Bridging Ordered Groups and Topology: Representation Theorems and Categorical Dualities

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Partly based on:

M. Abbadini, V. Marra, L. Spada. Stone-Gelfand duality for metrically complete lattice-ordered groups. *Advances in Mathematics*, 2025.

Categorical Dualities

Two worlds, one story.

Geometry $\longleftrightarrow$ **Algebra**

Categorical dualities are bridges between such worlds —
each side encodes the same information, but in a very different language.

Categorical dualities:

an improved version of representation theorems.

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Categorical dualities go further:

they make the correspondence **bijective**,
and let us use it **to its full power** —
translating problems, theorems, and intuitions across the mirror.

A short journey through three categorical dualities

1930s **Stone duality:** connects topology and logic.
(Boolean algebras $\leftrightarrow$ Stone spaces)

1940s **Yosida duality:** connects topology and analysis.
(Ordered vector spaces $\leftrightarrow$ compact Hausdorff spaces)

2020s **A new duality:** brings the two under one umbrella.
(Ordered groups $\leftrightarrow$ normal a-spaces)

I. Stone duality

Logic becomes algebra: Boolean algebras

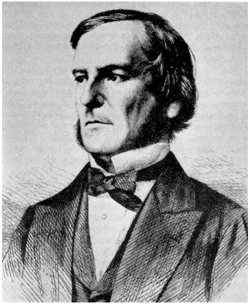
1847: **George Boole** formalized the rules of propositional reasoning into algebraic form.

A *Boolean algebra* is a set B (think of a set of propositions) equipped with

$$p \wedge q \text{ (and)}, \quad p \vee q \text{ (or)}, \quad \neg p \text{ (not)},$$

and two elements 0 (false) and 1 (true), satisfying familiar laws:

$$p \wedge \neg p = 0, \quad p \vee \neg p = 1, \quad p \wedge q = q \wedge p, \quad \dots$$



George Boole (1815–1864)

Example: the power set $\mathcal{P}(X)$ of a set X,

$$A \wedge B = A \cap B, \quad A \vee B = A \cup B, \quad \neg A = A^c, \quad 0 = \varnothing, \quad 1 = X.$$

Stone's representation: Boolean algebras as fields of sets

Start from a set X. Any family $\mathcal{F} \subseteq \mathcal{P}(X)$ that is closed under

$$A \cap B, \quad A \cup B, \quad A^c,$$

and contains $\varnothing$ and X, is a **Boolean algebra**.

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Stone's representation theorem (1936)

Every Boolean algebra is isomorphic to such a family of subsets of some set X (i.e. to a *field of sets*).

So, every Boolean algebra can be *realized concretely* as sets with $\cap$, $\cup$, and complement.

From representation to duality

So far, we have a **representation theorem.**

the base

Stone proved more: a **categorical duality**, for which we need *two ingredients*.

the ingredients

Canonical representation and bijective correspondence

For each Boolean algebra, there is a **canonical** representation: the unique one that is **separating** and **compact**.

Stone space $:=$ *compact* space whose *clopen* ($=$ *closed open*) sets separate the elements.

Examples: any finite discrete space.

Stone spaces $\xleftrightarrow{1:1}$ Boolean algebras.

Stone space $\mapsto$ Boolean alg. of *clopen* subsets.

Example: finite discrete space $\mapsto$ its power set.

a tomatopological sauce:
Stone sp. $\xleftrightarrow{1:1}$ Bool. alg.

Bijective correspondence on maps, reversed

Boolean homomorphisms

$A \to B$

$$\Big\updownarrow \ 1{:}1$$

continuous functions

Stone space of $B \to$ Stone space of A.

mozzarphisms:

$$A \longrightarrow B$$

$$X_A \longleftarrow X_B$$

Example:

$$X \hookrightarrow Y$$

between finite discrete spaces

$$\text{power set of } Y \longrightarrow \text{power set of } X$$

$$A \longmapsto A \cap X$$

Stone duality: bijection between Boolean algebras and Stone spaces + bijection between the interesting functions, but with opposite direction.

A *category* (Eilenberg, Mac Lane, 1945) consists of:

- a class of structures (e.g. Boolean algebras), and
- a class of maps between them (e.g. Boolean homomorphisms).

