



Putting the Cycle Back into Business Cycle Analysis

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Abstract

Are business cycles mainly a response to persistent exogenous shocks, or do they instead reflect a strong endogenous mechanism which produces recurrent boom-bust phenomena? In this paper we present new evidence in favor of the second interpretation and, most importantly, we highlight the set of key elements that influence our answer to this question. In particular, when adopting our most preferred estimation framework, we find support for the somewhat extreme notion that business cycles may be generated by stochastic limit cycle forces; that is, we find support for the notion that business cycles may primarily reflect an endogenous propagation mechanism buffeted only by temporary shocks. The three elements that tend to favor this type of interpretation of business cycles are: (i) slightly extending the frequency window one associates with business cycle phenomena, (ii) allowing for strategic complementarities across agents that arise due to financial frictions, and (iii) allowing for a locally unstable steady state in estimation. We document the sensitivity of our findings to each of these elements within the context of an extended New Keynesian model with real-financial linkages.

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Introduction

Market economies repeatedly go through periods where, for sustained periods of time, productive factors are used very intensively—with low rates of unemployment, high levels of hours worked per capita, and intensive use of productive capital—followed by periods where these utilization rates are reversed. The types of forces and mechanisms that drive these fluctuations remain a highly debated subject. As an organizational framework, two conceptual views are worth distinguishing. On the one hand, there is the view that business cycles are primarily driven by persistent exogenous shocks. In models reflecting this view, such shocks are generally propagated through a variety of endogenous mechanisms, including those that may ultimately prolong their effects such as adjustment costs, capital accumulation, and habit persistence. However, it generally remains the case that, if the persistence of the exogenous shocks in such models were substantially reduced, business cycle-type fluctuations would largely disappear. As a result, such models can be viewed as supporting the notion that persistent exogenous shocks are central to understanding business cycles. On the other hand, there is an alternative view according to which the bulk of business cycle fluctuations is believed to be the result of forces that are internal to the economy and that endogenously favor recurrent periods of boom and bust. According to this alternative view, even if shocks to the economy are not persistent, large and prolonged business cycles nevertheless arise due to the equilibrium incentives present in a decentralized economy.¹

Even though from a theoretical point of view both of the above possibilities are reasonable, there are at least two important empirical reasons why mainstream macroeconomics has broadly coalesced around the first class of explanations. The first reason is based on the estimation of a vast array of structural models that allow for internal propagation mechanisms and exogenous driving forces to compete in explaining observed fluctuations. Overwhelmingly, the results of such estimations suggest that persistent exogenous driving forces are required to explain the data, with estimated internal propagation mechanisms being far too weak to explain business cycle fluctuations without them.² The second reason is based on

¹While we present these two frameworks as distinct, there is actually a continuum between the two. To get a sense of whether a business cycle model relies more on persistent shocks versus internal propagation mechanisms in explaining the data, one could compute the fraction of some data feature explained by the model—for example, the auto-correlation or variance of a variable—that is lost if one reduces the persistence of the shocks to zero. If this fraction is close to 100%, then the model can be said to rely primarily on persistent exogenous shocks, while if this fraction is close to zero, we can say that the model relies primarily on endogenous mechanisms. Frisch [1933] first introduced this distinction between the “*propagation* problem” and the “*impulse* problem”.

²Certain credit cycle models, such as the seminal piece by Kiyotaki and Moore [1997], have internal propagation mechanism that can potentially be very strong. However, when such models are estimated, the implied parameters generally do not generate quantitatively meaningful endogenous cyclical behavior. Instead the estimated versions of these models most often maintain a reliance on persistent exogenous shocks

reduced-form evidence on the spectral properties of many macroeconomic aggregates. Since Granger [1966] and, more recently, Sargent [1987], it has been argued that the autocovariance patterns in the data are generally not supportive of strong internal boom-bust mechanisms, which would typically imply pronounced peaks in the spectral densities of macro aggregates at business cycle frequencies. However, such peaks, it has been argued, do not appear to be present in the data.

