Stone duality for finitely valued algebras with a near-unanimity term

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Goal: represent algebraic structures as algebras of continuous functions satisfying some local constraints.

$$\text{Algebraic structure} \qquad \longleftrightarrow \qquad \text{Space} + \text{local constraints}$$

1. Stone duality (1936): Boolean algebra $\cong$ algebra of continuous functions from a Stone space X to $\{0, 1\}$ with pointwise operations.
   Space: a Stone space.
   Local constraints: none.
2. Priestley duality (1969): bounded distributive lattice $\cong$ algebra of order-preserving continuous functions from a Priestley space X to $\{0 < 1\}$. (Priestley space $=$ Stone space with a partial order satisfying compatibility properties.)
   Space: a Stone space.
   Local constraints: the order.

$$\text{Algebraic structure} \qquad \longleftrightarrow \qquad \text{Space} + \text{local constraints}$$

We investigate the case where the constraints can be made *very* local: on subsets of cardinality ≤ 2. $\rightsquigarrow$ Easier to describe them.

$\mathbb{ISP}(\mathbf{L}) :=$ class of algebras **i**somorphic to a **s**ubalgebra of a **p**ower of $\mathbf{L}$.

Davey & Werner (1983): duality for classes of the form $\mathbb{ISP}(\mathbf{L})$ where $\mathbf{L}$ is a finite algebra with a near unanimity term. (E.g.: $\mathbf{L}$ has a lattice reduct.)

Example: for $\mathbf{L} = \{0, 1\}$ we get Stone/Priestley dualities.

Our main result: we give an analagous duality, where the generating algebra **L** is allowed to be <u>infinite</u>.

Our motivation: representation of *positive MV-algebras*, which are algebras related to Łukasiewicz many-valued logic: $\{0, 1\}$ is replaced by $[0, 1]$.

Examples of generating algebra $\mathbf{L}$:

- $\mathbf{L} = \{0, 1\}$, with the signature of Boolean algebras. $\rightsquigarrow$ Stone duality (1936).
- $\mathbf{L} = \{0, 1\}$, with the signature of bounded distributive lattices. $\rightsquigarrow$ Priestley duality (1969).
- $\mathbf{L} = [0, 1]$, with the signature of MV-algebras. $\rightsquigarrow$ Duality for weakly locally finite MV-algebras (Cignoli, Marra, 2012).
- $\mathbf{L} = [0, 1]$, with the signature of positive MV-algebras, i.e. $\{\oplus, \odot, \vee, \wedge, 0, 1\}$.
- $\mathbf{L} = \mathbb{R}$, with the signature of ℓ-groups with a designated constant 1.
- $\mathbf{L} = \mathbb{R}$, with the signature of ℓ-monoids with designated constants -1 and 1, i.e. $\{+, \vee, \wedge, 0, 1, -1\}$.

Hypotheses on the generating algebra $\mathbf{L}$ (think e.g. of $\mathbb{R}$):

1. **$\mathbf{L}$ has a majority term**. (E.g.: $\mathbf{L}$ has a lattice reduct.) (It can be generalized to near-unanimity terms.)
2. (**$\mathbf{L}$ is "indecomposable"**:) $\mathbf{L}$ is *hereditarily finitely subdirectly irreducible*, i.e. every subalgebra of $\mathbf{L}$ is finitely subdirectly irreducible. (E.g.: every subalgebra of $\mathbf{L}$ is simple, and $\mathbf{L}$ has two distinct constants.)
3. (**$\mathbf{L}$ is "rigid"**:) For each subalgebra $\mathbf{A}$ of $\mathbf{L}$, the inclusion $\mathbf{A} \hookrightarrow \mathbf{L}$ is the unique homomorphism from $\mathbf{A}$ to $\mathbf{L}$.

Under these hypotheses, we provide a duality for the category of algebras in $\mathbb{ISP}(\mathbf{L})$ that are *finitely* $\mathbf{L}$-*valued*.

We will represent an algebra via a

1. Stone space X, together with
2. a "local constraint" for each subset $I \subseteq X$ of cardinality ≤ 2.

For Priestley duality, $\{\text{constraints}\} = \text{order}$.

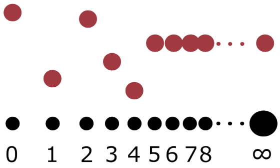
Example

$\mathbf{L} = \langle \mathbb{R}; +, \vee, \wedge, 0, 1, -1 \rangle$. (Commutative lattice-ordered monoid...)

$$\mathbf{A} := \{f \colon \mathbb{N} \to \mathbb{R} \mid f \text{ is eventually constant}\}.$$

$\mathbf{A} \overset{?}{\cong}$ {cont. functions over a Stone space satisfying local constraints}.

