Syllabus of

Categorical dualities in logic

Université catholique de Louvain, a.y. 2025–2026

Marco Abbadini

February 20, 2026

Contents

Introduction

John Baez and James Dolan [Baez and Dolan, 2001] remarked that

> "an equation is only interesting or useful to the extent that the two sides are different."

For instance, compare

$$2 = 2 \qquad \text{with} \qquad \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

The first equality is correct but uninformative: both sides express the same object in the same language. The second one is interesting precisely because it connects two apparently different descriptions, and allows us to switch freely between them depending on which is more convenient for a given computation.

In this course, we will study an analogous phenomenon, but at the level of *mathematical structures*. On one side, we will have algebraic structures that arise naturally in logic — most notably *Boolean algebras*, which provide an algebraic semantics for classical propositional logic. On the other side, we will encounter structures of a completely different nature — in this case, *Stone spaces*, which are certain topological spaces.

The connection between these two worlds is not a literal equality, but a *categorical duality* (a.k.a. categorical dual equivalence): a two-way translation that preserves all information. This means that we can translate a problem about Boolean algebras into a corresponding problem about Stone spaces, and vice versa. In practice, this is useful because on the side of Stone spaces many constructions are simpler and one can use geometric intuition.

A slogan to keep in mind is:

> *Categorical dualities in logic relate "algebras of formulas" to "spaces of models".*

We start with the simplest case: classical propositional logic, modeled by Boolean algebras and Stone spaces (Stone, 1936).

Later, we will see analogous dualities for other logics, such as intuitionistic propositional logic, and possibly touch the first-order setting (where quantifiers enter the picture).

Chapter 1

Classical propositional logic: Stone duality

Key definitions and theorems

In the first part of the course, we will see *Stone duality*: a connection (in the form of a categorical duality) between *Boolean algebras* and *Stone spaces*. Informally, Boolean algebras encode the *syntax* of classical propositional logic (algebras of formulas), while Stone spaces encode its *semantics* (spaces of models).

1.1 From classical propositional logic to Boolean algebras

1.1.1 Syntax: propositional languages and formulas

Definition 1.1 (Propositional language, propositional symbols)**.** A ***(propositional) language*** $\mathcal{L}$ is a set; its elements are called ***propositional symbols*** (or also *propositional variables*).

We use the fancy name "(propositional) language" for a plain set just to declare the usage we want to make out of it.

Propositional symbols are typically denoted by $p, q, r, \ldots$.

The connectives of classical propositional logic are

$$\vee, \quad \wedge, \quad \neg, \quad 0, \quad 1,$$

where $\vee$ is a binary operation denoting *or* (sometimes called *join*), $\wedge$ is a binary operation denoting *and* (sometimes called *meet*), $\neg$ is a unary operation denoting *not* (i.e. negation, or complement), 0 is a constant symbol denoting *false* (bottom) and 1 is a constant symbol denoting *true* (top).

Definition 1.2 (Formulas)**.** The set of ***formulas*** $\mathrm{Form}(\mathcal{L})$ is defined inductively as follows:

- every propositional symbol $p \in \mathcal{L}$ is a formula;
- if $\varphi, \psi \in \mathrm{Form}(\mathcal{L})$, then $(\varphi \vee \psi) \in \mathrm{Form}(\mathcal{L})$ and $(\varphi \wedge \psi) \in \mathrm{Form}(\mathcal{L})$;
- if $\varphi \in \mathrm{Form}(\mathcal{L})$, then $(\neg\varphi) \in \mathrm{Form}(\mathcal{L})$;
- 0 and 1 are formulas.

In other words, $\mathrm{Form}(\mathcal{L})$ is the smallest set containing $\mathcal{L}$ and closed under $\vee$, $\wedge$, $\neg$, 0 and 1.

We use the standard abbreviations for implications and bi-implication:

$$\varphi \to \psi \;\coloneqq\; \neg\varphi \vee \psi, \qquad \varphi \leftrightarrow \psi \;\coloneqq\; (\varphi \to \psi) \wedge (\psi \to \varphi).$$

While propositional symbols are usually denoted by $p, q, r, \ldots$, formulas are typically denoted by $\varphi, \psi, \sigma, \ldots$.

1.1.2 Semantics: valuations and truth tables

Set $2 \coloneqq \{0, 1\}$ (the "set of truth values").

Definition 1.3 (Valuations, interpretation of formulas)**.** A ***valuation*** on $\mathcal{L}$ is a function $v\colon \mathcal{L} \to 2$. Given a valuation v, its unique extension[a] to all formulas is a map

$$\bar{v}\colon \mathrm{Form}(\mathcal{L}) \longrightarrow 2$$

defined by recursion on the complexity of formulas:

$$\bar{v}(p) = v(p) \quad (p \in \mathcal{L}), \qquad \bar{v}(0) = 0, \qquad \bar{v}(1) = 1,$$

$$\bar{v}(\neg\varphi) = \neg\,\bar{v}(\varphi), \qquad \bar{v}(\varphi \vee \psi) = \bar{v}(\varphi) \vee \bar{v}(\psi), \qquad \bar{v}(\varphi \wedge \psi) = \bar{v}(\varphi) \wedge \bar{v}(\psi),$$

where on the right-hand side we use the usual Boolean operations on $2 = \{0, 1\}$, which you can find in Remark 1.4 below.

[a] This is a special case of a standard universal-algebraic fact: for any algebraic signature Σ, the forgetful functor $\mathsf{Alg}(\Sigma) \to \mathsf{Set}$ has a left adjoint sending a set X to the *free* Σ*-algebra* on X, whose underlying set can be described as the set of Σ-terms over X. For the signature $\{\vee, \wedge, \neg, 0, 1\}$, the left adjoint maps a set $\mathcal{L}$ to $\mathrm{Form}(\mathcal{L})$, the unit at $\mathcal{L}$ is the inclusion $\mathcal{L} \hookrightarrow \mathrm{Form}(\mathcal{L})$, and the extension $v \mapsto \bar{v}$ is given by the universal property of the unit.

Remark 1.4 (Boolean operations on $2 = \{0, 1\}$)**.** By convention,

$$0 \wedge 0 = 0, \;\; 0 \wedge 1 = 0, \;\; 1 \wedge 0 = 0, \;\; 1 \wedge 1 = 1,$$

$$0 \vee 0 = 0, \;\; 0 \vee 1 = 1, \;\; 1 \vee 0 = 1, \;\; 1 \vee 1 = 1,$$

$$\neg 0 = 1, \;\; \neg 1 = 0.$$

1.1.3 Semantic equivalence

Definition 1.5 (Semantic equivalence)**.** Let $\varphi, \psi \in \mathrm{Form}(\mathcal{L})$. We write $\varphi \equiv \psi$ and say that φ and ψ are ***semantically equivalent***[a] if

$$\forall v \colon \mathcal{L} \to 2, \qquad \bar{v}(\varphi) = \bar{v}(\psi).$$

[a] In Definition 1.5, "semantic" is in opposition to "syntactic": two formulas are *syntactically equivalent* if they are interprovable in a certain proof system, which we do not have the time to see here. Let me just mention that Stone's Representation Theorem, which will be seen later and which is the core of this chapter, can be seen as an algebraic way to affirm that the syntactic and semantic notions of equivalence coincide.

In other words, φ and ψ are equivalent if and only if they have the same truth table (a function from $\mathcal{L}$ to 2 corresponds to a row of a truth table).

Example 1.6. Let $\mathcal{L} = \{p, q\}$. Then $p \vee q \equiv q \vee p$, as can be checked via the truth table

p	q	$p \vee q$	$q \vee p$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

Example 1.7. If $\mathcal{L} = \{p\}$ has one variable, then every formula is equivalent to exactly one of

$$0, \quad p, \quad \neg p, \quad 1.$$

(For instance, $\neg\neg p \equiv p$, $p \vee \neg p \equiv 1$, and $p \wedge \neg p \equiv 0$.) Thus $\mathrm{Form}(\mathcal{L})/{\equiv}$ has 4 elements.

One may prove that, in a language $\mathcal{L} = \{p, q\}$ with two propositional symbols, there are 16 equivalence classes of formulas.[1]

1.1.4 Adding assumptions: theories and semantic equivalence modulo a theory

We can incorporate semantic assumptions by restricting the class of admissible valuations.

[1] More generally, if $\mathcal{L}$ is finite of cardinality n, then $|\mathrm{Form}(\mathcal{L})/{\equiv}| = 2^{2^n}$. Indeed, every formula determines a truth function $2^{\mathcal{L}} \to 2$, and vice versa one can prove that every function $2^{\mathcal{L}} \to 2$ is the truth function of a formula (this is called "functional completeness"), so that $|\mathrm{Form}(\mathcal{L})/{\equiv}| = 2^{2^{|\mathcal{L}|}}$. In particular, for $\mathcal{L} = \{p, q\}$ we have $|\mathrm{Form}(\mathcal{L})/{\equiv}| = 2^{2^2} = 16$.

Definition 1.8 (Propositional theory)**.** A ***(propositional) theory*** $\mathcal{T}$ in a propositional language $\mathcal{L}$ is a subset $\mathcal{T} \subseteq \mathrm{Form}(\mathcal{L})$.

Definition 1.9 (Model)**.** Let $\mathcal{T} \subseteq \mathrm{Form}(\mathcal{L})$ be a propositional theory in a propositional language. A ***model*** of $\mathcal{T}$ is a function $v\colon \mathcal{L} \to 2$ such that

$$\forall \sigma \in \mathcal{T}, \qquad \bar{v}(\sigma) = 1.$$

We denote by $\mathrm{Mod}(\mathcal{T})$ the set of models of $\mathcal{T}$.

