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Flutter of a beam in supersonic flow: truncated version of Timoshenko–Ehrenfest equation is sufficient

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Abstract This paper deals with flutter due to gas flow of a uniform and homogeneous beam with shear deformation and rotary inertia effects taken into account. It utilizes the truncated, consistent Timoshenko–Ehrenfest beam model in contrast to the original Timoshenko–Ehrenfest equations. Omission of the fourth order derivative term in the governing differential equation of the original equations of Timoshenko and Ehrenfest is shown to be more consistent than retaining it as is done in the original Timoshenko–Ehrenfest set. The original Timoshenko–Ehrenfest equations are exposed in almost all textbooks published after 1921. However, this paper shows that these original equations are unnecessarily overcomplicated, and simpler set is consistent. This truncation significantly simplifies the analytical and numerical analyses considerably and agrees with the principle of parsimony, i.e. the most acceptable explanation of a phenomenon is the simplest, involving the fewest entities and assumptions. The critical flutter velocities are compared with the results obtained by using the original set of Timoshenko–Ehrenfest equations.

Keywords Flutter instability · Supersonic flow · Timoshenko–ehrenfest theory · Truncated timoshenko–ehrenfest beam theory · Rotary inertia and shear deformation

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1 Introduction

Timoshenko and Ehrenefest cooperated in development of the beam theory that incorporates the effects of rotary inertia and shear deformation, as detailed in the recent monograph by Elishakoff (2020a). This theory, referred hereinafter as Timoshenko–Ehrenfest beam theory, was first presented in the book by Timoshenko (1916) in 1916, in the Russian language. In this book Stephen Prokofievich Timoshenko (1878–1972) mentioned that the theory was the result of his cooperation with Paul Ehrenfest (1880–1933), the famous Austrian-born Dutch physicist. Later on, Timoshenko published the same results twice again, both in English, namely, in 1920 and in 1921. Usually, this latter publication is referenced in overwhelming majority of papers and books. Later on, Timoshenko (1928, 1937, 1955, 1974, 1990) introduced the same material into his textbook on vibration, almost without any modification.

In 1985, Elishakoff and Lubliner (1985) suggested to drop the last term in original Thimoshenko-Ehrenfest equation, namely the fourth order derivative with respect to time, altogether, since it alters natural frequencies insignificantly. In Ref. (Elishakoff 2010) it was shown that this equation can be obtained by making relevant kinematic assumptions. Ref. (Elishakoff et al. 2015) demonstrated that the original Timoshenko–Ehrenfest equations were asymptotically inconsistent. It turned out that the simplest consistent equation was the Timoshenko–Ehrenfest equation without the fourth-order derivative in the respect of time as suggested by Elishakoff and Lubliner (1985), referred hereinafter as the truncated, consistent theory. Ref. (Elishakoff 2020b) discusses, in some detail, the question of priority associated with incorporating the effects of shear deformation and rotary inertia [see also Ref. (Challamel and Elishakoff 2019)]. We first study free vibrations of beams incorporating the effects of shear deformation and rotary inertia, and with knowledge of natural frequencies and normal modes, proceed to evaluate flutter velocities via Galerkin method [to get a broader view of non-local approaches an interested reader may consult (Paola et al. 2014, 2013; Paola and Zingales 2008; Cottone et al. 2009) and (Alotta et al. 2020)] when beam is in the supersonic flow.

2 Free vibration of Timoshenko–Ehrenfest beam with or without the 4-th derivative with respect to time

2.1 Governing differential equation for original Timoshenko–Ehrenfest theory

The governing differential equations for the original Timoshenko beam are two: one for the vertical displacement $w(x,t)$, and the other one for the rotation angle of the cross section $\vartheta(x,t)$

$$k'AG\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial \vartheta}{\partial x}\right)-\rho A\frac{\partial^2 w}{\partial t^2}=0 \tag{1a}$$

$$EI\frac{\partial^2 \vartheta}{\partial x^2}+k'AG\left(\frac{\partial w}{\partial x}-\vartheta\right)-\rho I\frac{\partial^2 \vartheta}{\partial t^2}=0 \tag{1b}$$

where E is the elastic modulus, G is the shear modulus, A is the area of the cross section, ρ is the mass density, I is the inertia moment of the cross section and k' is the shear coefficient.

From the first Eq. (1a) we define $\frac{\partial\vartheta}{\partial x}$ and we substitute the obtained expression in the differentiation of the second Eq. (1b) with respect to x; obtaining:

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\left(1+\frac{E}{k'G}\right)\frac{\partial^4 w}{\partial t^2\partial x^2}+\frac{\rho^2 I}{k'G}\frac{\partial^4 w}{\partial t^4}=0 \tag{2}$$

As is seen the governing differential equation contains the fourth order derivative of time. Equation (2) constitutes the original Timoshenko–Ehrenfest equation.

2.2 Governing differential equation for truncated, consistent version of the Timoshenko–Ehrenfest beam theory

Timoshenko (1920, 1921) concluded that the fourth term in Eq. (2) produces a small effect in the characteristic equation. On the basis of this affirmation, we recapitulate the derivation of consistent Timoshenko–Ehrenfest theory which does not contains the last term.

We first recall the Euler–Bernoulli free vibration differential equation:

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}=0 \tag{3}$$

Bresse (1859) and Rayleigh (1945) refined it by introducing rotary movements of the beam element in addition to the translatory ones. The angle of rotation equals the slope of the deflection curve, $\partial w/\partial x$; the corresponding angular acceleration is $\partial^3 w/\partial x\partial t^2$. As a result, the moment of inertia of the element about an axis through its center of mass equals $\rho I\left(\frac{\partial^3 w}{\partial x\partial t^2}\right)dx$. Incorporation of this moment, due to D'Alembert's principle leads to the following equation

$$-V_y+\frac{\partial M}{\partial x}-\rho I\frac{\partial^3 w}{\partial x\partial t^2}=0 \tag{4}$$

where V_y is the shear force. Substitution of V_y from Eq. (4) in the dynamic equilibrium condition for transverse vibration:

$$\frac{\partial V_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial M}{\partial x} - \rho I \frac{\partial^3 w}{\partial x \partial t^2}\right) = \rho A \frac{\partial^2 w}{\partial t^2} \tag{5}$$

Timoshenko and Ehrenfest (see Timoshenko (1916, 1921)) introduced the shear deformation saying that the slope of the deflection consists in two terms:

$$\frac{\partial w}{\partial x} = \vartheta + \beta \tag{6}$$

ϑ is the rotation of the cross section with shear deformation neglected and β the angle associated with the shear deformation at the neutral axis in the same cross section. This assumption leads us to:

$$M = EI \frac{\partial \vartheta}{\partial x} \tag{7a}$$

$$V_y = -k'AG\beta = -k'AG\left(\frac{\partial w}{\partial x} - \vartheta\right) \tag{7b}$$

Now, if one introduces Eqs. (7a) and (7b) into Eq. (4) one obtains:

$$EI \frac{\partial^2 \vartheta}{\partial x^2} + k'AG\left(\frac{\partial w}{\partial x} - \vartheta\right) - \rho I \frac{\partial^3 w}{\partial x \partial t^2} = 0 \tag{8}$$

Now, if one introduces Eq. (7b) into Eq. (5) the following result is derived:

$$k'AG\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \vartheta}{\partial x}\right) - \rho A \frac{\partial^2 w}{\partial t^2} = 0 \tag{9}$$

Last two equation represent the starting system for the free vibration differential equation:

$$k'AG\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \vartheta}{\partial x}\right) - \rho A \frac{\partial^2 w}{\partial t^2} = 0 \tag{10a}$$

$$EI \frac{\partial^2 \vartheta}{\partial x^2} + k'AG\left(\frac{\partial w}{\partial x} - \vartheta\right) - \rho I \frac{\partial^3 w}{\partial x \partial t^2} = 0 \tag{10b}$$

Now following the same idea of the previous section, we can join these two equations in only one, obtaining:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \rho I \left(1 + \frac{E}{k'G}\right) \frac{\partial^4 w}{\partial t^2 \partial x^2} = 0 \tag{11}$$

Actually, we put the governing differential equation in the following generic form:

$$\begin{aligned} &EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} \\ &+ \varepsilon\left[-\rho I \left(1 + \frac{E}{k'G}\right) \frac{\partial^4 w}{\partial t^2 \partial x^2} + \delta \frac{\rho^2 I}{k'G} \frac{\partial^4 w}{\partial t^4}\right] \\ &= 0 \end{aligned} \tag{12}$$

When $\varepsilon = 0$ we obtain the Bernoulli–Euler's beam theory. When $\varepsilon = 1$ we are introduced into two different Timoshenko–Ehrenfest beam's theories which are identified by a control parameter δ. When $\delta = 1$ we derive the original theory (Elishakoff 2020a; Timoshenko 1916, 1920, 1921) otherwise for $\delta = 0$ we obtain the truncated, consistent theory (Elishakoff et al. 2015).

2.3 Original Timoshenko–Ehrenfest theory

We think $w(x, t)$ as a product of two function: one in the time and one in the axial coordinate x. More specifically we think them in exponential form as follows:

$$w(x, t) = e^{\beta x} e^{i\omega t} \tag{13}$$

where ω is the frequency of the beam's vibration.

With this substitution and dividing by EI we obtain the following 4-th order equation:

$$\beta^4 + \frac{k'G\rho\omega^2 + \rho E\omega^2}{k'GE}\beta^2 + \frac{\rho^2 I\omega^4 - \rho A\omega^2 k'G}{k'GEI} = 0 \tag{14}$$

Denoting $d = \frac{k'G\rho\omega^2 + \rho E\omega^2}{k'GE}$, $e = \frac{\rho^2 I\omega^4 - \rho A\omega^2 k'G}{k'GEI}$ and $\beta^2 = z$, we obtain:

$$z^2 + dz + e = 0 \tag{15}$$

Solution of this equation are:

$$z_{1,2} = \frac{1}{2}\left(-d \pm \sqrt{\Delta}\right) \tag{16}$$

where $\Delta = d^2 - 4e = \left(\frac{\omega^2\rho}{E} - \frac{\omega^2\rho}{k'G}\right)^2 + \frac{4\omega^2\rho A}{EI}$. It is easy to see that this part is positive for any value of ω.

Now we study the sing of z_1 and z_2.

