Neuchatel-Zurich seminar on Coding Theory and Cryptography

October 7, 2015

CODE-BASED CRYPTOSYSTEMS WITH SHORT KEYS

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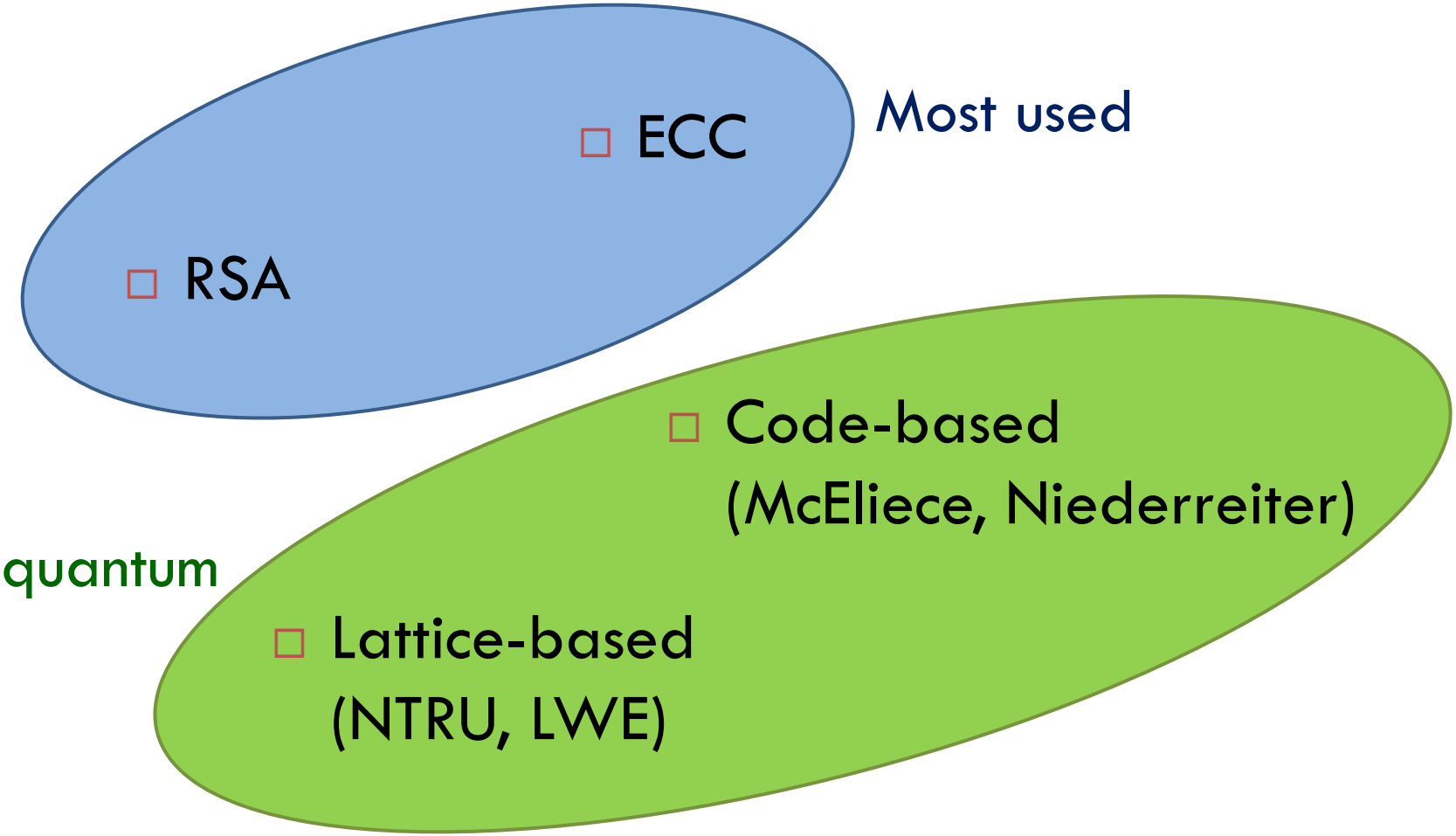
Asymmetric cryptography primitives



Do we need Post-quantum crypto?

- On a quantum computer, Shor's algorithm breaks RSA, ECC, and similar systems in polynomial time
- October 2011:
  - University of Southern California, Lockheed Martin and D-Wave Systems develop D-Wave One
- August 2012:
  - Harvard Researchers Use D-Wave quantum computer to fold proteins
- May 2013:
  - NASA and Google jointly order a 512-qubit D-Wave Two

Do we need Post-quantum crypto?

- According to Edward Snowden:
  - The National Security Agency has a $79.7 million research program (Penetrating Hard Targets) to build a "**cryptologically useful quantum computer**"

- RSA and ECC could become practically insecure in about 10-15 years or less

The (public) key size issue

- One of the main advantages of RSA is the short length of the public keys
  - RSA-2048 (recommended): 2x2048 bits = **512 bytes**

- In NTRU, the key is a polynomial with maximum degree $N - 1$ over a polynomial ring, hence a vector of size N over Z_q

- The key size is $N \cdot \lceil \log q \rceil$ bits
  - N=1171 and q=2048 (recommended): **1611 bytes**

Code-based cryptography

- Cryptographic primitives based on the decoding problem (put the adversary in the condition of decoding a random-like code)
- Everything started with the McEliece (1978) and Niederreiter (1986) public-key cryptosystems
- A large number of variants originated from them
- Some private-key cryptosystems were also derived
- The extension to digital signatures is still **challenging** (most concrete proposals: Courtois-Finiasz-Sendrier (CFS) and Kabatianskii-Krouk-Smeets (KKS) schemes)

McEliece cryptosystem

- Private key:

$$\{G, S, P\}$$

  - **G**: generator matrix of a *t*-error correcting (n, k) Goppa code
  - **S**: k x k non-singular dense matrix
  - **P**: n x n permutation matrix

- Public key:

$$G' = S \cdot G \cdot P$$

The private and public codes are permutation equivalent!

