

DRAFT! Advection, Diffusion & Reaction Modeling

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The movement of compounds in the environment is driven by two processes, advection and diffusion. Of course, these processes occur in three dimensions, but for this class we'll begin with one dimensional processes before getting to more complicated examples. The compounds are also subject to transformations or reactions. Thus, to understand the fate and transport of compounds in the environment, we need to understand the processes of advection, diffusion, and reaction. We'll begin with simple examples and then consider more complicated examples. We'll also consider the implications of these processes on the environment and human health.

Introduction

The Problem

The fate and transport of compounds in the environment is subject to a complex array of processes (Figure 2). However, we can simplify these processes into three subprocesses: advection, diffusion, and reaction. Of course, these processes occur in three dimensions but we'll focus on 1D and 2D for this session.

The implications of these three processes is profound – they provide the basic framework to understand the fate and transport of pollutants in the environment, the movement of nutrients in the soil, and the movement of solutes in the human body. Our understanding of these processes is critical to understanding issues of environmental quality and justice for human and non-humans alike. Moreover, we often use our understanding of these processes to develop of environmental policy, regulation, and mitigation.

Because these processes are complex and are often difficult to measure, we often rely on models to help us understand the movement of solutes in the environment. These models are based on the fundamental mathematical equations to describe advection, diffusion, and reaction.

The Processes

Advection and diffusion are two fundamental processes that govern the transport of solutes in the environment. Advection is the process of transport of a solute by the bulk motion of the fluid. Diffusion is the process of transport of a solute by random molecular motion.

In a simple example, if we have a dose of a solute, like a dye in

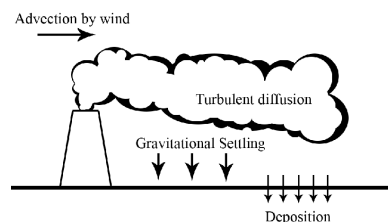
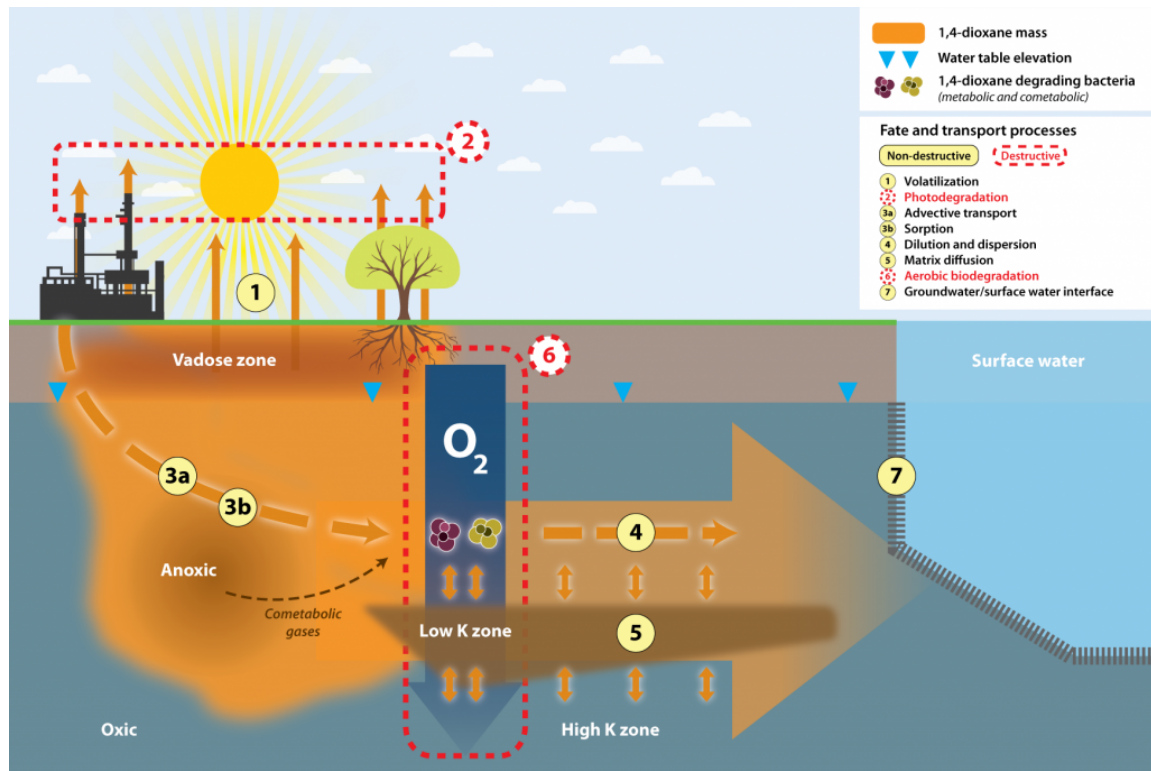


Figure 1: A simple diagram of advection and diffusion that includes how "solute" might be deposited downwind of a stationary source if air pollution. At the scale analyzed there, the term turbulent diffusion is different than molecular diffusion, but might be modelled in a similar way.



