Soft sheaf representations in Barr-exact categories

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We generalize a result for sheaves from varieties of universal algebras to Barr-exact categories.

1940s/1950s: sheaves introduced.
1960s: applications of sheaves to rings and modules (Grothendieck, Dauns & Hofmann, Pierce, ...)
1970s: sheaf representations of universal algebras (Comer, Cornish, Davey, Keimel, Wolf, ...)

$\rightsquigarrow$ A frame of commuting congruences of a universal algebra A yields a sheaf representation of A.

Example: let A be a Boolean algebra.

1. The whole poset of congruences on A is a frame of commuting congruences. This yields Stone duality: a sheaf representation of A over the Stone dual of A.
2. The poset $\{\Delta_A, A \times A\}$ is a frame of pairwise commuting congruences. (For simplicity: assume A to be non-singleton, so that $\Delta_A \neq A \times A$.) This yields a sheaf representation of A over a one-element space.

The bigger the frame, the bigger the space, the simpler the stalks.

Congruence $\sim$ on A	$\leftrightsquigarrow$	compact subspace K of X.
Quotient $A \to A/{\sim}$	$\leftrightsquigarrow$	restriction map from global sections on X to local sections on K.

Definition ($\sim$ Godement, 1958)

A sheaf $\Omega(X)^{\mathrm{op}} \to \mathsf{Set}$ on a compact Hausdorff space is **soft** if every local section on a compact subset of X can be extended to a global section.

Example: the sheaf of continuous real-valued functions on $[0,1]$

$$\begin{aligned} F \colon \Omega([0,1])^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ U &\longmapsto C(U, \mathbb{R}) \end{aligned}$$

is soft. Example: a section on $[\frac{1}{3}, \frac{2}{3}]$ is (roughly speaking) a continuous functions from $[\frac{1}{3}, \frac{2}{3}]$ to $[0,1]$ together with its local behaviour at $\frac{1}{3}^-$ and $\frac{2}{3}^+$ (a "stalk" at $[\frac{1}{3}, \frac{2}{3}]$).

Gehrke and van Gool (2018) identified soft sheaf representations as the sheaf representations corresponding to frames of pairwise commuting congruences.

$\mathcal{K}(X) :=$ poset of compact subsets of X ordered by inclusion.
$\mathsf{Con}(A) :=$ poset of congruences of A ordered by inclusion.
A *sheaf representation of A over X* is a sheaf F over X s.t. $F(X) \cong A$.

Theorem (Gehrke & van Gool, 2018)

Let X be a compact Hausdorff space and A a nonempty algebra in a variety $\mathcal{V}$. There is a bijection between:

1. *isomorphism classes of soft sheaf representations of A over X;*
2. *$(\wedge, \bigvee)$-preserving maps $\mathcal{K}(X)^{\mathrm{op}} \to \mathsf{Con}(A)$ with image consisting of pairwise commuting congruences.*

$1 \rightsquigarrow 2.$ To a soft sheaf representation $F\colon \Omega^{\mathrm{op}}(X) \to A$ one associates

$$\begin{aligned} \mathcal{K}(X)^{\mathrm{op}} &\longrightarrow \mathcal{V} \\ K &\longmapsto \text{alg. } F(K) \text{ of local sections,} \end{aligned} \qquad \begin{aligned} \mathcal{K}(X)^{\mathrm{op}} &\longrightarrow \mathsf{Con}(A) \\ K &\longmapsto \ker(F(X) \twoheadrightarrow F(K)). \end{aligned}$$

$2 \rightsquigarrow 1.$ To $\rho\colon \mathcal{K}(X)^{\mathrm{op}} \to \mathsf{Con}(A)$ one associates

$$\mathcal{K}(X)^{\mathrm{op}} \longrightarrow \mathcal{V} \qquad \Omega(X)^{\mathrm{op}} \longrightarrow \mathcal{V}$$

We replace “variety of finitary algebras” with a Barr-exact category.

Examples of Barr-exact categories: varieties of (possibly infinitary) algebras, toposes.

Definition (Gray, 1965)

A **C-valued sheaf on a space X** is a functor $F\colon \Omega(X)^{\mathrm{op}} \to \mathsf{C}$ s.t.

1. ($\exists!$ gluing on finite families)
   - $F(\emptyset)$ is a terminal object of C.
   - For all $U, V \in \Omega(X)$, the following is a pullback square in C:

$$\begin{tikzcd} F(U \cup V) \arrow[r, "\restriction_{U\cup V, U}"] \arrow[d, "\restriction_{U\cup V, V}"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(U) \arrow[d, "\restriction_{U, U\cap V}"] \\ F(V) \arrow[r, "\restriction_{V, U\cap V}"] & F(U \cap V) \end{tikzcd}$$

2. ($\exists!$ gluing on directed families) F preserves codirected limits, i.e.: for all directed $\mathcal{D} \subseteq \Omega(X)$, $F(\bigcup \mathcal{D}) \cong \lim_{U \in \mathcal{D}} F(U)$.

Softness? (Every local section on a compact subspace extends to a global section.)

Definition (Lurie, 2009 (HTT))

A **C-valued $\mathcal{K}$-sheaf on a space X** is a functor $F\colon \mathcal{K}(X)^{\mathrm{op}} \to \mathsf{C}$ s.t.

