LEZIONE 8ª

$+\infty - \infty$

$$a_n = 2n - n^2 \longrightarrow -\infty$$

$$\underset{+\infty}{2n} - \underset{(+\infty)}{n^2}$$

$$b_n = n^2 - 2n \longrightarrow +\infty$$

$$c_n = \sqrt{n+1} - \sqrt{n-1}$$

$$+\infty - \infty$$

$$= (\overset{a}{\sqrt{n+1}} - \overset{b}{\sqrt{n-1}}) \cdot \frac{(\overset{a}{\sqrt{n+1}} + \overset{b}{\sqrt{n-1}})}{\sqrt{n+1} + \sqrt{n-1}} = \frac{n+1-(n-1)}{\sqrt{n+1}+\sqrt{n-1}} = \frac{2}{\sqrt{n+1}+\sqrt{n-1}} \to 0$$

$$1^{+\infty}$$

$$e_n = \left(1+\frac{1}{n}\right)^n \qquad n\in\mathbb{N}$$

//

Lemma: Consideriamo le successioni (e_n) (E_n) di termini generali

$$e_n = \left(1+\frac{1}{n}\right)^n \qquad E_n = \left(1+\frac{1}{n}\right)^{n+1}$$

Sono vere le seg. affermazioni:

1. $1 < e_n < E_n \quad \forall n\in\mathbb{N}$.
2. (e_n) è strett. crescente
3. (E_n) è strett. decrescente. //

dim: 1 $\quad 1+\frac{1}{n} > 1 \qquad E_n = e_n \cdot \left(1+\frac{1}{n}\right) > e_n > 1$.

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② $e_{n-1} < e_n$ $n \geq 2$

$$\Updownarrow$$

$$\boxed{\frac{e_n}{e_{n-1}} > 1}$$

$$\boxed{\frac{e_n}{e_{n-1}}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n-1}\right)^{n-1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(\frac{n}{n-1}\right)^{n-1}} = \left(1+\frac{1}{n}\right)^n \cdot \left(\frac{n-1}{n}\right)^{n-1} = \left(1+\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^{n-1}$$

$$= \frac{\left(1+\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^n}{1-\frac{1}{n}} = \frac{\boxed{\left(1-\frac{1}{n^2}\right)^n}}{1-\frac{1}{n}}$$

dis. Bernoulli $(1+x)^n \geq 1+nx \quad x>-1 \quad n \geq 1$

$$\Rightarrow \boxed{(1+x)^n > 1+nx \quad x>-1 \quad x \neq 0 \quad n \geq 2}$$

$x = -\frac{1}{n^2}$

$$> \frac{1-n\cdot\frac{1}{n^2}}{1-\frac{1}{n}} = \boxed{1}$$

(3) $E_n < E_{n-1}$ $\quad n \geq 2$

$$\Updownarrow$$

$$\frac{E_{n-1}}{E_n} > 1$$

$$\frac{E_{n-1}}{E_n} = \frac{\left(1+\frac{1}{n-1}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{\left(\frac{n}{n-1}\right)^n \left(\frac{n}{n+1}\right)^n}{\left(1+\frac{1}{n}\right)} = \frac{\left(\frac{n^2}{n^2-1}\right)^n}{1+\frac{1}{n}}$$

Bernoulli

$$= \frac{\left(1+\frac{1}{n^2-1}\right)^n}{1+\frac{1}{n}} > \frac{1+n\frac{1}{n^2-1}}{1+\frac{1}{n}} > \frac{1+\frac{n}{n^2}}{1+\frac{1}{n}} = 1$$

$$n^2 > n^2-1 \iff \frac{1}{n^2-1} > \frac{1}{n^2}$$

$\blacksquare$

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$$e_1 < \underset{\equiv}{e_n} < E_n < E_1$$

Per il Teorema sulle successioni monotone limitate si ha

$$\lim e_n = \sup e_n \le \inf E_n = \lim E_n$$

$$\boxed{\begin{array}{l} E_n = e_n \left(1+\frac{1}{n}\right) \\ \downarrow \quad\ \downarrow \quad\quad \downarrow \\ \inf E_n = \sup e_n \cdot 1 \quad \Longrightarrow \quad \lim e_n = \lim E_n = \inf E_n = \sup e_n \end{array}}$$

