Università degli Studi di Padova

MODELS OF THEORETICAL PHYSICS

DI

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Introduction

The aim of this document is to collect the notes of *Models of Theoretical Physics* course, held by professors Marco Baiesi and Amos Maritan for the "Physics of data" curriculum in the academic year 2018-2019 (which is the first year of this new curriculum), to have them written in a neater way.

As just told, this document is far from pretendig to be perfect and his goal is to help studying in a neater and better way. For this reason, there may be some errors among it.

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I have tried to be as much ruthless as possible in finding and correcting errors and mistakes, and I apologize if some have survived.

I hope that the overall result will anyway be satisfactory.

Padua, October 2018
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Chapter 1

Methods to compute usefull integrals

(02/10/2018)

In this course we will use many kind of particular integrals. For this reason, in this first chapter we will see a brief introduction to all we need to calculate them.

1.1 Gaussian integrals

Let us consider the following integral:

$$Z(A) = \int d^n x \cdot e^{-A_2(\vec{x})} \tag{1.1}$$

where $Z(A)$ is the integral we want to calculate, $A_2(\vec{x})$ is a x quadratic form as the following:

$$A_2(\vec{x}) = \frac{1}{2} \sum_{i,j=1}^{n} x_i A_{ij} x_j \tag{1.2}$$

Here A is a matrix that can be identified as a metric matrix. In the case we consider A is symmetric with (generally) complex coefficients; furthermore, it has non-negative real parts and non-vanishing eigenvalues a_i.

$$Re(a_i) \geq 0 \qquad \text{and} \qquad a_i \neq 0_{\mathbb{C}}$$

(Note: if these conditions are not true the integral would be divergent...).

Let us bring $A \in \mathbb{R}$. In this case it can be digonalized with orthogonal transformation O, for which it holds:

$$\sum_i O_{ij} x_j = x'_j \qquad \text{and} \qquad |det(O)| = 1$$

This implies that the Jacobian matrix of the transormation J (which is O itself) has $|det(J)| = 1$: so when we use it to change coordinates in the integral, we can say that "it has no effect" (we multiply by 1). Furthermore, with that transformation the non-diagonal coefficients of the new matrix become all 0 (the matrix after the transformation is diagonal), so the new coordinates x'_j are indipendents: this mean that we can evaluate $Z(A)$ by dividing it in the integrals of every single x'_j.
We obtain:

$$Z(A) = (2\pi)^{\frac{n}{2}} \prod_{i=1}^{n} a_i^{-\frac{1}{2}} = (2\pi)^{\frac{n}{2}} \left(det(A)\right)^{-\frac{1}{2}} \tag{1.3}$$

(Note that for a diagonal matrix the product of the eigenvalues is equal to the determinant. Moreover, for Binet's theorem it holds $det(A \cdot O) = det(A) \cdot det(O)$ (true if A and O are square matrix with same dimensions)).
(Note: I think that what just written in the previous note modifies the conditions on $det(O)$, which I think would be $det(O) = +1$ and not $|det(O)| = 1$ otherwise the last formula is meaningless.
This have to be investigated.
However, the important key is that the integral with that tranformation does not become divergent.)

Since (1.1) and $det(A)$ are analytic functions of A's coefficients, we can extend (1.3) to $\mathbb{C}$.

Example:

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \quad \Rightarrow \quad (3-\lambda)^2 - 1 = 0$$

$$\rightarrow \quad \lambda_{1,2} = \frac{3+\sqrt{9-8}}{2} \quad \Rightarrow \quad \begin{matrix} \lambda_1 = 2 \\ \lambda_2 = 4 \end{matrix}$$

$$Z(A) = 2\pi \frac{1}{\sqrt{4 \cdot 2}} = \sqrt{\frac{\pi}{2}}$$

1.2 More general integrals

Let us try to generalize the previous agruments for a more general kind of integral:

$$Z(A) = \int d^n x \cdot e^{-A_2(\vec{x}) + \vec{b} \cdot \vec{x}} \tag{1.4}$$

With $\vec{b} \cdot \vec{x} = \sum_{i=1}^{n} b_i x_i$. We want to have the integral in the form of (1.1) so we expand the exponent near his maximum x^*.
To find x^* (signs are already changed):

$$\frac{\partial}{\partial x_i}(A_2(x) - \vec{b} \cdot \vec{x}) = 0 \quad \Rightarrow \quad \sum_j A_{ij} x_j = b_i$$

(Note: $A_2 = A/2$: in the derivative the factor 1/2 goes away because we are derivating a quadratic form (if one does the calculation, it can be seen easy).)

