Teaching and learning mathematics with understanding

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Chapter 2

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In order to prepare mathematically literate citizens for the 21st century, classrooms need to be restructured so mathematics can be learned with understanding. Teaching for understanding is not a new goal of instruction: School reform efforts since the turn of the 20th century have focused on ways to create learning environments so that students learn with understanding. In earlier reform movements, notions of understanding were often derived from ways that mathematicians understood and taught mathematics. What is different now is the availability of an emerging research base about teaching and learning that can be used to decide what it means to learn with understanding and to teach for understanding. This research base describes how students themselves construct meaning for mathematical concepts and processes and how classrooms support that kind of learning.

WHY UNDERSTANDING?

Perhaps the most important feature of learning with understanding is that such learning is generative. When students acquire knowledge with understanding, they can apply that knowledge to learn new topics and solve new and unfamiliar problems. When students do not understand, they perceive each topic as an isolated skill. They cannot apply their skills to solve problems not explicitly covered by instruction, nor extend their learning to new topics. In this day of rapidly changing technologies, we cannot anticipate all the skills that students will need over their lifetimes or the problems they will encounter. We need to prepare students to learn new skills and knowledge and to adapt their knowledge to solve new problems. Unless students learn with understanding, whatever knowledge they acquire is likely to be of little use to them outside the school.

WHAT IS UNDERSTANDING?

Understanding is not an all-or-none phenomenon. Virtually all complex ideas or processes can be understood at a number of levels and in quite different ways. Therefore, it is more appropriate to think of understanding as emerging or developing rather than presuming that someone either does or does not understand a given topic, idea, or process. As a consequence, we characterize understanding in terms of mental activity that contributes to the development of understanding rather than as a static attribute of an individual's knowledge.

HOW UNDERSTANDING IS DEVELOPED

We propose five forms of mental activity from which mathematical understanding emerges: (a) constructing relationships, (b) extending and applying mathematical knowledge, (c) reflecting about experiences, (d) articulating what one knows, and (e) making mathematical knowledge one's own. Although these various forms of mental activity are highly interrelated, for the sake of clarity we discuss each one separately.

Constructing Relationships

Things take meaning from the ways they are related to other things. People construct meaning for a new idea or process by relating it to ideas or processes that they already understand. Children begin to construct mathematical relations long before coming to school, and these early forms of knowledge can be used as a base to further expand their understanding of mathematics. Formal mathematical concepts, operations, and symbols, which form the basis of the school mathematics curriculum, can be given meaning by relating them to these earlier intuitions and ideas. For example, children as young as kindergarten and first grade intuitively solve a variety of problems involving joining, separating, or comparing quantities by acting out the problems with collections of objects. Extensions of these early forms of problem-solving strategies can be used as a basis to develop the mathematical concepts of addition, subtraction, multiplication, and division (see chapter 4, Carpenter et al., this volume).

Unless instruction helps children build on their informal knowledge and relate the mathematics they learn in school to it, they are likely to develop two separate systems of mathematical knowledge: one they use in school and one they use outside of school. For example, children often are not bothered by the fact that they get one answer when they calculate with paper and pencil and another when they figure out the same problem using counters or some other material. They do not see that the answers they get with procedures they learn in school should be the same as the answers they get when they solve problems in ways that make sense to them.

Extending and Applying Mathematical Knowledge

It is not sufficient, however, to think of the development of understanding simply as the appending of new concepts and processes to existing knowledge. Over the long run, developing understanding involves more than simply connecting new knowledge to prior knowledge; it also involves the creation of rich, integrated knowledge structures. This structuring of knowledge is one of the features that makes learning with understanding generative. When knowledge is highly structured, new knowledge can be related to and incorporated into existing networks of knowledge rather than connected on an element-by-element basis. When students see a number of critical relationships among concepts and processes, they are more likely to recognize how their existing knowledge might be related to new situations. Structured knowledge is less susceptible to forgetting. When knowledge is highly structured, there are multiple paths to retrieving it, whereas isolated bits of information are more difficult to remember.

Although developing structure is a hallmark of learning with understanding, the nature of that structure is also critical because not all relationships are mathematically fruitful. Learning with understanding involves developing relationships that reflect important mathematical principles. The examples in the chapters that follow illustrate how addition and subtraction of multidigit numbers is related to basic concepts of place value, fractions are related to the concept of division, knowledge of graphing is extended to more general forms of data representation and interpretation, and informal ideas about space can be developed into the mathematical structures of geometry.