1. ($\exists!$ gluing on finite families)
   - $F(\emptyset)$ is a terminal object of C.
   - For all $K, L \in \mathcal{K}(X)$, the following is a pullback square in C:

$$\begin{tikzcd} F(K \cup L) \arrow[r, "\restriction_{K\cup L,K}"] \arrow[d, "\restriction_{K\cup L,L}"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(K) \arrow[d, "\restriction_{K,K\cap L}"] \\ F(L) \arrow[r, "\restriction_{L,K\cap L}"] & F(K \cap L) \end{tikzcd}$$

2. F preserves directed colimits, i.e.: for all codirected $\mathcal{D} \subseteq \mathcal{K}(X)$, $F(\bigcap \mathcal{D}) \cong \mathrm{colim}_{K\in\mathcal{D}} F(K)$.

Theorem (Lurie, 2009)

Let X be a compact Hausdorff space and C a complete and cocomplete regular category where directed colimits commute with finite limits. There is a bijection between C-valued sheaves on X and C-valued $\mathcal{K}$-sheaves on X.

Idea:

1. An open is approximated by the compact sets contained in it.
2. A compact set is approximated by the open sets containing it.

Definition

Let C be a complete and cocomplete regular category.

1. A C-valued $\mathcal{K}$-sheaf $F\colon \mathcal{K}(X)^{\mathrm{op}} \to \mathsf{C}$ is **soft** if for every compact $K \subseteq X$ the restriction morphism $F(X) \to F(K)$ is regular epic.
2. A C-valued sheaf $F\colon \Omega(X)^{\mathrm{op}} \to \mathsf{C}$ over a compact Hausdorff space X is **soft** if for every compact $K \subseteq X$ the morphism $F(X) \to \operatorname{colim}_{U \in \Omega(X)\colon K \subseteq U} F(U)$ is regular epic.

For an object A, $\mathsf{Eq}(A) :=$ poset of internal equivalence relations on A.
We say that two equivalence relations R and S *commute* if $R \circ S = S \circ R$.

Theorem (A. & Reggio, 2023)

Let C *be a complete and cocomplete Barr-exact category where directed colimits commute with finite limits. Let* A *be an object of* C *such that the unique morphism* $A \to 1$ *is regular epic. Let* X *be a compact Hausdorff space. There is a bijection between:*

1. *isomorphism classes of soft sheaf representations of* A *over* X*;*
2. *isomorphism classes of soft* $\mathcal{K}$*-sheaf representations of* A *over* X*;*
3. $(\wedge, \bigvee)$*-preserving maps* $\mathcal{K}(X)^{\mathrm{op}} \to \mathsf{Eq}(A)$ *with image consisting of pairwise commuting internal equivalence relations.*

Our result holds also when X is a stably compact space (replace "compact" by "compact saturated"). Even further, one can go pointfree replacing $\Omega(X)$ with a stably continuous lattice and $\mathcal{K}(X)$ with the order-dual of its Lawson dual.

We weaken the notions of sheaves and $\mathcal{K}$-sheaves so to obtain perfectly dual notions.

Definition

Let X be a compact Hausdorff space and C a complete category. A **C-valued directed-sheaf on X** is a functor $F\colon \Omega(X)^{\mathrm{op}} \to \mathsf{C}$ that preserves codirected limits.

Definition

Let X be a compact Hausdorff space and C a cocomplete category. A **C-valued directed-$\mathcal{K}$-sheaf on X** is a functor $F\colon \mathcal{K}(X)^{\mathrm{op}} \to \mathsf{C}$ that preserves directed colimits.

These are dual notions: F is a C-valued directed-sheaf on X iff F^{op} is a C^{op}-valued codirected-$\mathcal{K}$-sheaf on (the de Groot dual of) X. If a property holds for all directed-sheaves, then the dual property holds for all directed $\mathcal{K}$-sheaves.

Example: Priestley duality for bounded distributive lattices.

The elements of a bounded distributive lattice are represented as continuous monotone functions from a Priestley space X to $\mathbf{2}$.

Priestley duality is not a sheaf representation: the gluing of two monotone functions might fail to be monotone. This is related to the failure of commutativity of congruences.

However, the gluing over a directed family preserves monotonicity.

Priestley duality is not a sheaf representation, but is a directed-sheaf representation. Directed-sheaf representations allow non-congruence-permutable algebras to be represented.

Theorem

Let X be a compact Hausdorff space and C a complete and cocomplete category. There is a bijection between C-valued directed-sheaves on X and C-valued directed-$\mathcal{K}$-sheaves on X.

Idea:

1. An open is approximated by the compact sets contained in it.
2. A compact set is approximated by the open sets containing it.

This generalizes to the pointfree context in the setting of continuous dcpos.

To sum up

1. From varieties of algebras to Barr-exact categories: bijection between
   - isomorphism classes of soft sheaf representations of A over X;
   - $(\wedge, \bigvee)$-preserving maps $\mathcal{K}(X)^{\mathrm{op}} \to \mathsf{Eq}(A)$ with image consisting of pairwise commuting internal equivalence relations.
2. We defined directed-sheaves and directed-$\mathcal{K}$-sheaves, which are dual notions: C-valued directed-sheaf $\leftrightsquigarrow$ C^{op}-valued directed-$\mathcal{K}$-sheaf.
3. Bijection between C-valued directed-sheaves and C-valued directed-$\mathcal{K}$-sheaves.

M. Abbadini, L. Reggio.
Barr-exact categories and soft sheaf representations.
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Thank you!