Example 1.10. Let $\mathcal{L} = \{\, p, q \,\}$ and let $\mathcal{T} = \{\, p \vee q \,\}$. Then $\mathrm{Mod}(\mathcal{T})$ consists of the three valuations

$$(p, q) = (1, 0),\ (1, 1),\ (0, 1).$$

Definition 1.11 (Semantic equivalence modulo a theory)**.** Let $\mathcal{T} \subseteq \mathrm{Form}(\mathcal{L})$ be a propositional theory and let $\varphi, \psi \in \mathrm{Form}(\mathcal{L})$. We say that φ and ψ are ***semantically equivalent modulo the theory*** $\mathcal{T}$ (or ***relative to*** $\mathcal{T}$, or ***semantically*** $\mathcal{T}$***-equivalent***), and write $\varphi \equiv_{\mathcal{T}} \psi$, if for every $v \in \mathrm{Mod}(\mathcal{T})$ we have

$$\bar{v}(\varphi) = \bar{v}(\psi).$$

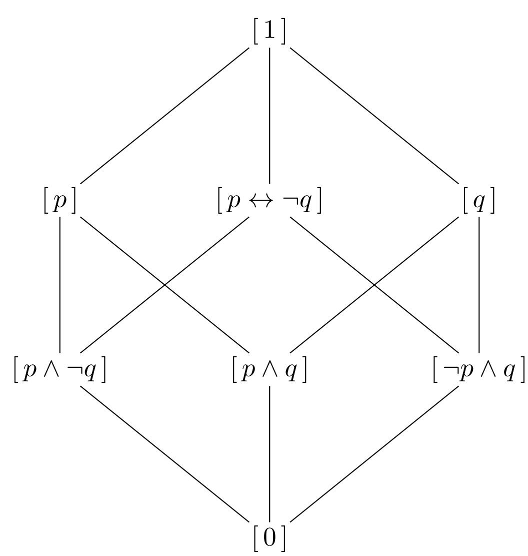
Example 1.12. Let $\mathcal{L} = \{\, p, q \,\}$ and let $\mathcal{T} = \{\, p \vee q \,\}$. One can check that $p \vee q \equiv_{\mathcal{T}} 1$, $\neg p \wedge \neg q \equiv_{\mathcal{T}} 0$ and $p \vee \neg q \equiv_{\mathcal{T}} p$.

The relation $\equiv_{\mathcal{T}}$ is again an equivalence relation, and we write $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ for the corresponding quotient set. It is convenient to partially order equivalence classes by *implication*:

$$\varphi \leq_{\mathcal{T}} \psi \quad :\Longleftrightarrow \quad \forall v \in \mathrm{Mod}(\mathcal{T}),\ \big(\bar{v}(\varphi) = 1 \Rightarrow \bar{v}(\psi) = 1\big).$$

(So "$\varphi \leq_{\mathcal{T}} \psi$" means that φ *implies* ψ on all models of $\mathcal{T}$.)

Example 1.13. Let $\mathcal{L} = \{\, p, q \,\}$ and $\mathcal{T} = \{\, p \vee q \,\}$. Then $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ has 8 elements and its Hasse diagram (with respect to $\leq_{\mathcal{T}}$) can be drawn as follows:



Boolean algebras (whose definition we will see soon) are meant to capture the algebraic structures of the form

$$\langle \mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}; \vee, \wedge, \neg, 0, 1 \rangle,$$

for $\mathcal{L}$ a propositional language and $\mathcal{T}$ a propositional theory in $\mathcal{L}$; here, $\vee, \wedge, \neg, 0, 1$ are defined on $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ by setting

$$[\,\varphi\,] \vee [\,\psi\,] \;\coloneqq\; [\,\varphi \vee \psi\,]\,, \qquad [\,\varphi\,] \wedge [\,\psi\,] \;\coloneqq\; [\,\varphi \wedge \psi\,]\,, \qquad \neg\,[\,\varphi\,] \;\coloneqq\; [\,\neg\varphi\,]\,, \qquad 0 \;\coloneqq\; [\,0\,]\,, \qquad 1 \;\coloneqq\; [\,1\,]\,.$$

(These are well-defined operations.)

In other words, Boolean algebras are meant to capture the algebras of formulas modulo a theory.

1.2 Boolean algebras

The quotient posets in Example 1.13 already display a key phenomenon: the logical connectives $\wedge$ and $\vee$ behave like "infimum" and "supremum" with respect to the implication order.

To recall the definition of infimum and supremum, let P be a poset. For $x, y \in P$, an *infimum* (or *greatest lower bound*) of $\{\,x, y\,\}$ is an element $x \wedge y \in P$ such that

$$x \wedge y \leq x, \qquad x \wedge y \leq y,$$

and, for every $z \in P$, if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$. Similarly, a *supremum* (or *least upper bound*) of $\{\,x, y\,\}$ is an element $x \vee y \in P$ such that

$$x \leq x \vee y, \qquad y \leq x \vee y,$$

and, for every $z \in P$, if $x \leq z$ and $y \leq z$ then $x \vee y \leq z$.

More generally, for any subset S of a poset P, an ***infimum*** of S (relative to P) is a greatest lower bound of S in P and a ***supremum*** of S (relative to P) is a smallest upper bound of S in P. Infima and suprema, if they exist, are unique. Note that being the infimum of $\varnothing$ relative to P means being the maximum of P, and being the supremum of $\varnothing$ relative to P means being the minimum of P.

One can show that, in the poset $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ ordered by implication, $[\,\varphi \wedge \psi\,]$ is the infimum of $[\,\varphi\,]$ and $[\,\psi\,]$, and $[\,\varphi \vee \psi\,]$ is the supremum of $[\,\varphi\,]$ and $[\,\psi\,]$.[2]

1.2.1 Order-theoretic definition of lattices

Definition 1.14 (Lattice: order theoretic presentation)**.** A ***lattice*** is a poset $(L, \leq)$ in which every pair of elements admits an infimum and a supremum.

Example 1.15 (Power-set lattice)**.** Let X be a set and consider the poset $(\mathscr{P}(X), \subseteq)$. Then $(\mathscr{P}(X), \subseteq)$ is a lattice, with

$$A \wedge B \coloneqq A \cap B \qquad \text{and} \qquad A \vee B \coloneqq A \cup B.$$

1.2.2 Equational definition of lattices

Lattices can also be presented *algebraically*, by taking $\wedge$ and $\vee$ as primitive operations and listing a small family of identities. This is useful because identities are stable under the kind of constructions we will use later (products, subalgebras, quotients).

Definition 1.16 (Lattice: equational presentation)**.** A ***lattice*** is a set L equipped with two binary operations

$$\wedge, \vee \colon L \times L \to L$$

[2] Proof: We prove the statement for $\wedge$; the case of $\vee$ is analogous. For every model $v \in \mathrm{Mod}(\mathcal{T})$, if $\bar{v}(\varphi \wedge \psi) = 1$ then $\bar{v}(\varphi) = 1$ and $\bar{v}(\psi) = 1$, hence $[\,\varphi \wedge \psi\,] \leq_{\mathcal{T}} [\,\varphi\,]$ and $[\,\varphi \wedge \psi\,] \leq_{\mathcal{T}} [\,\psi\,]$. Now let $[\,\rho\,]$ be any lower bound of $[\,\varphi\,]$ and $[\,\psi\,]$. This means that, for every $v \in \mathrm{Mod}(\mathcal{T})$, $\bar{v}(\rho) = 1$ implies both $\bar{v}(\varphi) = 1$ and $\bar{v}(\psi) = 1$. Therefore $\bar{v}(\rho) = 1$ implies $\bar{v}(\varphi \wedge \psi) = 1$, i.e. $[\,\rho\,] \leq_{\mathcal{T}} [\,\varphi \wedge \psi\,]$.

such that:

1. (commutativity) for all $a, b \in L$, $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$;
2. (associativity) for all $a, b, c \in L$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$;
3. (absorption) for all $a, b \in L$, $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

Remark 1.17. If L is a lattice in the equational sense, one can recover an order by setting

$$a \leq b \quad :\Longleftrightarrow \quad a \wedge b = a,$$

equivalently $a \vee b = b$. With this order, the operations $\wedge$ and $\vee$ are precisely infimum and supremum, so Definitions 1.14 and 1.16 are equivalent viewpoints.

1.2.3 Bounded lattices

Definition 1.18 (Bounded lattice)**.** A lattice is ***bounded*** if it has a least element 0 and a greatest element 1, i.e. elements such that $0 \leq a \leq 1$ for all a. Equivalently (in the equational presentation), 0 and 1 satisfy

1. For all a, $a \wedge 0 = 0$ (equivalently: $a \vee 0 = a$),
2. For all a, $a \vee 1 = 1$ (equivalently; $a \wedge 1 = a$).

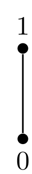
1.2.4 Distributive lattices

Definition 1.19 (Distributive lattice)**.** A lattice L is ***distributive*** if it satisfies any (and hence both) of the following equivalent[a] conditions:

1. for all $a, b, c \in L$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.
2. for all $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

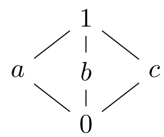
[a]For a proof of the equivalence between the two, see [Birkhoff, 1967, Sec. 6, Thm. 9, p. 11].

Example 1.20. The two-element chain is distributive:

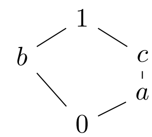


Example 1.21. For every set X, the power-set lattice $\mathscr{P}(X)$ is distributive.

Two small lattices play a special role as "minimal" obstructions to distributivity. They are usually denoted by M_3 (the *diamond*) and N_5 (the *pentagon*).



M_3



N_5

(1.1)

These are not distributive; one can easily verify that $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$.

In fact, one can prove that a lattice L is distributive if and only if it does not contain a sublattice isomorphic to M_3 or N_5, i.e., there is no injective map $M_3 \to L$ or $N_5 \to L$ preserving both $\wedge$ and $\vee$.[3]

1.2.5 Boolean algebras

Definition 1.22 (Boolean algebra)**.** A ***Boolean algebra*** is an algebraic structure

$$\langle B; \vee, \wedge, \neg, 0, 1 \rangle$$

(arities: 2, 2, 1, 0, 0, respectively) such that:

1. $\langle B; \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice;
2. for every $a \in B$,
$$a \wedge \neg a = 0 \qquad \text{and} \qquad a \vee \neg a = 1.$$

The element $\neg a$ is called the *complement* of a.

