Notes on Rep(G) as a unitary fusion category and techniques to explicitly compute its 6j/F symbols

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Abstract

The category of representations of a finite group G, denoted as Rep(G), is a unitary fusion category (UFC). This means that the Grothendieck ring of Rep(G) is guaranteed to admit unitary solutions to the pentagon equations. These solutions are known as the *associators* in the context of fusion categories. The entries of these associators are called F symbols or 6j symbols. These notes present a method to compute these explicit 6j symbols, and introduce a script written in GAP which executes the presented method. The motivation behind the project is that the access to the Grothendieck ring and 6j symbols of Rep(G) can allow us to work explicitly with a planar diaggrammatic algebra for an anyon model representing anyon fusion/splitting.

Contents

1 Prerequisites/Notes to Reader

This draft is a selection of my notes on Rep(G) as a unitary fusion category and a graphical technique to compute its F symbols, accumulated over my research assistantship with the Center for Topological Quantum Computing at New York University Abu Dhabi. This project was inspired because of Steve Simon's book Topological Quantum (2022). In particular, Steve comments in at the end of Chapter 20 in his book that there is a lack of thorough written work on Rep(G) aimed at physicists, and little documentation on how to compute 6j symbols of finite groups.

The following reading is best received by a reader somewhat familiar with the notion of *categorifying* a commutative fusion ring into a unitary fusion category (UFC). Most importantly, the pentagon axioms governing the associators of a UFC should be reviewed. Certain key ideas can be reviewed from (Baez, Simon). Furthermore, some elementary knowledge of constructing planar diagrammatic algebras from unitary fusion categories will also be helpful. This can be found in (Simon). Appendix A contains some of these prerequisites. In this article, we only engage with the representation theory of *finite* groups. To review aspects of representation theory relevant to this article, refer to Appendix B. Sections 2, 3 and 4.1-4.2 mainly serve as brief background to guide the reader through the algebraic structure of $Rep(G)$ and the rules for constructing planar diagrams from fusion categories.

The main discussion of the technique to compute associators of Rep(G) symbols is in section 4.3-4.4. It is first important to understand the objects in Rep(G) that fusion/splitting vertices represent, before attempting to understand the strategy to compute F symbols.

2 Fusion Ring Structure of the Isomorphism Classes of Irreducible Representations of a Finite Group

The set of isomorphism classes of irreducible complex representations of a finite group G (denoted Irr(G) from hereon) has the structure of a commutative fusion ring (see Appendix A for axioms).
Let $\mathcal{L} = \{(\rho_a, V_a), (\rho_b, V_b), (\rho_c, V_c), \dots\}$ be a set of representatives from each isomorphism class of irreducible representations of G. We can equip $\mathcal{L}$ with commutative multiplication and addition operators by defining these as the tensor product and direct sum of representations respectively. By Maschke's theorem, every representation of G is isomorphic to a direct sum of irreducible representations. We obtain fusion rules by inspecting the coefficients of irreducible representation spaces V_i in the direct sum decomposition of $V_a \otimes V_b$. Thus,

$$V_a \otimes V_b \cong \bigoplus_{i \in \mathcal{L}} c_i V_i \Rightarrow N_{ab}^i = c_i$$

It can be checked that the ring generated by $(\mathcal{L}, +, \times)$ along with fusion rules as per the aforementioned rule is indeed a commutative fusion ring (see Appendix B). The Grothendieck ring of the category Rep(G) refers to this fusion ring. Note: If all $N_{ab}^i \leq 1$ for $a, b, i \in \mathcal{L}$, then we say that Rep(G) (or its Grothendieck ring) is *multiplicity-free*.

3 Basics of Rep(G)- the category of representations of G

$Rep_{\mathbb{F}}(G)$ is the category of representations of a finite group G over a field $\mathbb{F}$. **In this exposition, we set $\mathbb{F} = \mathbb{C}$ and omit the $\mathbb{F}$ subscript.**

- The object class of $Rep(G)$ consists of pairs (ρ, V) where $\rho : G \to GL(V)$ is a representation of the group G and V is a vector space over the field $\mathbb{F}$.
- In Rep(G), the set Mor(V,W) of morphisms between objects (ρ^V, V) and (ρ^W, W) is the set of intertwining linear maps $T : V \to W$ (i.e. $T \in Mor(V, W) \iff \rho_g^V \circ T = T \circ \rho_g^W$ for all $g \in G$). Since the set of intertwining linear maps between two representation spaces is itself a complex vector space, Mor(V,W) is written as $Hom(V, W)$. In the language of module theory, $Hom(V, W)$ is the vector space of homomorphisms from the $\mathbb{C}[G]-$module corresponding to (ρ, V) to that of (ρ, W).
- As a consequence of Schur's lemma, if (ρ, V) is an irreducible representation of G, then $Hom(V, V) \cong \mathbb{C}$. Moreover, if W is an irreducible representation, then $Hom(V, W) = \{0\}$ if W is not isomorphic to V as a representation space. This also means that the set of isomorphism classes of irreducible representation spaces is the set of simple objects of $Rep(G)$. Moreover, this means that $Rep(G)$ is **K**-linear.
- Since the tensor product of representations is associative, $Rep(G)$ is always a monoidal category.
- The *monoidal unit* of $Rep(G)$ is the trivial irreducible representation. As a result, the endomorphism space of the monoidal unit is isomorphic to the base field $\mathbb{C}$.
- By Mashcke's theorem, $Rep(G)$ is a semi-simple category.
- Since the *Hom* spaces of $Rep(G)$ are vector spaces, $Rep(G)$ is a module category. As a consequence of the Freyd-Mitchell Embedding theorem, all module categories are Abelian. Thus, $Rep(G)$ is an Abelian category.

Remark 3.0.1 (Developing Anyon Models via Group Representation Theory)**.** An *anyon model* is a commutative fusion ring that admits a categorification into a UFC (i.e. admits unitary solutions to the pentagon equation). In general, it can be very difficult to check whether a commutative fusion ring admits a categorification into a UFC, especially if it contains multiplicities. So it is not possible, in general, to write down a set of fusion rules and be certain that it corresponds to an anyon model. However, we know that $Rep(G)$ is a unitary fusion category. Thus, there **always exist unitary solutions to the pentagon axiom** for $Rep(G)$. So, it is worthy of note that one has the ability to generate several valid anyon models by simply computing the Grothendieck rings of $Rep(G)$ for various finite groups G. Of course, this is subject to constraints on the available character table data for G (since we compute fusion rules via character tables, see Appendix B).

However, once we obtain an anyon model, in order to work with a planar diagrammatic algebra following the model, we explicitly require its associators. For a finite Abelian group G, the set of isomorphism classes of its irreducible representations is isomorphic to the group itself (considering the tensor product of representations as the law of composition). Thus, computing the associators for Rep(G) is a well known problem in the study of group cohomology - they are a 3-cocycle of G (i.e. in $H^3(G, U(1))$). The case for non-Abelian groups is more interesting, since the fusion ring is more complicated.

4 Planar Diagrams from a Unitary Fusion Category (UFC)

4.1 General Case

We can always create labelled, directed planar diagrams from a fusion category. Planar diagrams represents morphisms in the category. Here, we very briefly review how a planar diagrammatic algebra is developed for UFCs.

Let **C** be an arbitrary UFC. Note that the *Hom* spaces in **C** are finite dimensional Hilbert spaces over $\mathbb{C}$. By definition, a planar diagram either represents a linear map belonging to some *Hom* space in **C** or is a normalized complex number.