Generalized figure for the fate and transport of 1,4-dioxane: In the absence of water or soil moisture 1,4-dioxane volatilizes ① to the atmosphere where it is rapidly photodegraded ②. In the presence of water advective flow drives 1,4-dioxane into groundwater systems or plants via uptake through plant root systems ③a with little retardation from sorption into organic matter ③b. In the saturated zone, attenuation of 1,4-dioxane occurs via dilution and dispersion ④, matrix diffusion ⑤, or aerobic biodegradation mediated by microbes ⑥. Transport of undegraded 1,4-dioxane to surface water may occur through groundwater-surface water interfaces ⑦.

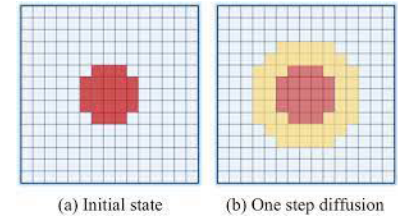
Figure 2: A diagram of the major processes that influence the fate and transport of a dioxane plume from a ce in the environment

(Source: <https://14d-1.itrcweb.org/environmental-fate-transport-and-investigative-str>)
1,4-Dioxane is often referred to as a "forever" compound.

the center of some media, we'll see it spread out by diffusion (Figure 3). If the media is moving, we'll see the dye move with the bulk motion of the media. If the dye reacts with the media, we'll see the dye disappear.

Besides examples we often see with respect to air pollution sources (Figure 1), we might think about solutes as pollutants or nutrients. For example, the movement of a nutrient in a river is driven by the bulk motion of the water (advection) and the random motion of the molecules (diffusion) and the reactions that might occur in the water column and sediments (Figure 4).

Figure 3: A simple diagram of 2D diffusion. To solve these equations, we'll use a numerical approach and discretize the media into a grid. We'll then solve the equations for each grid cell.



Nutrient Cycling vs. Spiraling

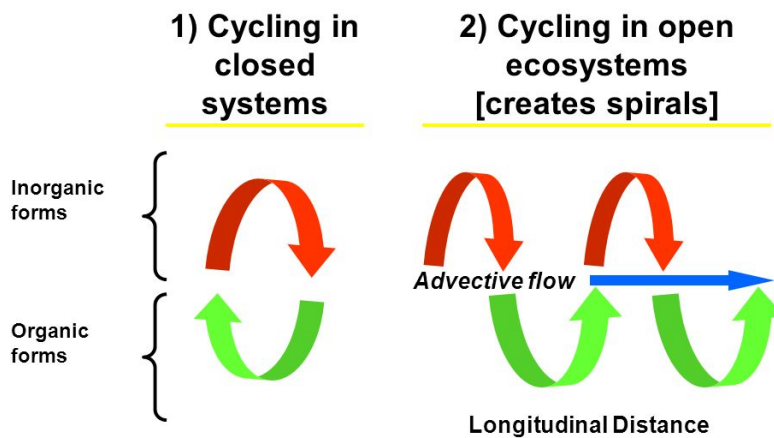


Figure 4: A simple diagram of advection and nutrient reactions (organic substances and inorganic substances) in a river. Although not shown, you might also think about how diffusion might influence the movement of nutrients in the river and in the sediments. The zone where water moves into the sediment bed is called the hyporheic zone. The porosity of the sediments will allow more advective flow that might influence the reaction capacity of solutes in the sediments.

Session Goals

We will not become experts in advection-diffusion-reaction modeling, but we will become familiar with the processes and the equations that describe them. Moreover, we'll see a bit more about how R can be used to model these processes. After this session, I hope you can do the following:

1. Describe the physical processes of advection and diffusion and solute reaction
2. Describe the equations used to model A-D-R.
3. Analyze 1-dimensional movement using advection equations in R.
4. Describe diffusion mathematically
5. Analyze 1-dimensional advection-diffusion using R.

6. Appreciate how two-dimensional analysis of advection-diffusion can be modeled in R.

In this session, we'll want to think about the movement of solutes in the porous media, i.e. a soil with air space, sediments with water between the particles. We will refer to the porosity as ξ , which is a proportion between 0 and 1.

