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THE GARCH OPTION PRICING MODEL

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This article develops an option pricing model and its corresponding delta formula in the context of the generalized autoregressive conditional heteroskedastic (GARCH) asset return process. The development utilizes the locally risk-neutral valuation relationship (LRNVR). The LRNVR is shown to hold under certain combinations of preference and distribution assumptions. The GARCH option pricing model is capable of reflecting the changes in the conditional volatility of the underlying asset in a parsimonious manner. Numerical analyses suggest that the GARCH model may be able to explain some well-documented systematic biases associated with the Black-Scholes model.

KEY WORDS: GARCH process, heteroskedasticity, local risk neutralization, Black-Scholes model, pricing measure, option delta, risk premium, Monte Carlo simulation

1. INTRODUCTION

Following the seminal work of Black and Scholes (1973) and Merton (1973), the option literature has developed into an important area of research. The heteroskedasticity of assets returns has attracted considerable attention. The option pricing models that deal with heteroskedasticity include the constant-elasticity-of-variance model by Cox (1975), the jump-diffusion model by Merton (1976), the compound-option model by Geske (1979), and the displaced-diffusion model by Rubinstein (1983). As opposed to the aforementioned models, Hull and White (1987) proposed a bivariate diffusion model for pricing options on assets with stochastic volatilities. In their model, an exogenous process is assumed to govern the evolution of asset volatilities. Other stochastic volatility option models similar to that of Hull and White include Johnson and Shanno (1987), Scott (1987), Wiggins (1987), and Stein and Stein (1991). The bivariate diffusion option pricing models all require conditions stronger than no arbitrage. Empirically, these models face the difficulty that the variance rate is not observable.

This article develops a pricing model for options on an asset whose continuously compounded returns follow the generalized autoregressive conditional heteroskedastic (GARCH) process. The GARCH process of Bollerslev (1986) and its variants have gained increasing prominence for modeling financial time series in recent years.[2] This article is an attempt to link this powerful econometric model with the contingent pricing literature. The

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[2]The GARCH process is a generalized version of the ARCH by Engle (1982). The GARCH(1,1) model was independently proposed by Taylor (1986).

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GARCH option pricing model has three distinctive features. First, the GARCH option price is a function of the risk premium embedded in the underlying asset. This contrasts with the standard preference-free option pricing result. Second, the GARCH option pricing model is non-Markovian.[3] In the option pricing literature, the underlying asset value is usually assumed to follow a diffusion process. The standard approach is thus of the Markovian nature. Third, the GARCH option pricing model can potentially explain some well-documented systematic biases associated with the Black-Scholes model. These biases include underpricing of out-of-the-money options (see Black 1975; Gultekin et al. 1982), underpricing of options on low-volatility securities (see Black and Scholes 1972; Gultekin et al. 1982; Whaley 1982), underpricing of short-maturity option (see Black 1975; Whaley 1982), and the U-shaped implied volatility curve in relation to exercise price (see Rubinstein 1985; Sheikh 1991). The GARCH option pricing model also subsumes the Black-Scholes model because the homoskedastic asset return process is a special case of the GARCH model.

The GARCH process was used by Engle and Mustafa (1992) to study options and their implied conditional volatilities. The first attempt to provide a rigorous theoretical foundation for option pricing in the GARCH framework can be found in Duan (1990). In this early attempt, the risk-neutral valuation relation was incorrectly applied to option pricing in the GARCH framework. Subsequently, Satchell and Timmermann (1992) and Amin and Ng (1993) proposed option pricing models in the GARCH framework which yield results invalidating the risk-neutral valuation relationship.[4] Both models assume joint log normality for the asset return and the state price density. In contrast to the standard result, the option price becomes a function of the expected return on the underlying asset.

The development of the GARCH option pricing model in this article differs from the previous models by exploring the extension of the risk neutralization in Rubinstein (1976) and Brennan (1979). Due to the complex nature of the GARCH process, a generalized version of risk neutralization, referred to as the locally risk-neutral valuation relationship (LRNVR), is called for. The LRNVR differs from its conventional counterpart in the aspect of variances. The LRNVR stipulates that the one-period ahead conditional variance is invariant with respect to a change to the risk-neutralized pricing measure. This is important because, in the context of the GARCH process, the unconditional variance or any conditional variance beyond one period is not invariant to the change in measures caused by risk neutralization. It is shown that the LRNVR holds under some familiar combinations of preference and distribution assumptions. With the LRNVR, the asset return process under the risk-neutralized pricing measure differs from the conventional GARCH process in an interesting way. The conditional variance process in the GARCH model of Bollerslev (1986) is known to be governed by the chi-square innovation. Local risk neutralization alters the conditional variance process. Under the risk-neutralized pricing measure, the conditional variance process continues to have the same form except the innovation is governed by the noncentral chi-square random variable and the noncentrality parameter equals the unit risk premium for the underlying asset.

The remainder of the article is organized as follows: Section 2 provides a description of the GARCH logarithmic asset returns and the definition of the LRNVR. Three sufficient

[3] The only Markovian GARCH process is GARCH(0,1) or ARCH(1). It is well known that a non-Markovian univariate process can be converted into a Markovian vector process through a change in dimension. The statement should thus be understood with the usual interpretation.

[4] Satchell and Timmermann's (1992) model was developed independent of Duan (1990). Amin and Ng (1993), on the other hand, developed their model along the line of Duan (1990) and pointed out a critical error in Theorem 2 of Duan (1990) and some subsequent versions.

conditions under which the LRNVR holds are laid out. The asset return process under the risk-neutralized pricing measure is also derived in this section. In Section 3, some general comparisons are made between the GARCH option pricing model and the Black-Scholes formula. A Monte Carlo simulation method is used in Section 4 to numerically examine the GARCH option pricing model. Section 5 concludes.

2. THE GARCH OPTION PRICING MODEL

Consider a discrete-time economy and let X_t be the asset price at time t. Its one-period rate of return is assumed to be conditionally lognormally distributed under probability measure P. That is,

$$\ell n \frac{X_t}{X_{t-1}} = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \varepsilon_t, \tag{2.1}$$

where ε_t has mean zero and conditional variance h_t under measure P; r is the constant one-period risk-free rate of return (continuously compounded) and λ the constant unit risk premium. Under conditional lognormality, one plus the conditionally expected rate of return equals $\exp(r + \lambda\sqrt{h_t})$. It thus suggests that λ can be interpreted as the unit risk premium.

We further assume that ε_t follows a GARCH(p, q) process of Bollerslev (1986) under measure P. Formally,

$$\varepsilon_t \mid \phi_{t-1} \sim N(0, h_t) \qquad \text{under measure } P. \tag{2.2}$$

$$h_t = \alpha_0 + \textstyle\sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i},$$

where ϕ_t is the information set (σ-field) of all information up to and including time t; $p \geq 0$, $q \geq 0$; $\alpha_0 > 0$, $\alpha_i \geq 0$, $i = 1, \ldots, q$; $\beta_i \geq 0$, $i = 1, \ldots, p$. In words, the conditional variance is a linear function of the past squared disturbances and the past conditional variances. Clearly, h_t is ϕ_t-predictable. The option pricing results to be developed in this section rely on conditional normality. Using an alternative specification for h_t such as the EGARCH of Nelson (1991) or that of Glosten et al. (1993) will not change the basic option pricing results as long as conditional normality remains in place.

