Positive subreducts of MV-algebras

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Positive MV-algebras

A *positive* referee report: ✓.

A *positive* COVID test: ✗.

A *positive* MV-algebra: ?

In this talk: I share some results about positive MV-algebras, introduced by (Cabrer, Jipsen, Kroupa, 2019), that shall help to develop their theory.

Positive MV-algebras

Positive fragment of Łukasiewicz propositional logic $\downarrow$ **Positive MV-algebras** / MV-algebras

Positive fragment of classical propositional logic $\downarrow$ Bounded distributive lattices / Boolean algebras

$$\frac{\textbf{Positive MV-algebras}}{\text{MV-algebras}} = \frac{\text{Bounded distributive lattices}}{\text{Boolean algebras}}$$

$\uparrow$

Łukasiewicz (**m**any-**v**alued) propositional logic (set of truth values: $[0, 1]$)

$\uparrow$

Classical propositional logic

Positive fragments

Examples of positive fragments:

- Positive modal algebras (Dunn, 1995) := "modal algebras without negation".
- Positive relational algebras := "relational algebras without complementation".

Positive MV-algebras: history and motivations

- (Cintula, Kroupa, 2013): positive fragment of Łukasiewicz logic in game theory.
- (Cabrer, Jipsen, Kroupa, 2019): introduced positive MV-algebras.
- (A., 2021): used positive MV-algebras to obtain a duality for Nachbin's compact ordered spaces ($\cong$ stably compact spaces).
- Just like Heyting algebras are based on bounded distributive lattices, an appropriate *many-valued* version of Heyting algebras (MV-intuitionistic logic) may be based on positive MV-algebras.
- Positive MV-algebras are a well-behaved study case for some results in duality theory and general algebra.

MV-algebras

Łukasiewicz logic

Łukasiewicz many-valued propositional logic [Łukasiewicz, 1920, Łukasiewicz, Tarski, 1930]: $[0,1]$ as the set of truth values.

Algebraic semantics

Algebraic semantics of Łukasiewicz logic = MV-algebras (Chang, 1958).

Consider $[0,1]$ with the operations:

- $x \oplus y := \min\{x+y, 1\}$.
  Example: $0.3 \oplus 0.2 = 0.5$, and $0.7 \oplus 0.8 = 1$.
- $\neg x := 1 - x$.
  Example: $\neg 0.3 = 0.7$.
- 0 as a constant.

MV-algebras

Definition (Chang, 1958, Mangani, 1973)

An *MV-algebra* (for *Many-Valued algebra*) is an algebra $\langle A; \oplus, \neg, 0\rangle$ s.t.

1. $\langle A; \oplus, 0\rangle$ is a commutative monoid;
2. $\neg\neg x = x$;
3. $x \oplus \neg 0 = \neg 0$;
4. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

MV-algebras are the algebras $\langle A; \oplus, \neg, 0\rangle$ satisfying all equations holding in $[0, 1]$, i.e.:

Theorem (Chang, 1959)

$[0, 1]$ *generates the variety of MV-algebras.*

I.e.: $\{\text{MV-algebras}\} = \text{HSP}([0, 1])$.

Examples of MV-algebras:

- $[0, 1]$.
- Subalgebras of $[0, 1]$, such as:
  1. $\{0, 1\}$ (here, $\oplus = \vee$);
  2. $Ł_3 := \{0, \frac{1}{2}, 1\}$.
  3. For every $n \geq 1$, $Ł_n := \{\frac{0}{n}, \frac{1}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}\}$.
  4. $\mathbb{Q} \cap [0, 1]$.
- Any Boolean algebra: set $\oplus = \vee$. One way to think of an MV-algebra is as a generalization of a Boolean algebra where the disjunction might fail to be idempotent.
- The set $[0, 1]^X$ of functions from a set X to $[0, 1]$.

Derived MV-terms

Every MV-algebra has a bounded distributive lattice reduct:

- $x \vee y := \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.
- $x \wedge y := \neg(\neg x \vee \neg y)$.
- $1 := \neg 0$.

The corresponding order on $[0,1]$ is the usual order. On $[0,1]^X$, it is the pointwise order.

One can also term-define the De Morgan dual of $\oplus$:

- $x \odot y := \neg(\neg x \oplus \neg y)$.
  In $[0,1]$: $x \odot y = \max\{x + y - 1, 0\}$.
  Example: $0.7 \odot 0.8 = 0.5$, and $0.3 \odot 0.2 = 0$.

