Data Structures and Algorithms with Python

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Lecture 4

Last time

Data structures and **object-oriented programming**

- Data structure design affects efficiency — choose right structure
- Built-in Python structures are good but do not cover everything
- OOP allows abstraction and modularization of programs

Plan for today:

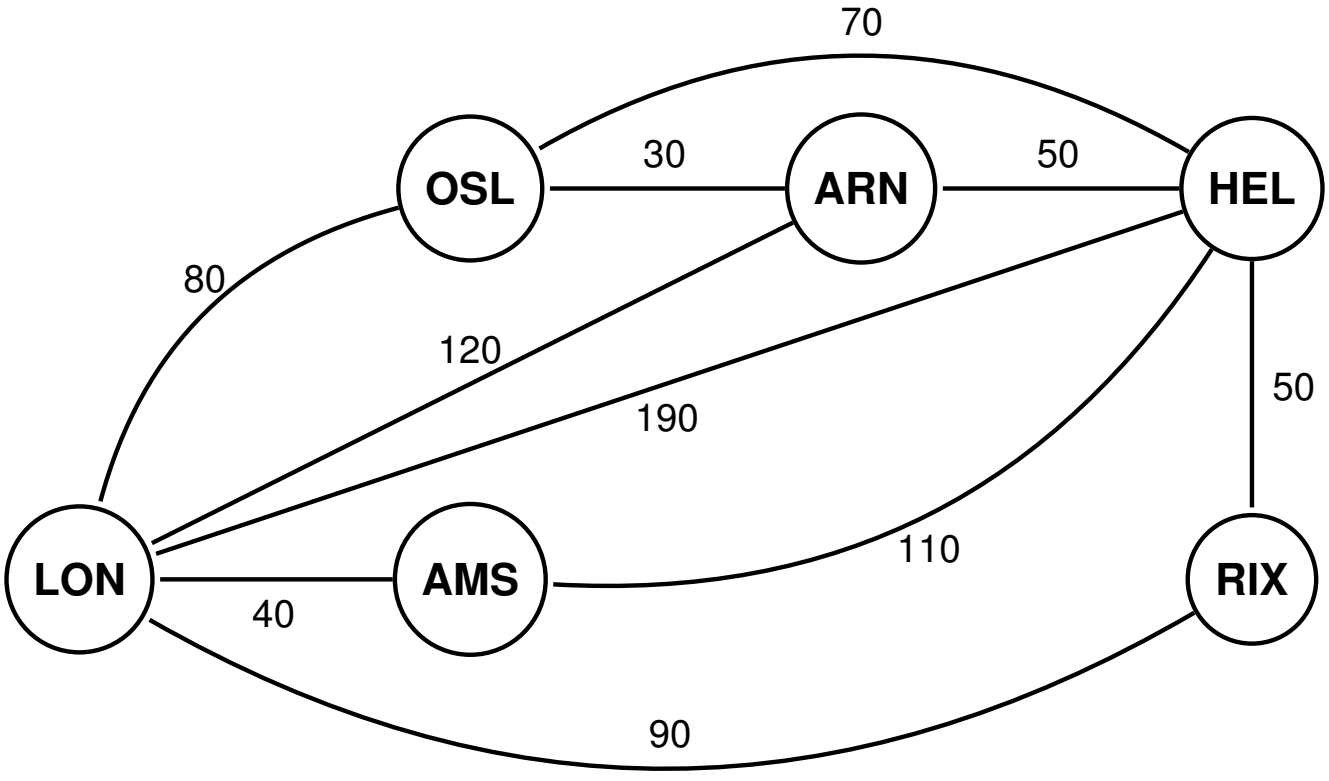
- Graphs
- Graph search

Graph search

Suppose you're travelling around Europe

- Know **flights** and their **prices**
- **Goal**: fewest stops between cities, cheapest price, ...?
- (Assume the price of flying from London to Helsinki via Stockholm is the sum of the two legs...)

