Journal of Sound and Vibration 490 (2021) 115708

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsv

Rigorous implementation of the Galerkin method for stepped structures needs generalized functions

Isaac Elishakoff[a,*], Marco Amato[b], Arvan Prakash Ankitha[a], Alessandro Marzani[b]

[a] Department of Ocean and Mechanical Engineering, Florida Atlantic University, Boca Raton, FL 33431- 0991, USA
[b] Department of Civil, Chemical, Environmental, and Materials Engineering, University of Bologna, Viale del Risorgimento 2, Bologna, Italy

ARTICLE INFO

Article history:
Received 19 November 2019
Revised 10 September 2020
Accepted 13 September 2020
Available online 25 September 2020

Keywords:
Galerkin's method
Unit step function
Dirac's delta function
Doublet function
Convergence
Stepped structures

ABSTRACT

In this paper we study free vibrations of stepped structures, specifically for longitudinal vibrations of bars and flexural vibrations of rectangular plates, providing two versions of the Galerkin method. Specifically, we first apply the straightforward version of the Galerkin method which stipulates the employment of the Galerkin procedure to be conducted in each subdomain, or step, of the structure. Second, the rigorous realization of the Galerkin method is presented where the structural parameters, like rigidity and mass, are treated as generalized functions over the entire domain. This latter implementation utilizes unit step functions, as well as the Dirac's delta function, and its derivative to treat the changes of the structural parameters across the steps.

It turns out that this rigorous implementation leads to additional terms that do not appear in the straightforward (or "naïve") realization of the Galerkin method. Both versions of Galerkin methods are compared with the exact solutions of the considered problems. It turns out that with increase of number of terms in the expansion, the rigorous, generalized-functions based Galerkin method *tends* to exact solution. In contrast the naïve realization of Galerkin's method, which is usually utilized in literature, *does not tend* to exact solution. This study demonstrates that extreme care must be taken when implementing the Galerkin's method for stepped structures, and only the rigorous version should be employed.

1. Introduction

Galerkin's method is a celebrated approximate technique suggested over a century ago (Galerkin [1]). Reviews of this method are provided by numerous authors. Interested readers can consult the works by Leipholz [2], Gander and Wanner [3], Repin [4] as well as the book by Mikhlin [5]. One can see also articles by Magen et al. [6], Sclavounos [7], Grinberg et al. [8], Wilson et al. [9], Wang et al. [10] and Izem et al. [11]. There are numerous papers devoted to its application to elastic structures. We are particularly interested in applications of this method to structures with discontinuities, and especially stepped structures. There are several studies focused on this topic. The interested readers can consult with the papers by Chehil and Jategaonkar [12], Maurini, Porfiri and Pouget [13], Sohrabian and Ahmadian [14], Al-said [15], Borneman, Hashemi and Alighanbari [16] and Pirmoradian, Keshmiri and Karimpour [17] as examples.

* Corresponding author.
E-mail address: elishako@fau.edu (I. Elishakoff).

https://doi.org/10.1016/j.jsv.2020.115708

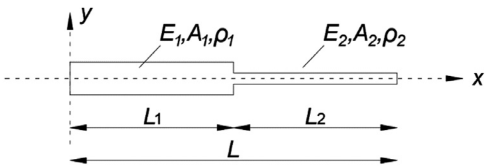

Fig. 1. A piecewise stepped bar of length L.

In the above works, the Galerkin's procedure is applied in the straightforward, naïve form. In the naïve form the integrations associated with each step of the structure are performed and the obtained results subsequently summed, as it will be illustrated in the following section. This paper shows that the above implementation can lead to large errors. The rigorous implementation is suggested instead, based upon the use of generalized functions existing over the entire domain of the structure. To show and prove this thesis, we consider two structural problems, namely the longitudinal vibrations in a stepped bar that is clamped at both its end cross-sections, and the flexural vibrations in a simply supported rectangular plate with a single step in thickness. For both examples we show that the naïve version of the Galerkin method does not tend to the exact solution when number of terms increases whereas the rigorous implementation does.

2. Longitudinal vibrations of the stepped bar

2.1. Governing differential equations

Consider the stepped bar shown in Fig. 1 with moduli of elasticity E_i, cross-sectional areas A_i and material densities ρ_i, and lengths L_i (i = 1,2). The governing differential equation for the non-uniform and/or inhomogeneous bar with piecewise constants E_i, A_i and ρ_i, reads:

$$\frac{\partial}{\partial x}\left(E_i A_i \frac{\partial v_i(x,t)}{\partial x}\right) = \rho_i A_i \frac{\partial^2 v_i(x,t)}{\partial t^2} \tag{1}$$

where $v_i(x, t)$ is the displacement in longitudinal direction along the i-th segment, x is the axial coordinate and t is the time. Specifically, for the two-segmented bar of Fig. 1 Eq. (1) leads to:

$$c_1^2 \frac{\partial^2 v_1}{\partial x^2} = \frac{\partial^2 v_1}{\partial t^2}, \quad 0 < x < L_1 \tag{2}$$

$$c_2^2 \frac{\partial^2 v_2}{\partial x^2} = \frac{\partial^2 v_2}{\partial t^2}, \quad L_1 < x < L \tag{3}$$

where $L = L_1 + L_2$ is the total length of the two-step bar, with $c_i = \sqrt{\frac{E_i}{\rho_i}}$ is the speed of propagation of longitudinal waves in the i-th bar. Consider the case of a clamped-clamped bar element, the boundary and continuity conditions read:

$$v_1(0,t) = v_2(L,t) = 0 \tag{4}$$

$$v_1(L_1,t) = v_2(L_1,t) \tag{5}$$

$$E_1 A_1 \frac{\partial v_1(L_1,t)}{\partial x} = E_2 A_2 \frac{\partial v_2(L_1,t)}{\partial x} \tag{6}$$

Consider now harmonic vibrations:

$$v_1(x,t) = V_1(x)\sin(\omega t) \tag{7}$$

$$v_2(x,t) = V_2(x)\sin(\omega t) \tag{8}$$

where $V_1(x)$ and $V_2(x)$ constitute mode shapes of the first and second step, respectively, and ω is the sought circular frequency, and subsisting them into Eqs. (2) and (3) leads to:

$$c_1^2 \frac{d^2 V_1(x)}{dx^2} + \omega^2 V_1(x) = 0 \tag{9}$$

$$c_2^2 \frac{d^2 V_2(x)}{dx^2} + \omega^2 V_2(x) = 0 \tag{10}$$

Solutions of Eqs. (9) and (10) read, respectively:

$$V_1 = D_1 \sin \beta_1 x + D_2 \cos \beta_1 x \tag{11}$$

$$V_2 = D_3 \sin \beta_2 x + D_4 \cos \beta_2 x \tag{12}$$

where $\beta_i = \frac{\omega}{c_i}$.

Substituting Eqs. (11)–(12) into Eqs. (4)–(6) leads to a homogeneous system whose non-trivial solutions can be obtained by imposing its determinant equal to zero:

$$\begin{vmatrix} \sin(\beta_1 L_1) & -\sin(\beta_2 L_1) & -\cos(\beta_2 L_1) \\ E_1 A_1 \beta_1 \cos(\beta_1 L_1) & -E_2 A_2 \beta_2 \cos(\beta_2 L_1) & E_2 A_2 \beta_2 \sin(\beta_2 L_1) \\ 0 & \sin(\beta_2 L) & \cos(\beta_2 L) \end{vmatrix} = 0 \tag{13}$$

which leads to the following characteristic transcendental equation:

$$\begin{aligned} &-E_2 A_2 \beta_2 [\sin(\beta_1 L_1)\cos(\beta_2 L_1)\cos(\beta_2 L) + \sin(\beta_1 L_1)\sin(\beta_2 L_1)\sin(\beta_2 L)] \\ &\quad -E_1 A_1 \beta_1 [\cos(\beta_1 L_1)\cos(\beta_2 L_1)\sin(\beta_2 L) - \sin(\beta_2 L_1)\cos(\beta_2 L)\cos(\beta_1 L_1)] = 0 \end{aligned} \tag{14}$$

2.2. Evaluation of exact solutions

We introduce the following parameter which contains the sought frequency ω:

$$z = \beta_1 L_1 = \frac{\omega L_1}{c_1} \tag{15}$$

Then,

$$\beta_2 L_1 = z\frac{c_1}{c_2} = z\sqrt{\frac{E_1}{E_2}\frac{\rho_2}{\rho_1}}; \quad \beta_1 L = z\frac{L}{L_1} = z\left(1 + \frac{L_2}{L_1}\right); \quad \beta_2 L = z\frac{c_1}{c_2}\frac{L}{L_1} = z\frac{c_1}{c_2}\left(1 + \frac{L_2}{L_1}\right) \tag{16}$$