Remark 1.23. All axioms in the definition of Boolean algebras are *equational*, i.e. they are identities of the form

$$\forall x_1, \ldots, x_n, \quad t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)$$

between terms.

Remark 1.24. A Boolean algebra is completely determined by its underlying partial order. This is because: $\vee, \wedge, 0, 1$ are the binary supremum, binary infimum, smallest element and greatest element, and, in every bounded distributive lattice L, if an element a has a complement (i.e., there is an element b such that $a \wedge b = 0$ and $a \vee b = 1$) it is unique.[4]

Example 1.25. The prototypical example of a Boolean algebra is $2 = \{0, 1\}$ with partial order $0 \leq 1$. The Boolean operations are those described in Remark 1.4.

Proposition 1.26 ($\mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}}$ is a Boolean algebra)**.** *Let $\mathcal{L}$ be a propositional language and $\mathcal{T} \subseteq \mathrm{Form}(\mathcal{L})$ a theory. Then the quotient set $\mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}}$ becomes a Boolean algebra by setting*

$$[\varphi] \vee [\psi] \coloneqq [\varphi \vee \psi], \qquad [\varphi] \wedge [\psi] \coloneqq [\varphi \wedge \psi], \qquad \neg [\varphi] \coloneqq [\neg \varphi], \qquad 0 \coloneqq [0], \qquad 1 \coloneqq [1].$$

Proof sketch. Step 1: the operations are well defined. We show this for $\vee$; the cases of $\wedge$ and $\neg$ are analogous. Suppose $\varphi \equiv_\mathcal{T} \varphi'$ and $\psi \equiv_\mathcal{T} \psi'$. Then for every $v \in \mathrm{Mod}(\mathcal{T})$ we have $\bar{v}(\varphi) = \bar{v}(\varphi')$ and $\bar{v}(\psi) = \bar{v}(\psi')$. Using the truth table for $\vee$ in $2 = \{0, 1\}$ we compute:

$$\bar{v}(\varphi \vee \psi) = \bar{v}(\varphi) \vee \bar{v}(\psi) = \bar{v}(\varphi') \vee \bar{v}(\psi') = \bar{v}(\varphi' \vee \psi').$$

Hence, $\varphi \vee \psi \equiv_\mathcal{T} \varphi' \vee \psi'$, and so $[\varphi] \vee [\psi]$ does not depend on the choice of representatives. The constants $0 = [0]$ and $1 = [1]$ are trivially well defined.

[3] For a textbook reference, see, e.g., [Davey and Priestley, 2002, 4.10].

[4] This is not true for arbitrary bounded lattice: for example, in the bounded lattice M_3 in (1.1), both b and c are complements of a, and in N_5 both a and c are complements of b.

Step 2: the Boolean algebra identities hold. As an example, we show that the axiom $a \wedge \neg a = 0$ holds. Fix $\varphi \in \mathrm{Form}(\mathcal{L})$. For every $v \in \mathrm{Mod}(\mathcal{T})$ we compute

$$\bar{v}(\varphi \wedge \neg\varphi) = \bar{v}(\varphi) \wedge \bar{v}(\neg\varphi) = \bar{v}(\varphi) \wedge \neg\bar{v}(\varphi) = 0 = \bar{v}(0),$$

where we used the recursive definition of $\bar{v}$ and the fact that $b \wedge \neg b = 0$ for all $b \in 2$. Hence, $\varphi \wedge \neg\varphi \equiv_{\mathcal{T}} 0$, and therefore

$$[\,\varphi\,] \wedge \neg\,[\,\varphi\,] = [\,0\,] = 0.$$

This is the general mechanism: any identity between Boolean terms can be checked pointwise in 2 under every valuation. To give more details, let $t_1(x_1, \dots, x_n) = t_2(x_1, \dots, x_n)$ be any Boolean algebra identity (an equation between terms built from $\vee, \wedge, \neg, 0, 1$). To check it in $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$, pick arbitrary formulas $\varphi_1, \dots, \varphi_n \in \mathrm{Form}(\mathcal{L})$ and consider the two formulas $t_1(\varphi_1, \dots, \varphi_n)$ and $t_2(\varphi_1, \dots, \varphi_n)$. For every $v \in \mathrm{Mod}(\mathcal{T})$, the evaluation map $\bar{v} \colon \mathrm{Form}(\mathcal{L}) \to 2$ respects the connectives, hence

$$\bar{v}\big(t_i(\varphi_1, \dots, \varphi_n)\big) = t_i\big(\bar{v}(\varphi_1), \dots, \bar{v}(\varphi_n)\big) \qquad (i = 1, 2).$$

Since the identity $t_1 = t_2$ holds in the two-element Boolean algebra 2, the right-hand sides are equal for all $v \in \mathrm{Mod}(\mathcal{T})$, so $t_1(\varphi_1, \dots, \varphi_n) \equiv_{\mathcal{T}} t_2(\varphi_1, \dots, \varphi_n)$. Therefore, the induced operations on equivalence classes satisfy all Boolean algebra axioms. $\blacksquare$

The Boolean algebra

$$\langle \mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}; \vee, \wedge, \neg, 0, 1 \rangle$$

is called the ***Lindenbaum–Tarski algebra*** of $\mathcal{T}$.

Proposition 1.26 states that every Lindenbaum–Tarski algebra is a Boolean algebra. Later on (Theorem 1.56), we will see that also the converse holds: every Boolean algebra is isomorphic to $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ for some language $\mathcal{L}$ and theory $\mathcal{T}$. This will be a consequence of Stone's Representation Theorem (or of the crucial lemmas used in proving Stone's Representation Theorem). It will provide a guarantee that the definition of Boolean algebras is "the correct one".

Example 1.27 (Power set Boolean algebras)**.** Let X be a set. The power set $\mathscr{P}(X)$ with the inclusion order is a Boolean algebra, with operations

$$A \wedge B \coloneqq A \cap B, \qquad A \vee B \coloneqq A \cup B, \qquad \neg A \coloneqq X \setminus A, \qquad 0 \coloneqq \varnothing, \qquad 1 \coloneqq X.$$

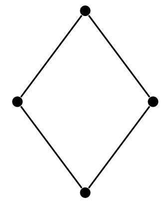
See the footnote for a proof.[5]

Below, we draw the Hasse diagrams of the power sets of sets of cardinality 0, 1, 2, and 3. These are the smallest Boolean algebras, and probably the only ones that one can draw without losing one's sight. (The next one has 16 elements.)

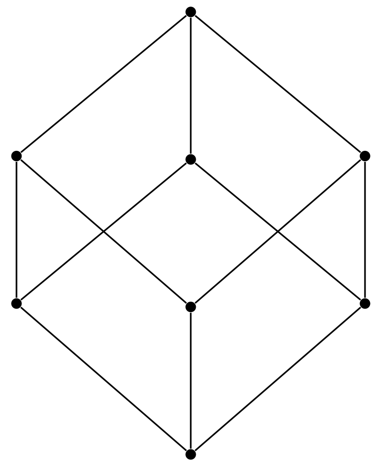
The *trivial Boolean algebra*, of cardinality 1, isomorphic to $\mathscr{P}(\varnothing)$

The Boolean algebra $2 = \{0, 1\}$, of cardinality 2, isomorphic to $\mathscr{P}(\{\,x\,\})$

[5] One way to prove that it is a Boolean algebra is by observing it is isomorphic to the power $\prod_{x \in X} 2$ of 2, that 2 is a Boolean algebra, and that equationally defined classes of algebras are closed under products (by the easy direction in Birkhoff's theorem).



A Boolean algebra of cardinality 4, isomorphic to $\mathscr{P}(\{x, y\})$



A Boolean algebra of cardinality 8, isomorphic to $\mathscr{P}(\{x, y, z\})$

There are examples of Boolean algebras that are not isomorphic to any power set. The following one is probably the simplest example.

Example 1.28 (The algebra of finite and cofinite subsets of $\mathbb{N}$)**.** Consider the collection

$$\mathrm{FC}(\mathbb{N}) \;\coloneqq\; \{\, A \subseteq \mathbb{N} \mid A \text{ is finite, or cofinite} \,\} \;\subseteq\; \mathscr{P}(\mathbb{N})$$

where "*A cofinite*" means that $\mathbb{N} \setminus A$ is finite. To show that this is indeed a Boolean algebra, the following is of help.

Remark 1.29 (Boolean subalgebras of power sets)**.** If $\mathcal{A} \subseteq \mathscr{P}(X)$ contains $\varnothing$ and X and is closed under $\cup$, $\cap$ and complement in X, then $\mathcal{A}$ is a Boolean algebra, as well.

This follows from the fact that Boolean algebras are equationally definable and hence closed under subalgebras.

Now one can prove that $\mathrm{FC}(\mathbb{N})$ is a Boolean algebra:

- $\varnothing$ and $\mathbb{N}$ belong to $\mathrm{FC}(\mathbb{N})$;
- $\mathrm{FC}(\mathbb{N})$ is closed under complement in $\mathbb{N}$: if A is finite then $\mathbb{N} \setminus A$ is cofinite, and if A is cofinite then $\mathbb{N} \setminus A$ is finite;
- $\mathrm{FC}(\mathbb{N})$ is closed under unions: if A and B are finite then $A \cup B$ is finite, while if at least one of A, B is cofinite then $A \cup B$ is cofinite.
- Analogously, $\mathrm{FC}(\mathbb{N})$ is closed under intersections.

The Boolean algebra $\mathrm{FC}(\mathbb{N})$ is infinite but much smaller than $\mathscr{P}(\mathbb{N})$: for instance, the set of even numbers is neither finite nor cofinite, hence it does not belong to $\mathrm{FC}(\mathbb{N})$. Moreover, $\mathrm{FC}(\mathbb{N})$ is *countably infinite* (i.e., in bijection with $\mathbb{N}$). This also shows that it is not isomorphic to any power set, since no power set is countably infinite: if X is finite then $\mathscr{P}(X)$ is finite, and if X is infinite then $\mathscr{P}(X)$ is uncountable.