To ensure covariance stationarity of the GARCH(p, q) process, $\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i$ is assumed to be less than 1.[5] The GARCH process specified in (2.1) and (2.2) reduces to the standard homoskedastic lognormal process in the Black-Scholes model if $p = 0$ and $q = 0$. This ensures that the Black-Scholes model is a special case.

In order to develop the GARCH option pricing model, the conventional risk-neutral valuation relationship has to be generalized to accommodate heteroskedasticity of the asset return process. We thus introduce a generalized version of this principle below.

DEFINITION 2.1. A pricing measure Q is said to satisfy the *locally risk-neutral valuation relationship* (LRNVR) if measure Q is mutually absolutely continuous with respect to

[5] See Theorem 1 of Bollerslev (1986). If the sum equals 1 in the case of the GARCH(1,1) process, the process is referred to as the integrated GARCH process. Nelson (1990) showed that the IGARCH process is stationary and ergodic although the variance is unbounded. In fact, the GARCH process may still be strictly stationary if the sum of the α_i's and β_i's is greater than 1. The sufficient and necessary condition for its strict stationarity is related to the top Lyapunov exponent of a particular sequence of random matrices (see Bougerol and Picard 1992).

measure P, $X_t/X_{t-1} \mid \phi_{t-1}$ distributes lognormally (under Q),

$$E^Q(X_t/X_{t-1} \mid \phi_{t-1}) = e^r,$$

and

$$\operatorname{Var}^Q(\ell n(X_t/X_{t-1}) \mid \phi_{t-1}) = \operatorname{Var}^P(\ell n(X_t/X_{t-1}) \mid \phi_{t-1})$$

almost surely with respect to measure P.

In the above definition of the LRNVR, the conditional variances under the two measures are required to be equal. This is desirable because one can observe and hence estimate the conditional variance under P. This property and the fact that the conditional mean can be replaced by the risk-free rate yield a well-specified model that does not locally depend on preferences. Local risk neutralization is, however, insufficient for eliminating the preference parameters. Under our model setup, it is nevertheless strong enough to reduce all preference consideration to the unit risk premium, λ. This assertion will be proved later in Theorem 2.2. In the definition, the conditional variance equality is an almost sure relationship. Since Q is absolutely continuous with respect to P, the almost sure relationship under P also holds true under Q. In the case of a homoskedastic lognormal process, i.e., $p = 0$ and $q = 0$, the conditional variances become the same constant and the LRNVR reduces to the conventional risk-neutral valuation relationship.

Rubinstein (1976) and Brennan (1979) proved that, under some combinations of preferences and distributions, the risk-neutral valuation relationship holds. In the following theorem, we prove the validity of the LRNVR under similar conditions.

THEOREM 2.1. *If the representative agent is an expected utility maximizer and the utility function is time separable and additive, then the LRNVR holds under any of the following three conditions:*

(i) The utility function is of constant relative risk aversion and changes in the logarithmic aggregate consumption are distributed normally with constant mean and variance under measure P

(ii) The utility function is of constant absolute risk aversion and changes in the aggregate consumption are distributed normally with constant mean and variance under measure P

(iii) The utility function is linear.

Proof. See Appendix. □

The constant mean and variance assumption for the aggregate consumption process in (i) and (ii) of Theorem 2.1 ensures that the implied interest rate is constant. Hence this guarantees the consistency with the constant interest rate assumption made earlier. Although it is possible to develop the model with stochastic interest rates, the resulting model will become considerably more complicated. The constant interest rate assumption allows for the comparison with the Black-Scholes model solely in the dimension of heteroskedasticity. It is worth noting that the condition in (ii) of Theorem 2.1 permits the aggregate consumption to become negative.

The implication of the LRNVR is presented in the following theorem.[6]

THEOREM 2.2. *The LRNVR implies that, under pricing measure Q,*

$$\ell n \frac{X_t}{X_{t-1}} = r - \frac{1}{2}h_t + \xi_t, \tag{2.3}$$

where

$$\xi_t \mid \phi_{t-1} \sim N(0, h_t)$$

and

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \left(\xi_{t-i} - \lambda\sqrt{h_{t-i}}\right)^2 + \sum_{i=1}^{p} \beta_i h_{t-i}.$$

Proof. See Appendix. □

Theorem 2.2 implies that the form of the GARCH(p, q) process remains largely intact with respect to local risk neutralization. The conditional variance process under the risk-neutralized pricing measure, is not a GARCH process. The variance innovation is governed by q noncentral chi-square random variables with one degree of freedom, whereas the GARCH process under P can be seen as the process governed by q central chi-square innovations. The common noncentrality parameter for the conditional variance process under Q is the unit risk premium λ. To see this, one needs to factor out $\sqrt{h_{t-i}}$ from the terms inside the parentheses and recognize that $\xi_{t-i}/\sqrt{h_{t-i}}$ is a standard normal random variable under measure Q. Theorem 2.2 suggests that the unit risk premium, λ, influences the conditional variance process globally although the risk has been locally neutralized under the pricing measure Q. In other words, local risk neutralization is not equivalent to global risk neutralization. The need to distinguish between local and global risk neutralizations disappears when the coefficients governing the variance innovation equal zero.

If an alternative GARCH specification for asset returns is chosen, say the EGARCH process, a result similar to Theorem 2.2 can immediately be obtained. Whenever the variable ε_t appears in the conditional variance equation, it should be replaced by $\xi_t - \lambda\sqrt{h_t}$ with everything else remaining unchanged. Although the pricing results are tailored specifically for the GARCH process of Bollerslev (1986), they are applicable, after minor modifications, to other asset return specifications.

Pricing contingent payoffs requires temporally aggregating one-period asset returns to arrive at a random terminal asset price at some future point in time. The terminal asset price is derived in the following corollary:

COROLLARY 2.1.

$$X_T = X_t \exp\left[(T-t)r - \frac{1}{2}\sum_{s=t+1}^{T} h_s + \sum_{s=t+1}^{T} \xi_s\right]. \tag{2.4}$$

[6]The author is grateful to an anonymous reviewer for pointing out an error in an earlier version of this article. The same error was also noted in Amin and Ng (1993).

Proof. Follows immediately from Theorem 2.2. □

The asset price, discounted at the risk-free rate, possesses the martingale property. The importance of the martingale property for the theory of contingent claim pricing was first established by Harrison and Kreps (1979) and later elaborated by Harrison and Pliska (1982).

