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RECITATION 3

This recitation covers some basic concepts in convex analysis.

PROBLEM 1: CONVEXITY DEFINITIONS

In this problem, we investigate the equivalence of two alternative definitions of a convex function. Recall that a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if (iff) $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have:

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

(a) Show that a differentiable function f is convex iff its gradient is monotone, i.e., it satisfies, $\forall \mathbf{x}, \mathbf{y} \in \mathrm{dom}(f)$,

$$\langle \nabla f(\mathbf{y}) - \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq 0 \tag{1}$$

HINT: Recall that $f(\mathbf{y}) = f(\mathbf{x}) + \int_0^1 \frac{d}{dt} f(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))dt$, $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

(b) Show that any differentiable function f is convex iff its epigraph, $\mathrm{epi}(f) = \{(\mathbf{x}, y) \mid \mathbf{x} \in \mathrm{dom}(f), y \in \mathbb{R}, f(\mathbf{x}) \leq y\}$ is convex.

PROBLEM 2: NON-CONVEX/CONCAVE FUNCTION

In this problem, we use an interesting function $f(\mathbf{x}) = 1 - \prod_{i=1}^n (1 - x_i)$[1] to illustrate some of the challenges we encounter when trying to find the optimal point of a non-convex/concave function. Throughout the exercise we consider the case $n = 2$.

(a) Plot the function defined $\forall \mathbf{x} \in \mathbb{R}^2$, $f(\mathbf{x}) = 1 - (1 - x_1)(1 - x_2)$ (in MATLAB or Python). You can use the following code:

```
x1 = -2:0.01:2;
x2 = -2:0.01:2;
[X, Y] = meshgrid(x1,x2);
f = @(x) 1 - prod(1-x);
for i=1:size(X,1)
for j=1:size(X,2)
Z(i,j) = f([X(i,j), Y(i,j)]);
end
end
figure, h = surfc(X,Y,Z)
set(h(1), 'edgecolor','none')
shg
```

Observe from the graph that the function f is neither convex nor concave.

(b) Derive the gradient of f, and compute the stationary point(s) of f. Can you deduce that f is not a convex nor a concave function?

(c) This function is an unusual example of a non-concave function for which the maximum can be efficiently approximated [1]. The main property that makes this possible is the following: f is concave along any positive direction. To see this recall first Taylor's theorem:

$$f(\mathbf{x}) = f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + \frac{1}{2}(\mathbf{x} - \mathbf{y})^T \nabla^2 f(\mathbf{z})(\mathbf{x} - \mathbf{y})$$

for some point $\mathbf{z}$ between $\mathbf{x}$ and $\mathbf{y}$. Show that f is concave along any positive direction; i.e., for $\mathbf{x} = \mathbf{y} + \mathbf{d}$ where $\mathbf{d} \in \mathbb{R}^{\mathbf{n}}_+$.

[1] For the interested reader, we note that this is the multilinear extension of the submodular function $f(S) = \min\{|S|, 1\}$ when restricted to the hypercube domain.

PROBLEM 3: BINARY LOGISTIC REGRESSION: GEOMETRIC PROPERTIES OF THE OBJECTIVE FUNCTION

Recall from previous week the logistic regression problem. We are given a collection of independent samples $(\mathbf{a}_1, b_1), (\mathbf{a}_2, b_2), \ldots, (\mathbf{a}_n, b_n) \in \mathbb{R}^p \times \{-1, 1\}$, which satisfy, for some unknown $\mathbf{x}^\natural \in \mathbb{R}^p$,

$$\mathbb{P}\{b_i = 1\} = 1 - \mathbb{P}\{b_i = -1\} = \frac{1}{1 + \exp\left(-\left\langle \mathbf{a}_i, \mathbf{x}^\natural \right\rangle\right)}.$$

The maximum-likelihood (ML) estimator of $\mathbf{x}^\natural$ is given by

$$\hat{\mathbf{x}} \in \arg\min \{ f(\mathbf{x}) | \mathbf{x} \in \mathbb{R}^p \},$$

where

$$f(\mathbf{x}) := \sum_{i=1}^n \log\left[1 + \exp\left(-b_i \left\langle \mathbf{a}_i, \mathbf{x} \right\rangle\right)\right].$$

(a) The ML estimate $\hat{x}$ can be computed by the gradient descent method, which starts with some $\mathbf{x}_0 \in \mathbb{R}^p$, and iterates as

$$\mathbf{x}_k = \mathbf{x}_{k-1} - \alpha_k \nabla f(\mathbf{x}_k), \quad k = 1, 2, 3, \ldots,$$

for some step sizes α_k. If α_k are properly chosen, we have $\mathbf{x}_k \to \hat{\mathbf{x}}$ as $k \to \infty$. Show that

$$\nabla f(\mathbf{x}) = -\sum_{i=1}^n \frac{b_i \exp(-b_i \langle \mathbf{a}_i, \mathbf{x} \rangle)}{1 + \exp(-b_i \langle \mathbf{a}_i, \mathbf{x} \rangle)} \mathbf{a}_i.$$