The Processes and the Equations to Describe Them

Advection and Convection: Material and Heat

Advection is the transport of a substance by bulk motion. Convection is the transfer of heat by the actual movement of the warmed matter. The equation that are used to describe advection and convection are similar, but the physical processes are different.

AN EQUATION THAT OFTEN CREATES ANXIETY

The advection equation is a partial differential equation that describes the movement of a substance in a fluid. The equation is derived from the conservation of mass. The equation is:

$$-\frac{1}{A_x \xi_x} \cdot \left(\frac{\partial}{\partial x} A_x \cdot \left(-D \cdot \frac{\partial \xi_x C}{\partial x} \right) - \frac{\partial}{\partial x} (A_x \cdot v \cdot \xi_x C) \right) \quad (1)$$

Here D is the "diffusion coefficient", v is the "advection rate" (or velocity), and A_x and ξ are the surface area and volume fraction, respectively.¹

Assuming that A , ξ , D and v are constant along x , we can rewrite this in a more general form:

$$D \frac{\partial^2 C}{\partial x^2} - v \frac{\partial C}{\partial x} \quad (2)$$

where $v = v/A_x \xi_x$ is the "velocity" of the fluid.

The movement of compounds in the environment is driven by two processes, advection and diffusion. Of course, these processes occur in three dimensions, but for this class we'll begin with one dimensional processes before getting to more complicated examples.

Nevertheless, let's look at the 3-D advection-diffusion-reaction equation in three dimensions:

$$\frac{\partial C}{\partial t} = \nabla \cdot (D \nabla C - vC) + R \quad (3)$$

where C is the concentration of the solute, t is time, v is the velocity of the fluid, D is the diffusion coefficient, and R is the reaction term.

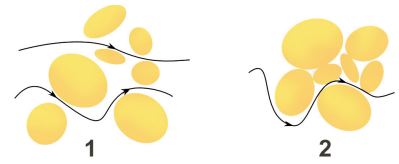


Figure 5: Notice how the porosity of the media can influence the path of the fluid. In ground water, this is measured as permeability and can be used to evaluate the flow characteristics in aquifers and oil fields. The permeability is a function of the porosity and the connectivity of the pores.

¹ What you look at this equation, think about how it makes you feel. It's a bit intimidating, to say the least. But that's only part of the story. I believe these equations, when put in front of is generates anxiety – and this anxiety can be a barrier to learning. While it's easy to claim we can just put this anxiety away seems to be a disservice and acknowledgement of our emotional responses. Before moving forward, let's take a moment to acknowledge the anxiety and see where in your body you are feeling that. Take a deep breath and let it out several times before moving forward.

Ok, what what is ∇ ? It's the gradient operator, which is a vector operator that operates on a scalar function to produce a vector whose magnitude is the maximum rate of change of the function at the point of the gradient and that points in the direction of that maximum rate of change. $\nabla \cdot$ represents divergence. In this equation, ∇C represents concentration gradient.

Advection

Advection is the process of transport of a solute by the bulk motion of the fluid. The rate of advection is proportional to the velocity of the fluid and the concentration of the solute. The rate of advection is given by the equation:

$$\frac{\partial C}{\partial t} + \nabla \cdot (\nu C) = 0 \quad (4)$$

where C is the concentration of the solute, t is time, and ν is the velocity of the fluid.

For one dimensional systems, the equation can be written as:

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = 0 \quad (5)$$

or

$$\frac{\partial C}{\partial t} + v_x \frac{\partial C}{\partial x} \quad (6)$$

where C is the concentration of the solute, t is time, ν is the velocity of the fluid, and x is the spatial coordinate.

The advection equation is not simple to solve numerically: the system is a hyperbolic partial differential equation, and interest typically centers on discontinuous "shock" solutions (which are notoriously difficult for numerical schemes to handle).

Diffusion

Diffusion is the process of transport of a solute by random molecular motion. The rate of diffusion is proportional to the concentration gradient of the solute. The rate of diffusion is given by the equation:

$$\frac{\partial C}{\partial t} = D \nabla^2 C \quad (7)$$

where D is the diffusion coefficient.

Advection-Diffusion Equation

The advection-diffusion equation is a combination of the advection and diffusion equations. The advection-diffusion equation is given by

the equation:

$$\frac{\partial C}{\partial t} + \nabla \cdot (\nu C) = D \nabla^2 C \quad (8)$$

or

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = D \frac{\partial^2 C}{\partial x^2} \quad (9)$$

where C is the concentration of the solute, t is time, ν is the velocity of the fluid, and D is the diffusion coefficient.

