Astrostatistics & Cosmology, ex. 1

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Introduction

In order to solve the exercise we need to perform a straightforward application of Bayes' theorem; since this is a one dimensional problem (in the sense that we only need to infer one parameter) there will be no need for special techniques, such as Markov chains.

Let's then setup the mathematics of the problem of inferring the coin bias as follows. We model the process of obtaining heads/tails when we toss a coin N times as a *binomial process*, in the sense that if the probability of obtaining e.g. heads is equal to H then the probability distribution of obtaining n heads in N tosses is given by the *binomial distribution*:

$$p(n, N|H) \propto H^n (1-H)^{N-n} \tag{1}$$

where the proportionality constant is the binomial coefficient of n over N, which we don't need to write explicitly here since we will simply normalize the final posterior distribution.

Notice that by using this pdf as our likelihood function we can easily simulate coin tosses once we specify the value of H; if instead H is unknown we need to perform an *inference* according to some rule. The frequentist prescription for this problem is to feed the experimental values of n and N, then treat the result as a function of H and maximize it (MLE paradigm).
The bayesian approach, instead, is to multiply the likelihood by an arbitrarily chosen prior, then normalize the result to obtain the posterior (which naturally encodes the idea of leftover uncertainty over H due to the finiteness of the available dataset).

This means that in order to solve the exercise we must first multiply the binomial likelihood (where n and N are fixed, and H is a random variable) by either the uniform or the gaussian prior, then normalize the result to obtain the posterior.
Mathematically speaking we write:

$$P(H|n, N) = \frac{P(n, N|H)P(H)}{\int_0^1 P(n, N|H)P(H) \, \mathrm{d}H} \tag{2}$$

which is nothing else than Bayes' theorem applied to the specific problem at hand.

Notice that the posterior distribution will be a 1D pdf defined over the $[0, 1]$ interval, as its only argument is the probability of obtaining heads in a single toss; this means that we can easily normalize it, for example by approximating the evidence integral as a Riemann sum:

$$\int_0^1 P(n, N|H)P(H) \, \mathrm{d}H \approx \sum_i P(n, N|H_i)P(H_i)\Delta H_i \tag{3}$$

where of course any other numerical scheme is a suitable alternative.

This technique is quite general, too; it applies to an arbitrary prior, something which isn't true e.g. if we use the conjugate prior technique.

Uniform prior

Let's start by fixing $H_{\text{true}} = 0.3$; this allows us to simulate the experimental data which one would need to collect in order to perform an "in real life" inference.
For example the first 10 obtained tosses may be:

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##  [1] "H" "T" "T" "T" "T" "T" "H" "H" "T" "T"
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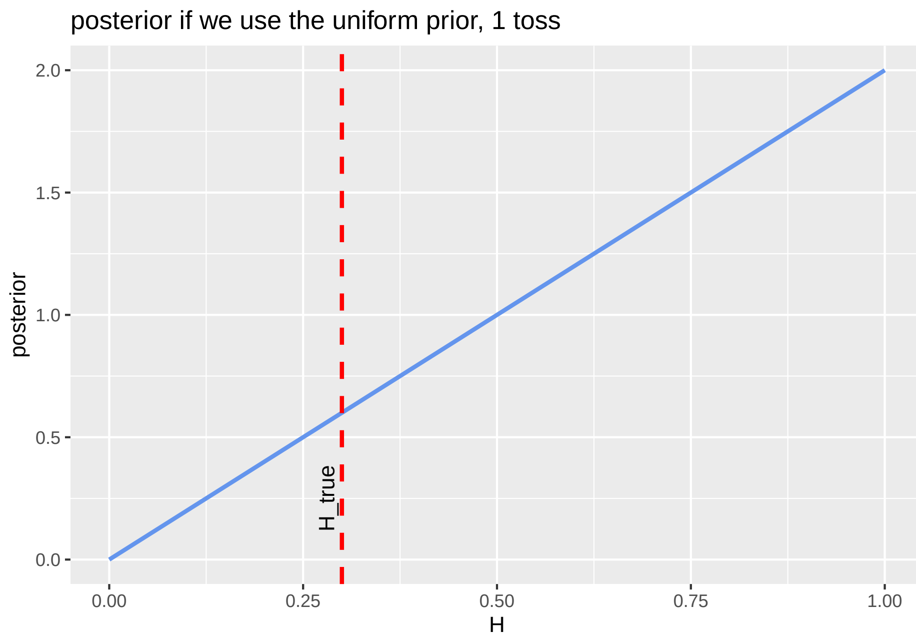
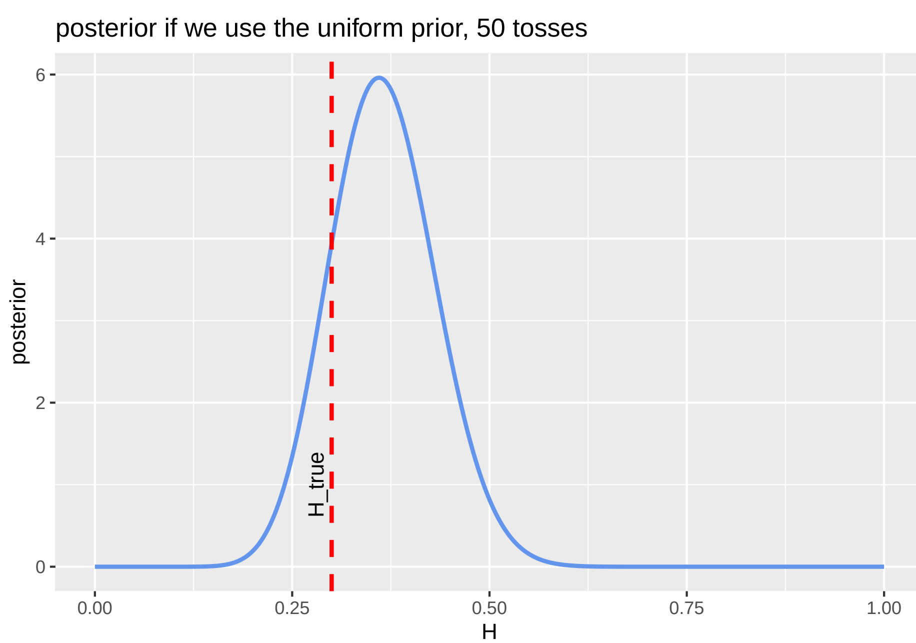
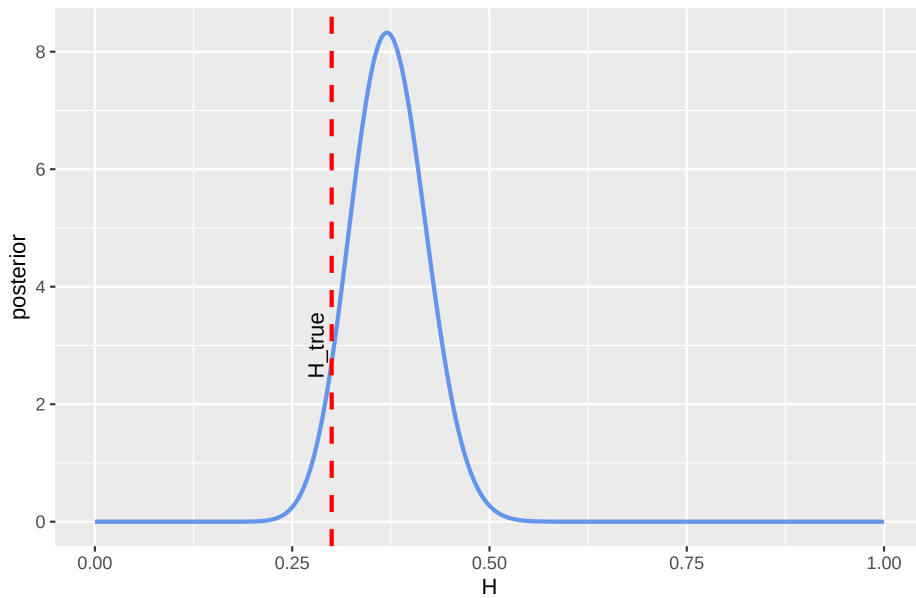
Now we can multiply the binomial likelihood by our prior, then normalize and plot the result; notice that since a uniform prior is simply a constant it suffices to normalize the likelihood itself.
We plot the posteriors directly because the prior is simply the $f(H) = 1$ flat (therefore boring) function.