- $z_2 < 0$ for any value of ω
- $z_1 > 0 \iff -d + \sqrt{d^2 - 4e} \iff \sqrt{d^2 - 4e} > d \iff e < 0$

  $e < 0$ means that $\omega^2 < \frac{k'AG}{\rho I}$

  We call $\omega = \sqrt{\frac{k'AG}{\rho I}} = \omega_{lim}$.

Note that $z_1 < 0$ implies that $\omega^2 > {\omega_{lim}}^2$

Now we define $\beta_3 = i\sqrt{abs(z_2)}$ and $\beta_4 = -i\sqrt{abs(z_2)}$ for any value of ω; and if $\omega < \omega_{lim}$ we set $\beta_1 = \sqrt{z_1}$ and $\beta_2 = -\sqrt{z_1}$ otherwise if $\omega > \omega_{lim}$ we set $\beta_1 = i\sqrt{abs(z_1)}$ and $\beta_2 = -i\sqrt{abs(z_1)}$.

In general, we can write the spatial part of equation (13) in the following form:

$$W(x) = C_1 e^{\beta_1 x} + C_2 e^{\beta_2 x} + C_3 e^{\beta_3 x} + C_4 e^{\beta_4 x} \tag{17}$$

In the specific case $\omega < \omega_{lim}$, using Euler's formula we obtain:

$$W(x) = P_1 \cosh(\lambda_1 x) + P_2 \sinh(\lambda_1 x) + P_3 \cos(\lambda_2 x) + P_4 \sin(\lambda_2 x) \tag{18}$$

where $\lambda_1 = \sqrt{\frac{-d+\sqrt{\Delta}}{2}}$ and $\lambda_2 = \sqrt{\frac{d+\sqrt{\Delta}}{2}}$. If one substitute equation (18) in the integral with respect to x of definition $\frac{\partial \vartheta}{\partial x}$ derived form ($1a$) obtains [see Ref. Pielorz (1996)]:

$$\begin{aligned}\vartheta(x,t) &= \left[\left(\lambda_1 + \frac{\rho\omega^2}{k'G\lambda_1}\right) P_1 \sinh(\lambda_1 x) + \left(\lambda_1 + \frac{\rho\omega^2}{k'G\lambda_1}\right) P_2 \cosh(\lambda_1 x)\right. \\ &\left. + \left(-\lambda_2 + \frac{\rho\omega^2}{k'G\lambda_2}\right) P_3 \sin(\lambda_2 x) + \left(\lambda_2 - \frac{\rho\omega^2}{k'G\lambda_2}\right) P_4 \cos(\lambda_2 x)\right] e^{i\omega t}\end{aligned} \tag{19}$$

If $\omega > \omega_{lim}$ with Euler's formula:

$$W(x) = Q_1 \cos(\lambda_1 x) + Q_2 \sin(\lambda_1 x) + Q_3 \cos(\lambda_2 x) + Q_4 \sin(\lambda_2 x) \tag{20}$$

where $\lambda_1 = \sqrt{\frac{d-\sqrt{\Delta}}{2}}$ and $\lambda_2 = \sqrt{\frac{d+\sqrt{\Delta}}{2}}$. If one substitutes equation (20) in the integral with respect to x of definition $\frac{\partial \vartheta}{\partial x}$ derived form ($1a$) obtains:

$$\begin{aligned}\vartheta(x,t) &= \left[\left(-\lambda_1 + \frac{\rho\omega^2}{k'G\lambda_1}\right) Q_1 \sin(\lambda_1 x) + \left(\lambda_1 - \frac{\rho\omega^2}{k'G\lambda_1}\right) Q_2 \cos(\lambda_1 x)\right. \\ &\left. + \left(-\lambda_2 + \frac{\rho\omega^2}{k'G\lambda_2}\right) Q_3 \sin(\lambda_2 x) + \left(\lambda_2 - \frac{\rho\omega^2}{k'G\lambda_2}\right) Q_4 \cos(\lambda_2 x)\right] e^{i\omega t}\end{aligned} \tag{21}$$

To find the four coefficients P_1, P_2, P_3 and P_4 or Q_1, Q_2, Q_3 and Q_4 in the equation (18) or (20) we have to satisfy the boundary conditions. For a clamped-free beam under consideration we demand: vertical displacement and rotation angle equal to zero in the left end and bending moment and shear force equal to zero in the right end.

For $x = 0$

$$w(0) = 0 \tag{22a}$$

$$\vartheta(0) = 0 \tag{22b}$$

- If $\omega < \omega_{lim}$

$$P_1 + P_3 = 0 \tag{23a}$$

$$\left(\lambda_1 + \frac{\rho\omega^2}{k'G\lambda_1}\right) P_2 + \left(\lambda_2 - \frac{\rho\omega^2}{k'G\lambda_2}\right) P_4 = 0 \tag{23b}$$

- If $\omega > \omega_{lim}$

$$Q_1 + Q_3 = 0 \tag{24a}$$

$$\left(\lambda_1 - \frac{\rho\omega^2}{k'G\lambda_1}\right) Q_2 + \left(\lambda_2 - \frac{\rho\omega^2}{k'G\lambda_2}\right) Q_4 = 0 \tag{24b}$$

For $x = L$

$$w'(L) - \vartheta(L) = 0 \tag{25a}$$

$$\vartheta'(L) = 0 \tag{25b}$$

- If $\omega < \omega_{lim}$

$$\begin{aligned}&-\frac{\rho\omega^2 P_1}{k'G\lambda_1} \sinh(\lambda_1 L) - \frac{\rho\omega^2 P_2}{k'G\lambda_1} \cosh(\lambda_1 L) \\ &\quad - \frac{\rho\omega^2 P_3}{k'G\lambda_1} \sin(\lambda_2 L) + \frac{\rho\omega^2 P_4}{k'G\lambda_1} \cos(\lambda_2 L) \\ &\quad = 0\end{aligned} \tag{26a}$$

$$\begin{aligned}&\left(\lambda_1 + \frac{\rho\omega^2}{k'G\lambda_1}\right) \lambda_1 P_1 \cosh(\lambda_1 L) + \left(\lambda_1 + \frac{\rho\omega^2}{k'G\lambda_1}\right) \\ &\quad \lambda_1 P_2 \sinh(\lambda_1 L) \\ &+ \left(\lambda_2 - \frac{\rho\omega^2}{k'G\lambda_2}\right) \lambda_2 P_3 \cos(\lambda_2 L) - \left(\lambda_2 - \frac{\rho\omega^2}{k'G\lambda_2}\right) \\ &\quad \lambda_2 P_4 \sin(\lambda_2 L) = 0\end{aligned} \tag{26b}$$

- If $\omega > \omega_{lim}$

$$-\frac{\rho\omega^2 Q_1}{k'G\lambda_1}\sin(\lambda_1 L)+\frac{\rho\omega^2 Q_2}{k'G\lambda_1}\cos(\lambda_1 L)-\frac{\rho\omega^2 Q_3}{k'G\lambda_1}\sin(\lambda_2 L)+\frac{\rho\omega^2 Q_4}{k'G\lambda_1}\cos(\lambda_2 L)=0 \quad (27a)$$

$$\left(-\lambda_1+\frac{\rho\omega^2}{k'G\lambda_1}\right)\lambda_1 Q_1\cos(\lambda_1 L)-\left(\lambda_1-\frac{\rho\omega^2}{k'G\lambda_1}\right)\lambda_1 Q_2\sin(\lambda_1 L)+\left(-\lambda_2+\frac{\rho\omega^2}{k'G\lambda_2}\right)\lambda_2 Q_3\cos(\lambda_2 L)-\left(\lambda_2-\frac{\rho\omega^2}{k'G\lambda_2}\right)\lambda_2 Q_4\sin(\lambda_2 L)=0 \quad (27b)$$

Now we obtained four equations for four unknowns; for a non-trivial solution we demand that the determinant of the matrix coefficient is equal to zero. This demand leads to a function in the single variable in terms of ω. When the function crosses the x axis we find the value of ω that makes the determinant equal to zero.

2.4 Truncated, consistent version of the Timoshenko–Ehrenfest beam theory

Following the same path that the previous section we obtain the second order polynomial equation in the variable z:

$$z^2+dz+e=0 \quad (28)$$

Which leads to the solutions:

$$z_{1,2}=\frac{1}{2}\left(-d\pm\sqrt{\Delta}\right) \quad (29)$$

where $\Delta=d^2-4e=\left(\frac{\omega^2\rho}{E}+\frac{\omega^2\rho}{k'G}\right)^2+\frac{4\omega^2\rho A}{EI}$. We see that the expression for Δ is positive for any value of ω.

Studying the signs of z_1 and z_2:

- $z_2<0$ for any value of ω
- $z_1>0 \iff -d+\sqrt{d^2-4e} \iff \sqrt{d^2-4e}>d \iff e<0$

$e<0$ for any value of ω

It is interesting to see that for the truncated, consistent version of the Timoshenko–Ehrenfest theory the term ω_{lim} does not appear.

Defining $\beta_1=\sqrt{z_1}, \beta_2=-\sqrt{z_1}, \beta_3=i\sqrt{abs(z_2)}$ and $\beta_4=-i\sqrt{abs(z_2)}$ we write the spatial part of equation (13) with Euler's transformation, obtaining:

$$W(x)=P_1\cosh(\lambda_1 x)+P_2\sinh(\lambda_1 x)+P_3\cos(\lambda_2 x)+P_4\sin(\lambda_2 x) \quad (30)$$

where $\lambda_1=\sqrt{\frac{-d+\sqrt{\Delta}}{2}}$ and $\lambda_2=\sqrt{\frac{d+\sqrt{\Delta}}{2}}$. Substitution of equation (30) in the result of integration of definition $\frac{\partial\vartheta}{\partial x}$ derived form (10a) with respect to x leads to:

$$\vartheta(x,t)=\left[\left(\lambda_1+\frac{\rho\omega^2}{k'G\lambda_1}\right)P_1\sinh(\lambda_1 x)+\left(\lambda_1+\frac{\rho\omega^2}{k'G\lambda_1}\right)P_2\cosh(\lambda_1 x)+\left(-\lambda_2+\frac{\rho\omega^2}{k'G\lambda_2}\right)P_3\sin(\lambda_2 x)+\left(\lambda_2-\frac{\rho\omega^2}{k'G\lambda_2}\right)P_4\cos(\lambda_2 x)\right]e^{i\omega t} \quad (31)$$

To find four coefficients of the equation we have to set the boundary conditions.