Advection-Diffusion-Reaction Equation

The advection-diffusion-reaction equation is a combination the advection, diffusion, and reaction equations. The advection-diffusion-reaction equation is given by the equation:

$$\frac{\partial C}{\partial t} + \nabla \cdot (\nu C) = D \nabla^2 C + R \quad (10)$$

where C is the concentration of the solute, t is time, ν is the velocity of the fluid, D is the diffusion coefficient, and R is the reaction term.

or in one-dimension:

$$\frac{\partial C}{\partial t} + \nu \frac{\partial C}{\partial x} = D \frac{\partial^2 C}{\partial x^2} + R \quad (11)$$

where C is the concentration of the solute, t is time, ν is the velocity of the fluid, D is the diffusion coefficient, and R is the reaction term.

Advection-Diffusion-Reaction in multi-phase systems and for shapes with variable geometry

The advection-diffusion-reaction equation can be extended to multi-phase systems and to shapes with variable geometry. The advection-diffusion-reaction equation for multi-phase systems and for shapes with variable geometry is given by the equation:

$$\frac{\partial C}{\partial t} + \nabla \cdot (\nu C) = D \nabla^2 C + R \quad (12)$$

where C is the concentration of the solute, t is time, ν is the velocity of the fluid, D is the diffusion coefficient, and R is the reaction term.

Applications using R

ReacTran Package

Soetaert et al. (2017) have developed a nice library in R that solves these equations using finite-difference solutions.

The ReacTran package is a collection of functions for modeling solute transport in 1D, 2D, and 3D. The package includes functions for solving the advection-diffusion equation, the advection-diffusion-reaction equation, and the advection-diffusion-reaction equation for multi-phase systems.

The package also includes functions for solving:

- Advection-diffusion equations in 1D, 2D, and 3D
- Advection-diffusion-reaction equations in 1D, 2D, and 3D
- Advection-diffusion-reaction equations for multi-phase systems

Using R as a Modelling Environment to Solve PDEs and ODEs

We can use R to solve the advection-diffusion equation. The 'deSolve' package to solve the ReacTran library functions decomposing the partial differential equation (PDE) into a discretized by space and solving by ordinary differential equations (ODE).

```
library(ReacTran) # Load the ReacTran package

## Loading required package: rootSolve
## Loading required package: deSolve
## Loading required package: shape

library(deSolve) # Load the deSolve package
library(xtable) # Load the xtable package to help format tables outputs
```

1D Transportation Model

The 'ReacTran' package provides a function to solve the advection-diffusion-reaction equation for a simple one-dimensional case.

```
tran.1D(C = 1, D = 0, flux.up = 1, v = 5, A = 1, dx = 1, full.output = TRUE)

## $dC
## [1] -4
##
## $C.up
## [1] 0.2
##
```

```
## $C.down
## [1] 1
##
## $dif.flux
## [1] 0 0
##
## $adv.flux
## [1] 1 5
##
## $flux
## [1] 1 5
##
## $flux.up
## [1] 1
##
## $flux.down
## [1] 5
```

Solving a 1-D reaction transport model

```
library(ReacTran)
out <- steady.1D(func = advModel, y = runif(25), params = parms, nspace= 1, positive = TRUE)
```

We can use R to solve the advection-diffusion-reaction equation. The following code uses the ‘deSolve’ package to solve the advection-diffusion-reaction equation for a simple one-dimensional case.

```
# Load the deSolve package
library(deSolve)

# Define the advection-diffusion-reaction equation
advection_diffusion_reaction <- function(t, C, parms) {
  with(as.list(parms), {
    dC <- D * (diff(C, lag = 2) - 2 * diff(C, lag = 1) + diff(C, lag = 0)) / dx^2 - k * C
    dC[1] <- 0
    dC[n] <- 0
    list(dC)
  })
}

# Set the parameters
parms <- list(
  D = 0.1, # Diffusion coefficient
```



```

dx = 0.1, # Spatial step
k = 0.01 # Reaction rate
)

# Set the initial conditions
C0 <- c(0, rep(0, 98), 1, rep(0, 98), 0)

# Set the times at which to evaluate the solution
times <- seq(0, 100, by = 1)

# Solve the advection-diffusion-reaction equation
out <- ode(y = C0, times = times, func = advection_diffusion_reaction, parms = parms)

# Plot the solution
plot(out, xlab = "Distance", ylab = "Concentration", type = "l")

```

1-D Reaction-Transport Model

For this example, we will solve the advection-diffusion-reaction equation for a simple one-dimensional case, where the reaction term is given by $R = kC$, and the initial concentration is given by $C(x,0) = 1$ for $x < 0.5$ and $C(x,0) = 0$ for $x \geq 0.5$.

