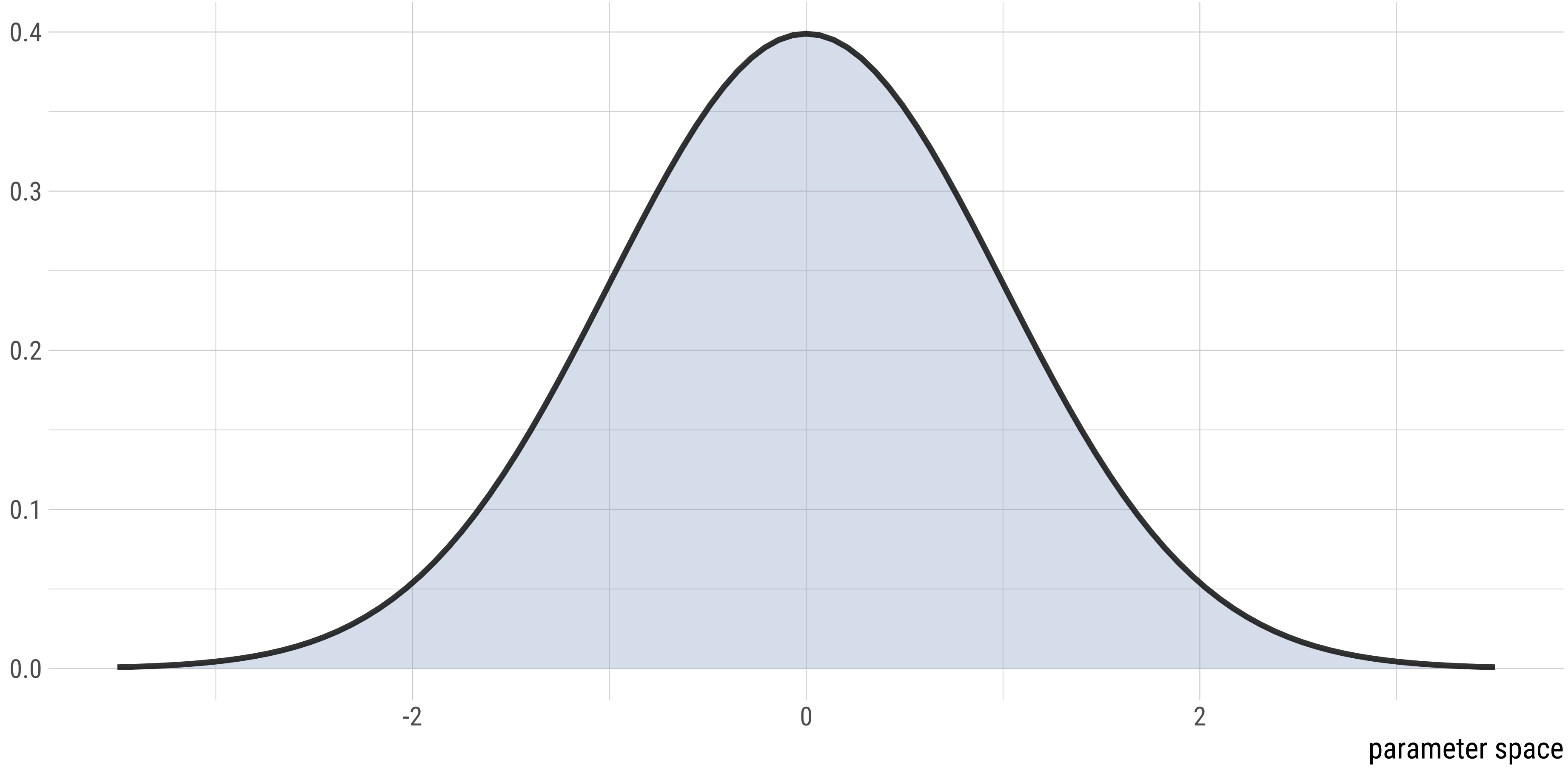
Here, we are dealing with the so-called **frequentist** approach to Statistics/Econometrics.

It assumes that there exists an underlying **true population parameter** in nature.

Therefore, while this **population parameter** value is fixed in nature, **samples** are variable.

And **using samples** is the best we can do.

Where does this come from?



Sampling distributions

Sampling distributions

Suppose we have a **population** consisting of *N = 2,930* observations, and we are interested in one variable called *X*.

Its population mean (μ) is 10,148.

Now suppose we select a **random sample** of size *n = 50* from this population.

Estimating the *mean* of this sample gives us $\bar{x} =$ 10,761.78.