Specifically, for $x=0$, we have

$$w(0)=0 \quad (32a)$$

$$\vartheta(0)=0 \quad (32b)$$

That implies that

$$P_1+P_3=0 \quad (33a)$$

$$\left(\lambda_1+\frac{\rho\omega^2}{k'G\lambda_1}\right)P_2+\left(\lambda_2-\frac{\rho\omega^2}{k'G\lambda_2}\right)P_4=0 \quad (33b)$$

When $x=L$

$$w'(L)-\vartheta(L)=0 \quad (34a)$$

$$\vartheta'(L)=0 \quad (34b)$$

That means

$$-\frac{\rho\omega^2 P_1}{k'G\lambda_1}\sinh(\lambda_1 L)-\frac{\rho\omega^2 P_2}{k'G\lambda_1}\cosh(\lambda_1 L)-\frac{\rho\omega^2 P_3}{k'G\lambda_1}\sin(\lambda_2 L)+\frac{\rho\omega^2 P_4}{k''G\lambda_1}\cos(\lambda_2 L)=0 \quad (35a)$$

$$\left(\lambda_1+\frac{\rho\omega^2}{k'G\lambda_1}\right)\lambda_1 P_1\cosh(\lambda_1 L)+\left(\lambda_1+\frac{\rho\omega^2}{k'G\lambda_1}\right)\lambda_1 P_2\sinh(\lambda_1 L)+\left(\lambda_2-\frac{\rho\omega^2}{k'G\lambda_2}\right)\lambda_2 P_3\cos(\lambda_2 L)-\left(\lambda_2-\frac{\rho\omega^2}{k'G\lambda_2}\right)\lambda_2 P_4\sin(\lambda_2 L)=0 \quad (35b)$$

2.5 Numerical example

We deal with a cantilever beam with the following data:

$$
\begin{aligned}
&E = 2.1 \cdot 10^{11} Pa \qquad L = 1m \\
&G = 6.3 \cdot 10^{10} Pa \qquad b = 0.02m \\
&\rho = 7860 \frac{kg}{m^3} \qquad h = 0.10m \\
&\nu = 0.2
\end{aligned} \tag{36}
$$

Normally, for rectangular section, the shear coefficient k' is taken as follows:

$$k' = \frac{10(1+\nu)}{12+11\nu} \tag{37}$$

2.5.1 *FEM solution*

In order to validate the theoretical results, we use the FEM software Strand7 which uses shear-deformable elements. Strand7 was developed by a group of academics from the University of Sydney and the University of New South Wales in 2000 [see Ref. Strand7 (2004)]. We implement a mash with fifty-one nodes.

2.5.2 *Original Timoshenko–Ehrenfest beam model*

With the beam's data, the value of ω_{lim} is:

$$\omega_{lim} = \sqrt{\frac{k'AG}{\rho I}} = 89528.0245 \frac{rad}{s} \tag{38}$$

It turns out this value is exceeding the value of the natural frequency ω_{10} obtained with FEM analysis so we concentrate on the first case relating to $\omega < \omega_{lim}$.

2.5.3 *Truncated, consistent Timoshenko–Ehrenfest beam model*

This model, as we showed, has not the ω_{lim} so we do not have to take care of this parameter.

In Table 1 we resume the results obtained with these three approaches.

In Table 2 we list the value of the relative error in percentual for the first ten natural frequencies between all the three different approaches using following expressions:

$$
\begin{aligned}
\varepsilon_1 &= \frac{\omega_{FEM} - \omega_{Original}}{\omega_{Original}} \cdot 100\% \\
\varepsilon_2 &= \frac{\omega_{FEM} - \omega_{Consistent}}{\omega_{Consistent}} \cdot 100\% \\
\varepsilon_3 &= \frac{\omega_{Original} - \omega_{Consistent}}{\omega_{Consistent}} \cdot 100\%
\end{aligned} \tag{39}
$$

Examination of Tables 1, 2 shows that these three different approaches yield similar results. In particular, the two theoretical ways, namely original and consistent, truncated versions of Timoshenko–Ehrenfest equation produce very close results. Thus, we verify that we can safely neglect the fourth derivative of the time in the original Timoshenko–Ehrenfest equations.

3 Dynamic stability of Timoshenko–Ehrenfest beam with or without the fourth derivative with respect to time

3.1 Piston theory

The solution of problems on the stability of elastic bodies in a flow of gas is made difficult by the

Table 1 Solutions obtained with different approaches

Mode number	FEM [rad/sec]	Original T-E theory [rad/sec]	Consistent T-E theory [rad/sec]
1	520.5223	519.6399	519.6313
2	3121.0979	3086.8280	3085.2128
3	8202.5919	8031.3652	8006.8717
4	14837.5883	14402.9084	14285.3683
5	22475.5719	21700.9585	21369.5008
6	30685.9550	29560.5088	28870.1051
7	39187.6272	37755.5844	36564.7675
8	47801.2661	46139.6594	44330.5689
9	56416.7271	54614.3405	52102.8447
10	64968.2880	63106.0902	59851.0564

Table 2 Relative error with different approaches

Mode number	FEM versus original T-E theory [%]	FEM versus consistent T-E theory [%]	Original versus consistent T-E theory [%]
1	0.1698	0.1715	0.0017
2	1.1102	1.1631	0.0524
3	2.1320	2.4444	0.3059
4	3.0180	3.8656	0.8228
5	3.5695	5.1759	1.5511
6	3.8073	6.2897	2.3914
7	3.7929	7.1732	3.2567
8	3.6013	7.8291	4.0809
9	3.3002	8.2796	4.8203
10	2.9509	8.5499	5.4386

complicated expression for the non-stationary aerodynamic forces in a distributed flow. Fortunately, at sufficiently high supersonic velocities, which nowadays present most interest, the aerodynamic side of the problem can be considerably simplified. Consider first the steady motion of a thin profile at supersonic velocity U. In contrast to the subsonic case, in which the velocity field at every point depends on the normal component of velocity at all points on the surface of the body, the disturbances here are transmitted only in a downstream direction. As the velocity U increases, the disturbances assume more local character, and in the limiting case of very high supersonic velocities each particle of gas moves only in a direction practically perpendicular to the velocity U.

Accepting this description of the phenomenon, we conclude that the excess pressure on the surface of the body can be calculated in the same way as the pressure on a piston moving in a one-dimensional duct [see Ref. Ashley and Zartarian (1956) and Ilyushin (1956)] is $P = \frac{\kappa p_\infty}{c_\infty}\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right)$ where κ is the polytropy index of the gas, p_∞ is the pressure in the undisturbed gas, c_∞ is the velocity of sound in undisturbed gas, U velocity of gas flow.

If we analyse this formula from a dimensional point of view, we can see that P has a dimension of pressure. We can convert this load to the distributed load $q(x,t)$ easily by multiplying the pressure by the width of the section b:

$$q(x,t) = Pb = \frac{\kappa p_\infty b}{c_\infty}\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right) \tag{40}$$

3.2 Governing differential equation for original Timoshenko–Ehrenfest beam in gas flow

The governing differential equation for the Timoshenko–Ehrenfest beam are two: one for the vertical displacement $w(x,t)$ and one for the rotation angle of the cross section $\vartheta(x,t)$

$$\rho A\frac{\partial^2 w}{\partial t^2} = k'AG\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\vartheta}{\partial x}\right) - q(x,t) \tag{41a}$$

$$\rho I\frac{\partial^2 \vartheta}{\partial t^2} = EI\frac{\partial^2 \vartheta}{\partial x^2} + k'AG\left(\frac{\partial w}{\partial x} - \vartheta\right) \tag{41b}$$

where E is the elastic modulus, G is the shear modulus, A is the area of the cross section, ρ is the mass density, I is the inertia moment of the cross section, k' is the shear coefficient and q the load.

Now we introduce the dimensionless parameters as follows:

$$
\begin{aligned}
&\xi = \frac{x}{L} \to x = \xi L \qquad && \mu = \frac{k'G}{E} \\
&v = \frac{w}{L} \to w = vL \qquad && \eta = \frac{AL^2}{I} = \frac{L^2}{r^2} \\
&\tau = t\sqrt{\frac{Gk'}{\rho L^2}} \to t = \tau\sqrt{\frac{\rho L^2}{Gk'}} \qquad && \gamma = \frac{\kappa p_\infty bUL}{k'AGc_\infty} \\
& && \alpha = \frac{\kappa p_\infty bL}{Ac_\infty\sqrt{k'G\rho}}
\end{aligned} \tag{42}
$$

where ξ is the non-dimensional axial coordinate, v is the non-dimensional transversal displacement, τ is the non-dimensional time, μ is a non-dimensional mechanical parameter related to the shear deformation, η is a non-dimensional geometrical parameter related to the squared length L over radius of inertia $r = \sqrt{I/A}$, γ is a non-dimensional parameter related to the gas velocity and α is the non-dimensional damping caused by the supersonic velocity.

Substituting the above parameters into (41a) and (41b) and merging the equations in only one we obtain:

$$
\begin{aligned}
&\frac{1}{\mu}\frac{\partial^4 v}{\partial\xi^4} + \frac{\partial^4 v}{\partial\tau^4} - \left(1 + \frac{1}{\mu}\right)\frac{\partial^4 v}{\partial\tau^2\partial\xi^2} - \frac{\gamma}{\mu}\frac{\partial^3 v}{\partial\xi^3} + \alpha\frac{\partial^3 v}{\partial\tau^3} \\
&\quad + \gamma\frac{\partial^3 v}{\partial\tau^2\partial\xi} - \frac{\alpha}{\mu}\frac{\partial^3 v}{\partial\tau\partial\xi^2} + \eta\frac{\partial^2 v}{\partial\tau^2} + \eta\gamma\frac{\partial v}{\partial\xi} + \eta\alpha\frac{\partial v}{\partial\tau} \\
&\quad = 0
\end{aligned} \tag{43}
$$

The problem consists in fixing geometric and material parameters to specific values in order to evaluate the critical velocity U_{cr} of the gas flow.