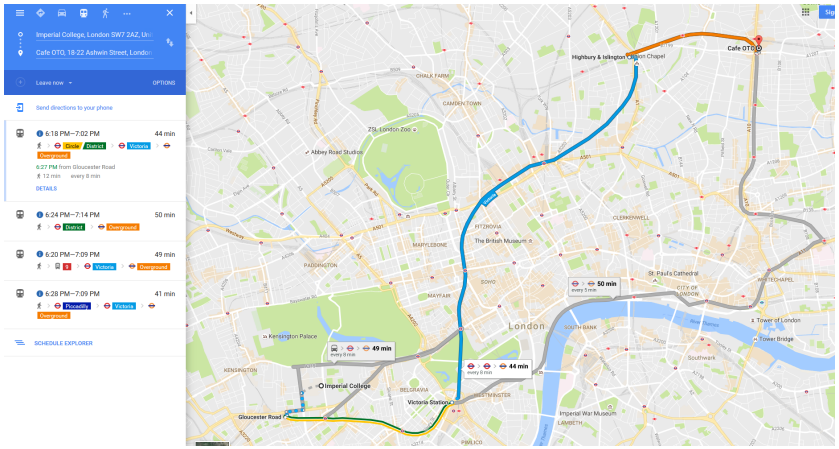
Defining graphs

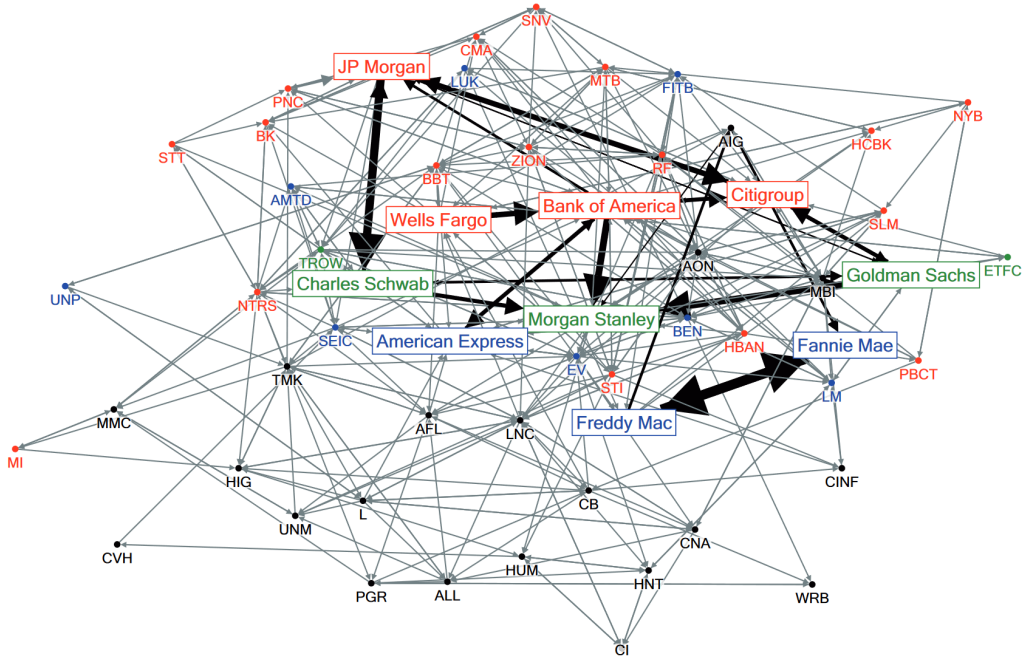


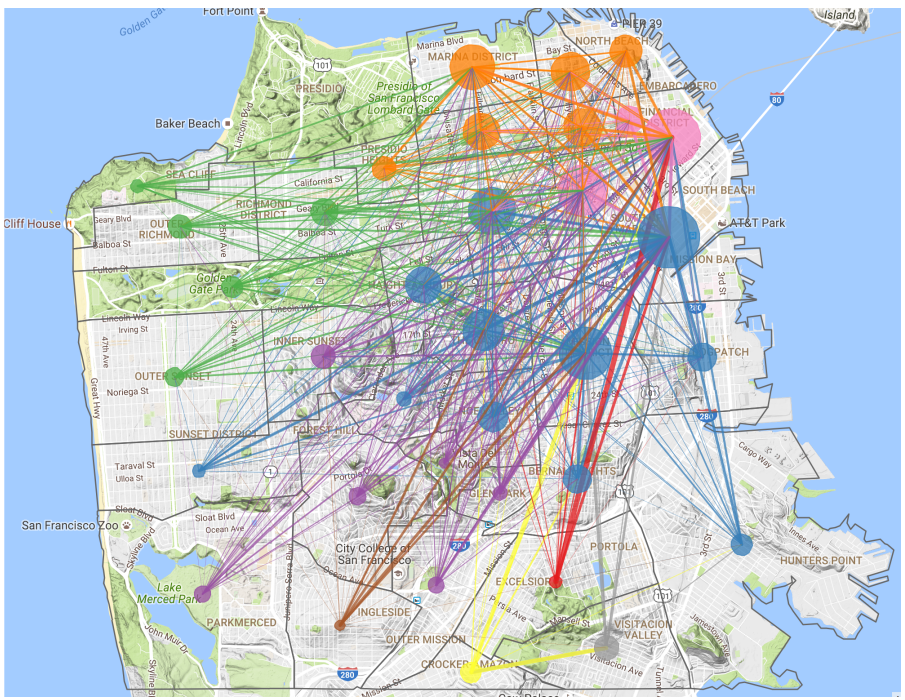
A set of **nodes** (or vertices) connected by **edges**

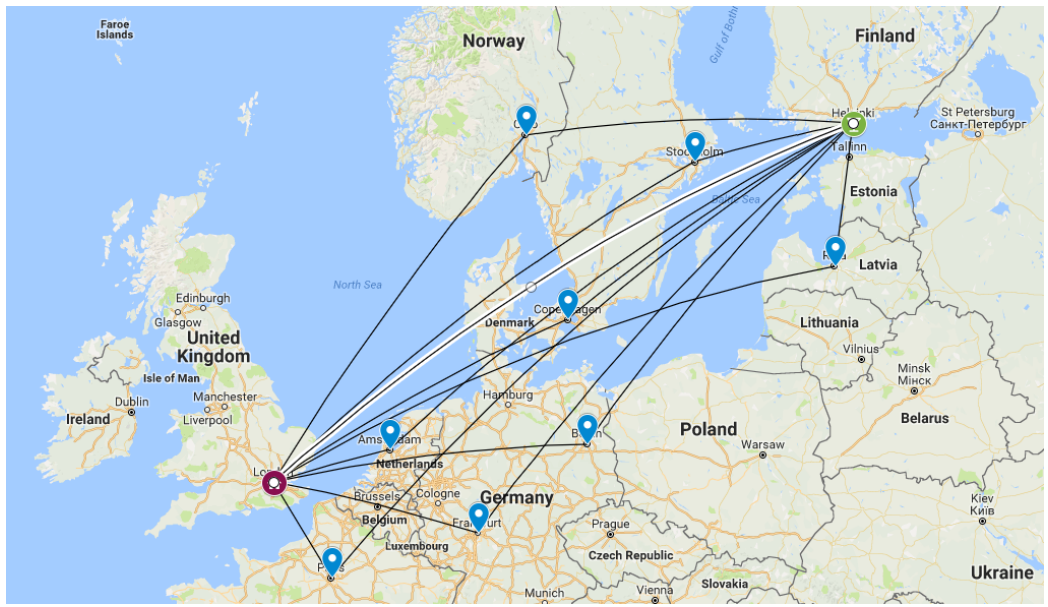
Edges may or may not be directed -> **directed** or **undirected** graph (Twitter vs Facebook?)

Edges may have **weights** (eg prices): weighted graph









Pics: Gmaps, Uber, Hautsch et al.

Graphs are everywhere

Represent situations with **interesting pairwise relationships**

Many practical applications, **some not obvious**

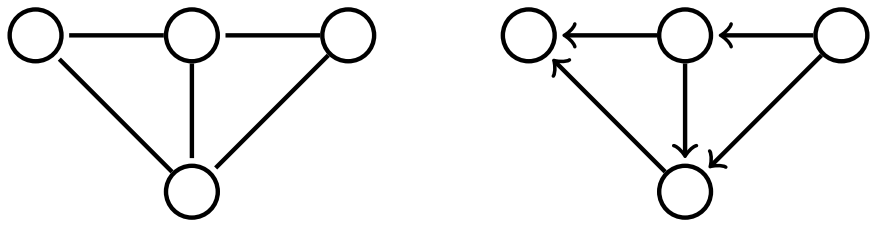
- Roads, social networks, the web, financial markets, complex projects etc

Need good algorithms!