Sampling distributions

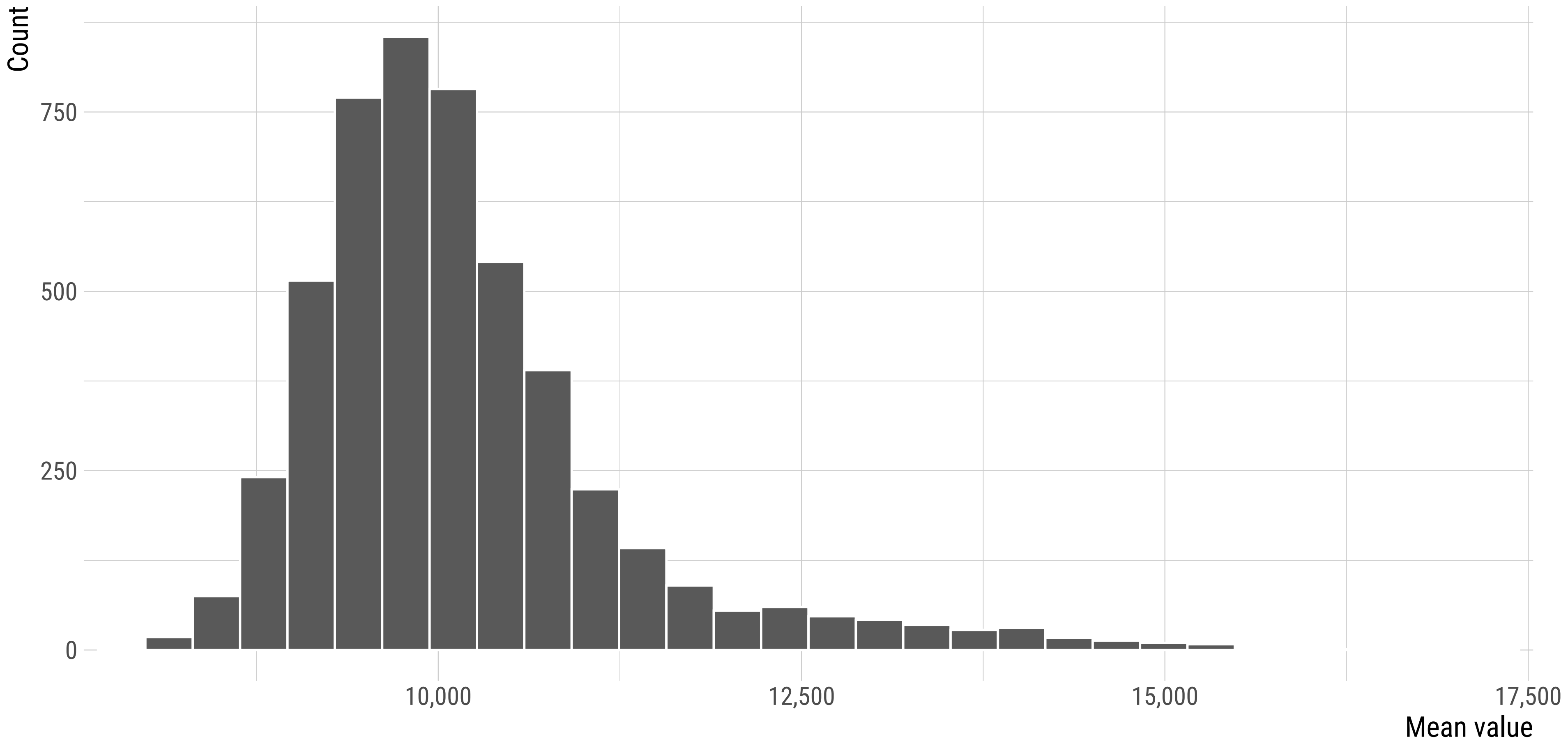
Then, we select another random *sample* of size *n = 50*.

Its mean is $\bar{x} = 12{,}611.9$.

A third random sample of size *n = 50* gives a sample mean of $\bar{x} = 9{,}058.66$.

What if we do this procedure *5,000* times?

