Università di Trento – Dip. di Ingegneria e Scienza dell'Informazione
CdL in Informatica, Ingegneria dell'informazione e delle comunicazioni e
Ingegneria dell'informazione e organizzazione d'impresa
a.a. 2017-18 – PIAZZA 2 – "Disequazioni"

1.1)(*) $\log_4 \frac{1}{16} = x \iff 4^x = \frac{1}{16} \iff 4^x = 4^{-2} \Rightarrow \underline{\underline{x=-2}}$.

$\log_2 \sqrt[4]{2} = x \iff 2^x = 2^{\frac{1}{4}} \Rightarrow \underline{\underline{x = \frac{1}{4}}}$.

$\log_{\sqrt{5}} 125 = x \iff (\sqrt{5})^x = 5^3 \iff 5^{\frac{x}{2}} = 5^3 \Rightarrow \underline{\underline{x=6.}}$

$\log_{\frac{1}{4}} \frac{4}{64} = x \iff \left(\frac{1}{4}\right)^x = \frac{1}{16} \iff \left(\frac{1}{4}\right)^x = \left(\frac{1}{4}\right)^2 \Rightarrow \underline{\underline{x=2.}}$

1.2) $x^2 - x + 2 = y \iff \left(x-\frac{1}{2}\right)^2 - \frac{1}{4} + 2 = y \iff \left(x-\frac{1}{2}\right)^2 + \frac{7}{4} = y$

$\iff \left(x-\frac{1}{2}\right)^2 = y - \frac{7}{4}$.

Allora, se $y < \frac{7}{4}$ $\nexists$ soluz. dell'eq. $x^2-x+2=y$;

se $y = \frac{7}{4}$ $\exists!$ soluzione dell'eq. $x^2-x+2=y$ $\left(x=\frac{1}{2}\right)$;

se $y > \frac{7}{4}$ $\exists$ono due soluzioni dell'eq. $x^2-x+2=y$

$\left(x_1 = \frac{1}{2} - \sqrt{y-\frac{7}{4}}\ ,\ x_2 = \frac{1}{2} + \sqrt{y - \frac{7}{4}}\right)$.

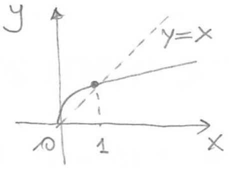
1.3) a) $\sqrt{|x-1|+2x}$ è ben definito $\forall x \in \mathbb{R}$: $|x-1|+2x \geq 0$.

Ora $|x-1|+2x \geq 0 \iff |x-1| \geq -2x$.

Poiché $|x-1| \geq 0$ $\forall x \in \mathbb{R}$, $|x-1| \geq -2x$ è sempre vero $\forall x \geq 0$ (essendo in questo caso $-2x \leq 0$).

D'altra parte, se $x<0$, allora $|x-1| \geq -2x \iff$
$x-1 \leq 2x$ o $x-1 \geq -2x$ ossia
$x \geq -1$ o $x \geq \frac{1}{3}$. In conclusione, $\sqrt{|x-1|+2x}$
è ben definito $\underline{\underline{\forall x \in [-1, +\infty[}}$.

b) $\sqrt[3]{\log(x-\sqrt{x})}$ è ben definito $\forall x \in \mathbb{R}$: $\begin{cases} x \geq 0 \\ x - \sqrt{x} > 0 \end{cases}$



$\iff \begin{cases} x \geq 0 \\ x > \sqrt{x} \end{cases} \Rightarrow \underline{\underline{]1, +\infty[}}$

(*) Si può anche scrivere
$\log_4 \frac{1}{16} = \log_4 4^{-2} = -2 \log_4 4 = \underline{\underline{-2}}$; $\log_2 \sqrt[4]{2} = \log_2 2^{1/4} = \frac{1}{4}\log_2 2 = \underline{\underline{\frac{1}{4}}}$; $\log_{\sqrt{5}} 125 = \log_{\sqrt{5}} (\sqrt{5})^6 = 6 \log_{\sqrt{5}} \sqrt{5} = \underline{\underline{6.}}$
$\log_{1/4} \frac{4}{64} = \log_{1/4} \frac{1}{16} = \log_{1/4} \left(\frac{1}{4}\right)^2 = 2 \log_{1/4} \frac{1}{4} = \underline{\underline{2}}$. □

c) $\sqrt[4]{1-\log x}$ è ben definito $\forall x \in \mathbb{R}$: $\begin{cases} x>0 \\ 1-\log x > 0 \end{cases}$

$$\Leftrightarrow \begin{cases} x>0 \\ \log x < \underset{\log e}{\overset{\parallel}{1}} \end{cases} \Rightarrow \underline{\underline{]0,e[}}.$$

d) $\log|e^x - \frac{1}{e}|$ è ben definito $\forall x \in \mathbb{R}$: $e^x - \frac{1}{e} \neq 0 \Leftrightarrow e^x \neq e^{-1}$

