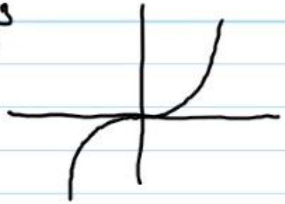
LEZIONE 12^a

DERIVAZIONE

$f(0) = 0$

$|x|$

x^3



$\sqrt[3]{x}$

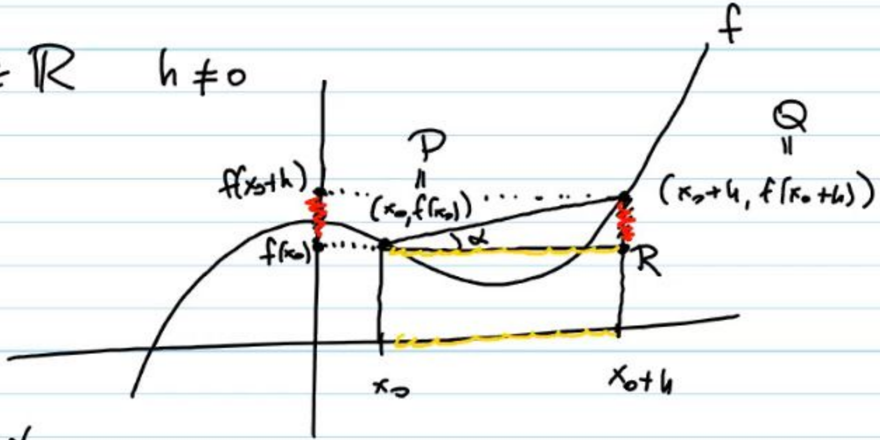
$\operatorname{sen} x$

RAPPORTO INCREMENTALE

$f: \mathbb{R} \to \mathbb{R} \qquad x_0 \in \mathbb{R} \qquad h \neq 0$

$$\frac{f(x_0+h) - f(x_0)}{h}$$

Significato geometrico



$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{\overline{QR}}{\overline{PR}} = \operatorname{tg} \alpha$$

$=$ coefficiente angolare retta passante per P e Q.

LEZIONE 12^a

DERIVAZIONE

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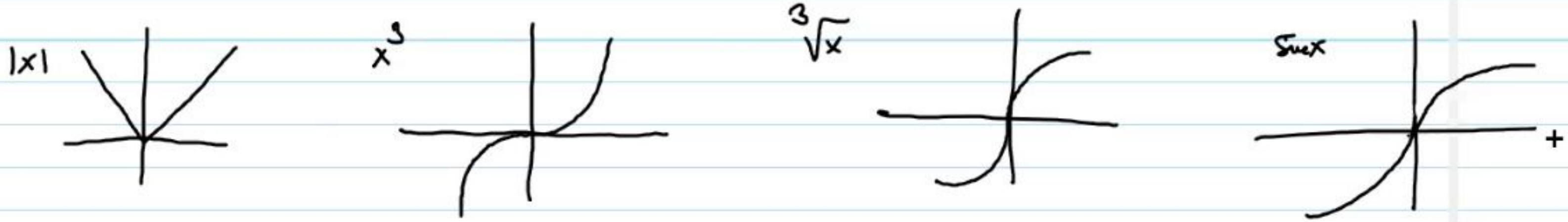

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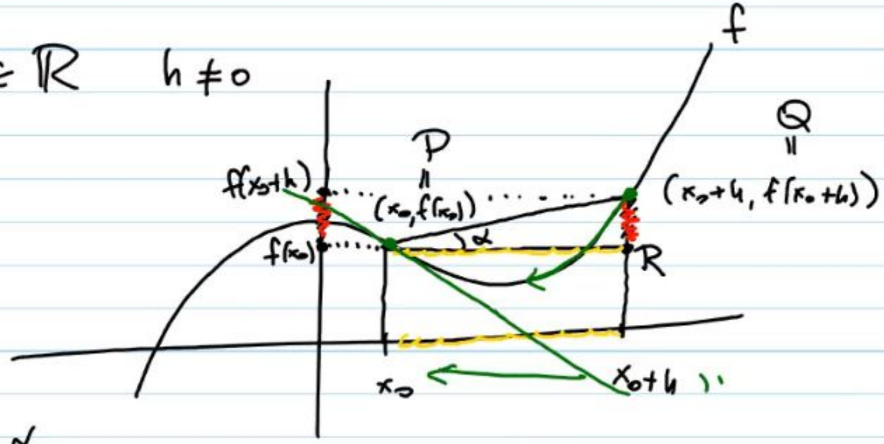


$$\frac{f(x_0+h) - f(x_0)}{h} = \frac{\overline{QR}}{\overline{PR}} = \operatorname{tg} \alpha$$

