An ε-Best-Arm Identification Algorithm for Fixed-Confidence and Beyond

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Motivation

Goal: Identify one item that has a good enough average return.

Typical approaches: control the error and minimize the sampling budget (**fixed-confidence**) or control the sampling budget and minimize the error (**fixed-budget**).

⚠ Too restrictive for many applications !

☞ This paper: **guarantees at any time** on the candidate answer !

ε-Best-arm identification (ε-BAI)

K arms: $\nu_i \in \mathcal{D}$ is the 1-sub-Gaussian distribution of arm $i \in [K]$ with mean μ_i.

Goal: identify one of the ε-good arms $\mathcal{I}_\varepsilon(\mu) = \{i \mid \mu_i \geq \mu_\star - \varepsilon\}$ with $\mu^\star = \max_i \mu_i$.

Algorithm: at time n,

- *Recommendation rule:* recommend the candidate answer $\hat{\imath}_n$
- **Sampling rule**: pull arm I_n and observe $X_n \sim \nu_{I_n}$.

Fixed-confidence: given an error/confidence pair $(\varepsilon, \delta) \in \mathbb{R}_+ \times (0,1)$, define a stopping time $\tau_{\varepsilon,\delta}$ which is (ε,δ)-PAC, i.e. $\mathbb{P}_\nu(\tau_{\varepsilon,\delta} < +\infty, \hat{\imath}_{\tau_{\varepsilon,\delta}} \notin \mathcal{I}_\varepsilon(\mu)) \leq \delta$, and

☞ Minimize the **expected sample complexity** $\mathbb{E}_\nu[\tau_{\varepsilon,\delta}]$.

Fixed-budget: given an error/budget pair $(\varepsilon, T) \in \mathbb{R}_+ \times \mathbb{N}$,

☞ Minimize the **probablity of ε-error** $\mathbb{P}_\nu(\hat{\imath}_T \notin \mathcal{I}_\varepsilon(\mu))$ at time T.

Anytime: Minimize the **expected simple regret** $\mathbb{E}_\nu[\mu^\star - \mu_{\hat{\imath}_n}]$ at any time n.

Lower bound on the expected sample complexity

? What is the best one could achieve ?

☞ Degenne and Koolen (2019): For all (ε,δ)-PAC algorithms and all Gaussian instances with $\mu \in \mathbb{R}^K$, $\liminf_{\delta \to 0} \mathbb{E}_\nu[\tau_{\varepsilon,\delta}]/\log(1/\delta) \geq T_\varepsilon(\mu)$ where

$$T_\varepsilon(\mu) = \min_{i \in \mathcal{I}_\varepsilon(\mu)} \min_{\beta \in (0,1)} T_{\varepsilon,\beta}(\mu, i) \quad \text{and} \quad T_{\varepsilon,\beta}(\mu,i)^{-1} = \max_{w \in \triangle_K, w_i = \beta} \min_{j \neq i} \frac{1}{2} \frac{(\mu_i - \mu_j + \varepsilon)^2}{1/\beta + 1/w_j} .$$

Top Two sampling rule: EB-TC$_{\varepsilon_0}$ with fixed β or IDS proportions

Input: slack $\varepsilon_0 > 0$, proportion $\beta \in (0,1)$ (only for fixed proportions).

Set $\hat{\imath}_n \in \arg\max_{i \in [K]} \mu_{n,i}$, $B_n = \hat{\imath}_n$ and $C_n \in \arg\max_{i \neq B_n} \frac{\mu_{n,B_n} - \mu_{n,i} + \varepsilon_0}{\sqrt{1/N_{n,B_n} + 1/N_{n,i}}}$;

Set [fixed] $\bar{\beta}_{n+1}(i,j) = \beta$ or [IDS] $\beta_n(i,j) = N_{n,j}/(N_{n,i} + N_{n,j})$ and update $\bar{\beta}_{n+1}(i,j)$;

Set $I_n = C_n$ if $N^{B_n}_{n,C_n} \leq (1 - \bar{\beta}_{n+1}(B_n, C_n)) T_{n+1}(B_n, C_n)$, otherwise set $I_n = B_n$;

Output: next arm to sample I_n and next recommendation $\hat{\imath}_n$.

Notation: $N_{n,i} = \sum_{t \in [n-1]} \mathbb{1}(I_t = i)$, $\mu_{n,i} = \sum_{t \in [n-1]} X_t \mathbb{1}(I_t = i)/N_{n,i}$, $T_n(i,j) = \sum_{t \in [n-1]} \mathbb{1}((B_t, C_t) = (i,j))$, $\bar{\beta}_n(i,j) = \sum_{t \in [n-1]} \beta_t(i,j) \mathbb{1}((B_t, C_t) = (i,j))/T_n(i,j)$, $N^i_{n,j} = \sum_{t \in [n-1]} \mathbb{1}((B_t, C_t, I_t) = (i,j,j))$ and $T_n(i) = \sum_{j \neq i} (T_n(i,j) + T_n(j,i))$.