Remark 1.30 (A different obstruction: completeness)**.** More generally, for any set X, one can consider the Boolean subalgebra

$$\mathrm{FC}(X) \coloneqq \{\, A \subseteq X \mid A \text{ is finite or cofinite} \,\} \subseteq \mathscr{P}(X).$$

If X is infinite, then $\mathrm{FC}(X)$ is *not complete* as a Boolean algebra: there are families that do not admit a supremum. For instance, choose a subset $Y \subseteq X$ such that both Y and $X \setminus Y$ are infinite, and consider the family $\{\, \{\, y \,\} \mid y \in Y \,\} \subseteq \mathrm{FC}(X)$. Its union in $\mathscr{P}(X)$ is Y, which does not belong to $\mathrm{FC}(X)$. Moreover, there is no least element of $\mathrm{FC}(X)$ containing all singletons $\{\, y \,\}$ (one can always remove a point from any upper bound and still obtain a cofinite upper bound), hence the family has no supremum in $\mathrm{FC}(X)$. By contrast, every power set $\mathscr{P}(X)$ is complete (arbitrary unions exist), so $\mathrm{FC}(X)$ for X infinite cannot be isomorphic to any power set.

1.3 Stone's Representation Theorem

Remark 1.29 gives a zoo of examples: every *Boolean subalgebra* of some power set is a Boolean algebra. Stone's Representation Theorem states that all Boolean algebras are of this form! We state it now, and we will prove it later Theorem 1.51.

Theorem (Stone's Representation Theorem for Boolean algebras)**.** *For every Boolean algebra B there is a set X such that B is isomorphic to a Boolean subalgebra of the power set $\mathscr{P}(X)$.*

This means that for every Boolean algebra B there are a set X and an injective map

$$\iota \colon B \hookrightarrow \mathscr{P}(X)$$

such that, under ι, the operations on B correspond to intersection, union, complement, empty set and whole set in X:

$$\iota(a \wedge b) = \iota(a) \cap \iota(b), \qquad \iota(a \vee b) = \iota(a) \cup \iota(b), \qquad \iota(\neg a) = X \setminus \iota(a), \qquad \iota(0) = \varnothing, \qquad \iota(1) = X.$$

The next goal is to prove Stone's Representation Theorem for Boolean algebras. We will first need to present auxiliary notions and lemmas.

Given a Boolean algebra B, how can we find a set X such that B embeds into the power set $\mathscr{P}(X)$ of X, as required by the statement of Stone's Representation Theorem?

Idea, from a logical perspective: In the special Boolean algebra $\mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}}$ (for $\mathcal{L}$ a language and $\mathcal{T}$ a theory), an equivalence class $[\,\varphi\,]$ can be identified with the set of models $v \in \mathrm{Mod}(\mathcal{T})$ such that $\bar{v}(\varphi) = 1$; i.e., a formula can be identified with the models that satisfy it. This gives a very concrete embedding into a power set: the power set of models of $\mathcal{T}$. So, the idea for $B = \mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}}$ is to take

$$X = \mathrm{Mod}(\mathcal{T}).$$

To translate this idea to a general Boolean algebra B, we note that a model

$$v \colon \mathcal{L} \longrightarrow 2$$

of $\mathcal{T}$ induces a function

$$\bar{v} \colon \mathrm{Form}(\mathcal{L}) \longrightarrow 2$$

which passes to the quotient

$$\begin{aligned} \mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}} &\longrightarrow 2 \\ [\varphi] &\longmapsto \bar{v}(\varphi). \end{aligned}$$

In fact, one can prove that the models of $\mathcal{T}$ are in bijection with the Boolean homomorphisms (i.e., functions preserving all Boolean connectives, see Definition 1.31 below) from $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ to 2. This suggests that, for a general Boolean algebra B, we shall take

$$X = \mathrm{hom}(B, 2),$$

the set of homomorphisms from B to 2.

Idea, from another perspective: If $B \subseteq \mathscr{P}(X)$ is a Boolean subalgebra of $\mathscr{P}(X)$, then every element $x \in X$ induces a function

$$\begin{aligned} B &\longrightarrow 2 \\ A &\longmapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

which is a Boolean homomorphism. This suggests a close relationship between X and $\mathrm{hom}(B, 2)$. This suggests, from yet another perspective, that to prove Stone's Representation Theorem we shall take

$$X = \mathrm{hom}(B, 2).$$

Definition 1.31 (Boolean homomorphism)**.** Let A and B be Boolean algebras. A ***Boolean homomorphism*** (or, simply, a *homomorphism*) $f\colon A \to B$ is a function such that, for all $a, b \in A$,

$$f(1) = 1, \qquad f(a \wedge b) = f(a) \wedge f(b), \qquad f(\neg a) = \neg f(a).$$

Remark 1.32. From the defining equations one immediately gets $f(0) = 0$ and $f(a \vee b) = f(a) \vee f(b)$. In other words, a Boolean homomorphism is a function that preserves all the basic logical operations: $\wedge$, $\vee$, $\neg$, 0, 1.

1.3.1 Ultrafilters

A homomorphism $f\colon B \to 2$ can be encoded via $f^{-1}[\{1\}]$. The subsets of B arising as $f^{-1}[\{1\}]$ for some homomorphism $f\colon B \to 2$ can be characterized as the *ultrafilters*. To define this, we first define filters (of which ultrafilters are special instances) and ideals.

Definition 1.33 (Filter)**.** A ***filter*** of a Boolean algebra B is a subset $F \subseteq B$ such that:

1. F is *upward closed*, i.e., if $x \in F$ and $x \leq y$, then $y \in F$;
2. F is *closed under finite meets*, i.e.,
   (a) if $x, y \in F$, then $x \wedge y \in F$;
   (b) $1 \in F$.

A filter F is *proper* if $0 \notin F$.

Turning the definition of a filter upside down, we get the following.

Definition 1.34 (Ideal)**.** An ***ideal*** of a Boolean algebra B is a subset $I \subseteq B$ such that:

1. I is *downward closed*, i.e., if $x \in I$ and $y \leq x$, then $y \in I$;
2. I is *closed under finite joins*, i.e.,
   (a) if $x, y \in I$, then $x \vee y \in I$;

(b) $0 \in I$.

Definition 1.35 (Ultrafilter)**.** An ***ultrafilter*** of a Boolean algebra is a filter whose complement is an ideal.

Spelling out the details, this means that an ultrafilter is a filter F such that

1. if $x \vee y \in F$, then $x \in F$ or $y \in F$;
2. $0 \notin F$.

There is also another useful characterization of ultrafilters.

Lemma 1.36 (Ultrafilters via negation)**.** *The ultrafilters of a Boolean algebra B are precisely the filters F such that, for every $x \in B$, exactly one between x and $\neg x$ belongs to F.*

Proof. ($\Rightarrow$). Suppose that F is an ultrafilter, and let $x \in B$. Since $1 = x \vee \neg x \in F$, we have $x \in F$ or $\neg x \in F$. Moreover, they cannot both lie in F because $x \wedge \neg x = 0 \notin F$.

($\Leftarrow$). Suppose that F is a filter such that, for every $x \in B$, exactly one between x and $\neg x$ belongs to F.

- Suppose $x \vee y \in F$, and let us prove that $x \in F$ or $y \in F$. If $x \in F$, we are done. Otherwise, from $x \notin F$ and from the hypothesis on F, we deduce $\neg x \in F$. Then, using distributivity,

$$F \ni (x \vee y) \wedge \neg x = (x \wedge \neg x) \vee (y \wedge \neg x) = y \wedge \neg x.$$

, By upward closure, this implies $y \in F$ (since $y \wedge \neg x \leq y$). $x \in F$ or $y \in F$.

- Since 1 belongs to F, its negation $\neg 1 = 0$ does not. ■

Proposition 1.37 (Ultrafilters encode homomorphisms to 2)**.** *For every Boolean algebra B, there is a bijection*

$$\begin{aligned} \hom(B, 2) &\longleftrightarrow \{\, \textit{ultrafilters of } B \,\}, \\ h &\longmapsto h^{-1}[\{\,1\,\}], \\ \left(\chi_U : x \mapsto \begin{cases} 1, & x \in U, \\ 0, & x \notin U, \end{cases}\right) &\longleftarrow U. \end{aligned}$$

Proof sketch. If $h\colon B \to 2$ is a homomorphism, set $U := h^{-1}[\{\,1\,\}]$. Then $1 \in U$, U is upward closed and closed under meets, so it is a filter; it is proper since $h(0) = 0$. Moreover, for each $x \in B$, $h(\neg x) = \neg h(x)$, and hence exactly one of x and $\neg x$ lands in 1. Thus, U satisfies the conditions in Lemma 1.36 and so is an ultrafilter.

Conversely, if U is an ultrafilter, define $h_U : B \to 2$ by $h_U(x) = 1$ if and only if $x \in U$. The ultrafilter axioms ensure that h_U preserves $\wedge$, $\neg$, and 1, hence h_U is a homomorphism. ■

Therefore, ultrafilters of B are encodings of homomorphisms from B to 2. Thus, we will use the set of ultrafilters of B as the set X such that B embeds into $\mathscr{P}(X)$.

Definition 1.38 (Principal filter)**.** The ***principal filter generated by an element*** a of a Boolean algebra B is

$$\uparrow a \;:=\; \{\, x \in B \mid a \leq x \,\}.$$

Therefore, a filter (and, in particular, an ultrafilter) is called ***principal*** if it has a minimum.

Remark 1.39. If B is finite, then every filter is principal.

We will soon see that $\uparrow a$ is an ultrafilter if and only if a satisfies the following simple condition.

Definition 1.40 (Atom)**.** An ***atom*** of a Boolean algebra B is a minimal element of $B \setminus \{ 0 \}$.

Proposition 1.41 (Atoms and ultrafilters)**.**

1. *For every Boolean algebra B we have a bijection*

$$\begin{aligned} \{ \text{atoms of } B \} &\longleftrightarrow \{ \text{principal ultrafilters of } B \} \\ a &\longmapsto \uparrow a \\ \min U &\longleftarrow U. \end{aligned} \tag{*}$$

2. *If B is finite, every ultrafilter is principal and so $*$ gives a bijection between atoms and ultrafilters.*

Proof. We only need to prove that the two functions are well defined; that they are mutually inverse will then be immediate.

