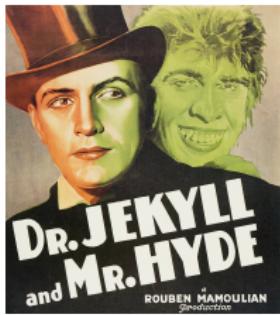


Bridging Ordered Groups and Topology: Representation Theorems and Categorical Dualities

Marco Abbadini



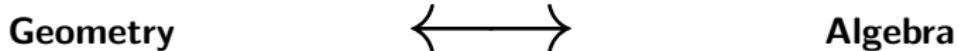
Partly based on:



- M. Abbadini, V. Marra, L. Spada. Stone-Gelfand duality for metrically complete lattice-ordered groups. *Advances in Mathematics*, 2025.

Categorical Dualities

Two worlds, one story.



Categorical dualities are bridges between such worlds — each side encodes the same information, but in a very different language.

Categorical dualities: an improved version of representation theorems.

Representation theorems show how abstract algebraic structures can be realized concretely via geometric or topological objects.

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Representation theorems show how abstract algebraic structures can be realized concretely via geometric or topological objects.

Categorical dualities go further:

they make the correspondence **bijection**,
and let us use it **to its full power** —
translating problems, theorems, and intuitions across the mirror.

A short journey through three categorical dualities

1930s Stone duality: connects topology and logic.

(Boolean algebras \leftrightarrow Stone spaces)

1940s Yosida duality: connects topology and analysis.

(Ordered vector spaces \leftrightarrow compact Hausdorff spaces)

2020s A new duality: brings the two under one umbrella.

(Ordered groups \leftrightarrow normal a-spaces)

I. Stone duality

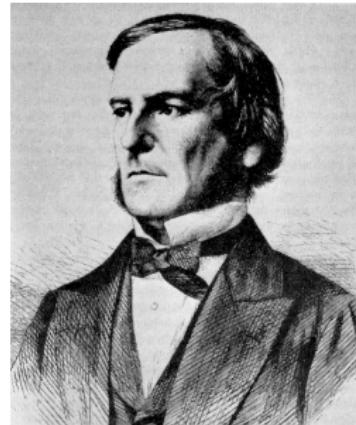
Logic becomes algebra: Boolean algebras

1847: **George Boole** formalized the rules of propositional reasoning into algebraic form.

A *Boolean algebra* is a set B (think of a set of propositions) equipped with

$$p \wedge q \text{ (and)}, \quad p \vee q \text{ (or)}, \quad \neg p \text{ (not)},$$

and two elements 0 (false) and 1 (true), satisfying familiar laws:



George Boole (1815–1864)

$$p \wedge \neg p = 0, \quad p \vee \neg p = 1, \quad p \wedge q = q \wedge p, \quad \dots$$

Example: the power set $\mathcal{P}(X)$ of a set X ,

$$A \wedge B = A \cap B, \quad A \vee B = A \cup B, \quad \neg A = A^c, \quad 0 = \emptyset, \quad 1 = X.$$

Stone's representation: Boolean algebras as fields of sets

Start from a set X . Any family $\mathcal{F} \subseteq \mathcal{P}(X)$ that is closed under

$$A \cap B, \quad A \cup B, \quad A^c,$$

and contains \emptyset and X , is a **Boolean algebra**.

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Stone's representation theorem (1936)

Every Boolean algebra is isomorphic to such a family of subsets of some set X (i.e. to a *field of sets*).



So, every Boolean algebra can be *realized concretely* as sets with \cap , \cup , and complement.

From representation to duality

So far, we have a
representation theorem.



the base

Stone proved more:
a categorical duality,
for which we need
two ingredients.



the ingredients

Canonical representation and bijective correspondence

For each Boolean algebra, there is a **canonical** representation: the unique one that is **separating** and **compact**.

Stone space := *compact* space whose *clopen* (= *closed open*) sets separate the elements.

Examples: any finite discrete space.

Stone spaces $\xleftrightarrow{1:1}$ Boolean algebras.

Stone space \mapsto Boolean alg. of *clopen* subsets.

