Categorical dualities in logic
Syllabus

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Contents

Introduction

John Baez and James Dolan [Baez and Dolan, 2001] remarked that

> "an equation is only interesting or useful to the extent that the two sides are different."

For instance, compare

$$2 = 2 \qquad \text{with} \qquad \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

The first equality is correct but uninformative: both sides express the same object in the same language. The second one is interesting precisely because it connects two apparently different descriptions, and allows us to switch freely between them depending on which is more convenient for a given computation.

In this course, we will study an analogous phenomenon, but at the level of *mathematical structures*. On one side, we will have algebraic structures that arise naturally in logic — most notably *Boolean algebras*, which provide an algebraic semantics for classical propositional logic. On the other side, we will encounter structures of a completely different nature — in this case, *Stone spaces*, which are certain topological spaces.

The connection between these two worlds is not a literal equality, but a *categorical duality* (a.k.a. categorical dual equivalence): a two-way translation that preserves all information. This means that we can translate a problem about Boolean algebras into a corresponding problem about Stone spaces, and vice versa. In practice this is useful because on the side of Stone spaces many constructions are simpler and one can use geometric intuition.

A slogan to keep in mind is:

> *Categorical dualities in logic relate "algebras of formulas" to "spaces of models".*

We start with the simplest case: classical propositional logic, modeled by Boolean algebras and Stone spaces (Stone, 1936).

Later we will see analogous dualities for other logics, such as intuitionistic propositional logic, and possibly touch the first-order setting (where quantifiers enter the picture).

Chapter 1

Classical propositional logic: Stone duality

In the first part of the course we will see *Stone duality*: a connection (in the form of a categorical duality), between *Boolean algebras* and *Stone spaces*. Informally, Boolean algebras encode the *syntax* of classical propositional logic (algebras of formulas), while Stone spaces encode its *semantics* (spaces of models).

1.1 From classical propositional logic to Boolean algebras

1.1.1 Syntax: propositional languages and formulas

A *(propositional) language* $\mathcal{L}$ is a set; its elements are called *propositional symbols* (or also *propositional variables*). They are typically denoted by $p, q, r, \dots$.

The connectives of classical propositional logic are

$$\vee, \quad \wedge, \quad \neg, \quad 0, \quad 1,$$

where $\vee$ is a binary operation denoting *or* (sometimes called *join*), $\wedge$ is a binary operation denoting *and* (sometimes called *meet*), $\neg$ is a unary operation denoting *not* (i.e. negation, or complement), 0 is a constant symbol denoting *false* (bottom) and 1 is a constant symbol denoting *true* (top).

Definition 1.1 (Formulas)**.** The set of formulas $\mathrm{Form}(\mathcal{L})$ is defined inductively as follows:

- every propositional symbol $p \in \mathcal{L}$ is a formula;
- if $\varphi, \psi \in \mathrm{Form}(\mathcal{L})$, then $(\varphi \vee \psi) \in \mathrm{Form}(\mathcal{L})$ and $(\varphi \wedge \psi) \in \mathrm{Form}(\mathcal{L})$;
- if $\varphi \in \mathrm{Form}(\mathcal{L})$, then $(\neg\varphi) \in \mathrm{Form}(\mathcal{L})$;
- 0 and 1 are formulas.

In other words, $\mathrm{Form}(\mathcal{L})$ is the smallest set containing $\mathcal{L}$ and closed under $\vee$, $\wedge$, $\neg$, 0 and 1.

Notation 1.2 (Derived connectives)**.** We use the standard abbreviations for implications and bi-implication:

$$\varphi \to \psi \;\coloneqq\; \neg\varphi \vee \psi, \qquad \varphi \leftrightarrow \psi \;\coloneqq\; (\varphi \to \psi) \wedge (\psi \to \varphi).$$

While propositional symbols are usually denoted by $p, q, r, \dots$, formulas are typically denoted by $\varphi, \psi, \sigma, \dots$.

1.1.2 Semantics: valuations and truth tables

Set $2 \coloneqq \{0,1\}$ (the "set of truth values").

Definition 1.3 (Valuations and interpretation of formulas)**.** A *valuation* on $\mathcal{L}$ is a function $v\colon \mathcal{L} \to 2$. Given a valuation v, its unique extension to all formulas is a map

$$\bar{v}\colon \mathrm{Form}(\mathcal{L}) \longrightarrow 2$$

defined by recursion on the complexity of formulas:

$$\bar{v}(p) = v(p) \quad (p \in \mathcal{L}), \qquad \bar{v}(0) = 0, \qquad \bar{v}(1) = 1,$$

$$\bar{v}(\neg\varphi) = \neg\,\bar{v}(\varphi), \qquad \bar{v}(\varphi \vee \psi) = \bar{v}(\varphi) \vee \bar{v}(\psi), \qquad \bar{v}(\varphi \wedge \psi) = \bar{v}(\varphi) \wedge \bar{v}(\psi),$$

where on the right-hand side we use the usual Boolean operations on $2 = \{0,1\}$, which you can find in Remark 1.4 below.

