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Galerkin's method revisited and corrected in the problem of Jaworski and Dowell

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ABSTRACT

This paper revisits the problem first studied by Jaworski and Dowell, namely, the free vibration of multi-step beams. Previous authors utilized approximate method of Ritz as well as the finite element method with attendant comparison with the experimental results. This study provides the exact solution for the Jaworski and Dowell problem in terms of Krylov-Duncan functions. Additionally, the Galerkin method is applied and contrasted with the exact solution. It is shown that the straightforward implementation of the Galerkin method, as it is usually performed in the literature, does not lead to results obtained by Jaworski and Dowell using the Ritz method. Moreover, the straightforward application of the Galerkin method does not tend to the results obtained by either exact solution or experiments. A modification of the Galerkin method is proposed by introducing generalized functions to describe both mass and stiffness of the stepped beam. Specifically, the unit step function, Dirac's delta function and the doublet function, are utilized for this purpose. With this modification, the Galerkin method yields results coinciding with those derived by the Ritz method, and turn out to be in close vicinity with those produced by the exact solution as well as experiments.

1. Introduction

Numerous structures, like aircraft wings, helicopter rotor blades, spacecraft antennae, robot arms can be modeled as beams. Free vibration of beams is a classic subject in structural mechanics originating at about 1735 when Daniel Bernoulli and Leonhard Euler investigated the vibration of uniform and homogeneous beams [5]. Since then, numerous papers, review papers and books, have been written on this subject. Among the several, it is important to note the work of Young and Felgar [21], which provides tables with the numerical solution for beams in different boundary conditions, the one by Duncan [4] in which normalized orthogonal deflection functions for beams can be found; and the books devoted to structural dynamics by Gorman [6] or by Karnovsky and Lebed [10].In another work, Karnovsky and Lebed [9] introduced also the Krylov-Duncan method as an approach to obtain the natural frequencies by solving an eigenvalue problem;

Beams with discontinuous cross-sectional areas, i.e. stepped beams, were also investigated by means of various approaches. Specifically, the Cauchy iteration method was applied by Taleb and Suppiger [18] yielding upper bounds for natural frequencies; lower bounds were found by Buckens [1] using a decomposition method; the variational component method with Lagrange multipliers was used by Klein [11] to satisfy geometric continuity conditions between different steps.

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Rayleigh-Ritz energy approach was used by Yuan and Dickinson [20] in conjunction with artificial spring constraints between beam components. This approach was utilized also by Maurizi and Belles [16].

The above papers resorted to approximate methods. Exact solution was provided by Levinson [13] for a single-stepped beam simply supported at its both ends. Jang and Bert [7] summarized results for a single, centrally located stepped beam. Naguleswaran [17] dealt with beams with up to three steps, whereas Lu et al [14], Mao [15], Duan and Wang [2] and Wang X-W. and Wang Y-L [19] dealt with the case of multiple stepped beams.

Jaworski and Dowell [8] conducted a thorough investigation of a beam with 13 steps using approximate methods. Namely, they applied the Rayleigh-Ritz method as well as the finite element method, and conducted extensive experiments to validate their results. For the implementation of the Rayleigh-Ritz method, the authors used the exact modes of vibration of cantilever beam, whereas they built FEM models using ANSYS and various element available in its library (BEAM4, BEAM188, SHELL93 and SOLID45). The above results were contrasted with the experimental results.

In this paper, we conducted two analyses that complement the work by Jaworski and Dowell [8]. Namely, we first used the so-called Krylov-Duncan functions to compute the exact natural frequencies of a stepped beam and next two versions of the Galerkin method. In particular, for the Galerkin method we implemented both the straightforward method, in which the basis functions exist and are evaluated for each step of the beam and next combined, and the rigorous version of the method which is based on generalized functions of the mass and stiffness of the beam which exist over the entire beam domain. In this latter case we used the Heaviside unit step function, the Dirac's delta function and its derivative, as well as the doublet function, to formulate the characteristic equation of the free vibration problem of the stepped beam.

2. Basic equations

We are interested to evaluate the natural frequencies of a multi-step beam as shown in Fig. 1.

The beam is a cantilever made of a single material so the elastic modulus E and the mass density ρ are constants. The beam is composed by two alternating sections, namely section A and section B.

We study the free vibrations of this beam in both vertical x-y and horizontal x-z planes as shown in Fig. 2.

