lezione 19

C. INTEGRAZIONE PER PARTI

$$(fg)' = f'g + fg'$$

$f, g : I \to \mathbb{R}$ derivabili in I

$$D^{-1}(f'g) \neq \phi \neq D^{-1}(fg')$$

$$\Rightarrow \boxed{D^{-1}(f'g) = fg - D^{-1}(fg')} \Leftarrow$$

Esempio: $\int x^2 \sinh x \, dx = \int \underset{g}{x^2} \, \underset{f'}{(\cosh x)'} \, dx$

$= x^2 \cosh x - \int \cosh x \cdot 2x \, dx$

$= x^2 \cosh x - 2 \int \underset{g}{x} \, \cosh x \, dx$, $\cosh x = (\underset{f}{\sinh x})'$

$= x^2 \cosh x - 2x \sinh x + 2 \int \sinh x \, dx$

$= x^2 \cosh x - 2x \sinh x + 2 \cosh x$

Con questo metodo si possono integrare in modo indefinito nei casi in cui l'integrando è del tipo $P(x)\,f(x)$

P è polinomio e $f(x)$ è combinazione di seni/coseni/esponenziali

Esempio:

$$\int \underset{f'}{e^x}\,\underset{g}{\sin x}\,dx = \int \sin x\,(e^x)'\,dx = e^x \sin x - \int \underset{f'}{e^x}\,\underset{g}{\cos x}\,dx$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$$

$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x) \iff \int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x)$$

D. Integrazione funzioni razionali

Esempio: $$\int \frac{2x-1}{x^2+2x+2}\,dx = \int \frac{2(x+1)-3}{(x+1)^2+1}\,dx \quad +$$

$$\boxed{x^2+2x+2 = (x+1)^2+1 \qquad 2x-1 = 2(x+1)-3}$$

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$$= \int \frac{2(x+1)}{(x+1)^2+1}\,dx - 3\underbrace{\int \frac{1}{(x+1)^2+1}\,dx}_{-3\,\mathrm{arctg}(x+1)}$$

$$= \underbrace{\int \frac{\left((x+1)^2+1\right)'}{(x+1)^2+1}\,dx}_{} - 3\,\mathrm{arctg}(x+1)$$

$$\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|$$

$$= \log\left(x^2+2x+2\right) - 3\,\mathrm{arctg}(x+1)$$

In generale se $a, b, u \in \mathbb{R}$ $v > 0$

$$\int \frac{2a(x-u)+b}{(x-u)^2+v^2}\,dx = \underline{a\log\left((x-u)^2+v^2\right) + \frac{b}{v}\,\mathrm{arctg}\left(\frac{x-u}{v}\right)}$$

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Strategia generale per integrare funzioni razionali

$$\frac{P(x)}{Q(x)} \qquad \text{grado} P = n \qquad \text{grado} Q = m \qquad n, m \in \mathbb{N}_0$$

1° PASSO Se $n \geq m$ allora faccio subito la divisione di polinomi

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

2° PASSO Strategia generale è usare il Teorema fondamentale dell'algebra che dice che un polinomio $Q(x)$ monico (cioè il termine di ordine più grande ha coefficiente uguale a 1) e a coefficienti reali ammette sempre la seguente unica fattorizzazione

$$Q(x) = (x-x_1)^{k_1}(x-x_2)^{k_2}\cdots(x-x_r)^{k_r}\left((x-u_1)^2+v_1^2\right)^{m_1}\left((x-u_2)^2+v_2^2\right)^{m_2}\cdots\left((x-u_c)^2+v_c^2\right)^{m_c}$$

$$k_i, m_j \in \mathbb{N}_0 \qquad k_1+k_2+\cdots+k_r+2(m_1+m_2+\cdots+m_c) = m$$

