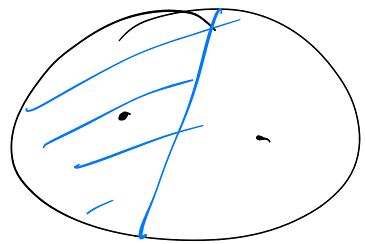
LAST TIME:

2026-02-25

4^{th} Lecture

Stone spaces: $\to$ totally separated (= clopens separate the points)
$\to$ compact



$$\text{Stone spaces} \underset{\text{Ult}}{\overset{\text{Clop}}{\rightleftarrows}} \text{Bool Alg}$$

$$B \overset{\eta_B}{\cong} \text{Clop}\,\text{Ult}(B)$$

$$X \simeq \text{Ult}\,\text{Clop}(X)$$

THM

Let X be a Stone space. The function

$$\varepsilon_X : X \longrightarrow \text{Ult}(\text{Clop}(X))$$
$$x \longmapsto \{A \in \text{Clop}(X) \mid x \in A\}$$

is a homeomorphism

Easy to see that is an ultrafilter X

PROOF

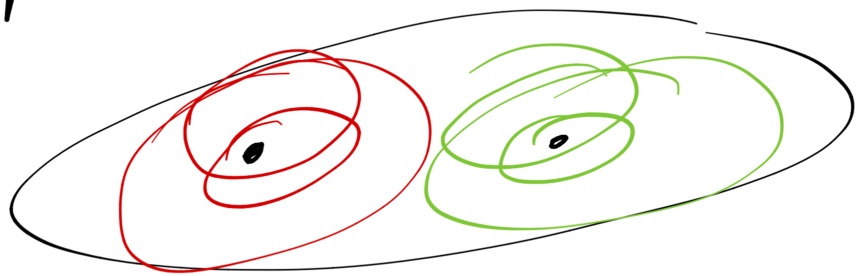
Enough → CONTINUOUS (IMMEDIATE)
↳ INJ. (TOTAL SEPARATION)
↳ SURJ. (COMPACTNESS)

CONTINUITY: enough to prove that the preimage of basic open set is open:

$$\mathcal{E}_X^{-1}\big[\underbrace{\eta_B(A)}_{\text{Clop}(X)}\big] = A \text{ is open.}$$

basic open set of $\mathrm{Ult}(X)$

INJ: Let $x \neq y$. By total separation, there is a clopen A s.t. $x \in A$, $y \notin A$

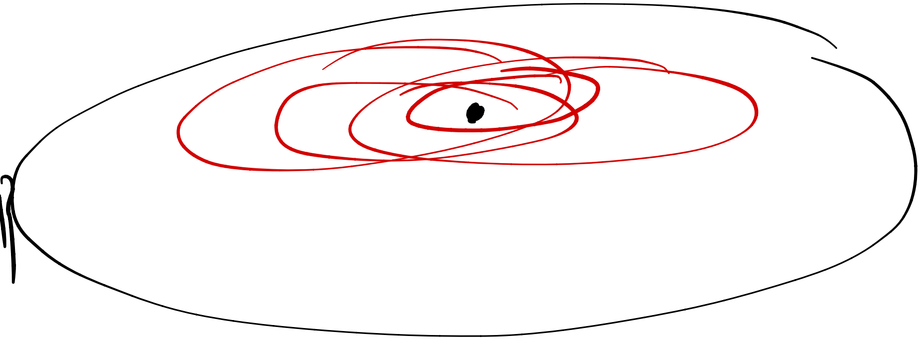


$$A \in \mathcal{E}_X(x)$$
$$\in \quad \neq$$
$$\mathcal{E}_X(y)$$

SURJ: Let $\mathcal{U} \in Ult(Clop(X))$

GOAL: find $x \in X$ s.t.

$$\mathcal{U} = \mathcal{E}_X(x) = \{A \in Clop(X) \mid x \in A\}$$



CLAIM: $\displaystyle\bigcap_{A \in \mathcal{U}} A \neq \emptyset$

The family $\{A\}_{A \in \mathcal{U}}$ has the FIP.
This proves the claim.

