LEZIONE 11ª

TECNOLOGIA DELL' o PICCOLO

f, g siano almeno definitivamente definite in $x_0 \in \overline{\mathbb{R}}$

DEF 11.1 " Si dice che f è o piccolo di g per $x \to x_0$ e si scrive

$$f(x) = o(g(x)) \quad x \to x_0$$

se esiste una funzione $\omega(x)$ tale che

$$\boxed{\begin{array}{l} f(x) = \omega(x)\, g(x) \\ \text{e} \quad \lim_{x \to x_0} \omega(x) = 0 \end{array}}$$ "

DEF 11.2 " (Quasi equivalente)

Supponiamo che $g(x) \neq 0$ almeno definitivamente in $x_0 \in \overline{\mathbb{R}}$

allora $\omega(x) = \dfrac{f(x)}{g(x)}$ $\Rightarrow$ $\boxed{\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = 0}$ "

$x_0 = 0$

Esempi : 1. $x^4 = o(x) \quad x \to 0$

$f(x) = g(x) \cdot \omega(x) \quad x \to 0$

$x^4 = x^3 \cdot x$

$\lim_{x \to 0} \omega(x) = \lim_{x \to 0} x^3 = 0$

con la def. q. equiv. ottengo $\lim_{x \to 0} \frac{x^4}{x} = \lim_{x \to 0} x^3 = 0$

2. $\underbrace{\operatorname{arctg}(x^3)} = o(x^2) \quad x \to 0$

$f(x) = g(x) \cdot \omega(x) \qquad \lim_{x \to 0} \omega(x) = 0$

con la def. q. e. $\lim_{x \to 0} \frac{\operatorname{arctg}(x^3)}{x^2} = \lim_{x \to 0} \underbrace{\frac{\operatorname{arctg}(x^3)}{x^3}}_{1} \cdot \underbrace{x}_{\to 0} = 0$

$\omega(x) = x$

$\operatorname{arctg}(x^2) = o(x) \quad x \to 0$

3. $$\frac{1}{\sin x} = o\left(\frac{1}{\operatorname{arctg}^2 x}\right) \quad x \to 0$$

$$f(x) \qquad g(x)$$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{\sin x} \cdot \operatorname{arctg}^2 x = \lim_{x\to 0} \underbrace{\frac{x}{\sin x}}_{\to 1} \underbrace{\frac{\operatorname{arctg}^2 x}{x^2}}_{\to 1} \cdot x = 0$$

Proprietà algebriche dell' o-piccolo

$$f_1(x) = o(g(x)) \quad x \to x_0 \qquad f_2(x) = o(g(x)) \quad x \to x_0$$

cosa possiamo dire per $f_1(x) + f_2(x)$? $f_1(x) \cdot f_2(x)$? $\frac{f_1(x)}{f_2(x)}$? $cf_1(x)$?

$\boxed{f_1 + f_2}$

$$f_1(x) = \omega_1(x)\, g(x) \quad x \to x_0 \qquad \omega_1(x) \to 0 \quad x \to x_0$$

$$f_2(x) = \omega_2(x)\, g(x) \quad x \to x_0 \qquad \omega_2(x) \to 0 \quad x \to x_0$$

$$f_1(x) + f_2(x) = \omega_1(x)\, g(x) + \omega_2(x)\, g(x) \quad x \to x_0$$

$$= g(x)\left(\omega_1(x) + \omega_2(x)\right) = \omega_3(x) \to 0 \quad x \to x_0$$

quindi $\quad f_1(x)+f_2(x) = o(g(x)) \quad x\to x_0$

$$\boxed{o(g(x)) + o(g(x)) = o(g(x)) \quad x\to x_0}$$

$\boxed{f_1 - f_2}$

$$f_1(x)-f_2(x) = g(x)\left(\omega_1(x)-\omega_2(x)\right)$$

$$\omega_3(x) \to 0 \quad x\to x_0$$

$$\boxed{o(g(x)) - o(g(x)) = o(g(x)) \quad x\to x_0}$$

$c\in\mathbb{R} \qquad \boxed{c f_1}$

$$c f_1(x) = c\,\omega_1(x)\,g(x) \quad x\to x_0 \qquad \omega_1(x)\to 0 \quad x\to x_0$$

