Università di Trento – Dip. di Ingegneria e Scienza dell'Informazione
CdL in Informatica, Ingegneria dell'informazione e delle comunicazioni e Ingegneria dell'informazione e organizzazione d'impresa
a.a. 2017-2018 – Foglio di esercizi 14
... "il mondo delle primitive – integrali e la funz. integrale – giusto qualche conto"

14.1) a)

$$f(x) = \begin{cases} \int_0^x (e^{-t^2}+t)\,dt & \text{se } x>0 \\ a^2 \sin x + b & \text{se } x \le 0. \end{cases}$$

Abbiamo che f è continua e derivabile $\forall x \neq 0$. Dobbiamo solo richiedere la continuità e derivabilità in $x=0$.

f è continua in $x=0 \iff \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$. Per questo deve essere $\underline{\underline{b=0}}$. Notiamo che per il TFC si ha $f'(x) = e^{-x^2}+x$ se $x>0$. (abb. fissato ora $b=0$)

D'altra parte $f'(x) = a^2\cos x$ se $x<0$. Poiché f è continua, e $f'_+(0) \overset{\text{coroll. de l'Hôpital}}{=} \lim_{x\to 0^+} f'(x) = 1$, $f'_-(0) \overset{\text{coroll. de l'Hôpital}}{=} \lim_{x\to 0^-} f'(x) = a^2$, f è derivabile in $x=0$ se e solo se $a^2=1$, quindi $\underline{\underline{a=\pm 1}}$. □

b)
$$\lim_{x\to 0^+} \frac{2}{x^2}\int_0^{\log^2(1+2x)} \sqrt[3]{1+t^2}\,dt = \lim_{x\to 0^+} \frac{\overbrace{2\int_0^{\log^2(1+2x)} \sqrt[3]{1+t^2}\,dt}^{=f(x)}}{\underbrace{(x^2)}_{=g(x)}} = \left[\frac{0}{0}\right]$$

Notiamo che il limite si presenta nella forma $\left[\frac{0}{0}\right]$, inoltre $g'(x) = 2x \neq 0$ in un intorno destro di 0. Oss. che

$$\frac{f'(x)}{g'(x)} \underset{\text{TFC}}{=} \frac{2\sqrt[3]{1+\log^4(1+2x)}\cdot \cancel{2}\overbrace{\log(1+2x)}^{\to 1}\cdot\frac{2}{1+2x}}{\cancel{2}x} \xrightarrow[x\to 0^+]{} 8.$$

Dal teorema di de l'Hôpital segue dunque che $\lim_{x\to 0^+} \frac{2}{x^2}\int_0^{\log^2(1+2x)} \sqrt[3]{1+t^2}\,dt = \underline{\underline{8}}$ ■

14.2) i) $F(x) = \frac{x}{2}\sqrt{4+x^2} + 2\log\left(\frac{x}{2}+\frac{1}{2}\sqrt{4+x^2}\right)$ è derivabile su $\mathbb{R}$ e

$$F'(x) = \frac{1}{2}\sqrt{4+x^2} + \frac{x}{2}\cdot\frac{1}{\cancel{2}}\frac{\cancel{2}x}{\sqrt{4+x^2}} + 2\left[\frac{\frac{1}{\cancel{2}}+\frac{1}{\cancel{2}}\frac{1}{2}\frac{2x}{\sqrt{4+x^2}}}{\frac{x}{\cancel{2}}+\frac{1}{\cancel{2}}\sqrt{4+x^2}}\right] =$$

$$= \frac{1}{2}\sqrt{4+x^2} + \frac{x^2}{2\sqrt{4+x^2}} + 2\left[\frac{1+\frac{x}{\sqrt{4+x^2}}}{x+\sqrt{4+x^2}}\right] =$$

$$= \frac{1}{2}\sqrt{4+x^2} + \frac{x^2}{2\sqrt{4+x^2}} + 2\left[\frac{1}{\sqrt{4+x^2}}\right] = \frac{(4+x^2)+x^2+4}{2\sqrt{4+x^2}}$$