$=$ coefficiente angolare retta passante per P e Q la derivata può non esistere: esempi

1. $f(x) = |x| \qquad x_0 = 0$

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

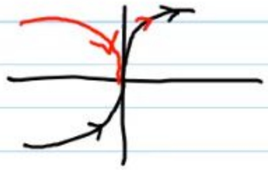
NON ESISTE!

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$\neq$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

$\sqrt[3]{|x|}$



2. $f(x) = \sqrt[3]{x} \qquad x_0 = 0$

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(h) - \cancel{f(0)}}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}} = +\infty$$

DEF 12.2. Si dice che f è DIFFERENZIABILE in x_0 se esiste $\alpha \in \mathbb{R}$ tale che

$$f(x_0+h) = f(x_0) + \alpha h + o(h) \qquad h \to 0 \qquad (DIF) \leftarrow$$

In tal caso α è detto il DIFFERENZIALE di f in x_0

Teorema "Una funzione f è differenziabile in x_0 se e solo se f è derivabile in x_0. Inoltre, se questo è vero allora $\alpha = f'(x_0)$."

Dim: ($\Rightarrow$) Supp f sia differenziabile in x_0 (DIF)

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{\alpha h + o(h)}{h} = \lim_{h\to 0}\left(\alpha + \frac{o(h)}{h}\right) = \alpha$$

($\Leftarrow$) Sia f derivabile in x_0 cioè $f'(x_0) \in \mathbb{R}$ esiste, devo dimostrare che vale (DIF) cioè

$$f(x_0+h) - f(x_0) - \underbrace{\alpha}_{= f'(x_0)} h = o(h) \qquad h \to 0$$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)-f'(x_0)\,h}{h} \stackrel{?}{=} 0$$

$$\lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0)\right) = 0 \ !$$

$\blacksquare$

Teorema "Ogni funzione f derivabile in x_0 è continua in x_0."

dim: $\lim_{x\to x_0} f(x) = f(x_0)$.

$h = x - x_0$ se $x\to x_0 \Rightarrow h\to 0$

$$\lim_{x\to x_0} f(x) = \lim_{h\to 0} f(x_0+h) \stackrel{(DF)}{=} \lim_{h\to 0}\Big(f(x_0)+\underbrace{f'(x_0)\,h}_{\to 0}+\underbrace{o(h)}_{\to 0}\Big)$$

$$= f(x_0)$$

$\blacksquare$

Regole della derivazione e derivate funzioni elementari

$$(\text{costante})' = 0$$

$$(e^x)' = e^x \qquad (a^x)' = a^x \cdot \log a \qquad (\log x)' = \frac{1}{x}$$

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$$

$$(\operatorname{arctg} x)' = \frac{1}{1+x^2} \qquad (\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\operatorname{tg} x)' = \frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x \qquad (x^\alpha)' = \alpha x^{\alpha-1} \quad \alpha \in \mathbb{R}\setminus\{0\}$$

$$(f \pm g)' = f' \pm g' \qquad (\alpha f)' = \alpha f' \quad \alpha \in \mathbb{R}$$

$$(f \cdot g)' = f'g + f \cdot g' \qquad \left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \qquad \big(f(g(x))\big)' = f'(g(x)) \cdot g'(x).$$

Esempio 1 Calcolare la derivata di $f(x) = x^3$

$\boxed{1^\circ \text{ modo}}$

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+h)^3 - x_0^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x_0^3} + 3x_0^2 h + 3x_0 h^2 + h^3 - \cancel{x_0^3}}{h} = \lim_{h\to 0}\Big(3x_0^2 + \underbrace{3x_0 h}_{\to 0} + \underbrace{h^2}_{\to 0}\Big)^{+} = 3x_0^2$$

$$f'(x) = (x^3)' = 3x^2$$

$\boxed{2^\circ \text{ modo}}$

$$f(x_0+h) = (x_0+h)^3 = x_0^3 + \underline{3x_0^2 h} + \underbrace{3x_0 h^2 + h^3} \qquad h \to 0$$

$$= \underbrace{x_0^3}_{= f(x_0)} + \underbrace{3x_0^2}_{\alpha} h + \underbrace{3x_0 h^2 + h^3}_{o(h)} \quad h \to 0$$