McEliece cryptosystem

- Encryption map:

$$\mathbf{x} = \mathbf{u} \cdot \mathbf{G'} + \mathbf{e}$$

- Decryption map:

$$\mathbf{x'} = \mathbf{x} \cdot \mathbf{P}^{-1} = \mathbf{u} \cdot \mathbf{S} \cdot \mathbf{G} + \mathbf{e} \cdot \mathbf{P}^{-1}$$

all errors are corrected, so we have:

$\mathbf{u'} = \mathbf{u} \cdot \mathbf{S}$ at the decoder output

$$\mathbf{u} = \mathbf{u'} \cdot \mathbf{S}^{-1}$$

McEliece cryptosystem

- Goppa codes are classically used as secret codes
- Any degree-*t* (irreducible) polynomial generates a different Goppa code (very large families of codes with the same parameters and correction capability)
- Their matrices are non-structured, thus their storage requires *kn* bits, which are reduced to *rk* bits with a CCA2 secure conversion
- The public key size grows quadratically with the code length

Niederreiter cryptosystem

- Exploits the same principle, but uses the code parity-check matrix (**H**) in the place of the generator matrix (**G**)
- Secret key: {**H**, **S**} → Public key: **H'** = **SH**
- Message mapped into a weight-*t* error vector (**e**)
- Encryption: $\mathbf{x} = \mathbf{H'e}^T$
- Decryption: $\mathbf{s} = \mathbf{S}^{-1}\mathbf{x} = \mathbf{He}^T$ → syndrome decoding (**e**)
- In this case there is no permutation (identity), since passing from **G** to **H** suffices to hide the Goppa code (indeed the permutation could be avoided also in McEliece)

Public key size

- Goppa code-based Niederreiter system
  - $n = 1632$
  - $k = 1269$
  - $t = 33$
- 80-bit security
- Key size = **57581** bytes

Permutation equivalence

- Many attempts of using **other families of codes** (RS, GRS, convolutional, RM, QC, QD, LDPC) have been made, aimed at reducing the public key size

- In most cases, they failed due to **permutation equivalence** between the private and the public code

- Permutation equivalence was exploited to recover the secret key from the public key

Permutation equivalence (2)

- Can we remove permutation equivalence?
- We need to replace **P** with a more general matrix **Q**
- This way, $\mathbf{G'} = \mathbf{S} \cdot \mathbf{G} \cdot \mathbf{Q}$ and the two codes are no longer permutation equivalent
- Encryption is unaffected
- Decryption: $\mathbf{x'} = \mathbf{x} \cdot \mathbf{Q}^{-1} = \mathbf{u} \cdot \mathbf{S} \cdot \mathbf{G} + \mathbf{e} \cdot \mathbf{Q}^{-1}$

Permutation equivalence (3)

- How can we guarantee that $\mathbf{e'} = \mathbf{e} \cdot \mathbf{Q}^{-1}$ is still correctable by the private code?
- We shall guarantee that **e'** has a low weight
- This is generally impossible with a randomly designed matrix **Q**
- But it becomes possible with a careful design of **Q** (and $\mathbf{Q}^{-1}$)

Design of **Q**: first approach

- Design $\mathbf{Q}^{-1}$ as an $n \times n$ sparse matrix, with average row and column weight equal to m:

$$1 < m \ll n$$

- This way, $w(\mathbf{e'}) \leq m \cdot w(\mathbf{e})$ and $w(\mathbf{e'}) \approx m \cdot w(\mathbf{e})$ due to the matrix sparse nature

- $w(\mathbf{e'})$ is always $\leq m \cdot w(\mathbf{e})$ with regular matrices (m integer)

- The same can be achieved with irregular matrices (m fractional), with some trick in the design of **Q**

Design of **Q**: second approach

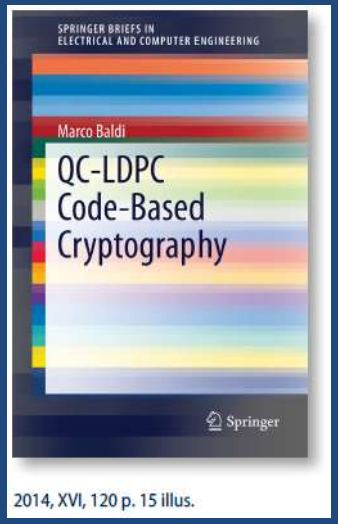
- Design $\mathbf{Q}^{-1}$ as an $n \times n$ sparse matrix $\mathbf{T}$, with average row and column weight equal to m, summed to a low rank matrix $\mathbf{R}$, such that:

$$\mathbf{e} \cdot \mathbf{Q}^{-1} = \mathbf{e} \cdot \mathbf{T} + \mathbf{e} \cdot \mathbf{R}$$

- Then:
  - Use only intentional error vectors $\mathbf{e}$ such that $\mathbf{e} \cdot \mathbf{R} = \mathbf{0}$ …or…
  - Make Bob informed of the value of $\mathbf{e} \cdot \mathbf{R}$

LDPC-CODE BASED CRYPTOSYSTEMS

(example of use of the first approach)



SpringerBriefs in Electrical and Computer Engineering
(preprint available on ResearchGate)

LDPC codes

- Low-Density Parity-Check (LDPC) codes are capacity-achieving codes under Belief Propagation (BP) decoding
- They allow a random-based design, which results in large families of codes with similar characteristics
- The low density of their matrices could be used to reduce the key size, but this exposes the system to key recovery attacks
- Hence, the public code cannot be an LDPC code, and permutation equivalence to the private code must be avoided

[1] C. Monico, J. Rosenthal, and A. Shokrollahi, "Using low density parity check codes in the McEliece cryptosystem," Proc. IEEE ISIT 2000, Sorrento, Italy, Jun. 2000, p. 215.