Sampling distributions

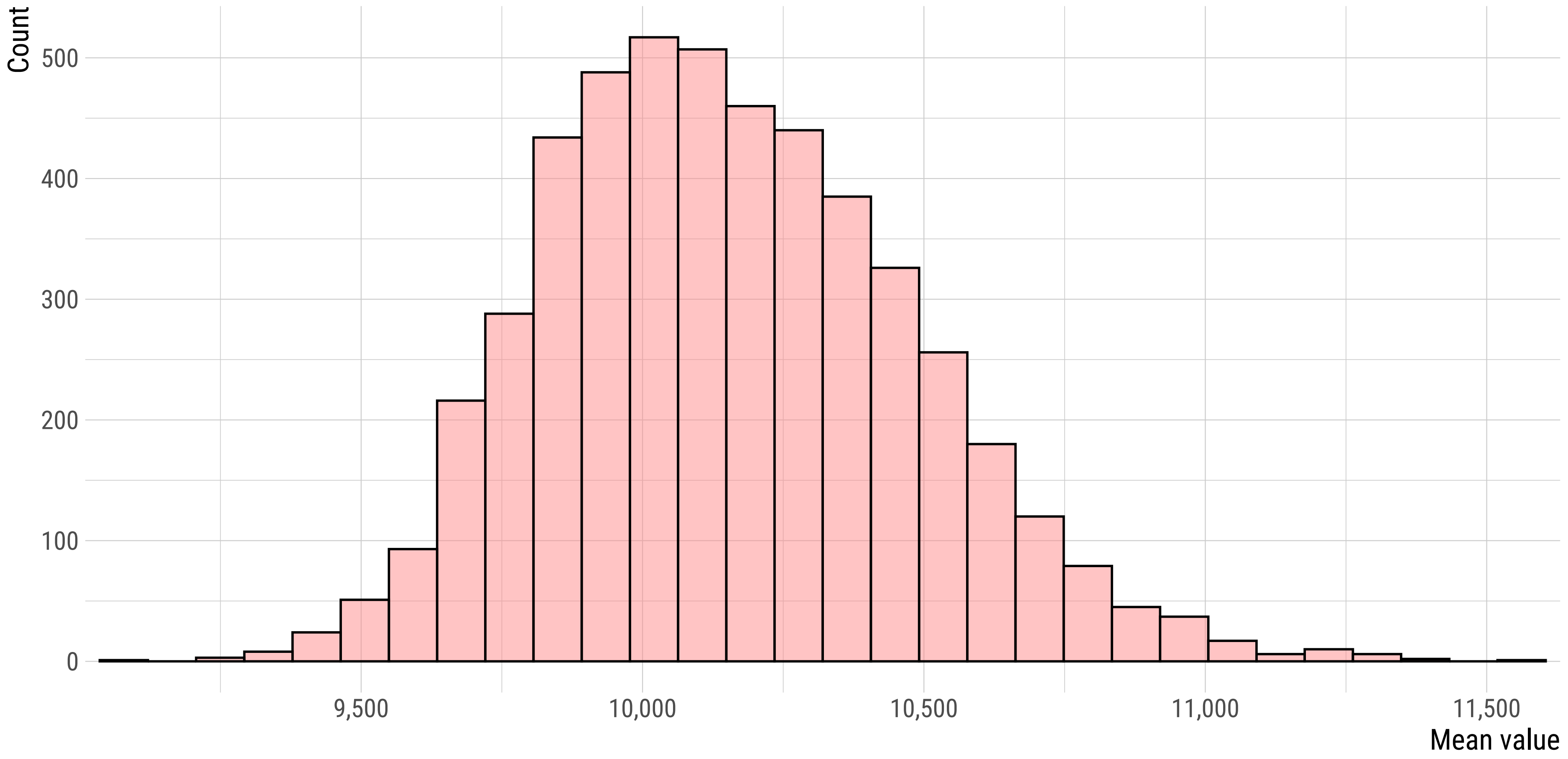


Sampling distributions

Now, suppose we increase the sample size to $n = 500$.

Once again, we repeat the procedure *5,000* times.