Example: finite discrete space \mapsto its power set.



a tomatological
sauce:
Stone sp. $\xleftrightarrow{1:1}$ Bool. alg.

Bijective correspondence on maps, reversed

Boolean homomorphisms

$$A \rightarrow B$$

$$\uparrow\downarrow \text{ 1:1}$$

continuous functions

Stone space of $B \rightarrow$ Stone space of A .



mozzarphisms:

Example:

$$A \longrightarrow B$$

$$X \hookrightarrow Y$$

$$X_A \longleftarrow X_B$$

between finite discrete spaces

power set of $Y \longrightarrow$ power set of X

$$A \longmapsto A \cap X$$

Stone duality: bijection between Boolean algebras and Stone spaces + bijection between the interesting functions, but with opposite direction.

A *category* (Eilenberg, Mac Lane, 1945) consists of:

- ▶ a class of structures (e.g. Boolean algebras), and
- ▶ a class of maps between them (e.g. Boolean homomorphisms).

A *categorical duality* is a bijective correspondence between two categories, but reversing the direction of the maps.

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Stone duality (1936)

There is a **categorical duality** between:

1. the category of **Boolean algebras** with Boolean homomorphisms, and
2. the category of **Stone spaces** with continuous functions.



Translating problems

Logic. Given propositional theories T in a language L and T' in a disjoint language L' : does adding T' create new theorems in L ?

Algebra. Is the coproduct map $A \rightarrow A + B$ between Boolean algebras injective?

Topology. Is the projection $X \times Y \rightarrow X$ surjective for Stone spaces?

Stone duality: all three are the same question — and the answer is immediate on the topological side.

Stone duality in one slide

- ▶ Boolean algebras \leftrightarrow Stone spaces (1:1)
- ▶ Boolean homomorphisms \leftrightarrow continuous maps (1:1, reversed)
- ▶ problems and results translate across the duality.

II. Yosida duality

The late 1930s / early 1940s

After Stone's work, mathematicians realized a certain pattern: various structures can be represented as spaces of (real-valued / complex-valued) *continuous functions* on a compact Hausdorff space.

This gave rise to several representation theorems in the late '30s and early '40s.

Some names in the game: Gelfand, Kolmogorov, Naimark, Kakutani, Yosida, the Krein brothers, Stone.

Some structures in the game: commutative C^* -algebras, Banach lattices and Kakutani's (M)-spaces, vector lattices, divisible lattice-ordered Abelian groups.

The most famous one

The best-known result of this family is the **Gelfand–Naimark representation theorem** (1943):

Each unital commutative C^* -algebra is the algebra of continuous *complex-valued* functions on a compact Hausdorff space.

Unital commutative C^* -algebras

↑ duality

Compact Hausdorff spaces

We will not enter C^* -algebras today. Instead, we look at a simpler *real-valued* cousin: Yosida duality (1941).

Continuous real functions are very structured

For a compact Hausdorff space X , the set $C(X, \mathbb{R})$ of real-valued continuous functions on X has many natural operations:

- ▶ pointwise sum $f + g$
- ▶ pointwise maximum $f \vee g$ and minimum $f \wedge g$
- ▶ scalar multiplication $\lambda \cdot f$
- ▶ constant functions 0 and 1

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These operations are enough to recover the space X from $C(X, \mathbb{R})$ equipped with this structure.

For the following definition, think of $C(X, \mathbb{R})$.

Definition (Riesz, 1928)

A *vector lattice* is a real vector space equipped with a lattice order (i.e. a partial order with binary suprema and infima) s.t.:

1. (Translation invariance) if $x \leq y$ then $x + z \leq y + z$;
2. (Positive homogeneity) if $x \leq y$ then $\lambda x \leq \lambda y$ for any real $\lambda \geq 0$.

For Yosida's duality a bit more structure is needed.

Definition

A *metrically complete unital vector lattice* is a vector lattice V equipped with a designated element $1 \in V$ such that the following defines a complete metric on V :

$$\text{dist}(v, w) := \inf \{\lambda \in \mathbb{R}_{\geq 0} \mid -\lambda 1 \leq v - w \leq \lambda 1\},$$

Example: in $C(X, \mathbb{R})$, let 1 be the function $X \rightarrow \mathbb{R}$ constantly equal to 1 ; the corresponding metric is the uniform metric.