[2] M. Baldi, F. Chiaraluce, "Cryptanalysis of a new instance of McEliece cryptosystem based on QC-LDPC codes," Proc. IEEE ISIT 2007, Nice, France (June 2007) 2591–2595

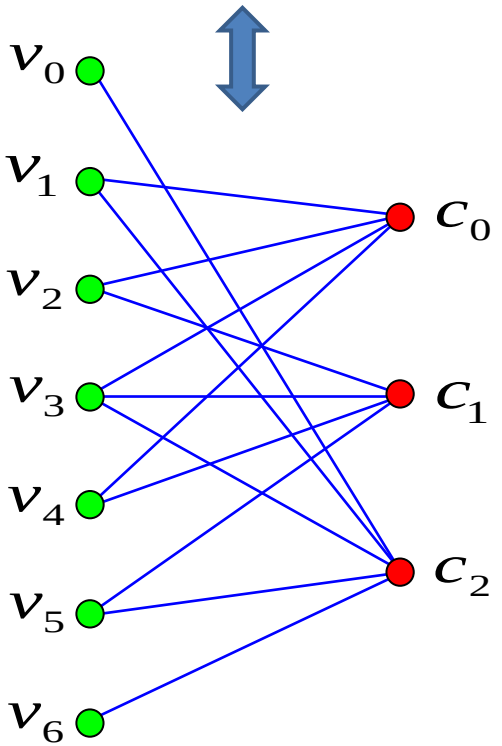
[3] A. Otmani, J.P. Tillich, L. Dallot, "Cryptanalysis of two McEliece cryptosystems based on quasi-cyclic codes," Proc. SCC 2008, Beijing, China (April 2008)

LDPC codes (2)

- LDPC codes are linear block codes
  - n: code length
  - k: code dimension
  - $r = n - k$: code redundancy
  - **G**: $k \times n$ generator matrix
  - **H**: $r \times n$ parity-check matrix
  - d_v: average **H** column weight
  - d_c: average **H** row weight

- LDPC codes have parity-check matrices with:
  - Low density of ones ($d_v \ll r$, $d_c \ll n$)
  - No more than one overlapping symbol 1 between any two rows/columns
  - No short cycles in the associated **Tanner graph**

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$



Bit flipping decoding

- Hard-decision decoding of LDPC codes is known as bit-flipping (BF) decoding
- During an iteration, every check node sends each neighboring variable node the binary **sum of all its neighboring variable nodes**, excluding that node
- In order to send a message back to each neighboring check node, a variable node **counts the number of unsatisfied parity-check sums** from the other check nodes
- If this number overcomes some **threshold**, the variable node **flips** its value and sends it back, otherwise, it sends its initial value unchanged
- BF is well suited when soft information from the channel is not available (as in the McEliece cryptosystem)

Decoding threshold

- Differently from algebraic codes, the **decoding radius** of LDPC codes is not easy to estimate
- Their error correction capability is statistical (with a high mean)
- For iterative decoders, the **decoding threshold** of large ensembles of codes can be estimated through density evolution techniques
- The decoding threshold of BF decoders can be found by iterating simple closed-form expressions

n [bits]		12288	15360	18432	21504	24576	27648	30720	33792	36864	39936	43008	46080	49152
$R = 2/3$	$d_v = 13$	190	237	285	333	380	428	476	523	571	619	666	714	762
	$d_v = 15$	192	240	288	336	384	432	479	527	575	622	670	718	766
n [bits]		16384	20480	24576	28672	32768	36864	40960	45056	49152	53248	57344	61440	65536
$R = 3/4$	$d_v = 13$	181	225	270	315	360	405	450	495	540	585	630	675	720
	$d_v = 15$	187	233	280	327	374	421	468	515	561	608	655	702	749

Quasi-Cyclic codes

- A linear block code is a **Quasi-Cyclic** (QC) code if:
  1. Its dimension and length are both multiple of an integer p ($k = k_0 p$ and $n = n_0 p$)
  2. Every cyclic shift of a codeword by n_0 positions yields another codeword

- The generator and parity-check matrices of a QC code can assume two alternative forms:
  - Circulant of blocks
  - Block of circulants

Rate $(n_0 - 1)/n_0$ random QC-LDPC codes

- A **Random Difference Family** (RDF) is a list of subsets of a finite group G such that every non-zero element of G appears no more than once as a difference of two elements in a subset

- An RDF can be used to obtain a QC-LDPC matrix free of length-4 cycles in the form:

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_0^c & \mathbf{H}_1^c & \cdots & \mathbf{H}_{n_0-1}^c \end{bmatrix}$$

- The codes in a family share the characteristics that mostly influence LDPC decoding, thus they have equivalent error correction performance

- Each of them is represented by single a row of **H** (**short keys**)

An example

- RDF over Z_{13}:
  - {1, 3, 8} (differences: 2, 11, 7, 6, 5, 8)
  - {5, 6, 9} (differences: 1, 12, 4, 9, 3, 10)
- Parity-check matrix ($n_0 = 2$, $p = 13$):

$$\mathbf{H} = \left[\begin{array}{cccccccccccc:cccccccccccc}
0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0\\
0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0\\
0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0\\
0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&1\\
0&0&0&0&0&1&0&1&0&0&0&0&1&1&0&0&0&0&0&0&0&0&1&1&0&0\\
1&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0\\
0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1\\
0&0&1&0&0&0&0&0&1&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&0&0&1&0&1&0&1&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&1&0&1&0&1&1&0&0&1&0&0&0&0&0&0&0\\
1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&1&1&0&0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&1&1&0&0&1&0&0&0&0&0\\
1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0
\end{array}\right]$$

Attacks

- In addition to classical attacks against McEliece, some specific attacks exist against QC-LDPC codes
- **Dual-code attacks**: search for low weight codewords in the dual of the public code in order to recover the secret (and sparse) H
- **QC code weakness**: exploit the QC nature to facilitate information set decoding (decode *one out of many*) and low weight codeword searches
- Their work factor depends on the complexity of information set decoding (**ISD**)

Dual code attacks

- Avoiding permutation equivalence is fundamental to counter these attacks
- We use $\mathbf{Q}^{-1}$ with row and column weight $m \ll n$
- $\mathbf{Q}$ and $\mathbf{Q}^{-1}$ are formed by n_0 x n_0 circulant blocks with size p to preserve the QC nature in the public code
- The public code has parity-check matrix $\mathbf{H'} = \mathbf{H}(\mathbf{Q}^{-1})^T$
- The row weight of $\mathbf{H'}$ is about m times that of $\mathbf{H}$

