Quantum-IMD

Marco Lewis

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Contents

1 CNOT Circuit

theory *CNOT*
  imports
  Isabelle-Marries-Dirac.Basics
  Isabelle-Marries-Dirac.Quantum
  Isabelle-Marries-Dirac.More-Tensor
  Isabelle-Marries-Dirac.Deutsch
begin

abbreviation ψ_{00} :: *complex Matrix.mat* **where**
$\psi_{00} \equiv$ *mat-of-cols-list 4* [[*1*,*0*,*0*,*0*]]

abbreviation ψ_{10} :: *complex Matrix.mat* **where**
$\psi_{10} \equiv$ *mat-of-cols-list 4* [[*0*,*0*,*1*,*0*]]

abbreviation ψ_{11} :: *complex Matrix.mat* **where**
$\psi_{11} \equiv$ *mat-of-cols-list 4* [[*0*,*0*,*0*,*1*]]

lemma ψ_{00}*-is-zero-zero*:
  shows $\psi_{00} = |zero\rangle \bigotimes |zero\rangle$
proof
  show *dim-row* ψ_{00} = *dim-row* ($|zero\rangle \bigotimes |zero\rangle$)
    by (*simp add*: *mat-of-cols-list-def*)
  show *dim-col* ψ_{00} = *dim-col* ($|zero\rangle \bigotimes |zero\rangle$)
    by (*simp add*: *mat-of-cols-list-def*)
  fix *i j*:: *nat*

assume *i* < *dim-row* (|*zero*⟩ $\bigotimes$ |*zero*⟩)
  and *j* < *dim-col* (|*zero*⟩ $\bigotimes$ |*zero*⟩)
then have *i* ∈ {*0*,*1*,*2*,*3*} **and** *j* = *0*
  using *mat-of-cols-list-def* **by** *auto*
show ψ_{00} $$ (*i*,*j*) = (|*zero*⟩ $\bigotimes$ |*zero*⟩) $$ (*i*,*j*)
  using *ket-zero-is-state*
  by *auto*
qed

lemma ψ_{00}-*is-state*:
  shows *state 2* ψ_{00}
proof
  show *dim-col* ψ_{00} = *1*
    by (*simp add*: *mat-of-cols-list-def*)
  show *dim-row* ψ_{00} = 2^2
    by (*simp add*: *mat-of-cols-list-def*)
  have ‖*Matrix.col* |*zero*⟩ *0*‖ = *1*
    using *ket-zero-is-state state.is-normal* **by** *auto*
  thus ‖*Matrix.col* ψ_{00} *0*‖ = *1*
    using *state.is-normal tensor-state2* ψ_{00}-*is-zero-zero*
    ket-zero-is-state **by** *force*
qed

1.1 First Operation

abbreviation *X-on-ctrl* **where** *X-on-ctrl* ≡ (*X* $\bigotimes$ *Id 1*)

lemma *X-tensor-id*:
  defines *d*: *v* ≡ *mat-of-cols-list 4* [[*0*,*0*,*1*,*0*],
                                    [*0*,*0*,*0*,*1*],
                                    [*1*,*0*,*0*,*0*],
                                    [*0*,*1*,*0*,*0*]]
  shows *X-on-ctrl* = *v*
proof
  show *dim-col X-on-ctrl* = *dim-col v*
    by (*simp add*: *d X-def Id-def mat-of-cols-list-def*)
  show *dim-row X-on-ctrl* = *dim-row v*
    by (*simp add*: *d X-def Id-def mat-of-cols-list-def*)
  fix *i j*:: *nat* **assume** *i* < *dim-row v*
            and *j* < *dim-col v*
  then have *i* ∈ {*0*..<*4*} ∧ *j* ∈ {*0*..<*4*}
    by (*auto simp add*: *d mat-of-cols-list-def*)
  thus *X-on-ctrl* $$ (*i*, *j*) = *v* $$ (*i*, *j*)
    by (*auto simp add*: *d Id-def X-def*
      mat-of-cols-list-def)
qed

lemma *X-on-ctrl-is-gate*:
  shows *gate 2 X-on-ctrl*

proof
  show *unitary X-on-ctrl*
    using *X-is-gate id-is-gate gate-def tensor-gate*
    by *blast*
  show *square-mat X-on-ctrl*
    using *X-is-gate id-is-gate gate-def tensor-gate*
    by *blast*
  show *dim-row X-on-ctrl* = 2^2
    using *X-tensor-id* **by** (*simp add*: *mat-of-cols-list-def*)
qed