Eq. (14) can be rewritten with respect to z as:

$$\begin{aligned} \phi_{C-C}(z) = \frac{E_2}{E_1}\frac{A_2}{A_1}\frac{c_1}{c_2}&\left[\sin(z)\sin\left(z\sqrt{\frac{E_1\rho_2}{E_2\rho_1}}\right)\sin\left(z\frac{c_1}{c_2}\left(1+\frac{L_2}{L_1}\right)\right) + \sin(z)\cos\left(z\sqrt{\frac{E_1\rho_2}{E_2\rho_1}}\right)\cos\left(z\frac{c_1}{c_2}\left(1+\frac{L_2}{L_1}\right)\right)\right] \\ &+ \left[\cos(z)\cos\left(z\sqrt{\frac{E_1\rho_2}{E_2\rho_1}}\right)\sin\left(z\frac{c_1}{c_2}\left(1+\frac{L_2}{L_1}\right)\right) - \cos(z)\sin\left(z\sqrt{\frac{E_1\rho_2}{E_2\rho_1}}\right)\cos\left(z\frac{c_1}{c_2}\left(1+\frac{L_2}{L_1}\right)\right)\right] = 0 \end{aligned} \tag{19}$$

We introduce the following notations:

$$M_c = \frac{E_2}{E_1}\frac{A_2}{A_1}\frac{c_1}{c_2} \tag{20}$$

$$B\left(z, \frac{E_2}{E_1}, \frac{\rho_2}{\rho_1}, \frac{L_2}{L_1}\right) = \left[\sin(z)\sin\left(z\sqrt{\frac{E_1\rho_2}{E_2\rho_1}}\right)\sin\left(z\frac{c_1}{c_2}\left(1+\frac{L_2}{L_1}\right)\right) + \sin(z)\cos\left(z\sqrt{\frac{E_1\rho_2}{E_2\rho_1}}\right)\cos\left(z\frac{c_1}{c_2}\left(1+\frac{L_2}{L_1}\right)\right)\right] \tag{21}$$

$$C\left(z, \frac{E_2}{E_1}, \frac{\rho_2}{\rho_1}, \frac{L_2}{L_1}\right) = \left[\sin(z)\cos\left(z\sqrt{\frac{E_1\rho_2}{E_2\rho_1}}\right)\sin\left(z\frac{c_1}{c_2}\left(1+\frac{L_2}{L_1}\right)\right) - \sin(z)\sin\left(z\sqrt{\frac{E_1\rho_2}{E_2\rho_1}}\right)\cos\left(z\frac{c_1}{c_2}\left(1+\frac{L_2}{L_1}\right)\right)\right] \tag{22}$$

which allow to rewrite Eq. (19) in the following synthetic form $\phi_{C-C}(z) = M_c B(z) + C(z)$.

2.3. Application of the Naïve Galerkin method

In order to apply the Galerkin method we have to select a set of comparison functions. We choose to use trigonometric comparison functions. The trigonometric comparison functions that satisfy the clamped-clamped boundary conditions are:

$$\psi_j(\xi) = \sin(j\pi\xi) \tag{23}$$

where $j = 1,2,3\ldots n$ denotes the number of the mode and $\xi = x/L$ is the non-dimensional axial coordinate. Now, we express the axial displacement in terms of comparison functions as:

$$V(\xi) = \sum_{j=1}^{n} p_j \psi_j(\xi) \tag{24}$$

We substitute this expansion into Eqs. (9) and (10); then we multiply each result by $\psi_k(\xi)$, integrate the result from zero to $\xi_1 = L_1/L$ for Eq. (9), and from ξ_1 to 1 for Eq. (10), and sum up the results to get:

$$\int_0^{\xi_1}\left[c_1^2\frac{d^2}{d\xi^2}\sum_{j=1}^{n}p_j\psi_j(\xi)+\omega^2L^2\sum_{j=1}^{n}p_j\psi_j(\xi)\right]\psi_k(\xi)d\xi+\int_{\xi_1}^{1}\left[c_2^2\frac{d^2}{d\xi^2}\sum_{j=1}^{n}p_j\psi_j(\xi)+\omega^2L^2\sum_{j=1}^{n}p_j\psi_j(\xi)\right]\psi_k(\xi)d\xi=0 \quad (25)$$

where $k = 1, 2, 3 \ldots n$ is a positive integer. The Eq. (25) can be rewritten in the following form:

$$\sum_{j=1}^{n}\left(K_{jk}+\omega^2M_{jk}\right)p_j=0 \quad (26)$$

where:

$$K_{jk}=\int_0^{\xi_1}c_1^2\frac{d^2\psi_j(\xi)}{d\xi^2}\psi_k(\xi)d\xi+\int_{\xi_1}^{1}c_2^2\frac{d^2\psi_j(\xi)}{d\xi^2}\psi_k(\xi)d\xi \quad (27)$$

$$M_{jk}=\int_0^{\xi_1}L^2\psi_j(\xi)\psi_k(\xi)d\xi+\int_{\xi_1}^{1}L^2\psi_j(\xi)\psi_k(\xi)d\xi \quad (28)$$

or in matrix notation as:

$$\left(K+\omega^2M\right)p=0 \quad (29)$$

which is a homogeneous linear system of dimension n in the unknowns ω and p. Eq. (29) has non-trivial solutions only when the determinant of the coefficient matrix is equal to zero, leading to the following eigenvalue problem in terms of the sought frequency ω^2:

$$\det\left(K+\omega^2M\right)=0 \quad (30)$$

2.4. Application of the Rigorous Galerkin method

Within rigorous implementation, we represent the axial rigidity and the mass over the entire domain of the system $0 \le x \le L$ using the following generalized functions:

$$D(x)=E(x)A(x)=D_1H(x)+(D_2-D_1)H(x-L_1) \quad (31)$$

$$M(x)=\rho(x)A(x)=M_1H(x)+(M_2-M_1)H(x-L_1) \quad (32)$$

where $H(x)$ is the unit step function or Heaviside function which has the following properties

$$H(x-\alpha)=\begin{cases}1, & if\ x>\alpha\\ 0, & otherwise\end{cases} \quad (33)$$

$$\frac{d}{dx}H(x)=\delta(x). \quad (34)$$

where $\delta(x)$ is the Dirac's delta function.

Now, rewriting Eq. (1) using the above considerations:

$$\frac{d}{dx}\left[(D_1H(x)+(D_2-D_1)H(x-L_1))\frac{dv}{dx}\right]=[M_1H(x)+(M_2-M_1)H(x-L_1)]\frac{d^2v}{dt^2} \quad (35)$$

calculating derivatives, introducing the non-dimensional axial coordinate $\xi = \frac{x}{L}$ and substituting Eqs. (7), (8) and (24), leads to the following model:

$$[D_1\delta(\xi)+(D_2-D_1)\delta(\xi-\xi_1)]\frac{d}{d\xi}\left(\sum_{j=1}^{n}p_j\psi_j(\xi)\right)+[D_1H(\xi)+(D_2-D_1)H(\xi-\xi_1)]\frac{d^2}{d\xi^2}\left(\sum_{j=1}^{n}p_j\psi_j(\xi)\right)$$
$$+[M_1H(\xi)+(M_2-M_1)H(\xi-\xi_1)]\omega^2L^2\sum_{j=1}^{n}p_j\psi_j(\xi)=0 \quad (36)$$

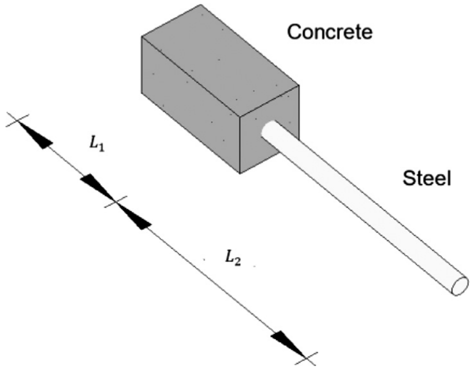

Fig. 2. A concrete-steel stepped bar.

