Vietoris endofunctor for closed relations and its de Vries dual

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Kripke semantics connects modal logic with the Vietoris endofunctor.

$$\begin{tikzcd} \mathsf{Stone} \arrow[r, bend left] \arrow[loop below, "\mathbb{V}"] \arrow[r, phantom, "\mathrm{op}"] & \mathsf{BA} \arrow[l, bend left] \arrow[loop below, "\mathbb{K}"] \end{tikzcd}$$

Algebras for $\mathbb{K}$ = modal algebras.

Coalgebras for $\mathbb{V}$ = descriptive frames.

This is the coalgebraic approach to modal logic.

$$
\begin{tikzcd}
\mathsf{Stone} \arrow[r, bend left, "\mathrm{op}"] \arrow[loop below, "\mathbb{V}"] & \mathsf{BA} \arrow[l, bend left] \arrow[loop below, "\mathbb{K}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Stone}^{\mathsf{R}} \arrow[r, bend left, "\mathrm{op}"] \arrow[loop below, "\mathbb{V}^{\mathsf{R}}"] & \mathsf{BA}^{\mathsf{S}} \arrow[l, bend left] \arrow[loop below, dashed, "\mathbb{K}^{?\mathsf{S}}"]
\end{tikzcd}
$$

1. Stone duality has a natural extension to closed relations [Celani, 2018].

2. Vietoris has a natural extension to closed relations [Goy, Petrișan, Aiguier, 2021]! Which arised in studies of the usual Vietoris on functions.

3. We provide a description of the dual of $\mathbb{V}^{\mathsf{R}}$.

Natural notions $\rightarrow$ hope for applications! Our application: a resolution of an open problem on de Vries duality [Bezhanishvili, Bezhanishvili, Harding, 2015], concerning the usual Vietoris on compact Hausdorff spaces and continuous functions.

$$\begin{tikzcd} \mathsf{Stone} \arrow[rr, bend left] \arrow[rr, phantom, "\mathrm{op}"] & & \mathsf{BA} \arrow[ll, bend left] \end{tikzcd}$$

Extended by Celani (2018) (see also Kurz, Moshier, Jung, 2023):

$$\begin{tikzcd} \mathsf{Stone}^{\mathsf{R}} \arrow[rr, bend left] \arrow[rr, phantom, "\mathrm{op}"] & & \mathsf{BA}^{\mathsf{S}} \arrow[ll, bend left] \end{tikzcd}$$

closed relations subordinations

Closed relation $R\colon X \looparrowright Y$ is a subset $R \subseteq X \times Y$ that is closed; equivalently, such that

- R[closed] is closed,
- R^{-1}[closed] is closed.

Why we got into closed relations:

Every compact Hausdorff space X is a continuous image of a Stone space (e.g., its Gleason cover). So it can be presented via

$$\text{Stone space} + \text{closed equivalence relation.}$$

$$\begin{tikzcd}
\mathsf{Stone} & \ni & \mathcal{G}(X) \arrow[r, hook] \arrow[d, two heads] & \mathcal{G}(Y) \arrow[d, two heads] & \in & \mathsf{Stone} \\
\mathsf{KHaus} & \ni & X \arrow[r, "f"] & Y & \in & \mathsf{KHaus}
\end{tikzcd}$$

Dual of a closed relation:

$$\begin{tikzcd} X \arrow[d, "R", "\shortmid"{marking}] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Clop}(X) \\ \mathrm{Clop}(Y) \arrow[u, "S"', "\shortmid"{marking}] \end{tikzcd}$$

For $V \in \mathrm{Clop}(Y)$ and $U \in \mathrm{Clop}(X)$:

$$V\,S\,U \iff R^{-1}[V] \subseteq U$$

Example:

$$\begin{tikzcd} X \arrow[d, "=", "\shortmid"{marking}] \\ X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Clop}(X) \\ \mathrm{Clop}(X) \arrow[u, "\subseteq"', "\shortmid"{marking}] \end{tikzcd}$$

Subordination := a relation $S\colon A \nrightarrow B$ such that

$$\left(\bigvee_{i=1}^{n} a_i\right) S \left(\bigwedge_{j=1}^{m} b_j\right) \iff \forall i,j \; a_i \; S \; b_j.$$

Theorem (Celani, 2018)

$\mathsf{Stone}^{\mathsf{R}}$ *(closed relations) is dual to* BA^{S} *(subordinations).*

Definition (Vietoris hyperspace)

The Vietoris hyperspace $\mathbb{V}(X)$ of a Stone space X is the set of closed subsets of X, equipped with the topology generated by the following subsets of $\mathbb{V}(X)$, for U clopen of X:

$$\begin{aligned} \Diamond U &\coloneqq \{K \in \mathbb{V}(X) \mid K \cap U \neq \varnothing\}, \\ \Box U &\coloneqq \{K \in \mathbb{V}(X) \mid K \subseteq U\}. \end{aligned}$$