$$= \frac{\cancel{2}(4+x^2)}{\cancel{2}\sqrt{4+x^2}} = \sqrt{4+x^2} = f(x) \quad \text{su } \mathbb{R}.$$ □

ii) $\displaystyle\int_0^1 f(x)\,dx = F(1)-F(0) = \underline{\underline{\frac{1}{2}\sqrt{5} + 2\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)}}.$ $\square$

iii) $\displaystyle\lim_{x\to 0} \frac{F(x)-2x}{x^3} \overset{=}{\left[\frac{0}{0}\right]}$

Oss. che $\displaystyle\frac{(F(x)-2x)'}{(x^3)'} = \frac{F'(x)-2}{3x^2} = \frac{f(x)-2}{3x^2} = \frac{\sqrt{4+x^2}-2}{3x^2} =$

$$= \frac{(\sqrt{4+x^2}-2)(\sqrt{4+x^2}+2)}{3x^2(\sqrt{4+x^2}+2)} = \frac{\cancel{4}+\cancel{x^2}-\cancel{4}}{3\cancel{x^2}(\sqrt{4+x^2}+2)} \xrightarrow[x\to 0]{} \frac{1}{12}$$

Per il teorema di de L'Hôpital segue che $\displaystyle\lim_{x\to 0}\frac{F(x)-2x}{x^3} = \underline{\underline{\frac{1}{12}}}$ ■

14.3) i) $f(x) = \cos 3x + e^{-2x}$ $\qquad \underline{\underline{F(x) = \frac{\sin 3x}{3} - \frac{e^{-2x}}{2} + c}}$ $\quad c\in\mathbb{R}$;

ii) $f(x) = \frac{(\log x)^3}{x} - x\,2^{x^2}$ $\qquad \underline{\underline{F(x) = \frac{(\log x)^4}{4} - \frac{2^{x^2}}{2\log 2} + c}}$ $\quad c\in\mathbb{R}$;

iii) $f(x) = \frac{1}{1+4x^2} + \frac{2}{1-x}$ $\qquad \underline{\underline{F(x) = \frac{\operatorname{arctg} 2x}{2} - 2\log|1-x| + c}}$ $\quad c\in\mathbb{R}$. ■

14.4) $\displaystyle\int \frac{\sqrt[3]{x}-\log^2 x}{2x}\,dx = \frac{1}{2}\int x^{-\frac{2}{3}}dx - \frac{1}{2}\int\frac{\log^2 x}{x}\,dx = \frac{1}{2}\cdot\frac{x^{\frac{1}{3}}}{\frac{1}{3}} - \frac{1}{2}\cdot\frac{\log^3 x}{3} + c$

$$= \underline{\underline{\frac{3}{2}x^{\frac{1}{3}} - \frac{1}{6}\log^3 x + c}} \qquad c\in\mathbb{R}.$$

$\displaystyle\int \frac{x}{x^2+1}\,dx = \underline{\underline{\frac{1}{2}\log(1+x^2) + c}},$ $\quad c\in\mathbb{R}$.

$\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\frac{1}{8}\int -8x\,(1-4x^2)^{-\frac{1}{2}}dx = -\frac{1}{8}\,\frac{(1-4x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$ $\quad c\in\mathbb{R}$

$$= \underline{\underline{-\frac{1}{4}(1-4x^2)^{\frac{1}{2}} + c}} \qquad c\in\mathbb{R}.$$

(*) $\longrightarrow$ $\displaystyle\int \frac{1}{\cos^2 2x}\,dx = \underline{\underline{\frac{1}{2}\operatorname{tg}(2x) + c}},$ $\quad c\in\mathbb{R}$. ■