Esempio 2. $f(x) = e^x$

$\boxed{1^\circ \text{ modo}}$

$$f'(x_0) = \lim_{h\to 0} \frac{e^{x_0+h} - e^{x_0}}{h} = \lim_{h\to 0} \frac{e^{x_0}\cdot e^h - e^{x_0}}{h}$$

$$= \lim_{h\to 0} e^{x_0} \underbrace{\frac{e^h - 1}{h}}_{\to 1} = e^{x_0}$$

$$x \longmapsto f'(x) = (e^x)' = e^x$$

derivate successive $\quad (f'(x))' = f''(x) \qquad f^{(2)}(x)$

$$\frac{d^2 f}{dx^2}(x)$$

$$D^2 f(x)$$

$\boxed{2^\circ \text{ modo}}$

$$f(x_0+h) = e^{x_0+h} = e^{x_0}\cdot e^h \qquad h\to 0$$

$$= e^{x_0}(1 + h + o(h)) \qquad h\to 0$$

$$= \underbrace{e^{x_0}}_{f(x_0)} + \underbrace{e^{x_0}}_{\alpha} h + \underbrace{e^{x_0}\, o(h)}_{o(h)} \qquad h\to 0$$

$$f(x_0) + \alpha h + o(h) \qquad h\to 0$$

Esempio 3 $\qquad f(x) = \log x$

$\boxed{1^\circ \text{ modo}}$

$$f'(x_0) = \lim_{h \to 0} \frac{\log(x_0+h) - \log(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{\log\left(x_0\left(1+\frac{h}{x_0}\right)\right) - \log(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{\log x_0} + \log\left(1+\frac{h}{x_0}\right) - \cancel{\log x_0}}{h}$$

$$= \lim_{h \to 0} \frac{\log\left(1+\frac{h}{x_0}\right)}{\frac{h}{x_0} \cdot x_0} = \frac{1}{x_0} \lim_{t \to 0} \frac{\log(1+t)}{t} \to 1$$

$$t = \frac{h}{x_0}$$

$$f'(x_0) = (\log x)'\Big|_{x=x_0} = \frac{1}{x_0}$$

$$\mathbb{R}_+ \ni x \longmapsto \log x \implies f'(x) = (\log x)' = \frac{1}{x}$$

$\boxed{2^\circ \text{ modo}}$

$$\log(x_0+h) = \log\left(x_0\left(1+\frac{h}{x_0}\right)\right) \qquad h\to 0$$

$$= \log x_0 + \log\left(1+\frac{h}{x_0}\right) \qquad h\to 0$$

$$= \underbrace{\log x_0}_{f(x_0)} + \underbrace{\frac{h}{x_0}}_{\alpha=\frac{1}{x_0}\cdot h} + o(h) \qquad h\to 0$$

Esempio 4. $\quad f(x)=\cos x$

$\boxed{1^\circ \text{ modo}}$

$$f'(x_0) = \lim_{h\to 0} \frac{\cos(x_0+h)-\cos x_0}{h} =$$

$$= \lim_{h\to 0} \frac{\cos x_0 \cos h - \sin x_0 \sin h - \cos x_0}{h}$$

$$= \lim_{h\to 0} \left(\cos x_0 \underbrace{\frac{\cos h - 1}{h^2}}_{\to -\frac{1}{2}} \cdot h - \sin x_0 \underbrace{\frac{\sin h}{h}}_{\to 1} \right) = -\sin x_0$$

$\boxed{2^\circ \text{ modo}}$

$$\cos(x_0+h) = \cos x_0 \cos h - \sin x_0 \sin h \qquad h \to 0$$

$$= \cos x_0 (1 + o(h)) - \sin x_0 (h + o(h)) \qquad h \to 0$$

$$= \cos x_0 + \underbrace{o(h) - \sin x_0 \cdot h + o(h)}_{o(h)} \qquad h \to 0$$

$$= \underbrace{\cos x_0} - \underbrace{\sin x_0 \cdot h} + o(h) \qquad h \to 0$$

$$f(x_0) + \alpha \cdot h + o(h) \qquad h \to 0$$

$$-\sin x_0$$

Regole

DERIVATA DELLA SOMMA $\qquad S(x) = f(x) + g(x)$

$$S'(x_0) = \lim_{h\to 0} \frac{S(x_0+h) - S(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0+h) + g(x_0+h) - f(x_0) - g(x_0)}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(x_0+h) - f(x_0)}{h} + \frac{g(x_0+h) - g(x_0)}{h} \right) = f'(x_0) + g'(x_0)$$