$$\Rightarrow \underline{\underline{\mathbb{R}\setminus\{-1\}}}.$$ ■

1.4) i) $2\log_{\frac{1}{2}}(x-1) - \log_{\frac{1}{2}} x^2 > 0 \Leftrightarrow \begin{cases} x-1>0 \\ x\neq 0 \\ \log_{\frac{1}{2}}(x-1)^2 > \log_{\frac{1}{2}} x^2 \end{cases}$

$$\Leftrightarrow \begin{cases} x>1 \\ (x-1)^2 < x^2 \end{cases} \Leftrightarrow \begin{cases} x>1 \\ \cancel{x^2}-2x+1 < \cancel{x^2} \end{cases}$$

$$\Leftrightarrow \begin{cases} x>1 \\ 2x>1 \end{cases} \Rightarrow \underline{\underline{S =]1,+\infty[}}.$$ □

ii) $\log_3\left(2x+\sqrt{x^2-1}\right) < \log_3 2x \Leftrightarrow \begin{cases} x^2-1 \geq 0 \\ x>0 \\ 2x+\sqrt{x^2-1} > 0 \\ 2x+\sqrt{x^2-1} < 2x \end{cases}$

$$\Leftrightarrow \begin{cases} x \geq 1 \\ \sqrt{x^2-1} < 0 \end{cases} \Rightarrow \underline{\underline{S=\emptyset}}.$$ □

iii) $\log(x-\sqrt{x}) < 1 \Leftrightarrow \begin{cases} x\geq 0 \\ x-\sqrt{x} > 0 \\ \log(x-\sqrt{x}) < \log e \end{cases} \Leftrightarrow \begin{cases} x \geq 0 \\ x > \sqrt{x} \\ x-\sqrt{x} < e \end{cases}$

$$\Leftrightarrow \begin{cases} x\geq 0 \\ x>1 \\ x-\sqrt{x}-e<0 \end{cases} \rightsquigarrow \sqrt{x}=t \quad t^2-t-e<0$$

$$t_{1/2} = \frac{1\pm\sqrt{1+4e}}{2}$$

$$\frac{1-\sqrt{1+4e}}{2} < t < \frac{1+\sqrt{1+4e}}{2} \rightsquigarrow \underline{\underline{S = \left]1, \left(\frac{1+\sqrt{1+4e}}{2}\right)^2\right[}}.$$ □

1.5) i) $|1-|x^2-1|| \leq 2 \Leftrightarrow -2 \leq 1-|x^2-1| \leq 2$

$$\Leftrightarrow -3 \leq -|x^2-1| \leq 1$$

$$\Longleftrightarrow \quad \underbrace{-1 \le}_{\text{sempre soddisfatta}} |x^2-1| \le 3 \quad \Longleftrightarrow \quad -3 \le x^2-1 \le 3$$

$$\Longleftrightarrow \quad \underbrace{-2 \le}_{\text{sempre soddisfatta}} x^2 \le 4$$

$$\Longrightarrow \quad S = [-2,2]. \qquad \square$$

$$e^{|x|} e^{1-x^2} > e \quad \Longleftrightarrow \quad e^{|x|+1-x^2} > e^1 \quad \Longleftrightarrow \quad |x| + \cancel{1} - x^2 > \cancel{1}$$

$$\Longleftrightarrow \begin{cases} x \geq 0 \\ x - x^2 > 0 \end{cases} \quad \text{o} \quad \begin{cases} x < 0 \\ -x - x^2 > 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x \geq 0 \\ x^2 - x < 0 \end{cases} \quad \text{o} \quad \begin{cases} x < 0 \\ x^2 + x < 0 \end{cases}$$

$$\Longrightarrow \quad x \in \left]0,1\right[\qquad \left]-1,0\right[$$

$$\Longrightarrow \quad S = \left]-1,1\right[\setminus \{0\}. \qquad \square$$

ii) $$(2-|x|)\, e^{x^3-1} < 0 \quad \underset{\left(e^{x^3-1}>0 \ \forall x \in \mathbb{R}\right)}{\Longleftrightarrow} \quad 2-|x| < 0 \quad \Longleftrightarrow \quad |x| > 2$$

$$S = \left]-\infty,-2\right[\cup \left]2,+\infty\right[. \qquad \square$$

$$(2-|x|) \cdot \log(x^2-1) < 0 \quad \Longleftrightarrow \quad \begin{cases} x^2-1>0 \\ 2-|x|<0 \\ \log(x^2-1)>0 \end{cases} \quad \text{o} \quad \begin{cases} x^2-1>0 \\ 2-|x|>0 \\ \log(x^2-1)<0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x<-1 \ \text{o} \ x>1 \\ x<-2 \ \text{o} \ x>2 \\ x^2-1>1 \end{cases} \quad \text{o} \quad \begin{matrix} x<-1 \ \text{o} \ x>1 \\ -2<x<2 \\ x^2-1<1 \end{matrix}$$

