

# Vietoris endofunctor for closed relations and its de Vries dual

Marco Abbadini

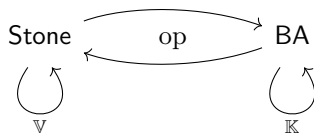
School of Computer Science, University of Birmingham, UK

Based on a homonymous paper with G. Bezhanishvili and L. Carai  
(*Topology Proceedings*, to appear. Available on arXiv.)

TACL 2024

4 July 2024

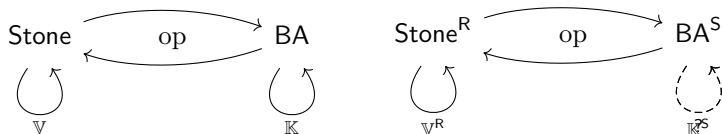
Kripke semantics connects modal logic with the Vietoris endofunctor.



Algebras for  $\mathbb{K}$  = modal algebras.

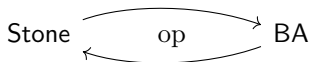
Coalgebras for  $\mathbb{V}$  = descriptive frames.

This is the coalgebraic approach to modal logic.

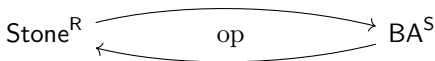


1. Stone duality has a natural extension to **closed relations** [Celani, 2018].
2. **Vietoris** has a natural **extension** to closed relations [Goy, Petrişan, Aiguier, 2021]! Which arised in studies of the usual Vietoris on functions.
3. We provide **a description of the dual of  $\mathbb{V}^R$** .

Natural notions  $\rightarrow$  hope for applications! Our application: a resolution of an open problem on de Vries duality [Bezhanishvili, Bezhanishvili, Harding, 2015], concerning the usual Vietoris on compact Hausdorff spaces and continuous functions.



Extended by Celani (2018) (see also Kurz, Moshier, Jung, 2023):



closed relations

subordinations

Closed relation  $R: X \multimap Y$  is a subset  $R \subseteq X \times Y$  that is closed; equivalently, such that

- ▶  $R[\text{closed}]$  is closed,
- ▶  $R^{-1}[\text{closed}]$  is closed.

Why we got into closed relations:

Every compact Hausdorff space  $X$  is a continuous image of a Stone space (e.g., its Gleason cover). So it can be presented via

Stone space + closed equivalence relation.

$$\begin{array}{ccccc}
 \text{Stone} & \ni & \mathcal{G}(X) & \xrightarrow{\varphi} & \mathcal{G}(Y) & \in & \text{Stone} \\
 & & \downarrow \wr & & \downarrow \wr & & \\
 \text{KHaus} & \ni & X & \xrightarrow{f} & Y & \in & \text{KHaus}
 \end{array}$$

Dual of a closed relation:

$$\begin{array}{ccc}
 X & & \text{Clop}(X) \\
 \downarrow \scriptstyle R & & \uparrow \scriptstyle S \\
 Y & & \text{Clop}(Y)
 \end{array}$$

For  $V \in \text{Clop}(Y)$  and  $U \in \text{Clop}(X)$ :

$$V S U \iff R^{-1}[V] \subseteq U$$

Example:

$$\begin{array}{ccc}
 X & & \text{Clop}(X) \\
 \downarrow \scriptstyle = & & \uparrow \scriptstyle \subseteq \\
 X & & \text{Clop}(X)
 \end{array}$$

Subordination  $\coloneqq$  a relation  $S: A \multimap B$  such that

$$\left( \bigvee_{i=1}^n a_i \right) S \left( \bigwedge_{j=1}^m b_j \right) \iff \forall i, j \ a_i S b_j.$$

Theorem (Celani, 2018)

$\text{Stone}^R$  (closed relations) is dual to  $\text{BA}^S$  (subordinations).

## Definition (Vietoris hyperspace)

The **Vietoris hyperspace**  $\mathbb{V}(X)$  of a Stone space  $X$  is the set of closed subsets of  $X$ , equipped with the topology generated by the following subsets of  $\mathbb{V}(X)$ , for  $U$  clopen of  $X$ :

$$\Diamond U := \{K \in \mathbb{V}(X) \mid K \cap U \neq \emptyset\},$$

$$\Box U := \{K \in \mathbb{V}(X) \mid K \subseteq U\}.$$

Vietoris functor on Stone:

$$\text{Stone} \xrightarrow{\mathbb{V}} \text{Stone}$$

$$\begin{array}{ccc} X & & \mathbb{V}(X) \\ \downarrow f & & \downarrow f[-] \\ Y & & \mathbb{V}(Y) \end{array}$$