One of the defining characteristics of learning with understanding is that knowledge is learned in ways that clarify how it can be used. It often has been assumed, however, that basic concepts and skills need to be learned before applications are introduced. This is a faulty assumption: Children use their intuitively acquired knowledge to solve problems long before they have been taught basic skills.

We have come to understand that applications provide a context for developing skills, so, in teaching for understanding, skills are linked to their application from the beginning. For example, in the projects described in Carpenter et al. (chapter 4, this volume), children start out solving problems involving joining, separating, and comparing—before they have learned about addition and subtraction. The operations of addition and subtraction are presented as ways of representing these problem situations.

In the classroom episode described in Lehrer, Jacobson, Kemeny, and Strom (chapter 5, this volume), children's natural language about shape and form eventually are transformed into mathematical propositions and definitions.

Reflecting About Experiences

Reflection involves the conscious examination of one's own actions and thoughts. Routine application of skills requires little reflection: One just follows a set of familiar procedures. Reflection, however, plays an important role in solving unfamiliar problems. Problem solving often involves consciously examining the relation between one's existing knowledge and the conditions of a problem situation. Students stand a better chance of acquiring this ability if reflection is a part of the knowledge-acquisition process.

To be reflective in their learning means that students consciously examine the knowledge they are acquiring and, in particular, the way it is related both to what they already know and to whatever other knowledge they are acquiring. But learning does not only occur with the addition of new concepts or skills: It also comes about through the reorganization of what one already knows. Reflecting about what one knows and how one knows can lead to this sort of reorganization.

Our notion of the emerging nature of understanding is seen in students' developing ability to reflect on their knowledge. Initially students have limited ability to reflect on their thinking. One characteristic of students' developing understanding is that they become increasingly able to reflect on their thinking.

Articulating What One Knows

The ability to communicate or articulate one's ideas is an important goal of education, and it also is a benchmark of understanding. Articulation involves the communication of one's knowledge, either verbally, in writing, or through some other means like pictures, diagrams, or models. Articulation requires reflection in that it involves lifting out the critical ideas of an activity so that the essence of the activity can be communicated. In the process, the activity becomes an object of thought. In other words, in order to articulate our ideas, we must reflect on them in order to identify and describe critical elements. Articulation requires reflection, and, in fact, articulation can be thought of as a public form of reflection.

As with reflection, students initially have difficulty articulating their ideas about an unfamiliar topic or task, but by struggling to articulate their ideas, especially with means like mathematical symbols or models, students develop the ability to reflect on and articulate their thinking.

Making Mathematical Knowledge One's Own

Understanding involves the construction of knowledge by individuals through their own activities so that they develop a personal investment in building knowledge. They cannot merely perceive their knowledge simply as something that someone else has told them or explained to them; they need to adopt a stance that knowledge is evolving and provisional. They will not view knowledge in this way, however, if they see it as someone else's knowledge, which they simply assimilate through listening, watching, and practicing.

This does not mean that students cannot learn by listening to teachers or other students, but they have to adapt what they hear to their own ends, not simply accept the reasoning because it is clearly articulated by an authority figure. Neither does this mean that understanding is entirely private. The development of students' personal involvement in learning with understanding is tied to classroom practices in which communication and negotiation of meanings are important facets.

In this more general sense, students author their own learning. They develop their own stances about different forms and practices of mathematics. For example, some students are fascinated by number, others by space, still others by questions of chance and uncertainty. Students who understand mathematics often define interests that guide their activity. Ideally, learning is guided by personal histories of aptitude and interest, not simply by curricular sequences.

An overarching goal of instruction is that students develop a predisposition to understand and that they strive to understand because understanding becomes important to them. This means that students themselves become reflective about the activities they engage in while learning or solving problems. They develop relationships that may give meaning to a new idea, and they critically examine their existing knowledge by looking for new and more productive relationships. They come to view learning as problem solving in which the goal is to extend their knowledge.

IS UNDERSTANDING THE SAME FOR EVERYONE?

In proposing the five forms of mental activity from which mathematical understanding emerges, we are not suggesting that all students learn in exactly the same way or that understanding always looks the same in all individuals. What we are proposing is that the development of understanding involves these forms of mental activity in some form. For an idea to be understood, it must be related to other ideas, but there are many ways that ideas might be related. We are not suggesting that relations must be formed in the same way or through the same activities, only that understanding depends on ideas being organized in some productive way that makes them accessible for solving problems. The ability to extend and apply knowledge is a hallmark of understanding, but this does not imply that all people extend and apply their knowledge in the same way. By the same token, reflection and articulation can take on a variety of forms, but we cannot conceive of understanding developing without some sort of reflection and articulation. Personal histories of developing understanding will vary. Such variation is often an important catalyst for conceptual change as students reconcile their own views with those of others.

