# Machine Learning Worksheet Solution 7

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# 1 Constrained Optimization

**Problem 1:**

Solving the following constrained optimization problem:

$$f_0(\mathbf{x}) = -(x_1 + x_2) \tag{1.1}$$
$$f_1(\mathbf{x}) = x_1^2 + x_2^2 - 1 \tag{1.2}$$

according to the recipe of the lecture. Calculate the Lagrange:

$$L(\mathbf{x}, \alpha) = f_0(\mathbf{x}) + \alpha f_1(\mathbf{x}) \tag{1.3}$$

Obtain the Lagrange Dual function $g(\alpha)$:

$$\nabla_x L = [2\alpha x_1 - 1, 2\alpha x_2 - 1]^T = \mathbf{0} \tag{1.4}$$
$$\implies x_1^* = x_2^* = \frac{1}{2\alpha} \tag{1.5}$$
$$g(\alpha) = L(\mathbf{x}^*, \alpha) = -(\alpha + \frac{1}{2\alpha}) \tag{1.6}$$

and solve the dual problem:

$$\frac{dg}{d\alpha} = -1 + \frac{1}{2\alpha^2} = 0 \implies \alpha = \pm\sqrt{\frac{1}{2}} \tag{1.7}$$
$$\implies x_1^* = x_2^* = \frac{\sqrt{2}}{2} \tag{1.8}$$

# 2 SVM

**Problem 2:**

**Problem 3:**

**Problem 4:**

(a)
In order to compute $Q$ based on the dual function of SVM, we can basically compare the two equations:

$$-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i y_j \alpha_i \alpha_j \mathbf{x}_i^T \mathbf{x}_j = \frac{1}{2}\alpha^T Q \alpha = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j q_{ij} \tag{2.1}$$

$$\implies q_{ij} = -y_i y_j \mathbf{x}_i^T \mathbf{x}_j \tag{2.2}$$

Or expressed by the Hadamard product: $Q = -\mathbf{y}\mathbf{y}^T \odot X^T X$.

(b)
Due to Schur's product theorem, the result of a hadamard product is positive semidefinite if the two multiplied matrices are positive semidefinite. Hence in order to prove that $Q$ is negative semidefinite, we can prove that $\mathbf{y}\mathbf{y}^T \odot X^T X$ is positive semidefinite which means that we have to prove that $\mathbf{y}\mathbf{y}^T$ and $X^T X$ are positive semidefinite.

For $\mathbf{y}\mathbf{y}^T$:

$$\alpha^T \mathbf{y}\mathbf{y}^T \alpha = (\alpha^T \mathbf{y})^2 \geq 0 \tag{2.3}$$

For $X^T X$

$$\alpha^T X^T X \alpha = (X\alpha)^T X\alpha \geq 0 \tag{2.4}$$