Vietoris functor on Stone:

$$\mathsf{Stone} \xrightarrow{\mathbb{V}} \mathsf{Stone}$$

$$\begin{tikzcd} X \arrow[d, "f"] & \mathbb{V}(X) \arrow[d, "{f[-]}"] \\ Y & \mathbb{V}(Y) \end{tikzcd}$$

Extension of $\mathbb{V}$ from Stone to $\mathsf{Stone}^{\mathsf{R}}$ [Goy, Petrișan, Aiguier, 2021]:

$$\mathsf{Stone} \xrightarrow{\mathbb{V}^{\mathsf{R}}} \mathsf{Stone}$$

$$\begin{tikzcd} X \arrow[d, "R"] & \mathbb{V}(X) \arrow[d, "\mathbb{V}^{\mathsf{R}}(R)"] \\ Y & \mathbb{V}(Y) \end{tikzcd}$$

(Egli-Milner:) For $K \in \mathbb{V}(X)$ and $L \in \mathbb{V}(Y)$,

$$K \ \mathbb{V}^{\mathsf{R}}(R)\ L \iff \begin{cases} \forall x \in K\ \exists y \in L: \ x\,R\,y, \\ \forall y \in L\ \exists x \in K: \ x\,R\,y. \end{cases}$$

It restricts to the usual Vietoris functor on continuous functions.

$$
\begin{tikzcd}
\mathsf{Stone}^{\mathsf{R}} \arrow[r, bend left] \arrow[loop below, "\mathbb{V}^{\mathsf{R}}"] \arrow[r, phantom, "\mathrm{op}"] & \mathsf{BA}^{\mathsf{S}} \arrow[l, bend left] \arrow[loop below, dashed, "?"]
\end{tikzcd}
$$

What is the dual of $\mathbb{V}^{\mathsf{R}}$?

$$\mathbb{V}^{\mathsf{R}} \colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$$

On objects: the same as for the dual of $\mathbb{V}$ on Stone (Abramsky, Johnstone, Kupke, Kurz, Venema, Vosmaer):

$$\begin{array}{cc} X & \mathbb{V}(X) \\ \hline A & \mathbb{K}(A) \end{array} \quad \text{Stone duality}$$

$$\mathbb{K}(A) = \frac{\mathrm{Free}_{\mathsf{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})}{\{\Box \text{ preserves finite meets}, \Diamond = \neg\Box_\neg\}}$$

On morphisms:

$$\begin{tikzcd} X \arrow[d, "R"] & & \mathbb{V}(X) \arrow[d, "\mathbb{V}^{\mathsf{R}}(R)"] \\ Y & & \mathbb{V}(Y) \end{tikzcd}$$

$$\begin{tikzcd} B & \mathbb{K}(B) = \mathrm{Free}_{\mathsf{BA}}(\{\Box_b, \Diamond_b \mid b \in B\})/{\sim} \\ A \arrow[u, "S"] & \mathbb{K}(A) = \mathrm{Free}_{\mathsf{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})/{\sim} \arrow[u, "\mathbb{K}^{\mathsf{S}}(S)?"] \end{tikzcd}$$

We shall describe when an element α of $\mathbb{K}(A)$ is $\mathbb{K}^{\mathsf{S}}(S)$-related with an element β of $\mathbb{K}(B)$.

Proposition

Given a Boolean algebra A. Every $\gamma \in \mathbb{K}(A)$ is (effectively) equal to

- (DNF) a finite join of elements of the form

$$\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b$$

  with each $a_i \leq b$;

- (CNF) a finite meet of elements of the form

$$\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m}$$

  with each $c \leq d_j$.

$$\begin{array}{ccc} A & & \mathbb{K}(A) \\ \Big\downarrow{\scriptstyle S} & & \Big\downarrow{\scriptstyle \mathbb{K}^{\mathsf{S}}(S)?} \\ B & & \mathbb{K}(B) \end{array}$$

Enough to describe when

$$(\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b) \, \mathbb{K}^{\mathsf{S}}(S) \, (\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m})$$

with $a_i \leq b$ and $c \leq d_j$.

$$
\begin{tikzcd}
A \arrow[d, hook, "\leq"] & \mathbb{K}(A) \arrow[d, hook, "\leq"] \\
A & \mathbb{K}(A)
\end{tikzcd}
$$

(With $a_i \leq b$ and $c \leq d_j$:)

$$(\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b) \leq (\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m})$$

$$\Updownarrow$$

$$(\exists i : a_i \leq c) \text{ or } (\exists j : b \leq d_j).$$

Key idea: $\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$.

E.g.: if A, B, C, D are clopens of a Stone space X with $A \subseteq C$ and $B \subseteq D$, then

$$\Diamond A \cap \Box B \subseteq \Diamond C \cup \Box D \iff A \subseteq C \text{ or } B \subseteq D.$$

$$
\begin{array}{ccc}
A & & \mathbb{K}(A) \\
\Big\downarrow S & & \Big\downarrow \mathbb{K}^{\mathsf{S}}(S) \\
B & & \mathbb{K}(B)
\end{array}
$$

(With $a_i \leq b$ and $c \leq d_j$:)

$$(\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b) \, \mathbb{K}^{\mathsf{S}}(S) \, (\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m})$$

$$\Updownarrow$$

$$(\exists i : a_i \; S \; c) \text{ or } (\exists j : b \; S \; d_j).$$

Key idea: **$\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$**.

