How to use `NonADE`

To check the associated double cover of a given square root in two variables for non-simple singularities, first copy the content of the `NonADE.txt` file and paste it in the free `Magma` Online Calculator (`http://magma.maths.usyd.edu.au/calc/`).
The function can be used via:

```
> NonADE(basering,polynomial);
```

Its first input is the ring of polynomials in three variables with coefficients in a field k, e.g., $k[x, y, z]$. Its second input is the homogeneous squarefree polynomial that defines the projective branch curve of the double cover. The output lists all non-simple singularities that the double cover has over k. The function returns the empty list if the double cover is smooth or has only simple singularities.
To see how `NonADE` is to be applied in practice, let us consider the following alphabet, which is discussed in Section 3.4 of the paper:

$$\begin{aligned} \mathcal{A} = \Big\{ & \sqrt{X+1},\ \sqrt{X-1},\ \sqrt{Y+1}, \\ & \sqrt{X+Y+1},\ \sqrt{16X+(4+Y)^2} \Big\}. \end{aligned}$$

We write $f_1, \ldots, f_5$ for the square root arguments. To show that $\mathcal{A}$ is not rationalizable, we consider $J = \{1, 2, 3, 4, 5\}$, define

$$f(X, Y) := \prod_{j \in J} f_j = (X+1)(X-1)(Y+1)(X+Y+1)(16X+(4+Y)^2),$$

and denote the associated double cover of $\sqrt{f}$ by $\overline{S}$. In other words, $\overline{S}$ is the hypersurface in the weighted projective space $\mathbb{P}_{\mathbb{C}}(1, 1, 1, 3)$, which is defined by the equation

$$u^2 = (x+z)(x-z)(y+z)(x+y+z)(16xz+(4z+y)^2).$$

The branch curve of $\overline{S}$ is, therefore, given by the projective curve $\overline{B} \subset \mathbb{P}^2_{\mathbb{C}}$, defined as the zeros of the polynomial

$$F(x, y, z) := (x+z)(x-z)(y+z)(x+y+z)(16xz+(4z+y)^2).$$

Since $\deg(f) = \deg(F) = 6$, we can prove that $\mathcal{A}$ is not simultaneously rationalizable by showing that $\overline{S}$ has at most simple singularities.
Before we can use `NonADE`, though, there is one important subtlety that we have to address first: not all singular points of $\overline{S}$ have rational coordinates. This is problematic because, while `Magma` does allow us to choose $k = \mathbb{Q}$ as our base field, it will not allow us to choose $k = \mathbb{C}$. That means, if we would determine the singular points of $\overline{S}$ over $k = \mathbb{Q}$, `NonADE` would only classify the singular points of $\overline{S}$ that have purely rational coordinates and would, therefore, miss two of the singular points that $\overline{S}$ has over the complex numbers.

We can resolve this problem by first computing the singular points with a different computer algebra software that will not only be sensitive to singularities over $\mathbb{Q}$ but to all singular points of $\overline{S}$ over $\mathbb{C}$. Since all singularities of a double cover stem from its branch locus, it suffices to look at the singularities of $\overline{B}$. For example, we can compute the singular points of $\overline{B}$ in `Mathematica` via

```
> F:= (x+z)*(x-z)*(y+z)*(x+y+z)*(16*x*z+(4*z+y)^2)
> Solve[F==0 && D[F,x]==0 && D[F,y]==0 && D[F,z]==0]
```

giving us the output

```
{{x -> -z, y -> 0}, {x -> -z, y -> -8 z}, {x -> -z, y -> -z},
{x -> z, y -> (-4 - 4 i) z}, {x -> z, y -> (-4 + 4 i) z},
{x -> z, y -> -2 z}, {x -> z, y -> -z},
{x -> -((9 z)/16), y -> -z}, {x -> -9 z, y -> 8 z},
{x -> 0, z -> 0}, {y -> 0, z -> 0}, {x -> 0, y -> -z},
{x -> 0, y -> 0, z -> 0}}
```

which tells us that $\overline{B}$ has twelve different singular points as a curve over $\mathbb{C}$. Notice that we ignore the trivial solution since $[0:0:0]$ is not an element of $\mathbb{P}^2_{\mathbb{C}}$. From the `Mathematica` output, we see that two of the singular points have an irrational number in their coordinates: the complex unit i. Therefore, the coordinates of the two corresponding singularities of $\overline{S}$ will also involve the complex unit.

Knowing which irrationalities occur, we can perform the remaining analysis in `Magma`: to ensure that `NonADE` takes all singular points of $\overline{S}$ into account, we have to adjoin i to the coefficient field $\mathbb{Q}$ of our base ring. Put differently, we have to pass from $\mathbb{Q}$ to the extension field $\mathbb{Q}(\sqrt{-1})$.

A convenient way to construct an extension field for $\mathbb{Q}$ is to consider a quotient ring of the polynomial ring $\mathbb{Q}[q]$ that corresponds to the irrational numbers we want to be contained in the extension field. In our example, we want to extend $\mathbb{Q}$ by i, i.e., we want to adjoin an element i that satisfies $i^2+1=0$. The field $\mathbb{Q}(\sqrt{-1})$ is, therefore, isomorphic to the quotient $\mathbb{Q}[q]/(q^2+1)$. In `Magma`, we can define $\mathbb{Q}(\sqrt{-1})$ via

```
> QQ:=Rationals();
> E<i>:=ext<QQ|[Polynomial([1,0,1])]>;
```

where `Polynomial([1,0,1])` specifies the coefficients of the polynomial $g(q)$ in the quotient $\mathbb{Q}[q]/g$—in our case $g = 1\cdot q^0 + 0\cdot q^1 + 1\cdot q^2$.

If the singular points would contain more than one irrationality, e.g., the imaginary unit i and, in addition, the irrational number $a := \sqrt{5}$, then the corresponding extension field $\mathbb{Q}(\sqrt{-1},\sqrt{5})$ can be created via

```
> QQ:=Rationals();
> E<i,a>:=ext<QQ|[Polynomial([1,0,1]), Polynomial([-5,0,1])]>;
```

Now, we can easily prove the non-rationalizability of $\mathcal{A}$ using `NonADE`:

```
> <paste content of NonADE.txt here>
> k:=Rationals();
> E<i>:=ext<k|[Polynomial([1,0,1])]>;
> R<x,y,z>:=PolynomialRing(E,3);
> F:=(x+z)*(x-z)*(y+z)*(x+y+z)*(16*x*z+(4*z+y)^2);
> NonADE(R,F);
[]
```

Since `NonADE` returns the empty list, we conclude that $\overline{S}$ has at most simple singularities. As a result, $\mathcal{A}$ is not rationalizable.

Note: The `Magma` code of this example can be found in the `Example.txt` file.