Security level and Key Size

- Minimum attack WF for $m = 7$:

p [bits]		4096	5120	6144	7168	8192	9216	10240	11264	12288	13312	14336	15360	16384
$n_0 = 3$	$d_v = 13$	2^{54}	2^{63}	2^{73}	2^{84}	2^{94}	2^{105}	2^{116}	2^{125}	2^{135}	2^{146}	2^{157}	2^{161}	2^{161}
	$d_v = 15$	2^{54}	2^{64}	2^{75}	2^{85}	2^{94}	2^{105}	2^{116}	2^{126}	2^{137}	2^{146}	2^{157}	2^{168}	2^{179}
$n_0 = 4$	$d_v = 13$	2^{60}	2^{73}	2^{85}	2^{98}	2^{109}	2^{121}	2^{134}	2^{146}	2^{153}	2^{154}	2^{154}	2^{154}	2^{154}
	$d_v = 15$	2^{62}	2^{75}	2^{88}	2^{100}	2^{113}	2^{127}	2^{138}	2^{152}	2^{165}	2^{176}	2^{176}	2^{176}	2^{176}

- Key size (bytes):

p [bits]	4096	5120	6144	7168	8192	9216	10240	11264	12288	13312	14336	15360	16384
$n_0 = 3$	1024	1280	1536	1792	2048	2304	2560	2816	3072	3328	3584	3840	4096
$n_0 = 4$	1536	1920	2304	2688	3072	3456	3840	4224	4608	4992	5376	5760	6144

[4] M. Baldi, M. Bianchi, F. Chiaraluce, ""Security and complexity of the McEliece cryptosystem based on QC-LDPC codes", IET Information Security, Vol. 7, No. 3, pp. 212-220, Sep. 2013.

Comparison with Goppa codes

- Comparison considering the Niederreiter version with 80-bit security (CCA2 secure conversion)

Solution	n	k	t	Key size [bytes]	Enc. compl.	Dec. compl.
Goppa based	1632	1269	33	57581	48	7890
QC-LDPC based	24576	18432	38	2304	1206	1790 (BF)

1/25 !

- For the **QC-LDPC** code-based system, the key size **grows linearly** with the code length, due to the quasi-cyclic nature of the codes, while with Goppa codes it grows **quadratically**

MDPC code-based variants

- An alternative is to use Moderate-Density Parity-Check (**MDPC**) codes in the place of LDPC codes
- This means to incorporate the density of $\mathbf{Q}^{-1}$ into the private code, which is no longer an LDPC code
- Then the public code can still be permutation equivalent to the private code
- QC-MDPC code based variants can be designed too

[5] R. Misoczki, J.-P. Tillich, N. Sendrier, P. S. L. M. Barreto, "MDPC-McEliece: New McEliece Variants from Moderate Density Parity-Check Codes", Proc. IEEE ISIT 2013, Istanbul, Turkey, pp 2069–2073.

MDPC-LDPC comparison

	QC-MDPC	QC-LDPC
SECURITY AGAINST KNOWN ATTACKS		
KEY SIZE		
COMPLEXITY		
SECURITY REDUCTION[1]		
SECURITY DECREASE WITH EVEN-SIZED CIRCULANTS		

[1] to the random linear code decoding problem

Irregular codes

- Irregular LDPC codes achieve higher error correction capability than regular ones
- This can be exploited to increase the system efficiency by reducing the code length…
- …although the QC structure and the need to avoid enumeration impose some constraints

160-bit security

QC-LDPC code type	n_0	d_v'	t	d_v	n	Key size (bytes)
regular	4	97	79	13	54616	5121
irregular	4	97	79	13	46448	4355

-15%

[6] M. Baldi, M. Bianchi, N. Maturo, F. Chiaraluce, "Improving the efficiency of the LDPC code-based McEliece cryptosystem through irregular codes", Proc. IEEE ISCC 2013, Split, Croatia, July 2013.

GRS-CODE BASED CRYPTOSYSTEMS

(example of use of the second approach)

Replacing Goppa with GRS codes

- GRS codes are **maximum distance separable** codes, thus have optimum error correction capability
- This would allow to reduce the public key size
- GRS codes are widespread, and already implemented in many practical systems
- On the other hand, they are more structured than Goppa codes (and wild Goppa codes)

Weakness of GRS codes

- When the public code is permutation equivalent to the private code, the latter can be recovered

- This was first shown by the **Sidelnikov-Shestakov attack** against the GRS code-based Niederreiter cryptosystem

Avoiding permutation equivalence

- Public parity-check matrix (Niederreiter):

$$\mathbf{H}' = \mathbf{S}^{-1} \cdot \mathbf{H} \cdot \mathbf{Q}^{-1}$$

- $\mathbf{Q}^{-1} = \mathbf{R} + \mathbf{T}$
- $\mathbf{R}$: dense $n \times n$ matrix with rank $z \ll n$
- $\mathbf{T}$: sparse $n \times n$ matrix with average row and column weight $m \ll n$
- All matrices are over $GF(q)$

[8] M. Baldi, M. Bianchi, F. Chiaraluce, J. Rosenthal, D. Schipani, "Enhanced public key security for the McEliece cryptosystem", Journal of Cryptology, Aug. 2014 (Online First).

Avoiding permutation equivalence (2)

- Example of construction of **R**:
  - take two matrices **a** and **b** defined over GF(q), having size $z \times n$ and rank z
  - Compute $\mathbf{R} = \mathbf{b}^T \mathbf{a}$

- Encryption:
  - Alice maps the message into an error vector **e** with weight $[t/m]$
  - Alice computes the ciphertext as $\mathbf{x} = \mathbf{H}' \cdot \mathbf{e}^T$

Avoiding permutation equivalence (3)