3° PASSO FORMULA DI HERMITE

$$\frac{P(x)}{Q(x)} \qquad \tilde{Q}(x) = (x-x_1)^{k_1-1}(x-x_2)^{k_2-1}\cdots(x-x_r)^{k_r-1}\left((x-u_1)^2+v_1^2\right)^{m_1-1}\cdots\left((x-u_c)^2+v_c^2\right)^{m_c-1}$$

allora esistono numeri reali $c_1, c_2, \dots, c_r, a_1, a_2, \dots, a_c, b_1, b_2, \dots, b_c$

ed un polinomio $\tilde{P}(x)$ di grado $\tilde{P}(x) =$ grado $\tilde{Q}(x) - 1$ tale che

$$\frac{P(x)}{Q(x)} = \frac{c_1}{x-x_1} + \frac{c_2}{x-x_2} + \dots + \frac{c_r}{x-x_r} + \frac{2a_1(x-u_1)+b_1}{(x-u_1)^2+v_1^2} + \dots + \frac{2a_c(x-u_c)+b_c}{(x-u_c)^2+v_c^2} + \underbrace{D\left(\frac{\tilde{P}(x)}{\tilde{Q}(x)}\right)}$$

Esempio: $$\int \frac{x^3-1}{4x^3-x}\,dx$$

1° PASSO DIVISIONE

Trucco: scrivo numeratore come denominatore

$$x^3-1 = \frac{4x^3}{4} - \frac{1}{4}x + \frac{1}{4}x - 1 = \frac{1}{4}(4x^3-x) + \frac{1}{4}x - 1 = \frac{1}{4}\left(\underline{(4x^3-x)} + x - 4\right)$$

$$\frac{x^3-1}{4x^3-x} = \frac{1}{4}\left(1+\frac{x-4}{x(4x^2-1)}\right)$$

$$\frac{P(x)}{Q(x)} = \frac{x-4}{x(4x^2-1)} \qquad Q(x) = x(4x^2-1) = x(2x-1)(2x+1)$$

$$(x-x_1)\quad (2x-x_2)\quad (2x-x_3)$$

$$x_1=0 \qquad x_2=1 \qquad x_3=-1$$

$$m_1=1 \qquad m_2=1 \qquad m_3=1$$

$$\frac{x-4}{x(4x^2-1)} = \frac{C_1}{x}+\frac{C_2}{2x-1}+\frac{C_3}{2x+1} = \frac{C_1(4x^2-1)+C_2x(2x+1)+C_3x(2x-1)}{x(2x-1)(2x+1)}$$

$$4C_1x^2 - C_1 + 2C_2x^2 + C_2x + 2C_3x^2 - C_3x = \underbrace{(4C_1+2C_2+2C_3)x^2 + (C_2-C_3)x - C_1}$$

$$\begin{cases} 4C_1+2C_2+2C_3 = 0 & \text{termine di grado 2} \\ C_2 - C_3 = 1 & \text{" " " 1} \\ -C_1 = -4 & \text{" " " 0} \end{cases}$$

$\boxed{C_1 = 4}$ dalla prima $2C_1 + C_2 + C_3 = 0 \Rightarrow C_2 + C_3 = -8$

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$c_2 + c_3 = -8$ sommo $2c_2 = -7 \Rightarrow \boxed{c_2 = -\frac{7}{2}}$

$\Rightarrow c_2 - c_3 = 1$

$c_2 - 1 = c_3 \Rightarrow \boxed{c_3 = -\frac{9}{2}}$

$$\int \frac{x^3-1}{4x^3-x}\,dx = \int \left(\frac{1}{4} + \frac{1}{x} - \frac{7}{8}\frac{1}{2x-1} - \frac{9}{8}\frac{1}{2x+1} \right) dx$$

$$= \frac{1}{4}x + \log|x| - \frac{7}{16}\log|2x-1| - \frac{9}{16}\log|2x+1|$$

$$= \frac{x}{4} + \frac{1}{16}\log\frac{x^{16}}{|2x-1|^7\,|2x+1|^9} = \frac{x}{4} + \frac{1}{16}\log\frac{x^{16}}{|4x^2-1|^7\,|2x+1|^2}$$