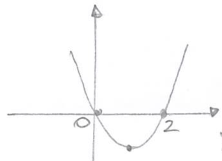
14.5) $\displaystyle\int_1^2 \frac{x^3-2x+e}{x}\,dx = \int_1^2\left(x^2-2+\frac{e}{x}\right)dx = \left[\frac{x^3}{3}-2x+e\log x\right]_1^2 = \left(\frac{8}{3}-4+e\log 2\right)-\left(\frac{1}{3}-2\right)$

$$= \underline{\underline{\frac{7}{3}-2+e\log 2}} = \underline{\underline{\frac{1}{3}+e\log 2}}.$$

$\displaystyle\int_0^2 \frac{9^x-1}{3^x+1}\,dx = \int_0^2 \frac{(3^x-1)\cancel{(3^x+1)}}{\cancel{(3^x+1)}}\,dx = \left[\frac{3^x}{\log 3}-x\right]_0^2 = \left(\frac{9}{\log 3}-2\right)-\left(\frac{1}{\log 3}\right) = \underline{\underline{\frac{8}{\log 3}-2}}$

(*) $\displaystyle\int \frac{-2}{\sqrt{1-4x^2}}\,dx = \underline{\underline{-\arcsin(2x)+c}},$ $\quad c\in\mathbb{R}$.

$$\int_{-1}^{4}|x^2-2x|\,dx=\int_{-1}^{0}(x^2-2x)\,dx+\int_{0}^{2}(-x^2+2x)\,dx$$



$x^2-2x=(x-1)^2-1$

$$+\int_{2}^{4}(x^2-2x)\,dx=\left[\frac{x^3}{3}-x^2\right]_{-1}^{0}+\left[-\frac{x^3}{3}+x^2\right]_{0}^{2}+\left[\frac{x^3}{3}-x^2\right]_{2}^{4}$$

$$=-\left(-\frac{1}{3}-1\right)+\left(-\frac{8}{3}+4\right)+\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-4\right)=\frac{49}{3}-7=\underline{\underline{\frac{28}{3}}}$$

$$\int_{-\pi}^{\pi/2}|\sin x|\,dx=\int_{-\pi}^{0}(-\sin x)\,dx+\int_{0}^{\pi/2}\sin x\,dx=\Big[\cos x\Big]_{-\pi}^{0}+\Big[-\cos x\Big]_{0}^{\pi/2}$$

$$=(1+1)+1=\underline{\underline{3}}\,.$$

14.6) $$\int x\arcsin x\,dx=\frac{x^2}{2}\arcsin x-\int\frac{x^2}{2}\frac{1}{\sqrt{1-x^2}}\,dx=$$

$$=\frac{x^2}{2}\arcsin x-\frac{1}{2}\int x\cdot\frac{x}{\sqrt{1-x^2}}\,dx=\frac{x^2}{2}\arcsin x-\frac{1}{2}\Big[-x\sqrt{1-x^2}+$$

$$+\int\sqrt{1-x^2}\,dx\Big]=\frac{x^2}{2}\arcsin x-\frac{1}{2}\left[-x\sqrt{1-x^2}+\frac{x\sqrt{1-x^2}+\arcsin x}{2}\right]+c$$

Ora $\int\sqrt{1-x^2}\,dx=\int\cos^2 t\,dt=\int\cos t\cos t\,dt=$

$x=\sin t$
$dx=\cos t$

$$=\sin t\cos t+\int\sin^2 t\,dt$$

$$=\sin t\cos t+t-\int\cos^2 t\,dt$$

$$\Rightarrow\int\cos^2 t\,dt=\frac{\sin t\cos t+t}{2}=$$

$$=\frac{x\sqrt{1-x^2}+\arcsin x}{2}$$

$$=\left(\frac{x^2}{2}-\frac{1}{4}\right)\arcsin x+\frac{1}{4}x\sqrt{1-x^2}+c$$

$$=\underline{\underline{\frac{1}{4}\left(x\sqrt{1-x^2}+(2x^2-1)\arcsin x\right)+c}},\ c\in\mathbb{R}.\quad\square$$