(ε,δ)-PAC sequential test

? How to obtain a (ε,δ)-PAC sequential test for 1-sub-Gaussian distributions ?

☞ **GLR$_\varepsilon$ stopping rule**: recommend $\hat{\imath}_n \in \arg\max_{i \in [K]} \mu_{n,i}$ and stop at time

$$\tau_{\varepsilon,\delta} = \inf \left\{ n > K \mid \min_{i \neq \hat{\imath}_n} \frac{\mu_{n,\hat{\imath}_n} - \mu_{n,i} + \varepsilon}{\sqrt{1/N_{n,\hat{\imath}_n} + 1/N_{n,i}}} \geq \sqrt{2c(n-1,\delta)} \right\} , \tag{1}$$

with $c(n,\delta) = 2\mathcal{C}_G(\log((K-1)/\delta)/2) + 4\log(4 + \log(n/2))$ and $\mathcal{C}_G(x) \approx x + \ln(x)$.

Asymptotic fixed-confidence guarantees

Theorem 1. *Let $\varepsilon \geq 0$ and $\varepsilon_0 > 0$. Combined with GLR$_\varepsilon$ stopping (1), the EB-TC$_{\varepsilon_0}$ algorithm satisfies that, for all $\nu \in \mathcal{D}^K$ with mean μ such that $|i^\star(\mu)| = 1$,*

- **IDS**: $\limsup_{\delta \to 0} \mathbb{E}_\nu[\tau_{\varepsilon,\delta}]/\log(1/\delta) \leq T_{\varepsilon_0}(\mu) D_{\varepsilon,\varepsilon_0}(\mu)$,
- **fixed** $\beta \in (0,1)$: $\limsup_{\delta \to 0} \mathbb{E}_\nu[\tau_{\varepsilon,\delta}]/\log(1/\delta) \leq T_{\varepsilon_0,\beta}(\mu) D_{\varepsilon,\varepsilon_0}(\mu)$,

where $D_{\varepsilon,\varepsilon_0}(\mu) = (1 + \max_{i \neq i^\star}(\varepsilon_0 - \varepsilon)/(\mu_\star - \mu_i + \varepsilon))^2$.

Corollary 1. *Let $\varepsilon > 0$. Combined with GLR$_\varepsilon$ stopping (1), the EB-TC$_\varepsilon$ algorithm with IDS (resp. fixed β) proportions is* **asymptotically** *(resp. β-)***optimal** *in fixed-confidence ε-BAI for Gaussian distributions.*

Finite fixed-confidence guarantees

Theorem 2. *Let $\delta \in (0,1)$ and $\varepsilon_0 > 0$. Combined with GLR$_{\varepsilon_0}$ stopping (1), the EB-TC$_{\varepsilon_0}$ algorithm with fixed $\beta = 1/2$ satisfies that, for all $\nu \in \mathcal{D}^K$ with mean μ,*

$$\mathbb{E}_\nu[\tau_{\varepsilon_0,\delta}] \leq \inf_{\varepsilon \in [0,\varepsilon_0]} \max\{T_{\mu,\varepsilon_0}(\delta,\varepsilon) + 1, \ S_{\mu,\varepsilon_0}(\varepsilon)\} + 2K^2 , \quad \textit{where}$$

$\limsup_{\delta \to 0} \frac{T_{\mu,\varepsilon_0}(\delta,0)}{\log(1/\delta)} \leq 2|i^\star(\mu)| T_{\varepsilon_0,1/2}(\mu)$ *and* $S_{\mu,\varepsilon_0}(\varepsilon_0/2) = \mathcal{O}(K^2 |\mathcal{I}_{\varepsilon_0/2}(\mu)| \varepsilon_0^{-2} \log \varepsilon_0^{-1})$.

Key technical tool

Lemma 1. *Let $\delta \in (0,1]$ and $n > K$. Assume there exists a sequence of events $(A_t(n,\delta))_{n \geq t > K}$ and positive reals $(D_i(n,\delta))_{i \in [K]}$ such that, for all $t \in \{K+1, \dots, n\}$, under the event $A_t(n,\delta)$, there exists $i_t \in \{B_t, C_t\}$, such that $T_t(i_t) \leq D_{i_t}(n,\delta)$. Then, we have $\sum_{t=K+1}^n \mathbb{1}(A_t(n,\delta)) \leq \sum_{i \in [K]} D_i(n,\delta)$.*

Beyond fixed-confidence guarantees

Anytime guarantees on the **probability of ε-error** and the **expected simple regret**.