	Value
F0	1.00
v	1.00
k	0.10
D	0.00
dx	1.00

```

advModel <- function(t, C, parms) {
  with(as.list(parms), {
    Tran <- tran.1D(C = C, D = D, flux.up = F0, v = v, dx = dx)
    Consumption = k * C
    dC <- Tran$dC - Consumption

    return(list(dC = dC, Consumption = Consumption, flux.up = Tran$flux.up, flux.down = Tran$flux.down))
  })
}

out <- steady.1D(func = advModel, y = runif(25), parms = parms, nspec= 1, positive = TRUE)

parms <- c(F0 = 1, v=1, k = 0.5, D=0, dx = 1)

```

```
out2 <- steady.1D(func = advModel, y = runif(25), parms = parms, nspec= 1, positive = TRUE)

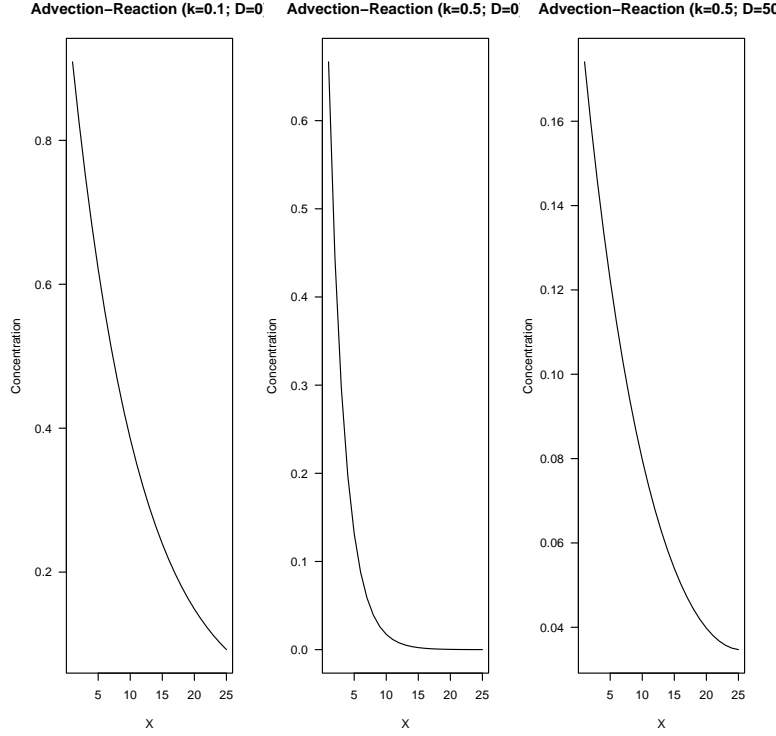
parms <- c(F0 = 1, v=1, k = 0.5, D=50, dx = 1)
out3 <- steady.1D(func = advModel, y = runif(25), parms = parms, nspec= 1, positive = TRUE)
```

We can look at the output, using a simple call of the object, but without more information, it's not clear what we are looking at.

```
out

## $y
## [1] 0.90909091 0.82644628 0.75131480 0.68301345 0.62092132 0.56447393
## [7] 0.51315812 0.46650738 0.42409762 0.38554329 0.35049390 0.31863081
## [13] 0.28966438 0.26333125 0.23939205 0.21762913 0.19784467 0.17985879
## [19] 0.16350799 0.14864363 0.13513057 0.12284597 0.11167816 0.10152560
## [25] 0.09229601
##
## $Consumption
## [1] 0.090909091 0.082644628 0.075131480 0.068301345 0.062092132 0.056447393
## [7] 0.051315812 0.046650738 0.042409762 0.038554329 0.035049390 0.031863081
## [13] 0.028966438 0.026333125 0.023939205 0.021762913 0.019784467 0.017985879
## [19] 0.016350799 0.014864363 0.013513057 0.012284597 0.011167816 0.010152560
## [25] 0.009229601
##
## $flux.up
## [1] 1
##
## $flux.down
## [1] 0.09229601
##
## attr(,"precis")
## [1] 3.328335e-01 1.891158e-09
## attr(,"steady")
## [1] TRUE
## attr(,"class")
## [1] "steady1D" "rootSolve" "list"
## attr(,"dimens")
## [1] 25
## attr(,"nspec")
## [1] 1
```

Thus, we might be better off plotting the output. I am not sure why the plot functions are ignoring my `par()` call, perhaps this will be fixed by version 0.9!