Remark 1.4 (Boolean operations on $2 = \{0,1\}$)**.** By convention,

$$0 \wedge 0 = 0, \;\; 0 \wedge 1 = 0, \;\; 1 \wedge 0 = 0, \;\; 1 \wedge 1 = 1,$$

$$0 \vee 0 = 0, \;\; 0 \vee 1 = 1, \;\; 1 \vee 0 = 1, \;\; 1 \vee 1 = 1,$$

$$\neg 0 = 1, \;\; \neg 1 = 0.$$

1.1.3 Semantic equivalence

Definition 1.5 (Semantic equivalence)**.** Let $\varphi, \psi \in \mathrm{Form}(\mathcal{L})$. We write $\varphi \equiv \psi$ and say that φ and ψ are *semantically equivalent*[1] if

$$\forall v\colon \mathcal{L} \to 2, \qquad \bar{v}(\varphi) = \bar{v}(\psi).$$

In other words, φ and ψ are equivalent if and only if they have the same truth table (a function from $\mathcal{L}$ to 2 corresponds to a row of a truth table).

Example 1.6. Let $\mathcal{L} = \{p, q\}$. Then $p \vee q \equiv q \vee p$, as can be checked via the truth table

p	q	$p \vee q$	$q \vee p$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

Example 1.7. If $\mathcal{L} = \{p\}$ has one variable, then every formula is equivalent to exactly one of

$$0, \quad p, \quad \neg p, \quad 1.$$

(For instance, $\neg\neg p \equiv p$, $p \vee \neg p \equiv 1$, and $p \wedge \neg p \equiv 0$.) Thus $\mathrm{Form}(\mathcal{L})/{\equiv}$ has 4 elements.

One may prove that, in a language $\mathcal{L} = \{p, q\}$ with two propositional symbols, there are 16 equivalence classes of formulas.[2]

[1] In Definition 1.5, "semantic" is in opposition to "syntactic": two formulas are *syntactically equivalent* if they are interprovable in a certain proof system, which we do not have the time to see here. Let me just mention that Stone's representation theorem, which will be seen later and which is the core of this chapter, can be also seen as affirming that the syntactic and semantic notions of equivalence coincide.

[2] More generally, if $\mathcal{L}$ is finite of cardinality n, then $|\mathrm{Form}(\mathcal{L})/{\equiv}| = 2^{2^n}$. Indeed, every formula determines a truth function $2^{\mathcal{L}} \to 2$, and vice versa one can prove that every function $2^{\mathcal{L}} \to 2$ is the truth function of a formula (this is called "functional completeness"), so that $|\mathrm{Form}(\mathcal{L})/{\equiv}| = 2^{2^{|\mathcal{L}|}}$. In particular, for $\mathcal{L} = \{p, q\}$ we have $|\mathrm{Form}(\mathcal{L})/{\equiv}| = 2^{2^2} = 16$.

1.1.4 Adding assumptions: theories and semantic equivalence modulo a theory

We can incorporate semantic assumptions by restricting the class of admissible valuations.

Definition 1.8 (Propositional theory)**.** A *(propositional) theory* $\mathcal{T}$ in a propositional language $\mathcal{L}$ is a subset $\mathcal{T} \subseteq \mathrm{Form}(\mathcal{L})$.

Definition 1.9 (Model of a theory)**.** A *model* of a propositional theory $\mathcal{T} \subseteq \mathrm{Form}(\mathcal{L})$ is a valuation $v\colon \mathcal{L} \to 2$ such that

$$\forall \sigma \in \mathcal{T}, \qquad \bar{v}(\sigma) = 1.$$

We denote by $\mathrm{Mod}(\mathcal{T})$ the set of models of $\mathcal{T}$.