COROLLARY 2.2. *The discount asset price process $e^{-rt}X_t$ is a Q-martingale.*

Proof. See Appendix. □

Under the GARCH(p, q) specification, a European call option with exercise price K maturing at time T has the time-t value equal to

COROLLARY 2.3.

$$C_t^{\mathrm{GH}} = e^{-(T-t)r} E^Q \left[\max(X_T - K, 0) \mid \phi_t\right]. \tag{2.5}$$

Under the GARCH(p, q) specification, ϕ_t is the σ-field generated by $\{X_t, \varepsilon_t, \ldots, \varepsilon_{t-q+1}, h_t, \ldots, h_{t-p+1}\}$. A substantial simplification of the information set can be obtained if one restricts the model to the popular GARCH(1, 1) specification. For the GARCH(1, 1) model, X_t and h_{t+1} together serve as the sufficient statistics for ϕ_t. In other words, the GARCH(1, 1) is not a univariate Markov process, but can be stated as a bivariate Markov process. The GARCH(1, 1) option pricing model explicitly reflects the state of the underlying asset price in two dimensions: price level and conditional volatility. This added dimension enables the model price to reflect high or low variance of the underlying asset when the state of the economy changes.

One important use of the option pricing model is delta hedging. The delta of an option is the first partial derivative of the option price with respect to the underlying asset price. To use such a technique in the GARCH framework, one must first derive the corresponding delta formula. Denote the GARCH option delta at time t by Δ_t^{GH}. The following result is in order:

COROLLARY 2.4.

$$\Delta_t^{\mathrm{GH}} = e^{-(T-t)r} E^Q \left[\frac{X_T}{X_t} 1_{\{X_T \geq K\}} \mid \phi_t\right], \tag{2.6}$$

where $1_{\{X_T \geq K\}}$ is an indicator function.

Proof. See Appendix. □

For a European put, its price and delta can be derived using the put-call parity relationship.

It can be shown that the GARCH option price and delta reduce to their Black-Scholes counterparts when the underlying process is homoskedastic. The analytic solution for the GARCH option price in (2.5) or delta in (2.6) is not available because the conditional distribution over more than one period cannot be analytically derived. A control-variate Monte Carlo simulation method can nevertheless be used to compute the GARCH option price and delta.

3. COMPARISON OF THE GARCH(1,1) OPTION PRICING MODEL AND THE BLACK-SCHOLES MODEL

In this section, we compare the Black-Scholes model with the GARCH option pricing model. Since the GARCH(1, 1) model is the most commonly used GARCH process, our discussion for the remainder of this article will be restricted to the GARCH(1, 1) model.

Although the homoskedastic process used by the Black-Scholes model is a special case of the GARCH process, the interpretation of the Black-Scholes model in the GARCH framework is considerably more complicated. Under the incorrect assumption of homoskedasticity when the real governing process is heteroskedastic, risk neutralization must be of a global nature to maintain model consistency. This incorrect assumption thus requires the asset volatility to remain unchanged with respect to risk neutralization. Using the Black-Scholes model when the underlying process follows a GARCH process should therefore be regarded as employing the stationary variance of the GARCH asset return process in the Black-Scholes formula.

The following theorem characterizes some properties of the GARCH process after local risk neutralization. It suggests that a correct use of local risk neutralization will alter some key characteristics of the GARCH process.

THEOREM 3.1. *Under pricing measure Q, if $|\lambda| < \sqrt{(1-\alpha_1-\beta_1)/\alpha_1}$, then*

(i) *The stationary variance of ξ_t equals $\alpha_0\left[1-(1+\lambda^2)\alpha_1-\beta_1\right]^{-1}$*
(ii) *ξ_t is leptokurtic*
(iii) $\mathrm{Cov}^Q(\xi_t/\sqrt{h_t}, h_{t+1}) = -2\lambda\alpha_0\alpha_1\left[1-(1+\lambda^2)\alpha_1-\beta_1\right]^{-1}$.

Proof. See Appendix. □

As stated earlier, the stationary variance of the GARCH return process, under the original probability measure P, is $\alpha_0(1-\alpha_1-\beta_1)^{-1}$. It is also true that the conditional variance is uncorrelated with the lagged asset return under measure P. By Theorem 3.1, local risk neutralization induces an increase in the stationary variance to $\alpha_0[1-\alpha_1(1+\lambda^2)-\beta_1]^{-1}$. It also causes the conditional variance to be negatively (positively) correlated with the lagged asset return if the risk premium λ is positive (negative).

As stated earlier, the Black-Scholes option price in the GARCH framework should be interpreted as using an incorrect assumption of homoskedasticity and hence an incorrect unconditional standard deviation for the risk-neutralized asset return process. Specifically, the Black-Scholes call option price and delta can be written as

$$C_t^{BS} = X_t N(d_t) - e^{-(T-t)r} K N(d_t - \sigma\sqrt{T-t}), \tag{3.1}$$

$$\Delta_t^{BS} = N(d_t), \tag{3.2}$$

where

$$d_t = \frac{\ell n(X_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \qquad \text{and} \qquad \sigma^2 = \alpha_0(1-\alpha_1-\beta_1)^{-1}.$$

Loosely speaking, the difference between the GARCH option price and its Black-Scholes counterpart can be regarded as using two different levels of asset return volatility. A closer examination of these two models reveals something that is considerably more complicated. The initial conditional variance for a GARCH process is unlikely to be equal to the unconditional variance σ^2. Even if the initial conditional variance is set at the unconditional variance of the original asset return process, local risk neutralization implies that the conditional variance under pricing measure Q will revert to an unconditional variance higher than σ^2. Since the conditional volatility is negatively (or positively) correlated with the lagged asset return, the reverting behavior is also different from the one usually associated with the standard GARCH model.

Since the risk-neutralized GARCH process is still leptokurtic, it will be more likely for an out-of-the-money option to finish in-the-money. This implies that the GARCH option price will be higher than its Black-Scholes counterpart. Leptokurtosis also implies that an in-the-money option is more likely to finish out-of-the-money. This does not, however, imply lower GARCH option prices for in-the-money options when compared to their Black-Scholes counterparts. This is true because there must be an offsetting increase in probability at the higher value end to make in-the-money options attain even higher values. Other than these general observations, the comparison between these two option pricing models can only be made numerically.

4. NUMERICAL RESULTS AND DISCUSSION

Monte Carlo simulation is used in the computation of the GARCH option prices and deltas. Use of the Monte Carlo method to compute option prices can be traced back to Boyle (1977). It is a convenient method for the GARCH option pricing model because the distribution for the temporally aggregated asset return cannot be derived analytically. To simulate the risk-neutralized GARCH(1, 1) asset returns for pricing options at time t, we recognize that the asset price X_t and the conditional volatility h_{t+1} can together serve as sufficient statistics. A control-variate technique is employed to improve the efficiency of the GARCH option price and delta estimators. The control variables are the Black-Scholes price and delta described in the preceding section.