Oxygen Consumption Porous Spherical Particle

We will be modeling the consumption of oxygen in a "sand-sized" porous spherical particle. The model is based on the following equation:

$$\frac{\partial C}{\partial t} = -v \frac{\partial C}{\partial x} - k(C)$$

where C is the concentration of oxygen, D is the diffusion coefficient, v is the velocity of the fluid, and $k(C)$ is the rate of oxygen consumption.

At this scale the velocity will be zero. Thus, we will rely on diffusion to for the oxygen movement to where it is consumed.

We start with defining the size and porosity of the particle and use R to create a grid to solve the advection-diffusion-reaction equation.

Parameter	Description	Typical Range	Modelled Value
R	Radius of the particle	0.005 – 0.2 cm	0.025 cm
Porosity	Proportion of void space	0.005 – 0.7	0.7

Table 1: Characteristics of the Particle

We will create a grid to model the particle with Radius R and N ($= 100$) grid points. We will also define the properties of the particle such as porosity, diffusion coefficient ($D = 400$), and the rate of

oxygen consumption, $R_{O_2} = 10^6$.

Although we are modeling a one-dimensional system, we will need to create a grid surface as a circle to effectively model the particle surface area changes as O_2 diffuses into the particle and is consumed by the reactions in the particle.

```
grid <- setup.grid.1D(x.up=0, L = R, N = N) # Grid definition
por.grid <- setup.prop.1D(value=por, grid=grid) # Porosity
D.grid <- setup.prop.1D(value=D, grid=grid) # Diffusion coefficient

sphere.surf <- function(x) 4*pi*x^2 # Surface area of a sphere
A.grid <- setup.prop.1D(func=sphere.surf, grid=grid) # Surface area
```

Finally, we need to define the O_2 concentration at the surface of the particle and the O_2 consumption rate of the particle.

Parameter	Description	Typical Range	Modelled Value
C_{ow}	Concentration of O_2 in Water	$0.1 - 0.3 \mu\text{mols}/\text{cm}^{-3}$	$0.25 \mu\text{mols}/\text{cm}^{-3}$
R_{O_2}	Rate of oxygen consumption	$10^5 - 10^6 \mu\text{mols}/\text{cm}^{-3} / \text{year}$	$10^6 \mu\text{mols}/\text{cm}^{-3} / \text{year}$
K_s	O_2 saturation	$0.001 - 0.01 \mu\text{mols}/\text{cm}^{-3}$	$0.005 \mu\text{mols}/\text{cm}^{-3}$

Note, we often measure oxygen using ppm (parts per million), but the model uses $\mu\text{mols}/\text{cm}^{-3}$. The conversion is $1 \text{ ppm} = 0.0224 \mu\text{mols}/\text{cm}^{-3}$, thus, we are modeling K_s within a range of 2.24 - 6.72 ppm, using $K_s = 0.22 \mu\text{mols}/\text{cm}^{-3}$.

Next we create a function to model the oxygen consumption in the particle that relies on We will use the `tran.1D` function to solve the advection-diffusion equation and the `steady.1D` function to solve the steady state solution of the advection-diffusion-reaction equation.

```
Aggregate.Model <- function(time, O2, pars) {
  tran <- tran.1D(C = O2, C.down = C.ow.O2, D = D.grid,
    A=A.grid, VF = por.grid, dx = grid)
  reac <- - R.O2 * (O2 / (Ks + O2))
  return(list(dCdt= tran$dC + reac, reac = reac,
    flux.up=tran$flux.up, flux.down=tran$flux.down))
}

O2.agg <- steady.1D(y = runif(N), func=Aggregate.Model,
  nspec=1, positive=TRUE, atol = 1e-10)
```

The plot shows the oxygen concentration in the particle. The concentration is highest at the surface and decreases as it moves into the particle. The concentration is zero at the center of the particle.