Example 1.10. Let $\mathcal{L} = \{p, q\}$ and let $\mathcal{T} = \{p \vee q\}$. Then $\mathrm{Mod}(\mathcal{T})$ consists of the three valuations

$$(p, q) = (1, 0),\ (1, 1),\ (0, 1).$$

Definition 1.11 (Semantic equivalence modulo a theory)**.** Let $\mathcal{T} \subseteq \mathrm{Form}(\mathcal{L})$ be a propositional theory and let $\varphi, \psi \in \mathrm{Form}(\mathcal{L})$. We say that φ and ψ are *semantically equivalent modulo the theory* $\mathcal{T}$ (or *relative* to $\mathcal{T}$, or *semantically* $\mathcal{T}$*-equivalent*), and write $\varphi \equiv_{\mathcal{T}} \psi$, if for every $v \in \mathrm{Mod}(\mathcal{T})$ we have

$$\bar{v}(\varphi) = \bar{v}(\psi).$$

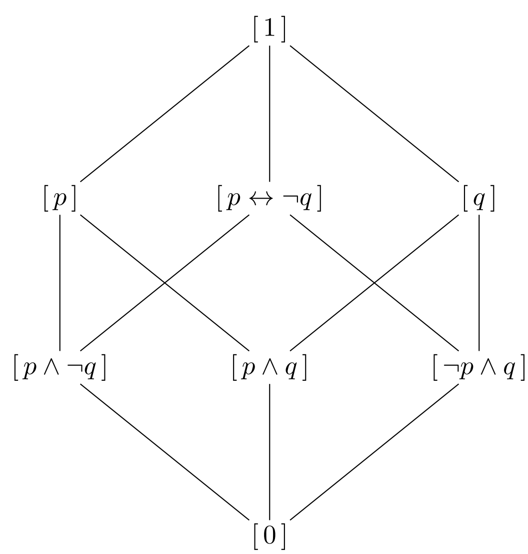
Example 1.12. Let $\mathcal{L} = \{p, q\}$ and let $\mathcal{T} = \{p \vee q\}$. One can check that $p \vee q \equiv_{\mathcal{T}} 1$, $\neg p \wedge \neg q \equiv_{\mathcal{T}} 0$ and $p \vee \neg q \equiv_{\mathcal{T}} p$.

The relation $\equiv_{\mathcal{T}}$ is again an equivalence relation, and we write $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ for the corresponding quotient set. It is convenient to partially order equivalence classes by *implication*:

$$\varphi \leq_{\mathcal{T}} \psi \quad :\Longleftrightarrow \quad \forall v \in \mathrm{Mod}(\mathcal{T}),\ \big(\bar{v}(\varphi) = 1 \Rightarrow \bar{v}(\psi) = 1\big).$$

(So "$\varphi \leq_{\mathcal{T}} \psi$" means that φ *implies* ψ on all models of $\mathcal{T}$.)

Example 1.13. Let $\mathcal{L} = \{p, q\}$ and $\mathcal{T} = \{p \vee q\}$. Then $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ has 8 elements and its Hasse diagram (with respect to $\leq_{\mathcal{T}}$) can be drawn as follows:



Boolean algebras (whose definition we will see soon) are meant to capture the algebraic structures of the form

$$\langle \mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}; \vee, \wedge, \neg, 0, 1\rangle,$$

for $\mathcal{L}$ a propositional language and $\mathcal{T}$ a propositional theory in $\mathcal{L}$; here, $\vee, \wedge, \neg, 0, 1$ are defined on $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ by setting

$$[\,\varphi\,] \vee [\,\psi\,] \;\coloneqq\; [\,\varphi \vee \psi\,], \qquad [\,\varphi\,] \wedge [\,\psi\,] \;\coloneqq\; [\,\varphi \wedge \psi\,], \qquad \neg\,[\,\varphi\,] \;\coloneqq\; [\,\neg\varphi\,], \qquad 0 \;\coloneqq\; [\,0\,], \qquad 1 \;\coloneqq\; [\,1\,].$$

(These are well-defined operations.)

In other words, Boolean algebras are meant to capture the algebras of formulas modulo a theory.

1.2 Boolean algebras

The quotient posets in Example 1.13 already display a key phenomenon: the logical connectives $\wedge$ and $\vee$ behave like "infimum" and "supremum" with respect to the implication order.

Definition 1.14 (Infimum, supremum)**.** Let $(P, \leq)$ be a poset. Given $x, y \in P$, an *infimum* (or *greatest lower bound*) of $\{x, y\}$ is an element $x \wedge y \in P$ such that

$$x \wedge y \leq x, \qquad x \wedge y \leq y,$$

and, for every $z \in P$, if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$. Dually, a *supremum* (or *least upper bound*) of $\{x, y\}$ is an element $x \vee y \in P$ such that

$$x \leq x \vee y, \qquad y \leq x \vee y,$$

and, for every $z \in P$, if $x \leq z$ and $y \leq z$ then $x \vee y \leq z$.

Infima and suprema, if exist, are unique.

One can show that, in the poset $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ ordered by implication, $[\,\varphi \wedge \psi\,]$ is the infimum of $[\,\varphi\,]$ and $[\,\psi\,]$, and $[\,\varphi \vee \psi\,]$ is the supremum of $[\,\varphi\,]$ and $[\,\psi\,]$.[3]

1.2.1 Order-theoretic definition of lattices

Definition 1.15 (Lattice)**.** A *lattice* is a poset $(L, \leq)$ in which every pair of elements admits an infimum and a supremum.