The object of this paper is to provide new impetus to the second class of explanations noted above. In particular, we provide both reduced-form and structural evidence in support of this view, and further show why certain procedures that are commonly used in macroeconomic research may have biased previous research against it. To this end, we proceed in three steps. The first step is purely empirical. We examine anew the spectral density properties of many trend-less macroeconomic aggregates, such as work hours, rates of unemployment and capacity utilization, and risk premia. We highlight a recurrent peak in several spectral densities in US macroeconomic and financial data at periodicities around 9 to 10 years. We complement this visual inspection with a set of tests aimed at documenting the statistical significance of this local peak. While the presence of such a peak does not necessarily imply strong endogenous cyclical forces,³ it is an important first step in our argument, as it runs counter to the notion that the spectral properties of the data rule out such a possibility. In addition to providing motivation for the following sections, these spectral densities play the central role in our later structural estimation exercises.

In a second step we present a simple framework aimed at highlighting key economic elements that favor the endogenous emergence of peaks in the spectral density of equilibrium outcomes. In particular, we focus on understanding the types of market forces—i.e., the types of interactions between agents—that would give rise to cyclical and spectral peaks in situations where individual-level decisions by themselves do not favor them. The framework emphasizes that there are two main ingredients that, in combination, favor spectral peaks: allowing for strategic complementarities across agents, and having some accumulation (i.e., stock) variable that, when high, tends to depress the individual value of accumulating more. We then discuss the extent to which such forces are present in existing models and use these properties to motivate the type of model we bring to the data in the subsequent section. We also discuss in this section the connection between models that are capable of producing a spectral peak and models in which limit cycles may emerge.

In the third step we present a dynamic stochastic general equilibrium model that builds on a simple New Keynesian model in a manner that incorporates the two elements discussed to explain business cycles features. The current paper offers insights to why this may be the case.

³One could in principle obtain such patterns with strongly cyclical exogenous forces.

above in the second step. In particular, the model contains real-financial linkages that produce complementarities, and an accumulated capital stock that exhibits diminishing returns. In the model, unemployment risk, default risk, and risk premia on borrowing are jointly determined. The model is aimed at capturing the common narrative of an accumulation-credit cycle, wherein booms are periods in which banks perceive lending to be safe and risk premia are correspondingly low, which allows households to spend more on durable goods and housing, which in turn contributes to making lending safe by keeping unemployment low.⁴ The model allows for endogenous propagation forces that can potentially generate boom-bust cycles as the unique equilibrium outcome.⁵ The model also features a persistent exogenous driving force, and in estimating the model we allow this stochastic force to compete with the endogenous forces in order to evaluate their respective roles. We use this model to illustrate the different inferences one would make regarding the relative importance of exogenous versus endogenous forces in driving business cycle movements depending on how one approaches estimation. In particular, we show that if one targets the spectral densities documented in our first step, and if one adopts an estimation method that allows for a locally unstable steady state, then the estimation results suggest that business cycles are mainly driven by internal forces buffeted by temporary shocks. In particular, our point estimates in this case actually suggest the presence of stochastic limit cycles, where the stochastic component is essentially an *i.i.d.* process.⁶ In contrast, if we use more standard estimation techniques, if we focus less on explaining business cycle properties, or if we restrict the presence of complementarities, then we find more evidence in favor of persistent exogenous forces as the main driver of business cycles.

While we view our results as providing novel support for the idea that business cycles may be largely driven by endogenous cyclical forces, we do not claim that these results—which are based on the estimation of only one model—constitute a definitive answer to this question. Instead we view an important contribution of the paper as illustrating why the current consensus regarding the importance of persistent exogenous shocks may reflect some arbitrary choices. In particular, we highlight how different estimation targets and different estimation methods can greatly influence one's inferences regarding the relative

⁴It should be emphasized that there are many possible sources of complementarity that could be embedded in the general class of models we describe in our second step and that could potentially generate similar dynamics to our accumulation-credit cycle model. While we focus on credit market frictions, as we believe they likely play a role in business cycle fluctuations (see Section 1.4), we by no means wish to rule out the possibility of other relevant sources of complementarity.

⁵Note that the endogenous boom-bust cycles that arise in our model do not reflect multiple equilibria or indeterminacy.

⁶While the idea that business cycles may possibly reflect stochastic limit cycle forces is not new, our empirical finding of support for such a view within the confines of a stochastic general equilibrium model with forward-looking agents appears unprecedented.

contribution of internal versus external mechanisms in driving business cycles. For example, we show how, when adopting an approach to estimation that allows for stochastic limit cycles, one finds little need for persistent exogenous shocks. We hope that these results will motivate exploration into other business cycle models in order to assess how sensitive inferences regarding the relative importance of internal versus external propagation are to three factors: focusing on frequencies that are slightly lower than the traditional focus of business cycle analysis, allowing for the possibility of locally unstable steady states, and finally, not unduly restricting strategic complementarities across agents. Our results provide a clear—even extreme—example of these sensitivities as we show how one’s inference can change from a situation where the economy appears to be driven primarily by exogenous shocks to one driven primarily by endogenous mechanisms depending on how these features are treated.