Esempio 2: $\displaystyle\int \frac{dx}{(x+1)(x^2+1)^2}$

$Q(x) = (x+1)(x^2+1)^2$ $\quad x_1 = -1 \quad k_1 = 1 \quad x+1$

$u_1 = 0 \quad v_1 = 1 \quad m_1 = 2 \quad x^2+1$

$\tilde{Q}(x) = x^2+1$ ha grado $\Rightarrow \tilde{P}$ deve avere grado 1 $\quad \tilde{P}(x) = Ax + B$

$$\frac{1}{(x+1)(x^2+1)^2} = \frac{c}{x+1} + \frac{2ax+b}{x^2+1} + D\left(\frac{Ax+B}{x^2+1}\right) = *$$

$$D\left(\frac{Ax+B}{x^2+1}\right) = \frac{A(x^2+1) - (Ax+B)2x}{(x^2+1)^2} = \frac{-Ax^2 - 2Bx + A}{(x^2+1)^2}$$

$$= \frac{c}{x+1} + \frac{2ax+b}{x^2+1} + \frac{-Ax^2 - 2Bx + A}{(x^2+1)^2}$$

$$= \frac{c(x^2+1)^2 + (2ax+b)(x+1)(x^2+1) + (-Ax^2-2Bx+A)(x+1)}{(x+1)(x^2+1)^2}$$

$$\begin{cases} c + 2a = 0 & 4^\circ \text{ [illegible]} \\ 2a + b - A = 0 & 3^\circ \text{ [illegible]} \\ 2c + 2a + b - 2B - A = 0 & 2^\circ \text{ [illegible]} \\ 2a + b + A - 2B = 0 & 1^\circ \text{ [illegible]} \\ c + b + A = 1 \end{cases} \iff \boxed{\begin{cases} a = -\frac{1}{8} \\ b = \frac{1}{2} \\ c = \frac{1}{4} \\ A = \frac{1}{4} \\ B = \frac{1}{4} \end{cases}}$$

$$\int \frac{dx}{(x+1)(x^2+1)^2} = \int \left(\frac{1}{4}\,\frac{1}{x+1} + \frac{2\left(-\frac{1}{8}\right)x + \frac{1}{2}}{x^2+1} + \frac{1}{4}\left(\frac{x+1}{x^2+1}\right)' \right) dx$$

$$= \frac{1}{4}\,\frac{x+1}{x^2+1} + \frac{1}{4}\log|x+1| - \frac{1}{8}\log(x^2+1) + \frac{1}{2}\operatorname{arctg} x$$

CAMBIO DELLE VARIABILI

Deriva dal Teorema delle sostituzioni, utilizziamo le stesse ipotesi, in più supponiamo che la funzione $g : I \to J$ sia anche iniettiva e quindi ha inversa $g^{-1} : J \to I$

$$y \to x$$

Dalla formula del Teorema di sostituzione possiamo dedurre

se $\hat{F} \in D^{-1}(f \circ g \cdot g') \Rightarrow \hat{F}\Big|_{x = g^{-1}(y)} = \hat{F} \circ g \in D^{-1} f$

Nella notazione di Leibniz viene una formula più suggestiva, cioè

$$I \ni x \longrightarrow y = y(x) = g(x) \in J$$

$$y \longmapsto x(y) = g^{-1}(y)$$

$$\boxed{\int (f \circ y)(x)\, \frac{dy(x)}{dx}\, dx \Big|_{x = x(y)} =} \int f(y)\, dy$$

$$\int f(y)\, dy = \int f(y(x))\, \frac{dy}{dx}\, dx \Big|_{x = x(y)}$$

Esempio:

$$\int \frac{1}{e^y + 1}\, dy \qquad x = e^y + 1 \Rightarrow y = \log(x-1) \qquad x > 1$$

$$x - 1 = e^y \qquad \frac{dy}{dx} = y' = \frac{1}{x-1}$$

$$= \int \frac{1}{x} \cdot \frac{1}{x-1}\, dx = \int \left(\frac{1}{x-1} - \frac{1}{x} \right) dx = \log(x-1) - \log|x|$$

$$= \log \left| \frac{x-1}{x} \right|$$

$$\int \frac{dy}{e^y+1} = \log \frac{e^y}{e^y+1} = \log \frac{1}{1+e^{-y}} = -\log(1+e^{-y})$$

Esempio 2: $\displaystyle\int \frac{y^2}{\sqrt{1-y^2}}\,dy$ $\qquad |y|<1$

$\sqrt{1-y^2}$ → suggerimento, le funzioni seno e coseno

$$\cos^2 x + \sin^2 x = 1 \Rightarrow \cos^2 x = 1-\sin^2 x$$

$$y = \sin x \Rightarrow x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$y' = \cos x$$

$$\sqrt{1-y^2} = \sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = |\cos x| = \cos x \qquad x \in (-\pi/2, \pi/2)$$

$$\int \frac{y^2}{\sqrt{1-y^2}}\,dy = \int \frac{\sin^2 x}{\cancel{\cos x}} \cdot \cancel{\cos x}\,dx = \int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) = \frac{1}{2}(x - \sin x \cos x) = \frac{1}{2}\left(x - \sin x\sqrt{1-\sin^2 x}\right)\Big|_{x=\arcsin y}$$

$$= \frac{1}{2}\left(\arcsin y - y\sqrt{1-y^2}\right)$$

Esempio 3: $\displaystyle\int \frac{dy}{(y+1)\sqrt{y^2+2y}}$ $\qquad y \geq 0$

$$y^2+2y = (y+1)^2 - 1 \Rightarrow x = y+1 \Rightarrow y = x-1 \qquad y' = 1$$

$$\int \frac{dy}{(y+1)\sqrt{y^2+2y}} = \int \frac{1}{x\sqrt{x^2-1}}\,dx$$

$$ch^2 t - sh^2 t = 1 \Rightarrow sh^2 t = ch^2 t - 1 \Rightarrow |\sinh t| = \sqrt{ch^2 t - 1}$$

$\Rightarrow$ suggerimento $x = \cosh t \qquad t \geq 0 \qquad x' = \sinh t$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{1}{\cosh t} \cdot \frac{1}{\cancel{\sinh t}}\,\cancel{\sinh t}\,dt = \int \frac{dt}{\cosh t}$$

$$= 2\int \frac{dt}{e^t + e^{-t}} \Rightarrow s = e^t \Rightarrow t = \log s \Rightarrow t' = \frac{1}{s}$$

$$= 2\int \frac{dt}{e^{t}+e^{-t}} = 2\int \frac{1}{s\left(s+\frac{1}{s}\right)}\,ds = 2\int \frac{1}{s^2+1}\,ds$$

$$= 2\,\operatorname{arctg} s$$

$$2\,\operatorname{arctg} s \Big|_{s=s(t)} = 2\,\operatorname{arctg} e^{t} \Big|_{t=\cosh^{-1}x} =$$

$$t = \cosh^{-1} x = \log\left(x+\sqrt{x^2-1}\right)$$

$$\Rightarrow \quad e^{t} = x+\sqrt{x^2-1}$$

$$= 2\,\operatorname{arctg}\left(x+\sqrt{x^2-1}\right) \Big|_{x=y+1} = 2\,\operatorname{arctg}\left(y+1+\sqrt{y^2+2y}\right)$$

$$\int \frac{1}{(y+1)\sqrt{y^2+2y}}\,dy = 2\,\operatorname{arctg}\left(y+1+\sqrt{y^2+2y}\right) +$$