The Euler-Bernoulli differential equation governing the flexural vibrations in one principal plane of the non-uniform beam reads:

$$\frac{\partial^2}{\partial x^2}\left(E(x)I(x)\frac{\partial^2 w}{\partial x^2}\right)+\rho(x)A(x)\frac{\partial^2 w}{\partial t^2}=0 \tag{1}$$

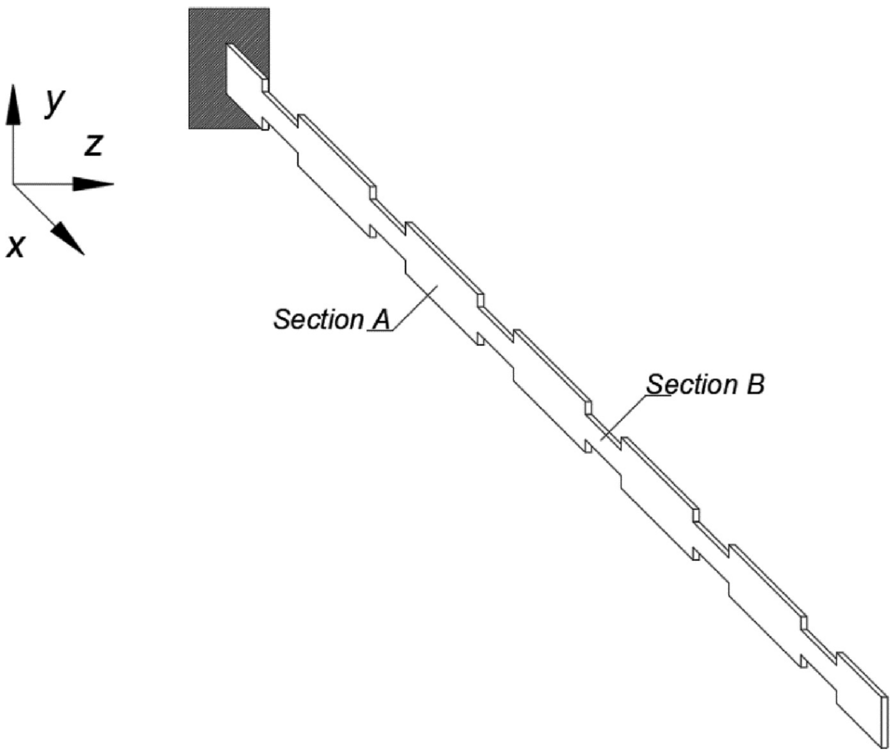


Fig. 1. Schematic of 13-stepped beam of length L.

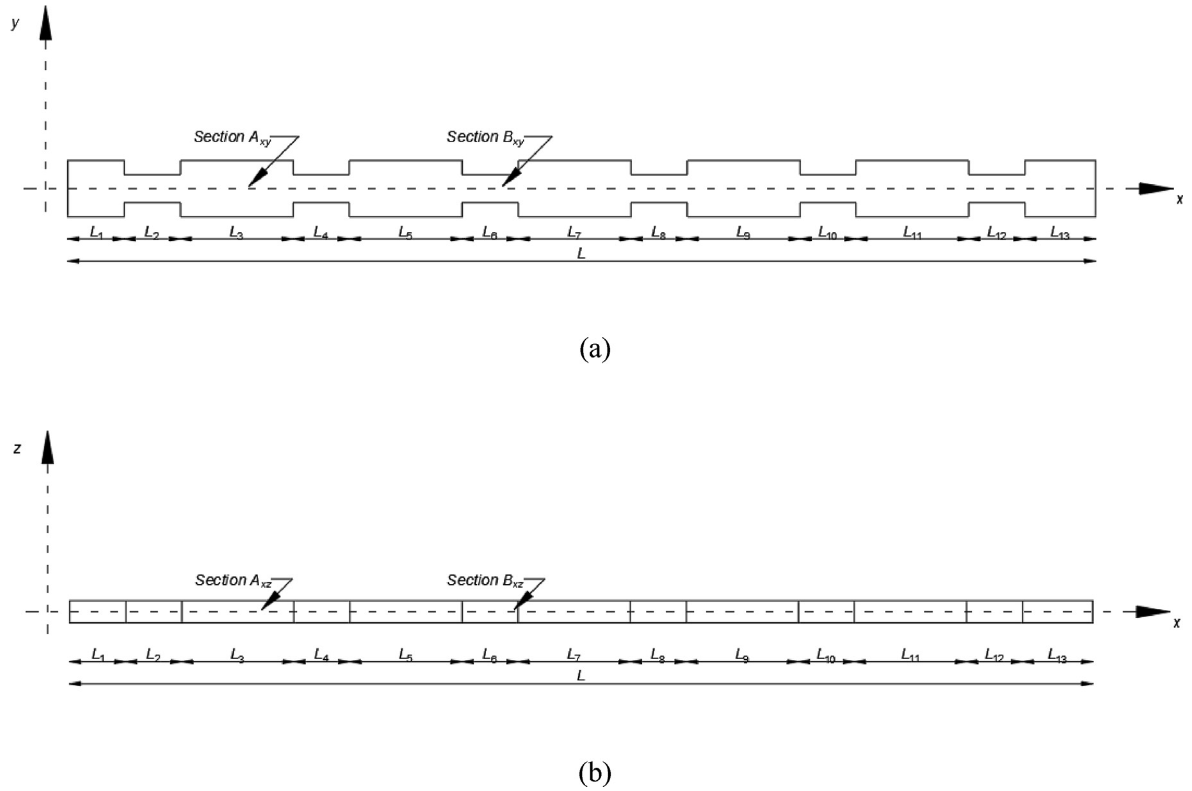


Fig. 2. Schematic of the stepped beam in the plane x-y (a) and x-z (b).