CRITICAL DIMENSIONS OF CLASSROOMS THAT PROMOTE UNDERSTANDING

What does all this mean for instruction? Essentially, for learning with understanding to occur on a widespread basis, classrooms need to provide students with opportunities to (a) develop appropriate relationships, (b) extend and apply their mathematical knowledge, (c) reflect about their own mathematical experiences, (d) articulate what they know, and (e) make mathematical knowledge their own. In order to organize a classroom that enables students to engage in these activities (discussed previously), there are at least three dimensions of instruction that need to be considered: (a) tasks or activities that students engage in and the problems that they solve; (b) tools that represent mathematical ideas and problem situations; and (c) normative practices, which are the standards regulating mathematical activity, agreed on by the students and teacher.

Tasks

Mathematics lessons frequently are planned and described in terms of the tasks students engage in. Tasks can range from simple drill-and-practice exercises to complex problem-solving tasks set in rich contexts. Almost any task can promote understanding. It is not the tasks themselves that determine whether students learn with understanding: The most challenging tasks can be taught so that students simply follow routines, and the most basic computational skills can be taught to foster understanding of fundamental mathematical concepts. For understanding to develop on a widespread basis, tasks must be engaged in for the purpose of fostering understanding, not simply for the purpose of completing the task. (For an example of how learning to use multidigit numbers within the context of computational activities becomes a task in which students' understanding of place value grows, see Carpenter et al., chapter 4, this volume.)

Tools

Tools are used to represent mathematical ideas and problem situations. They include such things as paper and pencil, manipulative materials, calculators and computers, and symbols. Problems are solved by manipulating these tools in ways that follow certain rules or principles (see Lajoie, chapter 7, this volume, on tools for data representation and visualization). Computational algorithms, for example, involve the manipulation of symbols to perform various arithmetic calculations. These same operations can be performed by representing the numbers with counters or base-10 blocks and combining, grouping, or partitioning the counters or blocks in appropriate ways. Connections with representational forms that have intuitive meaning for students can greatly help students give meaning to symbolic procedures. In Carpenter et al. and Sowder and Philipp (chapters 4 and 6, this volume), examples involving adding and subtracting whole numbers and dividing fractions illustrate how such connections can be developed.

Standard mathematical representations and procedures involve symbols and operations on those symbols that have been adopted over centuries and have been constructed for the purposes of efficiency and accuracy. The connections between symbols and symbolic procedures and the underlying mathematical concepts that they represent are not always apparent. As a consequence, practicing formal procedures involving abstract symbols does little to help students connect the symbols or procedures to anything that would give them meaning. One of the ways to resolve this dilemma is for students to link the critical steps in procedures with abstract symbols to representations that give them meaning (see Kaput, chapter 8, this volume).

Representations may be introduced by the teacher or constructed by students. Lehrer et al. (chapter 5, this volume) describe classroom episodes in which students invented representations for quantities, forms, measures, and large-scale space. Each form of representation provided opportunities for developing new mathematical knowledge. As noted in Romberg and Kaput (chapter 1, this volume), the syntactically guided manipulation of formal representations is an important goal of instruction and, if we want students to understand the representations they use, we should encourage them to reflect explicitly on the characteristics of those representations useful for understanding and communicating about mathematical ideas and for solving problems.

Normative Practices (Norms)

The norms in a particular class determine how students and the teacher are expected to act or respond to a particular situation. Normative practices form the basis for the way tasks and tools are used for learning, and they govern the nature of the arguments that students and teachers use to justify mathematical conjectures and conclusions. These norms can be manifest through overt expectations or through more subtle messages that permeate the classroom environments.