Theorem (A., Bezhanishvili, Carai, 2024)

The dual of the Vietoris endofunctor $\mathbb{V}^{\mathsf{R}} \colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$ *is the following endofunctor* $\mathbb{K}^{\mathsf{S}} \colon \mathsf{BA}^{\mathsf{S}} \to \mathsf{BA}^{\mathsf{S}}$*:*

- *On objects: it maps* A *to*

$$\mathbb{K}(A) \coloneqq \frac{\mathrm{Free}_{\mathsf{BA}}(\{\Box_a, \Diamond_a \mid a \in A\})}{\{\Box \text{ preserves finite meets}, \Diamond = \neg\Box_\neg\}}$$

- *On morphisms: it maps a subordination* $S \colon A \nrightarrow B$ *to the unique subordination* $\mathbb{K}^{\mathsf{S}}(S) \colon \mathbb{K}(A) \nrightarrow \mathbb{K}(B)$ *satisfying "*$\Diamond$*-with-*$\Diamond$ *or* $\Box$*-with-*$\Box$*".*

"$\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$": (With $a_i \leq b$ and $c \leq d_j$:)

$$(\Diamond_{a_1} \wedge \cdots \wedge \Diamond_{a_n} \wedge \Box_b) \; \mathbb{K}^{\mathsf{S}}(S) \; (\Diamond_c \vee \Box_{d_1} \vee \cdots \vee \Box_{d_m})$$

$$\Updownarrow$$

$$(\exists i : a_i \; S \; c) \text{ or } (\exists j : b \; S \; d_j).$$

An application

De Vries duality is a duality for compact Hausdorff spaces, which associates to a compact Hausdorff space X the Boolean algebra of regular opens, together with the binary relation $\prec$ of well-insideness: $U \prec V \iff \mathrm{cl}(U) \subseteq V$.

Question (Bezhanishvili, Bezhanishvili, Harding, 2015)

What is the De Vries dual of the Vietoris endofunctor on compact Hausdorff spaces?

We piggyback on the duality between $\mathbb{V}^{\mathsf{R}} \colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$ and $\mathbb{K}^{\mathsf{S}} \colon \mathsf{BA}^{\mathsf{S}} \to \mathsf{BA}^{\mathsf{S}}$.

Theorem (A., Bezhanishvili, Carai, 2024)

The de Vries dual of the Vietoris endofunctor on KHaus *is obtained by applying* $\mathbb{K}^{\mathsf{S}}$ *(= the dual of* $\mathbb{V}^{\mathsf{R}}\colon \mathsf{Stone}^{\mathsf{R}} \to \mathsf{Stone}^{\mathsf{R}}$*), followed by a(n appropriate) MacNeille completion.*

$$X \xrightarrow{R} Y \qquad\qquad (B, \prec_B) \xleftarrow{\quad S \quad} (A, \prec_A)$$

$$\mathbb{V}(X) \xrightarrow{\mathbb{V}^{\mathsf{R}}(R)} \mathbb{V}(Y) \qquad
\begin{tikzcd}
(\mathbb{K}(B), \mathbb{K}^{\mathsf{S}}(\prec_B)) \arrow[d, hook] & (\mathbb{K}(A), \mathbb{K}^{\mathsf{S}}(\prec_A)) \arrow[l, "\mathbb{K}^{\mathsf{S}}(S)"'] \arrow[d, hook] \\
\mathbf{M}(\mathbb{K}(B), \mathbb{K}^{\mathsf{S}}(\prec_B)) & \mathbf{M}(\mathbb{K}(A), \mathbb{K}^{\mathsf{S}}(\prec_A)) \arrow[l, "\mathbf{M}(\mathbb{K}^{\mathsf{S}}(S))"']
\end{tikzcd}$$

where $\mathbf{M}$ is an appropriate MacNeille completion functor.

Conclusions

Key ideas

1. Beyond functions; closed relations between Stone spaces ($\leftrightarrow$ subordinations between Boolean algebras).
   Especially: in dualities between "KHaus"-like and "lattice+proximity"-like structures.
2. "$\Diamond$-with-$\Diamond$ or $\Box$-with-$\Box$":

$$\left(\bigwedge_i \Diamond_{a_i}\right) \wedge \Box_b \leq \Diamond_c \vee \left(\bigvee_j \Box_{d_j}\right) \Leftrightarrow (\exists i : a_i \leq c) \text{ or } (\exists j : b \leq d_j).$$

3. Our packaging of these ideas:
   - Stone dual description of $\mathbb{V}^{\mathrm{R}} \colon \mathsf{Stone}^{\mathrm{R}} \to \mathsf{Stone}^{\mathrm{R}}$;
   - de Vries dual description of $\mathbb{V} \colon \mathsf{KHaus} \to \mathsf{KHaus}$ and for relations.

M. Abbadini, G. Bezhanishvili, L. Carai.
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