3.3 Consistent, truncated governing differential equation for Timoshenko–Ehrenfest beam in gas flow

In analogy with the original theory also the consistent, truncated theory is represented by two equations: one for the vertical displacement $w(x,t)$ and one for the rotation angle of the cross section $\vartheta(x,t)$

$$
\rho A\frac{\partial^2 w}{\partial t^2} = k'AG\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\vartheta}{\partial x}\right) - q(x,t) \tag{44a}
$$

$$
\rho I\frac{\partial^2 w}{\partial x\partial t^2} = EI\frac{\partial^2\vartheta}{\partial x^2} + k'AG\left(\frac{\partial w}{\partial x} - \vartheta\right) \tag{44b}
$$

where E is the elastic modulus, G is the shear modulus, A is the area of the cross section, ρ is the mass density, I is the inertia moment of the cross section, k' is the shear coefficient and q the load.

If one substitutes the parameters in (42) into (44a) and (44b) and merges the two equations in only one obtains:

$$
\begin{aligned}
&\frac{1}{\mu}\frac{\partial^4 v}{\partial\xi^4} - \left(1 + \frac{1}{\mu}\right)\frac{\partial^4 v}{\partial\tau^2\partial\xi^2} - \frac{\gamma}{\mu}\frac{\partial^3 v}{\partial\xi^3} - \frac{\alpha}{\mu}\frac{\partial^3 v}{\partial\tau\partial\xi^2} + \eta\frac{\partial^2 v}{\partial\tau^2} \\
&\quad + \eta\gamma\frac{\partial v}{\partial\xi} + \eta\alpha\frac{\partial v}{\partial\tau} \\
&\quad = 0
\end{aligned} \tag{45}
$$

Actually, we think the governing differential equation as:

$$
\begin{aligned}
&\frac{1}{\mu}\frac{\partial^4 v}{\partial\xi^4} + \eta\frac{\partial^2 v}{\partial\tau^2} + \eta\gamma\frac{\partial v}{\partial\xi} + \eta\alpha\frac{\partial v}{\partial\tau} \\
&+ \varepsilon\Bigg[-\left(1 + \frac{1}{\mu}\right)\frac{\partial^4 v}{\partial\tau^2\partial\xi^2} - \frac{\gamma}{\mu}\frac{\partial^3 v}{\partial\xi^3} - \frac{\alpha}{\mu}\frac{\partial^3 v}{\partial\tau\partial\xi^2} \\
&+ \delta\left(\frac{\partial^4 v}{\partial\tau^4} + \alpha\frac{\partial^3 v}{\partial\tau^3} + \gamma\frac{\partial^3 v}{\partial\tau^2\partial\xi}\right)\Bigg] = 0
\end{aligned} \tag{46}
$$

When $\varepsilon = 0$ equation (46) reduces to the Bernoulli–Euler's beam theory, when $\varepsilon = 1$ we are introduced into two different Timoshekno-Ehrenfest beam's theories which are identified by the artificial parameter δ. When $\delta = 1$ we have the original Timoshenko–Ehrenfest theory (Elishakoff 2020a; Timoshenko 1916, 1920, 1921) and when $\delta = 0$ we have the truncated, consistent theory (Elishakoff 2020a; Elishakoff et al. 2015).

3.4 Galerkin method

In order to apply the Galerkin procedure we have, at first, to approximate the variable $v(\xi,\tau)$ by a series expansion, following Bolotin (1963, p. 248); Dowell (1975, 2004) and Algazin and Kijko (2015):

$$
v(\xi,\tau) = \sum_{k=1}^{n}\psi_k(\xi)f_k(\tau) \tag{47}
$$

where $\psi_k(\xi)$ is a function of the non-dimensional axial coordinate and $f_k(\tau)$ of non-dimensional time τ.

We consider the following variation of the function $f_k(\tau)$ in non-dimensional time:

$$f_k(\tau) = Q_k e^{\Omega\tau} \tag{48}$$

We multiply the expression resulting from substitution Eq. (48) into Eq. (47) by $\psi_j(\xi)$ and integrate from zero to one with respect of ξ:

$$\begin{aligned}&\sum_{k=1}^{n} Q_k\left(\frac{1}{\mu}\int_0^1 \frac{d^4\psi_k(\xi)}{d\xi^4}\psi_j(\xi)d\xi - \varepsilon\frac{\gamma}{\mu}\int_0^1 \frac{d^3\psi_k(\xi)}{d\xi^3}\psi_j(\xi)d\xi\right.\\ &-\varepsilon\frac{\mu\Omega^2+\Omega^2+\alpha}{\mu}\int_0^1 \frac{d^2\psi_k(\xi)}{d\xi^2}\psi_j(\xi)d\xi + \gamma\left(\varepsilon\delta\Omega^2+\eta\right)\\ &\int_0^1 \frac{d\psi_k(\xi)}{d\xi}\psi_j(\xi)d\xi + \left(\varepsilon\delta\left(\Omega^4+\alpha\Omega^3\right)+\eta\Omega^2+\eta\alpha\Omega\right)\\ &\left.\int_0^1 \psi_k(\xi)\psi_j(\xi)d\xi\right) = 0\end{aligned} \tag{49}$$

We introduce following notations:

$$\begin{aligned}&A_{kj} = \int_0^1 \frac{d^4\psi_k(\xi)}{d\xi^4}\psi_j(\xi)d\xi \qquad C_{kj} = -\int_0^1 \frac{d^2\psi_k(\xi)}{d\xi^2}\psi_j(\xi)d\xi\\ &B_{kj} = -\int_0^1 \frac{d^3\psi_k(\xi)}{d\xi^3}\psi_j(\xi)d\xi \qquad D_{kj} = \int_0^1 \frac{d\psi_k(\xi)}{d\xi}\psi_j(\xi)d\xi\\ &E_{kj} = \int_0^1 \psi_k(\xi)\psi_j(\xi)d\xi\end{aligned} \tag{50}$$

The problem takes the following form, in matrix representation:

$$\begin{aligned}&\left(\frac{1}{\mu}\boldsymbol{A} + \varepsilon\frac{\gamma}{\mu}\boldsymbol{B} + \varepsilon\frac{\mu\Omega^2+\Omega^2+\alpha}{\mu}\boldsymbol{C} + \gamma\left(\varepsilon\delta\Omega^2+\eta\right)\boldsymbol{D}\right.\\ &+\left.\left(\varepsilon\delta\left(\Omega^4+\alpha\Omega^3\right)+\eta\Omega^2+\eta\alpha\Omega\right)\boldsymbol{E}\right)\boldsymbol{Q} = 0\end{aligned} \tag{51}$$

This system has non-trivial solution only if the determinant of coefficient matrix vanishes. In order to study the dynamic stability, we fix parameters related to the beam, so the variables of the system are the parameter related to the load: the velocity γ and the damping α. The unknown of the problem is the complex eigenfrequency Ω.

3.4.1 *Flutter of a beam in a gas flow: original Timoshenko–Ehrenfest equations*

We are interested to know the relationship between the critical velocity γ and the damping α in order to find the boundary of the stability of the system. This boundary is found via the Routh-Hurwitz criteria. The Routh criteria lead us to this plot where the upper side is unstable end the other stable.

We proceed numerically. We subdivide the interval of γ and α, both from 0 to 1, in one hundred points and evaluated the determinant of coefficient matrix of Eq. (51) for each combination of the one hundred points of γ and α. That means that we subdivide the γ, α plan in ten thousand points were created in which we study the stability. The determinant of coefficient matrix of Eq. (51) evaluated for each point is a polynomial in the variable Ω. For each obtained polynomial we apply the Routh-Hurwitz criteria to obtain stability boundary.

From Fig. 1 one can see that increasing the number of terms in the approximated method the critical velocity increase in function of the damping. We can see a great difference between the results obtained with two terms and those derived with the four, six or eight terms. Indeed, for $\alpha = 0.07$, say, two term-approximation yields $\gamma_{cr} = 0.3939$, four terms analysis results in$\gamma_{cr} = 0.4040$, whereas six terms treatment leads to value $\gamma_{cr} = 0.4141$; finally, eight terms of approximation results in $\gamma_{cr} = 0.4141$. The change observed between six and eight terms results in flutter velocity evaluation is very small: The associated curves are almost overlapping. In conclusion, it appears that using eight terms is sufficient to obtain an accurate stability boundary.

We illustrate the methodology as follows. Using eighth order Galerkin method and keeping the discretization for γ and just two points of α (0 and 0.5) we evaluate the determinant, which turns to be a polynomial in the variable Ω. Solution shows that Ω is complex, as expected.

We study the stability using the Routh-Hurwitz stability criterion. This test is an efficient recursive algorithm proposed by Routh in 1876 to determine whether all the roots of the characteristic polynomial of a linear system have negative real parts. Independently, Hurwitz, in 1895 1895 proposed to arrange the coefficients of the polynomial into a square matrix and showed that the polynomial is stable if and only if the sequence of determinants of its principal submatrix are all positive. The two procedure are equivalent. Both provide a way to determine if the equation of motion of a linear system have only stable solution without solving the system directly. Routh-Hurwitz criterion consist in building a table. When the table is