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Stone duality (1936)

There is a **categorical duality** between:

1. the category of **Boolean algebras** with Boolean homomorphisms, and
2. the category of **Stone spaces** with continuous functions.

Translating problems

Logic. Given propositional theories T in a language L and T' in a disjoint language L': does adding T' create new theorems in L?

Algebra. Is the coproduct map $A \to A + B$ between Boolean algebras injective?

Topology. Is the projection $X \times Y \to X$ surjective for Stone spaces?

Stone duality: all three are the same question — and the answer is immediate on the topological side.

Stone duality in one slide

- Boolean algebras $\leftrightarrow$ Stone spaces (1:1)
- Boolean homomorphisms $\leftrightarrow$ continuous maps (1:1, reversed)
- problems and results translate across the duality.

II. Yosida duality

The late 1930s / early 1940s

After Stone's work, mathematicians realized a certain pattern: various structures can be represented as spaces of (real-valued / complex-valued) *continuous functions* on a compact Hausdorff space.

This gave rise to several representation theorems in the late '30s and early '40s.

Some names in the game: Gelfand, Kolmogorov, Naimark, Kakutani, Yosida, the Krein brothers, Stone.

Some structures in the game: commutative C*-algebras, Banach lattices and Kakutani's (M)-spaces, vector lattices, divisible lattice-ordered Abelian groups.

The most famous one

The best–known result of this family is the **Gelfand–Naimark representation theorem** (1943):

Each unital commutative C^*-algebra is the algebra of continuous *complex–valued* functions on a compact Hausdorff space.

$$\textbf{Unital commutative } C^*\textbf{-algebras}$$

$$\updownarrow \text{ duality}$$

$$\textbf{Compact Hausdorff spaces}$$

We will not enter C^*-algebras today. Instead, we look at a simpler *real-valued* cousin: Yosida duality (1941).

Continuous real functions are very structured

For a compact Hausdorff space X, the set $C(X, \mathbb{R})$ of real–valued continuous functions on X has many natural operations:

- pointwise sum $f + g$
- pointwise maximum $f \vee g$ and minimum $f \wedge g$
- scalar multiplication $\lambda \cdot f$
- constant functions 0 and 1

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These operations are enough to recover the space X from $C(X, \mathbb{R})$ equipped with this structure.

For the following definition, think of $C(X, \mathbb{R})$.

Definition (Riesz, 1928)

A *vector lattice* is a real vector space equipped with a lattice order (i.e. a partial order with binary suprema and infima) s.t.:

1. (Translation invariance) if $x \leq y$ then $x + z \leq y + z$;
2. (Positive homogeneity) if $x \leq y$ then $\lambda x \leq \lambda y$ for any real $\lambda \geq 0$.

For Yosida's duality a bit more structure is needed.

Definition

A *metrically complete unital vector lattice* is a vector lattice V equipped with a designated element $1 \in V$ such that the following defines a complete metric on V:

$$\mathrm{dist}(v, w) := \inf \{\lambda \in \mathbb{R}_{\geq 0} \mid -\lambda 1 \leq v - w \leq \lambda 1\},$$

Example: in $C(X, \mathbb{R})$, let 1 be the function $X \to \mathbb{R}$ constantly equal to 1; the corresponding metric is the uniform metric.

Yosida proved that every metrically complete unital vector lattice is isomorphic to $C(X, \mathbb{R})$ for a unique compact Hausdorff space X (up to homeomorphism).

Yosida duality (1941)

The category of compact Hausdorff spaces is dually equivalent to the category of metrically complete unital vector lattices.

An inclusion of spaces $X \hookrightarrow Y$ gives a restriction function

$$\begin{aligned} C(Y, \mathbb{R}) &\longrightarrow C(X, \mathbb{R}) \\ f &\longmapsto f|_X, \end{aligned}$$

that preserves $+$, $\vee$, $\wedge$, scalar multiplication, the unit 1 (hence is a morphism of metrically complete unital vector lattices).

III. An umbrella for both

Two dualities, two mirrors

So far, we have seen two categorical dualities:

- **Stone duality** between Stone spaces and Boolean algebras (very discrete structures),
- **Yosida duality** between compact Hausdorff spaces and metrically complete unital vector lattices (very continuous structures).