$$\omega(x) = c\,\omega_1(x) \to 0 \quad x\to x_0$$

$$\boxed{c\, o(g(x)) = o(g(x)) \quad x\to x_0}$$

$\boxed{f_1\cdot f_2}$

$$f_1(x)\,f_2(x) = \omega_1(x)\,g(x)\,\omega_2(x)\cdot g(x) = \omega_1(x)\,\omega_2(x)\,g^2(x)$$

$$\omega(x)\to 0 \quad x\to x_0$$

$$\boxed{o(g)\cdot o(g) = o(g^2)}$$

$$\boxed{\frac{f_1}{f_2}} \qquad \frac{f_1(x)}{f_2(x)} = \frac{\omega_1(x)\,g(x)}{\omega_2(x)\,g(x)} = \frac{\omega_1(x)}{\omega_2(x)} \to \frac{0}{0} \qquad ?????$$

TRANSITIVITÀ di o-piccolo

Se $f(x) = o(g(x))$ $x \to x_0$ e

$g(x) = o(h(x))$ $x \to x_0$ allora

$$\boxed{f(x) = o(h(x)) \quad x \to x_0}$$

$f(x) = \omega_1(x)\, g(x)$ $x \to x_0$ $\qquad g(x) = \omega_2(x)\, h(x)$ $x \to x_0$

$f(x) = \omega_1(x)\, \omega_2(x)\, h(x)$ $x \to x_0$

$\omega(x) \to 0$ $x \to x_0$

OSS: Se ho che $f(x)$ è infinitesima per $x \to x_0$ $\Rightarrow$ $f(x) = \omega(x) \cdot 1$

$\boxed{f(x) = o(1)}$ $x \to x_0$

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Altre proprietà

ⓐ $f(x) = o(g_1(x) + g_2(x)) \quad x \to x_0$

$\Rightarrow \quad f(x) = (g_1(x) + g_2(x)) \cdot \omega(x) \qquad \omega(x) \to 0 \quad x \to x_0$

$= g_1(x)\,\omega(x) + g_2(x)\,\omega(x)$

$\to \quad f(x) = o(g_1(x)) + o(g_2(x)) \quad x \to x_0$

$$\boxed{o(g_1(x) + g_2(x)) = o(g_1(x)) + o(g_2(x)) \quad x \to x_0}$$

ⓑ $f(x) = o(c\,g(x)) \quad c \in \mathbb{R} \quad x \to x_0$

$= c\,g(x)\,\omega(x) \quad x \to x_0 \qquad \omega(x) \to 0 \quad x \to x_0$

$c\omega \to 0 \quad x \to x_0$

$$\boxed{o(c\,g(x)) = o(g(x)) \quad x \to x_0}$$

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Esempio 1. $\underbrace{\sin x^2}_{f(x)} = o(\underbrace{x}_{g(x)}) \quad x \to 0$

$$\lim_{x\to 0} \frac{\sin x^2}{x} = \lim_{x\to 0} \underbrace{\frac{\sin x^2}{x^2}}_{\to 1} \cdot x = 0$$

2. $\cancel{\sin x = o(x) \quad x \to 0}$

3. $\underbrace{x}_{f(x)} = o(\underbrace{\sqrt{x}}_{g(x)}) \quad x \to 0$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \lim_{x\to 0} \underbrace{\frac{x}{x}}_{\to 1} \cdot \sqrt{x} = 0$$

per la transitività 1 + 3 $\quad \sin x^2 = o(\sqrt{x}) \quad x \to 0$

Teorema 11.3 (Cancellazione)

"Se $f_1(x) = o(f(x))$ $x \to x_0$ $\quad g_1(x) = o(g(x))$ $x \to x_0$

$$\lim_{x\to x_0} \frac{f(x) + f_1(x)}{g(x) + g_1(x)} = \lim_{x\to x_0} \frac{f(x)}{g(x)} \text{ .}$$ "