Example 1.16 (Power-set lattice)**.** Let X be a set and consider the poset $(\mathcal{P}(X), \subseteq)$. Then $(\mathcal{P}(X), \subseteq)$ is a lattice, with

$$A \wedge B \coloneqq A \cap B \qquad \text{and} \qquad A \vee B \coloneqq A \cup B.$$

1.2.2 Equational definition of lattices

Lattices can also be presented *algebraically*, by taking $\wedge$ and $\vee$ as primitive operations and listing a small family of identities. This is useful because identities are stable under the kind of constructions we will use later (products, subalgebras, quotients).

Definition 1.17 (Lattice: equational presentation)**.** A *lattice* is a set L equipped with two binary operations

$$\wedge,\ \vee : L \times L \to L$$

such that:

[3] Proof: We prove the statement for $\wedge$; the case of $\vee$ is analogous. For every model $v \in \mathrm{Mod}(\mathcal{T})$, if $\bar{v}(\varphi \wedge \psi) = 1$ then $\bar{v}(\varphi) = 1$ and $\bar{v}(\psi) = 1$, hence $[\,\varphi \wedge \psi\,] \leq_{\mathcal{T}} [\,\varphi\,]$ and $[\,\varphi \wedge \psi\,] \leq_{\mathcal{T}} [\,\psi\,]$. Now let $[\,\rho\,]$ be any lower bound of $[\,\varphi\,]$ and $[\,\psi\,]$. This means that, for every $v \in \mathrm{Mod}(\mathcal{T})$, $\bar{v}(\rho) = 1$ implies both $\bar{v}(\varphi) = 1$ and $\bar{v}(\psi) = 1$. Therefore $\bar{v}(\rho) = 1$ implies $\bar{v}(\varphi \wedge \psi) = 1$, i.e. $[\,\rho\,] \leq_{\mathcal{T}} [\,\varphi \wedge \psi\,]$.

1. (commutativity) for all $a, b \in L$, $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$;
2. (associativity) for all $a, b, c \in L$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$;
3. (absorption) for all $a, b \in L$, $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

Remark 1.18. If L is a lattice in the equational sense, one can recover an order by setting

$$a \leq b \quad :\Longleftrightarrow \quad a \wedge b = a,$$

equivalently $a \vee b = b$. With this order, the operations $\wedge$ and $\vee$ are precisely infimum and supremum, so Definitions 1.15 and 1.17 are equivalent viewpoints.

1.2.3 Bounded lattices

Definition 1.19 (Bounded lattice)**.** A lattice is *bounded* if it has a least element 0 and a greatest element 1, i.e. elements such that $0 \leq a \leq 1$ for all a. Equivalently (in the equational presentation), 0 and 1 satisfy

1. For all a, $a \wedge 0 = 0$ (equivalently: $a \vee 0 = a$),
2. For all a, $a \vee 1 = 1$ (equivalently; $a \wedge 1 = a$).

1.2.4 Distributive lattices

Definition 1.20 (Distributive lattice)**.** A lattice L is *distributive* if it satisfies either (and hence both) of the following equivalent identities:

1. for all $a, b, c \in L$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.
2. for all $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

Example 1.21. For every set X, the power-set lattice $\mathcal{P}(X)$ is distributive.

Two small lattices play a special role as "minimal" obstructions to distributivity. They are usually denoted by M_3 (the *diamond*) and N_5 (the *pentagon*).

M_3 N_5 (1.1)

These are not distributive; one can easily verify that $a \vee (b \wedge c) \not(a \vee b) \wedge (a \vee c)$.

In fact, one can prove that a lattice L is distributive if and only if it does not contain a sublattice isomorphic to M_3 or N_5, i.e., there is no injective map $M_3 \to L$ or $N_5 \to L$ preserving both $\wedge$ and $\vee$.[4]

[4]For a textbook reference, see, e.g., [Davey and Priestley, 2002, 4.10].

1.2.5 Boolean algebras

Definition 1.22 (Boolean algebra)**.** A *Boolean algebra* is an algebraic structure

$$(B, \vee, \wedge, \neg, 0, 1)$$

such that:

1. $(B, \vee, \wedge, 0, 1)$ is a bounded distributive lattice;
2. for every $a \in B$,
$$a \wedge \neg a = 0 \qquad \text{and} \qquad a \vee \neg a = 1.$$

The element $\neg a$ is called the *complement* of a.