DERIVATA DEL PRODOTTO

$$P(x) = f(x)\cdot g(x)$$

$$P'(x_0) = \lim_{h\to 0} \frac{P(x_0+h) - P(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0+h)\,g(x_0+h) - f(x_0)\,g(x_0)}{h}$$

$$= \lim_{h\to 0} \frac{f(x_0+h)\,g(x_0+h) - f(x_0+h)\,g(x_0) + f(x_0+h)\,g(x_0) - f(x_0)\,g(x_0)}{h}$$

$$= \lim_{h\to 0} \left[f(x_0+h) \left(\frac{g(x_0+h) - g(x_0)}{h} \right) + g(x_0) \left(\frac{f(x_0+h) - f(x_0)}{h} \right) \right]$$

$$\boxed{f(x_0) \cdot g'(x_0) \quad + \quad g(x_0)\, f'(x_0)}$$

$\lim_{h\to 0} f(x_0+h) = f(x_0)$ <u>continuità</u> di f in x_0 che deriva dalla derivabilità di f in x_0.

DERIVATA DEL RECIPROCO

$$R(x) = \frac{1}{f(x)}$$

$$R'(x_0) = \lim_{h\to 0} \frac{R(x_0+h) - R(x_0)}{h} = \lim_{h\to 0} \frac{\frac{1}{f(x_0+h)} - \frac{1}{f(x_0)}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\frac{f(x_0) - f(x_0+h)}{f(x_0)\,f(x_0+h)} = \lim_{h\to 0}\left(-\frac{f(x_0+h)-f(x_0)}{h}\cdot\frac{1}{f(x_0)f(x_0+h)}\right)$$

$$-f'(x_0) \qquad \frac{1}{f(x_0)^2}$$

$$\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{f(x)^2}$$

DERIVATA DEL RAPPORTO

$$Q(x) = \frac{f(x)}{g(x)}$$

$$\left[\frac{f}{g}\right]' = \left[f\cdot\frac{1}{g}\right]' = f'\frac{1}{g} + f\cdot\left(\frac{1}{g}\right)' = f'\frac{1}{g} - f\frac{g'}{g^2}$$

$$= \frac{f'g - fg'}{g^2}$$

Esempio

$$(\operatorname{tg}x)' = \Bigg(\frac{\overset{f}{\sin x}}{\underset{g}{\cos x}}\Bigg)' = \frac{\cos^2 x - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\boxed{= \frac{1}{\cos^2 x} = 1 + (\operatorname{tg} x)^2}$$

esercizio per casa

$$(x^k)' = k\,x^{k-1}$$

$$(x^k)' = (x^{k-1}\cdot x)' = \text{LEIBNIZ}$$

DERIVATA DELLA COMPOSIZIONE

$$C(x) = f(g(x))$$

Ipotesi

g è derivabile in x_0 $\longrightarrow$ $\boxed{g(x_0+h) = g(x_0) + g'(x_0)h + o(h) \quad h\to 0}$

f è derivabile in $g(x_0)$ $\quad$ $\boxed{f(g(x_0)+k) = f(g(x_0)) + f'(g(x_0))\,k + o(k) \quad k\to 0}$

$$C(x_0+h) = f(g(x_0+h)) = f\big(g(x_0) + \underbrace{g'(x_0)h + o(h)}_{k}\big) \quad h\to 0$$

$$= \underbrace{f(g(x_0))}_{C(x_0)} + f'(g(x_0))\big(g'(x_0)h + o(h)\big) + o\big(g'(x_0)h + o(h)\big) \quad h\to 0$$

$$= C(x_0) + f'(g(x_0))\,g'(x_0)\,h + o(h) + \underbrace{o(g'(x_0)h)}_{o(h)} + \underbrace{o(o(h))}_{o(h)}$$

$$\underbrace{o(h) + o(h) + o(h)}_{o(h)} \qquad h \to 0$$

$$= C(x_0) + \underbrace{f'(g(x_0))\,g'(x_0)}_{\alpha}\,h + o(h) \qquad h \to 0$$