Abelian lattice-ordered groups

MV-algebras can be understood as intervals of Abelian lattice-ordered groups.

Definition

An *Abelian lattice-ordered group* (or *Abelian ℓ-group*, for short) is an Abelian group $\mathbf{G}$ equipped with a lattice order s.t.:

(Translation invariance) for all $x, y, z \in \mathbf{G}$, $x \leq y$ implies $x + z \leq y + z$.

Examples:

1. $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$, with the sum.
2. The set $\mathbb{R}^X$ of functions from a set X to $\mathbb{R}$.

MV-algebras as unit intervals

For **G** an Abelian ℓ-group and $1 \in \mathbf{G}$ *positive* (i.e. $1 \geq 0$), the interval

$$\Gamma(\mathbf{G}, 1) := \{x \in \mathbf{G} \mid 0 \leq x \leq 1\}$$

is an MV-algebra with

$$x \oplus y := (x + y) \wedge 1, \qquad \neg x := 1 - x, \qquad 0 := \text{identity element of } \mathbf{G}.$$

Examples:

1. $\Gamma(\mathbb{R}, 1) = [0, 1]$,
2. $\Gamma(\mathbb{Z}, 1) = \{0, 1\}$.
3. $\Gamma(\frac{1}{n}\mathbb{Z}, 1) = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\} = Ł_{n+1}$.
4. For a set X, $\Gamma(\mathbb{R}^X, 1) = [0, 1]^X$.

Mundici's equivalence

Theorem (Mundici, 1986)

Every MV-algebra arises in this way, i.e. it is isomorphic to $\Gamma(\mathbf{G}, 1)$ *for some Abelian* ℓ*-group* $\mathbf{G}$ *and some positive* $1 \in \mathbf{G}$.

Example: let $\mathbf{A}$ be the MV-algebra of functions from $\mathbb{N}$ to $[0,1]$.

$$\mathbf{A} \cong \Gamma(?).$$

$$\mathbf{A} \cong \Gamma(\{\text{functions from } \mathbb{N} \text{ to } \mathbb{R}\}, 1)$$

$$\mathbf{A} \cong \Gamma(\{\text{bounded functions from } \mathbb{N} \text{ to } \mathbb{R}\}, 1)$$

Mundici's equivalence

For each MV-algebra $\mathbf{A}$ there is a canonical $(\mathbf{G}, 1)$ s.t. $\mathbf{A} \cong \Gamma(\mathbf{G}, 1)$, characterized by the condition that 1 is a *strong unit*, i.e. for all $x \in \mathbf{G}$ there is $n \in \mathbb{N}$ s.t.

$$x \leq \underbrace{1 + \cdots + 1}_{n \text{ times}}.$$

Theorem (Mundici's equivalence, 1986)

The categories

1. *of MV-algebras and homomorphisms, and*
2. *of Abelian ℓ-groups with strong unit and unit-preserving homomorphisms*

are equivalent.

Positive MV-algebras

Positive MV-algebras

$$\frac{\textbf{Positive MV-algebras}}{\text{MV-algebras}} = \frac{\text{Bounded distributive lattices}}{\text{Boolean algebras}}.$$

Bounded distributive lattices as subreducts

Relationship between bounded distributive lattices and Boolean algebras?

The $\{\vee, \wedge, 0, 1\}$-reduct of any Boolean algebra is a bounded distributive lattice, as well as any subalgebra of this reduct.

$$\boxed{\begin{aligned}\text{Bounded distributive lattices} = {} & \text{subalgebras of } \{\vee, \wedge, 0, 1\}\text{-reducts} \\ & \text{of Boolean algebras.}\end{aligned}}$$

$\vee$, $\wedge$, 0, 1 are order-preserving, and they term-generate all order-preserving Boolean terms.

Bounded distributive lattices = positive subreducts of Boolean algebras.

Positive MV-algebras

Definition (Cabrer, Jipsen, Kroupa, 2019)

Positive MV-algebras $:=$ subalgebras of the $\{\oplus, \odot, \vee, \wedge, 0, 1\}$-reducts of MV-algebras.

$\oplus$, $\odot$, $\vee$, $\wedge$, 0, 1 are order-preserving in each coordinate. We leave out $\neg$, which is not order-preserving.