Every Stone space X is in particular compact Hausdorff:

1. Stone associates to it the Boolean algebra of clopens of X.
2. Yosida associates to it $C(X, \mathbb{R})$.

We are using two different dualities, obtaining for the same space two very different objects.

1. We recall a class of algebraic structures that can specializes to
   - something similar to vector lattices on one extreme,
   - something similar to Boolean algebras on the other.
2. We provide a duality for these structures, which specializes in two directions: to Stone duality on one extreme and to Yosida duality on the other.

A hint: both sides have "addition"

In Yosida duality, $C(X, \mathbb{R})$ has a genuine addition $f + g$, defined everywhere.

In a Boolean algebra, $x \vee y$ behaves *like* an addition — but only when x and y are disjoint.

Idea: embed Boolean algebras into algebraic structures where a *true* addition is always defined.

Example:

$$\{0, 1\} \hookrightarrow \mathbb{Z},$$

where the Boolean "join" becomes ordinary sum (with truncation hidden in the embedding).

One algebraic structure to unify them

A good abstraction is: **lattice-ordered Abelian group** (*Abelian ℓ–group*) (1940):

- an Abelian group $(G, +, 0, -)$,
- equipped with a lattice order $(\vee, \wedge)$,
- that is translation–invariant $(x \leq y \Rightarrow x + z \leq y + z)$.

These objects can specialize in two directions:

- if highly divisible $\Rightarrow$ they look like vector lattices,
- if poorly divisible $\Rightarrow$ they look like Boolean algebras.

(Connection to MV-algebras (Łukasiewicz many-valued logic).)

Definition

A *metrically complete unital Abelian ℓ-group* is a lattice-ordered Abelian group G equipped with a designated element $1 \in G$ such that the following defines a complete metric on G:

$$\mathrm{dist}(v, w) := \inf \left\{ \frac{p}{q} \in \mathbb{Q} \mid p \geq 0, q > 0, \text{ and } p(-1) \leq q(v - w) \leq p1 \right\}.$$

Example

Given a compact Hausdorff space X:

$$C(X, \mathbb{R}) := \{ f : X \to \mathbb{R} \mid f \text{ is continuous} \},$$

$$C\left(X, \frac{1}{n}\mathbb{Z}\right) := \left\{ f : X \to \frac{1}{n}\mathbb{Z} \mid f \text{ is continuous} \right\}.$$

We can also mix the values that the functions can take at different points...

Representation theorem

Representation theorem, Goodearl & Handelman 1980

- Let X be a compact Hausdorff space. For each $x \in X$, let D_x be either $D_x = \mathbb{R}$ or $D_x = \frac{1}{n}\mathbb{Z}$ (for some $n \in \mathbb{N} \setminus \{0\}$). Then,

$$\left\{ f : X \to \mathbb{R} \mid f \text{ continuous, } \forall x \in X \ f(x) \in D_x \right\}$$

  is a metrically complete unital Abelian ℓ-group.
- Every metrically complete unital Abelian ℓ-group can be represented in this way.

Example

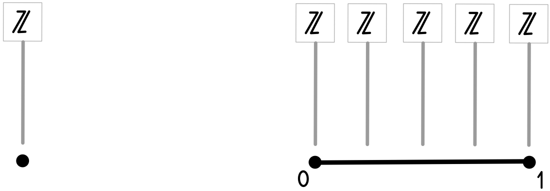
- If $D_x = \mathbb{R}$ for all $x \in X$, then we get $C(X, \mathbb{R})$.
- If $D_x = \mathbb{Z}$ for all $x \in X$, then we get $C(X, \mathbb{Z})$.

Our aim

Our aim (A., Marra, Spada): make the Goodearl-Handelman representation into a **categorical duality**, so that we have a representation of morphisms, and that we can transfer all problems that can be expressed in the categorical language.

Not a 1:1 correspondence

Goodearl & Handelman's representation is not a 1:1 correspondence.



In both cases, $C(X, \mathbb{Z}) \cong \mathbb{Z}$.