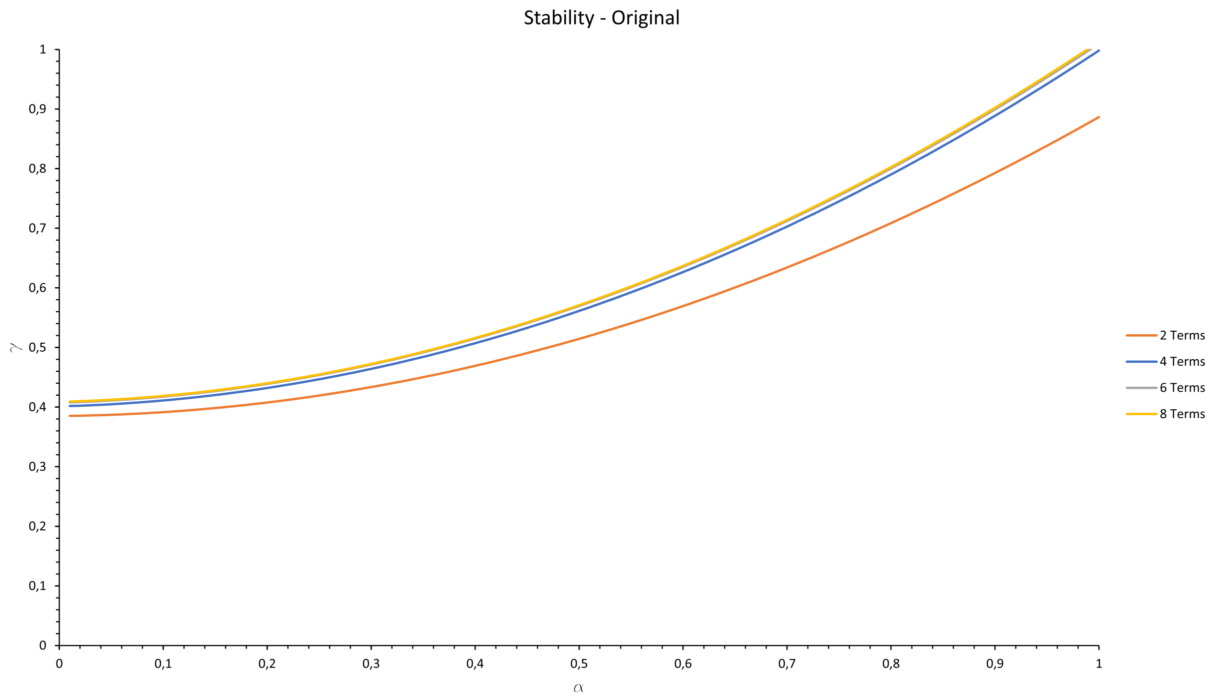

Fig. 1 Stability boundary for original Timoshenko–Ehrenfest theory

completed the number of sign changes in the first column shows the number of roots with non-negative real parts. When we have completed the table, we read the first column. Each change (permanence) in the sign of the coefficient correspond to the positive (negative) real part of the roots.

Let see what happens if we depict the real part and the imaginary part of the frequencies versus the velocity γ keeping the damping α constant first at zero and second at 0.5:

Figure 2a, b are the representation of the imaginary part of the complex frequency Ω in function of the velocity γ. We observe that these frequencies are getting closer with the increasing of the velocity. They are getting closer until reach the coalescence with a specific velocity called "critical velocity".

Figure 3a, b represent of the real part of the complex frequency Ω in function of the velocity γ. We see that these frequencies are overlapped for low value of the velocity. When the velocity increases, we see a divergence in the frequencies. The real part of the frequencies became greater than zero for a specific velocity called "critical velocity".

From these plots we can see that the stability condition is in imaginary part vs γ in the point of coalescence and in real part vs velocity when the real part is bigger than zero. The velocity in which these phenomena occur is called "critical velocity".

The critical velocity for the value of damping $\alpha = 0$ and $\alpha = 0.5$, which are the same in the reported pictures, are:

$$\begin{aligned} \gamma_{crit,\alpha=0} &= 0.4141 \\ \gamma_{crit,\alpha=0.5} &= 0.5657 \end{aligned} \tag{52}$$

An interesting observation is that in the first plot (imaginary part vs , $\alpha = 0$) we can check our derivation because when γ is equal to zero we should find the natural frequencies of our beam.

By 8th order Galerkin method we have the following non-dimensional frequencies:

$$\begin{aligned} \omega_1 &= 0.2034 \\ \omega_2 &= 1.2387 \end{aligned} \tag{53}$$

We convert the non-dimensional frequencies in dimensional frequencies dividing by $\sqrt{\frac{\rho L^2}{Gk'}}$, following the paper by Qian et al. (2019):

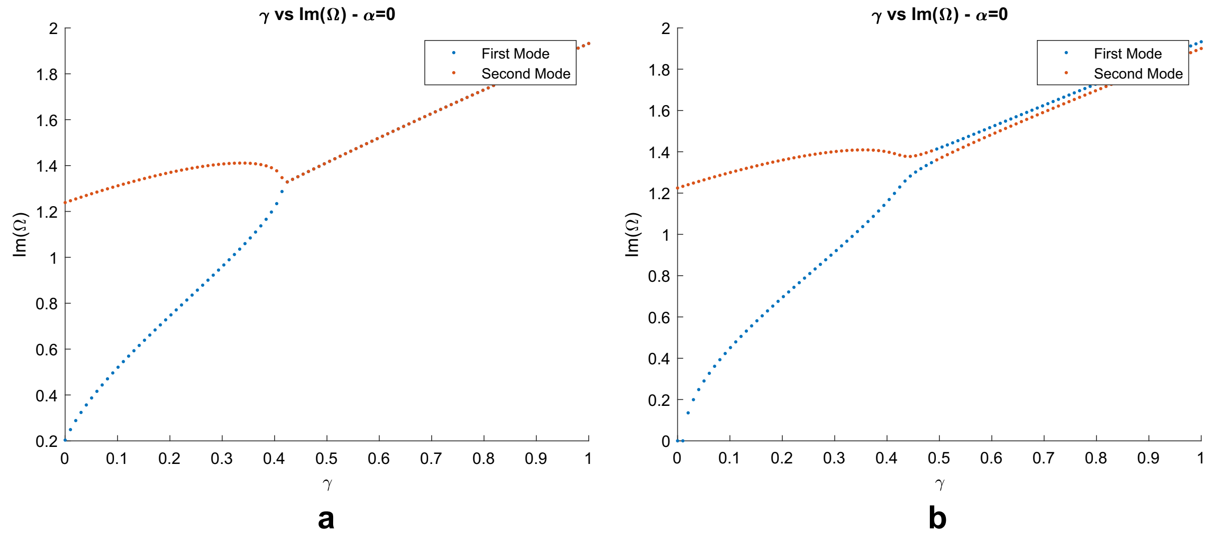

Fig. 2 **a**, **b** Imaginary part of frequencies vs velocity for original Timoshenko–Ehrenfest theory

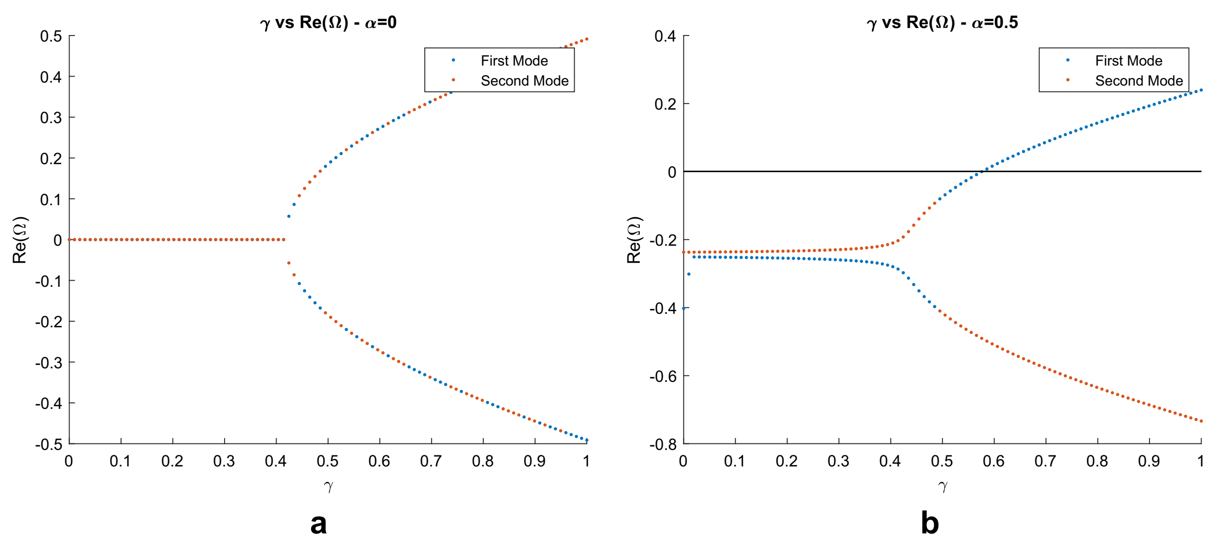

Fig. 3 **a**, **b** Real part of frequencies versus velocity for original Timoshenko–Ehrenfest theory

$$\begin{aligned} \omega_1 &= 525.5858\frac{rad}{s} \\ \omega_2 &= 3201.4080\frac{rad}{s} \end{aligned} \tag{54}$$

The relative error ε between Galerkin method and theoretical result is evaluated with the formula:

$$\varepsilon = \frac{\omega_{Galerkin} - \omega_{theoretical}}{\omega_{theoretical}} \cdot 100\% \tag{55}$$

leading to the following results:

$$\begin{aligned} \varepsilon_{\omega_1} &= 1.1442\% \\ \varepsilon_{\omega_2} &= 3.7119\% \end{aligned} \tag{56}$$

Another interesting observation is that we have the same critical velocity for $\alpha = 0$ and $\alpha = 0.5$ in the first plot.

Last two plots are real part vs imaginary part for $\alpha = 0$ and $\alpha = 0.5$. In these plots we see how the complex frequencies evolve with the increasing of the load γ. Is possible to see in Fig. 4a that for a low value of the load the two frequencies are different. When the

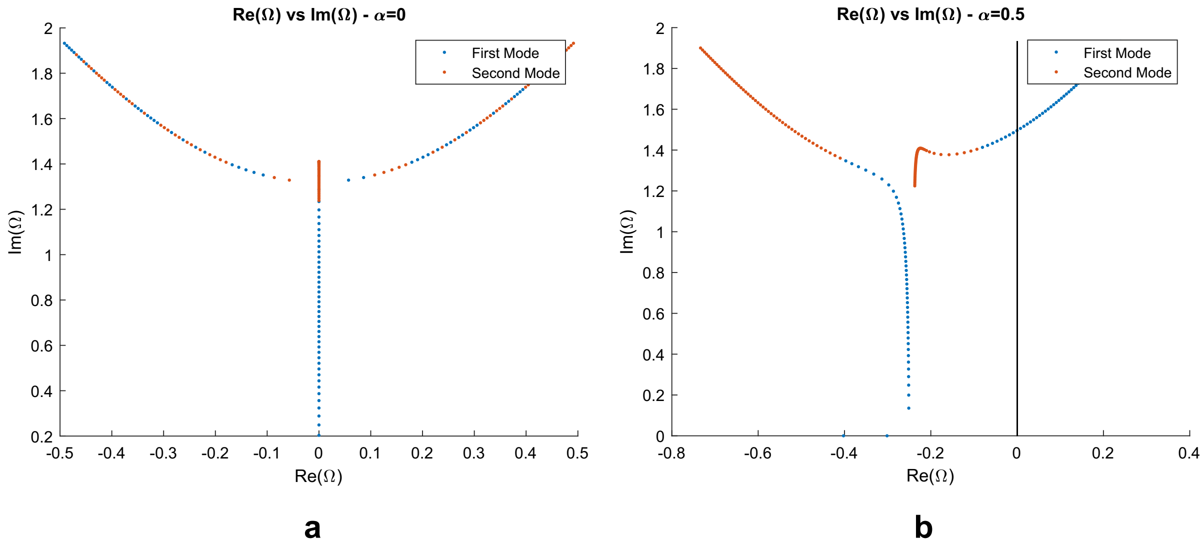

Fig. 4 **a**, **b** Real versus imaginary part of frequencies for original Timoshenko–Ehrenfest theory

real part is zero the imaginary part is clearly distinguished but for a specific value of the load, we have the coalescence of the imaginary part. This fact is possible to see because in the curved part of the plot the dots are alternated in colours. This behaviour is the same that we saw in Fig. 2a. From the Fig. 4b we see that when the real part became greater than zero, we have the instability and the imaginary part associated to this real part is the critical frequencies.

