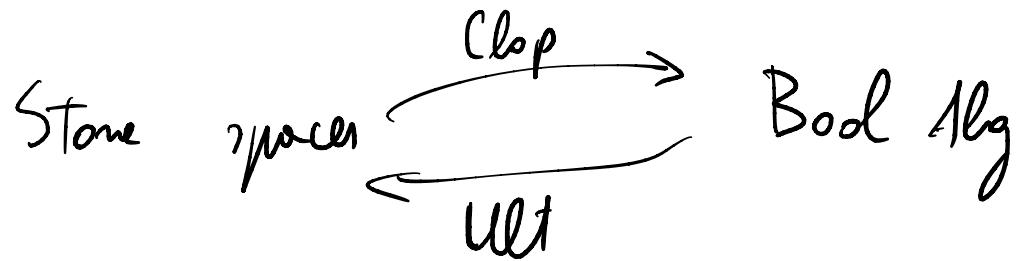
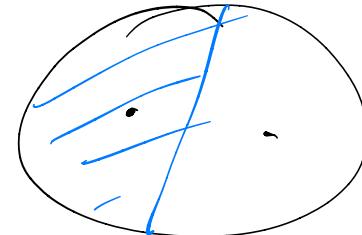


LAST TIME:

Stone spaces:
 ↗ totally separated (= clopens separate the points)
 ↗ compact



$$B \xrightarrow{\eta_B} \text{Clop Ult}(B)$$

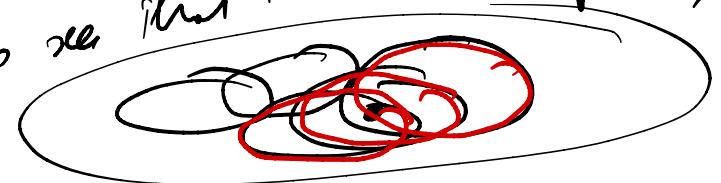
$$X \simeq \text{Ult}(\text{Clop}(X))$$

THM
 Let X be a Stone space. The function

$$\begin{aligned} \varepsilon_x: X &\longrightarrow \text{Ult}(\text{Clop}(X)) \\ x &\mapsto \{A \in \text{Clop}(X) \mid x \in A\} \end{aligned}$$

is a homeomorphism

Easy to see that in an ultrafilter x



PROOF

Enough \rightarrow CONTINUOUS (IMMEDIATE)
 \rightarrow INJ. (TOTAL SEPARATION)
 \rightarrow SURJ. (COMPACTNESS)

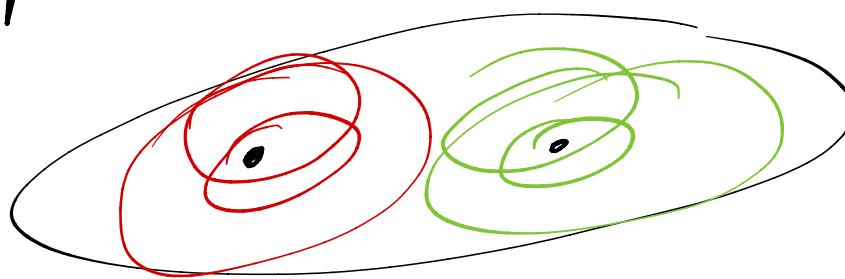
CONTINUITY: enough to prove that the preimage of basic open set

is open : $\mathcal{E}_x^{-1}[\eta_B(A)] = A$ is open.

$\nearrow \text{clsp}(x)$

basic open set of $\text{Ult}(x)$

INJ: Let $x \neq y$. By total separation, there is a clopen A s.t. $x \in A$
 $y \notin A$

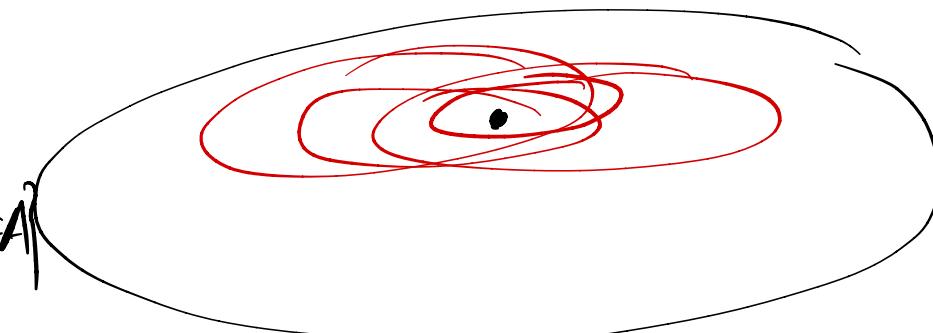


$A \in \mathcal{E}_x(x)$
 \cap $\not\in$
 $\mathcal{E}_x(y)$

SURJ: Let $\mathcal{U} \in \text{Ult}(\text{Cl}_\text{op}(x))$

GOAL: find $x \in X$ s.t.