- An oriented line labelled by object A is simply the identity morphism in $End_{\mathbf{C}}(A)$.
- A planar diagram containing an incoming arrow labelled by object $A \in Ob(\mathbf{C})$ and an outgoing arrow labelled by object $B \in \mathbf{C}$ represents a linear map in $Hom_{\mathbf{C}}(A, B)$.
- A diagram containing multiple incoming and outgoing lines labelled by $A_1, A_2 \ldots A_n$ and $B_1, B_2 \ldots B_m$ respectively corresponds to a morphism in $Hom_{\mathbf{C}}(A_1 \otimes A_2 \otimes \ldots A_n, B_1 \otimes B_2 \otimes \ldots B_m)$.
- A dotted line (or no line) represents the *monoidal unit* $\mathbf{1} \in \mathbf{C}$. This can be freely added into or removed from any diagram.
- A diagram containing no incoming or outgoing edges belongs to $End_C(\mathbf{1})$. By definition, this diagram evaluates to a normalized complex number.
- An oriented line segment labelled by an object $A \in Ob(\mathbf{C})$ is equivalent to a line segment with the opposite orientation labelled by $\bar{A}$, where $\bar{A} \in Ob(\mathbf{C})$ is the dual element of A.
- The diagram of the Hermitian conjugate of a morphism $f \in Hom_{\mathbf{C}}(A, B)$ is a mirrored diagram of f wherein all orientations have been reversed.
- The stacking of diagrams refers to the composition of morphisms. Diagrams can be stacked as long as the domains of the morphisms they represent agree. In a UFC, stacking the hermitian conjugate of a diagram upon itself is equivalent to the diagram of the identity morphism.
- The diagram of the tensor product morphism $f \otimes g$, where $f \in Hom_{\mathbf{C}}(A, B)$ and $g \in Hom_{\mathbf{C}}(C, D)$, is simply the diagrams representing f and g placed next to each other.
- Diagrams with a single outgoing and incoming edge labelled by distinct simple objects must vanish (Locality Principle/Schur's Lemma)

Example 4.1.1 (Fusion/Splitting Vertices and the *Jumping Jack* Diagrams)**.** For simple objects a, b, c in a unitary fusion category, $Hom(a \otimes b, c)$ and $Hom(c, a \otimes b)$ are N_{ab}^c dimensional Hilbert spaces.

Diagrams of form (i) and (ii) in figure 2 are referred to as fusion and splitting vertices respectively. The label μ at the vertex is an integer index that ranges over $\{1..N_{ab}^c\}$. By definition, the set of diagrams of form (i) and (ii) are an orthonormal basis of the spaces they respectively belong to. As is visually evident, diagrams (i) and (ii) are Hermitian conjugates of one another. Diagram (iii) (which resembles a jumping jack) is the outer product of diagram (i) and (ii). It is an idempotent, Hermitian operator in $End(a \otimes b)$. By definition, for various c such that $N_{ab}^c > 1$, diagrams of form (iii) form an orthonormal basis of $End(a \otimes b)$.

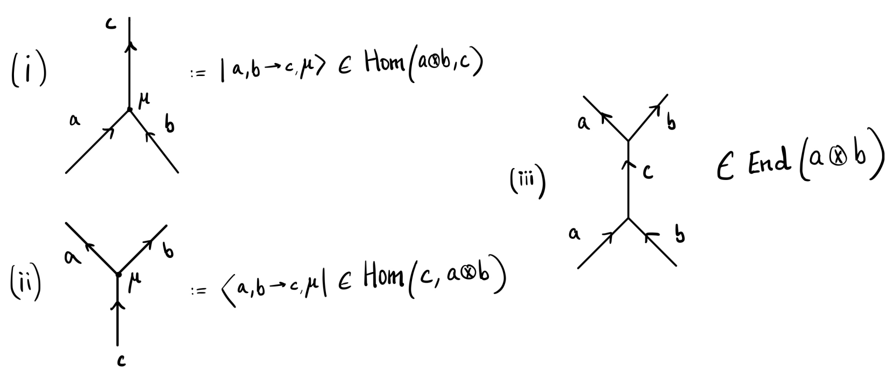

Figure 1: (i) A *fusion* vertex in $Hom(a \otimes b, c)$ (ii) A splitting vertex in $Hom(a \otimes b, c)$ and (iii) *Jumping Jack diagram* representing the outer product of diagrams (i) and (ii), which acts like the projector onto object c in $End(a \otimes b)$

Due to the properties of the jumping jack, the following diagrammatic manipulations are possible:

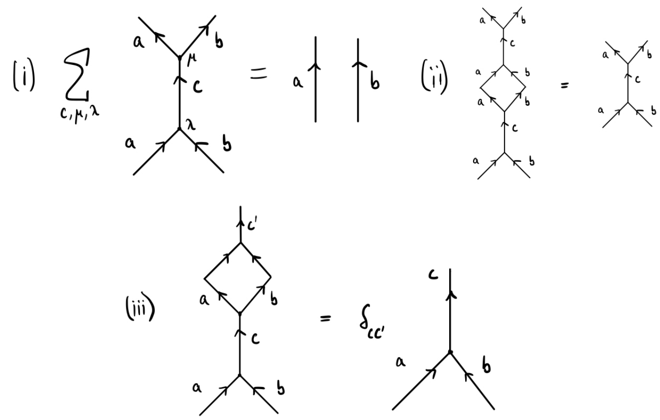

Figure 2: Diagrammatic representation of (i) the completeness property (ii) the idempotence property (iii) Projection onto object c in $End(a \otimes b)$

The completeness property is a consequence of the fact that the jumping jack diagrams form an orthonormal basis of $End(a \otimes b)$, and thus the sum of all diagrams of this type must be the identity operator on $End(a \otimes b)$, which is represented simply the right hand side of diagram (i) of figure 2 above.

(Further reading on rules of planar diagrammatics associated to UFCs can be found in Steve Simon's Topological Quantum Chapter 12)

4.2 Planar Diagrams from Multiplicity-Free Rep(G)

In Rep(G), a planar diagram containing an incoming line labelled by A and an outgoing line containing object B belongs to $Hom(A, B)$, and thus, corresponds to an interwining linear map from the representation space corresponding to A to that of B. For example, a diagram containing an incoming and outgoing line labelled by the same representation space V (as in figure 3) corresponds to some element of $End(V)$ (i.e. an intertwining linear map $T : V \to V$):

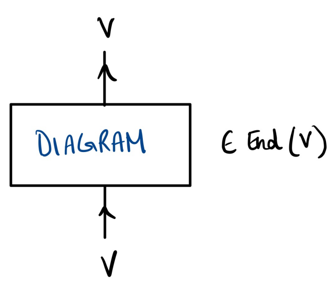

Figure 3: Diagram from a representation space V to itself

A diagram with incoming and outgoing edges labelled by representation spaces $V_1, V_2 \dots V_n$ and $W_1, W_2 \dots W_m$ respectively is an element of $Hom(V_1 \otimes V_2 \otimes \dots V_n, W_1 \otimes W_2 \dots W_m)$:

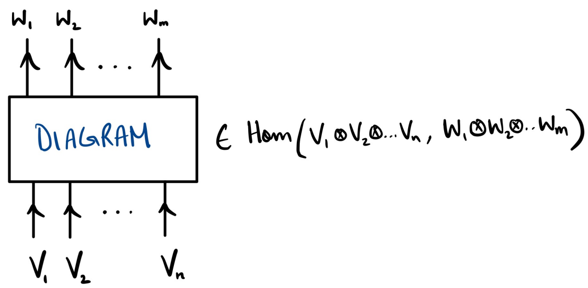

Figure 4: General Planar Diagram with labelled incoming and outgoing edges

4.3 Planar Diagrams Involving Irreducible Representations

In this section, we discuss directed planar diagrams labelled by isomorphism classes of irreducible representations. We will define the trivalent vertex and the *jumping jack* diagrams to set up a discussion of F moves. Throughout this section, let $\{(\rho_i, W_i)\}$ be a family of representatives of distinct isomorphism classes of irreducible representations of the group G. Let (ρ_0, W_0) correspond to the trivial irreducible representation.

Due to Schur's Lemma, if a diagram maps an irreducible representation space into a non-equivalent representation space, it must vanish (be equal to 0). If the incoming/outgoing labels are the same, then the diagram is the same as the action of a normalized complex number on an incoming labelled line.

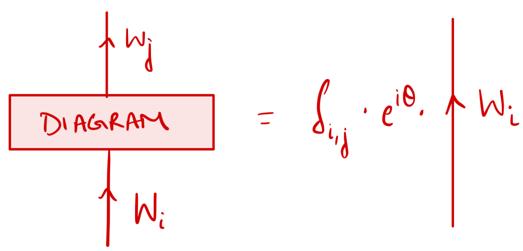

Figure 5: Locality Principle enforced by Schur's Lemma: No diagram can transmute a simple object

The trivial irreducible representation of G is the monoidal unit of $Rep(G)$ (also the identity element of its Grothendieck ring) and is thus represented by a dotted line or blank space. Thus a diagram with no incoming or outgoing edges lies in the endomorphism space of the trivial irreducible representation. Once again, due to Schur's lemma, such a diagram evaluates to a normalized complex number.