3.4.2 Flutter of a beam in gas flow: consistent Timoshenko–Ehrenfest model

For the consistent theory we proceed analogously. We, firstly, look forward to find the stability diagram. In the same way described in the previous section, we subdivide the interval of γ and α in one hundred points and we evaluated the determinant of coefficient matrix of Eq. (51) for each combination of γ and α in the variable Ω. For each polynomial we apply the Routh criteria to obtain stability curve.

From Fig. 5 one can see that the changing of the theory has not changed the trend of the chart. We still have the critical velocity which increase with the damping value. Moreover, the increasing of the number of terms in Galerkin method we obtain higher value of critical velocity but when we move from six to eight terms the result is almost the same so we get a stable boundary.

According to the previous section, we use eighth order Galerkin method and we keep the discretization for γ and just two points of α (0 and 0.5). We evaluate the determinant, the polynomial in the variable Ω and we solve for it.

We depict the real part and the imaginary part of the frequencies versus the velocity γ keeping constant the damping α firstly to zero and secondly to 0.5:

Figure 6a, b are the representation of the imaginary part of the complex frequency Ω in function of the velocity γ. We see that these frequencies, related to the first two mode shape, change their value with the damping. Specifically, with the increase of the damping these frequencies are getting closer.

Figure 7a, b are the representation of the real part of the complex frequency Ω in function of the velocity γ. For low value of the velocity, we see that the real part of the frequency is less than zero. When the critical velocity is reached, we see that the real part is become greater than zero and so the system flutters (Fig. 8).

From the charts above we define the stability condition is in “imaginary part vs γ” in the point of coalescence and in “real part vs velocity” when the real part is greater than zero. This specific velocity is defined as a “critical velocity”.

We report the critical velocity for value of damping $\alpha = 0$ and $\alpha = 0.5$:

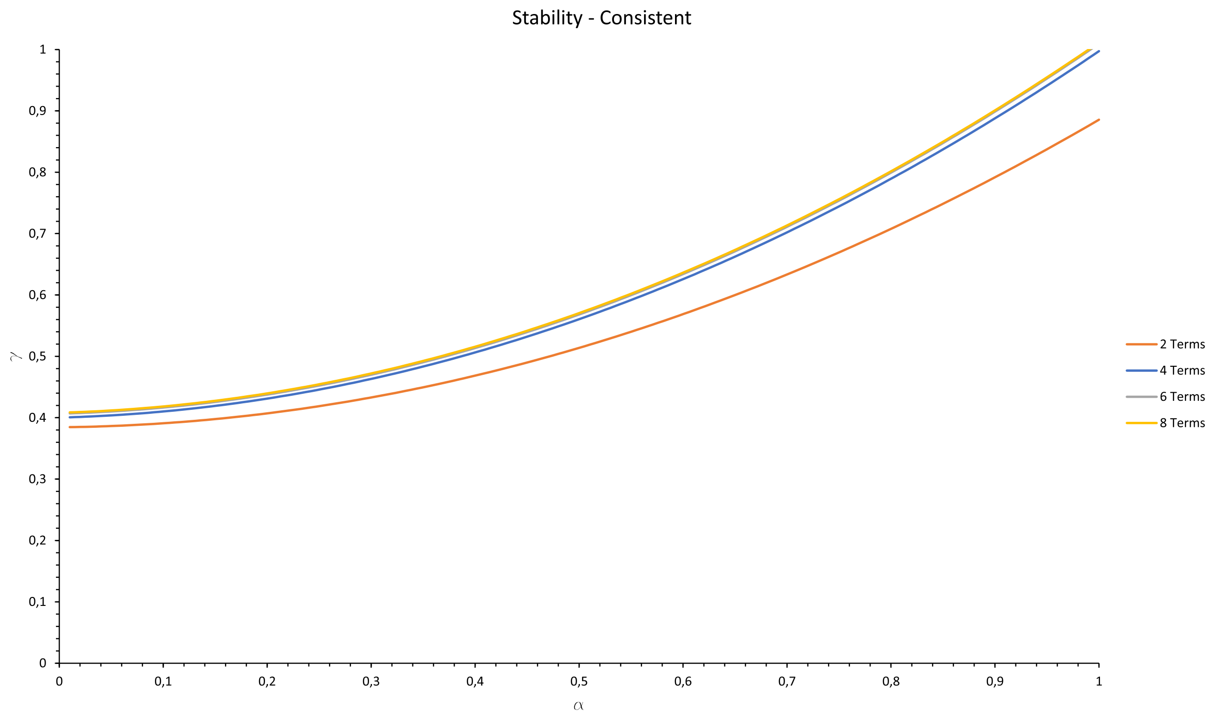

Fig. 5 Stability boundary for consistent Timoshenko–Ehrenfest theory

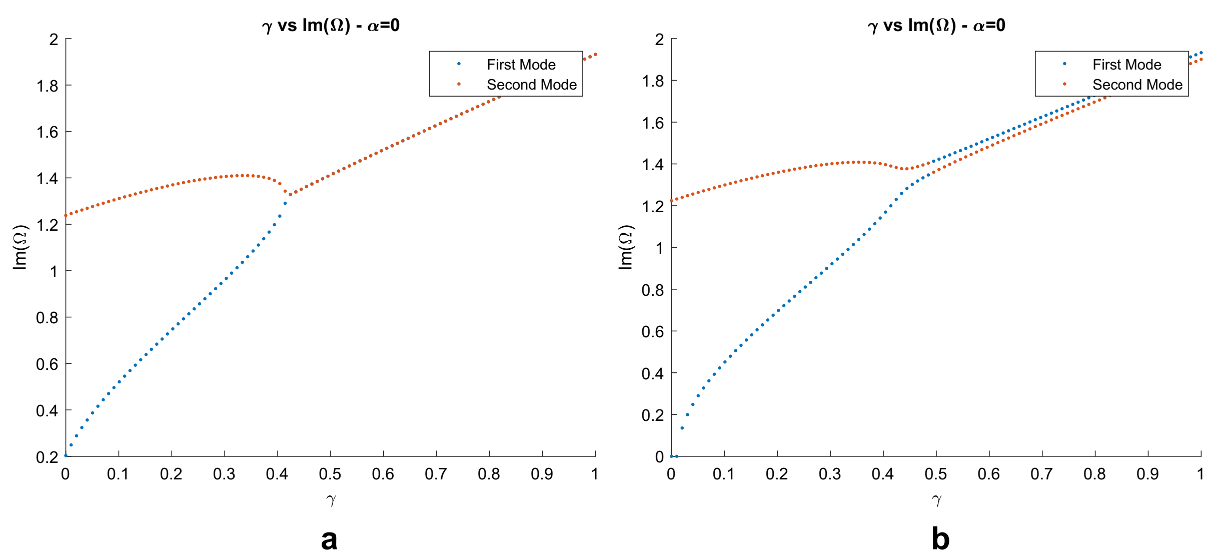

Fig. 6 **a**, **b** Imaginary part of frequencies versus velocity for consistent Timoshenko–Ehrenfest theory

$$\begin{aligned} \gamma_{crit,\alpha=0} &= 0.4141 \\ \gamma_{crit,\alpha=0.5} &= 0.5657 \end{aligned} \tag{57}$$

From the first plot (imaginary part vs , $\alpha = 0$) we check our derivation because when $\gamma = 0$ we have the natural frequency.

By Galerkin method with eight terms we obtain:

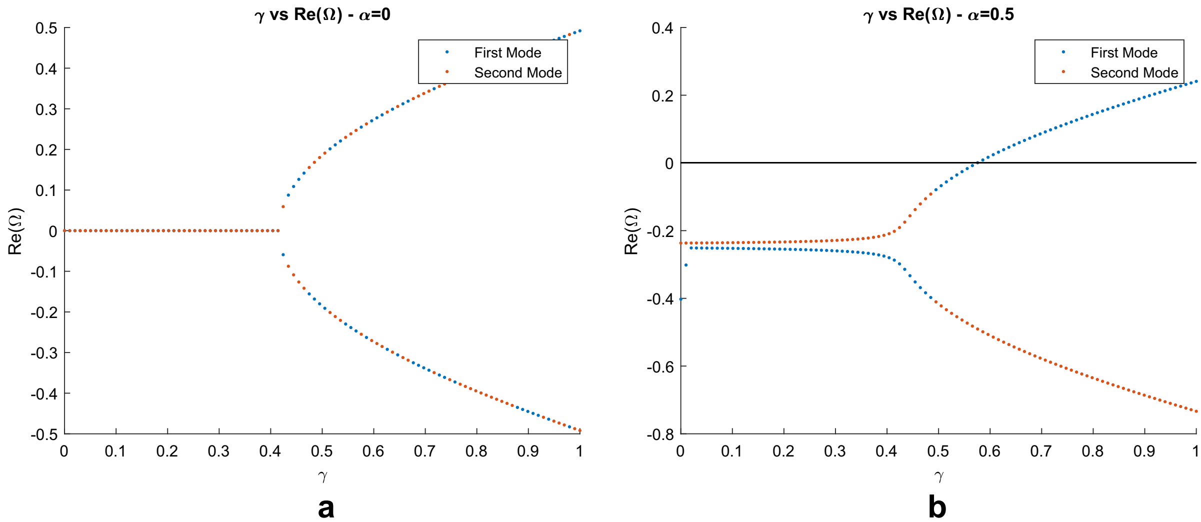

Fig. 7 **a**, **b** Real part of frequencies vs velocity for original Timoshenko–Ehrenfest theory