DEF: (Numero di NEPERO)

$$e \doteq \lim \left(1+\frac{1}{n}\right)^n$$

Si può dim. che $e \in \mathbb{R} \setminus \mathbb{Q}$

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$$\lim \left(1 + \frac{1}{n}\right)^{n^2} = +\infty$$

$$\lim \left(1 - \frac{1}{n^2}\right)^{n} = 1$$

$$\lim \left(1 - \frac{1}{n}\right)^{n^2} = 0$$

$$\left(1 + \frac{(-1)^n}{n}\right)^{n}$$ non ha limite!

$a_n \to \pm\infty$ $\qquad$ $a_n \to 0$

$$\downarrow$$

$$\left(1 + \frac{1}{a_n}\right)^{a_n} \to e \qquad (1 + a_n)^{\frac{1}{a_n}} \to e$$

$$\forall x \in \mathbb{R} \qquad \left(1 + \frac{x}{n}\right)^{n} \longrightarrow e^{x} \qquad \Longleftarrow \log x$$

Funzioni Continue per successioni $\iff$ $\boxed{\text{CPS}}$

DEF: "Sia $f: A \subseteq \mathbb{R} \to \mathbb{R}$, $a \in A$.

Allora diremo che f è CPS in a se

per ogni successione (a_n), con $a_n \in A$ $\forall n \in \mathbb{N}$, convergente ad a

si ha

$$\boxed{\lim f(a_n) = f(a)}.$$"

Se tale proprietà vale $\forall a \in A$ diremo che f è CPS (in A)."

- def. poco pratica
- $\lim f(a_n) = f(\lim a_n)$
- $A \subseteq \mathrm{dom}\, f$

$$\lim \left(f(a_n) - f(a)\right) = 0$$

$$|f(a_n) - f(a)| \to 0$$

Lemma: Sono CPS nei loro domini naturali le seguenti funzioni:

1. La funzione identica $\quad id_A : A \to \mathbb{R} \quad id_A(x) = x \quad \forall x \in A$

2. Funzioni affini $\quad a \in \mathbb{R} \quad F : \mathbb{R} \to \mathbb{R} \quad F(x) \doteq x + a$

3. Valore assoluto

4. Polinomi e razionali

5. La funzione reciproca $\quad x \mapsto \frac{1}{x} \quad x \in \mathbb{R} \setminus \{0\}$.

6. Potenze razionali $\quad x \mapsto x^t \quad$ con $t \in \mathbb{Q}$

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$t \in \mathbb{N}$

$t = \frac{p}{q} \in \mathbb{Q} \setminus \mathbb{N} \qquad p \in \mathbb{Z} \qquad q \in \mathbb{N}$

$\Rightarrow$ solo $t > 0 \iff p \in \mathbb{N}$.

(a_n)

$a_n^{\geq 0} \to a^{\geq 0}$

$a_n^t \longrightarrow a^t$

$a_n^t = \left(\sqrt[q]{a_n}\right)^p$

E' sufficiente dimostrare che $\sqrt[q]{a_n} \to \sqrt[q]{a}$

$a_n^{\geq 0} \to a \geq 0$

$a = 0 \qquad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ per cui $0 \leq \underset{a_n}{\underset{\|}{|a_n|}} < \varepsilon^q \quad \forall n > N$

