How to compute the geometric genus of a curve with `Magma`

Let $f \in \mathbb{C}[X]$ be a squarefree polynomial in one variable and write $W = \sqrt{f}$ for its square root. To compute the geometric genus of the associated curve $\overline{C}$ with `Magma`, open the free Online Calculator (http://magma.maths.usyd.edu.au/calc/) and enter the following:

```
> IP2<z,x,w> := ProjectivePlane(Rationals());
> f := <Define your polynomial here.  Use the variable x instead of X.>;
> d := Degree(f);
> g := z^(d-2)*w^2 - Numerator(z^d*Evaluate(f,x/z));
> Cbar := Curve(IP2,g);
> Genus(Cbar);
```

A short note one the base field: recall that, in characteristic 0, the geometric genus of a curve is invariant under base change, i.e., if a curve over $\mathbb{C}$ can be defined over $\mathbb{Q}$, its geometric genus as a curve over $\mathbb{C}$ is equal to its geometric genus as a curve over $\mathbb{Q}$. So whenever the associated curve of a given square root can be defined over $\mathbb{Q}$, we can safely compute its geometric genus over $\mathbb{Q}$ to obtain the geometric genus over $\mathbb{C}$. (In the sample code above, we specify $\mathbb{Q}$ as a base field in the definition of the projective plane by using the argument `Rationals()`.)

The `Examples.txt` file contains various different sample computations.

How to check a double cover for non-simple singularities with `NonADE`

To check the associated double cover of a given square root for non-simple singularities, first copy the content of the `NonADE.txt` file and execute it in the free `Magma` Online Calculator (http://magma.maths.usyd.edu.au/calc/).
The function can then be used via:

```
> NonADE(basering,polynomial);
```

Its first input is the ring of polynomials in three variables with coefficients in a field k, e.g., $k[x, y, z]$. Its second input is the homogeneous squarefree polynomial that defines the projective branch curve of the double cover whose singularities we want to study. The output lists all of the non-simple singularities that the double cover has over k. The function returns the empty list, if the double cover is smooth or has only simple singularities.
To see how `NonADE` is to be applied in practice, let us consider the following alphabet, which is discussed in Section 3.4 of the paper:

$$\mathcal{A} = \Big\{ \sqrt{X+1},\ \sqrt{X-1},\ \sqrt{Y+1},\\ \sqrt{X+Y+1},\ \sqrt{16X+(4+Y)^2} \Big\}.$$

We write $f_1, ..., f_5$ for the polynomial arguments of the square roots in $\mathcal{A}$. To show that $\mathcal{A}$ is not rationalizable, we consider the whole set of indices $J = \{1, 2, 3, 4, 5\}$, define

$$f(X, Y) := \prod_{j \in J} f_j = (X+1)(X-1)(Y+1)(X+Y+1)(16X + (4+Y)^2),$$

and denote the associated double cover of $\sqrt{f}$ by $\overline{S}$. In other words, $\overline{S}$ is the hypersurface in the weighted projective space $\mathbb{P}_{\mathbb{C}}(1, 1, 1, 3)$, which is defined by the equation

$$u^2 = (x+z)(x-z)(y+z)(x+y+z)(16xz + (4z+y)^2).$$

The branch curve of $\overline{S}$ is, therefore, given by the projective curve $\overline{B} \subset \mathbb{P}^2_C$, defined as the zeros of the polynomial

$$F(x, y, z) := (x+z)(x-z)(y+z)(x+y+z)(16xz + (4z+y)^2).$$

Since $\deg(f) = 6$, we can prove that $\mathcal{A}$ is not simultaneously rationalizable by showing that $\overline{S}$ has at most simple singularities. However, before we can use `NonADE`, there is one important subtlety that we have to address first: not all singular points of $\overline{B}$ have rational coordinates. This is problematic because `Magma` will not allow us to choose $k = \mathbb{C}$ for our base ring $k[x, y, z]$. It does, however, allow us to choose the field of rational numbers $k = \mathbb{Q}$. But if we would determine the singular points of $\overline{S}$ with this choice, `NonADE` would only classify the singular points of $\overline{S}$ that have purely rational coordinates and would, therefore, miss two of the singular points that $\overline{S}$ has over the complex numbers.