Yosida proved that every metrically complete unital vector lattice is isomorphic to $C(X, \mathbb{R})$ for a unique compact Hausdorff space X (up to homeomorphism).

Yosida duality (1941)

The category of compact Hausdorff spaces is dually equivalent to the category of metrically complete unital vector lattices.



An inclusion of spaces $X \hookrightarrow Y$ gives a restriction function

$$\begin{aligned} C(Y, \mathbb{R}) &\longrightarrow C(X, \mathbb{R}) \\ f &\longmapsto f|_X, \end{aligned}$$

that preserves $+$, \vee , \wedge , scalar multiplication, the unit 1 (hence is a morphism of metrically complete unital vector lattices).

III. An umbrella for both

Two dualities, two mirrors

So far, we have seen two categorical dualities:

- ▶ **Stone duality** between Stone spaces and Boolean algebras (very discrete structures),
- ▶ **Yosida duality** between compact Hausdorff spaces and metrically complete unital vector lattices (very continuous structures).

Every Stone space X is in particular compact Hausdorff:

1. Stone associates to it the Boolean algebra of clopens of X .
2. Yosida associates to it $C(X, \mathbb{R})$.

We are using two different dualities, obtaining for the same space two very different objects.

1. We recall a class of algebraic structures that can specialize to

 - ▶ something similar to vector lattices on one extreme,
 - ▶ something similar to Boolean algebras on the other.
2. We provide a duality for these structures, which specializes in two directions: to Stone duality on one extreme and to Yosida duality on the other.

A hint: both sides have “addition”

In Yosida duality, $C(X, \mathbb{R})$ has a genuine addition $f + g$, defined everywhere.

In a Boolean algebra, $x \vee y$ behaves *like* an addition — but only when x and y are disjoint.

Idea: embed Boolean algebras into algebraic structures where a *true* addition is always defined.

Example:

$$\{0, 1\} \hookrightarrow \mathbb{Z},$$

where the Boolean “join” becomes ordinary sum (with truncation hidden in the embedding).

One algebraic structure to unify them

A good abstraction is: **lattice-ordered Abelian group** (*Abelian ℓ -group*) (1940):

- ▶ an Abelian group $(G, +, 0, -)$,
- ▶ equipped with a lattice order (\vee, \wedge) ,
- ▶ that is translation-invariant $(x \leq y \Rightarrow x + z \leq y + z)$.

These objects can specialize in two directions:

- ▶ if highly divisible \Rightarrow they look like vector lattices,
- ▶ if poorly divisible \Rightarrow they look like Boolean algebras.

(Connection to MV-algebras ($\text{\L}ukasiewicz}$ many-valued logic).)

Definition

A *metrically complete unital Abelian ℓ -group* is a lattice-ordered Abelian group G equipped with a designated element $1 \in G$ such that the following defines a complete metric on G :

$$\text{dist}(v, w) := \inf \left\{ \frac{p}{q} \in \mathbb{Q} \mid p \geq 0, q > 0, \text{ and } p1 \leq q(v - w) \leq p1 \right\}.$$

Example

Given a compact Hausdorff space X :

$$C(X, \mathbb{R}) := \{f: X \rightarrow \mathbb{R} \mid f \text{ is continuous}\},$$

$$C\left(X, \frac{1}{n}\mathbb{Z}\right) := \left\{f: X \rightarrow \frac{1}{n}\mathbb{Z} \mid f \text{ is continuous}\right\}.$$

We can also mix the values that the functions can take at different points...

Representation theorem

Representation theorem, Goodearl & Handelman 1980

- ▶ Let X be a compact Hausdorff space. For each $x \in X$, let D_x be either $D_x = \mathbb{R}$ or $D_x = \frac{1}{n}\mathbb{Z}$ (for some $n \in \mathbb{N} \setminus \{0\}$). Then,

$$\{f: X \rightarrow \mathbb{R} \mid f \text{ continuous}, \forall x \in X f(x) \in D_x\}$$

is a metrically complete unital Abelian ℓ -group.