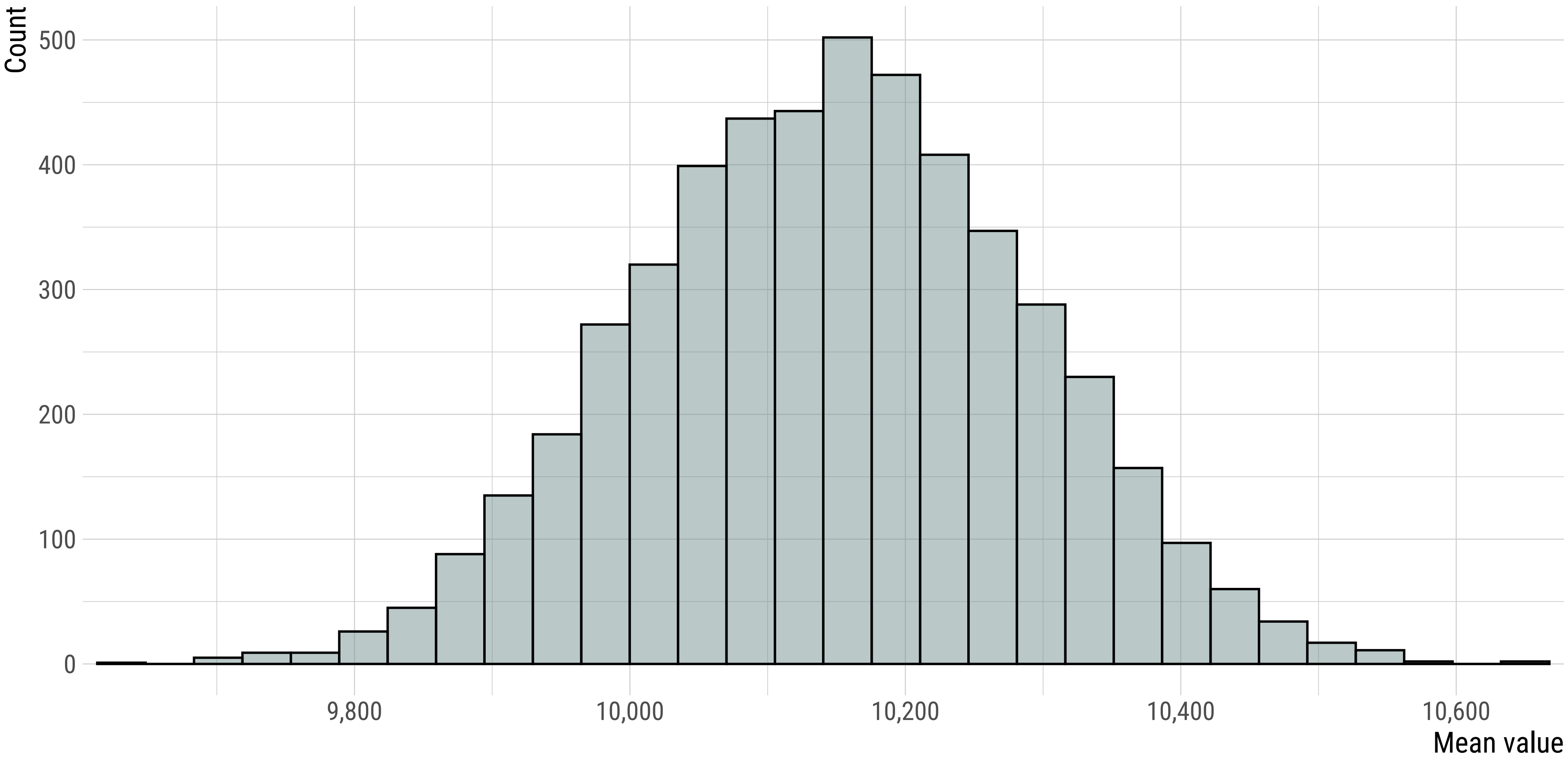
Sampling distributions



Sampling distributions

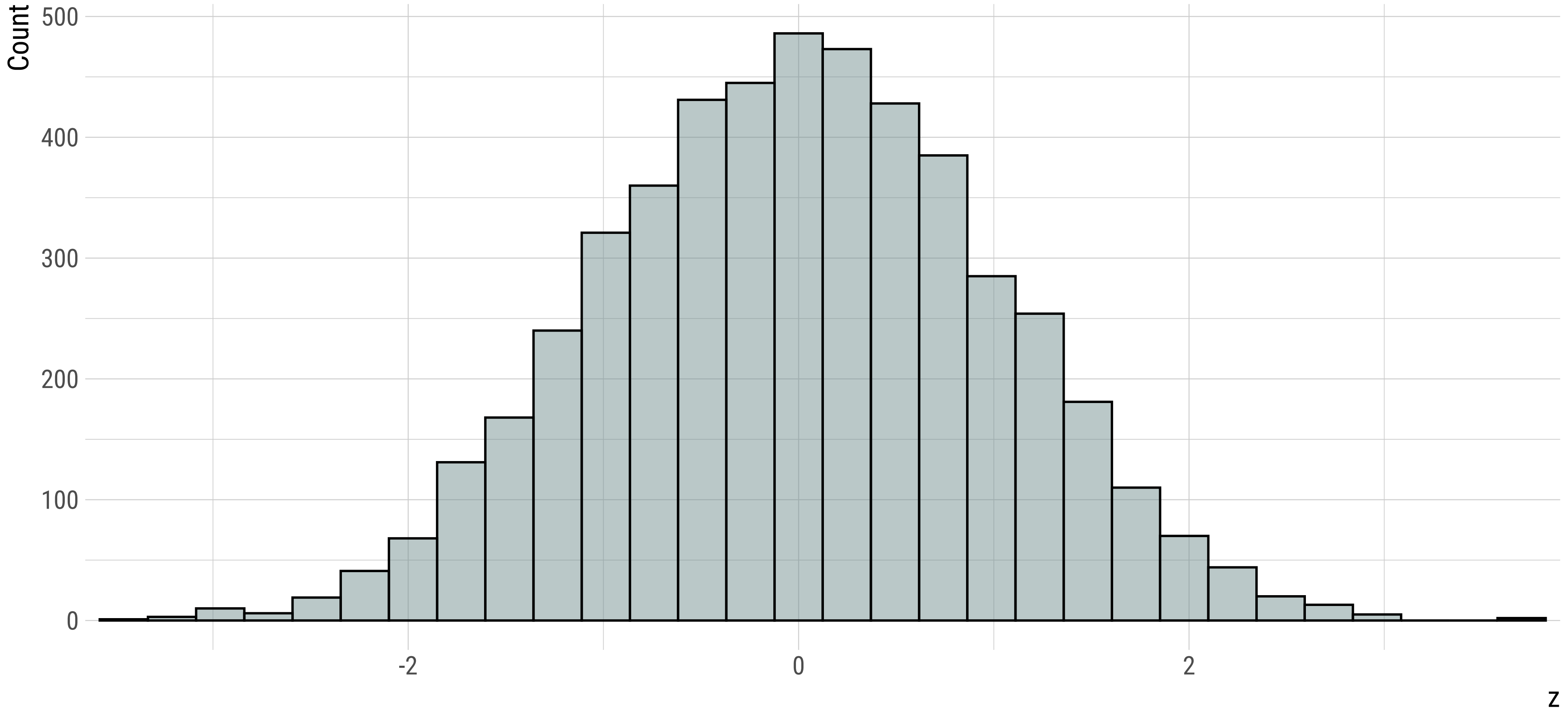
Next, we use samples of size $n = 1{,}500$.