$\Rightarrow \exists x \in \displaystyle\bigcap_{A \in \mathcal{U}} A$.

Then it is easily seen that

$\mathcal{U} = \mathcal{E}_X(x)$, i.e.

$\mathcal{U} = \{A \in Clop(X) \mid x \in A\}$

$\subseteq$ EASY

$\supseteq$) We prove the contrapositive:

$A \notin \mathcal{U} \Rightarrow X \setminus A \in \mathcal{U} \Rightarrow x \in X \setminus A \Rightarrow x \notin A$

REM Let X be a compact space.
Let $\{A_i\}_{i \in I}$ be a family of closed sets.
If $\{A_i\}_{i \in I}$ has the FINITE INTERSECTION PROPERTY (FIP): i.e. for every finite $J \subseteq I$, $\displaystyle\bigcap_{i \in J} A_i \neq \emptyset$.
then $\displaystyle\bigcap_{i \in I} A_i \neq \emptyset$

If
- each $A_i \neq \emptyset$
- $\{A_i\}_{i \in I}$ is closed under finite intersection

Then it has the FIP.

Bool. Alg $\Longleftrightarrow$ Stone spaces

From $B \cong \mathrm{Clop}\, \mathrm{Ult}(B)$ we deduce

If B is finite, then B is isomorphic to a power set of a finite set.

Indeed: B fin $\Rightarrow$ $\mathrm{Ult}(B)$ is finite

$\Downarrow$

$\mathrm{Ult}(B)$ is a finite discrete space

$\Downarrow$

$$\mathrm{Clop}(\mathrm{Ult}(B)) = \mathcal{P}(\mathrm{Ult}(B))$$

$\Updownarrow$

B

If two FINITE Bool. alg. have the same cardinality, they are isom.

$\mathcal{P}(X)$ $\quad$ $\mathcal{P}(Y)$ $\quad$ $2^n = 2^m \Rightarrow n = m$

Let $\mathcal{L} = \{p_0, p_1, p_2, \dots\}$. We will prove

"There is no propositional theory that encodes the information "exactly one among all p_i's holds""

$$\forall i \neq j \quad \neg(p_i \wedge p_j)$$

$$\bigvee_{i \in \mathbb{N}} p_i$$

REWRITE MORE FORMALLY:

"There is no propositional theory T such that

$$Mod(T) = \{ v : \mathcal{L} \to \{0,1\} \mid \text{exactly one among all } v(p_i)\text{'s is } 1 \}$$

REMARK

For any prop. theory T in $\mathcal{L}$, $Mod(T)$ is a Stone space, with clopens: for every formula $\varphi \in Form(\mathcal{L})$

$$\{v \in Mod(T) \mid \underbrace{v \models \varphi}_{\bar{v}(\varphi)=1}\}$$

SKETCH of PROOF

$$Mod(T) \longleftrightarrow Ult\left(Form(\mathcal{L})/\equiv_T \right)$$

$$v \longmapsto \{[\varphi] \mid v \models \varphi\}$$

$$\mathcal{L} \to \{0,1\} \longleftarrow U$$

$$p \mapsto \begin{cases} 1 & [p] \in U \\ 0 & [p] \notin U \end{cases}$$

If BWOC there was a theory T with the above property

$$\mathrm{Mod}(T) = \bigcup_{n\in\mathbb{N}} \underbrace{\{v \in \mathrm{Mod}(T) \mid v \models p_n\}}_{\text{clopens}}$$

By compactness, there would be $I \subseteq \mathbb{N}$ finite s.t.

$$\mathrm{Mod}(T) = \bigcup_{n\in I} \{v \in \mathrm{Mod}(T) \mid v \models p_n\}$$

$v_m \in$ $\quad \notin$

Take $m \in \mathbb{N}\setminus I$ $\quad v_m : \mathcal{L} \longrightarrow 2$

$$i \longmapsto \begin{cases} 1 & i = m \\ 0 & \text{if not} \end{cases}$$

$n \in I : \; v_m \not\models p_n$

One can prove, as a consequence of the fact that the spaces of models are compact.

