LEZIONE 14ª

COMPOSIZIONI & TAYLOR

$$f(x) = P_n(x) + o(x^n) \qquad x \to 0$$

$$g(x) = \underline{Q_n(x)} + o(x^n) \qquad x \to 0$$

Allora $f(g(x)) = P_n\left(Q_n(x)\right) + o(x^n) \qquad x \to 0$

questo vale se $g(0)=0$ ossia $Q_n(x)$ <u>non</u> ha il termine costante (NOTO)

<u>Esempio 1</u> $\qquad f(g(x)) = e^{\sin x}$ $\qquad \boxed{n = 4 \leftarrow \text{Taylor}}$

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + o(t^4)$$

$$t = \sin x$$

$$e^{\sin x} = 1 + \sin x + \frac{(\sin x)^2}{2} + \frac{\sin^3 x}{6} + \frac{\sin^4 x}{24} + o(\sin^4 x)$$

$$= 1 + \left(x - \frac{x^3}{6} + o(x^4)\right) + \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^4)\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6} + o(x^4)\right)^3 + \frac{1}{24}\left(x - \frac{x^2}{6} + o(x^4)\right)^4 + o(\sin^4 x)$$

$$e^{\sin x} = 1 + x - \cancel{\frac{x^3}{6}} + o(x^4) + \frac{1}{2}\left(x^2 - \frac{x^4}{3}\right) + \cancel{\frac{1}{6}x^3} + \frac{1}{24}x^4 + o(\sin^4 x)$$

$$\sin x = x + o(x) \longrightarrow \sin^4 x = (x + o(x))^4$$

$$= x^4 + x^3 o(x) + x^2 o(x)^2 + \dots + o(x)^4$$

$$= x^4 + o(x^4)$$

$$o(\sin^4 x) = o\left(x^4 + o(x^4)\right) = o(x^4) + o\underbrace{(o(x^4))}_{o(x^4)} = o(x^4)$$

$$\boxed{e^{\sin x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4)}$$

dim :

2 considerazioni generali

1ª considerazione $\boxed{f(g(x)) = P_n(Q_n(x)) + o(x^n) \qquad x \to 0}$

$$f(g(x)) = P_n(g(x)) + \boxed{o((g(x))^n)} \qquad x \to 0$$

segue da $f(t) = P_n(t) + o(t^n) \qquad t \to 0$

dimostriamo che $o((g(x))^n) = o(x^n) \qquad x \to 0$

$$o(t^n) = t^n \cdot \omega(t) \qquad \omega(t) \to 0 \text{ se } t \to 0$$

$$o((g(x)^n)) = (g(x))^n \cdot \underbrace{\omega(g(x))}$$

poiché questa deve essere infinitesima per $x \to 0$

$\Rightarrow g(x) \to 0 \quad x \to 0$ ossia $g(0) = 0$

$$= x^n \boxed{\left(\frac{g(x)}{x}\right)^n \underbrace{\omega(g(x))}_{\to 0}} = x^n \underbrace{\omega_1(x)}_{\to 0} \qquad x \to 0$$

costante alla peggio

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$$g(x) = Q_n(x) + o(x^n)$$

$g(0) = 0 \Rightarrow Q_n(x)$ non ha termine costante

$$\frac{g(x)}{x} = \frac{a_1 x + a_2 x^2 + \dots + a_n x^n}{x} = a_1 + a_2 x + \dots + a_n x^{n-1}$$

per cui

$$o\left((g(x))^n\right) = o(x^n)$$

grazie al fatto che $g(0) = 0$!