To prove that the left-to-right function is well defined, let us assume that a is an atom, and let us prove that $U := \uparrow a$ is an ultrafilter. It is a filter (easy). $0 \notin U$ because $a > 0$. Suppose $x \vee y \in \uparrow a$, i.e., $x \vee y \geq a$. Then, by distributivity,

$$a = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y).$$

Therefore, since the only elements below a are 0 and a, and the join of $(a \wedge x)$ and $(a \wedge y)$ is a, at least one of the two is a (otherwise they would both be 0 and so their join would be 0). If $a \wedge x = a$ then $x \in \uparrow a$, and if $a \wedge y = a$ then $y \in \uparrow a$. Thus, U is an ultrafilter.

To prove that the right-to-left function is well defined, let us assume that $\uparrow a$ is a principal ultrafilter, and let us prove that $\min U$ is an atom. We have $a \neq 0$ (otherwise 0 would belong to U). If $0 < x < a$, then $x \notin \uparrow a$, and hence $\neg x \in \uparrow a$ by Lemma 1.36. Thus $a \leq \neg x$, and since $x \leq a$ we get $x \leq \neg x$, hence $x = x \wedge \neg x = 0$, a contradiction. Therefore, there is no x with $0 < x < a$, so a is an atom. ■

Example 1.42. The trivial Boolean algebra of cardinality 1 has no ultrafilters (and no homomorphisms to 2). The Boolean algebra $2 = \{ 0, 1 \}$ has exactly one ultrafilter, namely $\{ 1 \}$. The Boolean algebra of cardinality 4 has exactly two ultrafilters, corresponding to its two atoms. The Boolean algebra of cardinality 8 has exactly three ultrafilters, corresponding to its three atoms.

In an infinite Boolean algebra, an ultrafilter may fail to be principal, i.e., to have a minimum. The following gives an example.

Example 1.43 (Ultrafilters of $\mathrm{FC}(\mathbb{N})$)**.** For each $n \in \mathbb{N}$, the set

$$U_n \;:=\; \{ A \in \mathrm{FC}(\mathbb{N}) \mid n \in A \}$$

is a (principal) ultrafilter on the finite-cofinite algebra $\mathrm{FC}(\mathbb{N})$ on $\mathbb{N}$.

In addition, the family of cofinite sets

$$U_\infty \;:=\; \{ A \in \mathrm{FC}(\mathbb{N}) \mid A \text{ is cofinite} \}$$

is an ultrafilter on $\mathrm{FC}(\mathbb{N})$; it is not principal, since it has no minimum. As a simple exercise, you can prove that it has no other ultrafilters, i.e., that

$$\mathrm{Ult}(\mathrm{FC}(\mathbb{N})) = \{ U_n \mid n \in \mathbb{N} \} \cup \{ U_\infty \}.$$

A proof is in the footnote.[6]

Remark 1.44 (Ultrafilters on a power set)**.** What are the ultrafilters of $\mathscr{P}(X)$, for X a set? First of all, for every $x \in X$, we have an ultrafilter

$$U_x := \{ A \in \mathscr{P}(X) \mid x \in A \}.$$

These are precisely the *principal* ultrafilters.

- If X is finite, these are the only ultrafilters (see Proposition 1.41).
- For X infinite, does $\mathscr{P}(X)$ have nonprincipal ultrafilters? This is a nontrivial question; we will see that, if we assume the axiom of choice, we can prove the existence of nonprincipal ultrafilters; this will follow from the Boolean Prime Ideal Theorem (Theorem 1.48) proved below. Do not hope to find an explicit description of any of them, though.

1.3.2 Stone's Representation Theorem

Let us now take $X = \mathrm{Ult}(B)$ and define the embedding $B \hookrightarrow \mathscr{P}(X)$ required by Stone's Representation Theorem.

Definition 1.45 (Stone map)**.** For a Boolean algebra B, we define the ***Stone map***

$$\begin{aligned} \eta_B \colon B &\longrightarrow \mathscr{P}(\mathrm{Ult}(B)) \\ b &\longmapsto \{ U \in \mathrm{Ult}(B) \mid b \in U \}. \end{aligned}$$

Logical intuition: the set $\eta_B(b)$ can be thought of as the set of "models" in which the "formula" b holds. To prove Stone's Representation Theorem it suffices to show that:

1. η_B is a Boolean homomorphism (easy),
2. η_B is injective (hard).

Lemma 1.46 (The Stone map is a homomorphism)**.** *The Stone map η_B is a Boolean homomorphism, i.e. for all $a, b \in B$,*

$$\eta_B(1) = \mathrm{Ult}(B), \qquad \eta_B(a \wedge b) = \eta_B(a) \cap \eta_B(b), \qquad \eta_B(\neg a) = \mathrm{Ult}(B) \setminus \eta_B(a),$$

(and hence also $\eta_B(a \vee b) = \eta_B(a) \cup \eta_B(b)$ and $\eta_B(0) = \varnothing$).

Proof. This are all straightforward computations. Details are in the footnote.[a] ■

[a] The equality $\eta_B(1) = \mathrm{Ult}(B)$ is immediate: every ultrafilter contains 1.
For the binary meet, let $U \in \mathrm{Ult}(B)$. Then

$$U \in \eta_B(a \wedge b) \iff a \wedge b \in U \iff (a \in U \text{ and } b \in U) \iff U \in \eta_B(a) \cap \eta_B(b),$$

where the middle equivalence uses upward closure and closure under binary meets of U.
For complements, using Lemma 1.36 we have

$$U \in \eta_B(\neg a) \iff \neg a \in U \iff a \notin U \iff U \notin \eta_B(a).$$

[6] Let U be an ultrafilter on $\mathrm{FC}(\mathbb{N})$. If U contains a finite set A, write A as a finite union of singletons: $A = \{ n_1 \} \cup \cdots \cup \{ n_k \}$. Since U is an ultrafilter, the primeness condition in Definition 1.35 implies that some singleton $\{ n_i \}$ belongs to U. Then U must coincide with U_{n_i}: if $B \in \mathrm{FC}(\mathbb{N})$ contains n_i, then $\{ n_i \} \subseteq B$, hence $B \in U$ by upward closure; if $n_i \notin B$, then $n_i \in \mathbb{N} \setminus B$ and $\mathbb{N} \setminus B \in U$, hence $B \notin U$. If instead U contains no finite set, then for every finite $F \subseteq \mathbb{N}$ we have $F \notin U$, hence $\mathbb{N} \setminus F \in U$ by Lemma 1.36. Thus U contains every cofinite set, i.e. $U_\infty \subseteq U$. Since U_∞ is itself an ultrafilter, maximality forces $U = U_\infty$.

Remark 1.47 ("Hard part")**.** To conclude the proof of Theorem 1.51, it remains to show that η_B is injective. This is a separation statement: given two distinct elements of B, we must find an ultrafilter containing one but not the other. The required separation will follow from the Boolean Prime Ideal Theorem, proved next.

Let us recall Zorn's Lemma:

Let $(P, \leq)$ be a partially ordered set. If

1. $P \neq \varnothing$,
2. and every nonempty chain[7] in P has an upper bound in P,

then P has a maximal element.

Recall that Zorn's lemma is not provable in ZF, and is equivalent to the Axiom of Choice over ZF. We will use it in the following, which in turn we will use to conclude the proof of Stone's Representation Theorem.

Theorem 1.48 (Boolean Prime Ideal Theorem)**.** *Let B be a Boolean algebra, let F be a filter, and let I be an ideal such that $F \cap I = \varnothing$. Then there is an ultrafilter U on B such that $F \subseteq U$ and $U \cap I = \varnothing$.*

Proof. Consider the poset

$$\mathcal{P} \;:=\; \{\, (G, J) \mid G \text{ is a filter},\ J \text{ is an ideal},\ F \subseteq G,\ I \subseteq J,\ G \cap J = \varnothing \,\},$$

ordered by componentwise inclusion: $(G, J) \leq (G', J')$ iff $G \subseteq G'$ and $J \subseteq J'$.

Nonemptiness. This poset is nonempty since $(F, I) \in \mathcal{P}$.

Nonempty chains have upper bounds. Let $\mathcal{C} \subseteq \mathcal{P}$ be a nonempty chain and set

$$G^* := \bigcup_{(G,J)\in\mathcal{C}} G, \qquad J^* := \bigcup_{(G,J)\in\mathcal{C}} J.$$

It is straightforward to prove that $(G^*, J*)$ belongs to $\mathcal{P}$: one checks that G^* is a filter, J^* is an ideal, $F \subseteq G^*$, $I \subseteq J^*$ and $G^* \cap J^* = \varnothing$. See the footnote for details. Thus $(G^*, J^*) \in \mathcal{P}$ is an upper bound of $\mathcal{C}$.

By Zorn's lemma, $\mathcal{P}$ has a maximal element (U, K). At this point, the key step is to show that the maximal pair (U, K) "splits" B: once we know that $U \cup K = B$, the disjointness $U \cap K = \varnothing$ forces $K = B \setminus U$. Since K is an ideal, this will imply that $B \setminus U$ is an ideal, and hence U is an ultrafilter.

We claim that $U \cup K = B$. Suppose not, and pick $x \in B \setminus (U \cup K)$.

Let U' be the filter generated by $U \cup \{\, x \,\}$. Concretely,

$$U' = \{\, y \in B \mid \exists u \in U,\ (u \wedge x) \leq y \,\}.$$

If $U' \cap K = \varnothing$, then $(U', K) \in \mathcal{P}$ strictly extends (U, K), contradicting maximality. Hence, there are $u \in U$ and $k \in K$ such that $u \wedge x \leq k$, and thus $u \wedge x \in K$.

Similarly, let K' be the ideal generated by $K \cup \{\, x \,\}$. Concretely,

$$K' = \{\, y \in B \mid \exists k \in K,\ y \leq (k \vee x) \,\}.$$

If $U \cap K' = \varnothing$, then $(U, K') \in \mathcal{P}$ strictly extends (U, K), again contradicting maximality. Hence, there are $u' \in U$ and $k' \in K$ such that $u' \leq k' \vee x$.