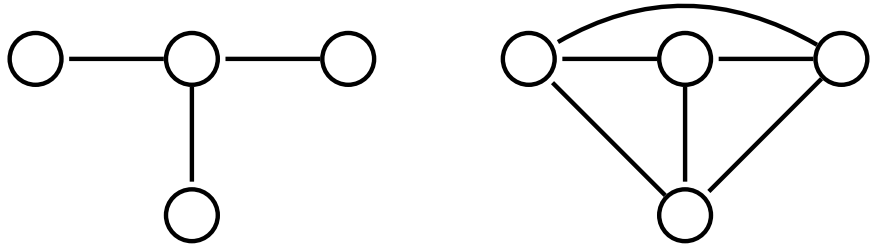
Graph: collection of vertices and edges

Graph $G = (V, E)$: $V =$ set of n **vertices**, $E =$ set of m **edges**

- Edges can be **directed** or **undirected**
- **Connected** graph: there is a **path** between any two vertices



How many edges can there be in a connected graph with n vertices?



Sparse vs. **dense** graphs:

- Between $n - 1$ vs $n(n-1)/2$ edges

Graph: collection of vertices and edges

Graph $G = (V, E)$: $V =$ set of n vertices, $E =$ set of m edges

How to represent a graph?

- **Adjacency matrix**: size $n \times n$
  - $A_{vw} = 1$ if there is an edge between vertices v, w (unweighted)
  - n^2 space required

	LON	*OSL*	*ARN*	*HEL*	*AMS*	*RIX*
LON	0	1	1	1	1	1
OSL	1	0	1	1	0	0
ARN	1	1	0	1	0	0
HEL	1	1	1	0	1	1
AMS	1	1	0	1	0	0
RIX	1	0	0	1	0	0

Graph: adjacency lists

How to represent a graph?

- **Adjacency list**: connectivity represented using lists of edges and vertices
  - List of vertices (each vertex points to edges starting from it); list of edges (each edge points to its endpoints)
  - Linear space requirement in m, n

- LON: {OSL,ARN,HEL,AMS,RIX}
- OSL: {LON,ARN,HEL}
- ARN: {LON,OSL,HEL}
- HEL: {LON,OSL,ARN,AMS,RIX}
- AMS: {LON,HEL}
- RIX: {LON,HEL}

Adjacency matrices vs lists:

- Dense vs sparse graphs
- We'll be using adjacency lists

Searching a graph

Small worlds phenomenon:

- Six degrees of separation (popularized through play by Guare, 1990)
- Hollywood: Kevin Bacon numbers

Searching a graph

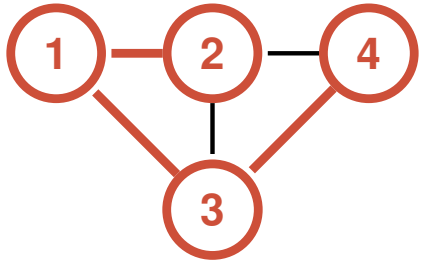
Graph search goal:

- Find everything findable from given start vertex
- Don't waste time finding the same thing many times

Algorithm (given graph G, vertex s):

- Initially mark s explored, other nodes unexplored
- While there are unexplored vertices:
  - Choose an edge (u, v) with u explored and v unexplored
  - Mark v explored

Claim: will explore every v with a path from s to v



How to choose which edge to explore?

Breadth-first search

- Explore nodes in layers
- Shortest paths
- Up next!

Depth-first search

- Explore aggressively like a maze
- Eg topological orderings
- Challenge problem...

Breadth-first search (BFS)

Algorithm: BFS(graph G, start vertex s)

- **Initialize**: mark all nodes unexplored except s explored
- Use Q = queue data structure, add s to Q
- **Main loop**: While Q is not empty:
  - Remove the first node of Q and call it v
  - For each edge (v, w): if w unexplored:
    - Mark w explored
    - Add w to Q

What is a **queue**?