THM (Compactness theorem of classical prop. logic)

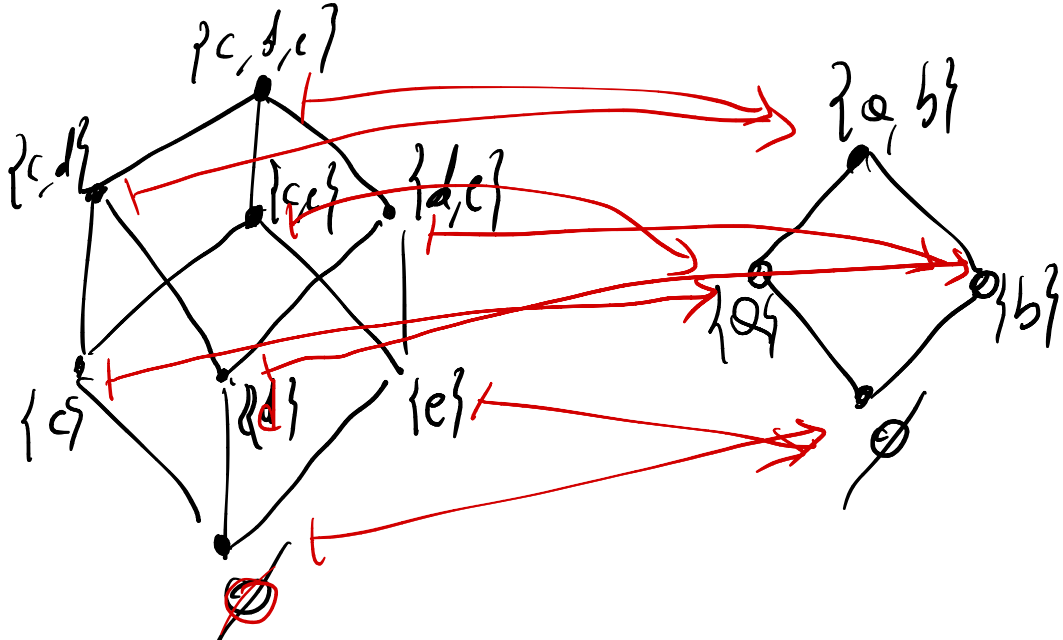
Let T be a prop. theory in a prop. language $\mathcal{L}$, $\varphi \in Form(\mathcal{L})$

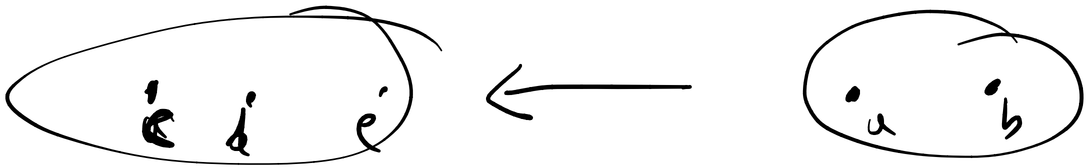
Suppose $T \models \varphi$ (i.e. for every model v of T, $v \models \varphi$ ($v(\varphi)=1$))

Then, there is a finite $T' \subseteq T$ s.t. $T' \models \varphi$

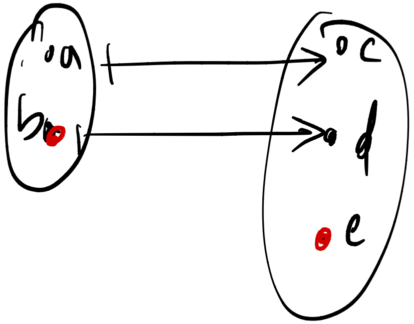
We skip the proof.