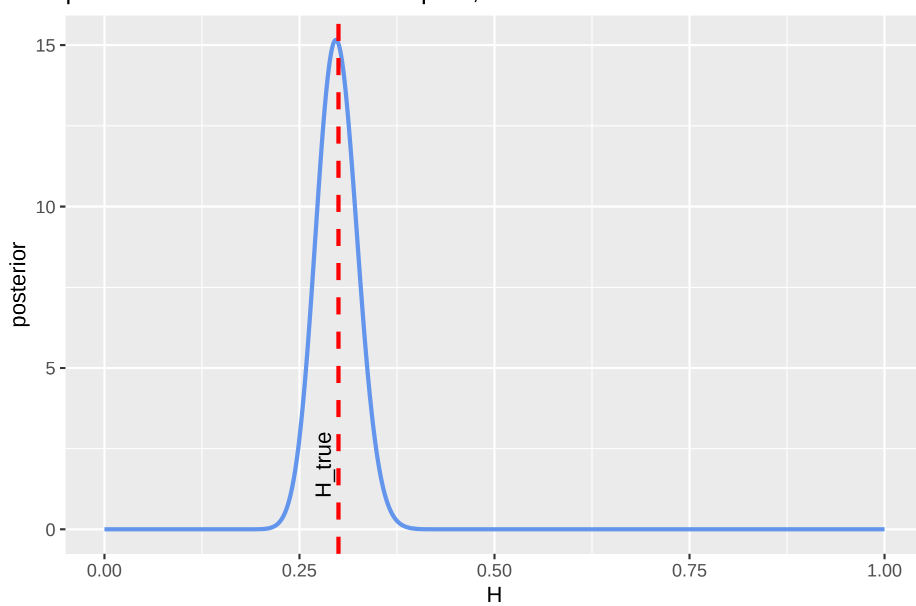


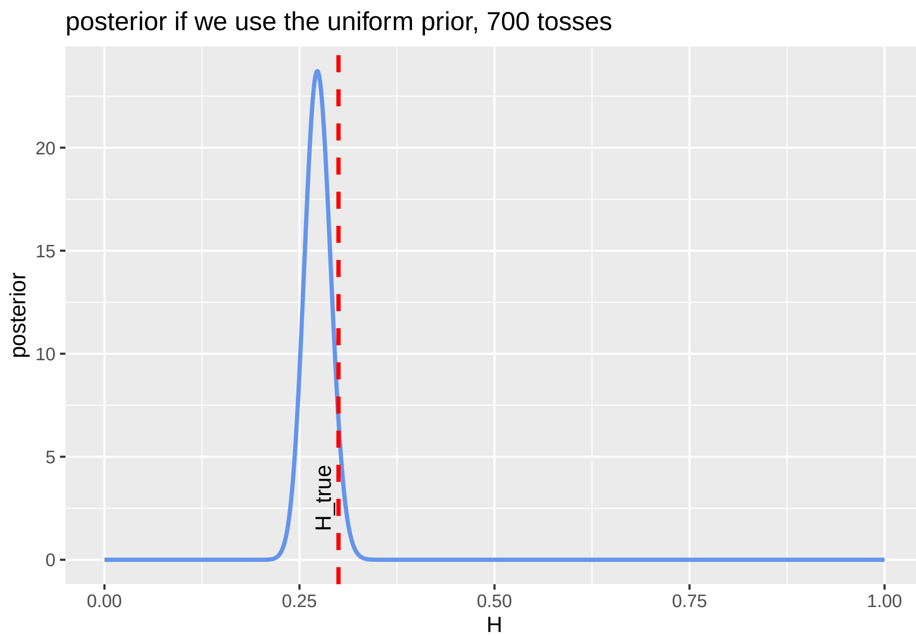


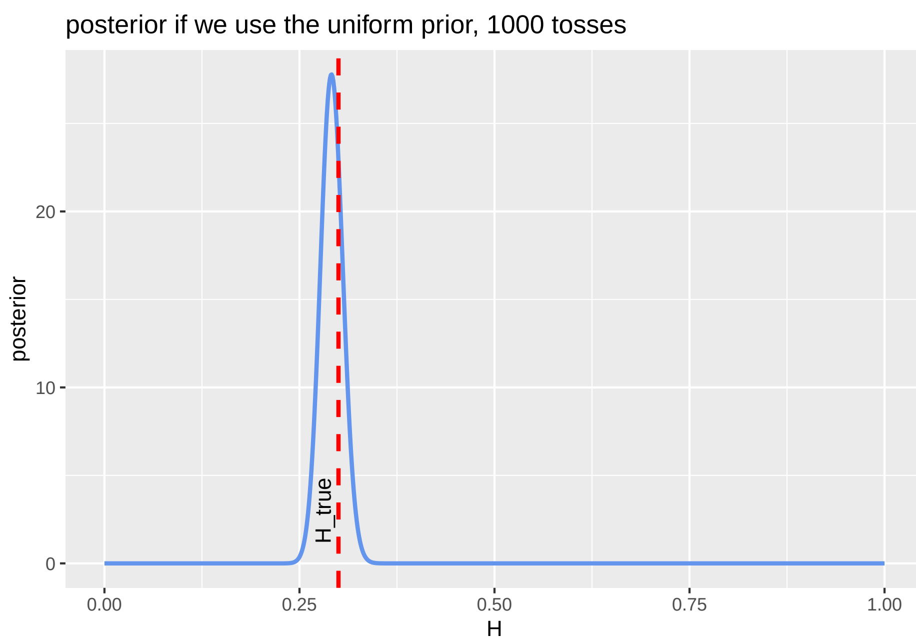
posterior if we use the uniform prior, 100 tosses