where $w(x,t)$ is the vertical displacement, $I(x)$ the moment of inertia, $A(x)$ the cross-sectional area, x the axial coordinate and t is the time. For each segment of the stepped beam, one can write:

$$E_jI_j\frac{\partial^4 w}{\partial x^4}+\rho_jA_j\frac{\partial^2 w}{\partial t^2}=0 \tag{2}$$

where j is an integer which identifies the segment of the beam. We rewrite the vertical displacement as follows:

$$w(x,t)=W(x)sin(\omega t) \tag{3}$$

where ω is the sought natural frequency of the beam. Substituting Eq. (3) in the differential Eq. (2) we easily obtain the following set of equations valid for any time instant:

$$\frac{d^4W}{dx^4}-{\alpha_j}^4W=0 \tag{4}$$

where α_j reads:

$$\alpha_j=\sqrt[4]{\frac{\rho_jA_j\omega^2}{E_jI_j}} \tag{5}$$

As well known, the mode shapes $W_j(x)$ are given by:

$$W_j(x)=D_{1,j}\sin(\alpha_jx)+D_{2,j}\cos(\alpha_jx)+D_{3,j}\cosh(\alpha_jx)+D_{4,j}\sinh(\alpha_jx) \tag{6}$$

These satisfy the differential equations in Eq. (4), where $D_{i,j}$ are constants of integration. We now take advantage of the Krylov-Duncan functions to rewrite Eq. (6). The Krylov-Duncan functions are four functions [12,3] defined as follows:

$$K_1(\alpha x)=\frac{1}{2}[\cosh(\alpha x)+\cos(\alpha x)] \tag{7.a}$$

$$K_2(\alpha x)=\frac{1}{2}[\sinh(\alpha x)+\sin(\alpha x)] \tag{7.b}$$

$$K_3(\alpha x) = \frac{1}{2}[\cosh(\alpha x) - \cos(\alpha x)] \tag{7.c}$$

$$K_4(\alpha x) = \frac{1}{2}[\sinh(\alpha x) - \sin(\alpha x)] \tag{7.d}$$

One notes that:

$$K_1(0) = 1 \tag{8.a}$$

$$K_2(0) = 0 \tag{8.b}$$

$$K_3(0) = 0 \tag{8.c}$$

$$K_4(0) = 0 \tag{8.d}$$

The second property of these functions is that the first derivative of K_i is equal to K_{i-1} (see Table 1):

We can use these functions into Eq. (6) in order to simplify the representation of the boundary conditions. This will lead us to the following equation:

$$W_j(x) = M_{1,j}K_1(\alpha_j x) + M_{2,j}K_2(\alpha_j x) + M_{3,j}K_3(\alpha_j x) + M_{4,j}K_4(\alpha_j x) \tag{9}$$

where $M_{i,j}$ are constant of integration.

3. Exact solution

The evaluation of the exact solution consists in the demand that not all four coefficients $M_{i,j}$ for each component vanish simultaneously. In our study we have 13 different segments for the multi-step beam resulting in 52 unknowns. The solution should satisfy continuity conditions between the segments and the boundary conditions at the outer sections of the beams (first and the 13th components).

For each discontinuity, we have four compatibility conditions namely continuity of vertical displacement, slope, bending moment and shear force, for a total of 48 equations of compatibility given the 12 discontinuities in the beam. In particular, they read:

$$W_j(x = L_j) = W_{j+1}(x = L_j) \tag{10.a}$$

$$\frac{dW_j}{dx}(x = L_j) = \frac{dW_{j+1}}{dx}(x = L_j) \tag{10.b}$$

$$E_jI_j\frac{d^2W_j}{dx^2}(x = L_j) = E_{j+1}I_{j+1}\frac{d^2W_{j+1}}{dx^2}(x = L_j) \tag{10.c}$$

$$E_jI_j\frac{d^3W_j}{dx^3}(x = L_j) = E_{j+1}I_{j+1}\frac{d^3W_{j+1}}{dx^3}(x = L_j) \tag{10.d}$$

By adding the 4 boundary conditions at the extremes of the beam we can formulate a problem with 52 equations for 52 unknowns. In particular, in the following, to compare our results with those of Jaworski and Dowell [8] we consider the case of the cantilever beam which boundary conditions read (see Table 2):

This system of equations has the following form:

$$\tilde{A}\bar{x} = \bar{0} \tag{11}$$

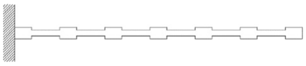
where $\tilde{A}$ is the coefficient matrix, $\bar{x}$ the vector of unknowns and $\bar{0}$ denotes the zero vector. The non-trivial solutions of the homogeneous system in Eq. (11) lead to the natural frequencies ω of the problem.