The general characteristics of the GARCH option pricing model in comparison to the Black-Scholes formula are reflected in Tables 4.1 and 4.2. The options used to generate these tables are the S&P 100 index calls of European style.[7] The S&P 100 index options are known to be most actively traded contracts. The GARCH-M model specified in (2.1) and (2.2) with $p = 1$ and $q = 1$ is fitted to the S&P 100 daily index series from January 2, 1986 to December 15, 1989. The estimated parameter values are $\alpha_0 = 1.524 \times 10^{-5}$,

[7] Strictly speaking, the GARCH option pricing model with the assumptions made in the article to ensure a constant interest rate can only be used for the valuation of individual equity options. The market portfolio (the S&P 100 index) is expected to be highly correlated with aggregate consumption. The market portfolio's return will therefore not follow a GARCH process. The numerical results in this section should therefore be interpreted as if the estimated parameter values are for individual stocks. The author appreciates this comment made by an anonymous reviewer.

TABLE 4.1
S&P100 Index Call Option Pricing Biases as a Percentage of the Black-Scholes Price for Different Maturities, Exercise Prices and Conditional Volatilities.[a]

Maturity	s/x	B-S	GARCH								
			$\sqrt{h_1}/\sigma = 0.8$			$\sqrt{h_1}/\sigma = 1.0$			$\sqrt{h_1}/\sigma = 1.2$		
		Price	Price	% bias	s.d.	Price	% bias	s.d.	Price	% bias	s.d.
	.80	.1027	.6892	571.2200	107.9300	.9495	824.7200	115.4700	1.6164	1474.2000	142.6000
	.90	18.2380	16.4340	−9.8893	1.6291	20.9300	14.7600	1.7680	26.9050	47.5210	2.0957
	.95	89.7900	75.4490	−15.9710	.4722	86.0280	−4.1893	.5102	99.2440	10.5290	.5855
$T = 30$	1.00	276.1100	251.5200	−8.9036	.1916	266.7500	−3.3897	.2058	284.3400	2.9828	.2339
	1.05	600.4100	583.9500	−2.7423	.1047	596.1300	−.7131	.1106	610.4400	1.6694	.1251
	1.10	1027.9000	1023.6000	−.4244	.0697	1030.2000	.2164	.0730	1037.9000	.9668	.0826
	1.20	2001.0000	2002.0000	.0518	.0417	2003.5000	.1240	.0442	2004.2000	.1628	.0498
	.80	13.0160	14.3700	10.4070	2.9874	15.7590	21.0770	3.0099	18.3170	40.7310	3.6271
	.90	118.5400	109.5700	−7.5685	.6037	116.0600	−2.0913	.6240	123.3200	4.0344	.6763
	.95	257.7900	241.3500	−6.3774	.3419	251.0200	−2.6268	.3533	259.9500	.8347	.3802
$T = 90$	1.00	478.0000	458.2200	−4.1371	.2180	468.9000	−1.9038	.2242	477.8600	−.0291	.2403
	1.05	779.6800	761.8800	−2.2836	.1531	772.3500	−.9402	.1572	780.4200	.0950	.1676
	1.10	1152.2000	1140.0000	−1.0631	.1159	1149.4000	−.2502	.1185	1155.6000	.2900	.1260
	1.20	2036.6000	2034.4000	−.1080	.0775	2040.5000	.1955	.0788	2042.3000	.2836	.0837
	.80	66.4460	65.8050	−.9638	1.1575	68.3570	2.8763	1.1727	71.6500	7.8323	1.2052
	.90	261.3000	252.5100	−3.3625	.4560	257.0900	−1.6128	.4642	264.6500	1.2818	.4730
	.95	438.1700	425.4700	−2.8984	.3213	431.7000	−1.4777	.3260	440.5700	.5483	.3332
$T = 180$	1.00	675.5400	661.5100	−2.0766	.2389	668.5000	−1.0419	.2420	677.9400	.3553	.2481
	1.05	970.4900	957.8900	−1.2987	.1866	964.2900	−.6391	.1886	973.6500	.3257	.1936
	1.10	1318.0000	1308.9000	−.6910	.1515	1313.9000	−.3147	.1526	1322.5000	.3437	.1567
	1.20	2133.4000	2131.7000	−.0764	.1088	2134.7000	.0637	.1098	2139.7000	.2959	.1128

[a] The riskless rate is fixed at 0%. The results are obtained using 50,000 Monte Carlo simulation runs. The values under "s.d." are the standard deviations of percentage biases from Monte Carlo simulation. Prices are reported as 10,000 times.

TABLE 4.2
S&P100 Index Call Option Delta Biases as a Percentage of the Black-Scholes Delta for Different Maturities, Exercise Prices and Conditional Volatilities.[a]

		B-S	GARCH								
			$\sqrt{h_1}/\sigma = 0.8$			$\sqrt{h_1}/\sigma = 1.0$			$\sqrt{h_1}/\sigma = 1.2$		
Maturity	s/x	Delta	Delta	% bias	s.d.	Delta	% bias	s.d.	Delta	% bias	s.d.
	.80	.0007	.0020	184.3500	29.0140	.0027	272.3800	33.5170	.0046	539.6500	44.5830
	.90	.0684	.0530	−22.4570	1.0397	.0622	−9.0391	1.0702	.0750	9.7047	1.1575
	.95	.2399	.2089	−12.8980	.4158	.2210	−7.8541	.4050	.2372	−1.1100	.4154
$T = 30$	1.00	.5138	.5130	−.1601	.1868	.5132	−.1153	.1899	.5170	.6285	.2029
	1.05	.7703	.7975	3.5301	.1240	.7874	2.2281	.1213	.7745	.5491	.1250
	1.10	.9211	.9385	1.8921	.0762	.9288	.8464	.0772	.9177	−.3643	.0801
	1.20	.9962	.9955	−.0721	.0260	.9937	−.2514	.0307	.9914	−.4832	.0341
	.80	.0358	.0334	−6.5577	1.8009	.0362	1.0530	1.8323	.0401	12.0250	1.8802
	.90	.2063	.1905	−7.6792	.5089	.1966	−4.7120	.5061	.2022	−1.9943	.5184
	.95	.3564	.3443	−3.4015	.3156	.3491	−2.0481	.3184	.3522	−1.1825	.3278
$T = 90$	1.00	.5239	.5227	−.2320	.2155	.5236	−.0472	.2171	.5213	−.4955	.2196
	1.05	.6798	.6880	1.2196	.1560	.6860	.9132	.1570	.6844	.6841	.1616
	1.10	.8038	.8161	1.5333	.1192	.8131	1.1571	.1192	.8081	.5426	.1208
	1.20	.9431	.9492	.6469	.0683	.9448	.1824	.0700	.9426	−.0550	.0714
	.80	.1090	.1035	−5.0268	.8782	.1026	−5.8582	.8926	.1083	−.6324	.9165
	.90	.2957	.2862	−3.2014	.4103	.2904	−1.7819	.4071	.2928	−.9889	.4123
	.95	.4137	.4089	−1.1624	.2959	.4114	−.5567	.2970	.4144	.1457	.3004
$T = 180$	1.00	.5338	.5354	.3002	.2267	.5343	.0881	.2270	.5345	.1298	.2311
	1.05	.6453	.6505	.8056	.1805	.6485	.5044	.1772	.6472	.2994	.1794
	1.10	.7412	.7489	1.0406	.1424	.7472	.8101	.1418	.7442	.3999	.1433
	1.20	.8771	.8844	.8278	.0954	.8819	.5436	.0964	.8787	.1785	.0981

[a]The riskless rate is fixed at 0%. The results are obtained using 50,000 Monte Carlo simulation runs. The values under "s.d." are the standard deviations of percentage biases from Monte Carlo simulation.