Set $w := u \wedge u' \in U$. Then $w \wedge x \leq u \wedge x \in K$, hence $w \wedge x \in K$ (since K is downward closed), and also $w \leq u' \leq k' \vee x$. Using distributivity,

$$w = w \wedge (k' \vee x) = (w \wedge k') \vee (w \wedge x).$$

Now $w \wedge k' \leq k'$, hence $w \wedge k' \in K$, and we already know $w \wedge x \in K$. Since K is closed under finite joins, we conclude $w \in K$. This contradicts $w \in U$ and $U \cap K = \varnothing$. Therefore $U \cup K = B$. This proves that U is an ultrafilter. It is clear that $F \subseteq U$ and $U \cap I = \varnothing$. ■

[7] A chain is a totally ordered poset. Thus, a *chain in P* is a subset S of P such that, for all $x, y \in S$, either $x \leq y$ or $y \leq x$.

Corollary 1.49 (Ultrafilters separate elements)**.** *If $a \not\leq b$ in B, then there is an ultrafilter $U \in \mathrm{Ult}(B)$ such that $a \in U$ and $b \notin U$.*

Proof. Let $F := \uparrow a$ be the principal filter generated by a, and let

$$I := \downarrow b := \{ x \in B \mid x \leq b \}$$

be the principal ideal generated by b. We notice that $F \cap I = \varnothing$, since otherwise there would be $x \in \uparrow a \cap \downarrow b$, which would imply $a \leq x \leq b$, which would contradict $a \not\leq b$. By Theorem 1.48 there is an ultrafilter U such that $F \subseteq U$ and $U \cap I = \varnothing$. In particular, $a \in U$ and $b \notin U$. ■

As a consequence:

Lemma 1.50 (The Stone map is injective)**.** *For every Boolean algebra B, the Stone map $\eta_B : B \to \mathscr{P}(\mathrm{Ult}(B))$ defined in Definition 1.45 is injective.*

Proof. Let $a \neq b$ in B. Then either $a \not\leq b$ or $b \not\leq a$. Assume $a \not\leq b$. (The proof in the other case is perfectly symmetrical.) By Corollary 1.49, there is $U \in \mathrm{Ult}(B)$ with $a \in U$ and $b \notin U$. Thus $U \in \eta_B(a)$ but $U \notin \eta_B(b)$, so $\eta_B(a) \neq \eta_B(b)$. ■

We are now ready to prove Stone's Representation Theorem for Boolean algebras [Stone, 1936].

Theorem 1.51 (Stone's Representation Theorem for Boolean algebras)**.** *For every Boolean algebra B there is a set X such that B is isomorphic to a Boolean subalgebra of the power set $\mathscr{P}(X)$.*

Proof. This follows from the fact that the Stone map $\eta_B : B \to \mathscr{P}(\mathrm{Ult}(B))$ is a Boolean homomorphism (Lemma 1.46) and is injective (Lemma 1.50). ■

Remark 1.52 (Choice)**.** To prove Stone's Representation Theorem we used the axiom of choice (since we used Zorn's lemma to prove the Boolean Prime Ideal Theorem). In fact, Stone's Representation Theorem is not provable in ZF alone. Over ZF, Stone's Representation Theorem for Boolean algebras is equivalent to the Boolean Prime Ideal Theorem; see, for instance, [Jech, 1973, Sec. 2.6]. The Boolean Prime Ideal Theorem is weaker than the Axiom of Choice, in the sense that it follows from Choice (via Zorn's lemma), but it does not imply Choice [Halpern and Lévy, 1971].

1.4 Boolean algebras are precisely the Lindenbaum–Tarski algebras

In Section 1.2.5 we saw that every propositional theory $\mathcal{T}$ gives rise to a Boolean algebra $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$, where $\equiv_{\mathcal{T}}$ denotes the semantic equivalence between formulas modulo $\mathcal{T}$: this Boolean algebra is called the Lindenbaum–Tarski algebra of $\mathcal{T}$.

Now we prove the converse statement announced earlier: *every Boolean algebra arises in this way.*

1.4.1 Warm-up examples

Example 1.53 (2 as a Lindenbaum–Tarski algebra)**.** Let $\mathcal{L} = \varnothing$ and $\mathcal{T} = \varnothing$. Then $\mathrm{Form}(\mathcal{L})$ only contains the constants $0, 1$ and the formulas built from them. Up to $\equiv_{\mathcal{T}}$ there are exactly two equivalence classes, so $\mathrm{Form}(\varnothing)/{\equiv_{\varnothing}} \cong 2$.

Example 1.54 (The 4-elements Boolean algebra as a Lindenbaum–Tarski algebra)**.** Let $\mathcal{L} = \{ p \}$ and $\mathcal{T} = \varnothing$. Then $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ is a Boolean algebra with 4 elements (it is isomorphic to $\mathscr{P}(\{x, y\})$).

Example 1.55 (The trivial Boolean algebra as a Lindenbaum–Tarski algebra)**.** If a theory $\mathcal{T}$ has no models (i.e. $\mathrm{Mod}(\mathcal{T}) = \varnothing$), then by definition every two formulas are $\equiv_{\mathcal{T}}$-equivalent. Hence,Hence, $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ is the trivial Boolean algebra. For instance, one can take any $\mathcal{L}$ and $\mathcal{T} = \{\, 0 \,\}$.

1.4.2 Every Boolean algebra is a Lindenbaum–Tarski algebra

We now show that *every* Boolean algebra can be presented as $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ for a suitable language $\mathcal{L}$ and theory $\mathcal{T}$.

Theorem 1.56 (Every Boolean algebra is a Lindenbaum–Tarski algebra)**.** *For every Boolean algebra B there is a propositional language $\mathcal{L}$ and a propositional theory $\mathcal{T}$ such that*

$$B \;\cong\; \mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}.$$

Proof. Let B be a Boolean algebra. We build a language whose propositional variables are *the elements of B themselves*:

$$\mathcal{L}_B \;\coloneqq\; B.$$

To avoid confusion, we will use $\wedge_B, \vee_B, \neg_B, 0_B, 1_B$ for the operations of the Boolean algebra B, while $\wedge, \vee, \neg, 0, 1$ are the connectives/constants in $\mathrm{Form}(\mathcal{L}_B)$.

Define a theory $\mathcal{T}_B$ that forces the propositional variables to behave like B:

$$\mathcal{T}_B \;\coloneqq\; \Big\{\; (\neg b) \leftrightarrow (\neg_B b) \;\Big|\; b \in B \;\Big\} \;\cup\; \Big\{\; (b \wedge c) \leftrightarrow (b \wedge_B c) \;\Big|\; b, c \in B \;\Big\} \;\cup\; \{\, 1 \leftrightarrow 1_B \,\}.$$

(Here $(\neg_B b)$, $(b \wedge_B c)$ and 1_B are elements of B, and hence propositional variables in the language $\mathcal{L}_B$. Moreover, the axioms for $\vee$ and 0 are not necessary since $\vee$ and 0 are definable from $\wedge, \neg, 1$.)

Now consider the assignment $e\colon \mathcal{L}_B \to B$ given by the identity map $e(b) = b$. As for truth assignments into 2, the map e extends uniquely (by recursion on formulas)[a] to an evaluation map

$$\bar{e}\colon \mathrm{Form}(\mathcal{L}_B) \to B,$$

defined by interpreting $\vee, \wedge, \neg, 0, 1$ as $\vee_B, \wedge_B, \neg_B, 0_B, 1_B$. In particular, $\bar{e}$ is a $\{\vee, \wedge, \neg, 0, 1\}$-homomorphism, and it is surjective because every $b \in B$ is the value of the variable b.

We claim that for all formulas $\varphi, \psi \in \mathrm{Form}(\mathcal{L}_B)$ we have

$$\varphi \equiv_{\mathcal{T}_B} \psi \qquad \Longleftrightarrow \qquad \bar{e}(\varphi) = \bar{e}(\psi) \;\text{ in } B.$$

Granting this for a moment, the isomorphism $B \cong \mathrm{Form}(\mathcal{L}_B)/{\equiv_{\mathcal{T}_B}}$ follows immediately: define $\pi\colon \mathrm{Form}(\mathcal{L}_B) \twoheadrightarrow \mathrm{Form}(\mathcal{L}_B)/{\equiv_{\mathcal{T}_B}}$ and set

$$\Phi\colon B \to \mathrm{Form}(\mathcal{L}_B)/{\equiv_{\mathcal{T}_B}}, \qquad \Phi(b) \coloneqq [\, b \,].$$

The claim says precisely that Φ is injective (two generators $[\, b\,], [\, c \,]$ coincide iff $b = c$ in B), and surjectivity is clear because every class is represented by some formula and $\bar{e}$ hits all elements of B. Moreover, Φ is a Boolean homomorphism by construction. For instance, $\Phi(b \wedge_B c) = [\, b \wedge_B c \,] = [\, b \wedge c \,] = [\, b \,] \wedge [\, c \,] = \Phi(b) \wedge \Phi(c)$ by definition of $\mathcal{T}_B$, and similarly for $\neg$ and 1. Hence, Φ is an isomorphism.

It remains to prove the claim. First, observe the key fact (which is a direct unpacking of the axioms of $\mathcal{T}_B$):

Models of $\mathcal{T}_B$ are the same thing as Boolean homomorphisms $h\colon B \to 2$.

Indeed, if $h\colon B \to 2$ is a Boolean homomorphism, then the valuation $v_h\colon \mathcal{L}_B \to 2$ given by $v_h(b) = h(b)$ satisfies the axioms of $\mathcal{T}_B$, so $v_h \in \mathrm{Mod}(\mathcal{T}_B)$. Conversely, if $v \in \mathrm{Mod}(\mathcal{T}_B)$, then the map $h_v\colon B \to 2$ defined by $h_v(b) \coloneqq v(b)$ is a Boolean homomorphism, because the axioms in $\mathcal{T}_B$ exactly say that v respects $\neg_B$, $\wedge_B$, and 1_B.