Sampling distributions



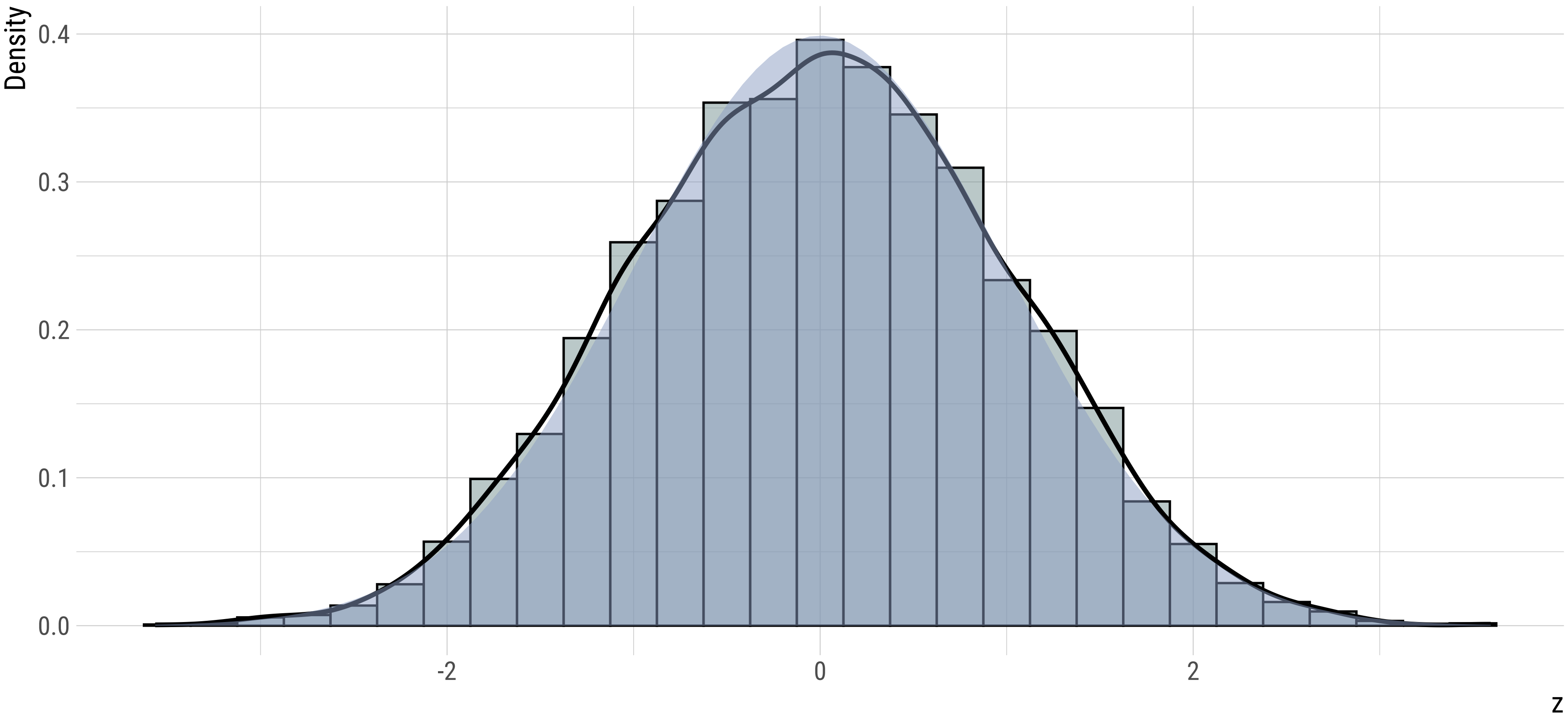
Sampling distributions

Standardizing the variable:



Sampling distributions

Sampling distributions



Confidence Intervals

Confidence intervals

In practical terms, a regression returns a **point estimate** of our desired parameter(s).

Supposedly, it **represents**, to the best of our efforts, the "true" population parameter.

But wouldn't it be better if we could have a **range** of values for β_i?

