Problem 2

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1 System

$$\frac{dx}{dt} = \alpha x - \beta xy \tag{1}$$

$$\frac{dy}{dt} = \theta xy - \gamma y \tag{2}$$

With this system we have the parameters α that corresponds to the growth rate of the preys, β is the predation rate, θ the growth rate of predators per prey eaten and γ the predator death rate. All of which are assumed to have positive values.

2 Zeros of the system

$$\frac{dx}{dt} = 0 \Leftrightarrow \alpha x = \beta xy \Leftrightarrow x \neq 0 \wedge y = \frac{\alpha}{\beta}$$

$$\frac{dy}{dt} = 0 \Leftrightarrow \theta xy = \gamma y \Leftrightarrow y \neq 0 \wedge x = \frac{\gamma}{\theta}$$

3 Jacobian matrix

For simplycity we rename the functions

$$f(t) := \frac{dx}{dt} = \alpha x - \beta xy \tag{3}$$

$$g(t) := \frac{dy}{dt} = \theta xy - \gamma y \tag{4}$$

So our Jacobian matrix can be written in a more friendly way

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \theta y & \theta x - \gamma \end{pmatrix}$$

3.1 Points of equilibrium

$$J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} \Rightarrow det(J(0,0)) = -\alpha\gamma$$

$$J\left(\frac{\gamma}{\theta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & -\beta\frac{\gamma}{\theta} \\ \theta\frac{\alpha}{\beta} & 0 \end{pmatrix} \Rightarrow det\left(J\left(\frac{\gamma}{\theta}, \frac{\alpha}{\beta}\right)\right) = \alpha\gamma$$

3.2 Not points of equilibrium

We don't need to confirm the points:

$$(x, y) = \left\{\left(0, \frac{\alpha}{\beta}\right), \left(\frac{\gamma}{\theta}, 0\right)\right\}$$

They were not part of the possible solutions, however if we calculated the determinant we would get:

$$J\left(0, \frac{\alpha}{\gamma}\right) = \begin{pmatrix} 0 & 0 \\ \theta\frac{\alpha}{\beta} & -\gamma \end{pmatrix} \Rightarrow det\left(J\left(0, \frac{\alpha}{\gamma}\right)\right) = 0$$

$$J\left(\frac{\gamma}{\theta}, 0\right) = \begin{pmatrix} \alpha & -\beta\frac{\gamma}{\theta} \\ 0 & 0 \end{pmatrix} \Rightarrow det\left(J\left(\frac{\gamma}{\theta}, 0\right)\right) = 0$$

This implies that the transformation has a dimensional deficiency, meaning that either the transformation expands or collapses the space we are working with, which in the study of the Jacobian, being a transformation related with the variations of your domain space, should not be zero. We can infer a necessary condition for equilibrium in a point represented as a vector of coordinates $\vec{p}$ to be: $det(J(\vec{p})) \neq 0$.

4 Characteristic polynomials

Now we look at the respective characteristic polynomials:

$$det(J(0,0) - \lambda I_n) = 0 \Leftrightarrow det\begin{pmatrix} \alpha - \lambda & 0 \\ 0 & -\gamma - \lambda \end{pmatrix} = 0$$

$$\Leftrightarrow (\alpha - \lambda)(-\gamma - \lambda) = 0 \Leftrightarrow \alpha - \lambda = 0 \vee -\gamma - \lambda = 0 \Rightarrow \lambda \in \{\alpha, -\gamma\}$$

Eigenvalues with oposite signs mean that the eigenvectors are transformed in oposite directions, therefore the point $(x, y) = (0, 0)$ is a saddle point.

$$det\left(J\left(\frac{\gamma}{\theta},\frac{\alpha}{\beta}\right)-\lambda I_n\right)=0\Leftrightarrow det\begin{pmatrix}-\lambda & -\beta\frac{\gamma}{\theta}\\ \theta\frac{\alpha}{\beta} & -\lambda\end{pmatrix}=0$$

$$\Leftrightarrow \lambda^2+\alpha\gamma=0\Leftrightarrow\lambda=\pm\sqrt{-\alpha\gamma}\Leftrightarrow\lambda=\pm i\sqrt{\alpha\gamma}$$

$$Re(\lambda)=0\Rightarrow stability$$

$$Im(\lambda)\neq 0\Rightarrow \text{periodic closed orbits}$$