Now suppose $\bar{e}(\varphi) = \bar{e}(\psi)$. Let $v \in \mathrm{Mod}(\mathcal{T}_B)$ and let $h_v : B \to 2$ be the corresponding homomorphism. By construction of evaluation-by-recursion, the interpretation of formulas agrees:

$$\bar{v} \;=\; h_v \circ \bar{e}.$$

Therefore $\bar{v}(\varphi) = h_v(\bar{e}(\varphi)) = h_v(\bar{e}(\psi)) = \bar{v}(\psi)$. Since this holds for all $v \in \mathrm{Mod}(\mathcal{T}_B)$, we get $\varphi \equiv_{\mathcal{T}_B} \psi$.

Conversely, suppose $\bar{e}(\varphi) \neq \bar{e}(\psi)$ in B. By the fact that ultrafilters separate distinct elements of B (i.e., that $\eta_B : B \to \mathscr{P}(\mathrm{Ult}(B))$ is injective, Lemma 1.50), there is a Boolean homomorphism $h : B \to 2$ such that

$$h(\bar{e}(\varphi)) \;\neq\; h(\bar{e}(\psi)).$$

Let $v_h \in \mathrm{Mod}(\mathcal{T}_B)$ be the corresponding model. Then again $\bar{v}_h = h \circ \bar{e}$, hence $\bar{v}_h(\varphi) \neq \bar{v}_h(\psi)$, so $\varphi \not\equiv_{\mathcal{T}_B} \psi$. This proves the claim and completes the proof. ∎

[a] Universal-algebra viewpoint: for the algebraic signature $\Sigma = \{0, 1, \neg, \wedge, \vee\}$, the forgetful functor $\mathsf{Alg}(\Sigma) \to \mathsf{Set}$ has a left adjoint sending a set $\mathcal{L}$ to the Σ-algebra of terms on $\mathcal{L}$. Here, $\bar{e}$ is exactly the unique Σ-homomorphism extending e, whose existence is guaranteed by the universal property of the unit of the adjunction.

1.5 Bijection between Boolean algebras and Stone spaces

Stone's Representation Theorem says that every Boolean algebra embeds into a power set. However, such a representation is typically *not unique*. There are two properties—*separation* and *compactness*—that single out the canonical representation

$$\eta_B : B \hookrightarrow \mathscr{P}(\mathrm{Ult}(B)).$$

Example 1.57 (Separated vs non-separated)**.** For instance, the two-element Boolean algebra 2 has the following two representations:

$$\begin{aligned} \iota_1 : 2 &\hookrightarrow \mathscr{P}(\{*\}) \\ 0 &\longmapsto \varnothing \\ 1 &\longmapsto \{*\} \end{aligned} \qquad\qquad \begin{aligned} \iota_2 : 2 &\hookrightarrow \mathscr{P}(\{x, y\}) \\ 0 &\longmapsto \varnothing \\ 1 &\longmapsto \{x, y\}. \end{aligned}$$

The representation ι_2 is not "*separated*": the only sets in its image are $\varnothing$ and $\{x, y\}$, so the image cannot distinguish x from y.

Example 1.58 (Non-compact vs compact)**.** Let $\mathrm{FC}(\mathbb{N})$ be the Boolean algebra of finite and cofinite subsets of $\mathbb{N}$. It has the following two representations:

$$\begin{aligned} \jmath_1 : \mathrm{FC}(\mathbb{N}) &\hookrightarrow \mathscr{P}(\mathbb{N}) \\ A &\longmapsto A \end{aligned} \qquad\qquad \begin{aligned} \jmath_2 : \mathrm{FC}(\mathbb{N}) &\hookrightarrow \mathscr{P}(\mathbb{N} \cup \{\infty\}) \\ A &\longmapsto \begin{cases} A, & \text{if } A \text{ is finite,} \\ A \cup \{\infty\}, & \text{if } A \text{ is cofinite.} \end{cases} \end{aligned}$$

The representation $\jmath_1$ is not "*compact*": the family of singletons $\{\{n\} \mid n \in \mathbb{N}\}$ consists of elements of $\mathrm{FC}(\mathbb{N})$ and covers $\mathbb{N}$, but it has no finite subcover.

On the other hand, $\jmath_2$ is *compact*. Indeed, let $\{\jmath_2(A_i)\}_{i \in I}$ be a cover of $\mathbb{N} \cup \{\infty\}$. Since ∞ is covered, there is $i_0 \in I$ such that $\infty \in \jmath_2(A_{i_0})$. By definition of $\jmath_2$, this means that A_{i_0} is cofinite in $\mathbb{N}$. Then $\jmath_2(A_{i_0})$ already covers all but finitely many natural numbers; the remaining finitely many points of $\mathbb{N}$ can be covered by finitely many singleton sets $\jmath_2(\{n\}) = \{n\}$. Hence, the original cover admits a finite subcover.

The previous examples show that representations into power sets can fail to be *separated* or *compact*. Spoiler: for every Boolean algebra B, the canonical representation

$$\eta_B \colon B \hookrightarrow \mathscr{P}(\mathrm{Ult}(B))$$

is (up to renaming the points of the underlying set) the unique one that is *both* separated and compact.

Now, both separation and compactness are strongly "topology-like" properties. So rather than treating them as ad hoc conditions on set-theoretic representations, we move to the topological setting: we equip $\mathrm{Ult}(B)$ with a natural topology generated by the basic subsets $\eta_B(b)$. With this topology, $\mathrm{Ult}(B)$ becomes a *Stone space*, i.e. a space that is totally separated and compact (definitions to follow).

Then, we will get a bijection between isomorphism classes of Boolean algebras and isomorphism classes of Stone spaces.

1.5.1 Stone spaces and the Boolean algebra of clopen sets

Let X be a topological space. A subset $C \subseteq X$ is called ***clopen*** if it is both closed and open in X. We write

$$\mathrm{Clop}(X)$$

for the set of clopen subsets of X. Notice that $\mathrm{Clop}(X)$ is a Boolean subalgebra of $\mathscr{P}(X)$.

Definition 1.59 (Stone space)**.** A ***Stone space***[a] is a topological space that is

1. ***totally separated***: for all $x, y \in X$ with $x \neq y$ there is a clopen subset C of X such that $x \in C$ and $y \notin C$;
2. ***compact***: every open cover of X has a finite subcover.

[a]Two further equivalent characterizations of Stone spaces are: (i) compact Hausdorff zero-dimensional spaces, and (ii) compact Hausdorff totally disconnected spaces.

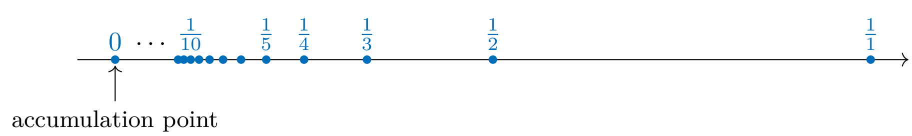
Notice that totally separated implies Hausdorff.

Example 1.60 (Examples of Stone spaces)**.**

1. (Finite discrete spaces) Every *finite discrete space* is a Stone space. In fact, these are the only finite Stone spaces.
2. (One-point compactification of $\mathbb{N}$) The simplest example of an infinite Stone space is called the *one-point* (or *Alexandroff*) *compactification* of $\mathbb{N}$. It is denoted $\alpha\mathbb{N}$, and is obtained by adding to the discrete space $\mathbb{N}$ an accumulation point ∞. More precisely, equip $\mathbb{N} \cup \{\infty\}$ with the topology whose open sets are:
   - every subset of $\mathbb{N}$, and
   - every cofinite subset U such that $\infty \in U$.

   For a visual rendering, this is homeomorphic to the following subspace of $\mathbb{R}$ via the map $n \mapsto \frac{1}{n}$ and $\infty \mapsto 0$:

   $$\{\,0\,\} \cup \left\{\, \frac{1}{n} \mid n \in \mathbb{N} \setminus \{\,0\,\} \right\}.$$



Example 1.61 (*Non*-examples of Stone spaces)**.**

1. Any infinite discrete space is *not* a Stone space, because it is not compact.
2. $[0, 1]$ is *not* a Stone space, because it is not totally separated.

1.5.2 The Stone topology on the set of ultrafilters

Definition 1.62 (Stone topology on the set of ultrafilters)**.** Let B be a Boolean algebra. The ***Stone topology*** on $\mathrm{Ult}(B)$ is the topology *generated* by the subsets

$$\eta_B(b) \qquad (b \in B).$$

That is, the open sets are the arbitrary unions of elements of the form[a]

$$\eta_B(b) \qquad (b \in B).$$

[a] In general, the topology generated by a set $\mathcal{A}$ of subsets of a set X is the set of arbitrary unions of finite intersections of elements in $\mathcal{A}$. Here, there is no need to take finite intersections because the family $\{ \eta_B(b) \mid b \in B \}$ is closed under finite intersections.

Thus, the family $\{ \eta_B(b) \mid b \in B \}$ forms a basis of the Stone topology on $\mathrm{Ult}(B)$, and hence the elements of the form $\eta_B(b)$ are called *basic open* sets of $\mathrm{Ult}(B)$.

Theorem 1.63 (The space of ultrafilters is a Stone space)**.** *For every Boolean algebra B, the space $\mathrm{Ult}(B)$ of ultrafilters equipped with the Stone topology is a Stone space.*

Proof. Totally separated. Let $U, V \in \mathrm{Ult}(B)$ with $U \neq V$. Then, there is b which belongs to exactly one of the two, say, for example, U (the other case is perfectly symmetrical). Then $U \in \eta_B(b)$ and $V \notin \eta_B(b)$. Moreover, $\eta_B(b)$ is clopen, since its complement is $\eta_B(\neg b)$, which is open. Hence, $\eta_B(b)$ is a clopen set separating U from V. Thus, $\mathrm{Ult}(B)$ is totally separated.