Stone proved a repr. of Bool. hom. (Logical meaning of Bool. hom.: interpretation of formulas)





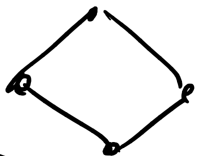
$3^2 = 9$



$f : \{a,b\} \longrightarrow \{c,d,e\}$

$\mathcal{P}(\{c,d,e\}) \longrightarrow \mathcal{P}(\{a,b\})$

$Z \longmapsto f^{-1}[Z]$

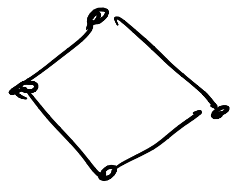
$FC(\mathbb{N}) \xrightarrow{\text{Bool. Alg}}$ 

$\mathbb{N} \cup \{\infty\} \xleftarrow[\text{continuous}]{} \{x,y\}$

$\infty \longleftarrow\!\shortmid x$
$ \longleftarrow\!\shortmid y \; : f$

$Clop(\mathbb{N} \cup \{\infty\}) \longrightarrow Clop(\{x,y\})$

$A \longmapsto f^{-1}[A]$

$$FC(\mathbb{N}) \longrightarrow$$ 

$$\begin{cases} \{x,y\} & \infty \in A \\ \emptyset & \infty \notin A \end{cases}$$

$$F \longmapsto \begin{cases} 1 & F \text{ cof.} \\ 0 & F \text{ finite} \end{cases}$$

BA := category of Bool. algs and Bool. hom.

Stone := " " Stone spaces and cont. maps

THM (Stone duality for Bool. algs.)

The categories BA and Stone are dually equivalent, i.e.

$$BA \simeq Stone^{op}$$

Roughly speaking: → "bijection on objects"

→ "bijection on morphisms, but reversing the direction".

Functors:

$$\mathrm{Clop} : \mathrm{Stone}^{op} \longrightarrow BA$$

$$X \longmapsto \mathrm{Clop}(X)$$

$$\begin{tikzcd} Y & \mathrm{Clop}(Y) \arrow[d] \\ X \arrow[u, "f"', "\mathrm{Stone}"] & \mathrm{Clop}(X) \end{tikzcd} \qquad f^{-1}[-] : Z \longmapsto f^{-1}[Z]$$

$$\mathrm{Ult} : BA \longrightarrow \mathrm{Stone}^{op}$$

$$B \longmapsto \mathrm{Ult}(B)$$

$$\begin{tikzcd} A \arrow[d, "f"'] & \mathrm{Ult}(A) & f^{-1}[U] \\ B \supseteq U & \mathrm{Ult}(B) \arrow[u, "f^{-1}[-]"'] & \ni U \arrow[u, mapsto] \end{tikzcd}$$

NAT. ISO :

$$\begin{tikzcd} \mathrm{Stone}^{op} \arrow[r, bend left, "\mathrm{Clop}"] & \mathrm{BA} \arrow[l, bend left, "\mathrm{Ult}"] \end{tikzcd}$$

UNIT:

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & \mathrm{Clop}\,\mathrm{Ult}(B) \arrow[d, "\mathrm{Clop}\,\mathrm{Ult}(f)"] \\ A \arrow[r] & \mathrm{Clop}\,\mathrm{Ult}(B) \end{tikzcd}$$

it is an iso, as we proved ~~[illegible]~~ some times.

COUNIT:

$$\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & \mathrm{Ult}\,\mathrm{Clop}(X) \arrow[d, "\mathrm{Ult}\,\mathrm{Clop}(f)"] \\ Y \arrow[r] & \mathrm{Ult}\,\mathrm{Clop}(X) \end{tikzcd}$$

NAT. TRANSF.

it is an iso because of the first theorem of today.

$$\begin{tikzcd} \mathrm{Stone}^{op} \arrow[r, bend left, "\mathrm{Clop}"] \arrow[r, phantom, "\perp"] & \mathrm{BA} \arrow[l, bend left, "\mathrm{Ult}"] \end{tikzcd}$$

It can be easily checked that it is an adj. $\square$

GUIDING QUESTION:

given A, B Bool. algs, is the coproduct map

$$a \neq 1 \longmapsto \neq 1$$

$$A \to A + B$$

injective?

Every equat. defined class of algebras is
- complete (EASY: limits are in Set)
- cocomplete (HARD: colimits NOT as in Set)

$$2 + 2 = 2$$

product map

$$X \times Y \to X$$

surj.