The matrix $\tilde{A}$ is sparse and the non-zero terms appear around the main diagonal, as represented in Fig. 3.

Table 1
Derivatives of Krylov-Duncan functions.

Krylov-Duncan function	$K_1(x)$	$K_2(x)$	$K_3(x)$	$K_4(x)$
First derivative	$K_4(x)$	$K_1(x)$	$K_2(x)$	$K_3(x)$
Second derivative	$K_3(x)$	$K_4(x)$	$K_1(x)$	$K_2(x)$
Third derivative	$K_2(x)$	$K_3(x)$	$K_4(x)$	$K_1(x)$

Table 2
Boundary conditions.

Constrain conditions		$x = 0$	$x = L$
	Cantilever	$W_1 = 0$ $\frac{dW_1}{dx} = 0$	$E_{13}I_{13}\frac{d^2W_{13}}{dx^2} = 0$ $E_{13}I_{13}\frac{d^3W_{13}}{dx^3} = 0$

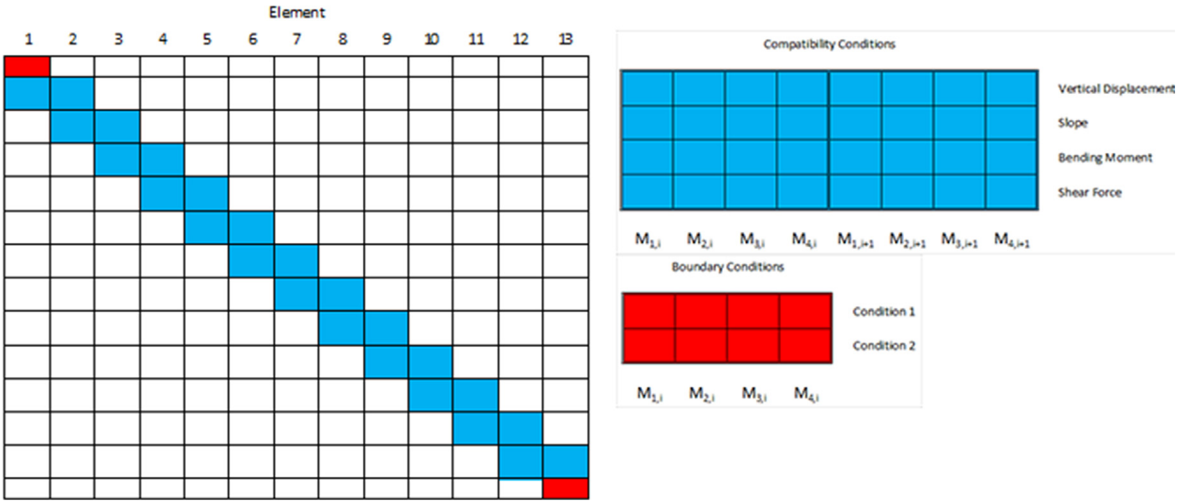


Fig. 3. Matrix structure for exact solution.

4. Straightforward Galerkin method

The Galerkin method is a numerical strategy to solve differential equations in a discrete manner:

$$E_jI_j\frac{d^4W}{dx^4} - \rho_jA_j\omega^2W = 0 \quad x_{j-1} < x < x_j \tag{12}$$

By introducing the axial coordinate in non-dimensional form Eq. (12) can be represented as:

$$E_jI_j\frac{d^4W}{d\xi^4} - \rho_jA_j\omega^2L^4W = 0 \quad \xi_{j-1} < \xi < \xi_j \tag{13}$$

In order to apply the Galerkin method in its straightforward version, we have to express the vertical displacement W in terms of the so-called comparison functions $\psi_p(\xi)$ as:

$$W(\xi) = \sum_{p=1}^{n} a_p\psi_p(\xi) \tag{14}$$

where a_p are unknown constants. Now we substitute the expression of $W(\xi)$ in the differential equations obtaining residuals $\varepsilon_j(\xi)$ since the functions $\psi_p(\xi)$ do not necessarily satisfy the differential equations:

$$E_jI_j\sum_{p=1}^{n} a_k\frac{d^4\psi_p(\xi)}{d\xi^4} - \rho_jA_j\omega^2L^4\sum_{p=1}^{n} a_p\psi_p(\xi) = \varepsilon_j(\xi) \quad \xi_{j-1} < \xi < \xi_j \tag{15}$$

We now multiply the error $\varepsilon_j(\xi)$ by $\psi_q(\xi)$, we sum it up for all the components and we integrate within j^{th} span:

$$\sum_{p=1}^{n}\left\{\sum_{j=1}^{13}\left[\int_{\xi_j}^{\xi_{j+1}} E_jI_j\frac{d^4\psi_p(\xi)}{d\xi^4}\psi_q(\xi)d\xi\right] - \omega^2\sum_{j=1}^{13}\left[\int_{\xi_j}^{\xi_{j+1}}\rho_jA_jL^4\psi_p(\xi)\psi_q(\xi)d\xi\right]\right\}a_p = 0 \tag{16}$$