$\alpha_1 = 0.1883$, $\beta_1 = 0.7162$ and $\lambda = 7.452 \times 10^{-3}$, respectively. The GARCH parameter values together imply that the annualized (based on 365 days) stationary standard deviation is 24.13%. The risk-free rate is fixed at 0% to make easier the interpretation of in-the-money or out-of-the-money options. The exercise price is set at \$1. The relative index values, denoted by s/x, range from 0.8 to 1.2. Fifty thousand simulation runs are carried out for a given set of GARCH option prices corresponding to different s/x ratios.

The GARCH option pricing model provides an opportunity for examining the impacts of the initial conditional variance on the price of an option. Intuitively, if the conditional volatility of the underlying asset at the time of option valuation is high (low), all options written on this asset should be relatively more (less) valuable. With the conditional variance incorporated, the GARCH option pricing model is expected to generate an important flexibility for valuing options in a frequently changing market. To shed light on the effect of conditional volatility, three levels of initial conditional standard deviations are studied. They are the stationary level, 20% below the stationary level, and 20% above, respectively.

The Black-Scholes model is known to underprice out-of-money options (see Black 1975; Gultekin et al. 1982). As compared to the GARCH option prices, Table 4.1 reveals that the Black-Scholes model can underprice or overprice an out-of-the-money option depending on the level of its initial conditional volatility. For deep out-of-the-money options, the Black-Scholes always underprices. Since the GARCH conditional variance process is known to generate low-variance states more frequently, underpricing by the Black-Scholes model is expected to be the norm rather than the exception. The Black-Scholes model is also known to underprice short-maturity options (see Black 1975; Whaley 1982). For out-of-the-money options, one can see this pattern present in Table 4.1. Moreover, the underpricing of the Black-Scholes model for out-of-the money options becomes more pronounced when the maturity of an option is shortened. The comparison of the GARCH option delta with its Black-Scholes counterpart is presented in Table 4.2. The results, by and large, exhibit patterns similar to the price comparison.

Rubinstein (1985) documented empirically that implied volatilities of traded options exhibit a systematic pattern with respect to different maturities and exercise prices. Sheikh (1991) analyzed S&P 100 index calls and arrived at a conclusion similar to that of Rubinstein. These findings suggest a U-shape implied volatility phenomenon, when the Black-Scholes model is used to invert the market prices of traded options. In Figures 4.1 and 4.2, we use the Black-Scholes formula to invert the GARCH option prices. Three implied volatility graphs corresponding to three different option maturities are plotted. The implied volatilities produced in this way also exhibit a U-shaped pattern.

Rubinstein (1985) and Sheikh (1991) also analyzed the effect of the time to maturity on the implied volatility relationship. They reported that, for at-the-money calls, the longer the time to maturity, the higher is the option's implied volatility. They also reported that a reversal occurs over a different time period. Figure 4.1 presents the results for the case of low initial conditional volatility, i.e., 20% below the stationary level. The implied volatility for the shortest-maturity option is clearly the lowest. When the initial conditional volatility is high, i.e., 20% above the stationary level, the resulting pattern is different and is shown in Figure 4.2. The implied volatility of the shortest-maturity option is now the highest for the at-the-money option. Since the estimated parameter values imply a model that is close to the integrated GARCH asset return process, the conditional volatility must exhibit a strong clustering phenomenon. The GARCH conditional variance process is also known to be asymmetrical and skewed to the lower end. These two facts together imply that the likelihood of observing an extended low-variance state is higher than others. For at-the-

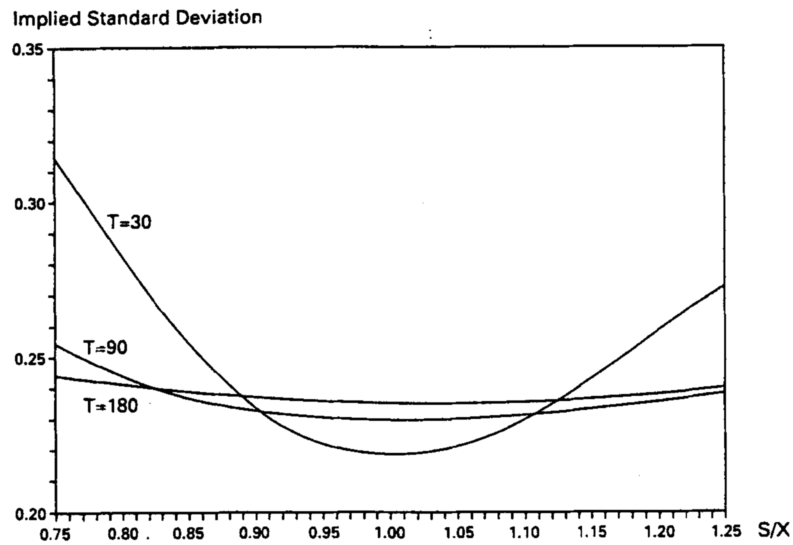

FIGURE 4.1. Low Conditional Volatility and its Effect on the Annualized Implied Volatility of the GARCH Option Price

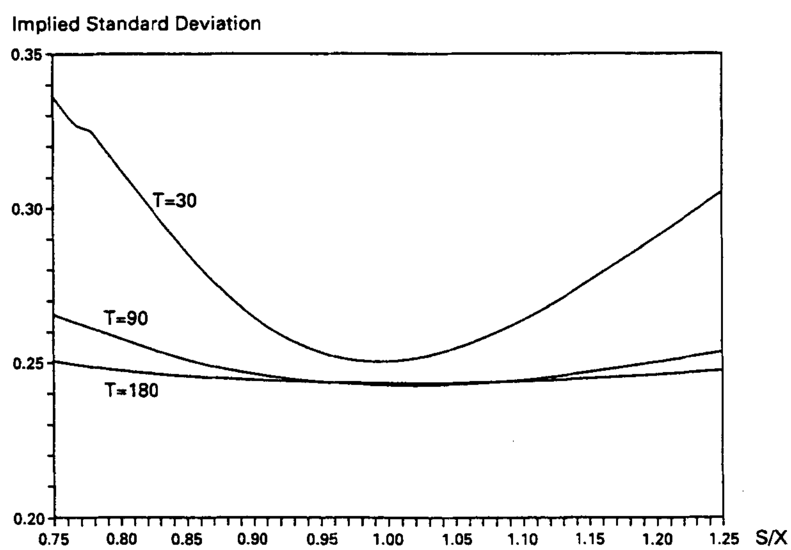

FIGURE 4.2. High Conditional Volatility and its Effect on the Annualized Implied Volatility of the GARCH Option Price

money calls, the longer-maturity option should frequently have a higher implied volatility with occasional periods of reversal. The reversal phenomenon reported by Rubinstein (1985) and Sheikh (1991) is therefore not too surprising, and indeed lends support to the GARCH option pricing model.