4.3.1 Fusion/Splitting Vertices and the Jumping Jack Diagrams in multiplicity-free Rep(G)

Let V_1 and V_2, and W_j be representatives of certain isomorphism classes of irreducible representation spaces of of a finite group G (assuming a multiplicity-free fusion ring). As defined in example 4.1.1, the fusion/splitting vertices and the jumping jack diagrams associated to Rep(G) must be morphisms in Rep(G) as identified in figure 6 below.

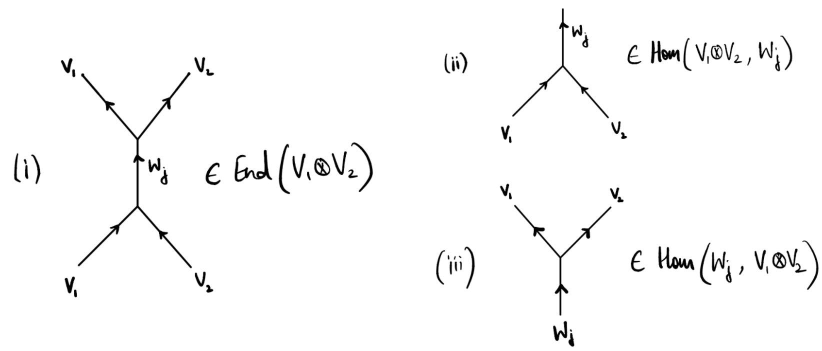

Figure 6: (i) A *jumping jack* diagram representing an intertwiner in $End(V_1 \otimes V_2)$, (ii) A trivalent vertex depicting *fusion* representing an intertwiner in $Hom(V_1 \otimes V_2, W_j)$ and (iii) A trivalent vertex depicting *splitting* representing an intertwiner $\in Hom(W_j, V_1 \otimes V_2)$

We now explicitly correlate these diagrams to objects in the category of representations of G. For this purpose, we introduce the following definition.

Definition 4.1 (Projector onto an Irreducible Representation)**.** Let (ρ, V) be an arbitrary representation space of G. Then, the projector onto the irreducible representation (ρ_j, W_j) in $End(V)$ is defined as an orthogonal projector $T_j : V \to V$ such that :

$$T_j|_W = \begin{cases} Id_{W_j} & \text{if } W \cong W_j, \\ 0 & \text{otherwise} \end{cases}$$

where W is an irreducible subrepresentation space of V. Note that $W \cong W_j$ refers to the isomorphism of representation spaces.

Definition 4.2 (Defining the Jumping Jack diagram)**.** The jumping jack diagram (i) of figure 6 is a projector onto W_j in $End(V_1 \otimes V_2)$.

To verify that this is a valid definition, we can check whether the properties of the jumping jack diagram are satisfied. Let the projectors onto W_j in $End(V_1 \otimes V_2)$ be denoted T_j, and ρ_1 and ρ_2 be the irreducible representations of G defined on the spaces V_1, V_2 respectively.

- **The diagram must be a morphism in** $End(V_1 \otimes V_2)$: We need to check if the projector onto the irreducible representation W_j in $V_1 \otimes V_2$, is an intertwining linear map $T_j : V_1 \otimes \to V_2$. Consider the maps $(\rho_1 \otimes \rho_2)(g)$ for all $g \in G$ and T_j expressed in a basis created by adjoining orthonormal bases of every irreducible subrepresentation space of $V_1 \otimes V_2$. Pick any one of such irreducible subrepresentation spaces, say W_i. Clearly, $T_j|_{W_i}$ commutes with $(\rho_1 \otimes \rho_2)(g)|_{W_i}$ for all $g \in G$ if $i = j$, since $T_j|_{W_j}$ is the identity map by definition. If $i \neq j$, then $T_j|_{W_i}$ is the zero map by definition. Thus once again, it must commute with $(\rho_1 \otimes \rho_2)(g)|_{W_i}$ for all $g \in G$. Therefore, $\rho_1 \otimes \rho_2$ and T_j commute over the entire vector space $V_1 \otimes V_2$.
- **Idempotent And Hermitian**: Since $T_j : V_1 \otimes V_2 \to V_1 \otimes V_2$ is an orthogonal projector, we have that $T_j \circ T_j = T_j$ and $T_j^* = T_j$.
- **Forming an orthonormal basis of** $End(V_1 \otimes V_2)$: Let W_i and W_k be two irreducible representation spaces of $End(V_1 \otimes V_2)$. It is easy to check that the projectors onto W_i and W_k are mutually orthogonal due to the trace inner product.
- **Completeness Property** The sum $\sum_j T_j$ of projectors onto irreducible subrepresentation spaces W_j of $V_1 \otimes V_2$ is simply the identity on every irreducible subrepresentation space. Thus, it is the identity map on $V_1 \otimes V_2$

Remark 4.3.1 (A Sketch to Compute the Jumping Jack in Rep(G))**.** If we pick a unitary basis B_1 of the representation space $V_1 \otimes V_2$ in which the matrices $(\rho_1 \otimes \rho_2)(g)$ are unitary for all $g \in G$, we can explicitly compute a matrix for T_j in this basis. First, we compute an orthonormal basis of every subrepresentation space of $V_1 \otimes V_2$, and adjoin these to form a basis B_2 of the whole space. Then, the matrix of $\rho_1 \otimes \rho_2$ in B_2 is *block diagonal*. In basis B_2, the matrix of T_j is the identity on the block associated to the irreducible representation space W_j, and the zero map for every other block. The explicit matrix of T_j in B_1 is its matrix in B_2 conjugated by the appropriate change of basis matrix.

Before associating the fusion/splitting vertex diagrams to objects in Rep(G), let us list the properties they should have:

- **The diagrams must be morphisms in the appropriate Hom spaces**. Thus, the fusion vertex (as represented in diagram (ii) in 6) is an intertwining linear map $F : V_1 \otimes V_2 \to W_j$. The splitting vertex represented in diagram (iii) is an intertwining linear map $S : W_j \to V_1 \otimes V_2$.
- **The fusion/splitting vertices are Hermitian conjugates of one another.** Thus, $F = S*$.
- **Inner Product Structure** The inner product of the fusion and splitting vertex must be the identity morphism. Thus, the composition $F \circ S = Id|_{W_j}$
- **Outer Product of Fusion/Splitting is the Jumping Jack** Thus, $S \circ F$ should be T_j.

Definition 4.3 (Defining the Fusion and Splitting Vertex in Rep(G))**.** Let (ρ_1, V_1) and $((\rho_2, V_2))$ be irreducible representations of G, and M be a basis of $V_1 \otimes V_2$ in which $\rho_1 \otimes \rho_2$ is a unitary representation. Then, the splitting vertex, represented in diagram (iii) in 6, is a linear map $S : W_j \to V_1 \otimes V_2$ whose columns consist of all the elements of an orthonormal basis B_j of the irreducible subrepresentation space $W_j' \subset V_1 \otimes V_2$ that is isomorphic to the representation space W_j. Here, elements of B_j must be expressed in the basis M. The fusion vertex, represented in diagram (ii), can simply be defined as $S : V_1 \otimes V_2 \to W_j$ wherein $S = F*$.

We can check that these definitions do indeed satisfy the aforementioned requirements. To do this, we sketch the process of computing the linear maps defined above, and show that these linear maps indeed are the desired intertwiners (which satisfy all other properties).

Remark 4.3.2 (Computing the Linear Maps Proposed as the Definition of Fusion/Splitting Vertices of Rep(G))**.** First, we pick a unitary basis B_1 of the representation space $V_1 \otimes V_2$ in which the matrices $(\rho_1 \otimes \rho_2)(g)$ are unitary for all $g \in G$. Let $\{(\rho_a, W_a), (\rho_b, W_b), \dots\}$ be the set of all irreducible subrepresentations in the direct sum decomposition of $V_1 \otimes V_2$.