$$\mathcal{U} = \mathcal{E}_x(x) = \{A \in \text{Cl}_\text{op}(x) \mid x \in A\}$$



CLAIM: $\bigcap_{A \in \mathcal{U}} A \neq \emptyset$

The family $\{A\}_{A \in \mathcal{U}}$ has the FIP.
This proves the claim.

$$\rightarrow \exists x \in \bigcap_{A \in \mathcal{U}} A$$

Then it is easily seen that

$$\mathcal{U} = \mathcal{E}_x(x), \text{ i.e.}$$

$$\mathcal{U} = \{A \in \text{Cl}_\text{op}(x) \mid x \in A\}$$

\subseteq EASY

? We prove the contrapositive:
 $A \notin \mathcal{U} \Rightarrow \forall A \in \mathcal{U} \Rightarrow x \notin A$

RETR

Let X be a compact space.
Let $\{A_i\}_{i \in I}$ be a family of closed sets.
If $\{A_i\}_{i \in I}$ has the FINITE INTERSECTION PROPERTY (FIP): i.e. for every finite $J \subseteq I$, $\bigcap_{i \in J} A_i \neq \emptyset$.
then $\bigcap_{i \in I} A_i \neq \emptyset$

If

- each $A_i \neq \emptyset$
- $\{A_i\}_{i \in I}$ is closed under finite intersection

 then it has the FIP.

$$A \notin \mathcal{U} \Rightarrow \forall A \in \mathcal{U} \Rightarrow x \notin A$$

Bool. Alg. \hookrightarrow Stone spaces

From $B \cong \text{Clop Ult}(B)$ we deduce

If B is finite, then B is isomorphic to a power set of a finite set.

Indeed: $B \text{ fin} \Rightarrow \text{Ult}(B) \text{ is finite}$

\Downarrow

$\text{Ult}(B)$ is a finite discrete space

\Downarrow

$\text{Clop}(\text{Ult}(B)) = \wp(\text{Ult}(B))$

\Downarrow

If two Bool. alg. have the same cardinality, they are isom.
 $\wp(X) \quad \wp(Y) \quad 2^n = 2^m \Rightarrow n = m$

Let $L = \{p_0, p_1, p_2, \dots\}$, we will prove

"There is no propositional theory that encodes the information "exactly one among all p_i 's holds"

$$\forall i \neq j \neg(p_i \wedge p_j)$$

$$\bigvee_{i \in N} p_i$$

REWRITE MORE FORMALLY:

"There is no propositional theory T such that

$$\text{Mod}(T) = \{v : L \rightarrow \{0,1\} \mid \text{exactly one among all } v(p_i), i \geq 1\}$$

REMARK

For any prop. theory T , $\text{Mod}(T)$ is a Stone space,
with clopens: for every formula $\varphi \in \text{Form}(\mathcal{L})$

$$\{\tau \in \text{Mod}(T) \mid \underbrace{\tau \models \varphi}_{\tilde{\tau}}\}$$

SKETCH of PROOF

$$\text{Mod}(T) \hookrightarrow \text{Ult}\left(\text{Form}(\mathcal{L}) / \equiv_T\right)$$

$$v \mapsto \{\tilde{\tau}[\varphi] \mid \tilde{\tau} \models \varphi\}$$

$$\begin{aligned} \mathcal{L} &\xrightarrow{\text{forget}} \\ n &\mapsto \begin{cases} 1 & [\tilde{\tau}] \in v \\ 0 & [\tilde{\tau}] \notin v \end{cases} \end{aligned}$$

If BWOC there was a theory T with the above property

$$\text{Mod}(T) = \bigcup_{n \in \mathbb{N}} \{v \in \text{Mod}(T) \mid v \models p_n\}$$

closures

By compactness, there would be $I \subseteq \mathbb{N}$ finite s.t.

$$\text{Mod}(T) = \bigcup_{n \in I} \{v \in \text{Mod}(T) \mid v \models p_n\}$$

Take $m \in \mathbb{N} \setminus I$

$$v_m: L \rightarrow 2$$
$$i \mapsto \begin{cases} 1 & i = m \\ 0 & \text{if not} \end{cases}$$

$$m \in I: v_m \not\models p_m$$

One can prove, as a consequence of the fact that the spaces of models are compact.