Remark 1.23. All axioms in the definition of Boolean algebras are *equational*, i.e. they are identities of the form
$$\forall x_1, \dots, x_n, \quad t_1(x_1, \dots, x_n) = t_2(x_1, \dots, x_n)$$
between algebraic terms.

Remark 1.24. A Boolean algebra is completely determined by its underlying partial order. This is because: $\vee, \wedge, 0, 1$ are the binary supremum, binary infimum, smallest element and greatest element, and moreover, in every bounded distributive lattice L, if an element a has a complement (i.e., there is an element b such that $a \wedge b = 0$ and $a \vee b = 1$) it is unique.[5]

Example 1.25. The prototypical example of a Boolean algebra is $2 = \{0, 1\}$ with partial order $0 \le 1$ is a Boolean algebra. The Boolean operations are those described in Remark 1.4.

Proposition 1.26 ($\mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}}$ is a Boolean algebra)**.** *Let $\mathcal{L}$ be a propositional language and $\mathcal{T} \subseteq \mathrm{Form}(\mathcal{L})$ a theory. Then the quotient set $\mathrm{Form}(\mathcal{L})/{\equiv_\mathcal{T}}$ becomes a Boolean algebra by setting*

$$[\,\varphi\,] \vee [\,\psi\,] \coloneqq [\,\varphi \vee \psi\,], \qquad [\,\varphi\,] \wedge [\,\psi\,] \coloneqq [\,\varphi \wedge \psi\,], \qquad \neg\,[\,\varphi\,] \coloneqq [\,\neg\varphi\,], \qquad 0 \coloneqq [\,0\,], \qquad 1 \coloneqq [\,1\,].$$

Proof sketch. Well-definedness follows from the fact that $\equiv_\mathcal{T}$ is a congruence with respect to the connectives. The Boolean identities hold because they are valid in the two-element Boolean algebra $2 = \{0, 1\}$ (i.e. they are tautologies), hence they hold for equivalence classes modulo $\equiv_\mathcal{T}$. □

Proof sketch. Step 1: the operations are well defined. We show this for $\vee$; the cases of $\wedge$ and $\neg$ are analogous. Suppose $\varphi \equiv_\mathcal{T} \varphi'$ and $\psi \equiv_\mathcal{T} \psi'$. Then for every $v \in \mathrm{Mod}(\mathcal{T})$ we have $\bar{v}(\varphi) = \bar{v}(\varphi')$ and $\bar{v}(\psi) = \bar{v}(\psi')$. Using the truth table for $\vee$ in $2 = \{0, 1\}$ we compute:

$$\bar{v}(\varphi \vee \psi) = \bar{v}(\varphi) \vee \bar{v}(\psi) = \bar{v}(\varphi') \vee \bar{v}(\psi') = \bar{v}(\varphi' \vee \psi').$$

Hence $\varphi \vee \psi \equiv_\mathcal{T} \varphi' \vee \psi'$, and so $[\,\varphi\,] \vee [\,\psi\,]$ does not depend on the choice of representatives. The constants $0 = [\,0\,]$ and $1 = [\,1\,]$ are trivially well defined.

Step 2: the Boolean algebra identities hold. Example (one Boolean identity: complements). Fix $\varphi \in$

[5] This is not true for arbirary bounded lattice: for example, in the bounded lattice M_3 in (1.1), both b and c are complements of a, and in N_5 both a and c are complements of b.

Form($\mathcal{L}$). For every $v \in \mathrm{Mod}(\mathcal{T})$ we compute

$$\bar{v}(\varphi \wedge \neg\varphi) = \bar{v}(\varphi) \wedge \bar{v}(\neg\varphi) = \bar{v}(\varphi) \wedge \neg\bar{v}(\varphi) = 0 = \bar{v}(0),$$

where we used the recursive definition of $\bar{v}$ and the fact that $b \wedge \neg b = 0$ for all $b \in 2$. Hence $\varphi \wedge \neg\varphi \equiv_{\mathcal{T}} 0$, and therefore

$$[\,\varphi\,] \wedge \neg\,[\,\varphi\,] = [\,0\,] = 0.$$

This is the general mechanism: any identity between Boolean terms can be checked pointwise in 2 under every valuation. To give more details, let $t_1(x_1, \dots, x_n) = t_2(x_1, \dots, x_n)$ be any Boolean algebra identity (an equation between terms built from $\vee, \wedge, \neg, 0, 1$). To check it in $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$, pick arbitrary formulas $\varphi_1, \dots, \varphi_n \in \mathrm{Form}(\mathcal{L})$ and consider the two formulas $t_1(\varphi_1, \dots, \varphi_n)$ and $t_2(\varphi_1, \dots, \varphi_n)$. For every $v \in \mathrm{Mod}(\mathcal{T})$, the evaluation map $\bar{v}\colon \mathrm{Form}(\mathcal{L}) \to 2$ respects the connectives, hence

$$\bar{v}\big(t_i(\varphi_1, \dots, \varphi_n)\big) = t_i\big(\bar{v}(\varphi_1), \dots, \bar{v}(\varphi_n)\big) \qquad (i = 1, 2).$$