Now, we compute an orthonormal basis of every subrepresentation space of $V_1 \otimes V_2$, and adjoin these to form a basis B_2 of the whole space. Let $U : V_1 \otimes V_2 \to \oplus_i W_i$ be the unitary intertwiner that acts like the change of basis from B_1 to B_2. Then, we have that for all $g \in G$:

$$\rho_1(g) \otimes \rho_2(g) = U^\dagger \begin{pmatrix} X_a(g) & 0 & 0\dots \\ 0 & X_b(g) & 0\dots \\ 0 & 0 & X_c(g)\dots \\ \vdots & \vdots & \vdots \end{pmatrix} U. \tag{4.3.1}$$

where $X_a(g), X_b(g), \dots$ are the matrices of $\rho_a(g), \rho_b(g) \dots$ in the basis B_2.

Now, observe that the columns of $U^\dagger$ (grouped by the dimensions of subrepresentations appearing in the block form) are precisely the set of intertwiners from the irreducible subrepresentation spaces $W_a, W_b \dots$ to $V_1 \otimes V_2$ respectively. For example, let d_a be the dimension of representation ρ_a. Let the first d_a columns of $U^\dagger$ be referred to as I_a. Then I_a is an intertwiner between ${}_a$ and ${}_1 \otimes {}_2$. To see this, first notice that

$$\rho_1(g) \otimes \rho_2(g) \cdot U^\dagger = U^\dagger \begin{pmatrix} X_a(g) & 0 & 0\dots \\ 0 & X_b(g) & 0\dots \\ 0 & 0 & X_c(g)\dots \\ \vdots & \vdots & \vdots \end{pmatrix} \tag{4.3.2}$$

Now, observe that $\rho_1 \otimes \rho_2(g) \circ I_a = I_a \circ X_a(g)$ for all $g \in G$. Thus, I_a is indeed the proposed intertwiner. Similarly, $\rho_1 \otimes \rho_2(g) \circ I_b = I_b \circ X_a(g)$ for all $g \in G$ where I_b are the columns $U^\dagger$ that correspond to the columns of X_b in the overall block diagonal matrix on the right hand side of the equation above.
Finally, note that the columns of $U^\dagger$ consists of **orthononormal basis vectors of the various irreducible subrepresentation spaces** contained in $V_1 \otimes V_2$, expressed in the B_1 basis.

The remark above shows that the linear maps chosen as candidates for the splitting/fusion vertices are indeed intertwiners (and Hermitian conjugates of one another by definition). However, it is simple to check that they satisfy the other properties as well.

- **Inner Product Structure:** The inner product of the fusion and splitting vertex as in diagram (ii) and (iii) in figure 6 must be the identity morphism on W_j. However, this is indeed the case, since the columns of the map $S : W_j \to V_1 \otimes V_2$ defined to be the splitting vertex are the columns a unitary matrix. Thus, $S \circ S* = Id|_{W_j}$.
- **Outer Product of Fusion/Splitting is the Jumping Jack** We can observe this following from the discussion in the remark above. The outer product of S with its Hermitian conjugate is of form:

$$U^\dagger \begin{pmatrix} 0 & 0 & 0\dots \\ 0 & 0 & 0\dots \\ 0 & 0 & Id_{W_j}\dots \\ \vdots & \vdots & \vdots \end{pmatrix} U \tag{4.3.3}$$

  which is precisely the matrix form of T_j in basis B_1.

4.4 F matrices in Rep(G)

We now move to a discussion of F matrices in Rep(G). For irreducible representation spaces W_a, W_b, W_c, W_e we can express $Hom_{Rep(G)}(W_a \otimes W_b \otimes W_c, We)$ in the left and right hand basis. The matrix (F_e^{abc}) is the transition matrix between different bases of $Hom_{Rep(G)}(W_a \otimes W_b \otimes W_c, We)$, arising from the different order of splitting. In particular, for mulitplicity-free Rep(G) without multiplicities we have,

$$|(a, b); d\rangle \otimes |d; c; e\rangle = \sum_{f \in \mathcal{L}} (F_e^{abc})_{fd} |a, f; e\rangle \otimes |(b; c); f\rangle$$

which can be represented diagrammatically as:

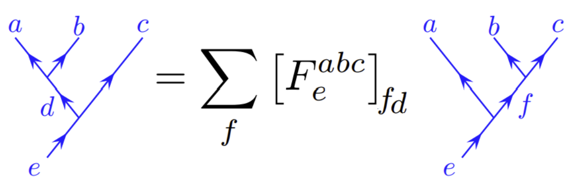

Figure 7: F matrix transitioning between different bases of $Hom(W_e, W_a \otimes W_b \otimes W_c)$

Owing to its structure as a monoidal category, there must always exist valid F matrices for Rep(G) which satisfy the pentagon equation. However, in order to compute these F symbols, we do not need to solve the pentagon equations. We can rely on the graphical calculus of Rep(G) we have been building so far.

4.4.1 Graphically computing F symbols for (multiplicity-free) Rep(G)

Let the intertwining linear map representing the fusion vertex belonging to $Hom(W_a \otimes W_b, W_c)$ be denoted F_{ab}^c. We start by constructing the left and right hand basis for $Hom(W_e, W_a \otimes W_b \otimes W_c)$. Note that a key idea which makes our construction possible is that the tensor product or composition of intertwiners is also an intertwiner.

A member of the left hand basis can be constructed as follows. Diagram 1 in figure 8 below is the fusion vertex belonging to $Hom(W_a \otimes W_b, W_d)$ tensored with the identity morphism on W_c. Thus, diagram 1 represents the intertwiner

$$F_{ab}^d \otimes Id|_{W_d}$$

Diagram 2 is the intertwiner $F_{dc}^e \in Hom(W_d \otimes W_c, W_e)$. The composition of diagram 2 with diagram 1 results in the left hand basis we desire. Thus the member of the left hand basis of $Hom(W_e, W_a \otimes W_b \otimes W_c)$ depicted in diagram 3 in figure 8 is

$$F_{dc}^e \circ (F_{ab}^d \otimes Id|_{W_d})$$

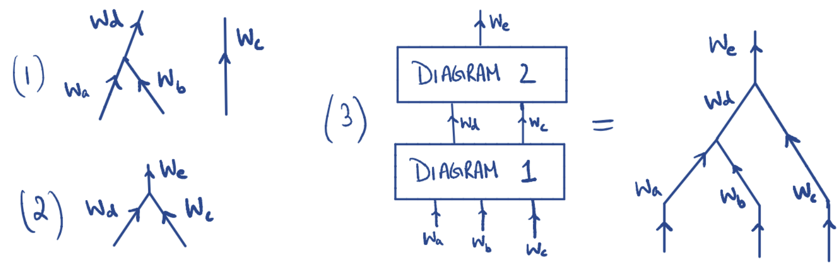

Figure 8: Constructing trees in the left hand basis of $Hom(We, W_a \otimes W_b \otimes W_c)$

A member of the right hand basis can be constructed similarly. Diagram 1 in figure 9 below is the identity map on W_a tensored with the intertwiner $F_{ab}^d \in Hom(W_b \otimes W_c, W_f)$. Thus, diagram 1 represents the intertwiner

$$Id|_{W_a} \otimes F_{bc}^f$$

Diagram 2 is the intertwiner $F_{af}^e \in Hom(W_a \otimes W_f, W_e)$. The composition of diagram 2 with diagram 1 results in the right hand basis we desire. Thus the member of the right hand basis of $Hom(W_e, W_a \otimes W_b \otimes W_c)$ depicted in diagram 3 in figure 9 is

$$F_{af}^e \circ (Id|_{W_a} \otimes F_{bc}^f)$$

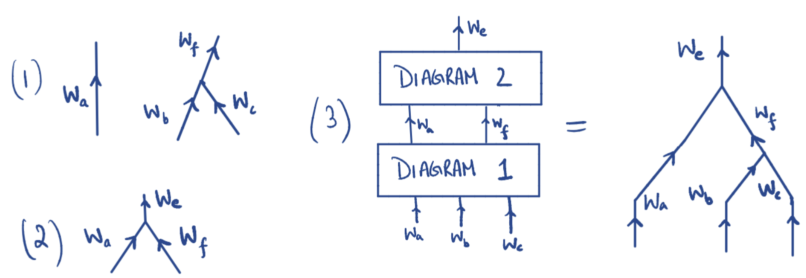

Figure 9: Constructing trees in the left hand basis of $Hom(We, W_a \otimes W_b \otimes W_c)$

Now, we are ready to compute the F symbol $(F_e^{abc})_{fd}$. We achieve this by computing the inner product of the left and right hand basis. We stack the right hand basis (diagram 3 in figure 9) on top of the Hermitian conjugate of the left hand basis (diagram 3 in figure 8). The resulting diagram can be manipulated via an F transformation on the bottom half. We sum over all valid choices i (when $N_{bc}^i \cdot N_{ai}^e = 1$). However, every diagram in the sum where $i \neq f$ vanishes due to Schur's Lemma. Therefore, the resulting diagram is simply the identity map on W_e multiplied by $(F_e^{abc})_{fd}$.