- Decryption:
  - Bob computes $\mathbf{x'} = \mathbf{S} \cdot \mathbf{x} = \mathbf{H} \cdot \mathbf{Q}^{-1} \cdot \mathbf{e}^T = \mathbf{H} \cdot (\mathbf{b}^T\mathbf{a} + \mathbf{T}) \cdot \mathbf{e}^T = \mathbf{H} \cdot \mathbf{b}^T \cdot \boldsymbol{\gamma} + \mathbf{H} \cdot \mathbf{T} \cdot \mathbf{e}^T$, where $\boldsymbol{\gamma} = \mathbf{a} \cdot \mathbf{e}^T$
  - We suppose that Bob knows $\boldsymbol{\gamma}$, then he computes $\mathbf{x''} = \mathbf{x'} - \mathbf{H} \cdot \mathbf{b}^T \cdot \boldsymbol{\gamma} = \mathbf{H} \cdot \mathbf{T} \cdot \mathbf{e}^T$
  - $\mathbf{e'} = \mathbf{T} \cdot \mathbf{e}^T$ has weight $\leq t$, thus $\mathbf{x''}$ is a correctable syndrome
  - Bob recovers $\mathbf{e'}$ by syndrome decoding through the private code
  - He multiplies the result by $\mathbf{T}^{-1}$ and demaps $\mathbf{e}$ into the secret message

Main issue

- How can Bob be informed of the value of $\boldsymbol{\gamma} = \mathbf{a} \cdot \mathbf{e}^T$?

- Two possibilities:
  - Alice knows $\mathbf{a}$ (which is made public), computes $\boldsymbol{\gamma}$ and sends it along with the ciphertext (or select only error vectors such that $\boldsymbol{\gamma}$ is known (all-zero)).
  - Alice does not know $\mathbf{a}$ and Bob has to guess the value of $\boldsymbol{\gamma}$

- Both them have pros and cons

A History of proposals and attacks

- M. Baldi, M. Bianchi, F. Chiaraluce, J. Rosenthal, D. Schipani, "A variant of the McEliece cryptosystem with increased public key security", Proc. WCC 2011, Paris, France, 11-15 Apr. 2011.
- J.-P. Tillich and A. Otmani, "Subcode vulnerability", private communication, 2011.
- M. Baldi, M. Bianchi, F. Chiaraluce, J. Rosenthal, D. Schipani, "Enhanced public key security for the McEliece cryptosystem", arXiv:1108.2462v2
- A. Couvreur, P. Gaborit, V. Gauthier, A. Otmani, J.-P. Tillich, "Distinguisher-based attacks on public-key cryptosystems using Reed–Solomon codes", Designs, Codes and Cryptography, Vol. 73, No. 2, pp 641-666, Nov. 2014.
- M. Baldi, M. Bianchi, F. Chiaraluce, J. Rosenthal, D. Schipani, "Enhanced public key security for the McEliece cryptosystem", Journal of Cryptology, Aug. 2014 (Online First).
- A. Couvreur, A. Otmani, J.-P. Tillich, V. Gauthier, "A Polynomial-Time Attack on the BBCRS Scheme", Public-Key Cryptography - PKC 2015, Vol. 9020 of LNCS, pp 175-193, 2015.
- M. Baldi, F. Chiaraluce, J. Rosenthal, D. Schipani, "An improved variant of McEliece cryptosystem based on Generalized Reed-Solomon codes", Poster at MEGA 2015.
- …

Subcode vulnerability

- When **a** is public, an attacker can look at $\mathbf{H}_S = \begin{bmatrix} \mathbf{H}' \\ \mathbf{a} \end{bmatrix}$
- For any codeword **c** in this subcode: $\mathbf{S}^{-1}\,\mathbf{H}\,\mathbf{T}\,\mathbf{c}^T = \mathbf{0}$
- Hence, the effect of the dense matrix **R** is removed
- When **T** is a permutation matrix, the subcode defined by $\mathbf{H}_S$ is permutation-equivalent to a subcode of the secret code
- The dimension of the subcode is $n - \mathrm{rank}\{\mathbf{H}_S\}$

Distinguishing attacks

- When $\mathbf{a}$ is private, Bob has to guess the value of $\boldsymbol{\gamma}$
- The number of attempts he needs increases as q^z
- Therefore only very small values of z ($z = 1$) are feasible
- When $z = 1$ and m is small, the system can be attacked by exploiting distinguishers
- These attacks, recently improved, force us to use very large values of m ($m \approx 2$) when $z = 1$

Avoiding attacks

- Publish $\mathbf{a}$ such that z can be increased, but avoid subcode attacks
- This could be achieved by reducing the dimension of the subcode to zero, which occurs for $z \geq k$
- Let us consider $z = k$ (can be extended to $z \geq k$): in this case $\mathbf{H}_S$ is a square invertible matrix
- The attacker could consider the system $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \mathbf{H}_S \cdot \mathbf{e}^T$ and solve for $\mathbf{e}$

Avoiding attacks (2)

- This further attacks is avoided if:
  - we introduce another secret matrix $\mathbf{X}$ and change the definition of $\mathbf{R}$ into $\mathbf{R} = \mathbf{b}^T \mathbf{X} \mathbf{a}$
  - we design $\mathbf{b}$ such that it has rank $z' < z$ and make a basis of the kernel of $\mathbf{b}^T$ public (through a $z' \times z$ matrix $\mathbf{B}$)
  - rather than sending $\mathbf{y}$ along with the ciphertext, Alice computes and sends $\mathbf{y}' = \mathbf{y} + \mathbf{v}$, where $\mathbf{v}$ is a $z \times 1$ vector in the kernel of $\mathbf{b}^T$ (that is, $\mathbf{b}^T \mathbf{v} = \mathbf{0}$)
  - $\mathbf{v}$ is obtained as a non-trivial random linear combination of the basis vectors

- This way, when Bob computes $\mathbf{b}^T \mathbf{y}'$ he still obtains $\mathbf{b}^T \mathbf{y}$, but the attack is avoided since $\mathbf{y}$ is hidden

Avoiding attacks (3)

- An attacker could exploit the matrix $\mathbf{H'}_S = \begin{bmatrix} \mathbf{H'} \\ \mathbf{B}^{\perp}\mathbf{a} \end{bmatrix}$
- Without $\mathbf{X}$, $\mathbf{H'}_S$ has the same kernel as $\mathbf{H}_S$, and it can be successfully exploited by the attacker
- With a random (non-singular) $\mathbf{X}$, this is no longer possible (already verified, paper in preparation)