(b) The ML estimate $\hat{x}$ can also be computed by the Newton method, which starts with some $\mathbf{x}_0 \in \mathbb{R}^p$, and iterates as

$$\mathbf{x}_k = \mathbf{x}_{k-1} - \beta_k \left[\nabla^2 f(\mathbf{x}_k)\right]^{-1} \nabla f(\mathbf{x}_k), \quad k = 1, 2, 3, \ldots,$$

for some step sizes β_k. If β_k are properly chosen, we have $\mathbf{x}_k \to \hat{\mathbf{x}}$ as $k \to \infty$. Show that

$$\nabla^2 f(\mathbf{x}) = \sum_{i=1}^n \frac{\exp\left(-b_i \left\langle \mathbf{a}_i, \mathbf{x} \right\rangle\right)}{\left[1 + \exp\left(-b_i \left\langle \mathbf{a}_i, \mathbf{x} \right\rangle\right)\right]^2} \mathbf{a}_i \mathbf{a}_i^{\mathrm{T}}.$$

(c) Show that $\nabla^2 f(\mathbf{x})$ may not be invertible, if $n < p$.

(d) Show that the function f is convex.

PROBLEM 4: FUNCTIONS OF COMPLEX VARIABLES (SELF STUDY)

Up to now, what we have seen are convex functions of *real* variables. In Signal Processing and Communication Engineering, however, many applications require minimizing a convex function of *complex* variables.

Take one-dimensional magnetic resonance imaging (MRI) as an example. Let $\boldsymbol{x}^\natural \in \mathbb{C}^p$ be an unknown signal. Denote by $\boldsymbol{A} \in \mathbb{C}^{n \times p}$ the so-called measurement matrix, determined by how the engineer measure the signal. The main task of MRI is to recover $\boldsymbol{x}^\natural$, given the measurement matrix $\boldsymbol{A}$ and the measurement outcome

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}^\natural + \boldsymbol{w} \in \mathbb{C}^n, \tag{2}$$

where $\boldsymbol{w} \in \mathbb{C}^n$ represents some unknown measurement noise.

The same model (2) also appears in wireless communication with multiple antennas, where $\boldsymbol{x}^\natural$ represents the signal to be transmitted, $\boldsymbol{A}$ represents the effect of a wireless channel, and $\boldsymbol{w}$ is the noise at the receiver, usually modeled as a Gaussian random vector. An important goal of the receiver is to recover $\boldsymbol{x}^\natural$ given $\boldsymbol{A}$ and $\boldsymbol{y}$, for further processing (e.g., demodulation, decoding, etc.).

One standard approach to recovering $\boldsymbol{x}^\natural$ is least squares (LS) estimation, which yields the following convex optimization problem

$$\hat{\boldsymbol{x}} \in \operatorname{argmin} \{ f(\boldsymbol{x}) \,|\, \boldsymbol{x} \in \mathcal{X} \}, \tag{3}$$

where we set

$$f(\boldsymbol{x}) := \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2^2, \tag{4}$$

and $\mathcal{X}$ is some closed convex set determined by the engineer's prior knowledge of $\boldsymbol{x}^\natural$. (3) is a standard convex optimization problem, if $\boldsymbol{y}$, $\boldsymbol{A}$, and $\boldsymbol{x}$ are all real.

The following questions show why computing ∇f can be an issue in the complex variable case, and present an elementary approach to solving this issue.

(a) Suppose that $\boldsymbol{x}^\natural$, $\boldsymbol{A}$, $\boldsymbol{w}$, and $\boldsymbol{y}$ are all real. Show that

$$\nabla f(\boldsymbol{x}) = -2\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}).$$

(b) Let us examine why the complex variable case deserves some additional care. Recall that a function $\varphi(z) := u(x,y) + \mathrm{i}v(x,y)$ of a complex variable $z = x + \mathrm{i}y$ is differentiable, if and only if the Cauchy-Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Using this fact, show that the standard definition of ∇f,

$$\nabla f(\boldsymbol{x}) := \left[\frac{\partial f(\boldsymbol{x})}{\partial(x_1)}, \frac{\partial f(\boldsymbol{x})}{\partial(x_2)}, \ldots, \frac{\partial f(\boldsymbol{x})}{\partial(x_p)}\right]^{\mathrm{T}},$$

is not applicable, where we write $\boldsymbol{x} = [x_1, x_2, \ldots, x_p]^{\mathrm{T}}$.