By defining:

$$K_{pq} = \sum_{j=1}^{13}\left[\int_{\xi_j}^{\xi_{j+1}} E_jI_j\frac{d^4\psi_p(\xi)}{d\xi^4}\psi_q(\xi)d\xi\right] \tag{17.a}$$

$$M_{pq} = \sum_{j=1}^{13} \left[\int_{\xi_j}^{\xi_{j+1}} \rho_j A_j L^4 \psi_p(\xi) \psi_q(\xi) d\xi \right] \tag{17.b}$$

We obtain:

$$\sum_{p=1}^{n} (K_{pq} - \omega^2 M_{pq}) a_p = 0 \tag{18}$$

Eq. (18) can be rewritten in matrix notation as:

$$(K - \omega^2 M) a = 0 \tag{19}$$

where K represent the stiffness matrix of the problem, M the mass matrix of the problem and a the vector of the unknown scale factors a_p.

This non-trivial solution of Eq. (19) lead to the eigenvalues ω^2 and the scale factors a_p of the problem.

5. Rigorous Galerkin method

The rigorous version of the Galerkin method does require generalized functions over the entire domain of the beam length ($0 < x < L$). Starting from Eqs. (1) and (3) we obtain:

$$\frac{d^2}{dx^2} \left(E(x) I(x) \frac{d^2 W}{dx^2} \right) \sin(\omega t) - \omega^2 \rho(x) A(x) W(x) \sin(\omega t) = 0 \tag{20}$$

Introducing a non-dimensional axial coordinate ξ and looking for a solution true for any time value, we obtain:

$$\frac{d^2}{d\xi^2} \left(E(\xi) I(\xi) \frac{d^2 W}{d\xi^2} \right) - \omega^2 L^4 \rho(\xi) A(\xi) W(\xi) = 0 \tag{21}$$

In order to implement the rigorous Galerkin method we represent the flexural rigidity and the mass of the system as generalized functions:

$$D(\xi) = E(\xi) I(\xi) = E_1 I_1 \cdot U(\xi) + \sum_{j=1}^{12} \left[\left(E_{j+1} I_{j+1} - E_j I_j \right) \cdot H\left(\xi - \xi_j \right) \right] \tag{22.a}$$

$$M(\xi) = \rho(\xi) A(\xi) = \rho_1 A_1 \cdot U(\xi) + \sum_{j=1}^{12} \left[\left(\rho_{j+1} A_{j+1} - \rho_j A_j \right) \cdot H\left(\xi - \xi_j \right) \right] \tag{22.b}$$

where $H(\xi - \xi_j)$ is the unit step function or Heaviside function which has the following properties:

$$H(\xi - \alpha) = \begin{cases} 1 \; if \, \xi > \alpha \\ 0 \; otherwise \end{cases} \tag{23.a}$$

$$\frac{d}{d\xi} H(\xi - \alpha) = \delta(\xi - \alpha) \tag{23.b}$$

$$\frac{d}{d\xi} \delta(\xi - \alpha) = \delta'(\xi - \alpha) \tag{23.c}$$

where $\delta(\xi)$ is the Dirac's delta function, and $\delta'(\xi - \alpha)$ is the doublet function. Now, rewriting the Eq. (21) with these considerations we obtain:

$$\frac{d^2}{d\xi^2} \left(D(\xi) \frac{d^2 W}{d\xi^2} \right) - \omega^2 L^4 M(\xi) W(\xi) = 0 \tag{24}$$

We evaluate the derivatives to get:

$$D(\xi) \frac{d^4 W}{d\xi^4} + 2 \frac{d}{d\xi} D(\xi) \frac{d^3 W}{d\xi^3} + \frac{d^2}{d\xi^2} D(\xi) \frac{d^2 W}{d\xi^2} - \omega^2 L^4 M(\xi) W(\xi) = 0 \tag{25}$$

We substitute the approximation in series of $W(\xi)$ (Eq. (14)) arriving at:

$$\sum_{p=1}^{n} \left[D(\xi) \frac{d^4 \psi_p(\xi)}{d\xi^4} + 2 \frac{d}{d\xi} D(\xi) \frac{d^3 \psi_p(\xi)}{d\xi^3} + \frac{d^2}{d\xi^2} D(\xi) \frac{d^2 \psi_p(\xi)}{d\xi^2} - \omega^2 L^4 M(\xi) \psi_p(\xi) \right] a_p = 0 \tag{26}$$

We next multiply the differential equation by $\psi_q(\xi)$ and we integrate it from zero to one, to get:

$$\sum_{p=1}^{n}\left[\int_0^1 D(\xi)\frac{d^4\psi_p(\xi)}{d\xi^4}\psi_q(\xi)d\xi+\int_0^1 2\frac{d}{d\xi}D(\xi)\frac{d^3\psi_p(\xi)}{d\xi^3}\psi_q(\xi)d\xi+\int_0^1 \frac{d^2}{d\xi^2}D(\xi)\frac{d^2\psi_p(\xi)}{d\xi^2}\psi_q(\xi)d\xi-\omega^2\int_0^1 L^4M(\xi)\psi_p(\xi)\psi_q(\xi)d\xi\right]a_p = 0 \tag{27}$$