We perform the Galerkin's procedure:

$$\int_0^1 [D_1 \cdot \delta(\xi) + (D_2 - D_1) \cdot \delta(\xi - \xi_1)] \frac{\mathrm{d}}{\mathrm{d}\xi} \sum_{j=1}^{n} p_j \psi_j(\xi) \psi_k(\xi) \mathrm{d}\xi$$
$$+ \int_0^1 [D_1 \cdot H(\xi) + (D_2 - D_1) \cdot H(\xi - \xi_1)] \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \sum_{j=1}^{n} p_j \psi_j(\xi) \psi_k(\xi) \mathrm{d}\xi$$
$$+ \int_0^1 [M_1 \cdot H(\xi) + (M_2 - M_1) \cdot H(\xi - \xi_1)] \omega^2 L^2 \sum_{j=1}^{n} p_j \psi_j(\xi) \psi_k(\xi) \mathrm{d}\xi = 0 \tag{37}$$

This equation can be rewritten in the following form:

$$\sum_{j=1}^{n} \left(N_{jk} + K_{jk} + \omega^2 M_{jk}\right) p_j = 0 \tag{38}$$

where

$$N_{jk} = \int_0^1 [D_1 \delta(\xi) + (D_2 - D_1) \delta(\xi - \xi_1)] \frac{\mathrm{d}\psi_j(\xi)}{\mathrm{d}\xi} \psi_k(\xi) \mathrm{d}\xi \tag{39}$$

$$K_{jk} = \int_0^1 [D_1 H(\xi) + (D_2 - D_1) H(\xi - \xi_1)] \frac{\mathrm{d}^2 \psi_j(\xi)}{\mathrm{d}\xi^2} \psi_k(\xi) \mathrm{d}\xi \tag{40}$$

$$M_{jk} = \int_0^1 [M_1 H(\xi) + (M_2 - M_1) H(\xi - \xi_1)] L^2 \psi_j(\xi) \psi_k(\xi) \mathrm{d}\xi \tag{41}$$

Introducing matrix notation, we have:

$$\left(N + K + \omega^2 M\right) p = 0 \tag{42}$$

Non trivial solutions of Eq. (42) can be found by solving the following eigenvalue problem in the unknown eigenfrequencies ω^2:

$$\det\left(N + K + \omega^2 M\right) = 0 \tag{43}$$

We observe that if we delete the matrix N we obtain the straightforward (naïve) Galerkin method. The full expression of this matrix N contains the terms N_{jk} appearing in the rigorous Galerkin method, but missing in the straightforward (naïve) version. In the next example we compare the exact frequencies of vibration with those of both the naïve and the rigorous application of the Galerkin method, and discuss the discrepancies.

2.5. Numerical example for a bar

We consider the concrete-steel bar, shown in Fig. 2

The geometrical and mechanical parameters of the system are described in Table 1.

The ratios in Eq. (44) reads:

$$\frac{E_2}{E_1} = 6.46, \quad \frac{\rho_2}{\rho_1} = 3.28, \quad \frac{L_2}{L_1} = 2, \quad \frac{A_2}{A_1} = 0.06 \tag{44}$$

Table 1
Parameters of each segment.

Concrete	50×50 cm	Steel	ϕ14 cm
E	$3.096 \cdot 10^{10}$ Pa	E	$2.0 \cdot 10^{11}$ Pa
ρ	$2.4 \cdot 10^{3}$ kg m^{-3}	ρ	$7.87 \cdot 10^{3}$ kg m^{-3}
A	0.25 m^2	A	$1.49 \cdot 10^{-2}$ m^2
L_1	1 m	L_2	2 m

Table 2
First four exact natural frequencies for the two-stepped bar with clamped-clamped boundary conditions.

Exact Solution for circular Frequency [rad s^{-1}]	
Mode	Clamped-Clamped
1	5144.3235
2	8559.5002
3	14,839.2830
4	17,857.6076

Table 3
First four natural frequencies for two-stepped bar in clamped-clamped boundary conditions obtained with naïve Galerkin method. For each frequency the relative error as defined per Eq. (45) is given in parentheses.

Clamped-Clamped Two-Stepped Bar NAÏVE GALERKIN METHOD								
Mode	Frequency [rad s^{-1}] (Relative Error ε [%])							
	1 Term	2 Terms	3 Terms	4 Terms	5 Terms	6 Terms	7 Terms	8 Terms
1	5018.5420 (−2.45%)	4930.1558 (−4.16%)	4851.9846 (−5.68%)	4834.2335 (−6.03%)	4833.2432 (−6.05%)	4820.9782 (−6.29%)	4815.1357 (−6.40%)	4814.9866 (−6.40%)
2		9501.2833 (11.00%)	9215.2323 (7.66%)	9123.8169 (6.59%)	9119.5767 (6.54%)	9113.0776 (6.47%)	9104.2562 (6.36%)	9103.8069 (6.36%)
3			14,689.0584 (−1.01%)	14,452.8971 (−2.60%)	14,318.3124 (−3.51%)	14,262.0080 (−3.89%)	14,257.8150 (−3.92%)	14,252.7848 (−3.95%)
4				19,723.2563 (10.45%)	19,141.3765 (7.19%)	18,692.1093 (4.67%)	18,546.0617 (3.86%)	18,544.4815 (3.85%)

Table 4
First four natural frequencies for two-stepped bar in clamped-clamped boundary conditions obtained with rigorous Galerkin method. For each frequency the relative error as defined per Eq. (45) is given in parentheses.

Clamped-Clamped Two-Stepped Bar RIGOROUS GALERKIN METHOD								
Mode	Frequency [rad s^{-1}] (Relative Error [%])							
	1 Term	2 Terms	3 Terms	4 Terms	5 Terms	6 Terms	7 Terms	8 Terms
1	5202.1240 (1.12%)	5180.7197 (0.71%)	5158.3163 (0.27%)	5152.1139 (0.15%)	5152.1106 (0.15%)	5150.2015 (0.11%)	5148.6875 (0.08%)	5148.6688 (0.08%)
2		8712.2687 (1.78%)	8708.2820 (1.74%)	8663.7780 (1.22%)	8648.6380 (1.04%)	8644.2586 (0.99%)	8624.4633 (0.76%)	8620.1693 (0.71%)
3			15,034.5161 (1.32%)	15,033.8675 (1.31%)	15,022.7849 (1.24%)	14,971.2199 (0.89%)	14,937.3245 (0.66%)	14,936.5568 (0.66%)
4				18,635.3520 (4.36%)	18,008.0035 (0.84%)	17,904.6285 (0.26%)	17,903.9560 (0.26%)	17,890.8099 (0.19%)

The exact characteristic Eq. (19) for the clamped-clamped bar leads to the following first four roots:

$$z_1 = 1.4323, \quad z_2 = 2.3832, \quad z_3 = 4.1316, \quad z_4 = 4.9720$$

Since we defined z as $\beta_1 L_1$, in order to obtain the circular frequency ω we need to multiply z by $\frac{c_1}{L_1}$.

A “rule of thumb” suggests that a recommendable approximation by Galerkin method is to use at least *2 N* terms to obtain the first N frequencies accurately. As in our case we are studying the first four natural frequencies, eight terms can be considered as upper bound of the series to lead accurate frequencies. Tables 3 and 4 show the results obtained by using the naïve form and the rigorous form, respectively, of the Galerkin method. The Tables report also the relative error ε

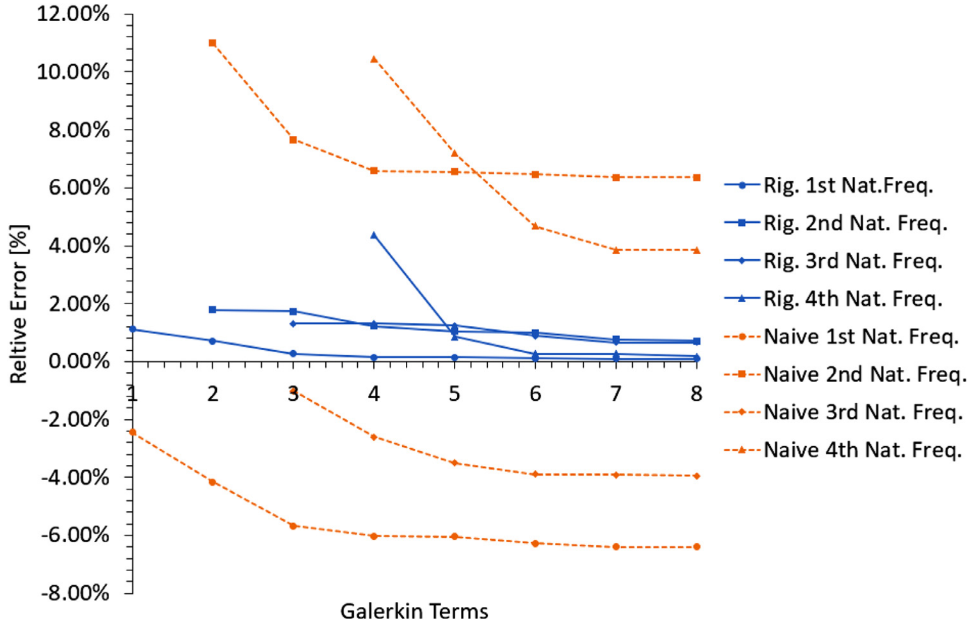

Fig. 3. Variation of the relative error between rigorous and naïve Galerkin method for the first four natural frequency of two-stepped bar in clamped-clamped boundary conditions with respect to the number of terms.

between the exact and the approximate Galerkin solutions defined as:

$$\varepsilon = \frac{\omega_{\text{approx}} - \omega_{\text{exact}}}{\omega_{\text{exact}}} \times 100\% \tag{45}$$

Fig. 3 portreys the relative error of the naïve and rigorous Galerkin methods solution with respect to the number of terms considered.