2ª considerazione

$$P_n(g(x)) = \underbrace{P_n\left(Q_n(x) + o(x^n)\right)}_{} = P_n(Q_n(x)) + o(x^n) \qquad x \to 0$$

$$P_n(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$$

$$P_n\left(Q_n(x) + o(x^n)\right) = a_0 + a_1\left(Q_n(x) + o(x^n)\right) + a_2\left(Q_n(x) + o(x^n)\right)^2 + \dots + a_n\left(Q(x) + o(x^n)\right)^n$$

$$= a_0 + a_1 Q_n(x) + a_2 Q_n(x)^2 + \dots + a_n Q_n(x)^n + o(x^n)$$

$$= P_n(Q_n(x)) + o(x^n)$$

Esempio 2 $\cos(\sin x)$ Taylor $n = 4$

$$\cos t = 1 - \frac{t^2}{2} + \frac{t^4}{24} + o(t^4) \qquad \sin x = x - \frac{x^3}{6} + o(x^4)$$

$$\cos(\sin x) = 1 - \frac{1}{2}\left(x - \frac{x^3}{6} + o(x^4)\right)^2 + \frac{1}{24}\left(x - \frac{x^3}{6} + o(x^4)\right)^4 + o(x^4)$$

$$= 1 - \frac{1}{2}\left(x^2 - \frac{x^4}{3}\right) + \frac{1}{24}x^4 + o(x^4)$$

$$= 1 - \frac{x^2}{2} + \underbrace{\left(\frac{1}{6} + \frac{1}{24}\right)}_{\frac{5}{24}} x^4 + o(x^4)$$

Esempio 3 $\sin(\cos x)$ questo non lo posso fare con la regola di prima perché $\cos(0) = 1 \neq 0$

Devo poter sviluppare il seno intorno al punto $x_0 = 1$ poiché $\cos(0) = 1$.

Esempio 4 $f(x) = e^x$ $x_0 = 6$

1° modo faccio le derivate $f^{(k)}(x) = (e^x)^{(k)} = e^x$

$$e^x = e^6 + \frac{e^6}{1!}(x-6) + \frac{e^6}{2!}(x-6)^2 + \frac{e^6}{3!}(x-6)^3 + \dots$$

2° modo

scrivo $x = 6 + h$ con $h \to 0$ $\qquad h = x - 6$

$$e^{6+h} = e^6 \cdot e^h = e^6 \left(1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \dots\right)$$

Esempio 5 $\qquad f(x) = \sin x \qquad x_0 = 1$

1° modo $\quad$ faccio le derivate

2° modo $\quad x = 1 + h \qquad h \to 0$

$$\sin(1+h) = \sin 1 \underline{\cos h} + \cos 1 \underline{\sin h}$$

$$= \sin 1 \left(1 - \frac{h^2}{2} + \frac{h^4}{24} + \dots\right) + \cos 1 \left(h - \frac{h^3}{6} + \dots\right)$$

Torniamo all'esercizio 3!

$$f(x) = \sin(\cos x) \qquad n = 4$$

$$\sin(\cos x) = \sin\Big(\underbrace{1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)}_{1+h}\Big)$$

= uso lo sviluppo fatto prima e trovo

$$= \sin 1 \left(1 - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)^2 + \frac{1}{24}\cancel{\left(- \cdots \right)}^4\right)$$

$$+ \cos 1 \left(\left(\cdots\right) - \frac{1}{6}\cancel{\left(\cdots\right)}^3\right)$$

$$= \sin\left(1 - \frac{1}{8}x^4\right) + \cos 1\left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + o(x^4)$$

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$$e^x = e^6 + \frac{e^6}{1!}(x-6) + \frac{e^6}{2!}(x-6)^2 + \frac{e^6}{3!}(x-6)^3 + \dots$$

$\boxed{\text{2° modo}}$

scrivo $x = 6+h$ con $h \to 0$ $\qquad h = x-6$

$$e^{6+h} = e^6 \cdot e^h = e^6\left(1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \dots\right)$$

Esempio 5 $\qquad f(x) = \sin x \qquad x_0 = 1$

1° modo $\quad$ fare le derivate

2° modo $\quad x = 1+h \quad h \to 0$

$$\sin(1+h) = \sin 1 \cos h + \cos 1 \sin h$$

$$= \sin 1\left(1 - \frac{h^2}{2} + \frac{h^4}{24} + \dots\right) + \cos 1\left(h - \frac{h^3}{6} + \dots\right)$$

Torniamo all'esercizio 3!

