Norm complete Abelian l-groups: equational axiomatization

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Main question

Is the category of norm-complete ℓ-groups equivalent to a variety of (possibly infinitary) algebras?

Definition

Variety of algebras:= category of $\mathcal{L}$-algebras (where $\mathcal{L}$ is a set of function symbols) satisfying a certain set of $\mathcal{L}$-equations.

$$\forall \underline{x} \quad \gamma(\underline{x}) = \eta(\underline{x}).$$

(We admit operations of infinite arity.)

Example of norm-complete ℓ-group

Let X be a compact Hausdorff space, and, for every $x \in X$, let us assign a set A_x such that either $A_x := \frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{N}_{>0}$, or $A_x = \mathbb{R}$. We can encode $(A_x)_{x \in X}$ via a function $\zeta \colon X \to \mathbb{N}$.

$$\mathscr{C}_\zeta(\mathrm{X}) :=$$

$$\{f \colon X \to \mathbb{R} \mid f \text{ continuous}, \forall x \in X \;\; f(x) \in A_x\} =$$

$$\{f \colon X \to \mathbb{R} \mid f \text{ continuous}, \forall x \in X \;\; \mathrm{den}(f(x)) \text{ divides } \zeta(x)\}.$$

$\mathscr{C}_\zeta(X)$, endowed with pointwise operations $+, \vee, \wedge, -, 0, 1$, is an *Abelian lattice-ordered group* (*ℓ-group*, for short):

1. $\langle \mathscr{C}_\zeta(X), 0, +, - \rangle$ is an Abelian group;
2. $\langle \mathscr{C}_\zeta(X), \vee, \wedge \rangle$ is a lattice;
3. the order is translation invariant:

$$\forall f, g, h \in \mathscr{C}_\zeta(X) \quad f \leqslant g \Rightarrow f + h \leqslant g + h.$$

- 1 is a *strong unit*:

  for all $f \in \mathscr{C}_\zeta(X)$, there exists $n \in \mathbb{N}$ s.t. $(-n)1 \leqslant f \leqslant (n)1$,

- $\mathscr{C}_\zeta(X)$ is *Archimedean*:

  for all $f, g \in \mathscr{C}_\zeta(X)$ such that $f \geqslant 0$ and $g \geqslant 0$ we have:

  if, for all $n \in \mathbb{N}$, $(n)f \leqslant g$, then $f = 0$.

- $\mathscr{C}_\zeta(X)$ is *norm-complete*, i.e., complete in the metric induced by the supremum norm

$$\|f\| := \inf\left\{\frac{p}{q} \in \mathbb{Q} \mid p, q \in \mathbb{N}, q \neq 0, (q)|f| \leqslant (p)1\right\}.$$

Norm-complete ℓ-group:= ℓ-group with strong unit, which is Archimedean and norm-complete.

$\mathscr{C}_\zeta(X)$ is a norm-complete ℓ-group, and, viceversa, every norm-complete ℓ-group is of this form, for some choice of X and ζ.

Morphisms of norm-complete ℓ-groups: functions that preserve $+, \vee, \wedge, -, 0, 1$.

Main question

Is the category of norm complete ℓ-groups equivalent to a variety of (possibly infinitary) algebras?

Answer (main result)

Yes.

In the following: we provide an explicit finite equational axiomatization of this infinitary variety.

Is the class of norm-complete ℓ-groups closed (in the class of $\{+, \vee, \wedge, -, 0, 1\}$-algebras) under...

1. **... products?**
   No, 1 is a *strong unit* of $\mathbb{R}$, but not of $\mathbb{R}^{\mathbb{N}}$. ✗
2. **... subalgebras?**
   No, $\mathbb{R}$ is *norm-complete*, but $\mathbb{Q} \subseteq \mathbb{R}$ is not. ✗
3. **... homomorphic images?**
   No, the image of a norm-complete ℓ-group might fail to be *Archimedean*. ✗

Idea: introduce some additional operations together with new axioms regulating them.

This might solve 2 and 3. But not 1.

To solve the problem given by the *strong unit*, we use the theory of MV-algebras.

Given an ℓ-group G with strong unit ($u\ell$-group, for short),

$$\Gamma(G) \coloneqq \{x \in G \mid 0 \leqslant x \leqslant 1\}.$$

For $x, y \in \Gamma(G)$,

$$x \oplus y \coloneqq (x + y) \wedge 1;$$

$$\neg x \coloneqq 1 - x.$$

An MV-algebra is a structure $(A, \oplus, \neg, 0)$ such that

$$(A, \oplus, 0) \text{ is a commutative monoid.} \tag{MV 1}$$

$$x \oplus \neg 0 = \neg 0. \tag{MV 2}$$

$$\neg(\neg x) = x. \tag{MV 3}$$

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x. \tag{MV 4}$$

Mundici showed that Γ establishes an equivalence between the category of $u\ell$-groups and the category of MV-algebras.

Idea

In addition to the operations of $u\ell$-groups, consider an operation γ ($\simeq \lim$) of countably infinite arity, together with some new axioms, so that

$$\gamma(x_1, x_2, x_3, \ldots) = \lim_{n\to\infty} x_n$$

for ‘enough’ Cauchy sequences $(x_1, x_2, x_3, \ldots)$.

Definition

A sequence $(x_1, x_2, x_3, \dots)$ in a metric space (X, d) is called *super-Cauchy* if, for every $n \geqslant 2$,

$$d(x_n, x_{n-1}) \leqslant \frac{1}{2^n}.$$

Every super-Cauchy sequence is Cauchy.

Lemma

(X, d) is complete if, and only if, every super-Cauchy sequence converges.