SVILUPPI per $x \to 0$

$$\boxed{\sin x = x + o(x)}$$

$$\operatorname{tg} x = x + o(x)$$

$$\to \quad \cos x = 1 + o(x)$$

$$\boxed{e^x = 1 + x + o(x)}$$

$$\operatorname{arctg} x = x + o(x)$$

$$\arcsin x = x + o(x)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2) \quad \leftarrow$$

$$\log(1+x) = x + o(x)$$

$$\sin x = x + o(x) \quad x \to 0$$

$$\underbrace{\sin x - x}_{f(x)} = \underset{g(x)}{o(x)} \quad x \to 0$$

$$\lim_{x \to 0} \frac{\sin x - x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x} - 1 \right) = 0$$

$$e^x = 1 + x + o(x) \quad x \to 0$$

$$e^x - 1 - x = o(x) \quad x \to 0$$

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x} = \lim_{x \to 0} \left(\frac{e^x - 1}{x} - 1 \right) = 0$$

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$\cos x = 1 + o(x) \quad x \to 0$

$\cos x = 1 - \frac{x^2}{2} + o(x^2) \quad x \to 0$

$x^2 = o(x) \quad x \to 0$

$f \qquad g$

$\frac{f}{g} = \frac{x^2}{x} = x \to 0 \quad x \to 0$

$-\frac{1}{2} x^2 = -\frac{1}{2} o(x) = \underline{o(x)} \quad x \to 0$

$$\boxed{o(x^2) = o(o(x)) = o(x) \quad x \to 0}$$

$\cos x = 1 - \frac{x^2}{2} + o(x^2) = 1 + o(x) + o(x^2) = 1 + \underbrace{o(x) + o(x)} = 1 + o(x)$

$x \to 0$

oss: $\boxed{x^\alpha = o(x^\beta)} \quad \forall \alpha > \beta \quad x \to 0$

$\lim_{x \to 0} \frac{x^\alpha}{x^\beta} = \lim_{x \to 0} x^{\alpha - \beta} = 0$

Exerciții:

1. $$\lim_{x\to 0} \frac{\operatorname{arctg} x + 3\sin x}{e^x - 1 + 2\arcsin x} = \frac{4}{3}$$

numărător

$\operatorname{arctg} x = x + o(x)$ $\qquad$ $\sin x = x + o(x)$

$3\sin x = 3x + 3\,o(x) = 3x + o(x)$

$$\operatorname{arctg} x + 3\sin x = \underbrace{x + o(x)} + 3x + o(x) = \boxed{4x + o(x)}$$

numitor

$e^x - 1 = x + o(x)$ $\qquad$ $\arcsin x = x + o(x)$

$2\arcsin x = 2x + 2\,o(x) = 2x + o(x)$

$$e^x - 1 + 2\arcsin x = \underbrace{x + o(x)} + 2x + o(x) = \boxed{3x + o(x)}$$

în final $$\lim_{x\to 0} \quad = \lim_{x\to 0} \frac{4x + o(x)}{3x + o(x)} = \frac{4}{3}.$$

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2. $$\lim_{x \to 0} \frac{e^x - \cos x}{2x + 3 \operatorname{arctg} x} = \frac{1}{5}$$

numeratore $\quad e^x = 1 + x + o(x)$

$$\cos x = 1 + o(x)$$

$$e^x - \cos x = \underbrace{1 + x + o(x) - 1} + o(x) = \boxed{x + o(x)}$$

denominatore

$$\operatorname{arctg} x = x + o(x)$$

$$3 \operatorname{arctg} x = 3x + o(x)$$

$$2x + 3 \operatorname{arctg} x = 2x + 3x + o(x) = 5x + o(x)$$

$$\lim_{x \to 0} \quad = \lim_{x \to 0} \frac{x + o(x)}{5x + o(x)} = \frac{1}{5}^{+}$$