Given a **confidence level** $(1 - \alpha)$, we can easily construct a **confidence interval** for β_i.

Confidence intervals

From **Stats**, we know:

$$\text{CI} = \bar{x} \pm t_c \cdot \sigma$$

$$\text{CI} = \bar{x} \pm t_c \cdot \frac{s}{\sqrt{n}}$$

And now:

$$\text{CI} = \hat{\beta}_k \pm t_c \cdot SE(\hat{\beta}_k)$$

where $t_c = t_{1-\alpha/2,\, n-k-1}$.

It denotes the $1 - \alpha/2$ **quantile** of a *t* distribution, with *n-k-1* **degrees-of-freedom**.

Confidence intervals

- The **standard error** (***SE***) of an estimate:

$$\mathrm{SE}\left(\hat{\beta}_2\right) = \sqrt{\frac{s_u^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}.$$

where $s_u^2 = \dfrac{\sum_i \hat{u}_i^2}{n-k-1}$ is the variance of u_i.

The standard error of an estimate is nothing but its **standard deviation**.

Confidence intervals

- **Informal interpretation:**
  - The confidence interval is a region in which we are able to place some **trust** for containing the parameter of interest.
- **Formal interpretation:**
  - With **repeated sampling** from the population, we can construct confidence intervals for each of these samples. Then $(1 - \alpha) \cdot 100$ percent of our intervals (*e.g.*, 95%) will contain the population parameter ***somewhere in this interval***.

Confidence intervals - An example

```
                        #>
#> ===============================================
#>                           Dependent variable:
#>                       ---------------------------
#>                                 lsalary
#> -----------------------------------------------
#> age                             -0.001
#>                                 (0.005)
#> lsales                         0.225***
#>                                 (0.028)
#> Constant                       5.005***
#>                                 (0.303)
#> -----------------------------------------------
#> Observations                      177
#> R2                               0.281
#> Adjusted R2                      0.273
#> Residual Std. Error       0.517 (df = 174)
#> F Statistic           34.004*** (df = 2; 174)
#> ===============================================
#> Note:               *p<0.1; **p<0.05; ***p<0.01
```

Confidence intervals - An example

From the previous regression output, we have:

- $\hat{\beta}_{lsales_i}$: 0.225
- $SE(\hat{\beta}_{lsales_i})$: 0.0277

In addition, the sample size (n) is 177.

Confidence intervals - An example

- Then, we can calculate a 95% confidence interval for β_{lsales_i}:

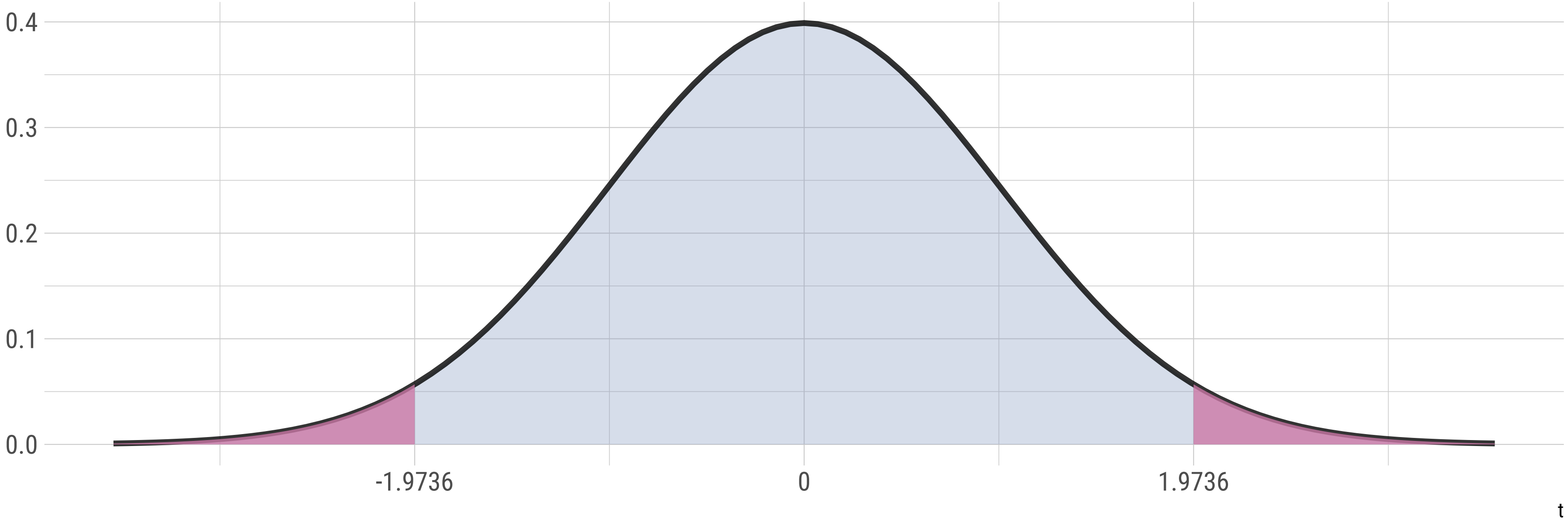
$$\text{CI} = \hat{\beta}_{lsales_i} \pm t_c \cdot SE(\hat{\beta}_{lsales_i})$$