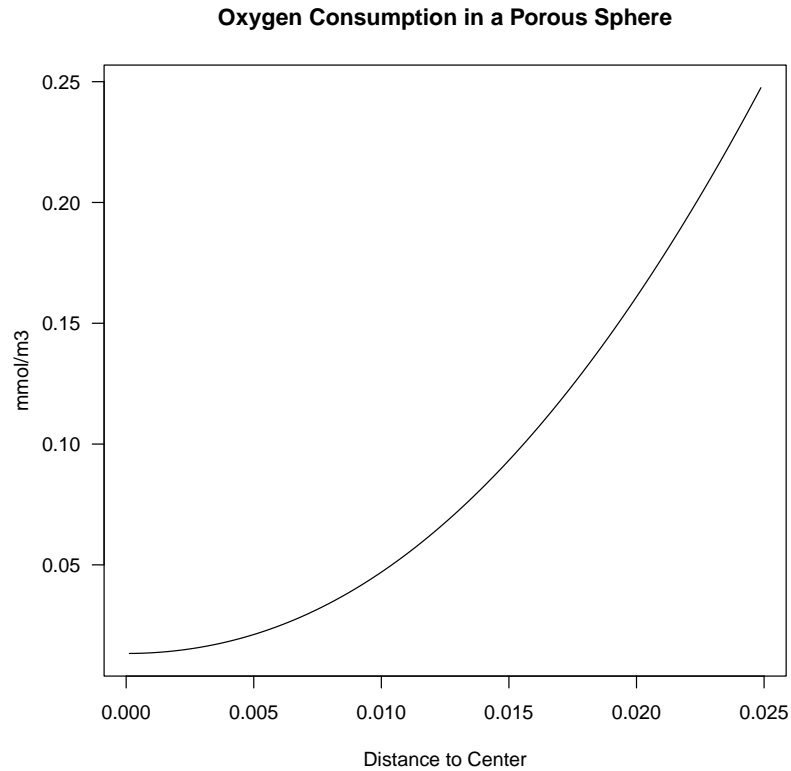


Figure 6: Oxygen Consumption in a Porous Sphere

```
diffusion2D <- function(t,conc,par){
  Conc <- matrix(nr=n,nc=n,data=conc) # vector to 2-D matrix
  dConc <- -r*Conc*Conc # consumption
  BND <- rep(1,n) # boundary concentration

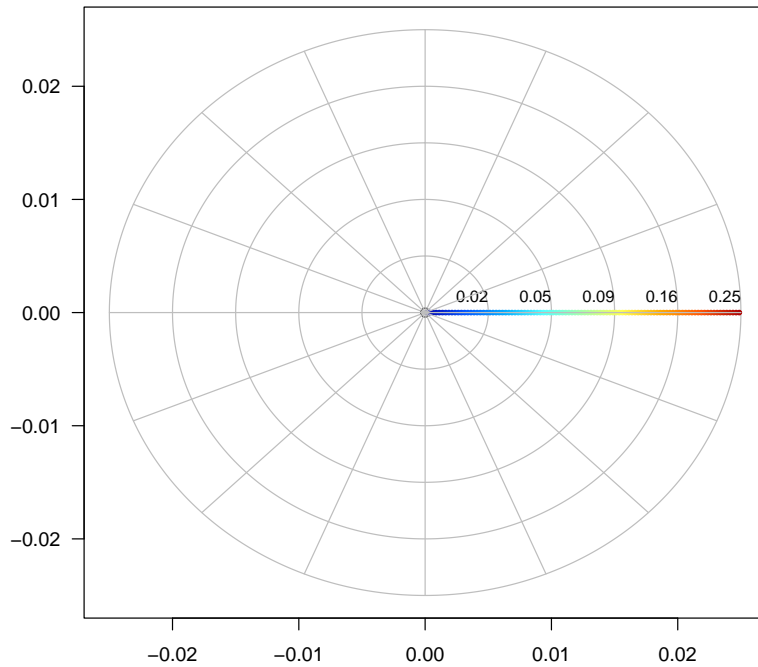
  # constant production in certain cells
  dConc[ii]<- dConc[ii]+ p

  #diffusion in X-direction; boundaries=imposed concentration

  Flux <- -Dx * rbind(rep(0,n),(Conc[2:n,]-Conc[1:(n-1),]),rep(0,n))/dx
  dConc <- dConc - (Flux[2:(n+1),]-Flux[1:n,])/dx

  #diffusion in Y-direction
  Flux <- -Dy * cbind(rep(0,n),(Conc[,2:n]-Conc[,1:(n-1)]),rep(0,n))/dy
  dConc <- dConc - (Flux[,2:(n+1)]-Flux[,1:n])/dy

  return(list(as.vector(dConc)))
}
```



```
}
```

After specifying the values of the parameters, 10 cells on the 2-D grid where there will be substance produced are randomly selected (ii).