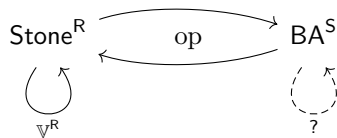


Extension of  $\mathbb{V}$  from Stone to  $\text{Stone}^R$  [Goy, Petrişan, Aiguier, 2021]:

$$\text{Stone} \xrightarrow{\mathbb{V}^R} \text{Stone}$$

$$\begin{array}{ccc} X & \mathbb{V}(X) & \\ \downarrow R & \downarrow \mathbb{V}^R(R) & \\ Y & \mathbb{V}(Y) & \end{array} \quad \begin{array}{l} \text{(Egli-Milner:)} \text{ For } K \in \mathbb{V}(X) \text{ and } L \in \mathbb{V}(Y), \\ K \mathbb{V}^R(R) L \iff \begin{cases} \forall x \in K \exists y \in L : x R y, \\ \forall y \in L \exists x \in K : x R y. \end{cases} \end{array}$$

It restricts to the usual Vietoris functor on continuous functions.



**What is the dual of  $\mathbb{V}^R$ ?**

$$\mathbb{V}^R: \mathbf{Stone}^R \rightarrow \mathbf{Stone}^R$$

On objects: the same as for the dual of  $\mathbb{V}$  on  $\mathbf{Stone}$  (Abramsky, Johnstone, Kupke, Kurz, Venema, Vosmaer):

$$\frac{X \quad \mathbb{V}(X)}{\quad \quad \quad} \quad \text{Stone duality}$$

$$A \quad \mathbb{K}(A)$$

$$\mathbb{K}(A) = \frac{\text{Free}_{\mathbf{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})}{\{\Box \text{ preserves finite meets, } \Diamond = \neg \Box \neg\}}$$

On morphisms:

$$\begin{array}{ccc}
 X & & \mathbb{V}(X) \\
 \downarrow \scriptstyle R & & \downarrow \scriptstyle \mathbb{V}^R(R) \\
 Y & & \mathbb{V}(Y)
 \end{array}$$

$$\begin{array}{ccc}
 B & \mathbb{K}(B) = \text{Free}_{\text{BA}}(\{\square_b, \diamond_b \mid b \in B\})/\sim & \\
 \uparrow \scriptstyle S & \uparrow \scriptstyle \mathbb{K}^S(S)? & \\
 A & \mathbb{K}(A) = \text{Free}_{\text{BA}}(\{\square_a, \diamond_a \mid a \in A\})/\sim &
 \end{array}$$

We shall describe when an element  $\alpha$  of  $\mathbb{K}(A)$  is  $\mathbb{K}^S(S)$ -related with an element  $\beta$  of  $\mathbb{K}(B)$ .

## Proposition

Given a Boolean algebra  $A$ . Every  $\gamma \in \mathbb{K}(A)$  is (effectively) equal to

- ▶ (DNF) a finite join of elements of the form

$$\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b$$

with each  $a_i \leq b$ ;

- ▶ (CNF) a finite meet of elements of the form

$$\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m}$$

with each  $c \leq d_j$ .

$$\begin{array}{ccc}
A & & \mathbb{K}(A) \\
\downarrow \scriptstyle S & & \downarrow \scriptstyle \mathbb{K}^S(S)? \\
B & & \mathbb{K}(B)
\end{array}$$

Enough to describe when

$$(\diamond_{a_1} \wedge \cdots \wedge \diamond_{a_n} \wedge \square_b) \mathbb{K}^S(S) (\diamond_c \vee \square_{d_1} \vee \cdots \vee \square_{d_m})$$

with  $a_i \leq b$  and  $c \leq d_j$ .

$$\begin{array}{ccc}
 A & & \mathbb{K}(A) \\
 \downarrow \leq & & \downarrow \leq \\
 A & & \mathbb{K}(A)
 \end{array}$$

(With  $a_i \leq b$  and  $c \leq d_j$ .)

$$(\diamond_{a_1} \wedge \cdots \wedge \diamond_{a_n} \wedge \square_b) \leq (\diamond_c \vee \square_{d_1} \vee \cdots \vee \square_{d_m})$$

$$\Updownarrow$$

$$(\exists i : a_i \leq c) \text{ or } (\exists j : b \leq d_j).$$

**Key idea:**  $\diamond$ -with- $\diamond$  or  $\square$ -with- $\square$ .

E.g.: if  $A, B, C, D$  are clopens of a Stone space  $X$  with  $A \subseteq C$  and  $B \subseteq D$ , then

$$\diamond A \cap \square B \subseteq \diamond C \cup \square D \iff A \subseteq C \text{ or } B \subseteq D.$$

$$\begin{array}{ccc}
 A & & \mathbb{K}(A) \\
 \downarrow s & & \downarrow \mathbb{K}^S(s) \\
 B & & \mathbb{K}(B)
 \end{array}$$

(With  $a_i \leq b$  and  $c \leq d_j$ .)