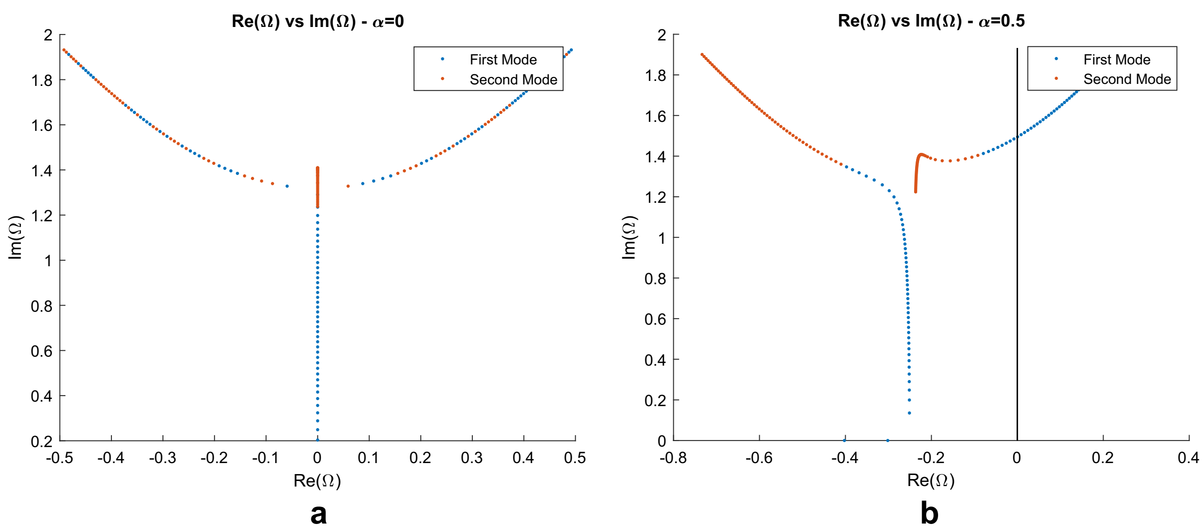

Fig. 8 **a**, **b** Real versus imaginary part of frequencies for original Timoshenko–Ehrenfest theory

$$\begin{aligned} \omega_1 &= 0.2034 \\ \omega_2 &= 1.2380 \end{aligned} \tag{58}$$

We convert the dimensionless frequencies in dimensional frequencies dividing by $\sqrt{\frac{\rho L^2}{Gk'}}$:

$$\begin{aligned} \omega_1 &= 525.5767 \frac{rad}{s} \\ \omega_2 &= 3199.4705 \frac{rad}{s} \end{aligned} \tag{59}$$

The relative error ε between the two approach is evaluated with the formula (55) leading us to:

$$\begin{aligned} \varepsilon_{\omega_1} &= 1.1442\% \\ \varepsilon_{\omega_2} &= 3.7034\% \end{aligned} \tag{60}$$

Last two plots are real part vs imaginary part for $\alpha = 0$ and $\alpha = 0.5$. In these charts we see the evolving of the complex frequencies in function of the load γ. In particular, when the level of the load is small, we can clearly distinguish the two frequencies. When the load increases the frequencies are getting closer. When the real part is greater than zero one achieves the phenomenon of the instability.

4 Conclusion

In this study we have analysed two versions of the Timoshenko–Ehrenfest beam's theory to understand the differences in terms of results. We applied these theories to free vibration analysis and flutter problems. We saw that for the natural frequencies the differences are negligible: we obtained less than 1% until the fourth frequency but anyway less than 6% until the tenth natural frequency. In the flutter problem, Galerkin method required at least the sixth terms to obtain an acceptable result but with the eighth is better. Comparison of the critical velocities obtained by the two theories in the eighth-order approximation eighth order, leads to the conclusion that the original Timoshenko–Ehrenfest beam model is not needed to accurately predict the critical velocities. Rather, the truncated, consistent theory suffices to accomplish this goal. This is in concordance with Sir Isaac Newton's statement that "We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances." Indeed, the fourth order time derivative in the original Timoshenko–Ehrenfest equatiosn constitutes the interaction between shear deformation and rotary inertia; this term is not needed,whereas the sspearte contributions of shear defiornation and rotary inertia are essential.

As a conclusion, each version performs in a similar fashion but the truncated, consistent version is much easier to implement because we are considering less terms in governing differential equations. These terms contained in the original formulation of the Timoshenko–Ehrenfest model as evaluated by Qian et al. (2019)—the only other reference that deals with flutter of the beam considering the shear deformation and rotary inertia– contain high power of the frequencies and as a consequence, turn out to be unnecessarily over-complicating the analysis. Further work, including the nonlocal effects [Ref. Paola et al. (2014), Paola et al. (2013), Paola and Zingales (2008)], and/or based upon fractional calculus [Refs. Cottone et al. (2009); Alotta et al. (2020)] are under way and will be reported elsewhere.

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Appendix

Two term Galerkin approximation for critical velocity for Bernoulli–Euler beam

For the specific case of "two terms Galerkin method" we provide the explicit formula for critical velocity in function of system parameters. We provide such a formula for the Bernoulli–Euler beam. Starting from formula (46) and choosing $\varepsilon = 0$ we get:

$$\frac{1}{\mu}\frac{\partial^4 v}{\partial \xi^4} + \eta\frac{\partial^2 v}{\partial \tau^2} + \eta\gamma\frac{\partial v}{\partial \xi} + \eta\alpha\frac{\partial v}{\partial \tau} \tag{61}$$

That lead us to rewrite Eqs. (51) in the following determinantal form:

$$\det\left(\Omega^2 + \boldsymbol{K}\Omega + \boldsymbol{M} + \gamma\boldsymbol{N}\right) = 0 \tag{62}$$

where $\boldsymbol{K} = \frac{\eta\alpha\boldsymbol{E}}{\eta\boldsymbol{E}}$, $\boldsymbol{M} = \frac{\boldsymbol{A}/\mu}{\eta\boldsymbol{E}} = \frac{\boldsymbol{A}}{\mu\eta\boldsymbol{E}}$ and $\boldsymbol{N} = \frac{\eta\boldsymbol{D}}{\eta\boldsymbol{E}}$. This determinant is expressed as follows:

$$\begin{aligned}
&\Omega^4 + (K_{11} + K_{22})\Omega^3 + (K_{11}K_{22} - K_{12}K_{21} + M_{11} + M_{22} + \gamma N_{11} + \gamma N_{22})\Omega^2 \\
&+ (\gamma K_{11}N_{22} - \gamma K_{12}N_{21} - \gamma K_{21}N_{12} + \gamma K_{22}N_{11} + K_{11}M_{22} - K_{12}M_{21} \\
&- K_{21}M_{12} + K_{22}M_{11})\Omega \\
&+ \left(\gamma^2 N_{11}N_{22} - \gamma^2 N_{12}N_{21} + \gamma M_{11}N_{22} - \gamma M_{12}N_{21} - \gamma M_{21}N_{12} + \gamma M_{22}N_{11} \right.\\
&\left. + M_{11}M_{22} - M_{12}M_{21}\right) = 0
\end{aligned} \tag{63}$$

It is in the form $a_0\Omega^4 + a_1\Omega^3 + a_2\Omega^2 + a_3\Omega + a_4 = 0$. According to Routh criteria, the fourth order polynomial equation is stable when:

$$(a_1a_2 - a_0a_3)a_3 - a_1^2a_4 > 0 \tag{64}$$

Which define the stability boundary. Denoting:

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= p_1 \\
a_2 &= \gamma r_1 + p_2 \\
a_3 &= \gamma r_2 + p_3 \\
a_4 &= \gamma^2 q_1 + \gamma r_3 + p_4
\end{aligned} \tag{65}$$

where:

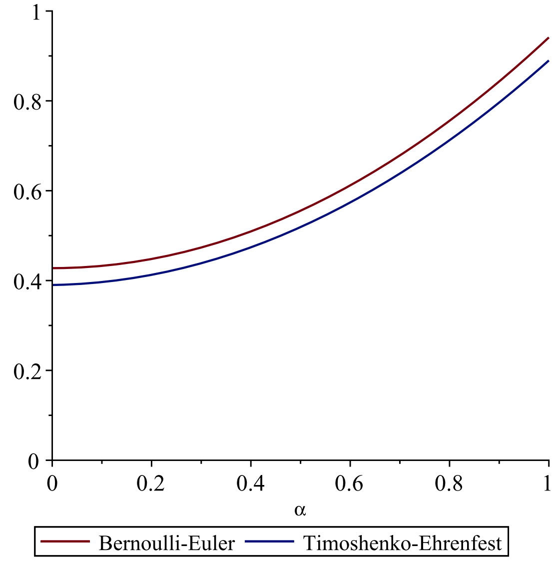

Fig. 9 Bernoulli–Euler versus Timoshenko–Ehrenfest stability plot

$$\begin{aligned}
p_1 &= K_{11} + K_{22} \\
p_2 &= K_{11}K_{22} - K_{12}K_{21} + M_{11} + M_{22} \\
p_3 &= K_{11}M_{22} - K_{12}M_{21} - K_{21}M_{12} + K_{22}M_{11} \\
p_4 &= M_{11}M_{22} - M_{12}M_{21} \\
r_1 &= N_{11} + N_{22} \\
r_2 &= K_{11}N_{22} - K_{12}N_{21} - K_{21}N_{12} + K_{22}N_{11} \\
r_3 &= M_{11}N_{22} - M_{12}N_{21} - M_{21}N_{12} + M_{22}N_{11} \\
q_1 &= N_{11}N_{22} - N_{12}N_{21}
\end{aligned} \tag{66}$$

Substituting these parameters in boundary equation obtained considering the inequality (64) as equality, we get the following second order equation:

$$\begin{aligned}
&\left(p_1r_1r_2 - r_2^2 - p_1^2q_1\right)\gamma^2 \\
&\quad + \left(p_1p_3 + p_1p_2r_2 - 2p_3r_2 - p_1^2r_3\right)\gamma \\
&\quad + \left(p_1p_2p_3 - p_3^2 - p_1^2p_4\right) \\
&\quad = 0
\end{aligned} \tag{67}$$

the positive root of Eq. (67) is the non-dimensional critical velocity γ; substituting in it reads the meaning of γ and α we can express obtain U_{cr} as follows:

$$\begin{aligned}
U_{cr} \cong\ & 0.26\frac{\kappa p_\infty bL}{A\rho c_\infty} - 19.55\frac{EIc_\infty}{bp_\infty \kappa L^3} \\
& - 4.39\cdot 10^{-8}\sqrt{3.51\cdot 10^{13}\frac{\kappa^2 p_\infty^2 b^2 L^2}{A^2\rho^2 c_\infty^2} + 3.82\cdot 10^{16}\frac{EI}{A\rho L^2} + 1.13\cdot 10^{19}\frac{E^2I^2c_\infty^2}{b^2p_\infty^2\kappa^2L^6}}
\end{aligned} \tag{68}$$

As expected, the obtained formula (68) does not contain any shear parameter. This formula represents the boundary of stability in function of the system parameters. It includes geometrical parameters, mechanical parameters and gas flow parameters.