We observe from Fig. 3. that rigorous implementation of Galerkin method tends to the exact solution for all the frequencies considered. For instance, with one term approximation the relative error of the first circular frequency is 1.12% and reduces to 0.08% considering eight terms. On the other side, the naïve implementation of Galerkin method leads to a relative error of −2.45% for one term and to −6.40% considering eight terms. In addition, it can be seen that while for the rigorous implementation all the frequencies share a common trend vs the number of terms adopted, in the naïve form for some frequencies the error reduces while for some others it grows. This leads to the conclusions that as the naïve implementation does not tend to the exact solution and that by increasing the number of terms retained in its use it is not always beneficial.

3. Stepped plate vibrations

3.1. Basic equations

In this section we consider a stepped thin plate simply supported at all its edges as shown in Fig. 4. We are interested in finding the exact and approximated Galerkin based frequencies of vibration with the idea to show the fact the naïve form does not tend to the exact solution.

This plate has a as a side-length in x direction and b as side-length in y direction. Each step is characterized by a_i as a length in x direction, h_i as thickness, elastic modulus E_i and ρ_i represent the mass density ($i = 1, 2$). Considering a Kirchhoff-Love plate model, the governing differential equations in each part reads:

$$D_1 \nabla^4 w_1 + \rho_1 h_1 \frac{\partial^2 w_1}{\partial t^2} = 0 \quad 0 < x < a_1 \tag{46.a}$$

$$D_2 \nabla^4 w_2 + \rho_2 h_2 \frac{\partial^2 w_2}{\partial t^2} = 0 \quad a_1 < x < a_2 \tag{46.b}$$

where $D_i = \frac{E_i {h_i}^3}{12(1-{\nu_i}^2)}$ represents the flexural stiffness of the i-th portion of the stepped plate and ν_i is the Poisson's ratio. The displacement function in the z-direction $w_i(x, y, t)$ is represented as follows:

$$w_i(x, y, t) = W_i(x, y) sin(\omega t) \tag{47}$$

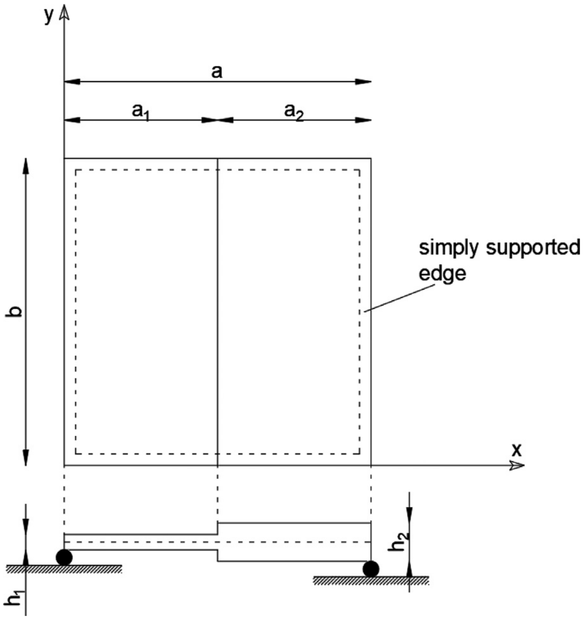

Fig. 4. All-Round Simply-Supported Stepped Plate under Study.

Inserting Eq. (47) into Eqs. (46.a) and (46.b) and dividing by D_i we obtain:

$$\left(\nabla^4 W_1(x,y) - \frac{\rho_1 h_1}{D_1}\omega^2 W_1(x,y)\right)\sin(\omega t) = 0 \quad 0 < x < a_1 \tag{48.a}$$

$$\left(\nabla^4 W_2(x,y) - \frac{\rho_2 h_2}{D_2}\omega^2 W_2(x,y)\right)\sin(\omega t) = 0 \quad a_1 < x < a \tag{48.b}$$

We are interested to a solution which is true for any value of time so the expressions in parentheses must vanish:

$$\nabla^4 W_1(x,y) - {k_1}^4 W_1(x,y) = 0 \quad 0 < x < a_1 \tag{49.a}$$

$$\nabla^4 W_2(x,y) - {k_2}^4 W_2(x,y) = 0 \quad a_1 < x < a \tag{49.b}$$

where ${k_i}^4$ is defined as ${k_i}^4 = \frac{\rho_i h_i}{D_i}\omega^2$.

The expression for the function $W_i(x, y)$, which describe a plate simply supported on all its edges was proposed independently by Voigt [18] and Lévy [19]. They consider, in the beginning, a plate with two opposite supported edges, for definiteness, let the simply supported edges be $y = 0$ and $y = b$, without specifying the boundary conditions along edges $x = 0$ and $x = a$. Following Voigt and Lévy, a solution of Eqs. (49.a) and (49.b) may be expressed in the form:

$$W_i(x,y) = \sum_{n=1}^{\infty} X_n(x) sin\left(\frac{n\pi y}{b}\right) \tag{50}$$

Substitution of Eq. (50) into Eqs. (49.a) and (49.b) results in:

$$\sum_{n=1}^{\infty}\left(\frac{d^4}{dx^4}X_{n,1}(x) - 2\frac{n^2\pi^2}{b^2}\frac{d^2}{dx^2}X_{n,1}(x) + \frac{n^4\pi^4}{b^4}X_{n,1}(x) - {k_1}^4 X_{n,1}(x)\right) sin\left(\frac{n\pi y}{b}\right) = 0 \quad 0 < x < a_1 \tag{51.a}$$

$$\sum_{n=1}^{\infty}\left(\frac{d^4}{dx^4}X_{n,2}(x) - 2\frac{n^2\pi^2}{b^2}\frac{d^2}{dx^2}X_{n,2}(x) + \frac{n^4\pi^4}{b^4}X_{n,2}(x) - {k_2}^4 X_{n,2}(x)\right) sin\left(\frac{n\pi y}{b}\right) = 0 \quad a_1 < x < a \tag{51.b}$$

The expression inside each pair of parentheses must vanish; namely:

$$\frac{d^4}{dx^4}X_{n,1}(x) - 2\frac{n^2\pi^2}{b^2}\frac{d^2}{dx^2}X_{n,1}(x) + \frac{n^4\pi^4}{b^4}X_{n,1}(x) - {k_1}^4 X_{n,1}(x) = 0 \quad 0 < x < a_1 \tag{52.a}$$

$$\frac{d^4}{dx^4}X_{n,2}(x) - 2\frac{n^2\pi^2}{b^2}\frac{d^2}{dx^2}X_{n,2}(x) + \frac{n^4\pi^4}{b^4}X_{n,2}(x) - {k_2}^4 X_{n,2}(x) = 0 \quad a_1 < x < a \tag{52.b}$$

The solution of Eqs. (52.a) and (52.b) is sought in the form $X_n(x) = Be^{rx}$ where r is the characteristic exponent, satisfying the following equation:

$$r^4 - 2\frac{n^2\pi^2}{b^2}r^2 + \frac{n^4\pi^4}{b^4} - k^4 = 0 \tag{53}$$

Solving first for r^2 yields:

$$r_{1,2}{}^2 = \frac{n^2\pi^2}{b^2} \pm k^2 \tag{54}$$

The four roots given in Eq. (53) are obtained as follows:

$$r_{1,2} = \pm R_n{}^+ = \pm\sqrt{\frac{n^2\pi^2}{b^2} + k^2} \tag{55.a}$$

$$r_{3,4} = \pm R_n{}^- = \pm\sqrt{\frac{n^2\pi^2}{b^2} - k^2} \ \ for\ k^2 < \frac{n^2\pi^2}{b^2} \tag{55.b}$$

$$r_{3,4} = \pm iQ_m = \pm i\sqrt{k^2 - \frac{n^2\pi^2}{b^2}} \ \ for\ k^2 > \frac{n^2\pi^2}{b^2} \tag{55.c}$$

$$r_{3,4} = 0 \ \ for\ k^2 = \frac{n^2\pi^2}{b^2} \tag{55.d}$$

The solution for Eqs. (52.a) and (52.b) may then be constructed as follows:

$$X_n(x) = A_1 \sinh\left(R_n{}^- x\right) + A_2 \cosh\left(R_n{}^- x\right) + A_3 \sinh\left(R_n{}^+ x\right) + A_4 \cosh\left(R_n{}^+ x\right) \quad for\ k^2 < \frac{n^2\pi^2}{b^2} \tag{56.a}$$

$$X_n(x) = A_1 \sin\left(Q_m x\right) + A_2 \cos\left(Q_m x\right) + A_3 \sinh\left(R_n{}^+ x\right) + A_4 \cosh\left(R_n{}^+ x\right) \quad for\ k^2 > \frac{n^2\pi^2}{b^2} \tag{56.b}$$

$$X_n(x) = A_1 \sinh\left(R_n{}^+ x\right) + A_2 \cosh\left(R_n{}^+ x\right) + A_3 + A_4 \quad for\ k^2 = \frac{n^2\pi^2}{b^2} \tag{56.c}$$

where A_i are unknown constants.