$$(\diamond_{a_1} \wedge \cdots \wedge \diamond_{a_n} \wedge \square_b) \mathbb{K}^S(S) (\diamond_c \vee \square_{d_1} \vee \cdots \vee \square_{d_m})$$

$$\Updownarrow$$

$$(\exists i : a_i \ S \ c) \text{ or } (\exists j : b \ S \ d_j).$$

**Key idea:**  $\diamond$ -with- $\diamond$  or  $\square$ -with- $\square$ .



## Theorem (A., Bezhanishvili, Carai, 2024)

The dual of the Vietoris endofunctor  $\mathbb{V}^R: \mathbf{Stone}^R \rightarrow \mathbf{Stone}^R$  is the following endofunctor  $\mathbb{K}^S: \mathbf{BA}^S \rightarrow \mathbf{BA}^S$ :

- On objects: it maps  $A$  to

$$\mathbb{K}(A) := \frac{\text{Free}_{\mathbf{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})}{\{\Box \text{ preserves finite meets, } \Diamond = \neg\Box\neg\}}$$

- On morphisms: it maps a subordination  $S: A \multimap B$  to the unique subordination  $\mathbb{K}^S(S): \mathbb{K}(A) \multimap \mathbb{K}(B)$  satisfying “ $\Diamond$ -with- $\Diamond$  or  $\Box$ -with- $\Box$ ”.

“ $\Diamond$ -with- $\Diamond$  or  $\Box$ -with- $\Box$ ”: (With  $a_i \leq b$  and  $c \leq d_j$ .)

$$(\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b) \mathbb{K}^S(S) (\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m})$$



$$(\exists i : a_i S c) \text{ or } (\exists j : b S d_j).$$

## An application

De Vries duality is a duality for compact Hausdorff spaces, which associates to a compact Hausdorff space  $X$  the Boolean algebra of regular opens, together with the binary relation  $\prec$  of well-insideness:  $U \prec V \iff \text{cl}(U) \subseteq V$ .

### Question (Bezhanishvili, Bezhanishvili, Harding, 2015)

What is the De Vries dual of the Vietoris endofunctor on compact Hausdorff spaces?

We piggyback on the duality between  $\mathbb{V}^R: \text{Stone}^R \rightarrow \text{Stone}^R$  and  $\mathbb{K}^S: \text{BA}^S \rightarrow \text{BA}^S$ .

## Theorem (A., Bezhanishvili, Carai, 2024)

*The de Vries dual of the Vietoris endofunctor on  $\mathbf{KHaus}$  is obtained by applying  $\mathbb{K}^S$  (= the dual of  $\mathbb{V}^R: \mathbf{Stone}^R \rightarrow \mathbf{Stone}^R$ ), followed by a(n appropriate) MacNeille completion.*

$$X \xrightarrow{R} Y$$

$$(B, \prec_B) \xleftarrow{S} (A, \prec_A)$$

$$\mathbb{V}(X) \xrightarrow{\mathbb{V}^R(R)} \mathbb{V}(Y)$$

$$\begin{array}{ccc} (\mathbb{K}(B), \mathbb{K}^S(\prec_B)) & \xleftarrow{\mathbb{K}^S(S)} & (\mathbb{K}(A), \mathbb{K}^S(\prec_A)) \\ \downarrow & & \downarrow \\ \mathbf{M}(\mathbb{K}(B), \mathbb{K}^S(\prec_B)) & \xleftarrow{\mathbf{M}(\mathbb{K}^S(S))} & \mathbf{M}(\mathbb{K}(A), \mathbb{K}^S(\prec_A)) \end{array}$$

where  $\mathbf{M}$  is an appropriate MacNeille completion functor.

## Conclusions

# Key ideas

1. Beyond functions; **closed relations** between Stone spaces ( $\leftrightarrow$  **subordinations** between Boolean algebras).

Especially: in dualities between “KHaus”-like and “lattice+proximity”-like structures.

2. “ $\diamond$ -with- $\diamond$  or  $\square$ -with- $\square$ ”:

$$\left( \bigwedge_i \diamond_{a_i} \right) \wedge \square_b \leq \diamond_c \vee \left( \bigvee_j \square_{d_j} \right) \Leftrightarrow (\exists i : a_i \leq c) \text{ or } (\exists j : b \leq d_j).$$

3. Our packaging of these ideas:

- ▶ **Stone dual** description of  $\mathbb{V}^R$ :  $\text{Stone}^R \rightarrow \text{Stone}^R$ ;
- ▶ **de Vries dual** description of  $\mathbb{V}$ :  $\text{KHaus} \rightarrow \text{KHaus}$  and for relations.



M. Abbadini, G. Bezhanishvili, L. Carai.

Vietoris endofunctor for closed relations and its de Vries dual.

*Topology Proceedings*, to appear. Available on arxiv:2308.16823.