posterior if we use the uniform prior, 300 tosses



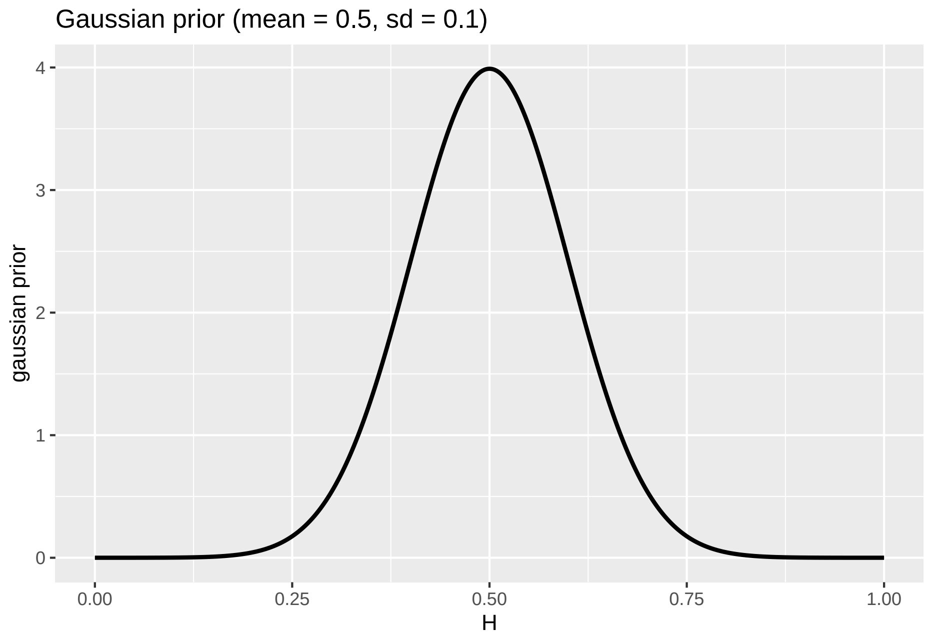




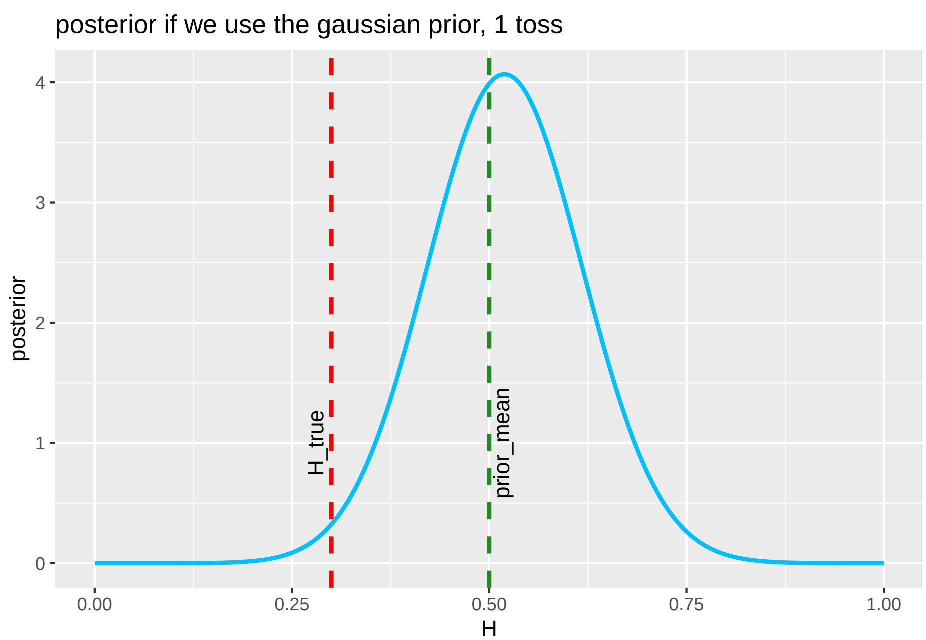
Notice that the more data we use the more satisfying the resulting posterior is. Indeed when we use more data the posterior's center becomes closer to the true value (which means we're getting closer and closer to the "correct" value), while at the same time the posterior itself becomes narrower (which means that the uncertainty with which we inferred H is decreasing, as it should).
As we know we can be confident that in the $N \to +\infty$ limit we will obtain H_{true}, and yet when the dataset is still small the posterior isn't very good (cfr the $N = 1$ plot above). One may wonder whether a smarter choice of the prior may accelerate the convergence; indeed this is exactly what we want to check in the next part of the exercise.

Gaussian prior

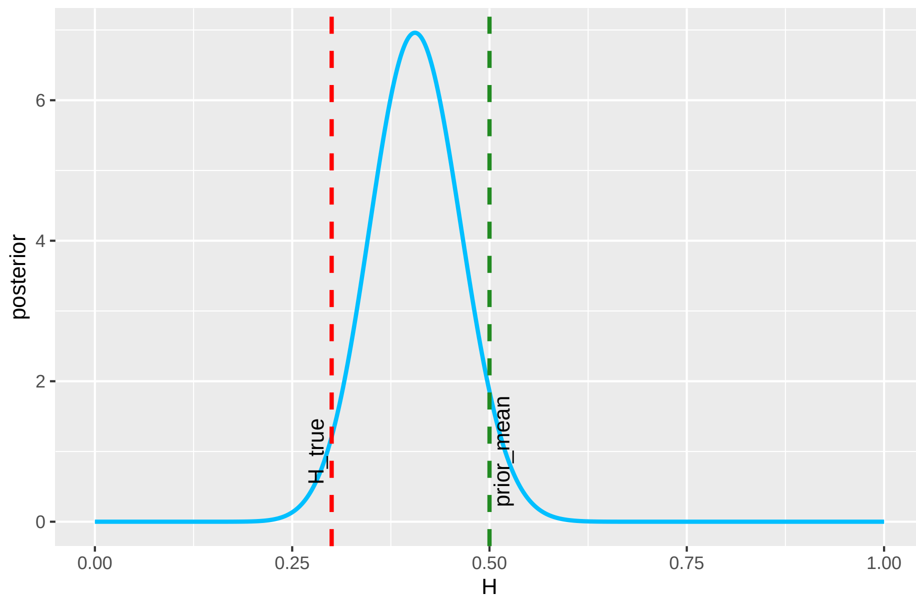
Let us now use a gaussian prior with $\mu = 0.5$, $\sigma = 0.1$:



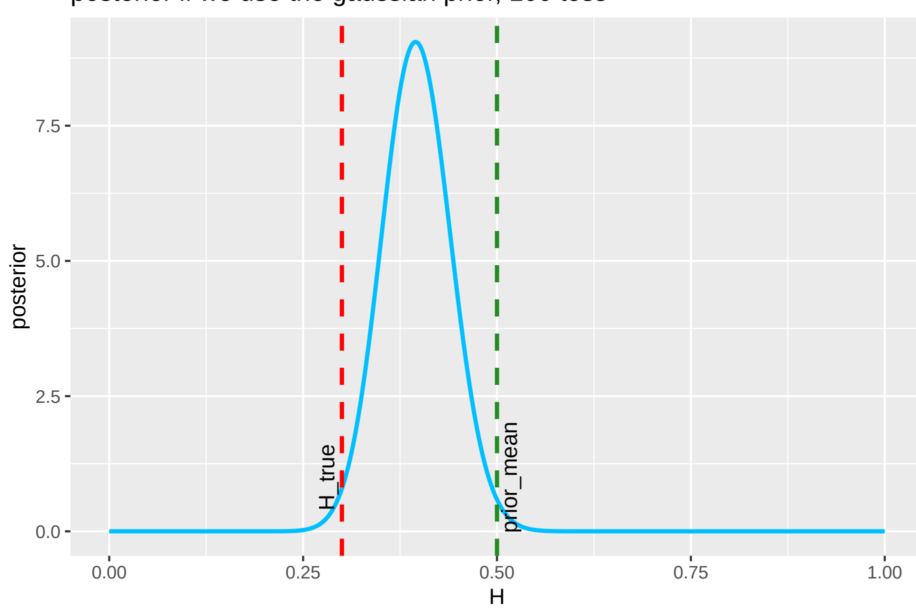
To obtain the posterior we multiply this function by the same binomial likelihood, then normalize the result.