BA	Stone
2	$\{*\}$
$+$	$\times$
2	$\{*\}$
$\parallel$	$\parallel$
2	$\{*\}$

1. Explain the logical reading of the quest.

2. ~~Ex~~ Limits in Stone are computed as in Set

3. inj $\Leftrightarrow$ surj. surj $\Leftrightarrow$ inj.

4. Solve the problem in Stone spaces, get back to BA (and logic)

---

1. $A \simeq \mathrm{Form}(\mathcal{L}_1)/_{\equiv_{T_1}}$ $B \simeq \mathrm{Form}(\mathcal{L}_2)/_{\equiv_{T_2}}$

Let T_1, T_2 be prop. theories in languages $\mathcal{L}_1, \mathcal{L}_2$, resp.

For all $\varphi \in \mathrm{Form}(\mathcal{L}_1)$ s.t. $T_1 \nvDash \varphi$ (i.e. there is a model v of T s.t. $\overline{v}(\varphi) = 0$)

$$\underbrace{T_1 \amalg T_2}_{\text{in } \mathcal{L}_1 \amalg \mathcal{L}_2} \nvDash \varphi \quad ?$$

$\mathcal{L}_1 = \{p, q\}$ $\qquad T_1 = \{p \vee q\}$ $\qquad T_1 \not\models p \wedge q$, because v: $\mathcal{L}_1 \longmapsto 2$, $p \longmapsto 1$, $q \longmapsto 0$

$\mathcal{L}_2 = \{r\}$ $\qquad T_2 = \{\neg r\}$

$T_1 \amalg T_2 \overset{?}{\not\models} \underline{p \wedge q}$, $v: p \longmapsto 0$, $q \longmapsto 0$, $r \longmapsto 0$

$\quad \overset{\shortparallel}{\underline{\{p \vee q, \neg r\}}}$

$$\mathrm{Form}(\mathcal{L}_1)/\equiv_{T_1} + \mathrm{Form}(\mathcal{L}_2)/\equiv_{T_2} = \mathrm{Form}(\mathcal{L}_1 \amalg \mathcal{L}_2)/\equiv_{(T_1 \amalg T_2)}$$

$$\mathrm{Mod}(T_1 \amalg T_2) \hookrightarrow \mathrm{Mod}(T_1) \times \mathrm{Mod}(T_2)$$

$A \longrightarrow A \times B$

$X \times Y \longrightarrow X$

2. LIMITS in Stone: easy: as in Set (as in Top)

Examples: Term obj: $\{*\}$

Bin. prod. $X_1 \times X_2$ cart. prod. with the product topology

Arb. prod. $\prod_i X_i$ cart. prod. with the product topology.

(Tychonoff's theorem: arbitrary product of compact spaces is compact)

$\Uparrow$

Ax. of Choice

Equalizer: $Z \hookrightarrow X \underset{g}{\overset{f}{\rightrightarrows}} Y$

$Eq(f,g) = \{x \in X \mid f(x) = g(x)\}$, equipped with the subspace topology.

This is a Stone space because:

- Enough to prove that

$\{x \in X \mid f(x) = g(x)\}$ is closed in X.

it is the preimage under

$$X \xrightarrow{\langle f, g\rangle} Y \times Y$$
$$x \longmapsto (f(x), g(x))$$

of the diagonal $\{(y,y) \mid y \in Y\}$, which is closed because Y is Hausdorff.

(Hausdorff = diagonal closed!

LEMMA:
Every closed subspace of a Stone space is a Stone space (with the subspace topology)

EX

THM

The functors

Stone $\to$ Set , Stone $\to$ Top

preserve limit

BTW, preserves finite coproducts, but not arbitrary coproducts. Also, it does not preserve pushouts and coequalizers.

Question from the students:

Does Stone $\to$ Set have a left adjoint?

(It cannot have a right adjoint, because it does not preserve colimits)

YES, it has a left adjoint Set $\to$ Stone, called the Stone-Cech compactification: a set X is mapped to $Ult(P(X))$

PROOF See above for the limit

$A \xrightarrow{inj?} A \times B$

$\boxed{X \times Y} \xrightarrow{surj?} X$

3.