Intended interpretation of γ on a norm-complete ℓ-group:

$$\gamma(x_1, x_2, x_3, \dots) = \lim_{n\to\infty} \rho_n(x_1, \dots, x_n)$$

where ρ_n is a term in the language of $u\ell$-groups—yet to be defined—such that

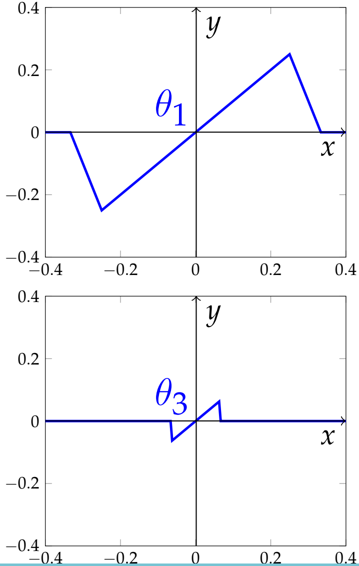
1. if $(x_1, x_2, x_3, \dots)$ is a super-Cauchy sequence, then, for all n,

$$\rho_n(x_1, \dots, x_n) = x_n;$$

2. for any $(x_1, x_2, x_3, \dots)$, the sequence $(\rho_n(x_1, \dots, x_n))_{n\geqslant 1}$ is super-Cauchy.

For $n \in \mathbb{N}_{>0}$, set

$$\theta_n(x) := \{\{[0 \wedge (-(2^{n+1}-1)x-1)] \vee x\} \wedge (-(2^{n+1}-1)x+1)\} \vee 0.$$



Let us define ρ_n as follows.

$$\begin{aligned}
\rho_1(x_1) &\coloneqq x_1;\\
\rho_2(x_1, x_2) &\coloneqq \rho_1(x_1) + \theta_1(x_2 - x_1);\\
\rho_3(x_1, x_2, x_3) &\coloneqq \rho_2(x_1, x_2) + \theta_2(x_3 - x_2);\\
&\vdots\\
\rho_n(x_1, \dots, x_n) &\coloneqq \rho_{n-1}(x_1, \dots, x_{n-1}) + \theta_{n-1}(x_n - x_{n-1}).
\end{aligned}$$

For every n, ρ_n is a term of $u\ell$-groups.

1. If $(x_n)_{n \in \mathbb{N}_{>0}}$ is a super-Cauchy sequence, then, for all n, $\rho_n(x_1, \dots, x_n) = x_n$.
2. For any $(x_n)_{n \in \mathbb{N}_{>0}}$ the sequence $(\rho_n(x_1, \dots, x_n))_{n \in \mathbb{N}_{>0}}$ is super-Cauchy.

Then, in any norm-complete ℓ-group, we can define

$$\gamma(x_1, x_2, x_3, \dots) \coloneqq \lim_{n \to \infty} \rho_n(x_1, \dots, x_n)$$

and γ maps super-Cauchy sequences to their limit.

Operations

Operations of $u\ell$-group, together with an operation γ of countably infinite arity.

Axioms

0. Axioms of ℓ-groups.
1. The element 1 is a strong unit.
2. $\gamma(x, x, x, \dots) = x$.
3. $\gamma(\theta_1(x), \theta_2(x), \theta_3(x), \dots) = 0$.
4. For each $n \in \mathbb{N}_{>0}$

$$\mathrm{d}(\gamma(x_1, x_2, x_3, \dots), \rho_n(x_1, \dots, x_n)) \leqslant \frac{1}{2^n},$$

i.e.

$$((2^n)|\gamma(x_1, x_2, x_3, \dots) - \rho_n(x_1, \dots, x_n))|) \vee 1 = 1.$$

Every norm-complete ℓ-group satisfies the axioms, with

$$\gamma(x_1, x_2, x_3, \dots) := \lim_{n\to\infty} \rho_n(x_1, \dots, x_n).$$

Lemma

The axioms

2. $\gamma(x, x, x, \dots) = x$;
3. $\gamma(\theta_1(x), \theta_2(x), \theta_3(x), \dots) = 0$.

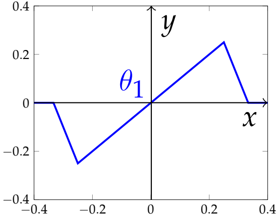
imply the Archimedean property.

Proof.

Let x be infinitesimal. Then

$$x \overset{2.}{=} \gamma(x, x, x, \dots) = \gamma(\theta_1(x), \theta_2(x), \theta_3(x), \dots) \overset{3.}{=} 0.$$

$\square$



The scheme of axioms

4. for each $n \in \mathbb{N}_{>0}$

$$\mathrm{d}(\gamma(x_1, x_2, x_3, \dots), \rho_n(x_1, \dots, x_n)) \leqslant \frac{1}{2^n}$$

'defines' $\gamma(x_1, x_2, x_3, \dots)$ as the limit of $(\rho_n(x_1, \dots, x_n))_{n \in \mathbb{N}_{>0}}$ and implies norm-completeness.

Let G_γ be the category of $\{+, \vee, \wedge, -, 0, 1, \gamma\}$-algebras satisfying Axioms 0, 1, 2, 3, 4.

Let G be the category of $u\ell$-groups.

Let $U \colon \mathsf{G}_\gamma \to \mathsf{G}$ be the forgetful functor (that forgets γ).

Theorem

The functor U is injective, full and faithful, and the objects in the image are precisely the norm-complete ℓ-groups.

Corollary

The category of norm-complete ℓ-groups is isomorphic to G_γ.

Conclusion

Theorem (Main result)

Up to an equivalence, the category of norm-complete ℓ-groups is a variety of infinitary algebras. Moreover, we have an explicit finite equational axiomatization of this variety.

Thank you.