It is important to note that the idea that macroeconomic fluctuations may predominantly reflect strong internal propagation mechanisms, and even the possibility of limit cycle forces, is not at all novel, having been advocated by many in the past, including early incarnations due to Kalecki [1937], Kaldor [1940], Hicks [1950] and Goodwin [1951].⁷ In the 1970s and 1980s, a larger literature emerged that examined the conditions under which qualitatively and quantitatively reasonable economic fluctuations might occur in purely deterministic settings (see, e.g., Benhabib and Nishimura [1979] and [1985], Day [1982] and [1983], Grandmont [1985], Boldrin and Montrucchio [1986], Day and Shafer [1987]; for surveys of the literature, see Boldrin and Woodford [1990] and Scheinkman [1990]). By the early 1990s, however, the interest in such models for understanding business cycle fluctuations greatly diminished and became quite removed from the mainstream research agenda.⁸ There are at least two reasons for this. First, if the economy exhibited a deterministic limit cycle, the cycles would be highly regular and predictable, which is inconsistent with the data. Second, the literature on limit cycles has generally made neither a clear empirical case nor a strong theoretical one for why they should be considered to be as or more relevant than the alternative explanations. An important contribution of this paper can therefore be seen as reviving the limit cycle view of fluctuations by offering new perspectives on these two arguments. In particular, with

⁷An earlier mention of self-sustaining cycles as a model for economic fluctuations is found in Le Corbeiller [1933] in the first volume of *Econometrica*.

⁸There are at least two strands of the macroeconomic literature that has productively continued to pursue ideas related to limit cycles: a literature on innovation cycles and growth (see, for example Shleifer [1986] and Matsuyama [1999]), and a literature on endogenous credit cycles in an OLG setting (see, for example, Azariadis and Smith [1998], Matsuyama [2007] and [2013], Myerson [2012] and Gu, Mattesini, Monnet, and Wright [2013]). One should also mention a large literature on endogenous business cycles under bounded rationality and learning, following early ideas of Grandmont [1998]. Hommes [2013] reviews this literature and the debate on endogenous business cycles versus exogenous shocks, and particularly the role of stochastic shocks in models with limit cycles and chaos.

respect to the first argument, we directly address the criticism of the excessive regularity of limit cycles by examining instead the notion of a stochastic limit cycle, where the system is subject to exogenous shocks, but where the deterministic part of the system admits a limit cycle. Such systems have been studied little by quantitative macro-economists, but recent solution techniques now make this a tractable endeavor.⁹

The remaining sections of the paper are organized as follows. In Section 1 we document the spectral properties of U.S. data on hours worked, unemployment, capacity utilization, and several indicators of financial conditions. These properties motivate our analysis and will be used later on in estimating our model. In Section 2 we present a simple mechanical set-up where agents both accumulate goods and interact strategically with one another. Following Cooper and John [1988], these strategic interactions can be characterized either by substitutability or complementarity. We use this framework to highlight when complementaries are likely to produce cyclical outcomes. In Section 3, we extend a standard three-equation New Keynesian model in a manner that allows for the features highlighted in model of Section 2. We present several estimations of this model to clarify what choices and restrictions would lead one to conclude that the economy is driven primarily by persistent exogenous shocks versus inferring that it is driven by a strong endogenous propagation mechanism, including the possibility of limit cycles. Finally, in the last section we offer concluding comments.

1 Motivating Observations

The object of this section is to re-assess certain properties of U.S. business cycles.¹⁰ In particular, we question the common view that business cycle fluctuations are best described as being a-cyclical, and provide evidence suggesting they may be better described as embedding or reflecting cyclical forces. Moreover, we examine whether there may be concurrent cyclical forces in financial markets, as would be implied by models of real-financial linkages. To illustrate what defines our notion of cyclicity, consider a stationary zero-mean series x_t (possibly expressed as deviations from an appropriate long-run trend), and let $\gamma_q \equiv \mathbb{E}[x_t x_{t+q}]$ be its auto-covariance function. Compare the following two cases. In the first, suppose that $\gamma_q > 0$ for all q , which would, for example, be the case if x_t follows a simple AR(1) process with positive auto-regressive parameter. In this case, positive values of x will, on average, be followed by positive values in each subsequent period. In this case, we refer to x_t as being a-cyclical, since a large boom today is not a predictor of a large recession to come;

⁹See Galizia [2018].