Although the selection of appropriate tasks and tools can facilitate the development of understanding, the normative practices of a class determine whether they will be used for that purpose. In classrooms that promote understanding, the norms indicate that tasks are viewed as problems to be solved, not exercises to be completed using specific procedures. Learning is viewed as problem solving rather than drill and practice. Students apply existing knowledge to generate new knowledge rather than assimilate facts and procedures. Tools are not used in a specified way to get answers: They are perceived as a means to solve problems with understanding and as a way to communicate problem-solving strategies. The classrooms are discourse communities in which all students discuss alternative strategies or different ways of viewing important mathematical ideas such as what is a triangle (see Lehrer et al. and Sowder and Philipp, chapters 5 and 6, this volume). Students expect that the teacher and their peers will want explanations as to why their conjectures and conclusions make sense and why a procedure they have used is valid for the given problem. In this way, mathematics becomes a language for thought rather than merely a collection of ways to get answers.

Structuring and Applying Knowledge

For students to learn with understanding, they must have opportunities to relate what they are learning to their existing knowledge in ways that support the extension and application of that knowledge. In classrooms where students learn with understanding, there are a number of ways that instruction can provide them the opportunities to structure their knowledge. For example, students may be asked specifically to identify relevant relationships. Students may be expected to specify explicit links between symbolic procedures and manipulations of physical materials, as in the Conceptually Based Instruction classes described in Carpenter et al. (chapter 4, this volume). The relationships may also be drawn in less direct ways, as when students compare and contrast alternative strategies that they have generated to solve a problem, as in the Cognitively Guided Instruction classes described in the same chapter.

It is critical that providing opportunities for students to develop structured knowledge is a major and continuing focus of instruction. Students cannot be expected to develop critical knowledge structures by practicing procedures. Watching a demonstration, listening to an explanation of how things are related, or even engaging in a few teacher-directed hands-on tasks is not enough. Students need time to develop knowledge structures, and instruction should offer students extended opportunities to develop relationships through the tasks that they engage in.

The selection and sequencing of tasks and tools is critical. They should not be selected exclusively on mathematical structure. We must take into account children's thinking, the knowledge they bring to a situation, and the way their thinking typically develops. A tacit assumption underlying much of the traditional mathematics curriculum has been that problem solving involves the application of skills, and, consequently, skills must be learned before students can profitably engage in problem solving. The examples in the chapters that follow document that this is not the case. In these examples, problem solving and the learning of basic concepts and skills are integrated; in fact, problems and applications provide the context for learning fundamental mathematical concepts and skills.

As noted earlier, there is an extensive body of research documenting that children acquire a great deal of intuitive or informal knowledge of mathematics and begin developing problem-solving abilities outside of school. The formal concepts and skills of the mathematics curriculum need to be related to these informal concepts and problem-solving skills, or students will not see how the mathematics they learn in school applies to solving problems in the world. Furthermore, this informal knowledge can provide a solid foundation for giving meaning to the abstract mathematical symbols, concepts, and skills that students learn in school.

Tasks and tools need to be selected such that mathematics instruction builds on children's informal mathematical knowledge and that problems and applications, and the related mathematical concepts and skills, are connected from the beginning.

Reflection and Articulation

One of the primary ways that learning with understanding occurs is through reflection. Initially, students generally use concrete tools as implements to solve a given task. As students reflect on the use of the tools, the manipulations of the physical materials become abstracted. Eventually students no longer have to actually manipulate the physical tools themselves; they can think directly about more-abstract symbolic representations of the tools (see Carpenter et al. and Lehrer et al., chapters 4 and 5, this volume). The process is recursive. The more-abstract representations become themselves objects of reflection, leading to an awareness of the underlying mathematical concepts that the tools, and the symbolic abstractions of the tools, embody. As the concepts and principles embodied in a given tool become objects of reflection, higher-level mathematical principles emerge, and so on. For example, students start out solving addition and subtraction problems by modeling the joining and separating action using counters. By reflecting on their procedures and their emerging knowledge of groupings of 10, they come to use more efficient procedures that involve the use of some sort of 10-structured material like base-10 blocks. As students describe and reflect about the solutions using materials grouped by 10, they become increasingly less dependent on the base-10 materials themselves; they start to use abstract representations of the base-10 materials (see Carpenter et al., chapter 4, this volume). As students compare different abstract strategies and reflect on these differences, they begin to see that certain procedures have advantages over others, and they begin to see explicitly how properties like commutativity and associativity are involved in their procedures.

Encouraging Reflection. The question is: How do we encourage this type of reflection? Providing explicit guidelines for encouraging reflection is difficult, but a critical factor is that teachers recognize and value reflection. When that is the case, teachers establish classroom norms that support reflection. A specific norm that plays a critical role in supporting reflection in the descriptions of classrooms that follow is the expectation that students articulate their thinking. Asking students why their solutions work, why a given solution is like another solution, how they decided to solve the problem as they did, and the like, not only helps to develop students' ability to articulate their thinking, it encourages them to reflect.