$$\Longleftrightarrow \quad x \in \left]-\infty,-2\right[\cup \left]2,+\infty\right[\qquad x \in \left]-\sqrt{2},-1\right[\cup \left]1,\sqrt{2}\right[$$

$$\Longrightarrow \quad S = \left]-\infty,-2\right[\cup \left]-\sqrt{2},-1\right[\cup \left]1,\sqrt{2}\right[\cup \left]2,+\infty\right[. \qquad \square$$

iii) $$t^2 - (\log_2 4)\, t + \log_3 \tfrac{1}{3} \geq 0 \quad \Longleftrightarrow \quad t^2 - 2t - 1 \geq 0$$

oss. $t^2-2t-1=0 \iff t_{1/2} = \dfrac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

$$\Longrightarrow \quad S = \left]-\infty, 1-\sqrt{2}\right] \cup \left[1+\sqrt{2}, +\infty\right[. \qquad \square$$

$$e^{2x} - (\log_2 4)\, e^x + \log_3 \tfrac{1}{3} < 0 \quad \Longleftrightarrow \quad \underbrace{1-\sqrt{2} < }_{\text{sempre}} e^x < 1+\sqrt{2}$$

$$\Longrightarrow \quad x \in \left]-\infty, \log(1+\sqrt{2})\right[. \qquad \blacksquare$$

1.6) a) $\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : e^x = y$ (F) $e^x > 0 \ \forall x \in \mathbb{R}$; quindi $\forall y \leq 0$
$\nexists x \in \mathbb{R} : e^x = y$. ☐

b) $\exists y \in \mathbb{R} : \forall x \in \mathbb{R}, e^x > y$ (V) Per esempio, $y = 0$. ☐

c) $\exists x \in \mathbb{R}, x > 0 : x^6 = 4$ (V) $x = \sqrt[6]{4}$. ☐

d) $\exists x \in \mathbb{R}, x < 0 : 2^x = 3^{-1}$ (V) $2^x = \frac{1}{3} \Leftrightarrow 2^x = 2^{\log_2 1/3}$
$\Leftrightarrow 2^x = 2^{-\log_2 3}$
$\Rightarrow x = -\log_2 3$ ■

1.7) i) $\log(x^2-1) = \log(x+1) + \log(x-1)$
$\forall x \in]-\infty, -1[\cup]1, +\infty[$ (F) ☐

ii) $\log(x^2-1) = \log(x+1) + \log(x-1)$
$\forall x \in]1, +\infty[$ (V) $\begin{cases} x^2-1 > 0 \\ x+1 > 0 \\ x-1 > 0 \end{cases} \Leftrightarrow x > 1$ ☐

iii) $\log(x^2-1) = [\log(x+1)][\log(x-1)]$
$\forall x \in]1, +\infty[$ (F) $\left(\begin{matrix} x=2 \\ \text{Falso} \end{matrix} \ \log(3) \overset{?}{=} (\cancel{\log 3})(\cancel{\log 1}) = 0 \right)$ ■

1.8) $A = \{x_n = \frac{3}{n} - \frac{1}{n^2} : n \in \mathbb{N}, n \geq 1\} = \{2, \frac{3}{2} - \frac{1}{4}, 1 - \frac{1}{9}, \frac{3}{4} - \frac{1}{16}, \dots\}$

i) $x_n = \frac{3}{n} - \frac{1}{n^2} \geq \frac{3}{n} - \frac{1}{n} = \frac{2}{n} > 0 \ \forall n$

$x_n = \frac{3}{n} - \frac{1}{n^2} \leq 2 \Leftrightarrow 3n - 1 \leq 2n^2 \Leftrightarrow 2n^2 - 3n + 1 \geq 0$
$\forall n \geq 1$

Ora $2n^2 - 3n + 1 = 0 \Leftrightarrow n_{1/2} = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4}$
$= \begin{cases} 1 \\ 1/2 \end{cases}$

Quindi $2n^2 - 3n + 1 \geq 0 \Leftrightarrow n \geq 1$ ☐

ii) $\underline{\inf A = 0}$. Abbiamo $0 < x_n \ \forall n \in \mathbb{N}, n \geq 1$. Dobbiamo provare che
$\forall \varepsilon > 0 \ \exists x_n \in A : x_n < 0 + \varepsilon$. Ora $\frac{3}{n} - \frac{1}{n^2} < \varepsilon$
$\Leftrightarrow 0 < \varepsilon n^2 - 3n + 1 \Leftrightarrow n > \frac{3 + \sqrt{9 - 4\varepsilon}}{2\varepsilon}$. Basta prendere

$\varepsilon n^2 - 3n + 1 = 0 \quad n_{1/2} = \frac{3 \pm \sqrt{9 - 4\varepsilon}}{2\varepsilon}$

$n > \frac{3}{\varepsilon}$.

$\underline{\underline{\sup A = \max A = 2}}$ $(x_1 = \max A)$. ☐

iii) $\inf A$ non è $\min A$; $\sup A = \max A = 2$. ■