5. CONCLUSION

This article develops a GARCH option pricing model using local risk neutralization. The GARCH option pricing model has a number of desirable features and presents a real possibility of correcting the pricing biases associated with the Black-Scholes model. The ultimate test of the model, however, must be its empirical performance. Since the empirical technology for the GARCH process has been well developed, a reasonable test of the model can therefore be designed and executed without too much difficulty.

Market prices for traded options are readily available. The GARCH option pricing model presents a possibility of inferring implied GARCH parameters from the market data. These inferred parameter values can be used in a spirit very similar to the implied volatility of the Black-Scholes model. Calculating the implied GARCH parameters with Monte Carlo simulation is, however, computationally demanding. To make the task manageable, one has to develop a numerical approximation procedure for computing the GARCH option price. Analytically approximating the distribution function of the terminal asset price under the locally risk-neutralized pricing measure seems a promising approach.

Finally, using the GARCH option pricing model in delta hedging adds another dimension of flexibility to the process of dynamic hedging. In a low-variance state, the GARCH delta hedging calls for a smaller option position, relative to the Black-Scholes model, whereas it calls for a larger position in a high-variance state. This additional dynamic feature may prove to be materially beneficial for risk management.

APPENDIX

Proof of Theorem 2.1. Let $U(C_t)$ and C_t denote the utility function and the aggregate consumption at time t, respectively. The parameter ρ is used to denote the impatience factor. The standard expected utility maximization argument leads to the following Euler equation:

$$X_{t-1} = E^P\left[e^{-\rho}\frac{U'(C_t)}{U'(C_{t-1})}X_t \,\Big|\, \phi_{t-1}\right].$$

To organize the proof, two intermediate lemmas are useful. Let $Y_t = \nu + Z_t$, where ν is the constant mean and Z_t distributes normally with zero mean and constant variance under P. Define a measure Q by $dQ = e^{(r-\rho)T+\sum_{s=1}^{T} Y_s} dP$. It is clear that measure Q is mutually absolutely continuous with respect to measure P.

LEMMA A.1. *If* $X_{t-1} = E^P(e^{-\rho+Y_t}X_t \mid \phi_{t-1})$, *then* Q *is a probability measure and, for any* ϕ_t*-measurable random variable* W_t, $E^Q(W_t \mid \phi_{t-1}) = E^P(W_t e^{(r-\rho)+Y_t} \mid \phi_{t-1})$.

Proof.

$$\int 1\, dQ = E^P(e^{(r-\rho)T+\sum_{s=1}^{T} Y_s} \mid \phi_0)$$

$$= E^P\left[e^{(r-\rho)(T-1)+\sum_{s=1}^{T-1} Y_s} e^r E^P(e^{-\rho+Y_T}|\phi_{T-1})|\phi_0\right]$$
$$= E^P[e^{(r-\rho)(T-1)+\sum_{s=1}^{T-1} Y_s} \mid \phi_0].$$

The last equality is due to

$$E^P(e^{-\rho+Y_T} \mid \phi_{T-1}) = e^{-r},$$

which is, of course, implied by the assumption of the lemma. Continuing the process, one obtains

$$\int 1\, dQ = 1$$

which confirms that Q is a probability measure. The second half of the lemma follows from the Radon-Nikodym theorem and the law of iterated expectations. □

LEMMA A.2. *If* $X_{t-1} = E^P(e^{-\rho+Y_t} X_t \mid \phi_{t-1})$, *then*

(a) $X_t/X_{t-1} \mid \phi_{t-1}$ *distributes lognormally under measure* Q.
(b) $E^Q(X_t/X_{t-1} \mid \phi_{t-1}) = e^r$.
(c) $\mathrm{Var}^Q(\ell n(X_t/X_{t-1}) \mid \phi_{t-1}) = \mathrm{Var}^P(\ell n(X_t/X_{t-1}) \mid \phi_{t-1})$ *almost surely with respect to measure* P.

Proof. Using Lemma A.1, we easily verify part (b).

$$\begin{aligned} E^Q\left(\frac{X_t}{X_{t-1}} \mid \phi_{t-1}\right) &= E^P\left(\frac{X_t}{X_{t-1}} e^{(r-\rho)+Y_t} \,\middle|\, \phi_{t-1}\right) \\ &= \frac{e^r}{X_{t-1}} E^P(X_t e^{-\rho+Y_t} \mid \phi_{t-1}) \\ &= e^r. \end{aligned}$$

To prove (a) and (c), we consider the conditional moment generating function of $W_t \equiv \ell n(X_t/X_{t-1})$ under Q:

$$E^Q(e^{cW_t} \mid \phi_{t-1}) = E^P(e^{cW_t+(r-\rho)+Y_t} \mid \phi_{t-1}).$$

Let $\mu_t = E^P(\ell n(X_t/X_{t-1}) \mid \phi_{t-1})$. Since under measure P, $W_t \mid \phi_{t-1} \sim N(\mu_t, h_t)$ and Y_t is ϕ_{t-1}-conditionally normal, it follows that $Y_t = \alpha + \beta W_t + U_t$, where W_t and U_t are independent. Thus,

$$\begin{aligned} E^Q(e^{cW_t} \mid \phi_{t-1}) &= e^{\alpha+(r-\rho)} E^P(e^{(\beta+c)W_t+U_t} \mid \phi_{t-1}) \\ &= e^{\alpha+(r-\rho)+E(U_t^2|\phi_{t-1})/2} E^P(e^{(\beta+c)W_t}|\phi_{t-1}) \\ &= e^{\alpha+(r-\rho)+E(U_t^2|\phi_{t-1})/2+\beta\mu_t+\beta^2 h_t/2+(\mu_t+\beta h_t)c+c^2 h_t/2}. \end{aligned}$$

Let $c = 0$ and use the fact that $E^Q(1 \mid \phi_{t-1}) = 1$ to yield

$$E^Q(e^{cW_t} \mid \phi_{t-1}) = e^{(\mu_t + \beta h_t)c + c^2 h_t/2}.$$

This in turn implies

$$\ell n \frac{X_t}{X_{t-1}} \bigg| \phi_{t-1} \sim N(\mu_t + \beta h_t, h_t)$$

under measure Q. So, the proofs for (a) and (c) are complete. □

With Lemma A.2 in place, the first two assertions in the theorem can be proved if, under measure P, logarithmic marginal rate of substitution can be expressed as the sum of a constant and a normally distributed disturbance with zero mean and constant variance.