$$\int x^4\log 2x\,dx=\frac{x^5}{5}\log 2x-\int\frac{\cancel{x^5}^{\,4}}{5}\cdot\frac{1}{\cancel{2}\cancel{x}}\cdot\cancel{2}\,dx$$

$$=\frac{x^5}{5}\log 2x-\frac{1}{5}\frac{x^5}{5}+c=\underline{\underline{\frac{x^5}{5}\left(\log 2x-\frac{1}{5}\right)+c}}\quad c\in\mathbb{R}.\quad\square$$

$$\int(2x+1)^2e^x\,dx=\int(4x^2+4x+1)e^x\,dx=4\int x^2e^x+4\int xe^x+\int e^x\,dx$$

$$=4\left[x^2e^x-\int 2xe^x\,dx\right]+4\int xe^x\,dx+e^x+c$$

$$=4x^2e^x-4\int xe^x\,dx+e^x+c=4x^2e^x-4\left[xe^x-\int e^x\,dx\right]$$

$$+e^x+c=\underline{\underline{e^x(4x^2-4x+5)+c}}\quad c\in\mathbb{R}.\quad\square$$

$$\int x^3e^{-x}\,dx = -x^3e^{-x} + \int 3x^2e^{-x}\,dx = -x^3e^{-x} - 3x^2e^{-x} + \int 6xe^{-x}dx$$
$$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} + 6\int e^{-x}\,dx$$
$$= \underline{\underline{e^{-x}\left[-x^3-3x^2-6x-6\right]+c}}\,, \qquad c\in\mathbb{R}.$$

$$\int \log(1+x^2)\,dx = x\log(1+x^2) - \int x\cdot\frac{2x}{1+x^2}\,dx = x\log(1+x^2) - 2\int\frac{x^2}{1+x^2}dx$$
$$= x\log(1+x^2) - 2\int\frac{x^2+1}{x^2+1}\,dx + 2\int\frac{1}{x^2+1}\,dx$$
$$= \underline{\underline{x\log(1+x^2) - 2x + 2\,\mathrm{arctg}\,x + c}}, \qquad c\in\mathbb{R}.$$

14.7)
$$\int\frac{x+1}{\sqrt{x}+1}\,dx \underset{\substack{\sqrt{x}=t\\ x=t^2\\ dx=2t\,dt}}{=} \int\frac{t^2+1}{t+1}\cdot 2t\,dt = 2\int\frac{t^3+t}{t+1}\,dt = 2\int(t^2-t+2)\,dt - 4\int\frac{1}{t+1}\,dt$$

$t^3+t \;:\; t+1 = t^2-t+2$
$\underline{t^3+t^2}$
$\quad /\ -t^2+t$
$\quad \underline{-t^2-t}$
$\quad \underline{+\quad +}$
$\qquad 2t$
$\qquad \underline{-\ 2t\pm 2}$
$\qquad /\ -2$

$$= 2\left[\frac{t^3}{3} - \frac{t^2}{2} + 2t\right] - 4\log(1+t) + c$$
$$= \underline{\underline{2\left[\frac{(\sqrt{x})^3}{3} - \frac{x}{2} + 2\sqrt{x}\right] - 4\log(1+\sqrt{x}) + c}} \qquad c\in\mathbb{R}.$$

$$\int\sqrt{4-x^2}\,dx \underset{\substack{x=2\sin t\\ dx=2\cos t}}{=} \int\sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt = 4\int\cos^2 t\,dt =$$
$$\underset{(14.6)}{=} \cancel{4}^{\,2}\left(\frac{\sin t\cos t + t}{\cancel{2}}\right) + c =$$
$$= \underline{\underline{\frac{x\sqrt{4-x^2}}{2} + 2\arcsin\frac{x}{2} + c}} \qquad c\in\mathbb{R}.$$

$$\int\frac{e^x}{e^{2x}+1}\,dx = \underline{\mathrm{arctg}\,e^x + c}\,, \quad c\in\mathbb{R}.$$