3.2. Exact solution for the stepped plate vibrations

In our problem, in total we have 8 unknowns: 4 constants for the first part of the plate and 4 for the second step.

The Eqs (56.a), (56.b) and (56.c) can be recast in terms of the natural frequencies. Thus, expression in Eqs (56.a)–(56.c) are valid when the following respective inequalities are satisfied:

$$\omega < \frac{n^2\pi^2}{b^2}\sqrt{\frac{Eh^2}{12\left(1-\nu^2\right)\rho}} \tag{57.a}$$

$$\omega > \frac{n^2\pi^2}{b^2}\sqrt{\frac{Eh^2}{12\left(1-\nu^2\right)\rho}} \tag{57.b}$$

$$\omega = \frac{n^2\pi^2}{b^2}\sqrt{\frac{Eh^2}{12\left(1-\nu^2\right)\rho}} \tag{57.c}$$

The boundary conditions are written as:

$$W_1(x=0) = 0 \tag{58.a}$$

$$M_x(x=0) = D_1\left[\frac{\partial^2 W_1}{\partial x^2} + \nu\frac{\partial^2 W_1}{\partial y^2}\right]_{x=0} = 0 \tag{58.b}$$

$$W_2(x=a) = 0 \tag{58.c}$$

$$M_x(x=a) = D_2\left[\frac{\partial^2 W_2}{\partial x^2} + \nu\frac{\partial^2 W_2}{\partial y^2}\right]_{x=a} = 0 \tag{58.d}$$

The continuity conditions between the two steps are written as:

$$W_1(x = a_1) = W_2(x = a_1) \tag{59.a}$$

$$W_1{}'(x = a_1) = W_2{}'(x = a_1) \tag{59.b}$$

$$M_x(x = a_1) = D_1\left[\frac{\partial^2 W_1}{\partial x^2} + \nu\frac{\partial^2 W_1}{\partial y^2}\right]_{x=a_1} = D_2\left[\frac{\partial^2 W_2}{\partial x^2} + \nu\frac{\partial^2 W_2}{\partial y^2}\right]_{x=a_1} = M_x(x = a_1) \tag{59.c}$$

$$V_x(x = a_1) = D_1\left[\frac{\partial^3 W_1}{\partial x^3} + (2-\nu)\frac{\partial^3 W_1}{\partial x \partial y^2}\right]_{x=a_1} = D_2\left[\frac{\partial^3 W_2}{\partial x^3} + (2-\nu)\frac{\partial^3 W_2}{\partial x \partial y^2}\right]_{x=a_1} = V_x(x = a_1) \tag{59.d}$$

The boundary condition Eqs. (58–59) lead thus to a homogeneous system of eight equation in the ω and A_i unknowns. Non-trivial solutions, i.e. the roots of the characteristic equation obtained by setting the determinant equal to zero, are the exact frequency of vibration of the problem.

3.3. Application of the naïve galerkin method for stepped plate vibrations

As was demonstrated by Elishakoff and Boutur [20], the literature applies, in overwhelming majority, if not all cases, the naïve version of the Galerkin method for stepped structures when applying the weighted residuals methodology. We first demonstrate this methodology. One starts with the following governing differential equation:

$$\frac{\partial^4}{\partial x^4}W(x,y) + 2\frac{\partial^4}{\partial x^2 \partial y^2}W(x,y) + \frac{\partial^4}{\partial y^4}W(x,y) - k^4(x)W(x,y) = 0 \tag{60}$$

expressing the function $W(x, y)$ in series as follows:

$$W(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn}\psi_{mn}(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn}\psi_m(x)\psi_n(y) \tag{61}$$

where $\psi_{mn}(x, y)$ is a comparison function that have to satisfy all the boundary conditions and the product of $\psi_m(x)$ and $\psi_n(y)$ represents a multiplicative form of the mode shape.

For the all-round simply supported plate on all edges, the case under study, the most employed and probably the best candidate comparison function is the product of two sinusoidal functions as follows, with attendant boundary conditions satisfied:

$$\psi_{mn}(x,y) = \psi_m(x)\psi_n(y) = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{62}$$

Substituting Eq. (61) in Eq. (60) results in:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\psi''''_m(x)\psi_n(y) + 2\psi''_m(x)\psi''_n(y) + \psi_m(x)\psi''''_n(y) - k^4(x)\psi_m(x)\psi_n(y)\right]p_{mn} = \varepsilon(x,y) \tag{63}$$

where $\varepsilon(x, y)$ is the residual or error. Now, multiplication of the residual by the comparison functions and integration over the plate's domain leads to:

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Bigg(&\int_0^a\int_0^b \psi''''_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy + \int_0^a\int_0^b 2\psi''_m(x)\psi''_n(y)\psi_q(x)\psi_n(y)dxdy\\ &+\int_0^a\int_0^b \psi_m(x)\psi''''_n(y)\psi_q(x)\psi_n(y)dxdy + \int_0^{a_1}\int_0^b -k^4{}_1\psi_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy\\ &+\int_{a_1}^a\int_0^b -k^4{}_2\psi_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy\Bigg)p_{mn} = 0\end{aligned} \tag{64}$$

It is instructive to introduce the following notation:

$$A_{mnq} = \int_0^a\int_0^b \psi''''_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy \tag{65.a}$$

$$B_{mnq} = \int_0^a\int_0^b 2\psi''_m(x)\psi''_n(y)\psi_q(x)\psi_n(y)dxdy \tag{65.b}$$

$$C_{mnq} = \int_0^a\int_0^b \psi_m(x)\psi''''_n(y)\psi_q(x)\psi_n(y)dxdy \tag{65.c}$$

$$M_{mnq} = \int_0^{a_1}\int_0^b -\frac{\rho_1 h_1}{D_1}\psi_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy + \int_{a_1}^a\int_0^b -\frac{\rho_2 h_2}{D_2}\psi_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy \tag{65.d}$$

In this form we obtain, for a specific value of n, the following matrix equation:

$$\left(A+B+C+\omega^2 M\right)p=0 \tag{66}$$

Denoting $K=A+B+C$ we reconduct the problem to a form similar to Eq. (29) which is a homogeneous linear system and it has non-trivial solution when the determinant of the expression between parentheses vanishes.

As was demonstrated by Elishakoff et al. [21] for the static problem of a beam under distributed load, such a naïve application of the Galerkin method might lead to erroneous results. One has to resort to another methodology, namely to so-called rigorous implementation of the Galerkin method.

3.4. Application of the rigorous galerkin method for stepped plate vibrations

Denoting $\rho h(x, y)$ as $M(x, y)$ and evaluating the Laplacian operator, the Eqs. (49.a) and (49.b) can be re-written as follows:

$$\frac{\partial^2}{\partial x^2}\left[D(x,y)\left(\frac{\partial^2 W(x,y)}{\partial x^2}+\nu\frac{\partial^2 W(x,y)}{\partial y^2}\right)\right]+2(1-\nu)\frac{\partial^2}{\partial x\partial y}\left[D(x,y)\left(\frac{\partial^2 W(x,y)}{\partial x\partial y}\right)\right]$$
$$+\frac{\partial^2}{\partial y^2}\left[D(x,y)\left(\frac{\partial^2 W(x,y)}{\partial y^2}+\nu\frac{\partial^2 W(x,y)}{\partial x^2}\right)\right]-M(x,y)\omega^2 W(x,y)=0 \tag{67}$$