$$\alpha \cdot h$$

$$\big(f(g(x))\big)' = f'(g(x)) \cdot g'(x)$$

REGOLA DELLA CATENA

DERIVAZIONE FUNZIONE INVERSA

f e g sono inverse una dell'altra allora

$$\left\| \begin{array}{l} g(f(x)) = x \\ f(g(x)) = x \end{array} \right\| \iff \begin{array}{l} g \circ f = id_A \\ f \circ g = id_B \end{array}$$

$$\left(f(g(x))\right)' = (x)' = 1$$

$$\downarrow$$

$$f'(g(x)) \cdot g'(x) = 1 \Rightarrow \boxed{g'(x) = \frac{1}{f'(g(x))}}$$

Esempi: 1. $f(x) = e^x \qquad g(x) = \log x$

$$f'(x) = (e^x)' = e^x$$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{e^{g(x)}} = \frac{1}{x}$$

2. $f(x) = \operatorname{tg} x \qquad g(x) = \operatorname{arctg} x \qquad f'(x) = \frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x$

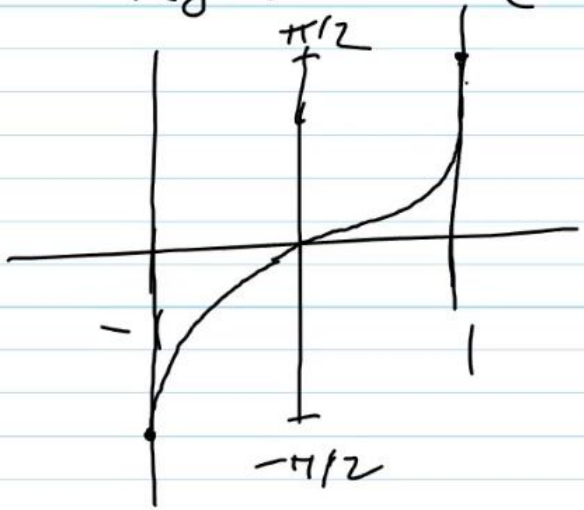
$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{1+\operatorname{tg}^2(\operatorname{arctg} x)} = \frac{1}{1+(\operatorname{tg}(\operatorname{arctg} x))^2} = \frac{1}{1+x^2}$$

Page 10
Page 11
Page 12
Page 13
Page 14
Page 15
Page 16
Page 17
17 / 17 11:03

3. $f(x) = \sin x \qquad g(x) = \arcsin x \qquad f'(x) = \cos x$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}}$$

$\cos x = \sqrt{1-\sin^2 x}$



$$= \frac{1}{\sqrt{1-x^2}}$$

4. $f(x) = x^{\alpha} \qquad \alpha \in \mathbb{R}\setminus\{0\}$

$$(f(x))' = (x^{\alpha})' = \left(e^{\lg x^{\alpha}}\right)' = \left(e^{\alpha \lg x}\right)'$$

$$= e^{\alpha \lg x} \cdot \alpha \frac{1}{x} = \alpha \frac{x^{\alpha}}{x} = \alpha x^{\alpha-1}$$

5.

- $f(x) = \cos(e^x)$

$$f'(x) = -\sin(e^x) \cdot e^x$$

- $f(x) = e^x \cos x$

$$f'(x) = e^x \cos x + e^x(-\sin x) = e^x(\cos x - \sin x)$$

- $f(x) = \operatorname{arctg} x^3$

$$f'(x) = \frac{1}{1+x^6} \cdot 3x^2$$

- $f(x) = x^2 \log(\sin x)$

$$f'(x) = 2x \log(\sin x) + x^2 \frac{1}{\sin x} \cos x$$

6. $f(x) = (\operatorname{arctg} x)^{\sin x}$ $\qquad f^{g} = e^{g \ln f}$

$$f'(x) = \left(e^{\sin x \log(\operatorname{arctg} x)}\right)'$$

$$= e^{\sin x \log(\operatorname{arctg} x)} \cdot \left(\sin x \log(\operatorname{arctg} x)\right)'$$

$$= \quad \| \quad \cdot \left(\cos x \log(\operatorname{arctg} x) + \sin x \frac{1}{\operatorname{arctg} x} \cdot \frac{1}{1+x^2}\right)$$