Theorem (Cintula, Kroupa, 2013)

$\oplus$, $\odot$, $\vee$, $\wedge$, 0, 1 *generate all order-preserving terms of MV-algebras.*

$$
\begin{aligned}
\text{Positive MV-algebras} &= \text{positive subreducts of MV-algebras} \\
&= \text{many-valued version of bounded distrib. lattices.}
\end{aligned}
$$

Examples of positive MV-algebras

Examples of positive MV-algebras:

- Every MV-algebra, such as $[0,1]$, $\{0,1\}$, $Ł_n$, $\mathbb{Q} \cap [0,1]$, $[0,1]^X$.
- Every bounded distributive lattice (set $\oplus := \vee$ and $\odot := \wedge$).

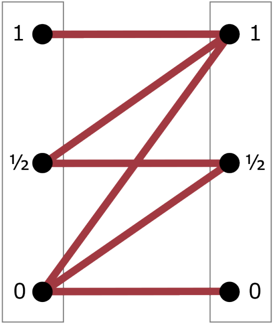
Positive MV-algebras are a common generalization of MV-algebras and bounded distributive lattices.

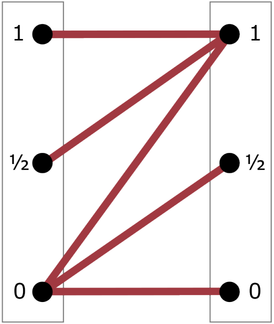
- For a poset X, the set of order-preserving functions from X to $[0,1]$.

Examples of positive MV-algebras

Some subreducts of the MV-algebra $Ł_3 \times Ł_3 = \{0, \frac{1}{2}, 1\} \times \{0, \frac{1}{2}, 1\}$:

- Order-preserving functions: $\{(a, b) \in Ł_3 \times Ł_3 \mid a \leq b\}$.
- Ordinal sum: $\{(a, b) \in Ł_3 \times Ł_3 \mid a = 0 \text{ or } b = 1\}$.





Positive MV-algebras as intervals

Unit intervals

MV-algebras = intervals of Abelian lattice-ordered groups.

Positive MV-algebras = intervals of certain lattice-ordered monoids.

Lattice-ordered monoids

Definition

A *cancellative commutative distributive ℓ-monoid* is a cancellative commutative monoid equipped with a distributive lattice-order s.t. the monoid operation $+$ distributes over the lattice operations $\vee$ and $\wedge$.

Examples of cancellative commutative distributive ℓ-monoids:

- $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$, every Abelian ℓ-group.
- The set of order-preserving functions from a poset X to $\mathbb{R}$.

Lattice-ordered monoids and positive MV-algebras

Given a cancellative commutative distributive ℓ-monoid $\mathbf{M}$ and a positive invertible element $1 \in \mathbf{M}$, the set

$$\Gamma(\mathbf{M}, 1) := \{x \in \mathbf{M} \mid 0 \leq x \leq 1\}$$

is a positive MV-algebra, with

- $x \oplus y := (x + y) \wedge 1$;
- $x \odot y := (x + y - 1) \vee 0$;
- $\vee$, $\wedge$, 0, 1 as in $\mathbf{M}$.

Positive MV-algebras as intervals

Theorem (A., Jipsen, Kroupa and Vannucci, 2022)

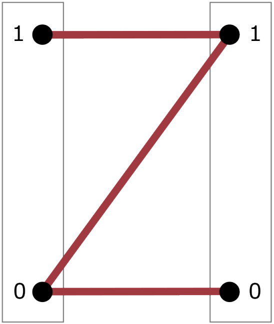
Every positive MV-algebra arises in this way, i.e. it is isomorphic to $\Gamma(\mathbf{M}, 1)$ for some cancellative commutative distributive ℓ-monoid $\mathbf{M}$ and some positive invertible $1 \in \mathbf{M}$.

Examples:

- $[0,1] \cong \Gamma(?)(\mathbb{R}, 1)$.
- $Ł_3 = \{0, \frac{1}{2}, 1\} \cong \Gamma(?)(\frac{1}{2}\mathbb{Z}, 1)$.

Positive MV-algebras as intervals

- The three-element bounded distributive lattice, as a positive MV-algebra (set $\oplus := \vee$ and $\odot := \wedge$)



is isomorphic to

$$\Gamma(?)(\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \leq b\}, (1, 1)),$$

i.e. the set of order-preserving functions from $\{x < y\}$ to $\mathbb{Z}$.