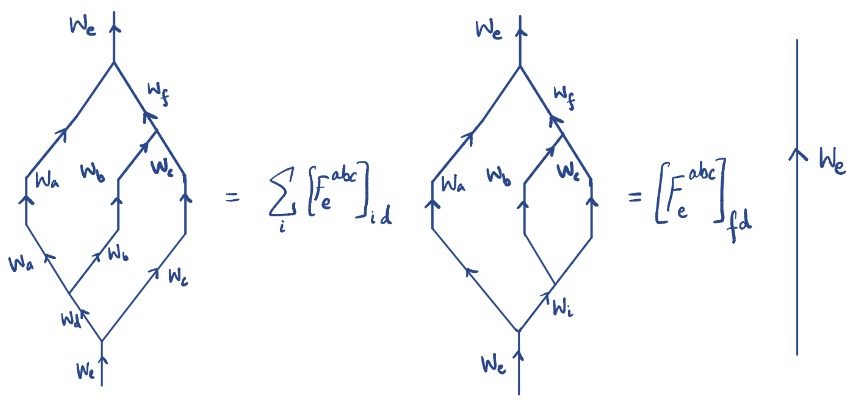

Figure 10: Computing the F symbol via the inner product of trees in the left and right hand basis

Thus, we can extract the F symbol by composing the right hand basis with the Hermitian conjugate of the left hand basis, and dividing the trace of the resulting intertwining map by the degree of W_e:

$$(F_e^{abc})_{fd} = \frac{tr(F_{af}^{e} \circ (Id|_{W_a} \otimes F_{bc}^{f}) \circ (F_{af}^{e} \circ (Id|_{W_a} \otimes F_{bc}^{f}))^{\dagger})}{dim(W_e)}$$

5 GAP Code to Compute 6j Symbols of Rep(G)

5.1 Setting up GAP

We have implemented the discussed method for computing 6j symbols for Rep(G) (multiplicity-free) in GAP, which is a programming language for computational discrete algebra. GAP was an ideal choice as it specializes in handling finite groups, computing character tables and explicit unitary representations.
In order to run our GAP scripts, you must first install GAP on your computer. Once installed, download and extract the GitHub zipped folder "SixJSolver.zip" on your computer.

The GitHub zipped folder "SixJSolver.zip" should contain 4 GAP files:

1. "G_fusion_printer.g"
2. "G_fusion_finder.g"
3. "matrix_methods.g"
4. "six_j_solver.g"

5.2 GAP Commands to Get Started

Note: Every GAP command should be followed by ;, or it will not execute. If you encounter any type of error, enter "quit;" to exit the error loop.

5.2.1 Folder Navigation

After installation of GAP, open the GAP terminal and navigate to the SixJSolver folder (the path will depend on where you have saved the downloaded folder). You can use the "DirectoryCurrent();" command to check the directory that your GAP terminal is in. To navigate to a different folder, use "ChangeDirectory("path to folder");". You must input this command every time you start a new GAP session, so it may be useful to store the path to the SixJSolver folder somewhere.

5.2.2 Defining the Group

After navigating to the correct folder, you should enter the name of the group which you are interested as the command "G:= name;". You can look up the names of common finite groups from the GAP library. For example, to set the group to symmetric group S_n, enter "G:=SymmetricGroup(n);" (where n must be a positive integer). For the alternating group A_n, enter "G:=AlternatingGroup(n);". For the Dihdral Group D_n (where 2n is the order of D_n), you must enter "G:=DihedralGroup(2n);". Thus, if you do not provide an even number as an argument, you will run into an error.

5.2.3 Running GAP files to compute fusion ring/6j symbols

After you have specified the group, you are ready to execute any of our GAP files. This is achieved by the command "Read("filename.g");". In order to visualize the fusion ring of a group, you should execute "Read("G_fusion_printer.g");". This will print LaTeXmarkup for the label set, along with all non-trivial fusion rules on the terminal, and also save it into a tex file. You can then either compile the tex file, or copy-paste the LaTeXmarkup into an online editor to view the fusion ring.

To compute the the 6j symbols of G, you need to execute "Read("six_j_solver.g");". This automatically runs "G_fusion_finder.g" and "matrix_methods.g" as subroutines. This prints both the fusion rules and the 6j symbols on the terminal, as well as in a text file.

To run the GAP files for a different group, simply rename the variable G as earlier, and run the Read(); command again.

5.2.4 GAP Scripts 1 and 2: Fusion Rule Calculators

G_fusion_printer.g" and "G_fusion_finder.g" extract the irreducible characters of a finite group and compute its fusion ring via the scalar product of characters. The irreducible representations are labelled as ""

5.2.5 GAP Script 3: Some Matrix Methods

The GAP file "matrix_mathods.g" section provides an overview of functions and utilities for evaluating cyclotomic expressions and performing matrix operations.

1. EvaluateCyclotomic Function: In GAP, sometimes real constants are expressed as expressions containing roots of unity. There isn't an automatic function to *cancel out* the imaginary part. This function evaluates the real part of an expression containing roots of unity through trigonometric expansion and returns it as float value.

- **Parameters:** `expr`, a cyclotomic expression.
- **Logic:**
  - Determines the order of the cyclotomic field using `Conductor`.
  - Extracts coefficients using `CoeffsCyc`.
  - Computes the real part by iterating over coefficients and applying cosine transformations.
- **Returns:** The real part of the evaluated cyclotomic expression.

2. HasZeroColumn Function: Checks if a given matrix contains any column where all entries are zero.

- **Parameters:** `mat`, a matrix (list of lists).
- **Logic:**
  - Iterates through each column and checks if all entries in that column evaluate to zero using `EvaluateCyclotomic`.
- **Returns:** `true` if a zero column exists; otherwise, `false`.

3. NormalizeColumns Function: Normalizes the columns of a matrix.

- **Parameters:** `mat`, a matrix (list of lists).
- **Logic:**
  - Checks if the input is a valid matrix using `IsMatrix`.
  - Iterates through columns, computes the norm of each column, and scales each entry appropriately.
- **Error Handling:** Throws an error if the input is not a matrix.
- **Returns:** A new matrix with normalized columns.

5.2.6 GAP Script 4

6 Appendix A: Fusion Rings/Pentagon Equation

Suggested Reading:

- Steve Simon Topological Quantum Chapters 8-12
- John Baez "Some Definitions Everyone Should Know"
- EGNO- Tensor Categories

This section contains some information about commutative fusion rings, and some details about the skeletal structure of unitary fusion categories.

Definition 6.1 (Commutative Fusion Ring)**.** Let $\mathcal{L} = \{a, b, c \dots\}$ be a finite set, and $\mathbb{Z}\mathcal{L}$ be a free $\mathbb{Z}-$module with the elements of $\mathcal{L}$ as its basis. Let there be a commutative and associative binary operation $\times$ called fusion defined on $\mathcal{L}$ as follows $\times : \mathbb{Z}\mathcal{L} \times \mathbb{Z}\mathcal{L} \to \mathbb{Z}\mathcal{L}$ by

$$a \times b = \sum_{c \in \mathcal{L}} N_{ab}^{c} c \tag{6.0.1}$$

for $a, b \in \mathcal{L}$ and $N_{ab}^{c} \in \mathbb{Z}_{\geq 0}$. Then $(\mathcal{L}, \times)$ is a fusion ring if it satisfies the following properties:

- Existence of Identity Element: $\exists! I \in \mathcal{L}$ such that $\forall c \in \mathcal{L}$, $I \times c = c$.
- Invertibility/Existence of Dual: $\forall a \in \mathcal{L}\ \exists! \bar{a} \in \mathcal{L}$ such that $N_{a\bar{a}}^{I} = 1$ and $N_{ab}^{I} = \mathbb{1}_{b\bar{a}}$.
- Existence of Fusion Channel: $\sum_{c \in \mathcal{L}} N_{ab}^{c} c \geq 1$.
- $N_{ab}^{c} < \infty$ for all $a, b, c \in \mathcal{L}$.
- There must be a bijection between the summands of $a \times b$ and those of $\bar{a} \times \bar{b}$ for all $a, b \in \mathcal{L}$

6.1 Change of Basis Matrices and the Pentagon Equation

Due to the isomorphic descriptions of fusion and splitting spaces V_{abc}^{e} and V_{e}^{abc} based on different choices of basis (i.e., order of fusion/splitting), we naturally require transition matrices to fluently switch between these equivalent descriptions of the same space. In the context of anyons, an F matrix is defined to be the unitary transition matrix between equivalent bases of the splitting space V_{e}^{abc}. In particular, as described above, two equivalent bases of V_{e}^{abc} are formed from vectors in $V_{d}^{ab} \otimes V_{c}^{de}$, representing all valid diagrams such as Fig 3(i), and by vectors in $V_{e}^{af} \otimes V_{f}^{bc}$ representing valid diagrams such as in Fig 3(ii).