(c) We show an elementary approach to deal with this differentiability issue. Notice that f can be viewed as a function of $2p$ *real* variables, if we use the decomposition $\boldsymbol{x} = \boldsymbol{x}_{\mathrm{R}} + \mathrm{i}\boldsymbol{x}_{\mathrm{I}}$, and write $f(\boldsymbol{x}) = \tilde{f}([\boldsymbol{x}_{\mathrm{R}}^{\mathrm{T}}, \boldsymbol{x}_{\mathrm{I}}^{\mathrm{T}}]^{\mathrm{T}})$, for some $\boldsymbol{x}_{\mathrm{R}}, \boldsymbol{x}_{\mathrm{I}} \in \mathbb{R}^p$. Then we have an equivalent optimization problem

$$\begin{bmatrix} \hat{\boldsymbol{x}}_{\mathrm{R}} \\ \hat{\boldsymbol{x}}_{\mathrm{I}} \end{bmatrix} \in \operatorname{argmin}\left\{\tilde{f}\left(\begin{bmatrix} \boldsymbol{x}_{\mathrm{R}} \\ \boldsymbol{x}_{\mathrm{I}} \end{bmatrix}\right) \middle| \begin{bmatrix} \boldsymbol{x}_{\mathrm{R}} \\ \boldsymbol{x}_{\mathrm{I}} \end{bmatrix} \in \tilde{\mathcal{X}}\right\}, \tag{5}$$

where

$$\tilde{\mathcal{X}} := \left\{\begin{bmatrix} \boldsymbol{x}_{\mathrm{R}} \\ \boldsymbol{x}_{\mathrm{I}} \end{bmatrix} \middle| \boldsymbol{x}_{\mathrm{R}} + \mathrm{i}\boldsymbol{x}_{\mathrm{I}} \in \mathcal{X}\right\} \subset \mathbb{R}^{2p}$$

Show that (5) is also a convex optimization problem; that is, show that $\tilde{f}$ is a convex function in $[\boldsymbol{x}_{\mathrm{R}}^{\mathrm{T}}, \boldsymbol{x}_{\mathrm{I}}^{\mathrm{T}}]^{\mathrm{T}}$, and $\tilde{\mathcal{X}}$ is a convex set.

(d) Then we can solve the standard optimization problem (5), and set $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}_{\mathrm{R}} + \mathrm{i}\hat{\boldsymbol{x}}_{\mathrm{I}}$. One algorithm that solves (5) is the projected gradient descent method, which starts with some $\boldsymbol{x}_0 \in \tilde{\mathcal{X}}$, and iterates as

$$\boldsymbol{x}_{t+1} = \Pi_{\tilde{\mathcal{X}}}\left(\boldsymbol{x}_t - \eta_t \tilde{f}'(\boldsymbol{x}_t)\right), \quad , t = 0, 1, 2, \ldots,$$

for some properly chosen step sizes $\eta_0, \eta_1, \ldots$. Show that for any $\boldsymbol{x} = \boldsymbol{x}_{\mathrm{R}} + \mathrm{i}\boldsymbol{x}_{\mathrm{I}} \in \mathbb{R}^{2p}$,

$$\begin{aligned}\nabla\tilde{f}(\begin{bmatrix} \boldsymbol{x}_{\mathrm{R}} \\ \boldsymbol{x}_{\mathrm{I}} \end{bmatrix}) = -2\Bigg\{&\begin{bmatrix} \boldsymbol{A}_{\mathrm{R}}^{\mathrm{T}} \\ -\boldsymbol{A}_{\mathrm{I}}^{\mathrm{T}} \end{bmatrix}\left(\boldsymbol{y}_{\mathrm{R}} - \begin{bmatrix} \boldsymbol{A}_{\mathrm{R}} & -\boldsymbol{A}_{\mathrm{I}} \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_{\mathrm{R}} \\ \boldsymbol{x}_{\mathrm{I}} \end{bmatrix}\right) + \\ &\begin{bmatrix} \boldsymbol{A}_{\mathrm{I}}^{\mathrm{T}} \\ \boldsymbol{A}_{\mathrm{R}}^{\mathrm{T}} \end{bmatrix}\left(\boldsymbol{y}_{\mathrm{I}} - \begin{bmatrix} \boldsymbol{A}_{\mathrm{I}} & \boldsymbol{A}_{\mathrm{R}} \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_{\mathrm{R}} \\ \boldsymbol{x}_{\mathrm{I}} \end{bmatrix}\right)\Bigg\},\end{aligned}$$

where we decompose $\boldsymbol{y} = \boldsymbol{y}_{\mathrm{R}} + \mathrm{i}\boldsymbol{y}_{\mathrm{I}}$ and $\boldsymbol{A} = \boldsymbol{A}_{\mathrm{R}} + \mathrm{i}\boldsymbol{A}_{\mathrm{I}}$, for $\boldsymbol{y}_{\mathrm{R}}, \boldsymbol{y}_{\mathrm{I}} \in \mathbb{R}^n$ and $\boldsymbol{A}_{\mathrm{R}}, \boldsymbol{A}_{\mathrm{I}} \in \mathbb{R}^{n\times p}$.

References

[1] Calinescu, G., Chekuri, C., Pál, M., and Vondrák, J. *Maximizing a monotone submodular function subject to a matroid constraint*. SIAM Journal on Computing 40, 6 (2011), 1740–1766.