- First in, first out (just like in a cafe)
- Remove from front, insert to back in constant time $O(1)$
- Here we keep a queue of nodes to visit next

BFS example

Algorithm: BFS(graph G, start vertex s)

- **Initialize**: mark all nodes unexplored except s explored
- Use Q = queue data structure, add s to Q
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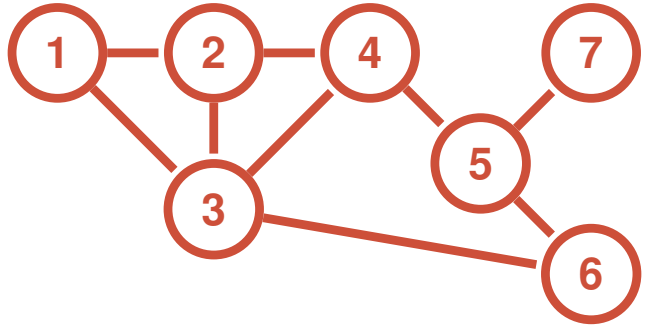


Queue of vertices:

BFS running time

Correctness: At the end, any given vertex v is explored if and only if there is a path s to any other v

Running time: $O(n_s + m_s)$, the number of nodes and edges reachable from s

- Each node will be added
- Loop through each edge, but how many times?

BFS for shortest paths

Suppose we want to measure actors' degrees of separation from Kevin Bacon

- Want the **smallest distance** (fewest number of edges) from s to v

We need some extra code: **initialize** $dist(v) = 0$ for $v = s$, ∞ otherwise

Main loop: when going through edge (v, w): if w unexplored, set $dist(w) = dist(v) + 1$

This will give the **shortest path**! **Why?**

BFS for connectivity

Connected components = the "pieces" of (undirected) graph G

Goal: compute all connected components

- Check if graph is disconnected
- Visualize graphs
- Clustering

Idea:

- If BFS finishes without exploring all nodes, restart BFS from unexplored node
- $O(m + n)$ time

Why study graph optimization?

Many problems revolve around **transitions from one system state to another**

- Graph search often useful
- Examples: travelling, scheduling, etc.

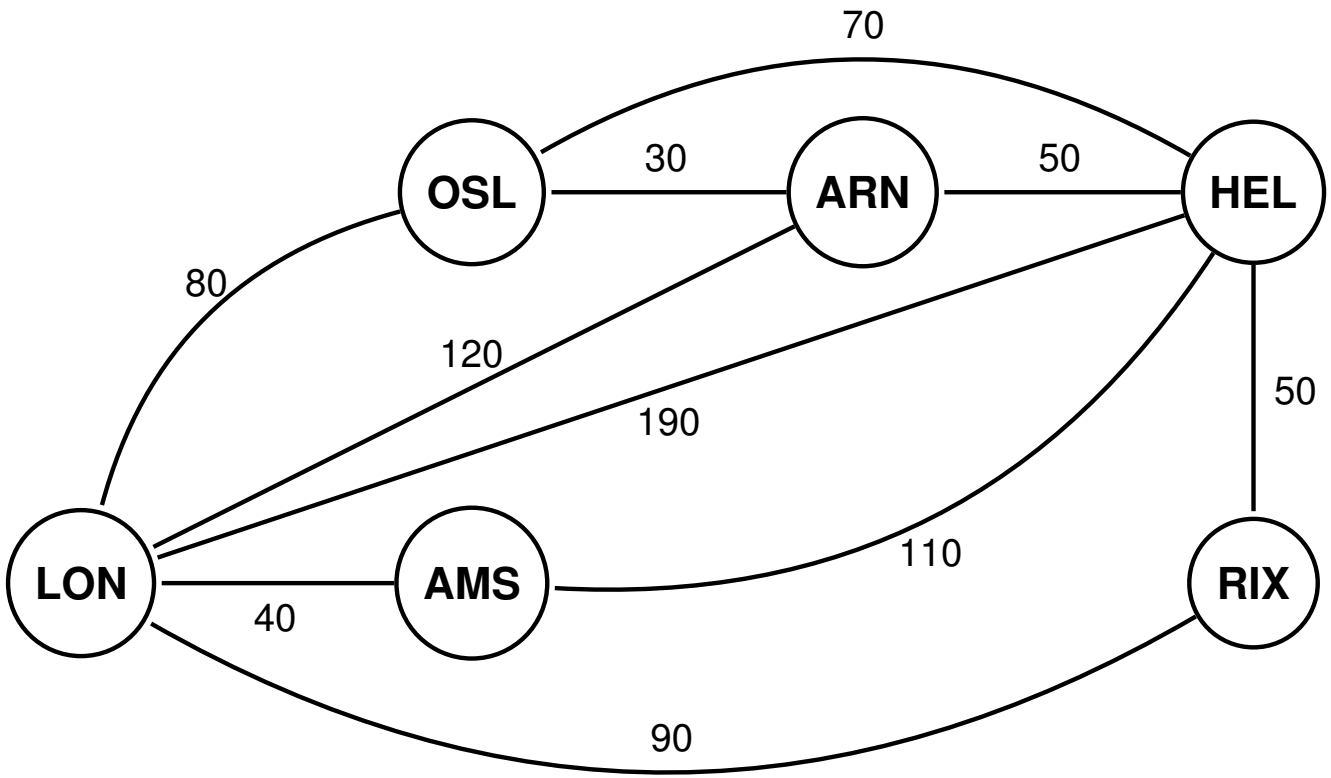
For **unweighted graphs**, many problems solved by **BFS** and **DFS**