Positive MV-algebras as intervals

- Let $\mathbf{L}$ be a bounded distributive lattice, and set $\oplus := \vee$ and $\odot := \wedge$.

$$\mathbf{L} \cong \Gamma(?).$$

Set $X :=$ Priestley dual of $\mathbf{L}$.
$\mathbf{L} \cong \{$order-preserving continuous functions $X \to \{0,1\}\}$.
Set $\mathbf{M} := \{$continuous order-preserving functions $X \to \mathbb{Z}\}$; let $1 \in \mathbf{M}$ be the function constantly equal to $1 \in \mathbb{Z}$. Then

$$\mathbf{L} \cong \{\text{order-preserving continuous functions } X \to \{0,1\}\} = \Gamma(\mathbf{M}, 1).$$

Positive Mundici's equivalence

For each positive MV-algebra $\mathbf{A}$ there is a canonical choice of $\mathbf{M}$ and $1 \in \mathbf{M}$ such that $\mathbf{A} \cong \Gamma(\mathbf{M}, 1)$. This is characterized by the condition that 1 is a *strong unit*, i.e. for all $x \in \mathbf{M}$ there is $n \in \mathbb{N}$ s.t.

$$\underbrace{(-1) + \cdots + (-1)}_{n \text{ times}} \leq x \leq \underbrace{1 + \cdots + 1}_{n \text{ times}}.$$

Theorem (Positive Mundici's equivalence) (A., Jipsen, Kroupa, Vannucci, 2022)

The categories

1. of positive MV-algebras and homomorphisms, and
2. cancellative commutative distributive ℓ-monoids with strong unit and unit-preserving homomorphisms

are equivalent.

Mundici's result follows as a restriction of this equivalence.

Axiomatization of positive MV-algebras

Axiomatizations

Boolean algebras	**Bounded distributive lattices**	**MV-algebras**	**Positive MV-algebras**
Variety	Variety	Variety	Not variety ✗ Quasivariety ✓
Generated by $\{0, 1\}$ as a quasivariety	Generated by $\{0, 1\}$ as a quasivariety	Generated by $[0, 1]$ as a quasivariety	Generated by $[0, 1]$ as a quasivariety ✓
Finitely axiomatized	Finitely axiomatized	Finitely axiomatized	Finitely axiomatized ✓

Finite axiomatization of positive MV-algebras

Theorem (A., Jipsen, Kroupa and Vannucci, 2022)

Positive MV-algebras are axiomatized by:

1. $\langle A; \oplus, 0\rangle$ *and* $\langle A; \odot, 1\rangle$ *are commutative monoids;*
2. $\langle A; \vee, \wedge\rangle$ *is a distributive lattice;*
3. *Both* $\oplus$ *and* $\odot$ *distribute over both* $\vee$ *and* $\wedge$*;*
4. *If* $x_0 = y_0 \oplus y_1$ *and* $x_1 = y_0 \odot y_1$*, then*
   - *(Modularity)* $(x_0 \odot z) \oplus x_1 = x_0 \odot (z \oplus x_1)$*;*
   - *(Absorption)* $((x_0 \odot z) \oplus x_1) \wedge z = x_0 \odot z$ *and* $(x_0 \odot (z \oplus x_1)) \vee z = z \oplus x_1$.
5. *(Cancellation) If* $x \oplus z = y \oplus z$ *and* $x \odot z = y \odot z$*, then* $x = y$.

(1–4) are equations, (5) is a quasi-equation.

Free MV-extension

Free MV-extension

By definition, every positive MV-algebra $\mathbf{A}$ embeds into some MV-algebra.

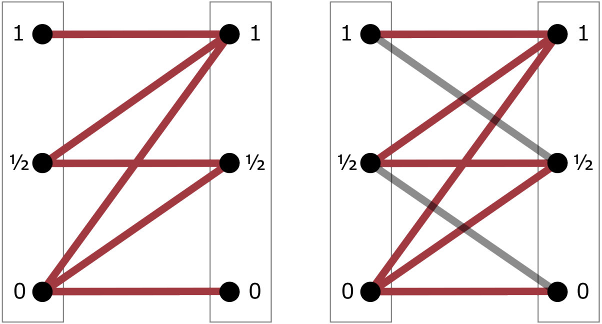
Is there a canonical embedding?

Yes, for general algebraic reasons: the forgetful functor

$$\{\text{MV-algebras}\} \longrightarrow \{\text{Positive MV-algebras}\}.$$

has a left adjoint (for general algebraic reasons). For each positive MV-algebra $\mathbf{A}$, the component $\eta_{\mathbf{A}} \colon \mathbf{A} \hookrightarrow \mathbf{B}$ of the unit is injective (fairly immediate), and we call $\eta_{\mathbf{A}}$ (or simply the MV-algebra $\mathbf{B}$) the *free MV-extension* of $\mathbf{A}$.