14 Package rootSolve : roots, gradients and steady-states in R 0.0
0.2 0.4 0.6 0.8 1.0 0.0 0.2 0.4 0.6 0.8 1.0 2-D diffusion+production x y
Figure 5: Steady-state solution of the nonlinear 2-Dimensional model

```
# parameters
dy <- dx <- 1 # grid size
Dy <- Dx <- 1.5 # diffusion coeff, X- and Y-direction
r <- 0.01 # 2-nd-order consumption rate (/time)
p <- 20 # 0-th order production rate (CONC/t)
n <- 100
# 10 random cells where substance is produced at rate p
ii <- trunc(cbind(runif(10)*n+1, runif(10)*n+1))
```

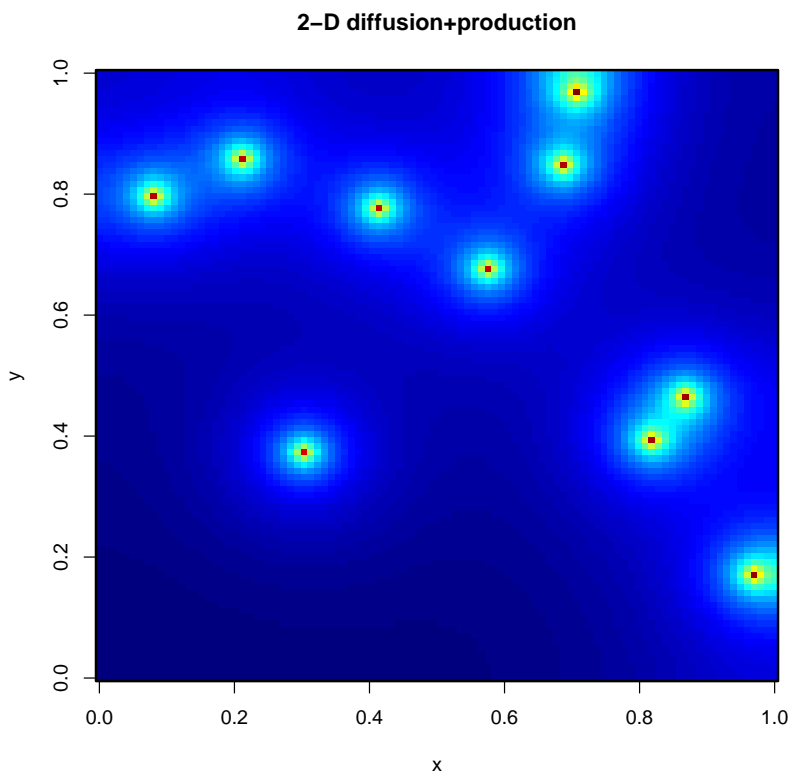
The steady-state is found using function `steady.2D`. It takes as arguments a.o. the dimensionality of the problem (`dimens`) and `lrw=1000000`, the length of the work array needed by the solver. If this value is set too small, the solver will return with the size needed. It takes about 0.5 second to solve this 10000 state variable model.

```
Conc0 <- matrix(nr=n,nc=n,10.)
print(system.time(
ST3 <- steady.2D(Conc0,func=diffusion2D,parms=NULL,pos=TRUE,dimens=c(n,n),
lrw=1000000,atol=1e-10,rtol=1e-10,ctol=1e-10)
))

##    user  system elapsed
##   0.231   0.027   0.260
```

user system elapsed 1.044 0.032 1.076 The S_3 image method is used to generate the steady-state plot.

```
image(ST3,main="2-D diffusion+production", xlab="x", ylab="y")
```



Considering 2D Models

2.4. Steady-state solution of 2-D PDEs Function `steady2D` efficiently finds the steady-state of 2-dimensional problems. Karline Soetaert 13

In the following model $\frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2} - r C^2 + pxy$ a substance C is consumed at a quadratic rate ($r C^2$), while dispersing in X- and Y-direction. At certain positions (x,y) the substance is produced (rate p). The model is solved on a square (100*100) grid. There are zero-flux boundary conditions at the 4 boundaries. The term $D_x \frac{\partial^2 C}{\partial x^2}$ is in fact shorthand for: $-\frac{\partial \text{Flux}}{\partial x}$ where $\text{Flux} = -D_x \frac{\partial C}{\partial x}$ i.e. it is the negative of the ux gradient, where the ux is due to diffusion. In the numerical approximation for the ux, the concentration gradient is approximated as the subtraction of two matrices, with the columns or rows shifted (e.g. `Conc[2:n,]-Conc[1:(n-1),]`). The ux gradient is then also approximated by subtracting entire matrices (e.g. `Flux[2:(n+1),]-Flux[1:(n),]`). This is very fast. The zero-flux at the boundaries is imposed by binding a column or row with 0's.

Conclusion

References

Soetaert, K., Meysman, F., and Soetaert, M. K. (2017). Package ‘reacTran’.