

Duality for metrically complete Abelian ℓ -groups

Joint work with V. Marra and L. Spada
(To appear soon on ArXiv)

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In the 30s and 40s, a series of papers ([Gelfand and Kolmogorov, 1939], [Krein and Krein, 1940], [Kakutani, 1941], [Stone, 1941], [Yosida, 1941], [Gelfand and Neumark, 1943]) showed that the opposite of the category

$\mathbf{KHaus} :=$ category of compact Hausdorff spaces and continuous maps

can be represented, to within an equivalence, in several useful ways that variously relate to known mathematical structures.

The common idea is to associate with a compact Hausdorff space X the set $C(X, \mathbb{R})$ of continuous real-valued functions (or the set $C(X, \mathbb{C})$ of continuous complex-valued functions).

The mathematical structures used in these representation theorems include rings, commutative C^* -algebras, Kakutani's (M) -spaces, vector lattices, divisible Abelian lattice-ordered groups.

For our work, the most relevant structures are the ones used by Yosida.

(For the following definition, think of $C(X, \mathbb{R})$.)

Definition (Riesz, 1928)

A *vector lattice* (a.k.a. Riesz space) is a real vector space equipped with a lattice order such that

1. (Translation invariance) if $x \leq y$ then $x + z \leq y + z$.
2. (Positive Homogeneity) if $x \leq y$ then $\lambda x \leq \lambda y$ for any real scalar $\lambda \geq 0$.

For Yosida's duality additional conditions on a vector lattice are needed.

Definition

A (*strong order*) *unit* of a vector lattice V is an element $1 \in V$ such that, for any $x \in V$ there is n such that $x \leq n1$ (where $n1 := \underbrace{1 + \cdots + 1}_{n \text{ times}}$).

A *unital vector lattice* is a vector lattice with a designated unit.

(Example: in $C(X, \mathbb{R})$, the constant function 1 is a unit (using compactness of X).)

A unital vector lattice can be equipped with a pseudometric induced by its unit 1:

$$d(v, w) := \inf \{ \lambda \in \mathbb{R} \mid \lambda \geq 0 \text{ and } |v - w| \leq \lambda 1 \},$$

where $|x| := x \vee -x$ is the absolute value of x .

In $C(X, \mathbb{R})$, this pseudometric is the uniform metric.

The pseudometric is a metric iff the vector lattice is *Archimedean*, i.e. for all $x, y \in V$, if for all $n \in \mathbb{N}$ $nx \leq y$, then $x \leq 0$.

Theorem (Yosida duality, 1941)

KHaus is dually equivalent to the category of Archimedean metrically complete unital vector lattices.

Every Archimedean metrically complete unital vector lattice is isomorphic to $C(X, \mathbb{R})$ for a unique compact Hausdorff space X .

In this work we extend Yosida's duality by replacing vector lattices with Abelian lattice-ordered group.

Definition

An *Abelian lattice-ordered group* (*Abelian ℓ -group*, for short) is an abelian group equipped with a lattice order such that $x \leq y$ implies $x + z \leq y + z$ (translation invariance).

Much of the theory of vector lattices goes through for Abelian ℓ -groups.

- ▶ In contrast with vector lattices, Abelian lattice-ordered groups have a **finitary language** (because we do not have multiplication by all real scalars).
- ▶ Abelian ℓ -groups are widely used in the study of **MV-algebras**, the algebraic semantics of Łukasiewicz many-valued logic.

A *unit* of an Abelian ℓ -group is defined as in the vector lattice case (for any x there is n such that $x \leq n1$). For the pseudometric induced by the unit:

$$d(v, w) := \inf \left\{ \frac{p}{q} \in \mathbb{Q} \mid p \geq 0, q > 0, \text{ and } q|v - w| \leq p1 \right\}.$$

The notion of being *Archimedean* is defined as in the vector lattice case ($(\forall n \in \mathbb{N} \ nx \leq y) \Rightarrow x \leq 0$), and it is equivalent to the above pseudometric being a metric.

FUNCTIONAL REPRESENTATION

Archimedean metrically complete unital Abelian ℓ -groups enjoy the following representation theorem.