(i) $\ell n(U'(C_t)/U'(C_{t-1})) = (\lambda_1 - 1)\ell n(C_t/C_{t-1})$ where λ_1 is the constant relative risk aversion coefficient. Since $\ell n(C_t/C_{t-1})$ distributes normally under P, the result is immediately established.

(ii) $\ell n(U'(C_t)/U'(c_{t-1})) = -\lambda_2(C_t - C_{t-1})$ where λ_2 is the constant absolute-risk-aversion coefficient. The assertion is true because $C_t - C_{t-1}$ distributes normally under P.

Statement (iii) of Theorem 2.1 trivially holds because the ratio of marginal utilities equals 1. Note that, under any of the above assumptions, the implied interest rate is constant. This thus ensures the consistency with the constant interest rate assumption made in the article. □

Proof of Theorem 2.2. Since $X_t/X_{t-1} \mid \phi_{t-1}$ distributes lognormally under measure Q, it can be written as

$$\ell n \frac{X_t}{X_{t-1}} = \nu_t + \xi_t,$$

where ν_t is the conditional mean and ξ_t is a Q-normal random variable. The conditional mean of ξ_t equals zero and its conditional variance is to be determined. First, we prove that $\nu_t = r - \frac{1}{2}h_t$.

$$\begin{aligned} E^Q\left(\frac{X_t}{X_{t-1}} \bigg| \phi_{t-1}\right) &= E^Q(e^{\nu_t + \xi_t} \mid \phi_{t-1}) \\ &= e^{\nu_t + h_t/2} \end{aligned}$$

where $h_t = \mathrm{Var}^P(\ell n(X_t/X_{t-1}) \mid \phi_{t-1}) = \mathrm{Var}^Q(\ell n(X_t/X_{t-1}) \mid \phi_{t-1})$ by the LRNVR. Since $E^Q(X_t/X_{t-1} \mid \phi_{t-1}) = e^r$ by the LRNVR, it follows that $\nu_t = r - \frac{1}{2}h_t$. It remains to prove that h_t can indeed be expressed as stated in Theorem 2.2. By the preceding result and (2.1), $r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \varepsilon_t = r - \frac{1}{2}h_t + \xi_t$. This implies that $\varepsilon_t = \xi_t - \lambda\sqrt{h_t}$. Substituting ε_t into the conditional variance equation yields the desirable result. □

Proof of Corollary 2.2. The proof follows from Corollary 2.1 and the fact that ξ_t is conditionally normal with mean zero and variance h_t under Q.

Proof of Corollary 2.4. Define

$$Y_{t,T} \equiv (T-t)r - \frac{1}{2}\sum_{s=t+1}^{T} h_s + \sum_{s=t+1}^{T} \xi_s.$$

By Corollaries 2.1 and 2.3

$$C_t(X_t) = e^{-(T-t)r} E^Q[\max(X_t e^{Y_{t,T}} - K, 0) \mid \phi_t].$$

Note that we have dropped the superscript GH to simplify the notation and added the argument X_t to reflect that the option price at time t is a function of X_t. Consider an arbitrary $h > 0$ and compute the following quantity:

$$\begin{aligned} & C_t(X_t+h) - C_t(X_t) \\ &= e^{-(T-t)r} E^Q\{\max[(X_t+h)e^{Y_{t,T}} - K, 0] - \max[X_t e^{Y_{t,T}} - K, 0] \mid \phi_t\} \\ &= e^{-(T-t)r} \int_{-\infty}^{\infty} \{\max[(X_t+h)e^y - K, 0] - \max[X_t e^y - K, 0]\} dF(y \mid \phi_t), \end{aligned}$$

where $F(y \mid \phi_t)$ denotes the conditional distribution function of $Y_{t,T}$ under measure Q. Using the fact that $h > 0$, the above quantity can be further simplified to

$$e^{-(T-t)r} \int_{ln(K/X_t)}^{\infty} h e^y dF(y \mid \phi_t) + e^{-(T-t)r} \int_{ln(K/(X_t+h))}^{ln(K/X_t)} [(X_t+h)e^y - K] dF(y \mid \phi_t).$$

Since $\frac{1}{h}\int_{ln(K/(X_t+h))}^{ln(K/X_t)} [(X_t+h)e^y - K] dF(y \mid \phi_t) \to 0$ as $h \to 0$, we have

$$\frac{1}{h}[C_t(X_t+h) - C_t(X_t)] \to e^{-(T-t)r} \int_{ln(K/X_t)}^{\infty} e^y \, dF(y \mid \phi_t).$$

A similar argument can be constructed for $h < 0$ to arrive at the same result. Together, we have

$$\begin{aligned} \Delta_t &= e^{-(T-t)r} \int_{ln(K/X_t)}^{\infty} e^y \, dF(y \mid \phi_t) \\ &= e^{-(T-t)r} E^Q\left[\frac{X_T}{X_t} 1_{\{X_T \geq K\}} \mid \phi_t\right]. \end{aligned}$$

□

Proof of Theorem 3.1. (i) Define $z_t = \xi_t/\sqrt{h_t} - \lambda$, $G_i = G_{i-1}(\alpha_1 z_{t-i}^2 + \beta_1)$, and $G_0 = 1$.

It follows from (2.3) with $p = 1$ and $q = 1$ that

$$\begin{aligned} h_t &= \alpha_0 + \alpha_1(\xi_{t-1} - \lambda\sqrt{h_{t-1}})^2 + \beta_1 h_{t-1} \\ &= \alpha_0 + h_{t-1}(\alpha_1 z_{t-1}^2 + \beta_1) \\ &= h_0 \prod_{i=0}^{t}(\alpha_1 z_{t-i}^2 + \beta_1) + \alpha_0 \left[1 + \sum_{k=1}^{t-1} \prod_{i=1}^{k} (\alpha_1 z_{t-i}^2 + \beta_1) \right] \\ &= h_0 G_t + \alpha_0 \sum_{k=0}^{t-1} G_k. \end{aligned}$$

Since z_t^2, for any t, is a noncentral chi-square random variable with the degrees of freedom 1 and the noncentrality parameter λ, $E^Q(z_t^2 \mid \phi_0) = 1 + \lambda^2$. Thus, for $t > k$,

$$\begin{aligned} E^Q(G_k \mid \phi_0) &= E^Q\left[G_{k-1}(\alpha_1 z_{t-k}^2 + \beta_1) \mid \phi_0\right] \\ &= E^Q(G_{k-1} \mid \phi_0) E^Q(\alpha_1 z_{t-k}^2 + \beta_1 \mid \phi_0) \\ &= E^Q(G_{k-1} \mid \phi_0)\left[\alpha_1(1+\lambda^2) + \beta_1\right] \\ &= \left[\alpha_1(1+\lambda^2) + \beta_1\right]^k. \end{aligned}$$