By defining:

$$K_{1,pq}=\int_0^1 D(\xi)\frac{d^4\psi_p(\xi)}{d\xi^4}\psi_q(\xi)d\xi \tag{28.a}$$

$$K_{2,pq}=\int_0^1 2\frac{d}{d\xi}D(\xi)\frac{d^3\psi_p(\xi)}{d\xi^3}\psi_q(\xi)d\xi \tag{28.b}$$

$$K_{3,pq}=\int_0^1 \frac{d^2}{d\xi^2}D(\xi)\frac{d^2\psi_p(\xi)}{d\xi^2}\psi_q(\xi)d\xi \tag{28.c}$$

$$M_{pq}=\int_0^1 L^4M(\xi)\psi_p(\xi)\psi_q(\xi)d\xi \tag{28.d}$$

We can rewrite Eq. (27) as:

$$\sum_{p=1}^{n}\left(K_{1,pq}+K_{2,pq}+K_{3,pq}-\omega^2M_{pq}\right)a_p=0 \tag{29}$$

or in more compact matrix form as:

$$\left(K_1+K_2+K_3-\omega^2M\right)a=0 \tag{30}$$

Non-trivial solutions of the equation:

$$\left(K-\omega^2M\right)a=0 \tag{31}$$

where $K=K_1+K_2+K_3$, lead to the frequencies of vibration ω^2 and the scale factors $a_p$of the problem.

We observe that the matrix K_1 coincides with the K matrix for the straightforward implementation of the method. Thus, the rigorous implementation of Galerkin method yields to two additional stiffness matrices, K_2 and K_3, which provide superior performances to the method w.r.t. its straightforward version.

6. Numerical investigation

6.1. Comparison functions for Galerkin method

To compare the straightforward and rigorous version of the Galerkin method with the exact solution of some beam problems, first comparison functions must be assumed. Comparison functions are supposed to represent well the solution of the differential equation while satisfying all the boundary conditions of the problem.

According to Jaworski and Dowell [8], good comparison functions for the problem at hand, consist in the mode shapes for the homogeneous cantilevered beam:

$$W_m(\xi)=\left(\frac{sin(\alpha_m)-sinh(\alpha_m)}{cos(\alpha_m)+cosh(\alpha_m)}\right)(sinh(\alpha_m\xi)-sin(\alpha_m\xi))+(cosh(\alpha_m\xi)-cos(\alpha_m\xi)) \tag{32}$$

These functions, however, are not perfect candidates because for large value of m becomes numerically unstable due to the difference between large values of the hyperbolic functions arguments. To overcome this problem, some authors have proposed to use the following expressions:

$$W_m(\xi)=sin(\alpha_m\xi)-cos(\alpha_m\xi)+e^{-\alpha_m\xi}+(-1)^{m+1}e^{-\alpha_m(1-\xi)}+O[\varepsilon] \tag{33}$$

where the order of error is $\varepsilon=e^{-\alpha_m}$, which is negligible for $n\geq 5$.

6.2. Examples

We consider the cantilever beam represented in Fig. 2a, composed by 13 segments, and the following geometrical and mechanical parameters (see Table 3):

Table 3
Mechanical and geometrical data.

	Section			
	A_{xy}	B_{xy}	A_{xz}	B_{xz}
E	$6.06 \cdot 10^{10}\ Pa$	$6.06 \cdot 10^{10}\ Pa$	$6.06 \cdot 10^{10}\ Pa$	$6.06 \cdot 10^{10}\ Pa$
ρ	$2664\ kg/m^3$	$2664\ kg/m^3$	$2664\ kg/m^3$	$2664\ kg/m^3$
b	$3.175\ mm$	$3.175\ mm$	$25.4\ mm$	$12,7\ mm$
h	$25.4\ mm$	$12,7\ mm$	$3.175\ mm$	$3.175\ mm$
A	$8.0645 \cdot 10^{-5}\ m^2$	$4.0322 \cdot 10^{-5}\ m^2$	$8.0645 \cdot 10^{-5}\ m^2$	$4.0322 \cdot 10^{-5}\ m^2$
I	$4.33574 \cdot 10^{-9}\ m^4$	$5.41968 \cdot 10^{-11}\ m^4$	$6.7746 \cdot 10^{-11}\ m^4$	$3.3873 \cdot 10^{-11}\ m^4$

The beam steps lengths are $L_1 = L_2 = L_4 = L_6 = L_8 = L_{10} = 25.40mm$ and $L_3 = L_5 = L_7 = L_9 = L_{11} = 50.80mm$, whereas the last segment $L_{13} = 31.75mm$. The total length of the beam is therefore $L = 463.55mm$.

We show in the following the first three frequencies of vibration for the two planes x-y and x-z, computed by using the exact solution, and both the straightforward and rigorous Galerkin method.