3. $$\lim_{x\to 0} \frac{\sin(2x) + 3\,\mathrm{tg}\,x}{\arcsin(7x) + 2\,\mathrm{arctg}\,x} = \frac{5}{9}$$

numeratore

$$\sin(2x) = 2x + o(2x) = 2x + o(x)$$

$$3\,\mathrm{tg}\,x = 3x + o(x)$$

$$\sin(2x) + 3\,\mathrm{tg}\,x = 2x + o(x) + 3x + o(x) = \boxed{5x + o(x)}$$

denominatore

$$\arcsin(7x) = 7x + o(7x) = 7x + o(x)$$

$$2\,\mathrm{arctg}\,x = 2x + o(x)$$

$$\arcsin(7x) + 2\,\mathrm{arctg}\,x = 7x + o(x) + 2x + o(x) = \boxed{9x + o(x)}$$

4. $$\lim_{x \to 0} \frac{\sin(2x^2) + 3\,\mathrm{tg}\,x}{\arcsin(7x) + 2\,\mathrm{arctg}(x^2)} = \frac{3}{7}.$$

numeratore

$$\sin x = x + o(x)$$

$$\sin(2x^2) = 2x^2 + o(2x^2) = 2x^2 + o(x^2) = 2x^2 + o(x)$$

$$= 2\,o(x) + o(x) = o(x) + o(x) = o(x)$$

$$3\,\mathrm{tg}\,x = 3\left(x + o(x)\right) = 3x + o(x)$$

$$\sin(2x^2) + 3\,\mathrm{tg}\,x = \boxed{3x + o(x)}$$

denominatore

$$\arcsin(7x) = 7x + o(7x) = 7x + o(x)$$

$$2\,\mathrm{arctg}(x^2) = 2\left(x^2 + o(x^2)\right) = 2\left(o(x) + o(x)\right) = 2\,o(x) = o(x)$$

$$\arcsin(7x) + 2\,\mathrm{arctg}(x^2) = \boxed{7x + o(x)}$$

Equivalenza asintotica

Siano f, g definite almeno definitivamente in un intorno di $x_0 \in \overline{\mathbb{R}}$

DEF 11.4. "Si dice che f è asintoticamente equivalente a g per $x \to x_0$ e si scrive $f(x) \sim g(x)$ $x \to x_0$ se esiste una funzione ω tale che

$$f(x) = \omega(x)\, g(x)$$

con $\lim\limits_{x \to x_0} \omega(x) = 1$ "

DEF 11.5 (Quasi equivalente)

" Se $g(x) \neq 0$ definitivamente per $x \to x_0$ (tranne al più in x_0) allora $f(x) \sim g(x)$ $x \to x_0$

se $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = 1$ "

Supponiamo $\begin{cases} f_1(x) = o(g(x)) & x \to x_0 \\ f_2(x) \sim f_1(x) & x \to x_0 \end{cases}$

allora $f_2(x) = o(g(x)) \quad x \to x_0$

dim:

$$f_1(x) = \omega_1(x)\, g(x) \qquad \omega_1 \to 0 \quad x \to x_0$$

$$f_2(x) = \omega_2(x)\, \underline{\underline{f_1(x)}} \qquad \omega_2 \to 1 \quad x \to x_0$$

$$= \underbrace{\omega_2(x) \cdot \omega_1(x)}\, g(x)$$

$$\omega_3(x) = \omega_2(x) \cdot \omega_1(x) \to 0 \quad x \to x_0$$

$$\Rightarrow \quad f_2(x) = o(g(x)) \quad x \to x_0 .$$

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Esempio 1. $e^{\sin x}$

brutalmente $\sin x \sim x \Rightarrow$ $\boxed{e^{\sin x} \sim e^{x} \sim 1+x}$

$e^{t} = 1 + t + o(t) \quad t \to 0$

Pongo $t = \sin x$

$\boxed{e^{\sin x}} = 1 + \sin x + o(\sin x) \quad x \to 0$

$\sin x = x + o(x) \quad x \to 0$

$= 1 + x + o(x) + o(x + o(x)) \quad x \to 0$

$= 1 + x + o(x) + o(x) + o(o(x)) = 1 + x + o(x) + o(x) + o(x)$

$\boxed{= 1 + x + o(x) \quad x \to 0}$

2. $x^2 \cdot o(x^3) = o(x^5)$

$x^2 \cdot x^3 \cdot \omega(x) = x^5 \cdot \omega(x) = o(x^5) \qquad x \to 0^{+}$