Security level and Key Size

- Goppa code-based (PK: **H'** over GF(2))

(a)	$n = 4096$									
k	3004	2884	2764	2644	2524	2404	2284	2164	2044	1924
t	91	101	111	121	131	141	151	161	171	181
WF	180	184	187	189	189	189	187	184	180	176
KS	400.4	426.7	449.4	468.6	484.3	496.5	505.2	510.4	512.0	510.1

$\log_2$ KiB

- GRS code-based (PK: {**H'**, **a**, **B**} over GF(512))

(b)	$n = 520$									
k	348	340	332	324	316	308	300	292	284	276
t	86	90	94	98	102	106	110	114	118	122
WF	180	181	182	183	183	183	183	183	182	181
KS	367.9	361.1	354.2	347.3	340.2	333.1	325.9	318.7	311.3	303.9

$\log_2$ KiB

-8%

Security level and Key Size

- Goppa code-based (PK: **H'** over GF(2))

(a)	$n = 8192$									
k	6957	6892	6827	6762	6697	6632	6567	6502	6437	6372
t	95	100	105	110	115	120	125	130	135	140
WF	261	267	273	279	285	290	295	299	303	307
KS	1048.8	1093.7	1137.6	1180.4	1222.2	1262.9	1302.7	1341.4	1379.0	1415.7

$\log_2$ KiB

- GRS code-based (PK: {**H'**, **a**, **B**} over GF(512))

(b)	$n = 796$									
k	564	556	548	540	532	524	516	508	500	492
t	116	120	124	128	132	136	140	144	148	152
WF	260	262	264	266	267	269	270	271	271	272
KS	901.9	891.8	881.7	871.4	861.1	850.7	840.3	829.7	819.1	808.4

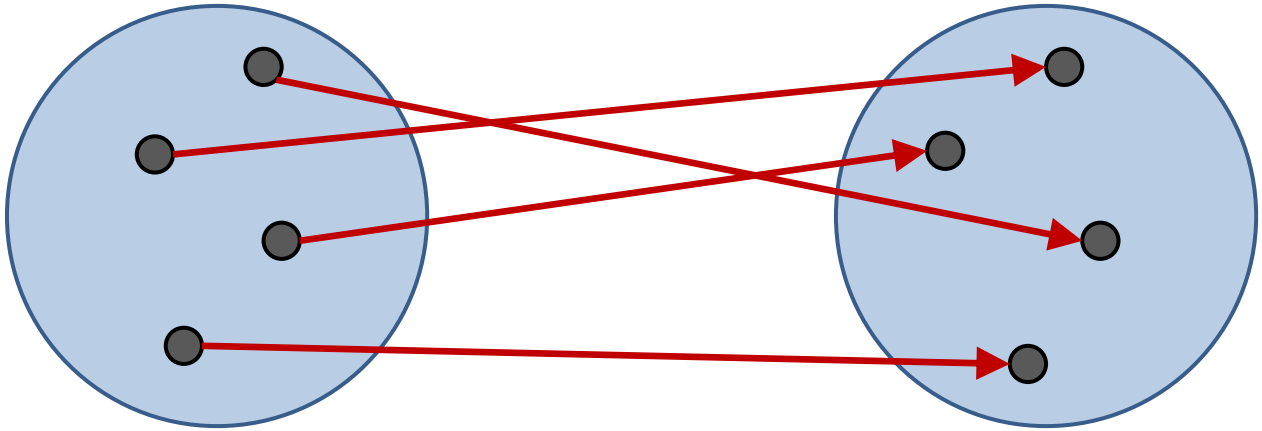
-14~18%

$\log_2$ KiB

DIGITAL SIGNATURE SCHEMES BASED ON SPARSE SYNDROMES

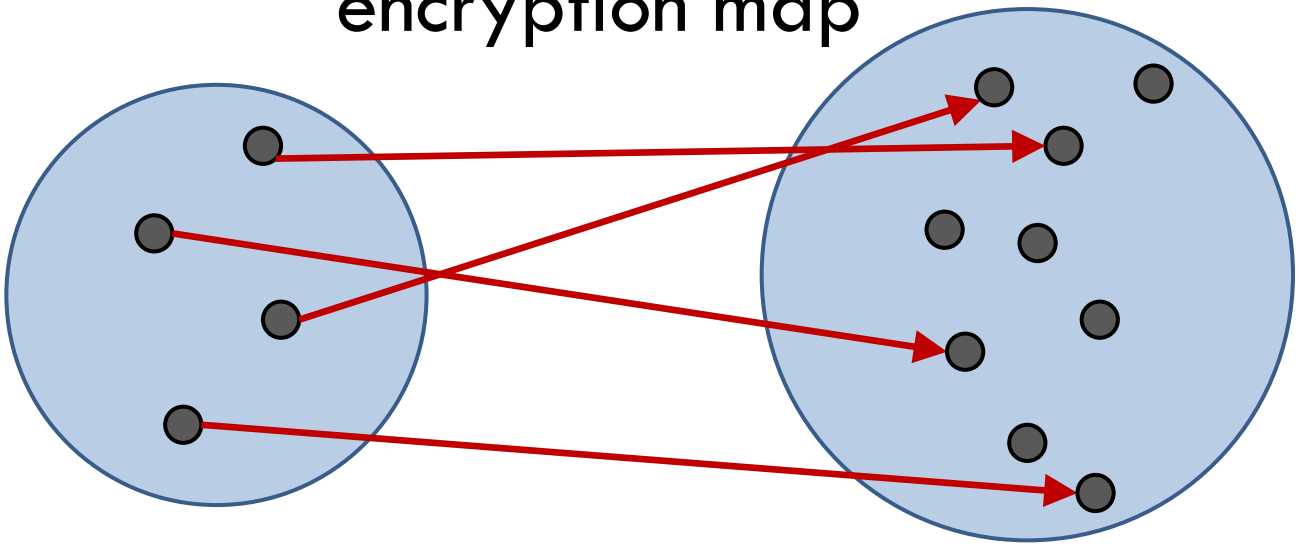
(another example of use of the second approach)

From PKC to Digital Signatures



RSA

encryption map



McEliece

Courtois-Finiasz-Sendrier (CFS)