$$\text{CI} = 0.225 \ \pm \ t_{1-0.05/2,\, 177-2-1} \ \cdot \ 0.0277$$

$$\text{CI} = 0.225 \ \pm \ t_{1-0.05/2,\, 174} \ \cdot \ 0.0277$$

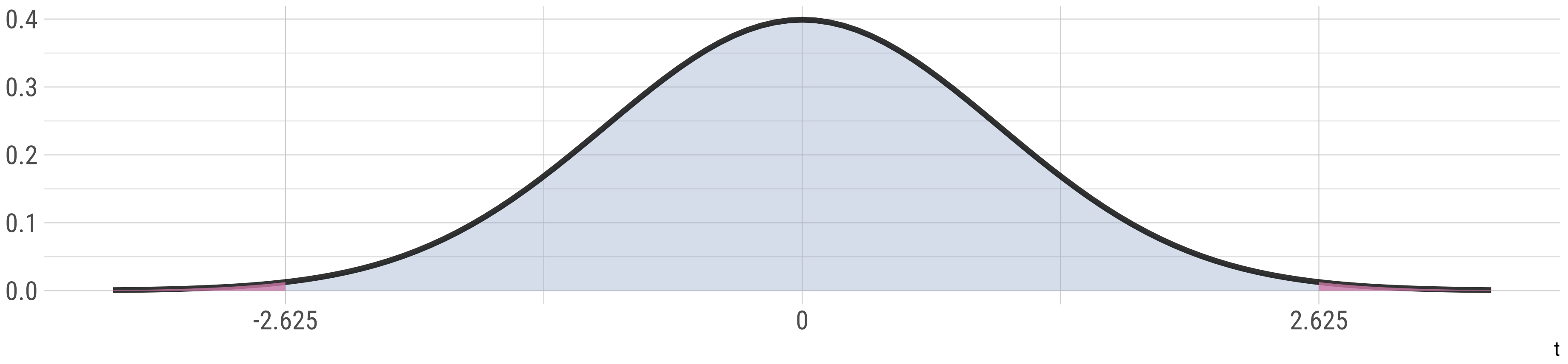
- $t_{1-0.05/2,\, 174} =$ `-1.973691`
- The interval is `[0.17, 0.28]`.

Confidence intervals - An example



With **repeated sampling** from the population, 95% of our intervals will contain the population parameter ***somewhere in this [0.17, 0.28] interval***.

Confidence intervals - An example



- If we estimate a 99% confidence interval, we have:

$$\mathbf{CI} = \mathbf{0.225} \pm t_{1-0.01/2,\,174} \cdot \mathbf{0.0277}$$

- $t_{1-0.01/2,\,174} =$ `2.604379`
- The interval is `[0.15, 0.29]`.

Hypothesis Testing

Hypothesis testing

- When doing *hypothesis testing*, our aim is to determine whether there is enough **statistical evidence** to reject a hypothesized value or range of values.

- In Econometrics, we usually run **two-sided (tailed)** tests about *regression parameters*.
  - $H_0 : \beta_i = 0$
  - $H_a : \beta_i \neq 0$

- The above testing procedure is a test of **statistical significance**.
  - If we **do not reject** H_0, the coefficient is not statistically significant.
  - If we **reject** H_0, we have enough evidence to support the coefficient's statistical significance.

Hypothesis testing

In `Stata`...

```
. reg wage educ exper tenure

      Source |       SS           df       MS      Number of obs   =       935
-------------+----------------------------------   F(3, 931)       =     53.00
       Model |  22278193.8         3  7426064.59   Prob > F        =    0.0000
    Residual |   130437974       931  140105.236   R-squared       =    0.1459
-------------+----------------------------------   Adj R-squared   =    0.1431
       Total |   152716168       934  163507.675   Root MSE        =    374.31

------------------------------------------------------------------------------
        wage | Coefficient  Std. err.      t    P>|t|     [95% conf. interval]
-------------+----------------------------------------------------------------
        educ |   74.41486   6.286993    11.84   0.000     62.07654    86.75318
       exper |   14.89164    3.25292     4.58   0.000     8.507732    21.27554
      tenure |   8.256811   2.497628     3.31   0.001     3.355178    13.15844
       _cons |  -276.2405   106.7018    -2.59   0.010    -485.6444   -66.83653
------------------------------------------------------------------------------
```

Hypothesis testing

- Where does the `11.8` t value come from?

$$t = \frac{\hat{\beta}_k - \beta_{H_0}}{SE(\hat{\beta}_k)} = \frac{74.4 - 0}{6.29} = 11.8283$$

- Where does the `4.58` t value come from?