THM (inj $\Leftrightarrow$ surj , surj $\Leftrightarrow$ inj)

Let X, Y be Stone space, and $f: X \to Y$ a cont. map.

1. f is inj. iff $f^{-1}[-]: Clop(Y) \to Clop(X)$ is surj.
2. f is surj iff $f^{-1}[-]: Clop(Y) \to Clop(X)$ is inj.

If $g: A \to B$ is a Bool. hom.

1. g is surj $\Leftrightarrow$ $g^{-1}[-]: Ult(B) \to Ult(A)$ is INJ
2. g is inj $\Leftrightarrow$ $g^{-1}[-]$: is SURJ.

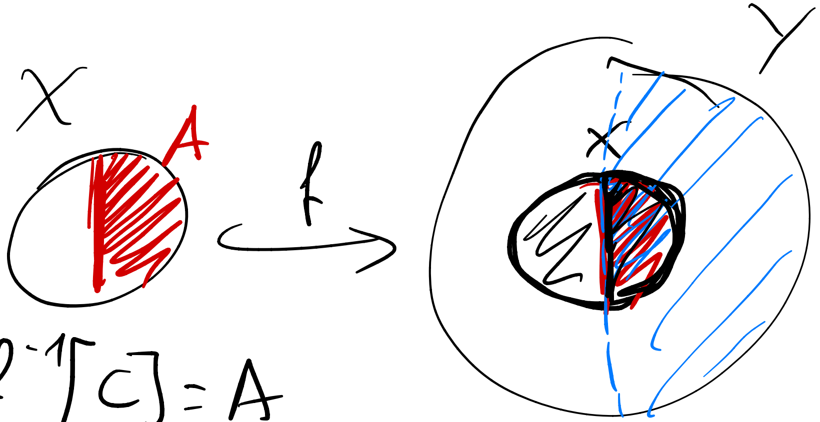
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Ult(P(X)) \arrow[d] \\ & Y \end{tikzcd}$$

PROOF

(1) $\Rightarrow)$ f is inj.

Let $A \in Clop(X)$.

Find a clopen C of Y s.t. $f^{-1}[C] = A$



$f[A]$

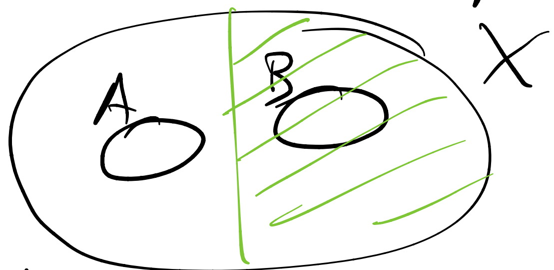
Apply the lemma with $f[A]$ and $f[X \setminus A]$

$\Rightarrow$ we have a clopen C s.t.

$f[A] \subseteq C$

$f[X \setminus A] \cap C = \emptyset$

LEMMA Let X be a Stone space



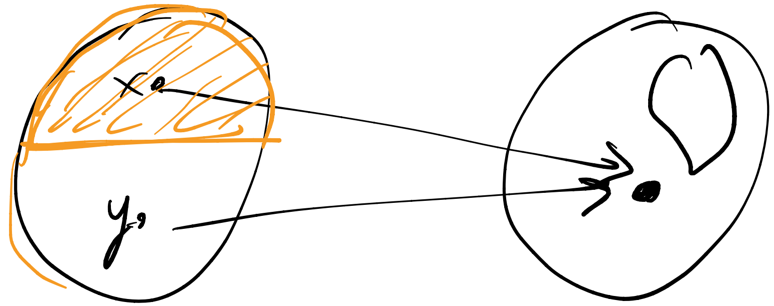
Let A, B closed disjoint subsets, Then there is a clopen C of X s.t. $B \subseteq C$ $A \cap C = \emptyset$.

EX

Then $A = f^{-1}[C]$. Thus $f^{-1}[-]$ is surj.

$(\Leftarrow)$ Let us prove the contrapositive

Suppose f not inj. GOAL: $f^{-1}[-]$ is not surj.



Take any clopen separating x and y. It is not the preimage of any clopen of Y.

② BE CONTINUED...