- Linear time
- DFS is challenge problem in the workshop
- **Weighted graphs** require different methods — Dijkstra's algorithm, Bellman-Ford algorithm

Graph search

Suppose you're travelling around Europe

- Know flights and their prices
- Assume the price of flying from London to Helsinki via Stockholm is the sum of the two legs (not true!)
- Cheapest price?



Shortest paths

Input: (directed) graph $G = (V, E)$

- V: set of n vertices, E: set of m edges
- Each edge e has length (cost) $c_e \geq 0$
- Start from vertex s

Output: for each vertex v in V:

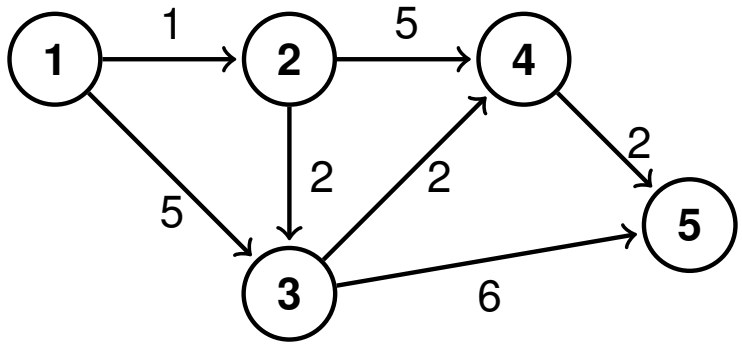
- $SP(v)$ = length of shortest $s - v$ path in G

Assume:

- That such paths exist (connected graph)
- $c_e \geq 0$ (important!!) – Dijkstra vs Bellman-Ford

Example

What is the **shortest path** from **s=1** to **v=1,2,3,4,5**?



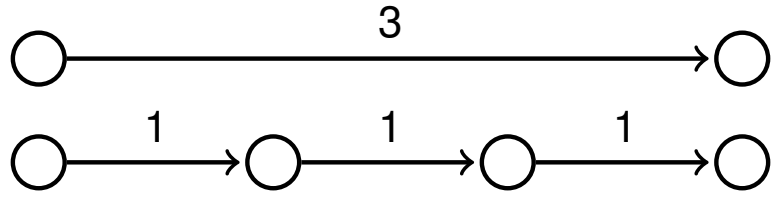
1. 0,1,5,6,11?
2. 0,1,3,6,8?
3. 0,1,5,5,8?
4. 0,1,3,5,7?

Why not just use BFS?

BFS already allows us to find the shortest paths?

- Assuming that all edges have length 1...

Write each edge as a series of length one edges?



- Graph becomes **impractical**...

Need a new algorithm: **Dijkstra's shortest-paths algorithm**

Dijkstra is extra...

Builds on BFS ideas

Running time $O(m \log n)$ – still blazingly fast

- With a clever use of **heap** data structure
- Essentially replace BFS queue with a **priority queue** based on distance scores

Workshop: graphs

After the break...

BFS, small world phenomenon and Kevin Bacon

Challenge: depth-first search (DFS)

Extra: Dijkstra...