Since the identity $t_1 = t_2$ holds in the two-element Boolean algebra 2, the right-hand sides are equal for all $v \in \mathrm{Mod}(\mathcal{T})$, so $t_1(\varphi_1, \dots, \varphi_n) \equiv_{\mathcal{T}} t_2(\varphi_1, \dots, \varphi_n)$. Therefore the induced operations on equivalence classes satisfy all Boolean algebra axioms. □

Definition 1.27 (Lindenbaum–Tarski algebra)**.** Let $\mathcal{T}$ be a propositional theory in a propositional language $\mathcal{L}$. The Boolean algebra

$$\langle \mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}; \vee, \wedge, \neg, 0, 1\rangle$$

with operations

$$[\,\varphi\,] \vee [\,\psi\,] \coloneqq [\,\varphi \vee \psi\,], \qquad [\,\varphi\,] \wedge [\,\psi\,] \coloneqq [\,\varphi \wedge \psi\,], \qquad \neg\,[\,\varphi\,] \coloneqq [\,\neg\varphi\,], \qquad 0 \coloneqq [\,0\,], \qquad 1 \coloneqq [\,1\,]$$

is called the *Lindembaum–Tarski algebra* of $\mathcal{T}$.

Later on, we will see that also the converse holds: every Boolean algebra is isomorphic to $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ for some language $\mathcal{L}$ and some theory $\mathcal{T}$. This will be a consequence of Stone's representation theorem (or of the crucial lemmas used in proving Stone's representation theorem). It will give a guarantee that the definition of Boolean algebras is "the correct one", meaning that they correctly axiomatize Boolean

Example 1.28 (Power set Boolean algebras)**.** Let X be a set. The power set $\mathscr{P}(X)$ with the inclusion order is a Boolean algebra, with operations

$$A \wedge B \coloneqq A \cap B, \qquad A \vee B \coloneqq A \cup B, \qquad \neg A \coloneqq X \setminus A, \qquad 0 \coloneqq \varnothing, \qquad 1 \coloneqq X.$$

Proof sketch. It is easily seen to be a bounded lattice. Distributivity can be checked by "element chasing". For instance, for $A, B, C \subseteq X$ we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

because, for every $x \in X$,

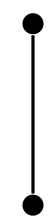
$$x \in A \cap (B \cup C) \iff x \in A \text{ and } (x \in B \text{ or } x \in C) \iff x \in A \cap B \text{ or } x \in A \cap C \iff x \in (A \cap B) \cup (A \cap C).$$

The complement axioms are immediate:

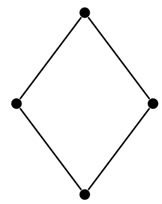
$$A \cap (X \setminus A) = \varnothing, \qquad A \cup (X \setminus A) = X.$$ □

Here below, we draw the Hasse diagrams of the power sets of sets of cardinality 0, 1, 2, 3. These are the smallest Boolean algebras, and probably the only ones that one can draw without losing one's sight. (The next one has 16 elements.)

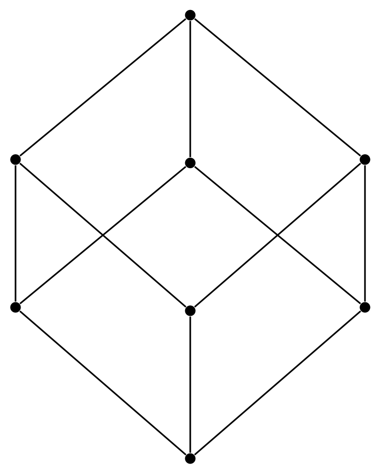
The Boolean algebra $\mathscr{P}(\varnothing)$, with cardinality 1



The Boolean algebra $\mathscr{P}(\{x\})$, with cardinality 2



The Boolean algebra $\mathscr{P}(\{x,y\})$, with cardinality 4



The Boolean algebra $\mathscr{P}(\{x,y,z\})$, with cardinality 8

There are examples of Boolean algebras that are not isomorphic to any power set. The following one is probably the simplest example.