$$\int\frac{1}{\cos x}\,dx \underset{\substack{t=\mathrm{tg}\frac{x}{2}\\ x=2\,\mathrm{arctg}\,t\\ dx=\frac{2}{1+t^2}dt}}{=} \int\frac{\cancel{1+t^2}}{1-t^2}\cdot\frac{2}{\cancel{1+t^2}}\,dt = \frac{\cancel{2}}{\cancel{2}}\left[\int\frac{1}{1-t} + \int\frac{1}{1+t}\right]dt$$

$$\cos t = \frac{1-t^2}{1+t^2}$$

$$= -\log|1-t| + \log|1+t| + c$$
$$= \log\left|\frac{1+t}{1-t}\right| + c$$
$$= \underline{\underline{\log\left|\frac{1+\mathrm{tg}\frac{x}{2}}{1-\mathrm{tg}\frac{x}{2}}\right| + c}}$$

$$\int\sqrt{2+x^2}\,dx \underset{\substack{x=\sqrt{2}\sinh t\\ dx=\sqrt{2}\cosh t}}{=} \int\sqrt{2+2\sinh^2 t}\cdot\sqrt{2}\cosh t\,dt = 2\int\cosh^2 t\,dt$$

Or $\int \cosh^2 t\,dt = \cosh t \sinh t - \int \sinh^2 t\,dt = \cosh t \sinh t + \int (1-\cosh^2 t)\,dt$

$$\Rightarrow \int \cosh^2 t\,dt = \frac{\cosh t \sinh t + t}{2} + c$$

Dunque $\int \sqrt{2+x^2}\,dx = \cancel{2}\left[\frac{\frac{x}{\sqrt{2}}\sqrt{1+\frac{x^2}{2}} + \sinh^{-1}\left(\frac{x}{\sqrt{2}}\right)}{\cancel{2}}\right] + c$

$$= \underline{\underline{\frac{x}{2}\sqrt{2+x^2} + \log\left(\frac{x}{\sqrt{2}} + \sqrt{1+\frac{x^2}{2}}\right) + c}}, \quad c \in \mathbb{R}.$$

14.8) $\int \frac{1}{2x-4x^2}\,dx = \frac{1}{2}\int \frac{1}{x-2x^2}\,dx = \frac{1}{2}\int \frac{1}{x(1-2x)}\,dx =$

$$= \frac{1}{2}\left[\int \frac{2}{1-2x}\,dx + \int \frac{1}{x}\,dx\right] = -\frac{1}{2}\log|1-2x| + \frac{1}{2}\log|x| + c$$

$$= \underline{\underline{\frac{1}{2}\log\left|\frac{x}{1-2x}\right| + c}}, \quad c \in \mathbb{R}.$$

$\int \frac{x+3}{x^2+2}\,dx = \int \frac{x}{x^2+2}\,dx + \frac{1}{2}\int \frac{3}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,dx$

$$= \frac{1}{2}\int \frac{2x}{x^2+2}\,dx + \frac{1}{\sqrt{2}}\cdot 3\int \frac{\frac{1}{\sqrt{2}}}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,dx$$

$$= \underline{\underline{\frac{1}{2}\log(x^2+2) + \frac{3}{\sqrt{2}}\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right) + c}}, \quad c \in \mathbb{R}.$$

$\int \frac{x+1}{x(x-2)}\,dx = -\frac{1}{2}\int \frac{1}{x}\,dx + \frac{3}{2}\int \frac{1}{x-2}\,dx$

$$= \underline{\underline{-\frac{1}{2}\log|x| + \frac{3}{2}\log|x-2| + c}}, \quad c \in \mathbb{R}$$