lemma ψ_{00}*-to-*ψ_{10}:
  shows $(X \bigotimes Id\ 1) * \psi_{00} = \psi_{10}$
proof
  fix *i j*:: *nat*
  assume $i <$ *dim-row* ψ_{10} **and** $j <$ *dim-col* ψ_{10}
  then have *a0*:$i \in \{0,1,2,3\} \wedge j = 0$
    by (*auto simp add*: *mat-of-cols-list-def*)
  then have $i <$ *dim-row* (*X-on-ctrl*) $\wedge\ j <$ *dim-col* ψ_{00}
    using *mat-of-cols-list-def X-tensor-id* **by** *auto*
  then have (*X-on-ctrl*$*\psi_{00}$) $$ (*i*,*j*)
      = ($\sum$ $k \in \{0\ ..<$ *dim-vec* $\psi_{00}\}$. (*Matrix.row* (*X-on-ctrl*) *i*) $ k $*$ (*Matrix.col* ψ_{00} *j*) $ *k*)
    by (*auto simp*: *times-mat-def scalar-prod-def*)
  thus (*X-on-ctrl* $* \psi_{00}$) $$ (*i*, *j*) = ψ_{10} $$ (*i*, *j*)
    using *mat-of-cols-list-def X-tensor-id a0*
    by (*auto simp*: *diff-divide-distrib*)
next
  show *dim-row* (*X-on-ctrl* $* \psi_{00}$) = *dim-row* ψ_{10}
    using *X-tensor-id mat-of-cols-list-def* **by** *simp*
  show *dim-col* (*X-on-ctrl* $* \psi_{00}$) = *dim-col* ψ_{10}
    using *X-tensor-id mat-of-cols-list-def* **by** *simp*
qed

lemma ψ_{10}*-is-state*:
  shows *state 2* ψ_{10}
  using *X-on-ctrl-is-gate* ψ_{00}*-is-state* ψ_{00}*-to-*ψ_{10}
  by (*metis gate-on-state-is-state*)

1.2 Second Operation

lemma ψ_{10}*-to-*ψ_{11}:
  shows *CNOT* $* \psi_{10} = \psi_{11}$
proof
  show *dim-row* (*CNOT* $* \psi_{10}$) = *dim-row* ψ_{11}
    by (*simp add*: *CNOT-def mat-of-cols-list-def*)
  show *dim-col* (*CNOT* $* \psi_{10}$) = *dim-col* ψ_{11}
    by (*simp add*: *CNOT-def mat-of-cols-list-def*)
  fix *i j*:: *nat*

assume $i < \textit{dim-row}\ \psi_{11}$ **and** $j < \textit{dim-col}\ \psi_{11}$
then have $\textit{asm}{:}i \in \{0,1,2,3\} \wedge j = 0$
by (*auto simp add*: *mat-of-cols-list-def*)
then have $i < \textit{dim-row}\ \textit{CNOT} \wedge j < \textit{dim-col}\ \psi_{10}$
by (*auto simp*: *mat-of-cols-list-def CNOT-def*)
then have $(\textit{CNOT} * \psi_{10})\ \$\$\ (i,j)$
$= (\sum\ k \in \{0\ ..< \textit{dim-vec}\ \psi_{10}\}.\ (\textit{Matrix.row}\ (\textit{CNOT})\ i)\ \$\ k * (\textit{Matrix.col}\ \psi_{10}\ j)\ \$\ k)$
by (*auto simp*: *times-mat-def scalar-prod-def*)
thus $(\textit{CNOT} * \psi_{10})\ \$\$\ (i,\ j) = \psi_{11}\ \$\$\ (i,\ j)$
using *mat-of-cols-list-def asm*
by (*auto simp add*: *CNOT-def*)
qed

lemma ψ_{11}-*is-state*:
shows *state 2* ψ_{11}
using *CNOT-is-gate* ψ_{10}-*is-state* ψ_{10}-*to*-ψ_{11}
by (*metis gate-on-state-is-state*)

1.3 Circuit

definition *circ*:: *complex Matrix.mat* **where**
$\textit{circ} \equiv \textit{CNOT} * ((\textit{X-on-ctrl}) * (\ |\textit{zero}\rangle \bigotimes |\textit{zero}\rangle))$

lemma *circ-result* [*simp*]:
shows $\textit{circ} = \psi_{11}$
using *circ-def* ψ_{00}-*is-zero-zero* ψ_{00}-*to*-ψ_{10} ψ_{10}-*to*-ψ_{11}
by *simp*

lemma *circ-res-is-state*:
shows *state 2 circ*
using ψ_{11}-*is-state* **by** *auto*

end