Definition 6.2 (F matrix)**.** The matrix (F_{e}^{abc}) is the transition matrix between different bases of V_{e}^{abc}, arising from the different order of splitting. In particular, in an anyon model without multiplicities (i.e., $N_{ab}^{c} \leq 1$ for all $a, b, c \in \mathcal{L}$),

$$|(a, b); d\rangle \otimes |d; c; e\rangle = \sum_{f \in \mathcal{L}} (F_{e}^{abc})_{fd} |a, f; e\rangle \otimes |(b; c); f\rangle$$

which can be represented diagrammatically as:

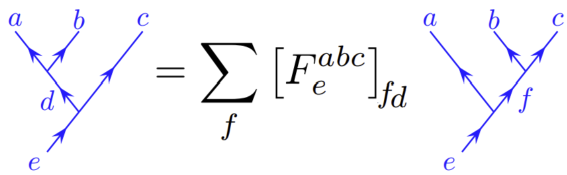

Figure 11: F matrix transitioning between different bases of V_e^{abc}

If a, b or c is the identity particle of the anyon model, then $(F_e^{abc}) = [1]$. Moreover, a change of basis as described above is known as an F move in the context of anyon fusion/splitting diagrammatics.

2.3.1 Pentagon Equation

Let us now consider the fusion space V_{abcd}^{e}. There are 5 ways, due to associativity, expression $a \times b \times c \times d$ without switching the order of anyons themselves:

- $((a \times b) \times c) \times d$
- $(a \times b) \times (c \times d)$
- $(a \times (b \times c)) \times d$
- $a \times (b \times c)) \times d)$
- $a \times (b \times (c \times d))$

Each of these different ways of evaluating the fusion outcomes of the 4 anyons gives rise to a different tensor description (and therefore basis) of the space V_{abcd}^{e}. Each of these bases can be related to one another through the F matrix. However, there are 2 *routes* that can be taken to transition between two bases. This is captured in the pentagon equation, in terms of splitting spaces:

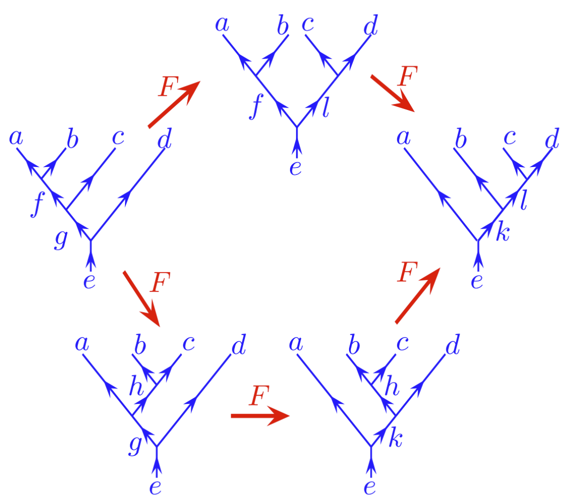

Figure 12: The Pentagon Equation- relates equivalent ways to transition from the left-most basis to the right-most basis

This equivalence can be written algebraically as:

$$\left[F_e^{fcd}\right]_{gl} \left[F_e^{abl}\right]_{fk} = \sum_h \left[F_g^{abc}\right]_{hf} \left[F_e^{ahd}\right]_{gk} \left[F_k^{bcd}\right]_{hl}. \tag{6.1.1}$$

By MacLane's Coherence Theorem, there are no further constraints that can be applied to the F matrices beyond the pentagon equation through the construction of any diagrams analogous to the pentagon equation

Theorem 6.1.1 (Ocneanu Rigidity)**.** *There are finitely many guage classes of solutions to the pentagon equations for a given anyon model.*

7 Appendix B: Constructing an Anyon Model from Irreducible Representations of a Finite Group

7.1 Defining a Fusion Ring

Let the set of all isomorphism classes of irreducible representations of a finite group G be denoted $\mathrm{Irr}(G)$. In this section, we describe how $\mathrm{Irr}(G)$ can be used to construct an anyon model. Recall: To construct an anyon model, we need to construct a fusion ring using a finite label set $\mathcal{L}$.

7.2 Defining a valid Label Set $\mathcal{L}$ from $\mathrm{Irr}(G)$

Let $\{\rho_a, \rho_b, \rho_c, \dots\}$ be representatives of all isomorphism classes of irreducible representations of a finite group G. We can construct $\mathcal{L} = \{a, b, c \dots\}$ by labelling the elements of $\mathrm{Rep}(G)$.

Theorem 7.2.1. *The label set $\mathcal{L}$ from the construction above is finite.*

Proof. $\mathcal{L} = |\mathrm{Irr}(G)|$. But $|\mathrm{Irr}(G)|$, the number of isomorphism classes of a finite group G, is equal to the number of conjugacy classes of G. Clearly, the number of conjugacy classes of a finite group G is bounded by its order $|G|$ (it is equal to $|G|$ when G is an Abelian group, wherein each element is in its own conjugacy class). Thus $|\mathcal{L}|$ is finite. □

7.3 Defining a Fusion Operation $\times$

Recall that the tensor product $\rho_a \otimes \rho_b$ of irreducible representations of a finite group G is also a representation (not necessarily irreducible) of G. Thus, by Maschke's theorem, $\rho_a \otimes \rho_b$ has a finite direct sum decomposition into irreducible representations. This can be expressed as the following isomorphism:

$$\rho_a \otimes \rho_b \cong \rho_c \oplus \rho_d \oplus \dots \tag{7.3.1}$$

where all terms on the right side of the equation are irreducible representations.

We can define fusion (a binary operator $\times : \mathbb{Z}\mathcal{L} \times \mathbb{Z}\mathcal{L} \to \mathbb{Z}\mathcal{L}$) on $\mathcal{L}$ as follows: Let N_{ab}^c be the number of times the irreducible representation ρ_c appears in the direct sum decomposition of the representation $\rho_a \otimes \rho_b$. Then we define $a \times b = \sum_{c \in \mathcal{L}} N_{ab}^c c$. We require this fusion operator to be associative and commutative. We can check this as follows:

- **Commutativity**: In order for $a \times b = b \times a$ to hold, we require the direct sum decomposition of $\rho_a \otimes \rho_b$ to the the same as $\rho_b \otimes \rho_a$. But this is indeed the case since $\rho_a(g) \otimes \rho_b(g) \cong \rho_a(g) \otimes \rho_b(g)$. Thus, by Eq. (7.3.1), we have the same direct product decomposition.
- **Associativity**: The tensor product of linear maps is associative. Thus, $(\rho_a(g) \otimes \rho_b(g)) \otimes \rho_c(g) \cong \rho_a(g) \otimes (\rho_b(g) \otimes \rho_c(g))$ for all $g \in G$. The direct sum decomposition of isomorphic representations is the same.

7.4 Verifying that $(\mathcal{L}, \times)$ is a fusion ring

We claim that $(\mathcal{L}, \times)$ is a fusion ring. To prove this we need to show that all the conditions in the definition of a fusion ring (definition 2.1) are satisfied. Namely, we must show the existence of a fusion channel, finiteness of N_{ab}^c for all $a, b, c \in \mathcal{L}$, existence of an identity element in $\mathcal{L}$, and unique existence of duals (inverses) for each anyon charge.

