Quantum-IMD

Marco Lewis

August 18, 2021

Contents

1 CNOT 1

1 CNOT

theory *CNOT*
  imports
  Isabelle-Marries-Dirac.Basics
  Isabelle-Marries-Dirac.Quantum
  Isabelle-Marries-Dirac.More-Tensor
begin

abbreviation *zero* **where** *zero* ≡ *unit-vec 2 0*
abbreviation *one* **where** *one* ≡ *unit-vec 2 1*

lemma *ket-zero-is-state*:
  shows *state 1* |*zero*⟩
  by (*simp add*: *state-def ket-vec-def cpx-vec-length-def numerals*(*2*))

lemma *ket-one-is-state*:
  shows *state 1* |*one*⟩
  by (*simp add*: *state-def ket-vec-def cpx-vec-length-def numerals*(*2*))

lemma *ket-zero-to-mat-of-cols-list* [*simp*]: |*zero*⟩ = *mat-of-cols-list 2* [[*1*, *0*]]
  by (*auto simp add*: *ket-vec-def mat-of-cols-list-def*)

lemma *ket-one-to-mat-of-cols-list* [*simp*]: |*one*⟩ = *mat-of-cols-list 2* [[*0*, *1*]]
  apply (*auto simp add*: *ket-vec-def unit-vec-def mat-of-cols-list-def*)
  using *less-2-cases* **by** *fastforce*

abbreviation *one-zero* **where** *one-zero* ≡ *unit-vec 4 2*
abbreviation *one-one* **where** *one-one* ≡ *unit-vec 4 3*

abbreviation *ψ00* :: *complex Matrix.mat* **where**

$\psi 00 \equiv$ *mat-of-cols-list 4* $[[1,0,0,0]]$

abbreviation $\psi 10$ :: *complex Matrix.mat* **where**
$\psi 10 \equiv$ *mat-of-cols-list 4* $[[0,0,1,0]]$

abbreviation $\psi 11$ :: *complex Matrix.mat* **where**
$\psi 11 \equiv$ *mat-of-cols-list 4* $[[0,0,0,1]]$

abbreviation *X-on-ctrl* **where** *X-on-ctrl* $\equiv (X \bigotimes Id\ 1)$

lemma *X-tensor-id*:
  defines *d*: $v \equiv$ *mat-of-cols-list 4* $[[0,0,1,0],$
$[0,0,0,1],$
$[1,0,0,0],$
$[0,1,0,0]]$
  shows *X-on-ctrl* $= v$
proof
  show *dim-col X-on-ctrl* = *dim-col v*
    by (*simp add*: *d X-def Id-def mat-of-cols-list-def*)
  show *dim-row X-on-ctrl* = *dim-row v*
    by (*simp add*: *d X-def Id-def mat-of-cols-list-def*)
  fix *i j*:: *nat* **assume** $i <$ *dim-row v* **and** $j <$ *dim-col v*
  then have $i \in \{0..<4\} \wedge j \in \{0..<4\}$
    by (*auto simp add*: *d mat-of-cols-list-def*)
  thus *X-on-ctrl* $$ $(i, j) = v$ $$ (i, j)
    by (*auto simp add*: *d Id-def X-def mat-of-cols-list-def*)
qed

lemma *X-on-fst-is-gate*:
  shows *gate 2 X-on-ctrl*
proof
  show *unitary X-on-ctrl*
    using *X-is-gate id-is-gate gate-def tensor-gate*
    by *blast*
  show *square-mat X-on-ctrl*
    using *X-is-gate id-is-gate gate-def tensor-gate*
    by *blast*
  show *dim-row X-on-ctrl* $= 2^2$
    using *X-tensor-id* **by** (*simp add*: *mat-of-cols-list-def*)
qed

lemma $\psi 00$*-to-*$\psi 10$:
  shows $(X \bigotimes Id\ 1) * \psi 00 = \psi 10$
proof
  fix *i j*:: *nat*

```
  assume i < dim-row ψ10 and j < dim-col ψ10
  then have a0:i ∈ {0,1,2,3} ∧ j = 0 by (auto simp add: mat-of-cols-list-def)
  then have i < dim-row (X-on-ctrl) ∧ j < dim-col ψ00
    using mat-of-cols-list-def X-tensor-id by auto
  then have (X-on-ctrl∗ψ00) $$ (i,j)
       = (∑ k ∈ {0 ..< dim-vec ψ00}. (Matrix.row (X-on-ctrl) i) $ k ∗ (Matrix.col
ψ00 j) $ k)
    by (auto simp: times-mat-def scalar-prod-def)
  thus (X-on-ctrl ∗ ψ00) $$ (i, j) = ψ10 $$ (i, j)
    using  mat-of-cols-list-def X-tensor-id a0
    by (auto simp: diff-divide-distrib)
next
  show dim-row (X-on-ctrl ∗ ψ00) = dim-row ψ10
    using X-tensor-id mat-of-cols-list-def by simp
  show dim-col (X-on-ctrl ∗ ψ00) = dim-col ψ10
    using X-tensor-id mat-of-cols-list-def by simp
qed


lemma ψ10-to-ψ11:
  shows CNOT ∗ ψ10 = ψ11
proof
  show dim-row (CNOT ∗ ψ10) = dim-row ψ11
    by (simp add: CNOT-def mat-of-cols-list-def)
  show dim-col (CNOT ∗ ψ10) = dim-col ψ11
    by (simp add: CNOT-def mat-of-cols-list-def)
  fix i j:: nat
  assume i < dim-row ψ11 and j < dim-col ψ11
  then have asm:i ∈ {0,1,2,3} ∧ j = 0
    by (auto simp add: mat-of-cols-list-def)
  then have i < dim-row CNOT ∧ j < dim-col ψ10
    by (auto simp: mat-of-cols-list-def CNOT-def)
  then have (CNOT ∗ ψ10) $$ (i,j)
        = (∑ k ∈ {0 ..< dim-vec ψ10}. (Matrix.row (CNOT) i) $ k ∗ (Matrix.col
ψ10 j) $ k)
    by (auto simp: times-mat-def scalar-prod-def)
  thus (CNOT ∗ ψ10) $$ (i, j) = ψ11 $$ (i, j)
    using mat-of-cols-list-def asm
    by (auto simp add:  CNOT-def)
qed


definition circ:: complex Matrix.mat where
circ ≡ (CNOT) ∗ ((X-on-ctrl) ∗ ψ00)

lemma circ-result [simp]:
  shows circ = ψ11
  using circ-def ψ00-to-ψ10 ψ10-to-ψ11 by simp
```

end