6.3. Exact solution

The first three frequencies of vibration in the $x - y$ and x-z planes, computed by using the exact formulation in section 3, are shown in Table 4.

6.4. Rigorous Galerkin method

The rigorous Galerkin method, for 1, 2, 3, 25, 50, 75 and 100 terms, leads to the frequencies in Tables 5 and 7 for the frequencies of vibration in the x-y and x-z plane, respectively.

The relative error between the natural frequencies computed via the Galerkin method and the exact ones, computed as:

$$\varepsilon = \frac{\omega_{Galerkin} - \omega_{Exact}}{\omega_{Exact}} \cdot 100\% \tag{34}$$

is reported in Table 6.

6.5. Straightforward Galerkin method

Similarly, we computed the frequencies of vibration for the straightforward implementation of the Galerkin method for 1, 2, 3, 25, 50, 75 and 100 terms. The frequencies of vibration in the x-y and x-z plane are reported in Table 9 and Table 11, respectively, whereas the relative errors with respect to the exact solutions are reported in Tables 10 and 12.

Table 4
Exact solution.

Exact solution [rad/sec]		
Mode	$x - y$ plane	$x - z$ plane
1	342.4121	67.5133
2	2166.4943	423.9471
3	6143.9243	1191.0450

Table 5
Frequencies of vibration for the x-y plane obtained with rigorous Galerkin method.

Frequencies [rad/s]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	532.3005	525.2584	525.2059	385.5929	362.8573	353.8210	352.2366
2	–	3303.9364	3302.5800	2590.9190	2296.1458	2238.5734	2229.1627
3	–	–	9288.0450	7360.6548	6511.9947	6377.2805	6322.6071

Table 6
Relative error between the rigorous Galerkin method and the exact solution for the frequencies of vibration in the x-y plane.

Relative error [%]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	53.41%	53.40%	53.38%	12.61%	5.97%	3.33%	2.87%
2	–	52.50%	52.44%	19.59%	5.98%	3.33%	2.89%
3	–	–	51.17%	19.80%	5.99%	3.80%	2.91%

Table 7
Frequencies of vibration for the x-z plane obtained with rigorous Galerkin method.

Frequencies [rad/s]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	71.1399	71.1379	71.1364	68.1456	67.9620	67.7800	67.7035
2	–	446.1477	446.0782	427.7656	426.7094	425.6417	425.1656
3	–	–	1250.6176	1208.3126	1199.1534	1195.9843	1194.5526

Table 8
Relative error between the rigorous Galerkin method and the exact solution for the frequencies of vibration in the x-z plane.

Relative error [%]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	5.37%	5.37%	5.37%	0.94%	0.66%	0.39%	0.28%
2	–	5.24%	5.22%	0.90%	0.65%	0.40%	0.29%
3	–	–	5.00%	1.45%	0.68%	0.41%	0.29%

Table 9
Frequencies of vibration for the x-y plane obtained with straightforward Galerkin method.

Frequencies [rad/s]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	531.0601	530.9327	530.7799	462.7421	427.8476	420.8202	419.3807
2	–	3335.5952	3334.2992	2884.2579	2697.5635	2652.9731	2645.0811
3	–	–	9357.7550	8163.1046	7594.1873	7490.8281	7450.9256

Table 10
Relative error between the straightforward Galerkin method and the exact solution for the frequencies of vibration in the x-y plane.

Relative error [%]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	55.09%	55.06%	55.01%	35.14%	24.95%	22.90%	22.48%
2	–	53.96%	53.90%	33.13%	24.51%	22.45%	22.09%
3	–	–	52.31%	32.86%	23.60%	21.92%	21.27%

6.6. Final comparison

Table 13 reports the frequencies of vibration computed on the previous subsections, in Hertz, along with those obtained experimentally by Jaworski and Dowell [8].

The relative errors between the first three columns and the fourth, evaluated with formula (35), are collected in Table 14:

Table 11
Frequencies of vibration for the x-z plane obtained with straightforward Galerkin method.

Frequencies [rad/s]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	71.5207	71.5207	71.5207	71.5207	71.5207	71.5207	71.5207
2	–	448.2448	448.2448	448.2448	448.2448	448.2448	448.2448
3	–	–	1255.2248	1255.2248	1255.2248	1255.2248	1255.2248

Table 12
Relative error between the straightforward Galerkin method and the exact solution for the frequencies of vibration in the x-z plane.

Relative error [%]							
Mode	1 Term	2 Terms	3 Terms	25 Terms	50 Terms	75 Terms	100 Terms
1	5.94%	5.94%	5.94%	5.94%	5.94%	5.94%	5.94%
2	–	5.73%	5.73%	5.73%	5.73%	5.73%	5.73%
3	–	–	5.39%	5.39%	5.39%	5.39%	5.39%

Table 13
Frequencies obtained by three methods compared with those proposed in Ref. Jaworski and Dowell [8].