In order to implement the rigorous Galerkin method we represent the flexural rigidity and the mass of the system as generalized functions in total analogy with Eqs. (31) and (32) obtaining:

$$\frac{\partial^2 D(x,y)}{\partial x^2}\frac{\partial^2 W(x,y)}{\partial x^2}+2\frac{\partial D(x,y)}{\partial x}\frac{\partial^3 W(x,y)}{\partial x^3}+2\frac{\partial D(x,y)}{\partial y}\frac{\partial^3 W(x,y)}{\partial y^3}+D(x,y)\frac{\partial^4 W(x,y)}{\partial x^4}+D(x,y)\frac{\partial^4 W(x,y)}{\partial y^4}$$
$$+2\frac{\partial^2 D(x,y)}{\partial x\partial y}\frac{\partial^2 W(x,y)}{\partial x\partial y}+2\frac{\partial D(x,y)}{\partial x}\frac{\partial^3 W(x,y)}{\partial x\partial y^2}+2\frac{\partial D(x,y)}{\partial y}\frac{\partial^3 W(x,y)}{\partial x^2\partial y}+2D(x,y)\frac{\partial^4 W(x,y)}{\partial x^2\partial y^2}-2\nu\frac{\partial^2 D(x,y)}{\partial x\partial y}\frac{\partial^2 W(x,y)}{\partial x\partial y}$$
$$+\nu\frac{\partial^2 D(x,y)}{\partial y^2}\frac{\partial^2 W(x,y)}{\partial x^2}+\frac{\partial^2 D(x,y)}{\partial y^2}\frac{\partial^2 W(x,y)}{\partial y^2}+\nu\frac{\partial^2 D(x,y)}{\partial x^2}\frac{\partial^2 W(x,y)}{\partial y^2}-M(x,y)\omega^2 W(x,y)=0 \tag{68}$$

Substituting Eq. (61) in Eq. (68) we get:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\partial^2 D(x,y)}{\partial x^2}\psi''_m(x)\psi_n(y)+2\frac{\partial D(x,y)}{\partial x}\psi'''_m(x)\psi_n(y)+2\frac{\partial D(x,y)}{\partial x}\psi_m(x)\psi'''_n(y)\right.$$
$$+D(x,y)\psi''''_m(x)\psi_n(y)+D(x,y)\psi_m(x)\psi''''_n(y)+2\frac{\partial^2 D(x,y)}{\partial x\partial y}\psi'_m(x)\psi'_n(y)$$
$$+2\frac{\partial D(x,y)}{\partial x}\psi'_m(x)\psi''_n(y)+2\frac{\partial D(x,y)}{\partial x}\psi''_m(x)\psi'_n(y)+2D(x,y)\psi''_m(x)\psi''_n(y)-2\nu\frac{\partial^2 D(x,y)}{\partial x\partial y}\psi'_m(x)\psi'_n(y)$$
$$+\nu\frac{\partial^2 D(x,y)}{\partial y^2}\psi''_m(x)\psi_n(y)+\frac{\partial^2 D(x,y)}{\partial y^2}\psi_m(x)\psi''_n(y)$$
$$\left.+\nu\frac{\partial^2 D(x,y)}{\partial x^2}\psi_m(x)\psi''_n(y)-\omega^2 M(x)\psi_m(x)\psi_n(y)\right]pmn=\varepsilon(x,y) \tag{69}$$

Now we multiply by the comparison functions and we integrate in both, x and y direction as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\int_0^a\int_0^b\frac{\partial^2 D(x,y)}{\partial x^2}\psi''_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy+\int_0^a\int_0^b 2\frac{\partial D(x,y)}{\partial x}\psi'''_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy\right.$$
$$+\int_0^a\int_0^b 2\frac{\partial D(x,y)}{\partial x}\psi_m(x)\psi'''_n(y)\psi_q(x)\psi_n(y)dxdy+\int_0^a\int_0^b D(x,y)\psi''''_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy$$
$$+\int_0^a\int_0^b D(x,y)\psi_m(x)\psi''''_n(y)\psi_q(x)\psi_n(y)dxdy+\int_0^a\int_0^b 2\frac{\partial^2 D(x,y)}{\partial x\partial y}\psi'_m(x)\psi'_n(y)\psi_q(x)\psi_n(y)dxdy$$
$$+\int_0^a\int_0^b 2\frac{\partial D(x,y)}{\partial x}\psi'_m(x)\psi''_n(y)\psi_q(x)\psi_n(y)dxdy+\int_0^a\int_0^b 2\frac{\partial D(x,y)}{\partial x}\psi''_m(x)\psi'_n(y)\psi_q(x)\psi_n(y)dxdy$$
$$+\int_0^a\int_0^b 2D(x,y)\psi''_m(x)\psi''_n(y)\psi_q(x)\psi_n(y)dxdy+\int_0^a\int_0^b -2\nu\frac{\partial^2 D(x,y)}{\partial x\partial y}\psi'_m(x)\psi'_n(y)\psi_q(x)\psi_n(y)dxdy$$
$$+\int_0^a\int_0^b \nu\frac{\partial^2 D(x,y)}{\partial y^2}\psi''_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy+\int_0^a\int_0^b\frac{\partial^2 D(x,y)}{\partial y^2}\psi_m(x)\psi''_n(y)\psi_q(x)\psi_n(y)dxdy$$
$$\left.+\int_0^a\int_0^b \nu\frac{\partial^2 D(x,y)}{\partial x^2}\psi_m(x)\psi''_n(y)\psi_q(x)\psi_n(y)dxdy+\int_0^a\int_0^b -\omega^2 M(x)\psi_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy\right]pmn=0 \tag{70}$$

We ntroduce the following notations:

$$A_{mnq}=\int_0^a\int_0^b\frac{\partial^2 D(x,y)}{\partial x^2}\psi''_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.a}$$

$$B_{mnq}=\int_0^a\int_0^b 2\frac{\partial D(x,y)}{\partial x}\psi'''_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.b}$$

$$C_{mnq}=\int_0^a\int_0^b 2\frac{\partial D(x,y)}{\partial x}\psi_m(x)\psi'''_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.c}$$

$$D_{mnq}=\int_0^a\int_0^b D(x,y)\psi''''_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.d}$$

$$E_{mnq}=\int_0^a\int_0^b D(x,y)\psi_m(x)\psi''''_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.e}$$

$$F_{mnq}=\int_0^a\int_0^b 2\frac{\partial^2 D(x,y)}{\partial x\partial y}\psi'_m(x)\psi'_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.f}$$

$$G_{mnq}=\int_0^a\int_0^b 2\frac{\partial D(x,y)}{\partial x}\psi'_m(x)\psi''_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.g}$$

$$H_{mnq}=\int_0^a\int_0^b 2\frac{\partial D(x,y)}{\partial x}\psi''_m(x)\psi'_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.h}$$

$$I_{mnq}=\int_0^a\int_0^b 2D(x,y)\psi''_m(x)\psi''_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.i}$$

$$L_{mnq}=\int_0^a\int_0^b -2\nu\frac{\partial^2 D(x,y)}{\partial x\partial y}\psi'_m(x)\psi'_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.j}$$

$$N_{mnq}=\int_0^a\int_0^b \nu\frac{\partial^2 D(x,y)}{\partial y^2}\psi''_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.k}$$

$$O_{mnq}=\int_0^a\int_0^b \frac{\partial^2 D(x,y)}{\partial y^2}\psi_m(x)\psi''_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.l}$$

$$P_{mnq}=\int_0^a\int_0^b \nu\frac{\partial^2 D(x,y)}{\partial x^2}\psi_m(x)\psi''_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.m}$$

$$M_{mnq}=\int_0^a\int_0^b -M(x)\psi_m(x)\psi_n(y)\psi_q(x)\psi_n(y)dxdy \tag{71.n}$$

Since Eqs. (71.a) and (71.b) present the second order derivative of the stiffness function $D(x)$, the first derivative of the Dirac delta function, namely the doublet function, must be introduced:

$$\frac{d}{dx}\delta(x)=\delta'(x) \tag{72}$$

The main property of the doublet function, applied to a generic function $f(x)$, reads as follows:

$$\int_{-\infty}^{\infty}\delta'(x)f(x)dx=-\int_{-\infty}^{\infty}\delta(x)f'(x)dx \tag{73}$$

In this form we obtain, for a specific value of n, the following matrix equation:

$$\left(A+B+C+D+E+F+G+H+I+L+N+O+P+\omega^2 M\right)p=0 \tag{74}$$

Denoting $K=A+B+C+D+E+F+G+H+I+L+N+O+P$ we define an eigenvalue problem in the form of Eq. (29) for the sough unknown ω^2.

3.5. Numerical example for stepped plate vibrations

We study stepped plate, a simply supported on all edges, and made of only one material, namely steel, with the mechanical properties defined in Table 5.

We define the plate under study through three characteristic dimensions ratios: $^{a}/_{b}$, $^{a_1}/_{a}$ and $^{h_1}/_{h_2}$. We report these parameters in Table 6.

Table 5
Material parameters of the plate.

Material: Steel	
Elastic modulus	$E = 207 \cdot 10^9$ Pa
Poisson's ratio	$\nu = 0.3$
Mass density	$\rho = 7800$ kg m^{-3}

Table 6
Geometrical parameters for the studied case

Geometrical parameters							
a	b	a_1	h_1	h_2	h_1/h_2	a_1/a	a/b
1.0 m	1.0 m	0.3 m	0.001 m	0.0015 m	0.67	0.30	1

Table 7
Exact solution.