$$f(x) = \sin(\cos x) \qquad n = 4$$

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$$\sin(\cos x) = \sin\Big(\underbrace{1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)}_{1 + h}\Big)$$

= uso lo sviluppo fatto prima e trovo

$$= \sin 1 \left(1 - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)^2 + \frac{1}{24}\cancel{\left(- \; - \; =\right)}^4\right)$$

$$+ \cos 1 \left(\left(\cdot \;\cdot\; \cdot\right) - \frac{1}{6}\cancel{\left(- \; - \; =\right)}^3\right)$$

$$= \sin\left(1 - \frac{1}{8}x^4\right) + \cos 1\left(-\frac{x^2}{2} + \frac{x^4}{24}\right) + o(x^4)$$

Funzioni IPERBOLICHE

$$\sinh x \doteq \frac{e^x - e^{-x}}{2}$$ SENO IPERBOLICO

$$\cosh x \doteq \frac{e^x + e^{-x}}{2}$$ COSENO IPERBOLICO

$$\tanh x \doteq \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$ TANGENTE IPERBOLICA $\forall x \in \mathbb{R}$

SIMMETRIE

$\cosh(-x) = \cosh(x)$ PARI

$\sinh(-x) = -\sinh(x)$ DISPARI

$\tanh(-x) = -\tanh(x)$ DISPARI

DERIVATE

$$(\sinh x)' = \frac{e^x + e^{-x}}{2} = \cosh x$$ $(e^{-x})' = -e^{-x}$

$$(\cosh x)' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$(\tanh x)' = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{(\cosh x)^2 - (\sinh x)^2}{(\cosh x)^2} = 1 - (\tanh x)^2$$

Sviluppi di Taylor intorno a $x_0 = 0$

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) =$$

$$= \frac{1}{2}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots\right)$$

$$= \left(1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots\right)$$ come il coseno tranne per i segni

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$ come il seno tranne per i segni

oss: $\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$

$$f(x) = f_{pari}(x) + f_{dispari}(x) \qquad f_{pari}(x) = \frac{f(x) + f(-x)}{2} \qquad f_{disp}(x) = \frac{f(x) - f(-x)}{2}$$

RELAZIONE IPERBOLICA FONDAMENTALE

$$(\sinh x)^2 = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + e^{-2x} - 2}{4}$$

$$(\cosh x)^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} + e^{-2x} + 2}{4}$$

$$\boxed{(\cosh x)^2 - (\sinh x)^2 = 1}$$

OSS: $$(\tanh x)^2 = \frac{(\cosh x)^2 - (\sinh x)^2}{(\cosh x)^2} = \frac{1}{(\cosh x)^2} = 1 - (\tanh x)^2$$

FORMULE DI DUPLICAZIONE

$$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2} = \frac{e^x + e^{-x}}{2} \cdot \frac{e^x - e^{-x}}{2}\,\underbrace{2 = 2}\,\sinh x \cosh x$$

fate a casa le altre possibili formulette analoghe alle trigonometriche

MONOTONIA

$\sinh x$ è strettamente crescente

$$= \underbrace{\frac{1}{2}e^{x}}_{\text{strett. crescente}} \underbrace{-\frac{1}{2}e^{-x}}_{\text{strett. crescente}}$$

$\Rightarrow$ somma di funzioni strett. crescenti $\Rightarrow$ è strett. crescente.