Theorem 3. *Let $\varepsilon_0 > 0$ and $p(x) = x - \log x$. The EB-TC$_{\varepsilon_0}$ algorithm with fixed proportions $\beta = 1/2$ satisfies that, for all $\nu \in \mathcal{D}^K$ with mean μ, for all $n > 5K^2/2$,*

$$\forall \varepsilon \geq 0, \quad \mathbb{P}_\nu(\hat{\imath}_n \notin \mathcal{I}_\varepsilon(\mu)) \leq K^2 e^2 (2 + \log n)^2 \exp\left(-p\left(\frac{n - 5K^2/2}{8H_{i_\mu(\varepsilon)}(\mu,\varepsilon_0)}\right)\right) ,$$

$$\mathbb{E}_\nu[\mu_\star - \mu_{\hat{\imath}_n}] \leq K^2 e^2 (2 + \log n)^2 \sum_{i \in [C_\mu - 1]} (\Delta_{i+1} - \Delta_i) \exp\left(-p\left(\frac{n - 5K^2/2}{8H_i(\mu,\varepsilon_0)}\right)\right) ,$$

where $(H_i(\mu,\varepsilon_0))_{i \in [C_\mu - 1]}$ are such that $H_1(\mu,\varepsilon_0) = K(2\Delta_{\min}^{-1} + 3\varepsilon_0^{-1})^2$ and $K/\Delta_{i+1}^{-2} \leq H_i(\mu,\varepsilon_0) \leq K \min_{j \in [i]} \max\{2\Delta_{j+1}^{-1}, 2\frac{\Delta_j/\varepsilon_0 + 1}{\Delta_{i+1} - \Delta_j} + 3\varepsilon_0^{-1}\}^2$ for all $i > 1$.

Notation: distinct mean gaps $0 = \Delta_1 < \Delta_2 < \cdots < \Delta_{C_\mu} < \Delta_{C_\mu+1} = +\infty$ where $C_\mu = |\{\mu_i \mid i \in [K]\}|$. For all $\varepsilon \geq 0$, let $i_\mu(\varepsilon) = i$ if $\varepsilon \in [\Delta_i, \Delta_{i+1})$.

Other guarantees: **unverifiable sample complexity** and **cumulative regret**.