Simply supported plate						
Mode	Half-waves in x direction	Half-waves in y direction	Frequency, current [rad s^{-1}]	Frequency, current [non – dimensional]	Frequency, Ref. 22 [non – dimensional]	Relative Error [%]
1	1	1	40.4486	2.6289	2.6289	0.0003%
2	2	1	100.5934	6.5379	6.5380	−0.0009%
3	1	2	104.0137	6.7602	6.7603	−0.0009%
4	2	2	164.9920	10.7234	10.7240	−0.0051%
5	3	1	206.6667	13.4320	13.4320	0.0003%
6	1	3	207.7331	13.5013	13.5010	0.0026%

Table 8
Natural frequencies and relative errors derived via rigorous Galerkin method

Simply supported plate										
Mode	1 Term	2 Terms	3 Terms	4 Terms	5 Terms	6 Terms	7 Terms	8 Terms	9 Terms	10 Terms
1	42.8955 (6.0495%)	42.1596 (4.2301%)	41.4168 (2.3937%)	41.0100 (1.3880%)	40.9660 (1.2792%)	40.9364 (1.2059%)	40.8207 (0.9199%)	40.7620 (0.7747%)	40.7619 (0.7746%)	40.7270 (0.6883%)
2		107.7234 (7.0879%)	104.4985 (3.8821%)	102.1641 (1.5614%)	101.5716 (0.9724%)	101.5716 (0.9724%)	101.3588 (0.7609%)	101.1555 (0.5588%)	101.1388 (0.5422%)	101.0984 (0.5020%)
3	108.9781 (4.7728%)	107.0490 (2.9181%)	105.5209 (1.4491%)	104.7518 (0.7096%)	104.6483 (0.6101%)	104.6207 (0.5836%)	104.4694 (0.4381%)	104.3832 (0.3552%)	104.3825 (0.3546%)	104.3427 (0.3163%)
4		173.5322 (5.1761%)	169.5910 (2.7874%)	166.6639 (1.0133%)	165.6704 (0.4112%)	165.6278 (0.3853%)	165.5171 (0.3183%)	165.3255 (0.2021%)	165.2829 (0.1763%)	165.2734 (0.1483%)
5			215.3084 (4.1815%)	210.8741 (2.0358%)	208.3348 (0.8071%)	207.3552 (0.3331%)	207.2139 (0.2648%)	207.1984 (0.2573%)	207.0735 (0.1969%)	206.9733 (0.1483%)
6	220.6238 (6.2054%)	215.8230 (3.8944%)	211.6194 (1.8708%)	209.1886 (0.7007%)	208.6383 (0.4358%)	208.6366 (0.4349%)	208.4123 (0.3270%)	208.2140 (0.2315%)	208.1980 (0.2238%)	208.1606 (0.2058%)

3.5.1. Exact solution

For the exact solution we substitute the sine function in y-direction and obtain the ordinary differential equation in the x-direction; satisfaction of boundary and continuity conditions yields to the frequency of vibrations, as in the paper by Xiang and Wang [22]. We report our exact solutions in Table 7 and compared them with those provided by reference [22]. For each mode we report the number of half-waves in both direction x and y, the circular frequency value and its dimensionless value as well as the error with respect to those in Ref. [22]. The non-dimensional frequencies are obtained by multiplying the circular frequencies by the coefficient $\gamma = (\frac{b}{\pi})^2\sqrt{\frac{\rho h_1}{D_1}}$, as in Ref. [22], whereas the relative error is evaluated as:

$$\varepsilon = \frac{\omega_{Exact,current} - \omega_{Ref.[22]}}{\omega_{Ref.[22]}} \times 100\% \tag{75}$$

3.5.2. Application of the Rigorous Galerkin method

We report the result of rigorous implementation of the Galerkin method obtained by employment of up to 10 terms. Results are summarized in Table 8, in terms of frequency values as well as the relative error according to Eq. (45).

One can observe that the relative error tends to diminish, as expected, for all the considered modes. It starts from a specific value and with increasing number of terms the error reduces to less than 1%.

Table 9
Natural frequencies and relative error derived via naïve Galerkin method.

Simply supported plate										
Mode	1 Term	2 Terms	3 Terms	4 Terms	5 Terms	6 Terms	7 Terms	8 Terms	9 Terms	10 Terms
1	42.3880 (4.7948%)	42.1571 (4.2240%)	42.1142 (4.1178%)	42.1106 (4.1088%)	42.1105 (4.1088%)	42.1101 (4.1077%)	42.1099 (4.1072%)	42.1099 (4.1072%)	42.1099 (4.1071%)	42.1098 (4.1071%)
2		99.2403 (−1.3451%)	98.3092 (−2.2707%)	98.2014 (−2.3779%)	98.1995 (−2.3798%)	98.1971 (−2.3821%)	98.1950 (−2.3842%)	98.1948 (−2.3844%)	98.1947 (−2.3845%)	98.1945 (−2.3847%)
3	105.9700 (1.8808%)	103.9772 (−0.0351%)	103.4244 (−0.5666%)	103.3628 (−0.6258%)	103.3628 (−0.6258%)	103.3565 (−0.6318%)	103.3533 (−0.6349%)	103.3532 (−0.6350%)	103.3529 (−0.6353%)	103.3524 (−0.6358%)
4		160.9462 (−2.4521%)	158.7705 (−3.7708%)	158.4516 (−3.9641%)	158.4403 (−3.9709%)	158.4362 (−3.9734%)	158.4308 (−3.9767%)	158.4300 (−3.9772%)	158.4298 (−3.9773%)	158.4293 (−3.9776%)
5			206.4340 (−0.1126%)	204.9155 (−0.8473%)	204.6357 (−0.9828%)	204.6047 (−0.9977%)	204.6046 (−0.9978%)	204.6029 (−0.9986%)	204.6015 (−0.9993%)	204.6014 (−0.9994%)
6	211.9401 (2.0252%)	203.1052 (−2.2278%)	199.6772 (−3.8780%)	199.1581 (−4.1279%)	199.1564 (−4.1287%)	199.1193 (−4.1466%)	199.0945 (−4.1585%)	199.0930 (−4.1592%)	199.0911 (−4.1602%)	199.0877 (−4.1618%)

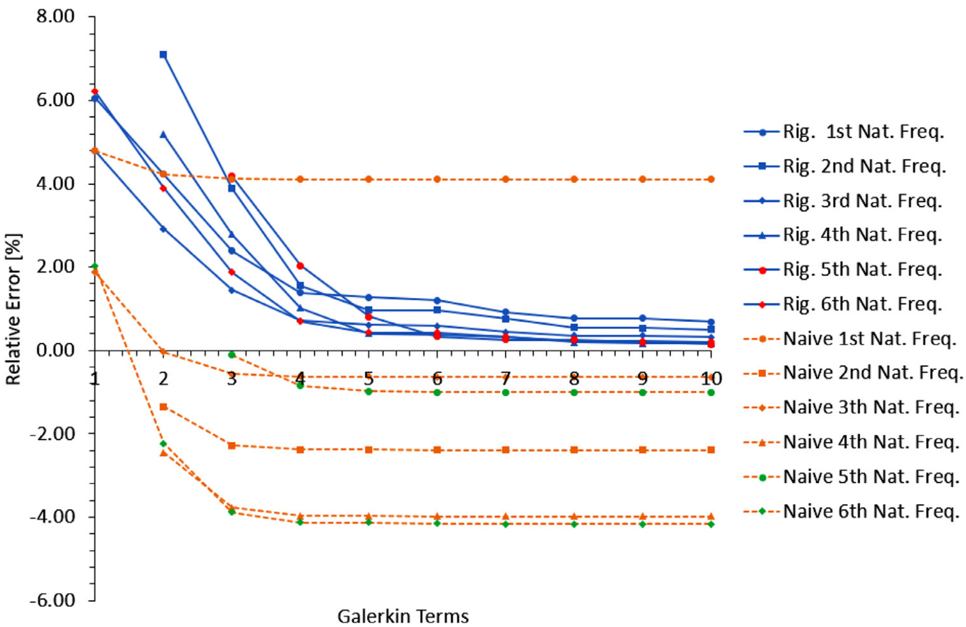

Fig. 5. Rigorous vs Naïve Galerkin method, 6th natural frequency, relative error.

3.5.3. *Naïve Galerkin method*

We report now the result of naïve implementation of the Galerkin method. Tables 9 reports the frequency values, as well as the relative error in comparison with the exact circular frequencies evaluated via Eq. (45), for up to 10 terms in the expansion.

We show in Fig. 5 the trend of the relative error between rigorous and naïve implementation of Galerkin method versus the exact solution. This plot represents the behavior of the relative error of the first six circular frequencies with the number of terms in the expansion. We observe that all the circular frequencies exhibit a decreasing trend as a function of the adopted number of approximating terms. However, we note that some frequencies are higher and remain higher than the exact ones w.r.t the increasing number of terms, some other are initially higher and ends to be lower than the exact one, whereas some others are lower and ends to be lower with respect to the exact ones.