- Close to the original McEliece Cryptosystem
- Based on Goppa codes

- Public:
  - A hash function $\mathcal{H}(\cdot)$
  - A function $\mathcal{F}(h)$ able to transform any hash digest h into a vector $\mathbf{s}$ such that $\mathbf{s'} = \mathbf{S}^{-1}\,\mathbf{s}$ is correctable syndrome through the code C

- Key generation:
  - The signer chooses a Goppa code able to correct t errors, having parity-check matrix $\mathbf{H}$
  - He chooses a scrambling matrix $\mathbf{S}$ and publishes $\mathbf{H'} = \mathbf{SH}$

CFS (2)

- Signing the document *D*:
  - The signer computes $\mathbf{s} = \mathcal{F}(\mathcal{H}(D))$ and $\mathbf{s'} = \mathbf{S}^{-1}\,\mathbf{s}$
  - He decodes the syndrome **s'** through the secret code
  - The error vector **e** is the signature

- Verification:
  - The verifier computes $\mathbf{s} = \mathcal{F}(\mathcal{H}(D))$
  - He checks that $\mathbf{H'}\,\mathbf{e}^T = \mathbf{S}\,\mathbf{H}\,\mathbf{e}^T = \mathbf{S}\,\mathbf{S}^{-1}\,\mathbf{s} = \mathbf{s}$

CFS (3)

- The main issue is to find an efficient function $\mathcal{F}(h)$
- In the original CFS there are two solutions:
  - Appending a counter to $h = \mathcal{H}(D)$ until a valid signature is generated
  - Performing complete decoding
- Both these methods require codes with very **special parameters**:
  - very high rate
  - very small error correction capability

Weaknesses

- Codes with small t and high rate could be decoded, with good probability, through the Generalized Birthday Paradox Algorithm (GBA)
- High rate Goppa codes have been discovered to produce public codes which are distinguishable from random codes
- The public key size and decoding complexity are very large

A CFS variant

- Main differences:
  - Only a subset of **sparse** syndromes is considered
  - Goppa codes are replaced with low-density generator-matrix (**LDGM**) codes

- Main advantages:
  - Significant reductions in the **public key size** are achieved
  - Classical attacks against the CFS scheme are inapplicable
  - Decoding is replaced by a straightforward vector operation

[9] M. Baldi, M. Bianchi, F. Chiaraluce, J. Rosenthal, D. Schipani, "Using LDGM Codes and Sparse Syndromes to Achieve Digital Signatures", Proc. PQCrypto 2013, Limoges, France, June 2013.

Rationale

- If we use a secret code in systematic form and sparse syndromes, we can obtain **sparse signatures**

- An attacker instead can only forge dense signatures

- Example:
  - secret code: $\mathbf{H} = [\mathbf{X} | \mathbf{I}]$, with $\mathbf{I}$ an $r \times r$ identity matrix
  - $\mathbf{s}$ is an $r \times 1$ sparse syndrome vector
  - the error vector $\mathbf{e} = [\mathbf{0} | \mathbf{s}^T]$ is sparse and verifies $\mathbf{H}\,\mathbf{e}^T = \mathbf{s}$

Key generation

- Private key: {**Q**, **H**, **S**}, with
  - **H**: $r \times n$ parity-check matrix of the secret code $C(n, k)$
  - $\mathbf{Q} = \mathbf{R} + \mathbf{T}$
  - $\mathbf{R} = \mathbf{a}^T \mathbf{b}$, having rank $z \ll n$
  - **T**: sparse random matrix with row and column weight m_T such that **Q** is full rank
  - **S**: sparse non-singular $n \times n$ matrix with average row and column weight $m_S \ll n$

- Public key: $\mathbf{H}' = \mathbf{Q}^{-1} \mathbf{H} \mathbf{S}^{-1}$

Signature generation

- Given the document M
- The signer computes $h = \mathcal{H}(M)$
- The signer finds $\mathbf{s} = \mathcal{F}(h)$, with weight w, such that $\mathbf{b}\,\mathbf{s} = \mathbf{0}$ (this requires 2^z attempts, on average)
- The signer computes the private syndrome $\mathbf{s}' = \mathbf{Q}\,\mathbf{s}$, with weight $\leq m_T w$
- The signer computes the private error vector $\mathbf{e} = [\mathbf{0} \,|\, \mathbf{s}'^T]$
- The signer selects a random codeword $\mathbf{c} \in C$ with small weight w_c
- The signer computes the public signature of M as

$$\mathbf{e}' = (\mathbf{e} + \mathbf{c})\,\mathbf{S}^T$$

Signature generation issues

- Without any random codeword **c**, the signing map becomes linear, and signatures can be easily forged
- With **c** having weight $w_c \ll n$, the map becomes affine, and summing two signatures does not result in a valid signature
- The signature should not change each time a document is signed, to avoid attacks exploiting many signatures of the same document
- It suffices to choose **c** as a deterministic function of M

Signature verification

- The verifier receives the message M, its signature $\mathbf{e}'$ and the parameters to use in $\mathcal{F}$
- He checks that the weight of $\mathbf{e}'$ is $\leq (m_T w + w_c) m_S$, otherwise the signature is discarded
- He computes $\mathbf{s}^* = \mathcal{F}(\mathcal{H}(M))$ and checks that it has weight w, otherwise the signature is discarded
- He computes $\mathbf{H}' \mathbf{e}'^T = \mathbf{Q}^{-1} \mathbf{H} \mathbf{S}^{-1} \mathbf{S} (\mathbf{e}^T + \mathbf{c}^T) = \mathbf{Q}^{-1} \mathbf{H} (\mathbf{e}^T + \mathbf{c}^T) = \mathbf{Q}^{-1} \mathbf{H} \mathbf{e}^T = \mathbf{Q}^{-1} \mathbf{s}' = \mathbf{s}$
- If $\mathbf{s} = \mathbf{s}^*$, the signature is accepted, otherwise it is discarded

LDGM codes

- LDGM codes are codes with a low density generator matrix **G**
- The row weight of **G** is $w_g \ll n$
- They are useful in this cryptosystem because:
  - Large random-based families of codes can be designed
  - Finding low weight codewords is very easy
  - Structured codes (e.g. QC) can be designed