$$\cosh x = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$$

$\cosh x$ è strett. crescente per $x \geq 0$

dim: $y > x \geq 0$ devo verificare che $\underbrace{\cosh y > \cosh x}$

$$\boxed{\frac{e^{y}+e^{-y}}{\cancel{2}} > \frac{e^{x}+e^{-x}}{\cancel{2}}}$$

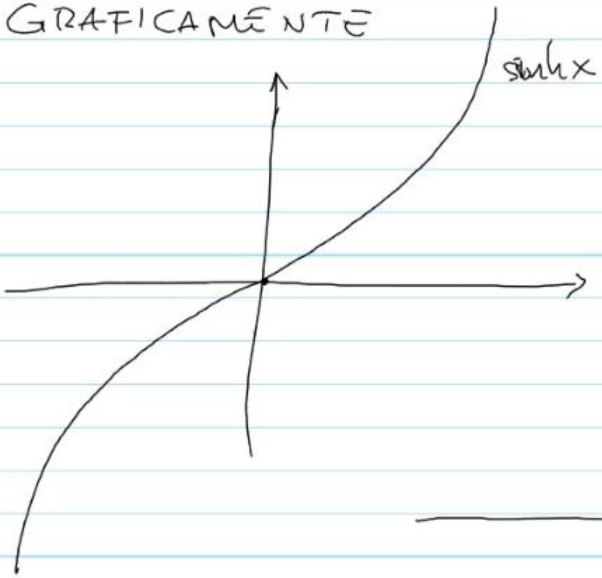
$b = e^{y}$ $\quad a = e^{x}$ $\Rightarrow b > a \geq 1$

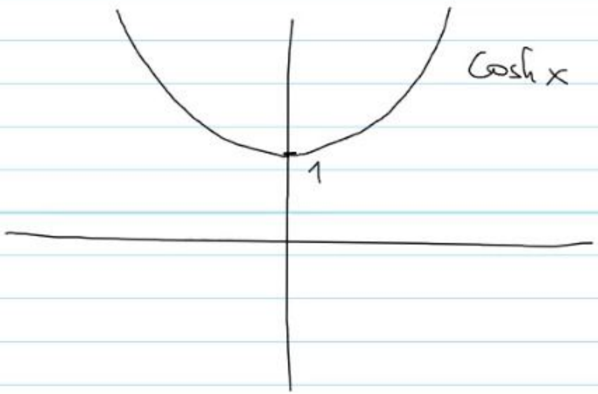
$$b + \frac{1}{b} > a + \frac{1}{a} \iff \frac{b^2+1}{b} > \frac{a^2+1}{a} \iff a(b^2+1) > b(a^2+1)$$

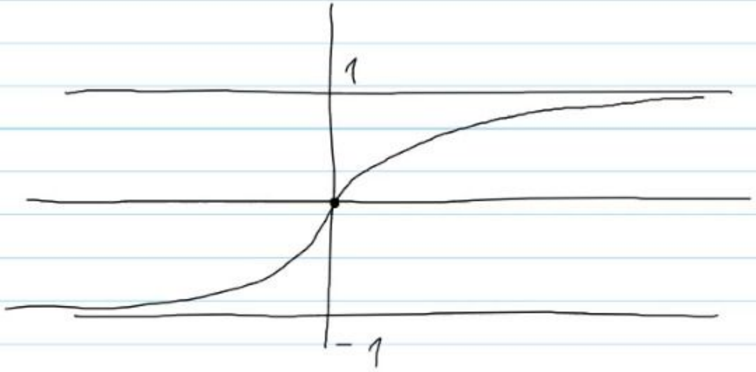
$$\iff \underline{\underline{ab^2}} + a > \underline{\underline{ba^2}} + b \iff ab(b-a) + a - b > 0$$

$$\iff (\underbrace{b-a}_{>0})(\underbrace{ab-1}_{>0}) > 0$$

GRAFICAMENTE







INVERSE

$f(x) = \sinh x \quad : \mathbb{R} \to \mathbb{R}$ bigettiva

ha allora una inversa $g : \mathbb{R} \to \mathbb{R}$

$$\boxed{g(x) = \operatorname{settsinh} x}$$ settore seno iperbolico

$f(x) = \cosh x \quad : \mathbb{R} \to \mathbb{R}$ poiché è pari non ha inversa su $\mathbb{R}$

se vedo però il coseno iperbolico come funzione tra $[0,+\infty)$ e $[1,+\infty)$

allora è surgettiva e quindi ha una inversa $g : [1,+\infty) \to [0,+\infty)$

$$g(x) = \operatorname{settcosh} x \qquad x \in [1,+\infty).$$

Lo stesso per la $\tanh x$ ha una inversa $g(x) = \operatorname{setttanh} x : (-1,1) \to \mathbb{R}$