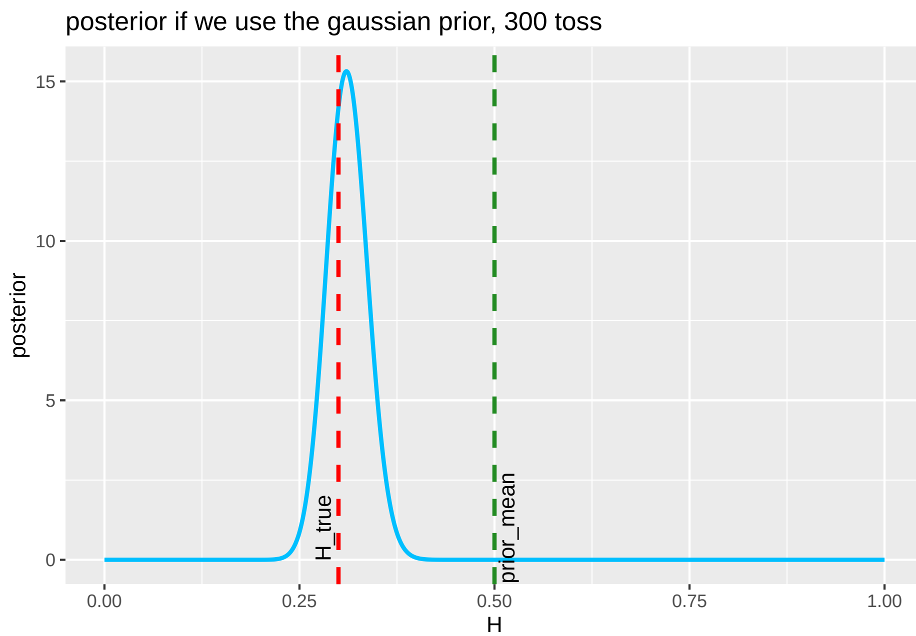


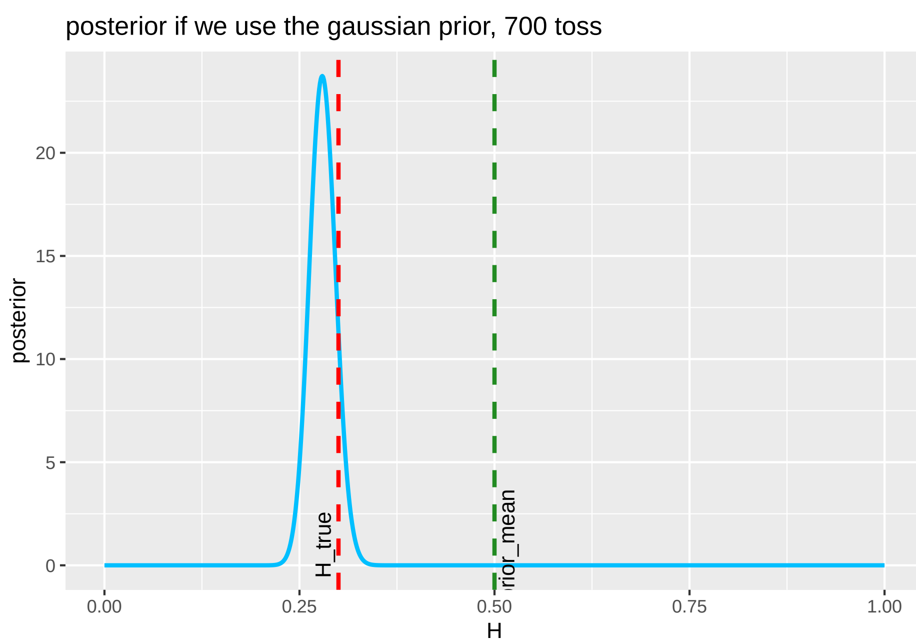
posterior if we use the gaussian prior, 50 toss