$\int \frac{3x^3+x^2}{x+1}\,dx = \int \left(3x^2-2x+2 - \frac{2}{x+1}\right)dx$

$$= \underline{\underline{x^3 - x^2 + 2x - 2\log|x+1| + c}}$$

$3x^3+x^2 \ : x+1 = 3x^2-2x+2$

$-3x^3+3x^2$

$/ \ -2x^2$

$\pm 2x^2 \mp 2x$

$/ \ 2x$

$-2x \pm 2$

$/ \ -2$

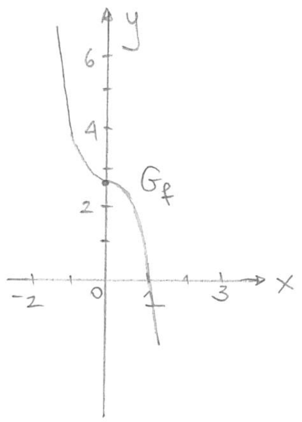
$\int \frac{1}{x^2+x+2}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}\,dx$

$$= \frac{4}{7}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)^2+1}\,dx = \frac{2}{\sqrt{7}}\int \frac{\frac{2}{\sqrt{7}}}{\left(\frac{2}{\sqrt{7}}\left(x+\frac{1}{2}\right)\right)^2+1}\,dx$$

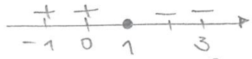
$$= \underline{\underline{\frac{2}{\sqrt{7}}\operatorname{arctg}\left(\frac{2}{\sqrt{7}}\left(x+\frac{1}{2}\right)\right) + c}}, \quad c \in \mathbb{R}.$$

14.9) $f(x) = -xe^{x^2}+e$ • dom $f = \mathbb{R}$ • $\lim_{x\to-\infty} f(x) = +\infty$

$\lim_{x\to+\infty} f(x) = -\infty$

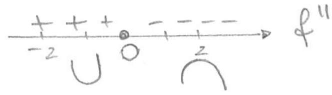


• f segno



• $f'(x) = -e^{x^2}-2x^2e^{x^2} = -e^{x^2}(1+2x^2) < 0$ $\quad f \searrow$

• $f''(x) = -2xe^{x^2}-4xe^{x^2}-4x^3e^{x^2} = -2xe^{x^2}(3+2x^2)$



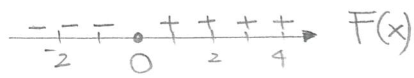
$$\text{area } E = \int_0^1 f(x)\,dx =$$

$$= -\int_0^1 \left(xe^{x^2}-e\right)dx = \left[-\frac{1}{2}e^{x^2}+ex\right]_0^1$$

$$= \left(-\frac{1}{2}e+e\right)-\left(-\frac{1}{2}\right) = \underline{\underline{\frac{e}{2}+\frac{1}{2}}}.$$

□

14.10) $F(x) = \int_0^x \underbrace{\left(e^{2t^2}-4e^{t^2}+5\right)}_{f(t)>0} dt$



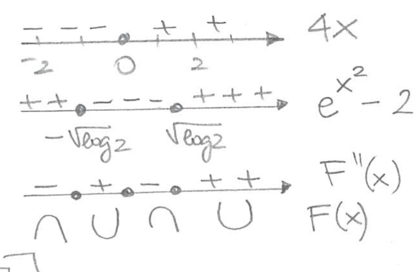
$F'(x) = e^{2x^2}-4e^{x^2}+5 > 0 \quad \forall x \in \mathbb{R}$

$F \nearrow$ su $\mathbb{R}$

$$F''(x) = 4xe^{2x^2}-8xe^{x^2}$$
$$= 4xe^{x^2}\left[e^{x^2}-2\right]$$

$x^2 = \log 2$

$x = \pm\sqrt{\log 2}$



F è concava su $]-\infty, -\sqrt{\log 2}]$, su $[0, \sqrt{\log 2}]$; è convessa su $[-\sqrt{\log 2}, 0]$ e su $[\sqrt{\log 2}, +\infty[$. I pt. $x = \pm\sqrt{\log 2}$, $x=0$ sono pt. di flesso.

■