[THM (Compactness theorem of classical prop. logic)]

Let T be a prop. theory in a prop. language L , $\varphi \in \text{Form}(L)$

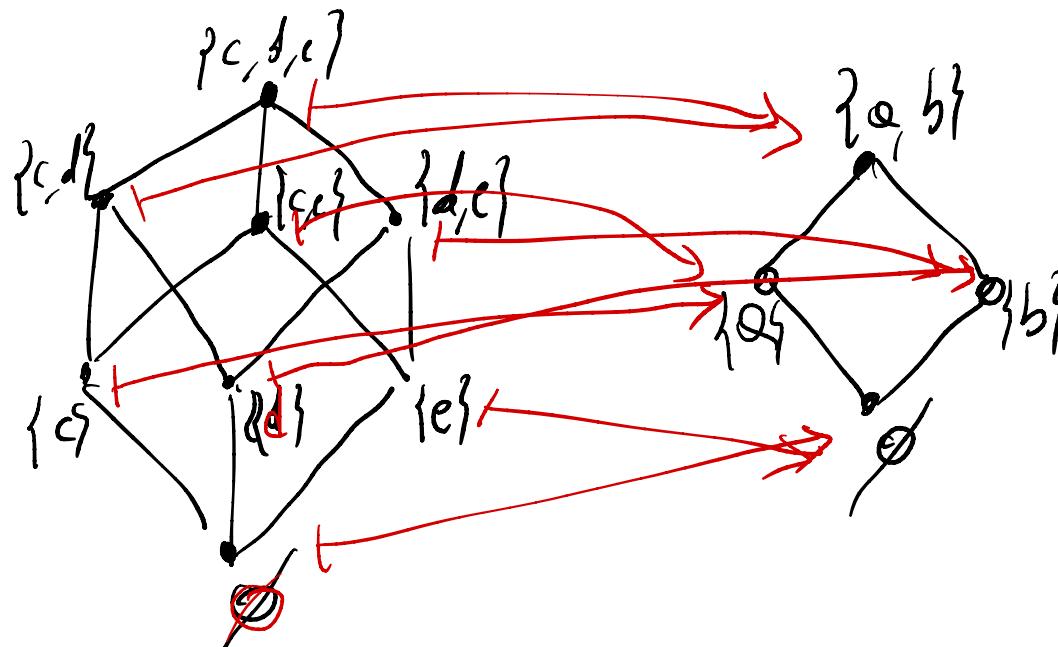
Suppose $T \models \varphi$ (i.e. for every model \mathcal{V} of T , $\mathcal{V} \models \varphi$
 $(\mathcal{V}(\varphi) = 1)$)

Then there is a finite $T' \subseteq T$ s.t. $T' \models \varphi$

We skip the proof.

(Logical meaning of Bool. hom.:
interpretation of formulas)

Stone model a. sign. of Bool. hom.

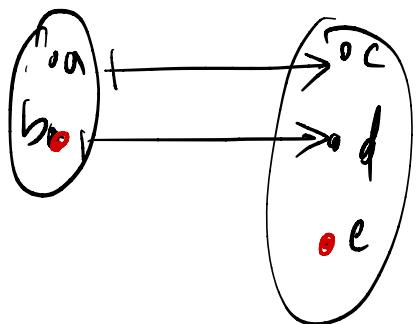


i j e



a b

$$z^2 = g$$



$$f: \{a, b\} \rightarrow \{c, d, e\}$$

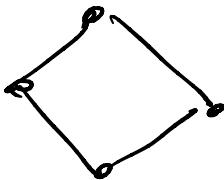
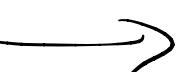
$$\rho(\{c, d, e\}) \rightarrow \rho(\{a, b\})$$

$$z \mapsto f^{-1}[z]$$

$$FC(N) \xrightarrow{\text{Bd. ? } \text{len}} \text{a polygon}$$

$$\begin{array}{ccc} N \cup \{\infty\} & \xleftarrow{\text{continuous}} & \{x, y\} \\ \infty & \xleftarrow{x: f} & y \end{array}$$

$$\begin{array}{ccc} \text{Chop}(N \cup \{\infty\}) & \rightarrow & \text{Chop}(\{x, y\}) \\ A & \xrightarrow{\quad} & f^{-1}[A] \end{array}$$