OSS:

$$\operatorname{setttanh} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$$

FORMULE ESPLICITE

$\sinh x = y \Rightarrow$ trovare x in funzione di y

$$\frac{e^x - e^{-x}}{2} = y \iff e^x - e^{-x} = 2y \qquad \text{poniamo } e^x = a$$

$$\iff a - \frac{1}{a} = 2y \iff a^2 - 2ay - 1 = 0$$

$$\Rightarrow a = y \pm \sqrt{1+y^2} \Rightarrow a = y + \sqrt{1+y^2} = e^x$$

$$x = \log\left(y + \sqrt{1+y^2}\right) = \operatorname{settsinh} y$$

concludendo

$$\boxed{\operatorname{settsinh} x = \log\left(x + \sqrt{1+x^2}\right)} \quad \forall x \in \mathbb{R}$$

ESERCITAZIONE

Esercizio 1

$$\lim_{x\to 0} \frac{\sin x \cdot \log(1+x^2) - \sin x^2 \cdot \log(1+x)}{x \operatorname{arctg} x^3}$$

Bisogna usare Taylor fino a $n=4$

$$\begin{cases} \operatorname{arctg} x^3 = x^3 + o(x^3) \\ x \operatorname{arctg} x^3 = x^4 + o(x^4) \end{cases} \qquad x \to 0$$

$$\begin{cases} \sin t = t - \frac{t^3}{6} + o(t^3) \\ \sin x^2 = x^2 + o(x^4) \end{cases}$$

$$\log(1+x) = x - \frac{x^2}{2} + o(x^2) \Rightarrow \log(1+x^2) = \underbrace{x^2 - \frac{x^4}{2} + o(x^4)}$$

$$\sin x^2 \log(1+x) = \left(x^2 + o(x^4)\right)\left(x - \frac{x^2}{2} + o(x^2)\right) = x^3 - \frac{x^4}{2} + o(x^4)$$

$$\sin x \log(1+x^2) = \left(x - \frac{x^3}{6} + o(x^4)\right)\left(x^2 - \frac{x^4}{2} + o(x^4)\right) = x^3 + o(x^4)$$

numeratore $\quad \sin x \log(1+x^2) - \operatorname{sen} x^2 \log(1+x)$

$$= x^3 + o(x^4) - x^3 + \frac{x^4}{2} + o(x^4) = \frac{x^4}{2} + o(x^4)$$

frazione $= \dfrac{\frac{x^4}{2} + o(x^4)}{x^4 + o(x^4)} = \dfrac{\cancel{x^4}\left(\frac{1}{2} + o(1)\right)}{\cancel{x^4}\left(1 + o(1)\right)} \Longrightarrow \dfrac{1}{2} \quad x \to 0$

Esercizio 2

$$\lim_{x \to 0} \frac{\sqrt{1+x^4} - \cosh x^2}{\sinh x^4 - x^4 \cosh x^2}$$

sviluppare tutto al termine successivo a x^4

$$\sinh x = x + \frac{x^3}{6} + o(x^3) \Rightarrow \boxed{\sinh x^4 = x^4 + \frac{x^{12}}{6} + o(x^{12})}$$

$o(x^{19})$

$$\cosh x^2 = 1 + \frac{x^4}{2} + o(x^6) \qquad \cosh t = 1 + \frac{t^2}{2} + \frac{t^4}{24} + \dots$$

$$\boxed{x^4 \cosh x^2 = x^4 + \frac{x^8}{2} + o(x^{10})}$$

$$\cancel{x^4} + \frac{x^{12}}{6} + o(x^{12}) - \cancel{x^4} - \frac{x^8}{2} + o(x^{10})$$