We can resolve this problem by first computing the singular points with a different computer algebra software that will not only be sensitive to singularities over $\mathbb{Q}$ but to all singular points of $\overline{S}$ over $\mathbb{C}$. Since all singularities of a double cover stem from its branch locus, we can simply check the singularities of $\overline{B}$. For example, we can compute the singular points of $\overline{B}$ in `Mathematica` via

```
> F:= (x+z)*(x-z)*(y+z)*(x+y+z)*(16*x*z+(4*z+y)^2)
> Solve[F==0 && D[F,x]==0 && D[F,y]==0 && D[F,z]==0]
```

giving us the output

```
{{x -> -z, y -> 0}, {x -> -z, y -> -8 z}, {x -> -z, y -> -z},
{x -> z, y -> (-4 - 4 i) z}, {x -> z, y -> (-4 + 4 i) z},
{x -> z, y -> -2 z}, {x -> z, y -> -z},{x -> -z, y -> 0},
{x -> -((9 z)/16), y -> -z}, {x -> -9 z, y -> 8 z},
{x -> 0, z -> 0}, {y -> 0, z -> 0}, {x -> 0, y -> -z},
{x -> 0, y -> 0, z -> 0}}
```

which tells us that $\overline{B}$ has twelve different singular points as a curve over $\mathbb{C}$. Notice that we ignore the trivial solution since $[0:0:0]$ is not an element of $\mathbb{P}^2_{\mathbb{C}}$. From the `Mathematica` output, we see that two of the singular points have irrational numbers in their coordinates, namely the complex unit i. Therefore, the coordinates of the two corresponding singularities of $\overline{S}$ will also involve the complex unit.

Now that we know the irrationalities that occur in the coordinates of the singular points of $\overline{S}$, we can perform the remaining analysis with `NonADE`: in order for `NonADE` to be able to consider all of the singular points of $\overline{S}$, we have to adjoin the imaginary unit to the coefficient field $\mathbb{Q}$ of our base ring $\mathbb{Q}[x, y, z]$. Put differently, we have to pass from $\mathbb{Q}$ to the extension field $\mathbb{Q}(\sqrt{-1})$. A convenient way to construct an extension field for $\mathbb{Q}$ is to consider a quotient ring of the polynomial ring $\mathbb{Q}[q]$ that corresponds to the irrational numbers we want to be contained in the extension field. In our example, $\mathbb{Q}$ does not contain the imaginary unit, i.e., it does not contain any element i with $i^2 + 1 = 0$. The sought-after extension field $\mathbb{Q}(\sqrt{-1})$ is, therefore, to be constructed as the quotient ring $\mathbb{Q}(\sqrt{-1}) = \mathbb{Q}[q]/(q^2 + 1)$. In `Magma`, $\mathbb{Q}(\sqrt{-1})$ is easily defined via

```
> QQ:=Rationals();
> E<i>:=ext<QQ|[Polynomial([1,0,1])]>;
```

where `Polynomial([1,0,1])` specifies the coefficients of the polynomial $g(q)$ in the quotient $\mathbb{Q}[q]/g$, in our case $g = 1 \cdot q^0 + 0 \cdot q^1 + 1 \cdot q^2$. If the coordinates of the singular points under consideration would contain more than one irrationality, for instance the imaginary unit i and, in addition, the irrational number $a := \sqrt{5}$, then the corresponding field extension $\mathbb{Q}(\sqrt{-1}, \sqrt{5})$ can be created via

```
> QQ:=Rationals();
> E<i,a>:=ext<QQ|[Polynomial([1,0,1]), Polynomial([-5,0,1])]>;
```

Now, we can easily prove the non-rationalizability of $\mathcal{A}$ using `NonADE`:

```
> k:=Rationals();
> E<i>:=ext<k|[Polynomial([1,0,1])]>;
> R<x,y,z>:=PolynomialRing(E,3);
> F:=(x+z)*(x-z)*(y+z)*(x+y+z)*(16*x*z+(4*z+y)^2);
> NonADE(R,F);
[]
```

Since `NonADE` returns the empty list, we can conclude that $\overline{S}$ has at most simple singularities. Hence, the alphabet $\mathcal{A}$ is not rationalizable.

The `Examples.txt` file contains even more sample computations.