$\text{FC}(N)$  $F+$

$$\begin{cases} 1 \\ 0 \end{cases}$$

$F_{\text{cof.}}$
 F_{fine}

 $\{x, y\}$
 \emptyset
 ∞ctd
 ∞fd

$\text{BA} :=$ category of Bdd. algs and Bdd. hom.

$\text{Stone} :=$ " Stone spaces and cont. maps

THM (Stone duality for Bdd. algs.)

The categories BA and Stone are dually equivalent, i.e.

$$\text{BA} \simeq \text{Stone}^{\text{op}}$$

Roughly speaking: \rightarrow "bijection on objects"

\rightarrow "bijection on morphism, but reversing the direction".

Functions:

$$\text{clap}: \text{Stone}^{\text{op}} \rightarrow \text{BA}$$
$$x \mapsto \text{clap}(x)$$

$$\begin{array}{ccc} Y & & \text{clap}(y) \\ \uparrow f & & \downarrow \\ X & & \text{clap}(x) \end{array}$$
$$f^{-1}[-]: z \mapsto f^{-1}[z]$$

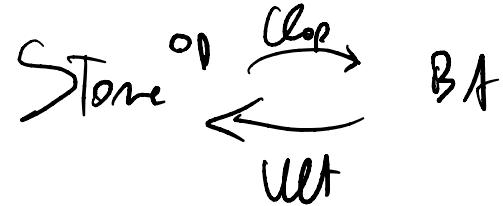
$$\text{Ult}: \text{BA} \rightarrow \text{Stone}^{\text{op}}$$
$$B \mapsto \text{Ult}(B)$$

$$\begin{array}{ccc} A & & \text{Ult}(A) & & f^{-1}[U] \\ \downarrow f & & \uparrow f^{-1}[-] & & \uparrow \\ B & & \text{Ult}(B) & \ni & U \end{array}$$

NAT.ISO :

$$\text{UNIT: } \eta_B : B \rightarrow \text{Clop Ult}(B)$$
$$f \downarrow \quad \quad \downarrow \text{Clop Ult}(f)$$
$$A \rightarrow \text{Clop Ult}(B)$$

$$\text{COUNIT: } \varepsilon_x : X \rightarrow \text{Ult Clop}(X)$$
$$f \downarrow \quad \quad \downarrow \text{Ult Clop}(f)$$
$$Y \rightarrow \text{Ult Clop}(X)$$



it is an ISO, as we proved
~~some times~~ some times.

NAT. TRANSF,

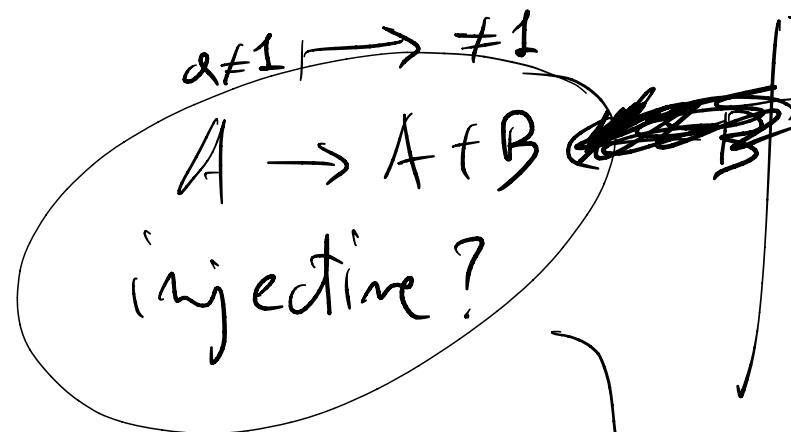
it is an iso because of the first theorem
of today.