$\iff \sqrt[q]{a_n} < \varepsilon \quad \forall n > N$

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$a_n^{\geq 0} \to a > 0$

$\exists N \in \mathbb{N}$ tale che $a_n > 0 \quad \forall n > N$

$\boxed{b_n \doteq \sqrt[q]{a_n} > 0}$ $\quad$ $\boxed{b \doteq \sqrt[q]{a} > 0}$

$$a_n - a = b_n^q - b^q = (b_n - b)\left(b_n^{q-1} + b_n^{q-2} b + b_n^{q-3} b^2 + \dots + b_n^2 b^{q-3} + b_n b^{q-2} + b^{q-1}\right)$$

$$= (b_n - b) \underbrace{\sum_{k=0}^{q-1} b_n^k b^{q-1-k}}$$

$$M_q = b^{q-1} + \sum_{k=1}^{q-1} \underbrace{b_n^k b^{q-1-k}} \quad > 0$$

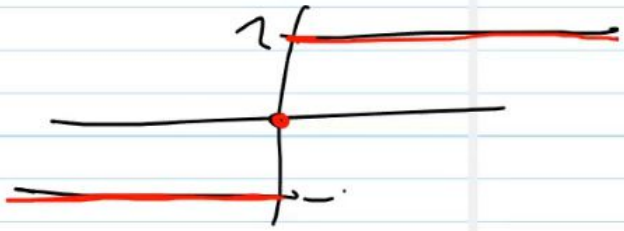
$$> b^{q-1}$$

$$b_n - b = \frac{b_n^q - b^q}{M_q} = \frac{a_n - a}{M_q}$$

$$|b_n - b| = \left|\sqrt[q]{a_n} - \sqrt[q]{a}\right| = \frac{|a_n - a|}{M_q} < \underbrace{\frac{1}{b^{q-1}}}_{\text{finito}} \underbrace{|a_n - a|}_{\downarrow 0} \to 0$$

Esempio di funzione <u>NON</u> CPS in un punto

$$\operatorname{segn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



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segn <u>non</u> è CPS in $x = 0$

$$a_n = \frac{1}{n} \longrightarrow 0 \quad \Rightarrow \quad \text{segn}(a_n) = 1$$

$$\text{“}f(a_n)\text{”} = \text{segn}(a_n) = 1 \quad ??? \quad \forall n \in \mathbb{N}$$

$$\downarrow$$

$$\text{segn}(0) = 0$$

se pongo $A = \mathbb{R}_+ \Rightarrow \text{segn}|_A$ è CPS ovunque in A

$$x \mapsto a^x \quad a > 0 \qquad\qquad x \mapsto e^x$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\log_a x \qquad\qquad\qquad\qquad \log x$$

$\cos x \quad \sin x$

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CONFRONTI E STIME ASINTOTICHE

$(a_n) \quad (b_n)$

$a_n \to \pm\infty \qquad b_n \to \pm\infty$

$$\lim \frac{a_n}{b_n} = \begin{cases} 0 & \Rightarrow (a_n) \text{ è infinito di ordine inferiore di } (b_n) \\ l \in \mathbb{R}\setminus\{0\} & \Rightarrow (a_n) \text{ ha lo stesso di ordine di infinito di } (b_n) \\ \pm\infty & \Rightarrow (a_n) \text{ ha ordine di infinito più grande di } (b_n) \\ \nexists & \text{non posso dire nulla} \end{cases}$$

$a_n \to 0 \qquad b_n \to 0$

$$\lim \frac{a_n}{b_n} = \begin{cases} \pm\infty & \Rightarrow (a_n) \text{ è infinitesima di ordine minore di } (b_n) \\ l \in \mathbb{R}\setminus\{0\} & \Rightarrow (a_n) \text{ è un infinitesima dello stesso ordine di } (b_n) \\ 0 & \Rightarrow (a_n) \text{ è infinitesima di ordine maggiore di } (b_n) \\ \nexists & \Rightarrow \text{non posso dire nulla} \end{cases}$$

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$$\lim \frac{a_n}{b_n} = 1$$

$\Rightarrow$ dirò che (a_n) è ASINTOTICA a (b_n)

$(a_n) \sim (b_n)$ $\quad$ $(a_n \sim b_n)$

Prop // Valgono i seguenti fatti:

1. Se $a_n \sim b_n$, le successioni hanno lo stesso carattere

2. Se $a_n \sim b_n \sim \dots \sim c_n \Rightarrow a_n \sim c_n$

3. Una espressione composta da un prodotto o quoziente di più fattori può essere stimata fattore per fattore

$$a_n \sim a_n' \qquad b_n \sim b_n' \qquad c_n \sim c_n' \Rightarrow \frac{a_n b_n}{c^n} \sim \frac{a_n' b_n'}{c_n'} \, .$$ //

oss $\quad a_n \sim b_n \qquad a_n = b_n c_n \qquad c_n \to 1$

es: $$2n^2+3n+1 = 2n^2\left(1+\frac{3}{2n}+\frac{1}{2n^2}\right)$$

$$\frac{3}{2n} \to 0 \qquad \frac{1}{2n^2} \to 0$$

Teorema (CRITERIO DEL RAPPORTO)

Sia (a_n) con $a_n>0 \ \forall n\in\mathbb{N}$. Se esiste

$$\lim \frac{a_{n+1}}{a_n} = \ell \in \overline{\mathbb{R}}$$

allora

1. Se $\ell < 1 \Rightarrow a_n \to 0$
2. Se $\ell > 1$ (incluso $\ell = +\infty$) $\Rightarrow a_n \to +\infty$
3. Se $\ell = 1$ non posso dire nulla

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Teorema (GERARCHIA INFINITI)

1/ Vale la seguente lista di risultati

1. $\lim \dfrac{\log_a n}{n^\alpha} = 0 \qquad \forall a>1 \quad \forall \alpha>0$

2. $\lim \dfrac{n^\alpha}{a^n} = 0$ $\qquad \dfrac{\pm\infty}{\pm\infty}$

3. $\lim \dfrac{a^n}{n!} = 0$

4. $\lim \dfrac{n!}{n^n} = 0$ //

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$+\infty^0$

$$\lim \sqrt[n]{n} = \lim n^{\frac{1}{n}} = \lim e^{\frac{1}{n}\log n} = e^0 = 1$$

$$0 < x = e^{\log x} = a^{\log_a x}$$

FORMULA DI STIRLING

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n}$$

$$\lim e_n = e = \lim E_n$$

$$e_n < e < E_n \qquad \forall n \in \mathbb{N}^+$$

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$$\left(1+\frac{1}{n}\right)^{n} < e < \left(1+\frac{1}{n}\right)^{n+1}$$

$n=1$ $$2 < e < 2^2$$

$n=2$ $$\frac{3^2}{2^2} < e < \frac{3^3}{2^3}$$

$n=3$ $$\frac{4^3}{3^3} < e < \frac{4^4}{3^4}$$

$\vdots$

$n-1$ $$\frac{n^{n-1}}{(n-1)^{n-1}} = \left(1+\frac{1}{n-1}\right)^{n-1} < e < \left(1+\frac{1}{n-1}\right)^{n} = \frac{n^n}{(n-1)^n}$$

$$\Rightarrow \cancel{2}\cdot\frac{\cancel{3^2}}{\cancel{2^2}}\cdot\frac{\cancel{4^3}}{\cancel{3^3}}\cdot\ldots\cancel{(n-1)^{n-2}}\,\frac{n^{n-1}}{(n-1)^{\cancel{n-1}}} < e^{n-1} < \cancel{2^2}\cdot\frac{\cancel{3^3}}{\cancel{2^3}}\cdot\frac{4^4}{3^4}\cdots\frac{n^n}{(n-1)^n}$$

$$\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{4}\cdot\ldots+\frac{n^{n-1}}{n-1} < e^{n-1} < \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdots\frac{n^n}{n-1}$$

$$\frac{n^n}{n!} = \frac{n}{n}\,\frac{n^{n-1}}{(n-1)!} < e^{n-1} < \frac{n^n}{(n-1)!}\,\frac{n}{n} = \frac{n^{n+1}}{n!} \quad \forall n \geq 2$$

moltiplico per $n!\, e^{1-n} > 0$

$$e^{1-n} n^n < n! < n^{n+1} e^{1-n}$$

$$e\,(e^{-n} n^n) < n! < (e\,n)(e^{-n} n^n)$$

$$\sqrt{2\pi n}$$