At this point, we should distinguish between two types of reflection, both of which are important: (a) reflection by students about what they are doing and why, as tasks are being carried out; and (b) reflection about tasks and their solutions after the tasks have been completed. Discussion of alternative strategies that students have used to solve a given problem involves reflection about a task after it has been completed. The probing questions that a teacher might ask at this point address this type of reflection.

Discussing alternative strategies, however, addresses more than the issue of reflection on completed tasks. When students know that they are expected to explain their responses, they are more likely to reflect on a task as they are carrying it out. Reflection on the task while carrying it out can also be encouraged directly by asking students to articulate what they are doing during the process of solving a problem. One possibility is for teachers to talk to students as they are solving a problem, asking the students to explain assumptions and why they are pursuing the strategy that they have chosen. Questions like "What are you doing?; Why are you doing that?" and "How will that help you to solve the problem?" encourage reflection. Being asked such questions on a regular basis helps students internalize them, so that they will ask themselves the same questions as they think about a given task.

Another way to encourage reflection on tasks in progress is to have students work in small cooperative groups. When students are actually solving a problem together, they must articulate their assumptions, conjectures, and plans to one another. For this kind of reflection and articulation to occur, however, classroom norms must establish that these kinds of interactions are what cooperative group work is all about.

A Basis for Articulation. For articulation to be meaningful to all the participants in a class, there must be a common basis for communication. The selection of appropriate tools can fill this role, but teachers must ensure that everyone has a consistent interpretation of the tools and their use. Manipulative materials can provide common referents for discussion, but students do not always impart the same meanings to manipulations of physical materials that knowledgeable adults do. It is important that discussions include opportunities for students to articulate how they are thinking about and using tools.

Notations, (e.g., those developed for representing quantity, two-dimensional graphs, etc.) can provide a common basis for discussion, and they can help students to clarify their thinking. Notations thus play a dual role, first, as a window (for teachers and others) into the evolution of student thinking and, second, as a tool for thought. Notations are records that communicate about thinking. Appropriate notational systems allow students to articulate their thinking in very precise ways, and the precision demanded by the notational system can make students sharpen their thinking so that it can be articulated.

Classroom Norms

One norm that underlies teaching for understanding is that students apply existing knowledge in the generation of new knowledge. Learning is not perceived by either the teacher or students as assimilation and practice. Learning is viewed as problem solving, and students are expected to actively work to relate new concepts and procedures to their existing knowledge.

A specific class norm that supports this conception of learning is that students regularly discuss alternative strategies (which they have generated to solve a given problem) with the teacher, with other students, and within the context of whole-class discussion. It is not enough to have an answer to a problem; students are expected to be able to articulate the strategy they used to solve the problem and explain why it works. This means discussing how the solution is related to the parameters of the problem and how the procedures used in the solution are related to underlying mathematical concepts or some external representation that has established meaning.

In discussing alternative strategies, students not only explain their own solutions and their own thinking, but they also discuss how strategies used by different students are alike and different. In other words, they consider the connections between alternative solutions. This is one of the important ways in which relationships are made explicit. As students report and discuss solutions representing different levels of abstraction and understanding, they have the opportunity to link more-abstract strategies with more-basic strategies. For example, when some students solve a problem using manipulative materials and other students solve the problem with symbolic representations, the discussion of the relationship between the two strategies draws attention to connections that give the abstract symbolic procedures meaning (see the class interaction at the beginning of Carpenter et al., chapter 4, this volume).

Making Knowledge One's Own

As with reflection and articulation, classroom norms play a central role in helping students develop a personal sense of ownership of their knowledge. Again, specifying guidelines is difficult, but it is critical that teachers place a high value on the individual student's involvement and autonomy. All students should have opportunity to discuss their ideas, and each student's ideas should be taken seriously by everyone else in the class. The overriding goal of the classroom should be the development of understanding.

Reflection is inherently personal, and encouraging reflection is critical in helping students develop a sense of ownership of their knowledge. Students need to be given some control over the tasks they engage in and the tools they use to solve them so that they believe they have control over their own learning.

TEACHERS AND UNDERSTANDING

Inherent in much of the previous discussion is the assumption that understanding is a goal not only for students but also for teachers. Understanding plays a critical role in the solution of any complex problem, and teaching certainly involves solving complex problems. Our conception of teacher understanding is based on the same principles as our conception of student understanding.