Example 1.29 (The algebra of finite and cofinite subsets of $\mathbb{N}$)**.** Consider the collection

$$\mathscr{P}_{\text{fin, cofin}}(\mathbb{N}) \;\coloneqq\; \{\, A \subseteq \mathbb{N} \mid A \text{ is finite, or } \mathbb{N} \setminus A \text{ is finite}\,\} \;\subseteq\; \mathscr{P}(\mathbb{N}).$$

Elements of $\mathscr{P}_{\text{fin, cofin}}(\mathbb{N})$ are called *finite* or *cofinite* subsets of $\mathbb{N}$.

To show that this is indeed a Boolean algebra, the following (which is not difficult to prove) is of help.

Remark 1.30 (Boolean subalgebras of power sets)**.** If $\mathcal{A} \subseteq \mathscr{P}(X)$ contains $\varnothing$ and X and is closed under $\cup$, $\cap$ and complement in X, then $\mathcal{A}$ is a Boolean algebra, as well.

Now one can prove that $\mathscr{P}_{\text{fin, cofin}}(\mathbb{N})$ is a Boolean algebra:

- $\varnothing$ and $\mathbb{N}$ belong to $\mathscr{P}_{\text{fin, cofin}}(\mathbb{N})$;
- $\mathscr{P}_{\text{fin, cofin}}(\mathbb{N})$ is closed under complement in $\mathbb{N}$: if A is finite then $\mathbb{N} \setminus A$ is cofinite, and if A is cofinite then $\mathbb{N} \setminus A$ is finite;
- $\mathscr{P}_{\text{fin, cofin}}(\mathbb{N})$ is closed under unions: if A and B are finite then $A \cup B$ is finite, while if at least one of A, B is cofinite then $A \cup B$ is cofinite.

- Analogously, $\mathscr{P}_{\mathrm{fin,\,cofin}}(\mathbb{N})$ is closed under intersections.

The Boolean algebra $\mathscr{P}_{\mathrm{fin,\,cofin}}(\mathbb{N})$ is infinite but much smaller than $\mathscr{P}(\mathbb{N})$: for instance, the set of even numbers is neither finite nor cofinite, hence it does not belong to $\mathscr{P}_{\mathrm{fin,\,cofin}}(\mathbb{N})$. Moreover, $\mathscr{P}_{\mathrm{fin,\,cofin}}(\mathbb{N})$ is *countable*. This also shows that it is not isomorphic to any power set, since no power set is countable.

1.3 Stone's Representation Theorem

Remark 1.30 gives a zoo of examples: every *Boolean subalgebra* of some power set is a Boolean algebra. Stone's representation theorem states that all Boolean algebras are of this form!:

Theorem 1.31 (Stone's representation theorem for Boolean algebras [Stone, 1936])**.** *Every Boolean algebra B is isomorphic to a Boolean subalgebra of a power set Boolean algebra, i.e., there are a set X and an injective map*

$$\iota\colon B \hookrightarrow \mathscr{P}(X)$$

such that, under ι, the operations on B correspond to intersection, union, complement, empty set and whole set in X:

$$\iota(a \wedge b) = \iota(a) \cap \iota(b), \qquad \iota(a \vee b) = \iota(a) \cup \iota(b), \qquad \iota(\neg a) = X \setminus \iota(a), \qquad \iota(0) = \varnothing, \qquad \iota(1) = X.$$

The next goal is to prove Stone's representation theorem for Boolean algebras. We will first need to present auxiliary notions and lemmas.

Given a Boolean algebra B, how can we find a set X such that B embeds into the power set $\mathscr{P}(X)$ of X, as required by the statement of Stone's Representation Theorem?

Idea, from a logical perspective: In the special Boolean algebra $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ (for $\mathcal{L}$ a language and $\mathcal{T}$ a theory), an equivalence class $[\,\varphi\,]$ can be identified with the set of models $v \in \mathrm{Mod}(\mathcal{T})$ such that $\bar{v}(\varphi) = 1$; i.e., a formula can be identified with the models that satisfy it. This gives a very concrete embedding into a power set: the power set of models of $\mathcal{T}$. So, the idea for $B = \mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ is to take

$$X = \mathrm{Mod}(\mathcal{T}).$$

To translate this idea to a general Boolean algebra B, we note that a model

$$v\colon \mathcal{L} \longrightarrow 2$$

of $\mathcal{T}$ induces a function

$$\bar{v}\colon \mathrm{Form}(\mathcal{L}) \longrightarrow 2$$

which passes to the quotient

$$\begin{aligned} \mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}} &\longrightarrow 2 \\ [\varphi] &\longmapsto \bar{v}(\varphi). \end{aligned}$$

In fact, one can prove that the models of $\mathcal{T}$ are in bijection with the Boolean homomorphisms (i.e., functions preserving all Boolean connectives, see Definition 1.32 below) from $\mathrm{Form}(\mathcal{L})/{\equiv_{\mathcal{T}}}$ to 2. This suggests that, for a general Boolean algebra B, we shall take

$$X = \hom(B, 2),$$

the set of homomorphisms from B to 2.