- ▶ Every metrically complete unital Abelian ℓ -group can be represented in this way.

Example

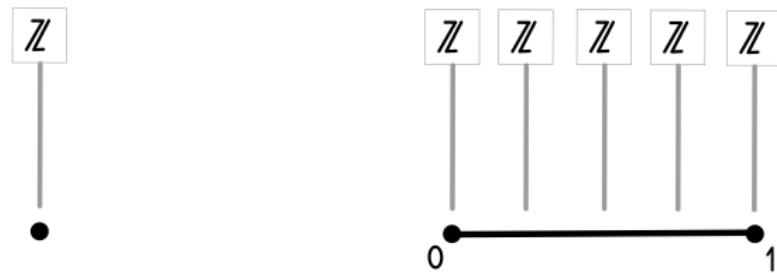
- ▶ If $D_x = \mathbb{R}$ for all $x \in X$, then we get $C(X, \mathbb{R})$.
- ▶ If $D_x = \mathbb{Z}$ for all $x \in X$, then we get $C(X, \mathbb{Z})$.

Our aim

Our aim (A., Marra, Spada): make the Goodearl-Handelman representation into a **categorical duality**, so that we have a representation of morphisms, and that we can transfer all problems that can be expressed in the categorical language.

Not a 1:1 correspondence

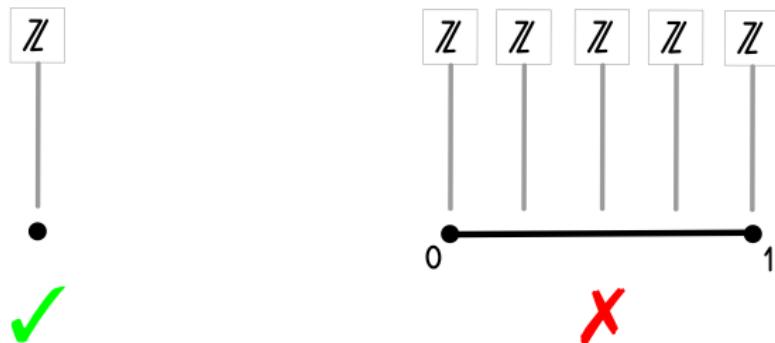
Goodearl & Handelman's representation is not a 1:1 correspondence.



In both cases, $C(X, \mathbb{Z}) \cong \mathbb{Z}$.

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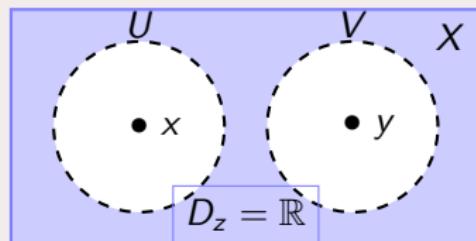
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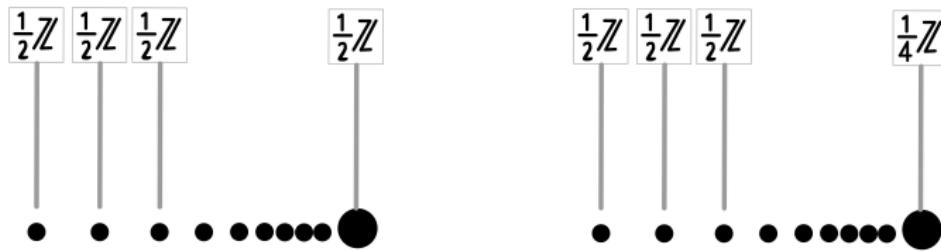
In both cases, $C(X, \mathbb{Z}) \cong \mathbb{Z}$.

Remedy: Separation

For $x \neq y \in X$, there are disjoint open sets $U \ni x$ and $V \ni y$ s.t., for all $z \in X \setminus (U \cup V)$, $D_z = \mathbb{R}$.