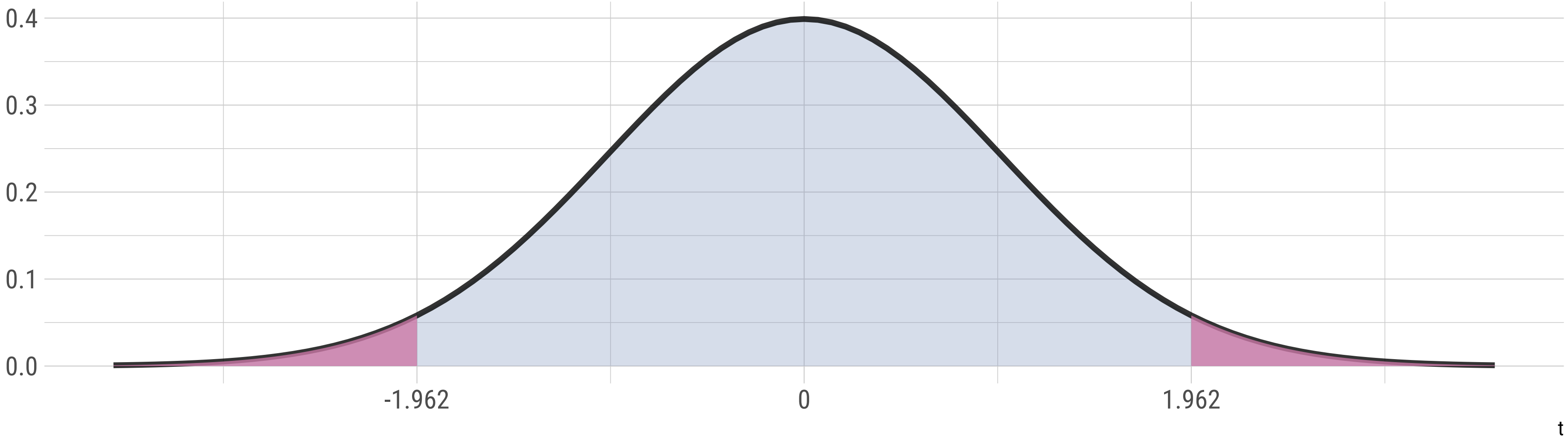
$$t = \frac{\hat{\beta}_k - \beta_{H_0}}{SE(\hat{\beta}_k)} = \frac{14.9 - 0}{3.25} = 4.584615$$

Hypothesis testing

What are we supposed to do with these test statistics?

- t_{educ} = `11.8`
- t_{exper} = `4.58`
- t_{tenure} = `3.31`

- $t_{critical\ value} = t_{.05/2,\ 931}$ = `1.962515`



Hypothesis testing

Interpretation

At 5% of significance, we have enough evidence to **reject the null hypothesis** that `educ` is not statistically significant.

At 5% of significance, we have enough evidence to **reject the null hypothesis** that `exper` is not statistically significant.

At 5% of significance, we have enough evidence to **reject the null hypothesis** that `tenure` is not statistically significant.

Therefore, all coefficients are (individually) **statistically significant**.

The F-test

Sometimes, a coefficient on a **specific variable** may not be *statistically significant*.

However, it may be of use in the **model's context**.

Thus, a test of **joint** significance is appropriate to evaluate whether **all slope coefficients** are *jointly* significant within the model.

$$F = \frac{R^2_{\text{unr}} - R^2_{\text{rest}}}{1 - R^2_{\text{unr}}} \cdot \frac{(n-k-1)}{q}$$

The F-test

Still with our **wage** model:

Suppose we want to test whether `educ` and `exper` are **jointly** significant.

For the purpose of this test, our previous model is the **unrestricted** (full) model.

Then, we estimate a **restricted** model, excluding `educ` and `exper`.

- Its R-squared is **0.0165**; while the unrestricted's is **0.146**.

We have imposed **2** restrictions to the full model. Thus, $q=2$.

And the **sample size** is $n=935$, which gives $n-k-1 = 931$ for the full model.

The F-test

$$F = \frac{R^2_{\text{unr}} - R^2_{\text{rest}}}{1 - R^2_{\text{unr}}} \cdot \frac{(n-k-1)}{q}$$

$$= \frac{0.146 - 0.0165}{1 - 0.146} \cdot \frac{935 - 3 - 1}{2} = 70.588$$

- 70.588 is the **test statistic** for the F-test
- Then, we compare the above value with the **critical values** given by the F-distribution table.
- Right-tail critical value:
    - $F_{1-.05/2,\,2,\,931} = 3.703535$
    - Thus, we **reject the null hypothesis**, meaning that we have enough evidence to infer that `educ` and `exper` are **jointly significant** in this model.

Next time: Inference in practice