What is the free MV-extension of the following positive MV-algebra?

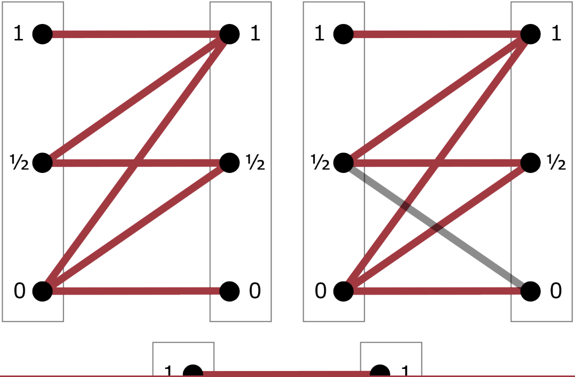


$Ł_3 \times Ł_3$? $[0,1] \times [0,1]$? Something else?

Canonical embedding

Theorem (A., Jipsen, Kroupa, Vannucci, 2022)

Let $\mathbf{A} \hookrightarrow \mathbf{B}$ be an embedding of a positive MV-algebra $\mathbf{A}$ into an MV-algebra $\mathbf{B}$ (i.e. $\mathbf{A}$ is a positive subreduct of $\mathbf{B}$), and let $\mathbf{C}$ be the MV-subalgebra of $\mathbf{B}$ generated by $\mathbf{A}$. The embedding $\mathbf{A} \hookrightarrow \mathbf{C}$ is the free MV-extension of $\mathbf{A}$.



In other words: an embedding of a positive MV-algebra into an MV-algebra is free iff it is MV-generating.

Free MV-extension

There is a unique generating embedding (the universal one):

Theorem (Equivalent reformulation)

Let $\mathbf{A}$ *be a positive MV-algebra, let* $f\colon \mathbf{A} \hookrightarrow \mathbf{B_1}$ *and* $g\colon \mathbf{A} \hookrightarrow \mathbf{B_2}$ *be two injective positive MV-homomorphisms into MV-algebras, and suppose that the images of* f *and* g *generate* $\mathbf{B_1}$ *and* $\mathbf{B_2}$ *as MV-algebras. Then* $\mathbf{B_1}$ *and* $\mathbf{B_2}$ *are isomorphic over* $\mathbf{A}$*.*

Free MV-extension

Universal property:

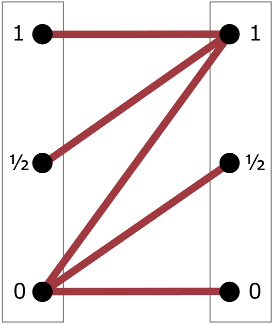
$$\begin{tikzcd} \mathbf{A} \arrow[r, hook, "\eta_A"] \arrow[dr] & \mathbf{B} \arrow[d, dashed, "\exists!"] \\ & \mathbf{C} \end{tikzcd}$$

Theorem (Equivalent reformulation)

Given MV-algebras $\mathbf{B}$ *and* $\mathbf{C}$ *and a partial function* $f\colon \mathbf{A} \subseteq \mathbf{B} \to \mathbf{C}$ *such that* $\mathbf{A}$ *MV-generates* $\mathbf{B}$ *and is closed under* $\oplus$, $\odot$, $\vee$, $\wedge$, 0 *and* 1, *and* $f\colon \mathbf{A} \to \mathbf{C}$ *preserves these operations,* f *extends uniquely to an MV-homomorphism* $\mathbf{B} \to \mathbf{C}$.

Computational advantage:

The set of homomorphisms of MV-algebras from $Ł_3 \times Ł_3$ to an MV-algebra $\mathbf{C}$ is in bijection with the set of homomorphisms of positive MV-algebras from the algebra below to $\mathbf{C}$.



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Free MV-extension

(For a general fact holding in subreducts of prevarieties (work in progress with C. van Alten),) this is equivalent to the following:

Theorem

Every MV-equation is equivalent (in MV-algebras) to a system of equations in the positive fragment.