Theorem (Goodearl-Handelman 1980)

- *Let X be a compact Hausdorff space. For each $x \in X$ let A_x be either $A_x = \mathbb{R}$ or $A_x = \frac{1}{n}\mathbb{Z}$. Then, the algebra of functions*

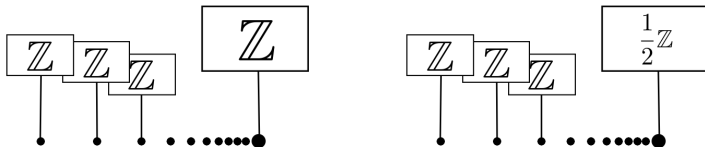
$$\{f: X \rightarrow \mathbb{R} \mid f \text{ cont.}, f(x) \in A_x \text{ for all } x \in X\},$$

is an Archimedean metrically complete unital Abelian ℓ -group.

- *Every Archimedean metrically complete unital Abelian ℓ -group can be represented in this way.*

NOT A 1:1 CORRESPONDENCE

This is not a 1:1 correspondence; two “non-isomorphic labelled spaces” may give isomorphic Archimedean metrically complete unital Abelian ℓ -group.



In both cases

$$\begin{aligned} \{f: X \rightarrow \mathbb{R} \mid f \text{ cont.}, f(x) \in A_x \text{ for all } x \in X\} = \\ = \{f: X \rightarrow \mathbb{Z} \mid f \text{ is definitely constant}\}. \end{aligned}$$

NOT A 1:1 CORRESPONDENCE

\mathbb{Z}

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0 ————— 1

In both cases

$$\begin{aligned} \{f: X \rightarrow \mathbb{R} \mid f \text{ cont.}, f(x) \in A_x \text{ for all } x \in X\} = \\ = \{f: X \rightarrow \mathbb{Z} \mid f \text{ is constant}\} \cong \mathbb{Z}. \end{aligned}$$

OUR AIM

Our aim: make the Goodearl-Handelman representation into a **categorical duality**.

We can encode a label A_x of Goodearl-Handelman via a natural number.

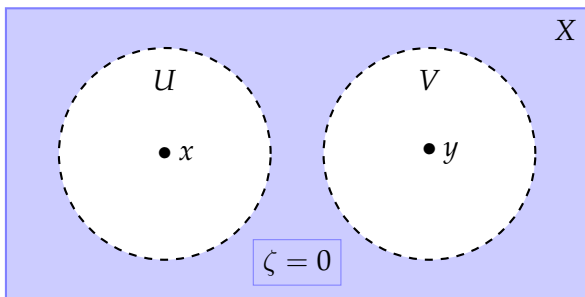
$$\begin{array}{ccc} \frac{1}{n}\mathbb{Z} & \rightsquigarrow & n. \\ \mathbb{R} & \rightsquigarrow & 0. \end{array}$$

Instead of a system of labels $(A_x)_{x \in X}$ we have a function $\zeta: X \rightarrow \mathbb{N}$.

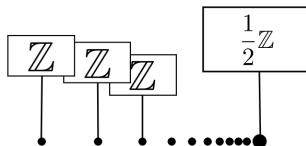
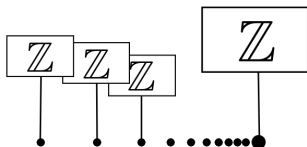
Definition

An *a-normal space* (for ‘arithmetically normal space’) is a pair (X, ζ) where X is a compact Hausdorff space and $\zeta: X \rightarrow \mathbb{N}$ is a function such that

1. (Continuity) For every $n \in \mathbb{N}$, $\zeta^{-1}[\text{div } n]$ is closed.
2. (Separation) For any two distinct points x and y of X , there exist two disjoint open neighbourhoods U and V of x and y , respectively, such that, for every $z \in X \setminus (U \cup V)$, $\zeta(z) = 0$.



EXAMPLES



In the second example, $\zeta^{-1}[\text{div } 1]$ ($= \zeta^{-1}[1]$) is not closed.

EXAMPLES

\mathbb{Z}



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In the second example, the elements $0, 1 \in [0, 1]$ cannot be separated by opens U and V so that $\zeta(x) = 0$ for all $x \in [0, 1] \setminus (U \cup V)$.

MAIN RESULT

Theorem

The category of **a-normal spaces** is dually equivalent to the category of **Archimedean metrically complete unital Abelian ℓ -groups**.

Thank you for listening!