$$= -\frac{x^8}{2} + o(x^{10})$$

Quindi il numeratore lo stesso sviluppo fino a $n=8$

$$\cosh x^2 = 1 + \frac{x^4}{2} + \frac{x^8}{24} + o(x^8)$$

$$\sqrt{1+t} = 1 + \frac{t}{2} - \frac{1}{8}t^2 + o(t^2)$$

$$\sqrt{1+x^4} = 1 + \frac{x^4}{2} - \frac{x^8}{8} + o(x^8)$$

$$\text{Numeratore} = \cancel{1} + \cancel{\frac{x^4}{2}} \underbrace{- \frac{x^8}{8} + o(x^8) - \cancel{1} - \cancel{\frac{x^4}{2}} - \frac{x^8}{24}} + o(x^8)$$

$$= -\frac{1}{6}x^8 + o(x^8)$$

$$\text{frazione} = \frac{-\frac{1}{6}x^8 + o(x^8)}{-\frac{1}{2}x^8 + o(x^8)} = \frac{\cancel{x^8}\left(-\frac{1}{6} + o(1)\right)}{\cancel{x^8}\left(-\frac{1}{2} + o(1)\right)} \longrightarrow \frac{1}{3}$$

Esercizio 3

$$\lim_{x \to 0} \frac{\operatorname{arctg}(x+x^3) - \sin(x-x^3)}{x\cosh(x+x^2) - \sinh x}$$

$$\sinh x = x + \frac{x^3}{6} + o(x^3)$$

$$\cosh(x+x^2) = 1 + \frac{(x+x^2)^2}{2} + o\left((x+x^2)^3\right) \qquad \cosh t = 1 + \frac{t^2}{2} + \frac{t^4}{24} + \dots$$

$$= 1 + \frac{1}{2}(x^2 + x^4 + 2x^3) + o(x^3)$$

$$= 1 + \frac{1}{2}x^2 + x^3 + o(x^3)$$

Denominatore $x\cosh(x+x^2) - \sinh x = \cancel{x} + \frac{x^3}{2} + x^4 + o(x^4) - \cancel{x} - \frac{1}{6}x^3 + o(x^3)$

$$= \frac{1}{3}x^3 + o(x^3)$$

$\operatorname{arctg} t \sim t \quad \Rightarrow \quad \operatorname{arctg}(x+x^3) \sim x+x^3$

$\cancel{\operatorname{arctg}(x+x^3) = x + x^3 + o(x^3)}$

NON È VERO !!!

$$\boxed{\operatorname{arctg} t = t + o(t)} \quad \Rightarrow \quad \operatorname{arctg}(x+x^3) = x + x^3 + \underset{\substack{\parallel \\ o(x)}}{o(x+x^3)}$$

$$= \underbrace{x + x^3 + o(x) = x + o(x)}_{\text{non è Taylor con } n=3\,!}$$

$$\operatorname{arctg} t = t - \frac{t^3}{3} + o(t^3)$$

ORA POSSO SOSTITUIRE $t = x+x^3$!

$$\operatorname{arctg}(x+x^3) = x + x^3 - \frac{1}{3}(x+x^3)^3 + \underbrace{o\big((x+x^3)^3\big)}_{o(x^3)}$$

$$= x + x^3 - \frac{1}{3}x^3 + o(x^3) = x + \frac{2}{3}x^3 + o(x^3)$$

$$\sin t = t - \frac{t^3}{6} + o(t^3)$$

pongo se $t = x - x^3$

$$\Rightarrow \sin(x - x^3) = x - x^3 - \frac{1}{6}(x - x^3)^3 + o(x^3)$$

$$= x - x^3 - \frac{1}{6}x^3 + o(x^3)$$

$$= x - \frac{7}{6}x^3 + o(x^3)$$

Numeratore $\quad \operatorname{arctg}(x + x^3) - \sin(x - x^3)$

$$= \cancel{x} + \frac{2}{3}x^3 + o(x^3) - \cancel{x} + \frac{7}{6}x^3 + o(x^3)$$

$$= \frac{11}{6}x^3 + o(x^3)$$

frazione $$= \frac{\frac{11}{6}x^3 + o(x^3)}{\frac{1}{3}x^3 + o(x^3)} = \frac{\frac{11}{6} + o(1)}{\frac{1}{3} + o(1)} \Rightarrow \frac{11}{2}$$