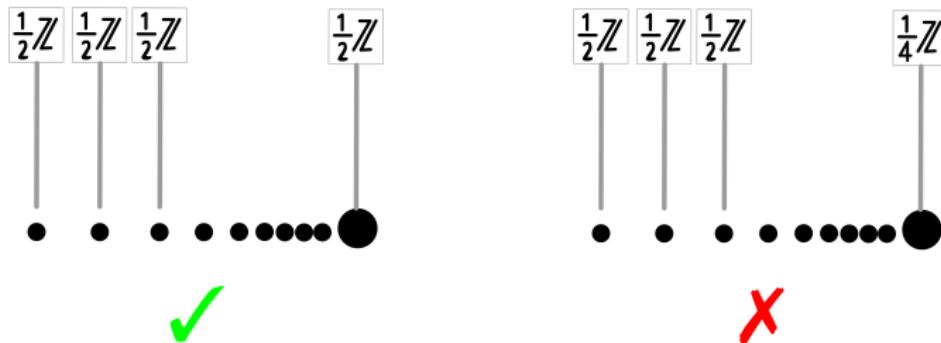
Not a 1:1 correspondence



In both cases

$$\begin{aligned} & \left\{ f: X \rightarrow \mathbb{R} \mid f \text{ cont. and, for all } x \in X, f(x) \in D_x \right\} = \\ &= \left\{ f: X \rightarrow \frac{1}{2}\mathbb{Z} \mid f \text{ is eventually constant} \right\}. \end{aligned}$$

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Remedy: Semicontinuity

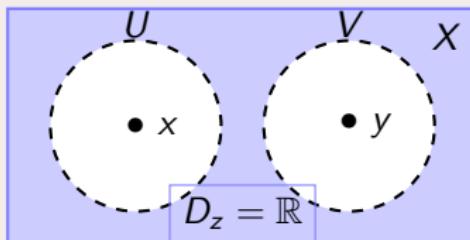
For every $n \in \mathbb{N}_{>0}$, $\{x \in X \mid D_x \subseteq \frac{1}{n}\mathbb{Z}\}$ is closed.

Normal a-spaces

Definition

A *normal a-space* (for “arithmetically normal arithmetic space”) is a compact Hausdorff space X equipped with a family $(D_x)_{x \in X}$ with $D_x \in \{\mathbb{R}\} \cup \left\{ \frac{1}{n} \mathbb{Z} \mid n \in \mathbb{N}_{>0} \right\}$ such that

1. (Separation) For $x \neq y \in X$, there are disjoint open sets $U \ni x$ and $V \ni y$ s.t., for all $z \in X \setminus (U \cup V)$, $D_z = \mathbb{R}$.



2. (Semicontinuity) For every $n \in \mathbb{N}_{>0}$, $\{x \in X \mid D_x \subseteq \frac{1}{n} \mathbb{Z}\}$ is closed.

Example: $[0, 1]$ with $D_{\frac{p}{q}} = \frac{1}{q} \mathbb{Z}$ and $D_x = \mathbb{R}$ otherwise.

A new duality (A., Marra, Spada, 2025)

The category of **metrically complete unital Abelian ℓ -groups** (and homomorphisms) is dually equivalent to the category of **normal a-spaces** (and continuous “denominator-decreasing” maps).



M. Abbadini, V. Marra, L. Spada.

Stone–Gelfand duality for metrically complete lattice-ordered groups.
Advances in Mathematics, 2025.

Case $D_x = \mathbb{R}$ for all x : we get an arbitrary compact Hausdorff space X , and $C(X, \mathbb{R})$: Yosida duality.

Case $D_x = \mathbb{Z}$ for all x : we get an arbitrary Stone space X , and $C(X, \mathbb{Z})$, which is very close to the Boolean algebra $C(X, \{0,1\})$: Stone duality.

A concluding reflection

We've crossed a small universe of mirrors: dualities between distant worlds that seem to have nothing in common, and yet contain the same information.

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Baez, Dolan (2001):

“... an equation is only interesting or useful to the extent that the two sides are different!”

The equation

$$e^{i\omega} = \cos(\omega) + i \sin(\omega)$$

is far more interesting and useful than

$$1 = 1,$$

precisely because its two sides have such different shapes.

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Thank you!