So the solution is:

$$x^* = \sum_i (A^{-1})_{ij} b_j$$

Now we change variable $\vec{x} \quad \Rightarrow \quad \vec{y}$ in order to have the integral in the form of (1.1). $\vec{y}$ is the deviation from the min of the exponential:

$$\vec{y} = \vec{x} - \vec{x^*} \quad \Rightarrow \quad x_i = x_i^* + y_i$$

$$\Rightarrow \quad -A_2(\vec{x}) + \vec{b} \cdot \vec{x} = w_2(\vec{b}) - A_2(\vec{y})$$

Notice that $w_2(\vec{b})$ is a costant, so it can be eventually take outside the integral.

$$w_2(\vec{b}) = \frac{1}{2}\sum_{i,j=0}^{n} b_i(A^{-1})_{ij}b_j = \frac{1}{2}\vec{b}\cdot\vec{x}^*$$

With those changes, we can write:

$$Z(A,\vec{b}) = e^{w_2(\vec{b})}\int d^n y\cdot e^{-A_2(\vec{y})} = e^{w_2(\vec{b})}(2\pi)^{\frac{n}{2}}(det(A))^{-\frac{1}{2}} \tag{1.5}$$

Example (as before)

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$Z(A) = \int\int e^{-\frac{1}{2}(x,y)A\binom{x}{y}+\binom{x}{0}}dxdy$$

$$A^{-1} = \frac{1}{det(A)}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$w_2(\vec{b}) = \frac{1}{2}(1,0)\frac{1}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{16}(1,0)\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{3}{16}$$

$$\Rightarrow \quad Z(A,\vec{b}) = e^{\frac{3}{16}}Z(A) = e^{\frac{3}{16}}\frac{\pi}{\sqrt{2}}$$

1.3 Gaussian expectation values

We want now to explore briefly how the expectation values of polynomials with a gaussian distribution works.

$$\langle x_{k_1}, x_{k_2}, \ldots, x_{k_m}\rangle = \underbrace{Z^{-1}(A,\vec{0})}_{Normalization factor}\int d^n x x_{k_1}, x_{k_2}, \ldots, x_{k_m} e^{-A_2(\vec{x})} \tag{1.6}$$

Where the "normalization factor" is that for which $\langle 1\rangle = 1$.
Note: since here we use angular brackets to denote an average made with gaussian weights.

From (1.4) we have:

$$\frac{\partial}{\partial b_k}Z(A,\vec{b}) = \int d^n x\, x_k\, e^{-A_2(\vec{x})+\vec{b}\cdot\vec{x}}$$

In this sense we say that $\vec{b}$ is "completed" to $\vec{x}$ and that Z is the generating function. In the case of only 1 variable average note that if $\vec{b} = 0$ the result is 0, as one expect for a gaussian variable.
Generalizing (1.6) using what just shown, we obtain:

$$\langle x_{k_1}, x_{k_2}, \ldots, x_{k_l}\rangle = (2\pi)^{-\frac{n}{2}}(detA)^{\frac{1}{2}}\left[\frac{\partial}{\partial b_{k_1}}\frac{\partial}{\partial b_{k_2}}\cdots\frac{\partial}{\partial b_{k_m}}Z(A,\vec{b})\right]_{b=0} \tag{1.7}$$

NOT SURE ARGUMENT
We can also notice that if we want to evaluate the expectation value of a power series of x $\langle F(x)\rangle$ we can write the following, thanks to the linearity of the integral (assuming that x is not a vector):

$$\langle F(x)\rangle = ... \left[F\left(\frac{\partial}{\partial b}\right) e^{w_2(b)}\right]_{b=0}$$

The "..." indicates that every power of x in F(x) has its own normalization factor.

From (1.7) we can obtain one form of Wick's theorem.
If there is one $\frac{\partial}{\partial b_i}$ in (1.7), it pulls down some b_j because $w_2(\vec{b}) = \frac{1}{2}\sum_{i,j=0}^{n} b_i (A^{-1})_{ij} b_j$. Without another $\frac{\partial}{\partial b_k}$ (also with $k \neq i$), setting $\vec{b} = \vec{0}$ makes all components $b_j \to 0$ in $\langle ...\rangle$.

$$\lim_{b_1 \to 0} \langle b1, ...\rangle \to 0$$

$\Rightarrow$ Another $\frac{\partial}{\partial b_k}$ is needed to ensure the average does not tend to 0.
$\Rightarrow$ l index in (1.7) must be even.