Idea, from another perspective: If $B \subseteq \mathscr{P}(X)$ is a Boolean subalgebra of $\mathscr{P}(X)$, then every element $x \in X$ induces a function

$$\begin{aligned} B &\longrightarrow 2 \\ A &\longmapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

which is a Boolean homomorphism. This suggests a close relationship between X and $\hom(B,2)$. This suggests, from yet another perspective, that to prove Stone's Representation theorem we shall take

$$X = \hom(B,2).$$

Definition 1.32 (Boolean homomorphism)**.** Let A and B be Boolean algebras. A *Boolean homomorphism* (or, simply, a *homomorphism*) $f\colon A \to B$ is a function such that, for all $a, b \in A$,

$$f(1) = 1, \qquad f(a \wedge b) = f(a) \wedge f(b), \qquad f(\neg a) = \neg f(a).$$

Remark 1.33. From the defining equations one immediately gets $f(0) = 0$ and $f(a \vee b) = f(a) \vee f(b)$. In other words, a Boolean homomorphism is a function that preserves all the basic logical operations: $\wedge$, $\vee$, $\neg$, 0, 1.

1.3.1 Ultrafilters

A homomorphism $f\colon B \to 2$ can be encoded via $f^{-1}[\{1\}]$.

Definition 1.34 (Filter)**.** Let B be a Boolean algebra. A *filter* of B is a subset $F \subseteq B$ such that:

1. F is *upward closed*, i.e., if $x \in F$ and $x \leq y$, then $y \in F$;
2. F is *closed under finite meets*, i.e.,
   (a) if $x, y \in F$, then $x \wedge y \in F$;
   (b) $1 \in F$.

A filter F is *proper* if $0 \notin F$.

Definition 1.35 (Ideal)**.** Let B be a Boolean algebra. An *ideal* of B is a subset $I \subseteq B$ such that:

1. I is *downward closed*, i.e., if $x \in I$ and $y \leq x$, then $y \in I$;
2. I is *closed under finite joins*, i.e.,
   (a) if $x, y \in I$, then $x \vee y \in I$;
   (b) $0 \in I$.

Definition 1.36 (Ultrafilter)**.** An *ultrafilter* of B is a filter U of B satisfying any (hence all) of the following equivalent conditions:

1. for every $x \in B$, exactly one of x and $\neg x$ belongs to U;
2. U is proper and is maximal among proper filters (ordered by inclusion);
3. $B \setminus U$ is an ideal. This amounts to the following conditions[6]:
   (a) $0 \notin U$;
   (b) if $x \vee y \in U$, then $x \in U$ or $y \in U$.

Proposition 1.37. *For every Boolean algebra B there is a bijection*

$$\begin{aligned} \hom(B,2) &\longleftrightarrow \{\textit{ultrafilters of } B\}, \\ h &\longmapsto h^{-1}[\{1\}], \\ \left(x \mapsto \begin{cases} 1, & x \in U, \\ 0, & x \notin U, \end{cases} \right) &\longleftarrow U. \end{aligned}$$

[6] Note that the fact that $B \setminus U$ is downward closed follows from the fact that U is upward closed.

Sketch. If $h\colon B \to 2$ is a homomorphism, set $U \coloneqq h^{-1}[\{1\}]$. Then $1 \in U$, U is upward closed and closed under meets, so it is a filter; it is proper since $h(0) = 0$. Moreover, for each $x \in B$, $h(\neg x) = \neg h(x)$, and hence exactly one of x and $\neg x$ lands in 1. Thus, U satisfies Definition 1.36(1) and so is an ultrafilter.

Conversely, if U is an ultrafilter, define $h_U\colon B \to 2$ by $h_U(x) = 1$ if and only if $x \in U$. The ultrafilter axioms ensure that h_U preserves $\wedge$, $\neg$, and 1, hence h_U is a homomorphism. $\square$

Therefore, ultrafilters of B are encodings of homomorphisms from B to 2. Thus, we will use the set of ultrafilters of B as the set X such that B embeds into $\mathscr{P}(X)$.

Bibliography

[Baez and Dolan, 2001] Baez, J. C. and Dolan, J. (2001). From finite sets to Feynman diagrams. In *Mathematics unlimited—2001 and beyond*, pages 29–50. Springer, Berlin. 2

[Davey and Priestley, 2002] Davey, B. A. and Priestley, H. A. (2002). *Introduction to lattices and order.* Cambridge: Cambridge University Press, 2nd ed. edition. 7

[Stone, 1936] Stone, M. H. (1936). The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40(1):37–111. 11