In particular we observed that for the sixth natural frequency with one term of approximation rigorous version has 6.21% of error and naïve implementation 2.02%, then with ten terms in the expansion the rigorous method reached the convergence at 0.21% of error, on the other hand naïve implementation is far from the exact solution with an error of −4.16%.

4. Discussion and conclusion

Theoretical and numerical results derived in this study show that the naïve implementation of Galerkin method does not tend to exact solution for stepped structures, whereas rigorous version leads to exact solution. Rigorous Galerkin method has additional terms in contrast with the naïve version. These terms lead to a bounded error in the computation of the frequencies of vibration and more importantly the *convergence* to the exact solution. This study demonstrates that straightforward, naïve Galerkin method should be *abandoned* altogether for stepped structures, and *rigorous* Galerkin method should be *adopted*.

In order to stress the novelty and importance of the proposed technique, some relevant comments appear to be instructive. The application of the generalized functions in vibration problems is not new. For example, papers by Eftekhari [23], Chicurel-Uziel [24], Jones [25], Soedel and Powder [26] and Caddemi Calio [27]. Still, the present paper is the first one that deals with both stepped bars and plates in conjunction with generalized functions, namely the Heaviside step function, Dirac's delta function, and doublet function. This study clearly shows that naïve version of the Galerkin method is inapplicable to stepped structures. This observation appears to be of extreme importance due to following consideration; engineers and researchers usually use low order approximations. Even, Leipholtz [28,29] who dealt with mathematical aspects of the convergence of the Galerkin method, still, in his numerous papers resorted to one or two-term Galerkin approximations, according to the testimony of Gladwell [30]. If engineer resorts to the low-order approximation of Galerkin method for the stepped structure, he/she will be unable to detect lack of convergence to the exact solution, as was demonstrated in this study. This is despite the fact that there are papers that deal with multi-term implementation of the method [31–34]. Thus, application of naïve methodology can lead to incorrect design of the structure, whereas the rigorous implementation will lead to exact solution, and thus, to rigorous design. In this study we dealt only with variation of thickness in one of the coordinate directions; it is hoped that generalizations of this study for other types of steps, for example, along the diagonal, or with a certain angle to the rectangular plates' sides will be endeavored in the future by current authors or other investigators.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Isaac Elishakoff: Conceptualization, Investigation, Methodology, Writing - original draft. **Marco Amato:** Investigation, Software, Methodology. **Arvan Prakash Ankitha:** Investigation, Software, Writing - original draft. **Alessandro Marzani:** Investigation, Writing - original draft.

Acknowledgement

This study was conducted when Prakash Ankitha Arvan and Marco Amato served as Visiting Scholars at the Florida Atlantic University and part of their respective Master's thesis work. We thank the reviewers for constructive comments that led to the improvement of the manuscript and its more lucid presentation.

References

[1] B.G. Galerkin, Series occurring in some problems of elastic stability of rods and plates, Eng Bull 19 (1915) 14.
[2] H.H.E. Leipholz, Use of Galerkin's method for vibration problems, Shock Vib. Dig. 8 (1976) 3–18.
[3] M.J. Gander, G. Wanner, From Euler, Ritz, and Galerkin to modern computing, Siam Rev. 54 (2012) 627–666.
[4] S. Repin, One Hundred Years of the Galerkin Method, Comput. Methods Appl. Math. 17 (2017) 351–357.
[5] S.G. Mikhlin, The Numerical Performance of Variational Methods, Wolters-Noordhoff Publishing, 1971.
[6] M. Magen, D. Pnueli, Y. Zvirin, The stability chart of double-diffusive processes with parallel flows—Construction by the Galerkin and continuation methods, J. Eng. Math. 20 (1986) 127–144.
[7] P.D. Sclavounos, A note on a Galerkin technique for integral equations in potential flows, J. Eng. Math. 21 (1987) 101–114.
[8] L. Grinberg, G.E. Karniadakis, A scalable domain decomposition method for ultra-parallel arterial flow simulations, Commun. Comput. Phys. 4 (2008) 1151–1169.
[9] H.B. Wilson, R.W. Scharstein, Computing elliptic membrane high frequencies by Mathieu and Galerkin methods, J. Eng. Math. 57 (2007) 41–55.
[10] J. Wang, X. Ye, A weak Galerkin finite element method for the Stokes equations, Adv. Comput. Math. 42 (2016) 155–174.
[11] N. Izem, M. Seaid, M. Wakrim, A high-order nodal discontinuous galerkin method for 1D morphodynamic modelling, Int. J. Comput. Appl. (2012) 41.
[12] D.S. Chehil, R. Jategaonkar, Determination of natural frequencies of a beam with varying section properties, J. Sound Vib. 115 (1987) 423–436.
[13] C. Maurini, M. Porfiri, J. Pouget, Numerical methods for modal analysis of stepped piezoelectric beams, J. Sound Vib. 298 (2006) 918–933.
[14] M. Soheylian, H. Ahmadian, Dynamic stability of step beam carrying concentrated masses under follower force, (2007).
[15] S.M. Al-Said, Crack detection in stepped beam carrying slowly moving mass, J. Vib. Control. 14 (2008) 1903–1920.
[16] S.R. Borneman, S.M. Hashemi, H. Alighanbari, Vibration Analysis of Cracked Stepped Laminated Composite Beams, Int. J. Veh. Struct. Syst. (2009) 1.
[17] M. Pirmoradian, M. Keshmiri, H. Karimpour, On the parametric excitation of a Timoshenko beam due to intermittent passage of moving masses: instability and resonance analysis, Acta Mech 226 (2015) 1241–1253.
[18] W. Voigt, Bemerkung zu dem Problem der transversalen Schwingungen rechteckiger Platten, Nachrichten von Der Königl, Gesellschaft Der Wissenschaften Und Der Georg. Zu Göttingen (1893) 225–230.
[19] M. Levy, Sur l'équilibre élastique d'une plaque rectangulaire, Comptes Rendus Acad. Sci 129 (1899) 535–539 Paris.
[20] I. Elishakoff, D. Boutur, Rigorous Implementation of the Galerkin Method for Uniform and Stepped Columns, AIAA J 58 (2020) 2261–2268.
[21] I. Elishakoff, A.P. Ankitha, A. Marzani, Rigorous versus na\"\ive implementation of the Galerkin method for stepped beams, Acta Mech 230 (2019) 3861–3873.
[22] Y. Xiang, C.M. Wang, Exact buckling and vibration solutions for stepped rectangular plates, J, Sound Vib 250 (2002) 503–517.
[23] S.A. Eftekhari, A note on mathematical treatment of the Dirac-delta function in the differential quadrature bending and forced vibration analysis of beams and rectangular plates subjected to concentrated loads, Appl. Math. Model. 39 (2015) 6223–6242.

[24] E. Chicurel-Uziel, Dirac delta representation by exact parametric equations.: application to impulsive vibration systems, J. Sound Vib 305 (2007) 134–150.
[25] A.T. Jones, Vibration of beams immersed in a liquid, Exp. Mech. 10 (1970) 84–88.
[26] W. Soedel, D.P. Powder, A general Dirac delta function method for calculating the vibration response of plates to loads along arbitrarily curved lines, J. Sound Vib 65 (1979) 29–35.
[27] S. Caddemi, I. Calio, Exact closed-form solution for the vibration modes of the Euler–Bernoulli beam with multiple open cracks, J. Sound Vib 327 (2009) 473–489.
[28] H.H.E. Leipholz, On the convergence of Ritz and Galerkin's method in the case of certain nonconservative systems and using admissible coordinate functions, Acta Mech 19 (1974) 57–76.
[29] H.H.E. Leipholz, R. Mandadi, On the non-uniform convergence of Galerkin's method, J. Sound Vib 57 (1978) 483–498.
[30] G.M.L. Gladwell, Follower forces: Leipholz's early researches in elastic stability, Can. J. Civ. Eng. 17 (1990) 277–286.
[31] A. Cohen, R. DeVore, C. Schwab, Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs, Found. Comput. Math. 10 (2010) 615–646.
[32] I. Elishakoff, M. Zingales, Coincidence of boobnov-galerkin and closed-form solutions in an applied mechanics problem, J. Appl. Mech. 70 (2003) 777–779.
[33] I. Elishakoff, L.H.N. Lee, On equivalence of the Galerkin and Fourier series methods for one class of problems, JSV 109 (1986) 174–177.
[34] I. Elishakoff, M. Zingales, Convergence of Boobnov-Galerkin method exemplified, AIAA J 42 (2004) 1931–1933.