It can be easily checked that it is
an adj.
□

GUIDING QUESTION:

given A, B Bool algs, is the coproduct map



Every equat. defined class of algebra

is complete (EASY: limits are)
in Set

↪ cocomplete (HARD: colimits)

NOT
in Set

$$2 + 2 = 2$$

product map

$$X \times Y \rightarrow X$$

? unj.

B_A	stars
2	{*}
+	X
2	{*}
"	
2	{*}

1. Explain the logical reading of the quest.
 2. ~~Ex~~ Lint₁ in Stone are computed on in S₁
 3. inj \leftrightarrow surj. surj \leftrightarrow inj.
 4. Solve The problem in Stone spaces, get back to BA (and Log_2)
-

$$A \simeq \text{Form}(\mathcal{L}_1) \not\equiv_{T_1} B \simeq \text{Form}(\mathcal{L}_2) \not\equiv_{T_2}$$

Let T_1, T_2 be m.p. theories in languages $\mathcal{L}_1, \mathcal{L}_2$, resp.

For all $\varphi \in \text{Form}(\mathcal{L}_1)$, s.t. $T_1 \models \varphi$ (i.e. there is a model \mathfrak{U} of T)
 s.t. $\overline{\mathfrak{U}}(\varphi) = 0$

$$\underbrace{T_1 \amalg T_2 \not\models \varphi}_{\text{in } \mathcal{L}_1 \amalg \mathcal{L}_2} ?$$

in $\mathcal{L}_1 \amalg \mathcal{L}_2$

$$\mathcal{L}_1 = \{p, q\}$$

$$T_1 = \{p \vee q\}$$

$$T_1 \not\models p \wedge q, \text{ because } v: \begin{array}{l} p \mapsto 1 \\ q \mapsto 0 \end{array}$$

$$\mathcal{L}_2 = \{r\}$$

$$T_2 = \{\neg r\}$$

$$T_1 \amalg T_2 \not\models \underline{p \wedge q}, \text{ in } v: \begin{array}{l} p \mapsto 0 \\ q \mapsto 0 \\ r \mapsto 0 \end{array}$$

$\{p \vee q, \neg r\}$

$$\frac{\text{For}(\mathcal{L}_1) \models_{T_1} \text{For}(\mathcal{L}_2) \models_{T_2}}{\text{For}(\mathcal{L}_1 \amalg \mathcal{L}_2) \models_{(T_1 \amalg T_2)}}$$

$$\text{Mod}(T_1 \amalg T_2) \hookrightarrow \text{Mod}(T_1) \times \text{Mod}(T_2)$$

$$A \rightarrow A \times B$$

$$(X \times Y) \rightarrow X$$

2. LIMITS in Stone: easy: or in Set (as in Top)

Examples: Term obj: $\{x\}$

Bim. prod. $X_1 \times X_2$ cart. prod. with the product topology

Arbr. prod. $\prod_i X_i$ cart. prod. with the product topology.

(Tychonoff's theorem: arbitrary product of
compact spaces is compact)

↑

Ax. of Choice

Equivalenz: $Z \hookrightarrow X \rightrightarrows Y$

$\text{Eq}(f, g) = \{x \in X \mid f(x) = g(x)\}$, equipped with the
subspace topology.

This is a Stone space because:

- Enough to prove that

$\{x \in X \mid f(x) = g(x)\}$ is closed
in X .

it is the preimage ~~of~~ under

$$X \xrightarrow{\langle f, g \rangle} Y \times Y$$
$$x \mapsto (f(x), g(x))$$

of the diagonal $\{(y, y) \mid y \in Y\}$, ~~which~~

is closed because

Y is Hausdorff.

(Hausdorff = diagonal closed)

LEMMA:
Every closed subspace
of a Stone space
is a Stone space
(with the subspace
Topology) \square

THM

The functors

$\text{Stone} \rightarrow \text{Set}$, $\text{Stone} \rightarrow \text{Top}$

preserve limit

BTW,

merges finite colimit (but not arbitrary colimit)

Proof See above for the limit

3.

THM ($\text{inj} \leftrightarrow \text{surj}$, $\text{surj} \leftrightarrow \text{inj}$)

Let X, Y be Stone space, and $f: X \rightarrow Y$ a cont. map.

① f is inj. iff $f^{-1}[-]: \text{Clop}(Y) \rightarrow \text{Clop}(X)$ is surj.