Example: for all x, y, z in an MV-algebra:

$$x = \neg y \iff \begin{cases} x \oplus y = 1; \\ x \odot y = 0; \end{cases}$$

$$x \oplus \neg y = z \iff \begin{cases} 1 = z \oplus y; \\ x \wedge y = z \odot y. \end{cases}$$

Digression: characterization of unique embedding

Fact

Each cancellative commutative monoid $\mathbf{M}$ has a **unique** (up to iso) generating embedding $\mathbf{M} \hookrightarrow \mathbf{G}$ into an Abelian group.

Uniqueness up to iso means:

$$\begin{tikzcd} \mathbf{M} \arrow[r, hook, "f_1"] \arrow[dr, hook, "f_2"'] & \mathbf{G}_1 \arrow[d, dashed, "\text{iso}"] \\ & \mathbf{G}_2 \end{tikzcd}$$

Example: $\mathbb{N} \hookrightarrow \mathbb{Z}$.

It is not difficult to prove that every $g \in \mathbf{G}_1$ is a difference $g = x - y$ of elements of $\mathbf{M}$. Then, set $\psi(g) := f(x) - f(y) \in \mathbf{G}_2$.

Is this a well-defined function?

Suppose $x - y = x' - y'$ and let us prove $f(x) - f(y) = f(x') - f(y')$.

$$
\begin{aligned}
x - y = x' - y' &\iff x + y' = x' + y \\
&\implies f(x + y') = f(x' + y) \\
&\iff f(x) + f(y') = f(x') + f(y) \\
&\iff f(x) - f(y) = f(x') - f(y').
\end{aligned}
$$

Further, one proves that ψ is a group isomorphism that extends f.

Key facts used:

1. If **M** is a generating submonoid of an Abelian group **G**, then every element of **G** is a difference of two elements of **M**.
2. For all x, y in an Abelian group,

$$x - y = x' - y' \iff x + y' = x' + y.$$

Fact

Any equation between two terms in the language of Abelian groups is equivalent to an equation in the language of monoids.

Example: for all x, y, z in an Abelian group

$$-x + y - z = x \iff y = x + x + z.$$

(Thanks to the cancellation property.)

Groups and monoids

For arbitrary (i.e. non-necessarily commutative) monoids/groups, we do **not** have this uniqueness of embeddings.

For example, the monoid $\{x, y, z\}^*$ of words on three letters has distinct non-isomorphic generating embeddings into groups:

- into the free group $\mathrm{Free}(\{x, y, z\})$ on three elements ($x \mapsto x$, $y \mapsto y$, $z \mapsto z$) and
- into the free group $\mathrm{Free}(\{x, y\})$ on two elements ($x \mapsto x$, $y \mapsto y$, $z \mapsto xy^{-1}x$).

The equation $z = xy^{-1}x$ cannot be expressed via an equation in the language of monoids.

Fact

Each cancellative commutative distributive ℓ-monoid $\mathbf{M}$ has a **unique** (up to iso) generating embedding $\mathbf{M} \hookrightarrow \mathbf{G}$ into an Abelian ℓ-group.

Key facts used:

1. If $\mathbf{M} \hookrightarrow \mathbf{G}$ is a generating sub-ℓ-monoid of an Abelian ℓ-group $\mathbf{G}$, then every element of $\mathbf{G}$ is a difference of elements of $\mathbf{M}$.
2. For all x, y in an Abelian ℓ-group,

$$x - y = x' - y' \iff x + y' = x' + y.$$

Fact

Any equation between two terms in the language of Abelian ℓ-groups is equivalent to an equation in the language of ℓ-monoids.

Fact

Each bounded distributive lattice $\mathbf{L}$ has a **unique** (up to iso) generating embedding $\mathbf{L} \hookrightarrow \mathbf{B}$ into a Boolean algebra.

This embedding is called the *free Boolean extension*.

1. If $\mathbf{L} \hookrightarrow \mathbf{B}$ is a generating bounded sublattice of a Boolean algebra $\mathbf{B}$, then every element of $\mathbf{B}$ is a join of finitely many differences of elements of $\mathbf{L}$:
$$z = \bigvee_{i=1}^{n} x_i \wedge \neg y_i.$$
2. Every equation between joins of differences is equivalent to a system of equations in the language of bounded distributive lattices.

Distributivity in a lattice: For all a, b, c, $a = b$ iff $a \vee c = b \vee c$ and $a \wedge c = b \wedge c$.

$$x \wedge \neg y = z \iff \begin{cases} (x \wedge \neg y) \vee y = z \vee y \\ (x \wedge \neg y) \wedge y = z \wedge y \end{cases} \iff \begin{cases} x \vee y = z \vee y \\ 0 = z \wedge y. \end{cases}$$

Fact

Any equation between two terms in the language of Boolean algebras is equivalent to a system of equations in the language of bounded distributive lattices.