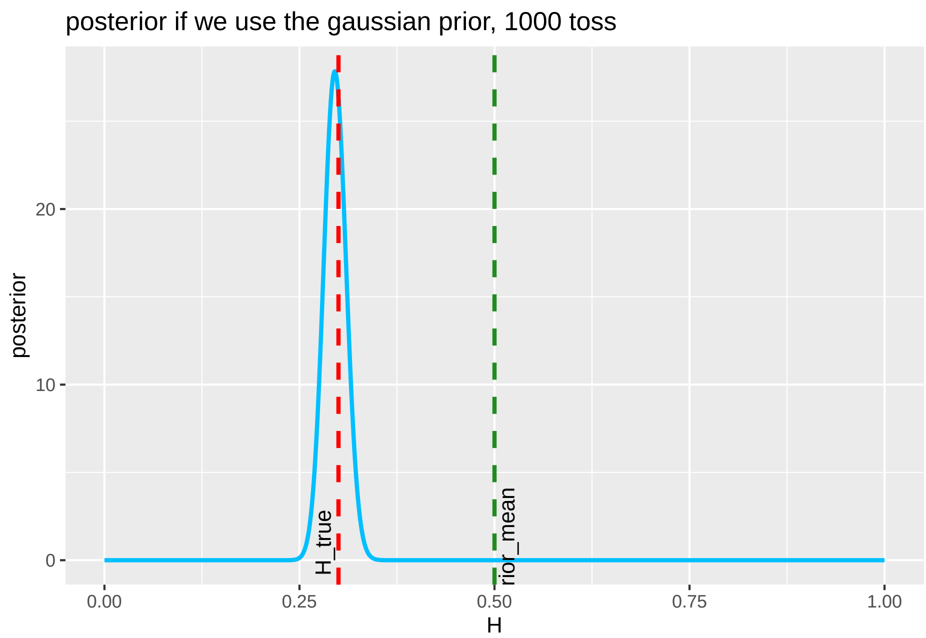


posterior if we use the gaussian prior, 100 toss



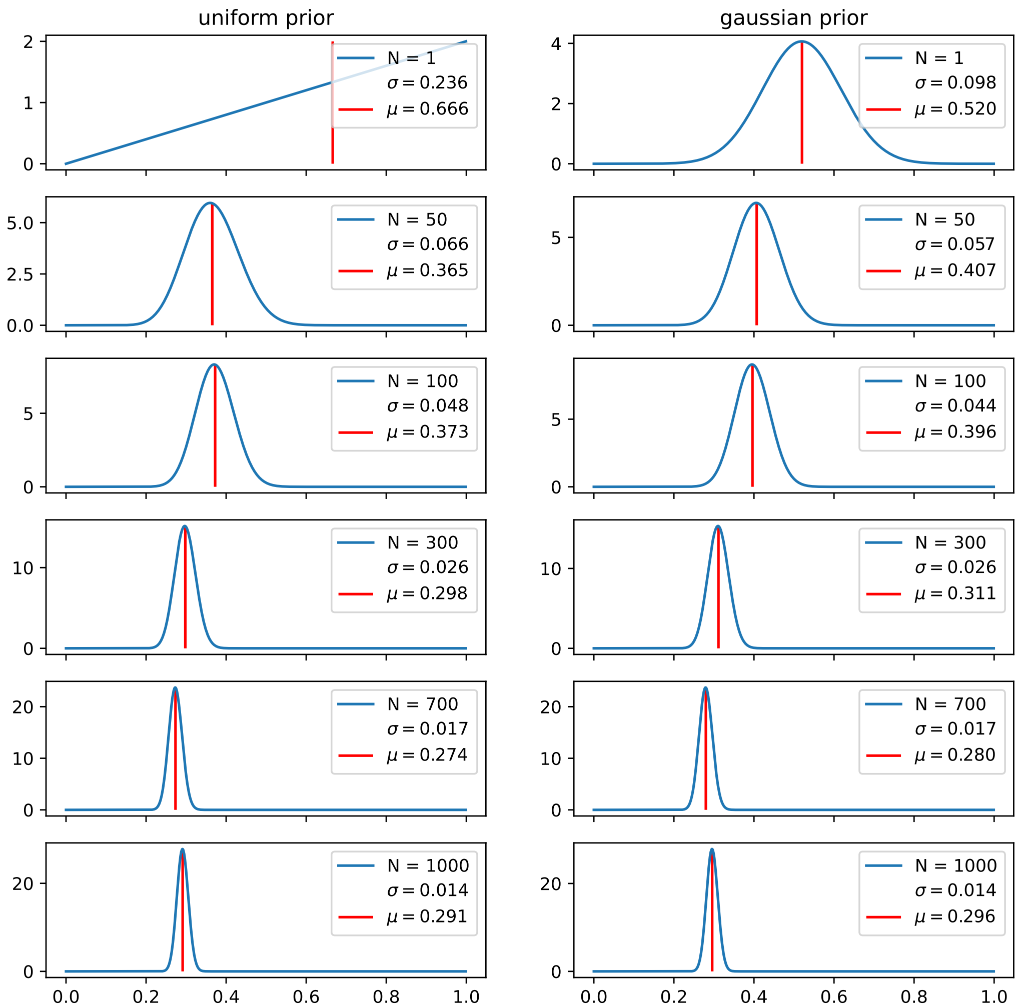

posterior if we use the gaussian prior, 300 toss
posterior
15
10
5
0
H_true
prior_mean
0.00
0.25
0.50
0.75
1.00
H


posterior if we use the gaussian prior, 700 toss
posterior
20
15
10
5
0
H_true
0.00
0.25
0.50
0.75
1.00
H



Comparison between the two priors

Having plotted both sequences of posteriors let's plot them together side by side as a means of comparing them.



The left column depicts the evolution of the posterior when we use the uniform prior; similarly the right one contains the gaussian prior-based posteriors. For each of these distribution the mean and variance values (evaluated with a simple numerical scheme) are reported; this is useful because the asymptotic distribution is guaranteed to be centered on 0.3 (the true value of H) with negligible variance *irrespective of the prior*.

We notice that if we use the uniform prior the posterior's mean converges a bit faster, which is an intuitive result: an uniformative prior immediately gives more importance to the data, whereas if we start with a prior peaked around a wrong value for H we will need more data to "fix" our distribution first. Therefore when we use the gaussian prior the process of learning consists of moving the mean to the left, then decreasing the width; with the other prior, instead, we can get closer to the correct mean sooner. Of course this is only a slight difference, since a lot of data is available almost immediately - and therefore convergence to the asymptotic distribution is reached easily in both cases.

Conclusion

We simulated a binomial process in order to study a simple but significant example of Bayesian parameter inference. In particular we were able to verify two important results, which are a general consequence of known theoretical results:

- starting with a “wrong” prior slows down convergence because some data will be spent initially to “fix” the distribution, whereas starting with a prior which is closer to the asymptotic distribution (or at least more similar to it than the available alternatives) speeds up convergence. Indeed we expect that if we modified the gaussian prior so that the mean was much closer to H_{true} we would reach convergence even faster.
- when the dataset is very large the likelihood dominates, in the sense that the final posterior is approximately insensitive to the chosen prior.