The first two conditions (existence and finiteness of fusion channels) are very easy to check due to Maschke's theorem: every representation has a finite direct sum decomposition into irreducible representations. So the tensor product of any two representations is either itself an irreducible representation, or it is a finite direct sum of irreducible representations. This implies that $1 \leq \sum_{c \in \mathcal{L}} N_{ab}^c \leq \infty$, which satisfies the first two conditions.

7.4.1 Existence of Identity Element (Trivial Anyon Charge):

For any finite group G, we have the trivial representation $T : G \to \mathrm{GL}_1(V)$ given by $T(g) = [1]\ \forall g \in G$ is an irreducible representation. Thus, the label of the trivial representation is present in $\mathcal{L}$. We claim that this acts as the identity element.

Theorem 7.4.1. *Let $T \in \mathcal{L}$ be the label of the trivial representation of G. Then $R \times T = R$ for any label R.*

Proof. Note that $\rho_R(g) \otimes \rho_T(g) \cong \rho_R(g)$ for all $g \in G$. Since ρ_R is an irreducible representation, we have that $R \times T = R$. □

Thus, the label of the trivial representation is the identity element of the fusion ring.

7.4.2 Existence and Uniqueness of Duals

Definition 7.1 (Complex Conjugate Representation)**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of the finite group G. Then we define the conjugate representation $\overline{\rho}$ to be a group homomorphism $\overline{\rho} : G \to \mathrm{GL}(V)$ defined as $\overline{\rho}(g) = \overline{\rho(g)}$. Here $\overline{A}$ represents the element-by-element complex conjugate of the matrix A.

We can verify that $\overline{\rho}$ is indeed a representation of G:

$$\overline{\rho}(g_1) \circ \overline{\rho}(g_2) = \overline{\rho(g_1)} \circ \overline{\rho(g_2)} = \overline{\rho(g_1) \circ \rho(g_2)} = \overline{\rho(g_1 * g_2)} = \overline{\rho}(g_1 * g_2)$$

Theorem 7.4.2. *Let $\overline{\rho}$ be the conjugate representation of ρ. If ρ is irreducible, then $\overline{\rho}$ is also irreducible.*

Proof. We use the fact that for any representation $\mathcal{R}$, $\langle \chi^{\mathcal{R}}, \chi^{\mathcal{R}} \rangle = 1 \iff \mathcal{R}$ is an irreducible representation. Furthermore, in our case,

$$\chi^{\overline{\rho}}(g) = \mathrm{tr}(\overline{\rho(g)}) = \overline{\mathrm{tr}(\rho(g)} = \overline{\chi^{\rho}(g)}.$$

Thus, we have that:

$$\langle \chi^{\overline{\rho}}, \chi^{\overline{\rho}} \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^{\overline{\rho}}(g)} \chi^{\overline{\rho}}(g) = \frac{1}{|G|} \sum_{g \in G} \chi^{\rho}(g) \overline{\chi^{\rho}(g)} = \langle \chi^{\rho}, \chi^{\rho} \rangle.$$

But since ρ is an irreducible representation, $\langle \chi^{\rho}, \chi^{\rho} \rangle = 1$. Thus, $\langle \chi^{\overline{\rho}}, \chi^{\overline{\rho}} \rangle = 1$ also, and we have that $\overline{\rho}$ is irreducible. □

Theorem 7.4.3 (Decomposition)**.** *Let ρ be an irreducible representation of a finite group G and $\overline{\rho}$ be its conjugate representation. Then the trivial representation appears exactly once in the direct sum decomposition (into irreducible representations) of the representation $\rho \otimes \overline{\rho}$.*

Proof. We can compute exactly the number of times that any given irreducible representation occurs in the direct sum decomposition of a representation via the Hermitian product on characters. This is explained in detail in section 4.4. The number of times the trivial representation T occurs in the decomposition of $\rho \otimes \overline{\rho}$ is

$$\left\langle \chi^{\rho \otimes \overline{\rho}}, \chi^{T} \right\rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^{\rho \otimes \overline{\rho}}(g)} \chi^{T}(g).$$

But $\chi^T(g) = \mathrm{tr}(T(g)) = 1$ for all $g \in G$. Moreover, since the character of the tensor product of two representations is the product of the characters of the respective representations,

$$\chi^{\rho \otimes \overline{\rho}}(g) = \chi^{\rho}(g) \times \chi^{\overline{\rho}}(g) = \chi^{\rho}(g) \times \overline{\chi^{\rho}(g)}.$$

Thus, we have:

$$\left\langle \chi^{\rho \otimes \overline{\rho}}, \chi^{T} \right\rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^{\rho \otimes \overline{\rho}}(g)} \times 1 = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^{\rho}(g) \times \overline{\chi^{\rho}(g)}} = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^{\rho}(g)} \times \chi^{\rho}(g) = \langle \chi^{\rho}, \chi^{\rho} \rangle = 1.$$

□

Theorem 7.4.4 (Existence of Dual Anyon Label)**.** *For any $a \in \mathcal{L}$, $\exists \overline{a} \in \mathcal{L}$ such that $N^I_{a\overline{a}} = 1$ where I is the identity element (label of the Trivial Representation).*

Proof. We have shown that for any $a \in \mathcal{L}$, the corresponding irreducible representation ρ_a has a complex conjugate representation $\overline{\rho_a}$ which is also an irreducible representation, and thus must correspond to an element (possibly a itself) in the label set $\mathcal{L}$. Denote this element as $\overline{a}$. We have also shown that the trivial representation T appears once in the decomposition of $\rho_a \otimes \overline{\rho_a}$ into irreducible representations. Thus, we have found $\overline{a} \in \mathcal{L}$ such that $N^I_{a\overline{a}} = 1$ where I is the label of the trivial representation. ☐

Remark 7.4.1 (Self-Dual Anyon Charges)**.** If the character $\chi^{\mathcal{R}}_a$ of an irreducible representation $\mathcal{R}_a$ of a finite group is real-valued, then the complex conjugate representation $\overline{\mathcal{R}_a} = \mathcal{R}_a$. If a is the label of $\mathcal{R}_a$, then the anyon label a is self-dual (i.e., $N^I_{aa} = 1$)

Theorem 7.4.5 (Uniqueness of Dual Anyon Label)**.** *Let $a \in \mathcal{L}$, and let $\overline{a}$ be such that $N_{a\overline{a}^I=1}$ where I is the label of the trivial representation. Then $N^I_{ab} = 0$ if $b \neq \overline{a}$ where I is the label of the Trivial Rep.*

Proof. Let ρ_a be the irreducible representation corresponding to a. Let $b \in \mathcal{L}$ be such that $N^I_{ab} \neq 0$. If ρ_b is the irreducible representation corresponding to b, then we have:

$$\langle \chi^{\rho_a \otimes \rho_b}, \chi^{\rho_T} \rangle > 0$$

$$\implies \frac{1}{|G|} \sum_{g \in G} \overline{\chi^{\rho_a \otimes \rho_b}(g)} \times 1 = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^{\rho_a}(g)} \times \overline{\chi^{\rho_b}(g)} = \langle \chi^{\rho_a}, \chi^{\rho_x} \rangle > 0$$

where ρ_x is some representation of G such that $\chi^{\rho_x}(g) = \overline{\chi^{\rho_b}(g)} \;\; \forall g \in G$. Observe that the complex conjugate representation $\overline{\rho_b}$ of ρ_b has the same character as ρ_x. Recall that representations are isomorphic iff they have the same character. Thus, $\rho_x \cong \overline{\rho_b}$. By theorem 4.3, ρ_x is an irreducible representation. Due to the orthonormality relations on irreducible characters,

$$\langle \chi^{\rho_a}, \chi^{\rho_x} \rangle = 1 > 0 \iff \rho_a \cong \rho_x \iff \rho_a \cong \overline{\rho_b} \iff \rho_b \cong \overline{\rho_a}.$$

Since we have shown in the previous theorem that $\overline{\rho_a}$ corresponds to the label $\overline{a}$ such that $N_{a\overline{a}} = 1$, we have that $b = \overline{a}$. ☐