② f is surj iff $f^{-1}[-]: \text{Clop}(Y) \rightarrow \text{Clop}(X)$ is inj.

$$\begin{array}{ccc} A & \xrightarrow{\text{inj?}} & A \times B \\ \boxed{X \times Y} & \xrightarrow{\text{surj?}} & X \end{array}$$

Question from the students:

Does $\text{Stone} \rightarrow \text{Set}$ have a left adjoint?

(It cannot have a right adjoint, because it does not preserve colimits)

YES, it has a left adjoint $\text{Set} \rightarrow \text{Stone}$, called the Stone-Cech compactification: and X is mapped to $\text{Ult}(\wp(X))$

If $g: A \rightarrow B$ is a Bdd. hom.

① g is surj $\Leftrightarrow g^{-1}[-]: \text{Ult}(B) \rightarrow \text{Ult}(A)$ is INJ

② g is inj $\Leftrightarrow g^{-1}[-]: \text{Ult}(B) \rightarrow \text{Ult}(A)$ is SURJ.

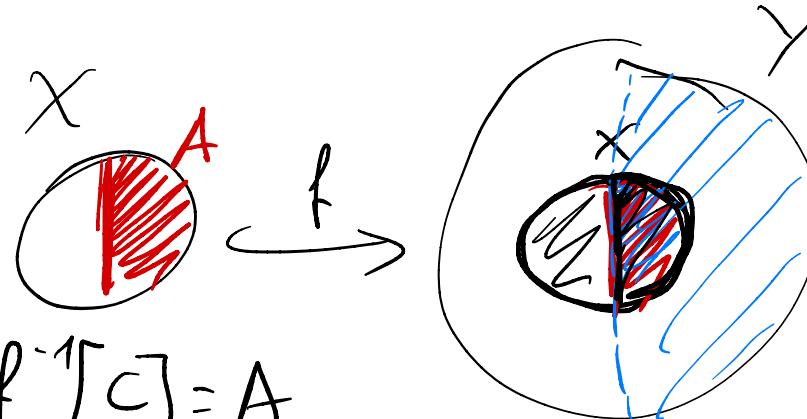
$$\begin{array}{c} X \rightarrow \text{Ult}(\wp(X)) \\ \downarrow \\ Y \end{array}$$

PROOF

① \Rightarrow) f is inj.

Let $A \in \text{Clop}(X)$.

Find a clopen C of Y , s.t. $f^{-1}[C] = A$



$f[A]$

Apply the lemma with

$f[A]$ and $f[X \setminus A]$

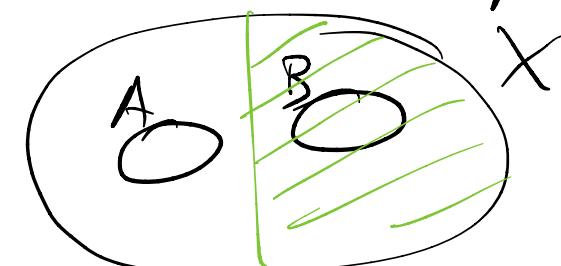
\Rightarrow we have a clopen
 C , s.t.

$f[A] \subseteq C$

$f[X \setminus A] \cap C = \emptyset$

[LEMMA]

Let X be a Stone space



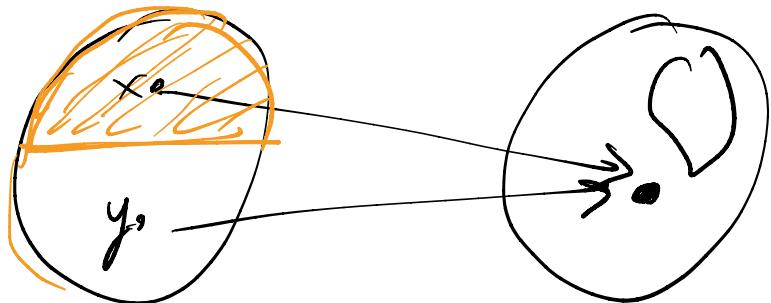
Let A, B closed disjoint subsets,
Then there is a clopen C of X
s.t. $BSC \cap C = \emptyset$.

EX

Then $A = f^{-1}[c]$. Thus $f^{-1}[-J]$ is inj.

\Leftarrow) Let us prove the contrapositive

Suppose f not inj. GOAL: $f^{-1}[-J]$ is not inj.



Take any clopen separating x and y . It is not the preimage of any clopen of Y .

② BE CONTINUED...