Compact. To check the compactness of a space, it is enough to check that any cover by opens in a given basis has a finite subcover. Suppose that $\mathrm{Ult}(B)$ is covered by basic opens:

$$\mathrm{Ult}(B) \ = \ \bigcup_{j \in J} \eta_B(b_j). \tag{1.2}$$

Consider the ideal $I \subseteq B$ generated by the set $\{ b_j \mid j \in J \}$, i.e.

$$I = {\downarrow}\{ b_{j_1} \vee \cdots \vee b_{j_n} \mid n \in \mathbb{N}, j_1, \ldots, j_n \in J \}.$$

We claim that $1 \in I$. Indeed, if, by way of contradiction, we had $1 \notin I$, then the filter $\{1\}$ would be disjoint from I; then, by the Boolean Prime Ideal Theorem (Theorem 1.48), there would be an ultrafilter U extending $\{1\}$ and disjoint from I. But by hypothesis ((1.2)), we deduce that there is $j \in J$ such that $U \in \eta_B(b_j)$, i.e., $b_j \in U$. This contradicts the facts that $b_j \in I$ and that I and U are disjoint. This proves the claim $1 \in I$.

Since $1 \in I$, by definition of I there are $j_1, \ldots, j_n \in J$ such that

$$1 \ \leq \ b_{j_1} \vee \cdots \vee b_{j_n},$$

and hence

$$1 \ = \ b_{j_1} \vee \cdots \vee b_{j_n}.$$

Applying η_B on both sides, and using the fact that it η_B a Boolean homomorphism (Lemma 1.46), we get

$$\mathrm{Ult}(B) = \eta_B(1) = \eta_B(b_{j_1} \vee \cdots \vee b_{j_n}) = \eta_B(b_{j_1}) \cup \cdots \cup \eta_B(b_{j_n}),$$

so we have found a finite subcover. ∎

In the proof of Theorem 1.63 we used the (immediate) fact that the subsets of $\mathrm{Ult}(B)$ of the form $\eta_B(b)$

are clopen; in fact, the sets of the form $\eta_B(b)$ are all and the only clopens, as shown in the following.

Lemma 1.64 (In the space of ultrafilters: clopen = basic open)**.** *Let B be a Boolean algebra. The clopens of* $\mathrm{Ult}(B)$ *are precisely the subsets of the form $\eta_B(b)$, for some $b \in B$.*

Proof. ($\Leftarrow$). For every $b \in B$, $\eta_B(b)$ is clopen because it is open and its complement is $\eta_B(\neg b)$ (since η_B is a Boolean homomorphism by Lemma 1.46), which is open, too.

($\Rightarrow$). Let C be a clopen of $\mathrm{Ult}(B)$. Since C is open, we can write

$$C = \bigcup_{i \in I} \eta_B(b_i)$$

for some family $(b_i)_{i \in I}$ in B, and automatically each $\eta_B(b_i) \subseteq C$. Because C is closed in the space $\mathrm{Ult}(B)$, which is compact by Theorem 1.63, the subspace C is compact. Thus the open cover $\{ \eta_B(b_i) \mid i \in I \}$ of C has a finite subcover: there is a finite $I_0 \subseteq I$ such that

$$C = \bigcup_{i \in I_0} \eta_B(b_i).$$

Since η_B is a homomorphism,

$$C = \eta_B\left(\bigvee_{i \in I_0} b_i\right),$$

so C lies in the image of η_B. ■

1.5.3 Bijection between Boolean algebras and Stone spaces

Theorem 1.65 ($B \cong \mathrm{Clop}(\mathrm{Ult}(B))$)**.** *For every Boolean algebra B, the map*

$$\begin{aligned} \eta_B \colon B &\longrightarrow \mathrm{Clop}(\mathrm{Ult}(B)), \\ b &\longmapsto \{ U \in \mathrm{Ult}(B) \mid b \in U \} \end{aligned}$$

is an isomorphism of Boolean algebras.

Proof. By Lemma 1.64, for every b the set $\eta_B = \{ U \in \mathrm{Ult}(B) \mid b \in U \}$ is indeed clopen, and the map $\eta_B \colon B \longrightarrow \mathrm{Clop}(\mathrm{Ult}(B))$ is surjective. By Lemmas 1.46 and 1.50, $\eta_B \colon B \longrightarrow \mathrm{Clop}(\mathrm{Ult}(B))$ is an injective homomorphism. ■

It follows immediately:

Corollary 1.66 (Finite Boolean algebras are power sets)**.**

1. *Every finite Boolean algebra is isomorphic to the power set of some finite set.*
2. *Every finite Boolean algebra has cardinality 2^n for some $n \in \mathbb{N}$.*
3. *Any two finite Boolean algebras with the same cardinality are isomorphic.*

Theorem 1.67 ($X \cong \mathrm{Ult}(\mathrm{Clop}(X))$)**.** *For every Stone space X, the map*

$$\begin{aligned} \varepsilon_X \colon X &\longrightarrow \mathrm{Ult}(\mathrm{Clop}(X)) \\ x &\longmapsto \{\, C \in \mathrm{Clop}(X) \mid x \in C \,\} \end{aligned}$$

is a homeomorphism (where $\mathrm{Ult}(\mathrm{Clop}(X))$ *carries the Stone topology).*

Proof. First of all, the map is well-defined because, for every $x \in X$, the set $\varepsilon_X(x) = \{\, C \in \mathrm{Clop}(X) \mid x \in C \,\}$ is easily seen[a] to be an ultrafilter of $\mathrm{Clop}(X)$.

Since $\varepsilon_X \colon X \to \mathrm{Ult}(\mathrm{Clop}(X))$ is a function between compact Hausdorff spaces, to prove that it is a homeomorphism it is enough[b] to prove that it is

1. continuous (this is a straightforward computation),
2. injective (this will follow from total separation),
3. surjective (this will follow from compactness).

Continuity. This is a straightforward computation.[c]

Injectivity. If $x \neq y$, since X is totally separated there is a clopen $C \subseteq X$ such that $x \in C$ and $y \notin C$. Then $C \in \varepsilon_X(x)$ but $C \notin \varepsilon_X(y)$, and hence $\varepsilon_X(x) \neq \varepsilon_X(y)$.

Surjectivity. Let $U \in \mathrm{Ult}(\mathrm{Clop}(X))$. Since U is proper, $\varnothing \notin U$. Moreover, U is closed under finite intersections, and so the family $\{\, C \mid C \in U \,\}$ has the finite intersection property (i.e., every intersection of finitely many of its elements is nonempty). Each $C \in U$ is closed, and X is compact, hence[d]

$$\bigcap_{C \in U} C \;\neq\; \varnothing.$$

Pick x in this intersection. We claim $U = \varepsilon_X(x)$. The inclusion $U \subseteq \varepsilon_X(x)$ is obvious because for all $C \in U$ we have $x \in C$. To prove the reverse inclusion ($\varepsilon_X(x) \subseteq U$), let $D \in \mathrm{Clop}(X)$. Reasoning by contraposition, suppose $D \notin U$ and let us prove $D \notin \varepsilon_X(x)$. Since U is an ultrafilter, exactly one of D and $X \setminus D$ belongs to U; hence, $X \setminus D \in U$. Thus, $x \in X \setminus D$ (since x belongs to the intersection of all clopens in U), and so $x \notin D$, i.e. $D \notin \varepsilon_X(x)$. Thus $U = \varepsilon_X(x)$. Therefore, ε_X is surjective. ■

[a] Indeed, it is easily seen that $\varepsilon_X(x)$ is a filter. It is also clear that $\varnothing \notin \varepsilon_X(x)$. Finally, if $C \cup D \in \varepsilon_X(x)$ then $x \in C \cup D$, hence $x \in C$ or $x \in D$, i.e. $C \in \varepsilon_X(x)$ or $D \in \varepsilon_X(x)$. So $\varepsilon_X(x)$ is an ultrafilter.

[b] Indeed, every continuous bijection between compact Hausdorff spaces is a homeomorphism. This follows from the fact that every continuous function between compact Hausdorff spaces is closed, i.e. the image of a closed subset is closed. This latter fact, in turn, follows from the facts that (i) a subset of a compact Hausdorff space is closed if and only if it is compact, and (ii) every continuous function maps a compact set to a compact set.

[c] Indeed, the Stone topology on $\mathrm{Ult}(\mathrm{Clop}(X))$ has a basis of opens of the form

$$\eta_{\mathrm{Clop}(X)}(C) \;=\; \{\, U \in \mathrm{Ult}(\mathrm{Clop}(X)) \mid C \in U \,\}, \qquad (C \in \mathrm{Clop}(X)).$$

For such a basic open we have

$$\varepsilon_X^{-1}\big[\eta_{\mathrm{Clop}(X)}(C)\big] \;=\; \{\, x \in X \mid \varepsilon_X(x) \in \eta_{\mathrm{Clop}(X)}(C) \,\} \;=\; \{\, x \in X \mid C \in \varepsilon_X(x) \,\} \;=\; \{\, x \in X \mid x \in C \,\} \;=\; C,$$

which is open. Hence, ε_X is continuous.

[d] It is easily seen that, if a family of closed subsets of a compact space has the finite intersection property, then the intersection of all its elements is nonempty.

Bibliography

[Baez and Dolan, 2001] Baez, J. C. and Dolan, J. (2001). From finite sets to Feynman diagrams. In *Mathematics unlimited—2001 and beyond*, pages 29–50. Springer, Berlin. 2

[Birkhoff, 1967] Birkhoff, G. (1967). *Lattice theory. Third ed.*, volume 25 of *Colloq. Publ., Am. Math. Soc.* American Mathematical Society (AMS), Providence, RI. 8

[Davey and Priestley, 2002] Davey, B. A. and Priestley, H. A. (2002). *Introduction to Lattices and Order*. Cambridge University Press, Cambridge, 2 edition. 9

[Halpern and Lévy, 1971] Halpern, J. D. and Lévy, A. (1971). The boolean prime ideal theorem does not imply the axiom of choice. In Scott, D. S., editor, *Axiomatic Set Theory*, volume 13 of *Proceedings of Symposia in Pure Mathematics*, pages 83–134. American Mathematical Society, Providence, RI. 18

[Jech, 1973] Jech, T. (1973). *The Axiom of Choice*, volume 75 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam. 18

[Stone, 1936] Stone, M. H. (1936). The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40(1):37–111. 18