This in turn implies that

$$E^Q(h_t \mid \phi_0) = h_0\left[\alpha_1(1+\lambda^2)+\beta_1\right]^t + \alpha_0 \sum_{k=0}^{t-1}\left[\alpha_1(1+\lambda^2)+\beta_1\right]^k.$$

The stationary variance can be computed by letting $t \to \infty$. The condition of the theorem ensures that $|\alpha_1(1+\lambda^2)+\beta_1| < 1$ and implies that

$$\begin{aligned} E^Q(h_t) &= \alpha_0 \sum_{k=0}^{\infty}\left[\alpha_1(1+\lambda^2)+\beta_1\right]^k \\ &= \alpha_0\left[1 - \alpha_1(1+\lambda^2) - \beta_1\right]^{-1}. \end{aligned}$$

(ii) By the definition of z_t, we have $E^Q(z_t^4 \mid \phi_0) = 3 + 6\lambda^2 + \lambda^4$. For any $t > k$,

$$\begin{aligned} E^Q(G_k^2 \mid \phi_0) &= E^Q\left[G_{k-1}^2(\alpha_1 z_{t-k}^2 + \beta_1)^2 \mid \phi_0\right] \\ &= E^Q(G_{k-1}^2 \mid \phi_0) E^Q\left[(\alpha_1 z_{t-k}^2 + \beta_1)^2 \mid \phi_0\right] \\ &= E^Q(G_{k-1}^2 \mid \phi_0)\left[\alpha_1^2(3+6\lambda^2+\lambda^4) + 2\alpha_1\beta_1(1+\lambda^2) + \beta_1^2\right] \\ &= \left[\alpha_1^2(3+6\lambda^2+\lambda^4) + 2\alpha_1\beta_1(1+\lambda^2) + \beta_1^2\right]^k. \end{aligned}$$

Define $u = \alpha_1^2(3+6\lambda^2+\lambda^4) + 2\alpha_1\beta_1(1+\lambda^2) + \beta_1^2$ and $v = \alpha_1(1+\lambda^2) + \beta_1$. It can be easily verified that $u = v^2 + 2(1+2\lambda^2)\alpha_1^2$ and $u > v$. For $k > j$,

$$\begin{aligned}
E^Q(G_k G_j \mid \phi_0) &= E^Q\left[G_j^2 \prod_{i=j}^{k}(\alpha_1 z_{t-2k+i}^2 + \beta_1) \mid \phi_0\right] \\
&= E^Q(G_j^2 \mid \phi_0) E^Q\left[\prod_{i=j}^{k}(\alpha_1 z_{t-2k+i}^2 + \beta_1) \mid \phi_0\right] \\
&= u^j v^{k-j}.
\end{aligned}$$

Thus,

$$\begin{aligned}
E^Q(h_t^2 \mid \phi_0) &= h_0^2 E^Q(G_t^2 \mid \phi_0) + 2\alpha_0 h_0 \sum_{k=0}^{t-1} E^Q(G_t G_k \mid \phi_0) \\
&\quad + \alpha_0^2 E^Q\left[\left(\sum_{k=0}^{t-1} G_k\right)^2 \mid \phi_0\right] \\
&= h_0^2 u^t + 2\alpha_0 h_0 \sum_{k=0}^{t-1} u^k v^{t-k} + \alpha_0^2\left[\sum_{k=0}^{t-1} u^k + 2\sum_{k=0}^{t-1}\sum_{j=0}^{k-1} u^j v^{k-j}\right] \\
&= h_0^2 u^t + 2\alpha_0 h_0 \frac{v(u^t - v^t)}{u - v} \\
&\quad + \alpha_0^2\left[\frac{1-u^t}{1-u} + 2\frac{v}{u-v}\left(\frac{1-u^t}{1-u} - \frac{1-v^t}{1-v}\right)\right].
\end{aligned}$$

It follows that, as $t \to \infty$,

$$E^Q(h_t^2 \mid \phi_0) \to \begin{cases} \frac{(1+v)\alpha_0^2}{(1-u)(1-v)} & \text{for } u < 1, \\ \infty & \text{for } u \geq 1. \end{cases}$$

The random variable ξ_t is leptokurtic if $E^Q(\xi_t^4) > 3\left[E^Q(\xi_t^2)\right]^2$. Recall that h_t is ϕ_{t-1}-measurable and $\xi_t/\sqrt{h_t}$ is a standard normal random variable.

$$\begin{aligned}
E^Q(\xi_t^4) &= E^Q\left[E^Q(\xi_t^4 \mid \phi_{t-1})\right] \\
&= E^Q\left\{h_t^2 E^Q\left[(\xi_t/\sqrt{h_t})^4 \mid \phi_{t-1}\right]\right\} \\
&= 3E^Q(h_t^2).
\end{aligned}$$

The random variable ξ_t is clearly leptokurtic when $u \geq 1$. For $u < 1$,

$$E^Q(\xi_t^4) = 3\frac{(1+v)\alpha_0^2}{(1-u)(1-v)} = 3\frac{1-v^2}{1-u}\left[E^Q(\xi_t^2)\right]^2.$$

Since $v < u$ and $u < 1$, it follows that $(1-v^2)/(1-u) > 1$ and the proof is complete.

(iii) Using the conditional variance process in (2.3) with $p = 1$ and $q = 1$, the product of ξ_t and h_{t+1} can be written as

$$\frac{\xi_t}{\sqrt{h_t}}h_{t+1} = \alpha_0\frac{\xi_t}{\sqrt{h_t}} + \alpha_1\frac{\xi_t}{\sqrt{h_t}}\left(\xi_t - \lambda\sqrt{h_t}\right)^2 + \beta_1\frac{\xi_t}{\sqrt{h_t}}h_t.$$

Thus,

$$\begin{aligned} E^Q\left(\frac{\xi_t}{\sqrt{h_t}}h_{t+1} \mid \phi_{t-1}\right) &= \alpha_1 E^Q\left[\frac{\xi_t}{\sqrt{h_t}}(\xi_t - \lambda\sqrt{h_t})^2 \mid \phi_{t-1}\right] \\ &= -2\lambda\alpha_1 h_t \end{aligned}$$

and

$$\begin{aligned} \operatorname{Cov}^Q\left(\frac{\xi_t}{\sqrt{h_t}}, h_{t+1}\right) &= E^Q\left(\frac{\xi_t}{\sqrt{h_t}}h_{t+1}\right) \\ &= E^Q\left[E^Q\left(\frac{\xi_t}{\sqrt{h_t}}h_{t+1} \mid \phi_{t-1}\right)\right] \\ &= E^Q(-2\lambda\alpha_1 h_t) \\ &= -2\lambda\alpha_0\alpha_1\left[1 - (1+\lambda^2)\alpha_1 - \beta_1\right]^{-1}. \qquad \square \end{aligned}$$

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