Fact

Each positive MV-algebra $\mathbf{A}$ has a **unique** (up to iso) generating embedding $\mathbf{A} \hookrightarrow \mathbf{B}$ into an MV-algebra.

We called this embedding the *free MV-extension* of $\mathbf{A}$.

1. If $\mathbf{A} \hookrightarrow \mathbf{B}$ is a generating positive MV-subalgebra of an MV-algebra $\mathbf{B}$, then every element of $\mathbf{B}$ is a sum of finitely many "differences" of elements of $\mathbf{A}$:

$$z = \bigoplus_{i=1}^{n} x_i \odot \neg y_i.$$

2. Every equation between sums of differences is equivalent to a system of equations in the language of positive MV-algebras.

Cancellation: For all a, b, c in an MV-algebra, $a = b$ iff $a \oplus c = b \oplus c$ and $a \odot c = b \odot c$.

$$x \odot \neg y = z \iff \begin{cases} (x \odot \neg y) \oplus y = z \oplus y \\ (x \odot \neg y) \odot y = z \odot y. \end{cases} \iff \begin{cases} x \vee y = z \oplus y \\ 0 = z \odot y. \end{cases}$$

Fact

Any equation between two terms in the language of MV-algebras is equivalent to a system of equations in the language of positive MV-algebras.

Definition

A *prevariety* is a class $\mathcal{V}$ of algebras closed under subalgebras and products.

Examples: any variety, any quasivariety.

Setting

- An algebraic language $\mathcal{L}_+$ and a sublanguage $\mathcal{L}_- \subseteq \mathcal{L}_+$.
- Two prevarieties $\mathcal{V}_+$ and $\mathcal{V}_-$ for $\mathcal{L}_+$ and $\mathcal{L}_-$, respectively.

We assume "$\mathcal{V}_+ \subseteq \mathcal{V}_-$" i.e.: $\mathcal{V}_-$ contains all $\mathcal{L}_-$-reducts of algebras in $\mathcal{V}_+$.

For example:

1. $\mathcal{V}_+ = \{\text{Abelian groups}\}$, $\mathcal{V}_- = \{\text{cancellative commutative monoids}\}$.
2. $\mathcal{V}_+ = \{\text{Abelian groups}\}$, $\mathcal{V}_- = \{\text{commutative monoids}\}$.
3. $\mathcal{V}_+ = \{\text{groups}\}$, $\mathcal{V}_- = \{\text{monoids}\}$.
4. $\mathcal{V}_+ = \{\text{MV-algebras}\}$, $\mathcal{V}_- = \{\text{positive MV-algebras}\}$.

Definition

Unique embeddability property :=
Given $\mathbf{A} \in \mathcal{V}_-$, $\mathbf{B}, \mathbf{C} \in \mathcal{V}_+$, and injective $\mathcal{V}_-$-homomorpisms $f\colon \mathbf{A} \hookrightarrow \mathbf{B}$ and $g\colon \mathbf{A} \hookrightarrow \mathbf{C}$ whose images $\mathcal{V}_+$-generate $\mathbf{B}$ and $\mathbf{C}$ respectively, there is a $\mathcal{V}_+$-isomorphism $h\colon \mathbf{B} \to \mathbf{C}$ making the following diagram commute.

$$\begin{tikzcd} \mathbf{A} \arrow[r, hook, "f"] \arrow[dr, hook, "g"'] & \mathbf{B} \arrow[d, dashed, "h"] \\ & \mathbf{C} \end{tikzcd}$$

We have the unique embeddability property for $\mathcal{V}_+ = \{\text{Abelian groups}\}$ and $\mathcal{V}_- = \{\text{commutative monoids}\}$.

We do not have the unique embeddability property for $\mathcal{V}_+ = \{\text{groups}\}$ and $\mathcal{V}_- = \{\text{monoids}\}$.

Definition

Expressibility property $:=$
for each pair $(\sigma(x_1, \ldots, x_n), \rho(x_1, \ldots, x_n))$ of terms in $\mathcal{L}_+$, there is a (finite) set of pairs $(\alpha_i(x_1, \ldots, x_n), \beta_i(x_1, \ldots, x_n))_i$ of terms in $\mathcal{L}_-$ s.t., for all $\mathbf{A} \in \mathcal{V}_+$ and $x_1, \ldots, x_n \in \mathbf{A}$,

$$\sigma(x_1, \ldots, x_n) = \rho(x_1, \ldots, x_n) \;\Leftrightarrow\; \forall i \;\; \alpha_i(x_1, \ldots, x_n) = \beta_i(x_1, \ldots, x_n).$$

I.e.: every equation in $\mathcal{L}_+$ is equivalent to a system of equations in $\mathcal{L}_-$.