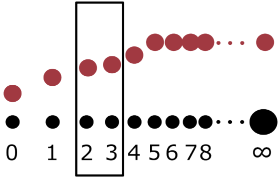
Stone space: $\alpha\mathbb{N} = \mathbb{N} \cup \{\infty\}$, the one-point compactification of $\mathbb{N}$.



A continuous function from $\alpha\mathbb{N}$ to the discrete space $\mathbb{R}$.

$\mathbf{A} \cong$ algebra of continuous functions from $\alpha\mathbb{N}$ to the discrete space $\mathbb{R}$.
(No local constraints.)

$\mathbf{B} := \{f\colon \mathbb{N} \to \mathbb{R} \mid f \text{ is eventually constant and order-preserving}\}$.



A continuous order-preserving function from $\alpha\mathbb{N}$ to the discrete space $\mathbb{R}$.

$\mathbf{B} \cong$ algebra of *order-preserving* continuous functions from $\alpha\mathbb{N}$ to the discrete space $\mathbb{R}$.

Order-preservation is given by a family of binary constraints.

Constraint on $\{2, 3\}$: $\{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$.

We equip $\mathbf{L}$ with the discrete topology. A continuous function from a Stone space X to $\mathbf{L}$ has finite image.

"$I \subseteq_2 X$" stands for "I is a subset of X of cardinality at most 2".

Definition

A *Priestley* $\mathbf{L}$*-space* consists of a Stone space X and, for each $I \subseteq_2 X$, of a subalgebra $\mathbf{A}_I$ of $\mathbf{L}^I$ s.t.

1. (**Local-to-global extension**) For all $I \subseteq_2 X$, every $f \in \mathbf{A}_I$ has a continuous extension $g \colon X \to \mathbf{L}$ that satisfies all constraints, i.e. s.t., for all $J \subseteq_2 X$, $g|_J \in \mathbf{A}_J$.
2. (**Separation**) For all distinct $x, y \in X$, there is $f \in \mathbf{A}_{\{x,y\}}$ s.t. $f(x) \neq f(y)$.

E.g.: $X = \alpha\mathbb{N}$. For $I \subseteq_2 \alpha\mathbb{N}$, $\mathbf{A}_I := \{f \colon I \to \mathbb{R} \mid f \text{ is order-preserving}\}$.

Definition

An algebra $\mathbf{A}$ in $\mathbb{ISP}(\mathbf{L})$ is said to be *finitely* $\mathbf{L}$*-valued* for each $a \in \mathbf{A}$ the set

$$\{h(a) \colon h \colon \mathbf{A} \to \mathbf{L} \text{ homomorphism}\}$$

is finite.

I.e., each element of $\mathbf{A}$, thought of as a function $X \to \mathbf{L}$, has finite image.

Example: For $\mathbf{L} = \langle \mathbb{R}; +, \vee, \wedge, 0, 1, -1 \rangle$, the following are finitely $\mathbf{L}$-valued:

- $\mathbb{R}$,
- any finite power $\mathbb{R}^n$ of $\mathbb{R}$,
- any subalgebra of a finite power $\mathbb{R}^n$ of $\mathbb{R}$,
- $\{f \colon \mathbb{N} \to \mathbb{R} \mid f \text{ is eventually constant}\}$.

Hypotheses on $\mathbf{L}$ (think of 2, or $[0,1]$, or $\mathbb{R}$):

1. **$\mathbf{L}$ has a majority term**. (E.g.: $\mathbf{L}$ has a lattice reduct.)
2. (**$\mathbf{L}$ is "indecomposable"**:) $\mathbf{L}$ is *hereditarily finitely subdirectly irreducible*, i.e. every subalgebra of $\mathbf{L}$ is finitely subdirectly irreducible. (E.g.: every subalgebra of $\mathbf{L}$ is simple, and $\mathbf{L}$ has two distinct constants.)
3. (**$\mathbf{L}$ is "rigid"**:) For each subalgebra $\mathbf{A}$ of $\mathbf{L}$, the inclusion $\mathbf{A} \hookrightarrow \mathbf{L}$ is the unique homomorphism from $\mathbf{A}$ to $\mathbf{L}$.

Theorem (Main result)

Suppose $\mathbf{L}$ satisifes (1–3). The category of finitely $\mathbf{L}$-valued algebras in $\mathbb{ISP}(\mathbf{L})$ (and homomorphisms) is dually equivalent to the category of Priestley $\mathbf{L}$-spaces (and appropriate morphisms).

Thank you!