Two Term Galerkin approximation for critical velocity for consistent Timoshenko–Ehrenfest beam theory

As we shown in the previous section, we deal with two terms Galerkin method in order to obtain an approximate formulation for the critical velocity. Starting from Eq. (46) we set $\varepsilon = 1$ and $\delta = 0$ obtaining:

$$\begin{aligned}
&\frac{1}{\mu}\frac{\partial^4 v}{\partial \xi^4} - \left(1 + \frac{1}{\mu}\right)\frac{\partial^4 v}{\partial \tau^2 \partial \xi^2} - \frac{\gamma}{\mu}\frac{\partial^3 v}{\partial \xi^3} - \frac{\alpha}{\mu}\frac{\partial^3 v}{\partial \tau \partial \xi^2} + \eta\frac{\partial^2 v}{\partial \tau^2} \\
&\quad + \eta\gamma\frac{\partial v}{\partial \xi} + \eta\alpha\frac{\partial v}{\partial \tau} \\
&\quad = 0
\end{aligned} \tag{69}$$

Rewriting equations (51) we obtain the following determinantal form:

$$\det\left(\Omega^2 + \boldsymbol{K}\Omega + \boldsymbol{M} + \gamma\boldsymbol{N}\right) = 0 \tag{70}$$

where $\boldsymbol{K} = \frac{\eta\alpha\boldsymbol{E}}{\boldsymbol{C}+\frac{1}{\mu}\boldsymbol{C}+\eta\boldsymbol{E}}$, $\boldsymbol{M} = \frac{\frac{1}{\mu}\boldsymbol{A}+\frac{\alpha}{\mu}\boldsymbol{C}}{\boldsymbol{C}+\frac{1}{\mu}\boldsymbol{C}+\eta\boldsymbol{E}}$ and $\boldsymbol{N} = \frac{\frac{1}{\mu}\boldsymbol{B}+\gamma\boldsymbol{D}}{\boldsymbol{C}+\frac{1}{\mu}\boldsymbol{C}+\eta\boldsymbol{E}}$.
Equation (70) is in the same form of previous section but the three matrixes involved has a different definition. As a result, we can follow the previous section reaching the solution for the non-dimensional critical velocity γ:

$$\begin{aligned}
\gamma_{1,2} =\ & \frac{-\left(p_1p_3 + p_1p_2r_2 - 2p_3r_2 - p_1^2r_3\right)}{2\cdot\left(p_1r_1r_2 - r_2^2 - p_1^2q_1\right)} \\
& \pm\frac{\sqrt{\left(p_1p_3 + p_1p_2r_2 - 2p_3r_2 - p_1^2r_3\right)^2 - 4\cdot\left(p_1r_1r_2 - r_2^2 - p_1^2q_1\right)\cdot\left(p_1p_2p_3 - p_3^2 - p_1^2p_4\right)}}{2\cdot\left(p_1r_1r_2 - r_2^2 - p_1^2q_1\right)}
\end{aligned} \tag{71}$$

Explicit expression for this formula is not provided because of its length. By the way we provide a comparison between Bernoulli–Euler result and Timoshenko–Ehrenfest result. We compare the plot of non-dimensional velocity in function of the non-dimensional damping. The plot is reported in Fig. 9.

This plot represents the stability curve. In this plot we see that the Bernoulli–Euler beam is more stable in comparison of the Timoshenko–Ehrenfest one. Is required a highest value of velocity to let the system flutter.

References

Algazin, S.D., Kijko, I.A.: Aeroelastic Vibrations and Stability of Plates and Shells. Walter de Gruyter GmbH, Berlin (2015)

Alotta, G., Di Paola, M., Pinnola, F. P., Zingales, M.: A fractional nonlocal approach to nonlinear blood flow in small-lumen arterial vessels. Meccanica, pp. 1–16, (2020)

Ashley, H., Zartarian, G.: Piston theory-a new aerodynamic tool for the aeroelastician. J. Aeronaut. Sci. **23**(12), 1109–1118 (1956)

Bolotin, V.V.: Nonconservative Problems of the Theory of Elastic Stability. The Macmillan Company, New York (1963)

Bresse, J.A.C.: Cours de mécanique appliquée – Résistance des matériaux et stabilité desconstructions. Gauthier-Villars, Paris (1859).. ((**in French**))

Challamel, N., Elishakoff, I.: A brief history of first-order shear-deformable beam and plate models. Mech. Res. Commun. **102**, 103389 (2019)

Cottone, G., Di Paola, M., Zingales, M.: Fractional mechanical model for the dynamics of non-local continuum. In: Advances in numerical methods, pp. 389–423. Springer, Boston (2009)

Di Paola, M., Zingales, M.: Long-range cohesive interactions of non-local continuum faced by fractional calculus. Int. J. Solids Struct. **45**(21), 5642–5659 (2008)

Di Paola, M., Failla, G., Pirrotta, A., Sofi, A., Zingales, M.: The mechanically based non-local elasticity: an overview of main results and future challenges. Philos. Trans. R. Soc. A Math. Phys. Eng. Sci. **371**(1993), 20120433 (2013)

Dowell, E.H.: Aeroelasticity of Plates and Shells. Noordhoff International Publishing, Leyden (1975)

Dowell, E.H., Clark, R., Cox, D., Curtiss, H.C., Jr., Edwards, J.W., Hall, K.C., Peters, D.A., Scanlan, R., Simiu, E., Sisto, F., Strganac, T.W.: A *Modern Course in Aeroelasticity*. Kluwer Academic Publisher, Dordrecht (2004)

Elishakoff, I.: An equation which is simpler and more consistent than bresse-timoshenko equations. In: Gilat, R., Sills-Banks, L. (eds.) Advances in Mathematical Modeling and Experimental Methods for Materials and Structures, pp. 249–254. Springer Verlag, Berlin (2010)

Elishakoff, I.: Handbook on Timoshenko-Ehrenfest Beam and Uflyand-Mindlin Plate Theories. World-Scientific, Singapore (2020a)

Elishakoff, I.: Who developed the so-called timoshenko beam theory. Math. Mech. Solids **25**(1), 97–116 (2020b)

Elishakoff, I., Lubliner, E.: Random vibration of a structure via classical and nonclassical theories. In: Eggwertz, S., Lind, N.C. (eds.) Probabilistic Methods in the Mechanics of Solids and Structures, pp. 455–467. Springer, Berlin (1985)

Elishakoff, I., Kaplunov, J., Nolde, E.: Celebrating the centenary of Timoshenko's study of effects of shear deformation and rotary inertia. Appl. Mech. Rev. **67**, 060802 (2015)

Hurwitz, A.: Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitz. Math. Ann. **46**(2), 276–284 (1895). ((**German**))

Ilyushin, A.A.: Law of plane sections in aerodynamics of high supersonic speed. J. Appl. Math. Mech. **20**(6), 733–735 (1956). ((**Russian**))

Rayleigh Lord (J. W. S. Strutt), *The Theory of Sound,* London: Macmillan, 1877–1878 (see also Dover, New York, 1945).

Paola, M.D., Failla, G., Zingales, M.: Mechanically based nonlocal Euler-Bernoulli beam model. J. Nanomech. Micromech. **4**(1), A4013002 (2014)

Pielorz, A.: Discrete-continuous models in the analysis of low structures subject to kinematic Excitations caused by transversal waves. J. Theor. Appl. Mech. **34**(3), 547–566 (1996)

qian y-j., yang x-d., zhang w., liang f., yang t-z. ren y.: flutter mechanism of Timoshenko beams in supersonic flow, J. Aerosp. Eng. Vol. 32(4), 2019.

Routh, E.J.: A Treatise on the Stability of a Given State of Motion, Particularly Steady Motion. Macmillan and co., London (1877)

Strand7, Theoretical Manual – Theoretical background to the Strand7 finite element analysis system, (2004)

Timoshenko S.P.: A Course of Elasticity Theory. Part 2: Rods and Plates, St. Petersburg: A.E. Collins Publishers, 1916 (in Russian), (2nd Edition, Kiev: "Naukova Dumka" Publishers, pp. 337–338, 341, 1972).

Timoshenko, S.P.: On the differential equation for the flexural vibrations of prismatical rods. Glasnik Hrvatskoga Prirodaslovnoga Društva **32**, 55–57 (1920)

Timoshenko, S.P.: On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bar. Philosophical Magazine Series **41/245**(6), 744–746 (1921)

Timoshenko, S.P.: Vibration Problems in Engineering, p. 231. Constable and Company, London (1928)

Timoshenko, S.P.: Vibration Problems in Engineering, 2nd edn. Van Nostrand Reinhold Company, New York (1937)

Timoshenko, S.P., Young, D.H.: Vibration Problems in Engineering, 3rd edn. Van Nostrand Reinhold Company, New York (1955)

Timoshenko, S.P., Young, D.H., Weaver, W., Jr.: Vibration Problems in Engineering. Wiley, Fourth Edition, New York (1974)

Weaver, W., Jr., Timoshenko, S.P., Young, D.H.: Vibration Problems in Engineering. Wiley, Fifth Edition, New York (1990)