We focus on two components of teachers' understanding and the relations between them: (a) understanding of mathematics and (b) understanding students' thinking. In order to provide instruction of the kind envisioned in this book, teachers need to understand the mathematics they are teaching, and they need to understand their own students' thinking. The mathematics to be taught and the tasks and tools to be used might be specified by an instructional program, but without requisite understanding of mathematics and students, teachers will be relegated to the routine presentation of (someone else's) ideas neither written nor adapted explicitly for their own students. In short, their teaching will be dominated by curriculum scripts, and they will not be able to establish the classroom norms necessary for learning with understanding to occur. They will not be able to engage students in productive discussion of alternative strategies because they will not understand the students' responses; neither will they be able readily to recognize student understanding when it occurs.

Understanding mathematics for instruction involves more than understanding mathematics taught in university mathematics content courses. It entails understanding how mathematics is reflected in the goals of instruction and in different instructional practices. Knowledge of mathematics must also be linked to knowledge of students' thinking, so that teachers have conceptions of typical trajectories of student learning and can use this knowledge to recognize landmarks of understanding in individuals.

Teachers need to reflect on their practices and on ways to structure their classroom environment so that it supports students' learning with understanding. They need to recognize that their own knowledge of mathematics and of students' thinking, as well as any student's understanding, is not static.

Teachers must also take responsibility for their own continuing learning about mathematics and students. Class norms and instructional practices should be designed to further not only students' learning with understanding, but also teachers' knowledge of mathematics and of students' thinking. Tasks and tools should be selected to provide a window on students' thinking, not just so that the teacher can provide more appropriate instruction for specific students, but also so the teacher can construct better models for understanding students' thinking in general.

CONCLUSION

For students and teachers, the development of understanding is an ongoing and continuous process and one that should pervade everything that happens in mathematics classrooms. For many years there has been a debate on whether an individual should learn skills with understanding from the outset or whether he or she should acquire a certain level of skill mastery first and then develop an understanding of why skills work the way they do. A mounting body of evidence supports the importance of learning with understanding from the beginning. When students learn skills without understanding, the rote application of the skills often interferes with students' subsequent attempts to develop understanding. When students learn skills in relation to developing an understanding, however, not only does understanding develop, but mastery of skills is also facilitated.

If we have learned one thing through our studies, it is that the development of understanding takes time and requires effort by both teachers and students. Learning with understanding will occur on a widespread basis only when it becomes the ongoing focus of instruction, when students are given time to develop relationships and learn to use their knowledge, when students reflect about their own thinking and articulate their own ideas, and when students make mathematical knowledge their own.

We do not have precise prescriptions for how classrooms should be organized to accomplish these goals. In this chapter we have provided some issues and components to consider when thinking about instruction, but ultimately responsibility for learning with understanding rests with the teachers and students themselves. Teachers must come to understand what it means for their students to learn with understanding and must appreciate and value learning with understanding. Like their teachers, students must come to value understanding and make understanding the goal of their learning. That is the ultimate goal of instruction.

In the chapters that follow, we provide examples of instruction that provides opportunity for the development of understanding as we have characterized it. There are a number of similarities among the examples, but we are not suggesting that understanding will occur only in classes similar to the ones described. We do, however, argue that for learning with understanding to occur, instruction needs to provide students the opportunity to develop productive relationships, extend and apply their knowledge, reflect about their experiences, articulate what they know, and make knowledge their own.

NOTES

1. Authors contributed equally to the writing of this chapter.

FOR FURTHER READING

Anderson, J. (1990). *Cognitive, psychology and its implications*. Cambridge, MA: Harvard University Press.

Hiebert, J., Carpenter, T.P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., Wearne, D. (1997). *Making sense,: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.

Kaput, J. (1992). Technology and mathematics education. In D.Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515–556). New York: Macmillan.

Lehrer, R., & Chazan, D. (1998). *Designing learning environments for developing understanding of geometry and space*. Mahwah, NJ: Lawrence Erlbaum Associates.

Schoenfeld, A. (1995). *Mathematical problem solving*. New York: Academic Press.

Schon, D. (1987). *Educating the reflective frractitioner*. San Francisco: Jossey-Bass.

Wertsch, L. (1991). *Voices of the mind*. Cambridge, MA: Harvard University Press.

Yakel, E., & Cobb, P. (1996) Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal of Research in Mathematics Education, 27,* 458–477.