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3. $$\frac{1}{\sqrt{x}}\, o(\sqrt{x}) = o(1) \qquad x\to 0$$

$$\frac{1}{\sqrt{x}}\cdot \sqrt{x}\;\omega(x) = \omega(x) \to 0$$

$$\frac{1}{\sqrt{x}}\, o(\sqrt{x}) = o\left(\frac{1}{\sqrt{x}}\cdot \sqrt{x}\right) = o(1) \qquad x\to 0$$

4. $$\sin^2 x = (x+o(x))^2 = x^2 + 2x\cdot o(x) + (o(x))^2$$

$$= x^2 + 2\cdot o(x^2) + o(x)\cdot o(x)$$

$$= x^2 + \underbrace{o(x^2) + o(x^2)} = x^2 + o(x^2)$$

$$\sin^2 x = x^2 + o(x^2) \qquad x\to 0$$

$$\underbrace{\sin^2 x - x^2}_{f(x)} = o(\underset{g(x)}{x^2})$$

$$\lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2} = \lim_{x\to 0}\left(\left(\frac{\sin x}{x}\right)^2 - 1\right) = 0$$

5. $\sin x^2$ $\qquad \sin t = t + o(t) \quad t \to 0$

$t = x^2 \quad x \to 0 \qquad \sin x^2 = x^2 + o(x^2) \quad x \to 0$

6. $\log\left(1 + tg^3(2x)\right) \quad x \to 0$ $\qquad tg\, x \sim x \quad x \to 0$

$$\log\left(1 + (2x)^3\right) = \log(1 + 8x^3) \sim 8x^3 \quad x \to 0$$

$\log(1+t) = t + o(t) \quad t \to 0$

$t = tg^3 2x$

$$\log\left(1 + tg^3(2x)\right) = tg^3(2x) + o\left(tg^3(2x)\right) \quad x \to 0$$

$$= \left(2x + o(x)\right)^3 + o\left(\left(2x + o(x)\right)^3\right) \quad x \to 0$$

$$= 8x^3 + 12x^2 o(x) + \underline{6x \cdot o(x)^2} + o(x)^3 + o\left(8x^3 + 12x^2 \cdot o(x) + 6x\, o(x)^2 + o(x)^3\right)$$

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$$= 8x^3 + o(x^3) + o\left(8x^3 + o(x^3)\right)$$

$$= 8x^3 + o(x^3) + o(x^3) + \underbrace{o\left(o(x^3)\right)}_{= o(x^3)}$$

$$= 8x^3 + \underbrace{o(x^3)} \qquad x \to 0$$

7. $e^{x+\sin x}$ $\qquad e^{x+\sin x} \sim e^{2x} \sim \boxed{1+2x}$ $\qquad x \to 0$

$$e^t = 1 + t + o(t) \qquad t \to 0$$

$$t = x + \sin x \to 0 \qquad x \to 0$$

$$o(\sin x) = o(x + o(x)) = o(x) + o(o(x)) = o(x)$$

$$e^{x+\sin x} = 1 + x + \sin x + o(x + \sin x)$$

$$= 1 + x + x + o(x) + o(x) + o(\sin x) = \boxed{1+2x+o(x)}$$

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8. $$\lim_{x\to 0} \frac{e^{\operatorname{arctg} x} + \arcsin(3x) - \cos(x^2) + \log(1+x^5 \operatorname{tg} x)}{\operatorname{arctg}(3x+x^2) + \sin x} = 1$$

$e^{\operatorname{arctg} x} \sim e^x \sim 1+x \qquad x\to 0$

$\arcsin(3x) \sim 3x \qquad x \to 0$

$\cos(x^2) \sim 1 \qquad x\to 0$

$\sin x \sim x \qquad x \to 0$

$\operatorname{arctg}(3x+x^2) \sim 3x+x^2 \sim 3x \qquad x\to 0$

$\log(1+x^5 \operatorname{tg} x) \sim x^5 \operatorname{tg} x \sim x^6 \qquad x\to 0$

$1+x+3x-1+x^6 = 4x+x^6 \sim 4x \qquad \Longrightarrow \sim \frac{\cancel{4x}}{\cancel{4x}} = 1$

$3x + x = 4x$