Attacks

- The signature $\mathbf{e}'$ is an error vector corresponding to the public syndrome $\mathbf{s}$ through the public code parity-check matrix $\mathbf{H}'$
- If $\mathbf{e}'$ has a low weight it is difficult to find, otherwise signatures could be forged
- If $\mathbf{e}'$ has a too low weight the supports of $\mathbf{e}$ and $\mathbf{c}$ could be almost disjoint, and the link between the support of $\mathbf{s}$ and that of $\mathbf{e}'$ could be discovered
- Hence, the density of $\mathbf{e}'$ must be:
  - sufficiently low to avoid forgeries
  - sufficiently high to avoid support decompositions

Examples

SL (bits)	n	k	p	w	w_g	w_c	z	m_T	m_S	A_{w_c}	N_s	S_k (KiB)
80	9800	4900	50	18	20	160	2	1	9	$2^{82.76}$	$2^{166.10}$	117
120	24960	10000	80	23	25	325	2	1	14	$2^{140.19}$	$2^{242.51}$	570
160	46000	16000	100	29	31	465	2	1	20	$2^{169.23}$	$2^{326.49}$	1685

- For **80-bit security**, the original CFS system needs a Goppa code with $n = 2^{21}$ and $r = 2^{10}$, which gives a key size of **52.5 MiB**
- By using the parallel CFS, the same security level is obtained with key sizes between **1.25 MiB** and **20 MiB**
- The proposed system requires a public key of only **117 KiB** to achieve 80-bit security (by using QC-LDGM codes)

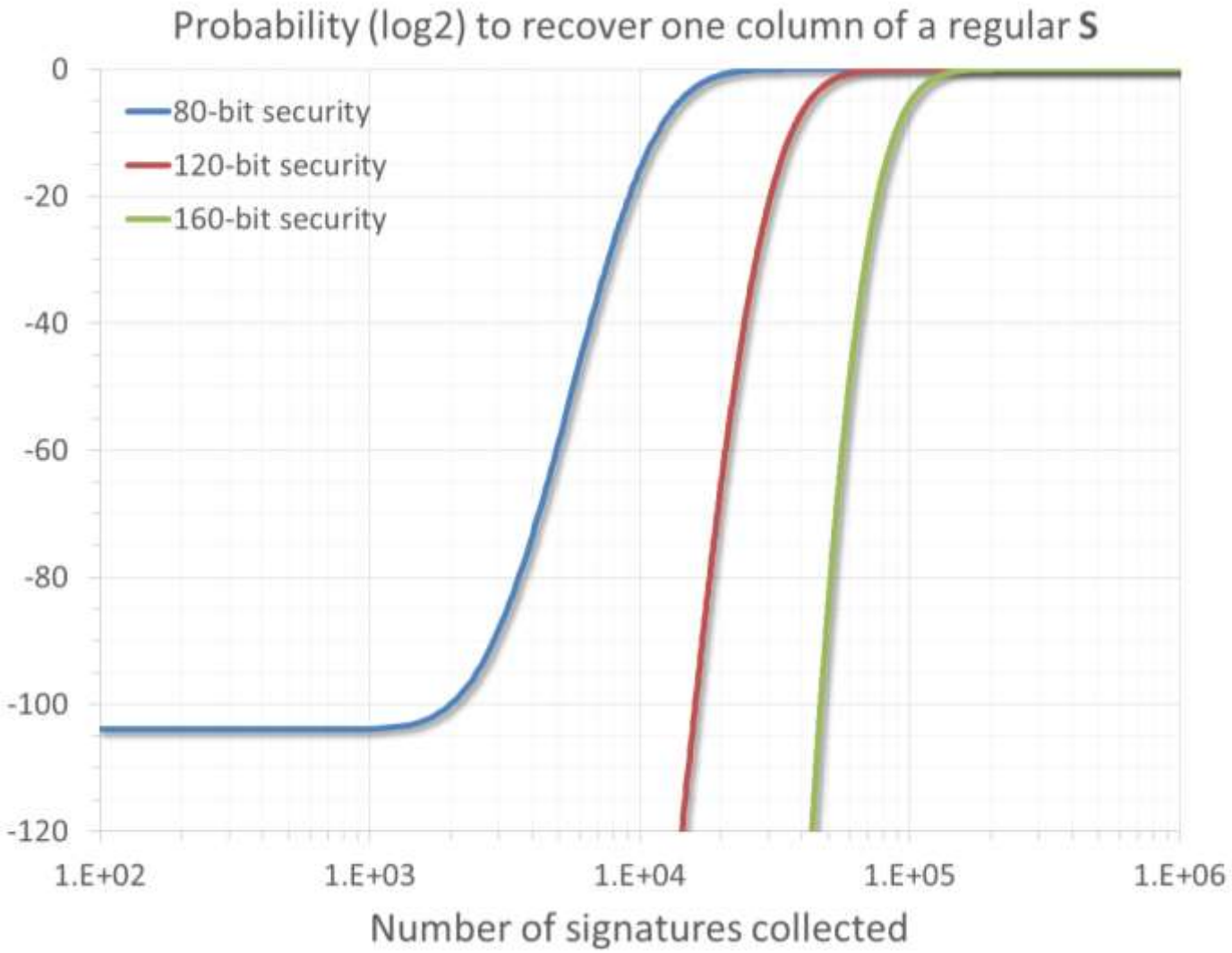
Attacks to regular **S**

- If the matrix **S** is (sparse and) regular, statistical arguments could be used to analyze large number of intercepted signatures (pointed out by *J. P. Tillich*, analysis in progress)

- This way, an attacker could discover which columns of **S** have a symbol 1 in the same row

- By iterating the procedure, the structure of the matrix **S** could be recovered (except for a permutation)

Attacks to regular **S**



Attacks to regular **S**

- An attacker has probability of success equal to the inverse of the security level (SL) when he collects at least L_{min} signatures

SL	L_{min}
80	3565
120	14296
160	37947

- This can be avoided by using an **irregular matrix S** with the same average weight (already verified, paper in preparation)