7.5 Calculating the Fusion Coefficients

We now turn to the matter of computing fusion coefficients (the set $\{N^c_{ab} \,|\, a, b, c \in \mathcal{L}\}$) for the anyon model constructed above. Recall that, by definition, N^c_{ab} is the number of times the irreducible representation ρ_c appears in the direct sum decomposition of $\rho_a \otimes \rho_b$ into irreducible representations.
Consider, now that:

$$\rho_a \otimes \rho_b \cong n_1\rho_1 \oplus n_2\rho_2 \oplus \dots$$

where the right hand side is a direct sum of irreducible representations and $n_i\rho_i$ represents n_i direct sum copies of the irreducible representation ρ_i. Since the character of isomorphic representations is the same,

$$\chi^{\rho_a \otimes \rho_b}(g) = \chi^{n_1\rho_1 \oplus n_2\rho_2 \oplus \cdots}(g) \;\; \forall g \in G.$$

By the elementary properties of the character, $\chi^{\rho \oplus \rho'}(g) = \chi^{\rho}(g) + \chi^{\rho'}(g)$. Thus:

$$\chi^{\rho_a \otimes \rho_b}(g) = \sum_i n_i \chi^{\rho_i}(g) \;\; \forall g \in G.$$

$$\implies \langle \chi^{\rho_a \otimes \rho_b}, \chi^{\rho_i} \rangle = \Big\langle \sum_j n_j \chi^{\rho_j}, \chi^{\rho_{x_i}} \Big\rangle.$$

By the orthonormality relations on the characters of irreducible representations,

$$\Big\langle \sum_j n_j \chi^{\rho_j}, \chi^{\rho_{x_i}} \Big\rangle = n_i \langle \chi^{\rho_i}, \chi^{\rho_i} \rangle = n_i.$$

Therefore,

$$n_i = \langle \chi^{\rho_a \otimes \rho_b}, \chi^{\rho_i} \rangle$$

which can be simplified as

$$n_i = \langle \chi^{\rho_a} \cdot \chi^{\rho_b}, \chi^{\rho_i} \rangle$$

since $\chi^{\rho_a \otimes \rho_b} = \chi^{\rho_a} \cdot \chi^{\rho_b}$. Thus, in general,

$$N_{ab}^c = \langle \chi^{\rho_a} \cdot \chi^{\rho_b}, \chi^{\rho_c} \rangle \tag{7.5.1}$$

where a, b and c are labels of the irreducible representations ρ_a, ρ_b and ρ_c respectively.

Example 7.5.1 (Rep(S_3)). Consider the group $S_3 = \{1, x, x^2, y, xy, x^2y \,|\, xyxy = 1, x^3 = 1, y^2 = 1\}$. S_3 has 3 conjugacy classes, which consist of the identity, the transpositions and the 3-cycles respectively. Thus, $|\text{Rep}(S_3)| = 3$. The irreducible representations of S_3 are the trivial representation T, the sign representation S and the standard (2-D) representation V wherein V_x is a rotation by $2\pi/3$ radians, and V_y is a reflection is also an irreducible representation. We can compute the characters directly from the matrix representations:

$$\chi^T(\sigma) = 1 \;\; \forall \sigma \in S_3$$

$$\chi^S(\sigma) = \begin{cases} 1, & \text{if } \sigma \text{ is an even permutation} \\ -1, & \text{if } \sigma \text{ is an odd permutation} \end{cases}$$

Moreover, $V(1) = I_2$, $V(x) = \begin{pmatrix} \cos(\frac{2\pi}{3}) & -\sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, $\quad V(y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We can compute $\chi^V(g)$ by taking the trace of these matrices:

$$\chi^V(g) = \begin{cases} 2, & \text{if } g = 1 \\ -1, & \text{if } g \text{ is a 3-cycle i.e.} g \in \{x, x^2\} \\ 0, & \text{if } g \text{ is a transposition i.e.} g \in \{y, xy, x^2y\}. \end{cases}$$

We can write this as a **character table**, which contains explicit values of the irreducible characters evaluated for group elements:

Element	Id	x, x^2	y, xy, x^2y
χ_I	1	1	1
χ_S	1	1	-1
χ_V	2	-1	0

Since characters are constant on conjugacy classes, we only specify the value of each irreducible character on each conjugacy class.

From this character table, it is very easy to compute the fusion coefficients of the anyon model derived from Rep(S_3). Let $\mathcal{L} = \{I, S, V\}$ where I, S and V are the labels of the trivial representation, sign representation and standard representation respectively. The only non-trivial fusion coefficients are N_{VV}^x, N_{SV}^y and N_{SS}^z for $x, y, z \in \mathcal{L}$. They can simply be computed through equation 4.4.1. To demonstrate, we have worked through the examples for N_{VV}^x:

$$N_{VV}^I = \langle (\chi^V)^2, \chi^V \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^V(g)^2} \cdot \chi^I(g) = \frac{1}{|G|} \sum_{g \in G} (\chi^V(g))^2) = \frac{1}{6} \cdot (1 \cdot 4 + 2 \cdot 1 + 3 \cdot 0) = 1$$

$$N_{VV}^S = \langle (\chi^V)^2, \chi^V \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^V(g)^2} \cdot \chi^S(g) = \frac{1}{|G|} \sum_{g \in G} (\chi^V(g))^2 \cdot \chi^S(g) = \frac{1}{6} \cdot (1 \cdot 4 + 2 \cdot 1 + 3 \cdot 0) = 1$$

$$N_{VV}^V = \langle (\chi^V)^2, \chi^V \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{(\chi^V(g))^2} \cdot \chi^V(g) = \frac{1}{|G|} \sum_{g \in G} (\chi^V(g))^3 = \frac{1}{6} \cdot (1 \cdot 8 + 2 \cdot -1 + 3 \cdot 0) = 1.$$

After computing all the fusion coefficients, the fusion rules should come out to be:

- $I \times I = I, \quad I \times S = S, \quad I \times V = V$
- $S \times S = I, \quad S \times V = V$
- $V \times V = I + S + V.$

Example 7.5.2 (Self-Dual Anyon Labels in *Rep*(S_n))**.** If we construct an anyon model from Rep(G) as described in sections 4.1-4.3, then we may wonder when an anyon charge $a \in \mathcal{L}$ in this anyon model might be self-dual (i.e. $N_{aa}^I = 1$). By equation 4.4.1, in order for this to occur, $\langle \chi^{\rho_a} \cdot \chi^{\rho_a}, \chi^T \rangle = 1$ where χ^T is the character of the trivial representation. The anyon model constructed from Rep(S_n) is a special case wherein every anyon label is actually self-dual. We will prove this using the following theorem:

Theorem 7.5.1. *If g and g^{-1} belong to the same conjugacy class of G for all $g \in G$, then for any irreducible representation ρ_a of G, we have that $\langle \chi^{\rho_a} \cdot \chi^{\rho_a}, \chi^T \rangle = 1$.*

Proof.

$$\langle \chi^{\rho_a} \cdot \chi^{\rho_a}, \chi^T \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^{\rho_a(g)} \cdot \chi^{\rho_a}(g)} \cdot 1$$

$$= \langle \chi^{\rho_a} \cdot \chi^{\rho_a}, \chi^T \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^{\rho_a(g)}} \cdot \overline{\chi^{\rho_a}(g)}$$

$$= \frac{1}{|G|} \sum_{g \in G} \overline{\chi^{\rho_a}(g)} \cdot \chi^{\rho_a}(g^{-1}).$$

Recall that characters are constant on conjugacy classes. So, if g and g^{-1} belong to the same conjugacy class of G, then $\chi^{\rho_a}(g^{-1}) = \chi^{\rho_a}(g)$. Thus,

$$\langle \chi^{\rho_a} \cdot \chi^{\rho_a}, \chi^T \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi^{\rho_a}(g)} \cdot \chi^{\rho_a}(g) = \langle \chi^{\rho_a}, \chi^{\rho_a} \rangle = 1.$$

$\square$

This means that if a finite group G has the property that g and g^{-1} are in the same conjugacy class for all $g \in G$, then $N_{aa}^I = 1$ for all $a \in \mathcal{L}$ (where $\mathcal{L}$ corresponds to labels of isomorphism classes in Irr(G)). In particular, the group S_n has this property: each permutation and its inverse have the same disjoint cycle structure and thus belong to the same conjugacy class. Thus anyon labels associated to Rep(S_n) are self-dual.