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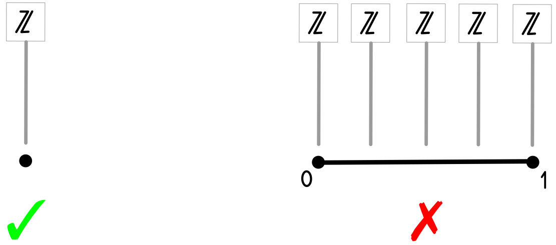

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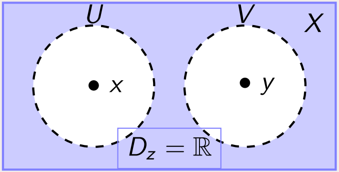
Remedy: Separation

For $x \neq y \in X$, there are disjoint open sets $U \ni x$ and $V \ni y$ s.t., for all $z \in X \setminus (U \cup V)$, $D_z = \mathbb{R}$.



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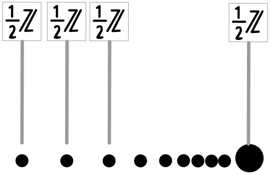

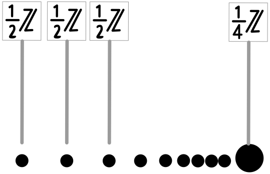

In both cases

$$\{f \colon X \to \mathbb{R} \mid f \text{ cont. and, for all } x \in X,\ f(x) \in D_x\} =$$

$$= \left\{f \colon X \to \frac{1}{2}\mathbb{Z} \mid f \text{ is eventually constant}\right\}.$$

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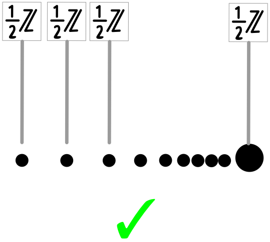

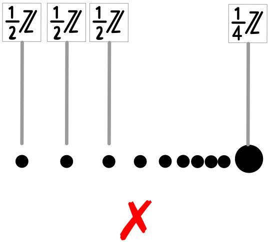

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Remedy: Semicontinuity

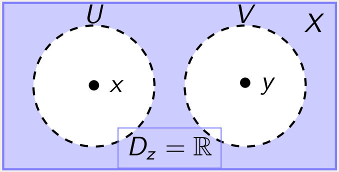
For every $n \in \mathbb{N}_{>0}$, $\left\{x \in X \mid D_x \subseteq \frac{1}{n}\mathbb{Z}\right\}$ is closed.

Normal a-spaces

Definition

A *normal a-space* (for "arithmetically normal arithmetic space") is a compact Hausdorff space X equipped with a family $(D_x)_{x \in X}$ with $D_x \in \{\mathbb{R}\} \cup \left\{\frac{1}{n}\mathbb{Z} \mid n \in \mathbb{N}_{>0}\right\}$ such that

1. (Separation) For $x \neq y \in X$, there are disjoint open sets $U \ni x$ and $V \ni y$ s.t., for all $z \in X \setminus (U \cup V)$, $D_z = \mathbb{R}$.



2. (Semicontinuity) For every $n \in \mathbb{N}_{>0}$, $\left\{x \in X \mid D_x \subseteq \frac{1}{n}\mathbb{Z}\right\}$ is closed.

Example: $[0,1]$ with $D_{\frac{p}{q}} = \frac{1}{q}\mathbb{Z}$ and $D_x = \mathbb{R}$ otherwise.

A new duality (A., Marra, Spada, 2025)

The category of **metrically complete unital Abelian ℓ-groups** (and homomorphisms) is dually equivalent to the category of **normal a-spaces** (and continuous “denominator-decreasing” maps).

M. Abbadini, V. Marra, L. Spada.
Stone-Gelfand duality for metrically complete lattice-ordered groups.
Advances in Mathematics, 2025.

Case $D_x = \mathbb{R}$ for all x: we get an arbitrary compact Hausdorff space X, and $C(X, \mathbb{R})$: Yosida duality.

Case $D_x = \mathbb{Z}$ for all x: we get an arbitrary Stone space X, and $C(X, \mathbb{Z})$, which is very close to the Boolean algebra $C(X, \{0, 1\})$: Stone duality.

A concluding reflection

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Baez, Dolan (2001):

> *"... an equation is only interesting or useful to the extent that the two sides are different!"*

The equation

$$e^{i\omega} = \cos(\omega) + i\sin(\omega)$$

is far more interesting and useful than

$$1 = 1,$$

precisely because its two sides have such different shapes.

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Thank you!