Frequency		Exact solution	Rigorous Galerkin method (100 Terms)	Straightforward Galerkin method (100 Terms)	Experimental results, Jaworski and Dowell [8]
x-y plane	1	54.4965	56.0601	66.7464	49.38
	2	344.8075	354.7814	420.9767	–
	3	977.8336	1006.2718	1185.8489	–
x-z plane	1	10.7451	10.7753	11.3828	10.63
	2	67.4731	67.6670	71.3402	66.75
	3	189.5603	190.1185	199.7748	–

Table 14
Relative errors between all the methods.

Frequency		Exact solution vs paper by Jaworski/Dowell	Rigorous Galerkin method vs paper by Jaworski/Dowell	Straightforward Galerkin method vs paper by Jaworski/Dowell
In Plane	1	10.36%	13.53%	35.17%
Out of Plane	1	1.08%	1.37%	7.08%
	2	1.08%	1.37%	6.88%

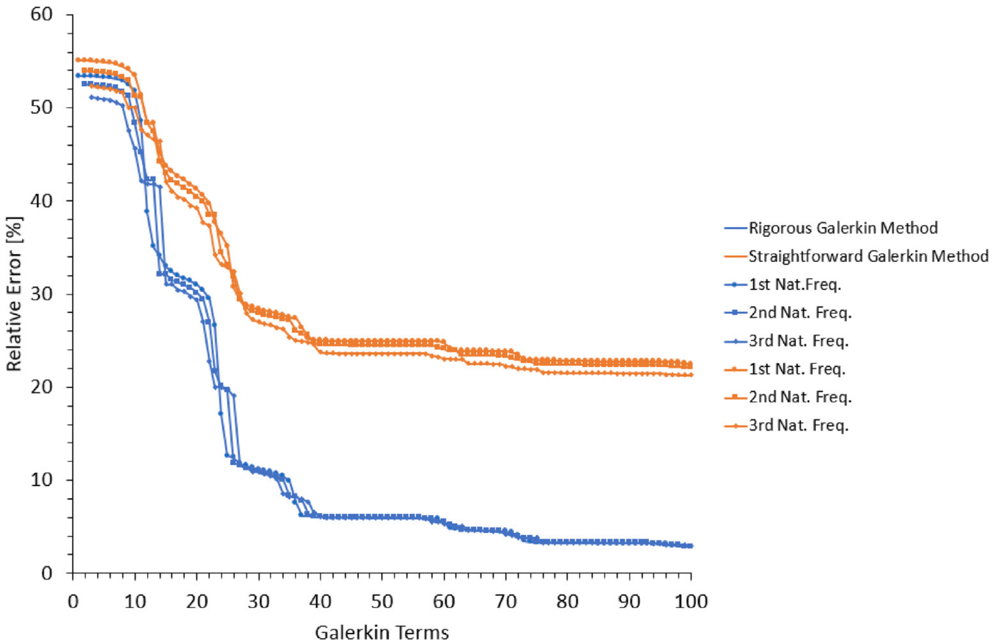


Fig. 4. Rigorous vs straightforward Galerkin method – relative error.

$$\varepsilon = \frac{\omega - \omega_{exact}}{\omega_{exact}} \cdot 100\% \tag{35}$$

The Table 14 shows that the rigorous Galerkin method tends to the results yielded by the experimental study carried out by Jaworski and Dowell [8]. Moreover, Tables 6 and 8 clearly demonstrate that the proposed approach tends to the exact solution too. This aspect is demonstrated also by Fig. 4 which shows that the error decreases with increase of the number of terms. On the other hand, straightforward Galerkin method does not converge to the exact solution. The experimental frequencies reported by Jaworski and Dowell [8] are close to the results obtained with Krylov-Duncan functions and rigorous Galerkin method but not with its straightforward implementation.

7. Conclusions

In this study we have analyzed the free vibrations of a cantilever homogeneous non-uniform beam with different methods. In particular, we have compared two different versions of Galerkin method, namely the straightforward version which is generally used in literature and the rigorous one proposed in this work, with the exact solution based on the Krylov-Duncan functions. While the straightforward approach considers basis functions with domain existing only over each segment, the rigorous implementation is based on two generalized functions, one for the stiffness and one for the mass of the beam, existing over the entire length of the beam. In this case, the abrupt changes of cross-sections are taken into account via the Heaviside function and its derivatives. The study shows that the proposed rigorous implementation does convergence to the derived exact solutions whereas the straightforward version does not.

8. Dedication

This paper is dedicated to the blessed memory of Professor Simon G. Braun, the former Editor-in-Chief of this journal, and the former colleague of one of the authors (I.E.), at the Technion--Israel Institute of Technology, Haifa, Israel.

CRediT authorship contribution statement

Isaac Elishakoff: Conceptualization, Methodology, Validation, Formal analysis, Investigation. **Marco Amato:** Methodology, Validation, Formal analysis, Investigation, Software, Data curation. **Alessandro Marzani:** Methodology, Validation, Formal analysis, Investigation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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