For Abelian groups and commutative monoids we have the expressibility property.

For groups and monoids we do **not** have the expressibility property.

Theorem (ongoing joint work with C. van Alten)

Unique embeddability property $\iff$ expressibility property.

Main usage: proving the unique embeddability property by showing that equations in the richer language can be expressed in the poorer language (e.g.: $x - y = x' - y'$ iff $x + y' = y + x'$).

For positive MV-algebras:

Every equation in the language of Abelian ℓ-groups can be rewritten in the language of ℓ-monoids.	$\rightarrow$	Each cancellative commutative distributive ℓ-monoid has a unique generating embedding into an Abelian ℓ-group.
		$\downarrow$
Every equation in the language of MV-algebras can be rewritten in the language of positive MV-algebras.	$\leftarrow$	Each positive MV-algebra has a unique generating embedding into an MV-algebra.

Recap

Recap

Definition

Positive MV-algebras $:= \{\oplus, \odot, \vee, \wedge, 0, 1\}$-subreducts of MV-algebras.

1. Not a variety. ✗
2. Quasivariety, generated by $[0, 1]$. ✓
3. Finite quasi-equational axiomatization. ✓
4. Intervals of certain ℓ-monoids. ✓
5. Unique embedding into MV-algebras. (Embedding + MV-generating $\Rightarrow$ Free.) ✓

Future directions

Future directions

1. What makes Mundici's equivalence work?
   Goal: to obtain an equivalence à la Mundici between
   - certain algebras in the signature $\{\oplus, \odot, 0, 1\}$, and
   - certain algebras in the signature $\{0, +, 1, \tau_0, \tau_1\}$, where τ_0 and τ_1 are unary symbols to be thought of as $\tau_0(x) = x \vee 0$ and $\tau_1(x) = x \wedge 1$.

   I would like to do it without assuming the cancellation property so that (not necessarily distributive) bounded lattices can be seen as intervals of monoids.
   (Side question: is there a *unique* generating embedding of $\{\oplus, \odot, 0, 1\}$-subreducts of MV-algebras into MV-algebras? Equivalently, is every equation in the language of MV-algebras equivalent to an equation in $\{\oplus, \odot, 0, 1\}$?)
   Yet a further step would be to go to the non-commutative case.

Future directions

2. (Jointly with A. Přenosil): MV-version of Blok-Esakia theorem.
   - Consider a notion of modal MV-algebras which is an MV-version of S4 in the sense that the Gödel–McKinsey–Tarski translation ($x \to y = \Box(\neg x + y)$) connects the logic MV.S4 and the intuitionistic version of Łukasiewicz (= logic of positive MV-algebras where the product is residuated).
   - Then, try to extend this to some sort of Blok-Esakia style bijection between the extensions of MV.S4.Grz (whatever this is) and the intuitionistic version of Łukasiewicz.

Future directions

3. (From an input of a referee:) Does the characterization of the unique embeddability property extend beyond algebraic structures to a general model-theoretic setting? (Replacing equations by atomic formulae.)

Future directions

4. (From an input of L. Carai) Duality à la Baker-Beynon for positive MV-algebras?

Recap

Definition

Positive MV-algebras $:= \{\oplus, \odot, \vee, \wedge, 0, 1\}$-subreducts of MV-algebras.

1. Not a variety. ✗
2. Quasivariety, generated by $[0,1]$. ✓
3. Finite quasi-equational axiomatization. ✓
4. Intervals of certain ℓ-monoids. ✓
5. Unique embedding into MV-algebras. (Embedding + MV-generating $\Rightarrow$ Free.) ✓

Based on:

- M. A., P. Jipsen, T. Kroupa, and S. Vannucci. *A finite axiomatization of positive MV-algebras*. Algebra Universalis, 83:28, 2022.
- M. A. *On the axiomatisability of the dual of compact ordered spaces*. PhD thesis, University of Milan, 2021. (Ch. 4)
- M. A. *Equivalence à la Mundici for commutative lattice-ordered monoids*. Algebra Universalis, 82:45, 2021.

Thank you!