Experiments

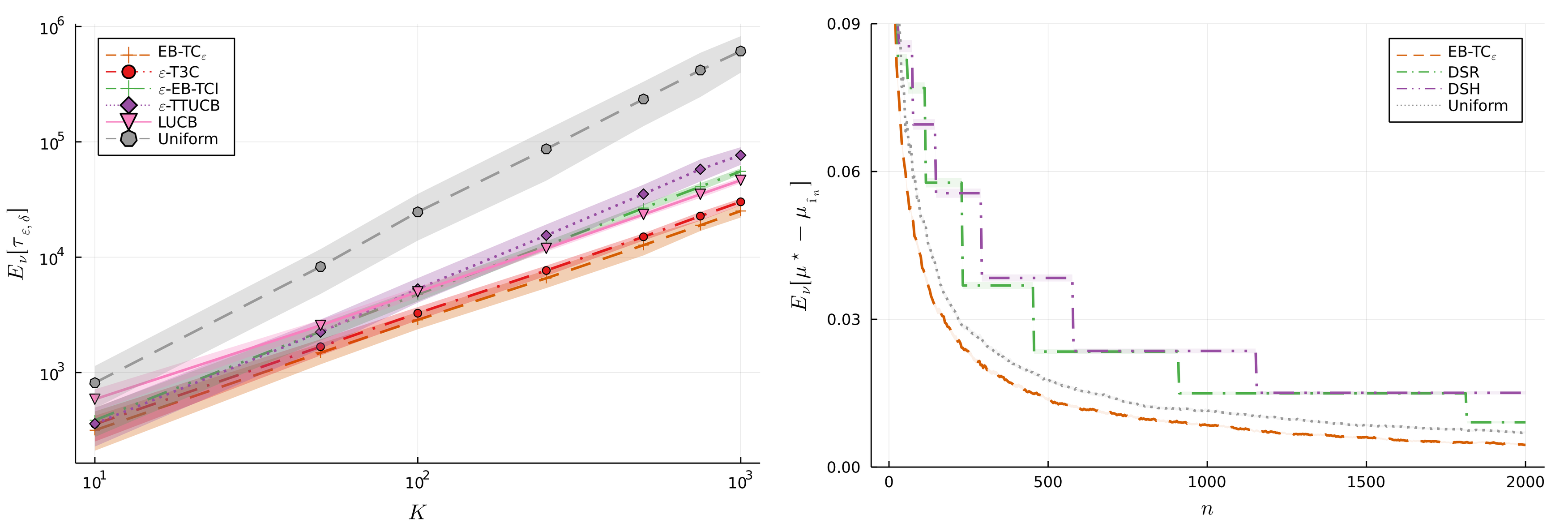

Figure 1: (a) Stopping time on instances $\mu_i = 1 - ((i-1)/(K-1))^{0.3}$ for varying K. (b) Simple regret on instance $\mu = (0.6, 0.6, 0.55, 0.45, 0.3, 0.2)$ fors EB-TC$_{\varepsilon_0}$ with slack $\varepsilon_0 = 0.1$ and fixed $\beta = 1/2$.

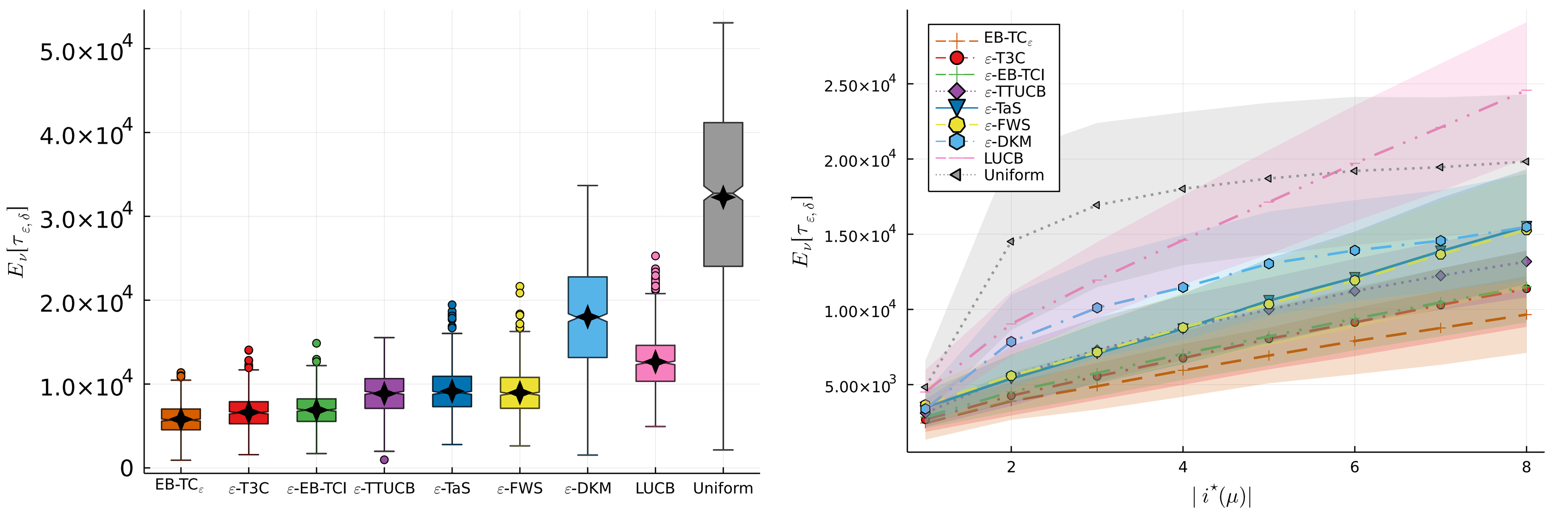

Figure 2: Stopping time on (a) random instances ($K = 20$) with $\mu_1 = 1$, $\mu_i \sim \mathcal{U}([0, 0.9])$ for all $i \geq 6$, otherwise $\mu_i \sim \mathcal{U}([0.9, 1])$ and (b) "two-groups" instances ($K = 10$) with $(\mu^\star, \Delta_2) = (0.6, 0.2)$.

Note: GLR$_\varepsilon$ stopping (1) with $(\varepsilon,\delta) = (10^{-1}, 10^{-2})$. T3C, EB-TCI, TTUCB, TaS, FWS, DKM are modified for ε-BAI.

Conclusion

1. Good empirical performance as regards the sample complexity and the simple regret. Easy to implement and computationally inexpensive algorithm.
2. Asymptotic and finite confidence upper bound on the expected sample complexity. Asymptotic (β-)optimality in ε-BAI for Gaussian distributions.
3. Anytime upper bounds on the uniform ε-error and the expected simple regret.