Theorem 1.3.1 (Wick's theorem) *Using arguments just above, for every $k_p k_q$ from the list $\{k_1, k_2, ..., k_l\}$, with associated $(A^{-1})_{k_p k_q}$, the average of l (multiplied) variables can be calculated as following:*

$$\begin{aligned}\langle x_{k_1}, x_{k_2}, ..., x_{k_l}\rangle &= \sum_{all\ possible\ pairings\ P\ of\ \{k_1,...,k_l\}} A^{-1}_{k_{P_1} k_{P_2}} ... A^{-1}_{k_{P_{l-1}} k_{P_l}} \\ &= \sum_{all\ possible\ pairings\ P\ of\ \{k_1,...,k_l\}} \langle x_{k_{P_1}} x_{k_{P_2}}\rangle \cdot ... \cdot \langle x_{k_{P_{l-1}}} x_{k_{P_l}}\rangle\end{aligned}$$

Example:
Let us see an application 1-dim of the theorem: with $x, b \in \mathbb{R}$

$$\begin{aligned}\langle x^2\rangle &= \frac{1}{Z(A,0)} \frac{\partial^2}{\partial b^2} \int_{-\infty}^{\infty} x^2\, e^{-\frac{A}{2}x + bx} dx \Big|_{b=0} \\ &= \frac{\partial^2}{\partial b^2} e^{\frac{b^2}{2A}}\Big|_{b=0} = \frac{\partial}{\partial b}\left[\frac{b}{A} e^{\frac{b^2}{2A}}\right]_{b=0} = \frac{1}{A}\end{aligned}$$

$$\begin{aligned}\langle x^4\rangle &= \frac{1}{Z(A,0)} \frac{\partial^4}{\partial b^4} \int_{-\infty}^{\infty} x^4\, e^{-\frac{A}{2}x + bx} dx \Big|_{b=0} = \frac{\partial^4}{\partial b^4} e^{\frac{b^2}{2A}}\Big|_{b=0} = \frac{\partial^3}{\partial b^3}\left[\frac{b}{A} e^{\frac{b^2}{2A}}\right]_{b=0} \\ &= \frac{\partial^2}{\partial b^2}\left[\frac{1}{A} e^{\frac{b^2}{2A}} + \frac{b}{A} \cdot \frac{b}{A} e^{\frac{b^2}{2A}}\right]_{b=0} = \frac{\partial}{\partial b}\left[\frac{1}{A} \cdot \frac{b}{A} e^{\frac{b^2}{2A}} + 2\frac{b}{A^2} e^{\frac{b^2}{2A}} + \frac{b^2}{A^2} \cdot \frac{b}{A} e^{\frac{b^2}{2A}}\right]_{b=o} \\ &= \left[\frac{1}{A^2} e^{\frac{b^2}{2A}} + \frac{b}{A^2} \cdot \frac{b}{A} e^{\frac{b^2}{2A}} + \frac{2}{A^2} e^{\frac{b^2}{2A}} + \frac{2b}{A^2} \cdot \frac{b}{A} e^{\frac{b^2}{2A}} + \frac{3b^2}{A^3} e^{\frac{b^2}{2A}} + \frac{b^3}{A^3} \cdot \frac{b}{A} e^{\frac{b^2}{2A}}\right]_{b=0} \\ &= \frac{3}{A^2}\end{aligned}$$

Remembering that $\langle x^2 \rangle = \frac{1}{A}$ we can write:

$$\langle x^4 \rangle = 3\langle x^2 \rangle^2$$

Using Wick's theorem we can rewrite this last result: With $l = 4$ and A as following:

$$A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

we can write:

$$\langle x_1 x_2 x_3 x_4 \rangle = \langle x_1 x_2 \rangle + \langle x_3 x_4 \rangle + \langle x_1 x_3 \rangle \langle x_2 x_4 \rangle + \langle x_1 x_4 \rangle \langle x_2 x_3 \rangle$$

Now come back to the 1-dim case, using $x = x_{1,2,3,4}$, obtaining:

$$\langle x^4 \rangle = \langle x^2 \rangle \cdot \langle x^2 \rangle \cdot 3 = 3\langle x^2 \rangle^2$$

which is the same result as before.