¹⁰Our analysis is primarily focused on the U.S., but in Appendix D we also document that the properties we emphasize are present in several other industrialized economies.

that is, there is no sense in which a current boom is sowing the seeds of a subsequent bust. In contrast, suppose instead that there is some n for which γ_{kn} is negative when k is odd and positive when k is even. In this case, positive values of x in one period are, on average, followed by negative values n periods in the future and positive values n periods after that. We refer to such a process as being cyclical, since a large boom today is predicted to be followed by a large bust in the future, with subsequent echo effects.

As the above example illustrates, this notion of cyclicity is simply a property of the auto-covariance function: if the auto-covariance function displays oscillations, then we will say that the series exhibits cyclicity; if it is not, then we will say it is a-cyclical. It should be emphasized that, according to this definition, cyclicity does not imply the presence of deterministic cycles of constant length, nor does it imply that the cycle is highly predictable. In particular, even if a series exhibits cyclicity, both the realized amplitude and realized length of cycles will generally be random.

There are two well known practical difficulties that arise when attempting to assess whether macroeconomic phenomena exhibit cyclicity. First, there are no generally accepted macroeconomic series that are thought to reflect only cyclical forces. Instead, most macroeconomic series are believed to reflect both business cycle forces and lower frequency forces unrelated to business cycles (e.g., demographic factors, trend growth). Therefore, in order to evaluate business cycle properties, one needs to find a way to extract properties of the data that are unlikely to be contaminated by lower frequency forces that are not of direct interest. For this reason, simple plots of the auto-covariance function of the data are not usually very informative. Second, since we are only interested in evaluating cyclical forces that are potentially attributable to business cycle phenomena, one needs to define the relevant range of periodicities to focus upon.

To address these two issues, we proceed in steps. First, we first exploit NBER recession dates to examine the probability distribution governing the arrival of recessions; in particular, we look at the conditional probability of the economy being in recession at time $t + k$ given that it was in recession at time t . Using this information, we propose a range of periodicities where one should look for cyclical forces.¹¹ In a second step, we look at the spectral densities of a set of non-trending variables that can be reasonably taken to be stationary (based on unit root tests). For example, this is the case for most labor market variables such as hours worked per capita, employment rates, job finding rates, and unemployment rates, and is also

¹¹As we shall see, the proposed range of periodicities suggested by our analysis is slightly longer than that usually associated with business cycle phenomena. However, given that our conclusions come directly from recession dates, it would be difficult to argue that the phenomena we capture are somehow distinct from the business cycle (e.g., that they may better be thought of as some alternative “medium-run” cyclical phenomena as, for example, in Comin and Gertler [2006]).

the case for many financial variables. Our interest in such series derives from the fact that it is not necessary to transform them before looking at their spectral densities. This is especially attractive since standard data transformations could spuriously induce an appearance of cyclicalities in a-cyclical data. The reason we focus on spectral densities, meanwhile, is that it is a representation of the data for which cyclical properties may potentially be inferred even in the presence of low-frequency confounders. To evaluate whether a series displays cyclicalities relevant for business cycle analysis, we look to see whether it exhibits a local peak (or hump) in the relevant range, as such a peak is an indicator that the series exhibits cyclicalities at that periodicity. In addition to plotting estimated spectral densities so as to allow for an easy visual inspection for the presence of a peak, we provide a set of tests to evaluate whether any such peaks are statistically significant.

1.1 Conditional Probabilities of NBER Recession Dates

In this subsection we use NBER recession dates to suggest what range of periodicities may be associated with cyclicalities in macroeconomic data.¹² One of the attractive features of NBER recession dates is that they are a synthesis of a number of different variables that are, by design, associated with standard ideas about business cycle activity. In panel (a) of Figure 1, we plot the probability that the economy will be declared by the NBER dating committee to be in a recession at some point in an x -quarter window around time $t + k$, given that it is in a (NBER) recession at time t . The Figure plots this probability as we vary k between 12 and 90 quarters, using all NBER recession dates from 1946Q1 to 2017Q2. We look in a $2x$ -quarter window around date $t + k$ since NBER recessions are rather short-lived, showing results for different window widths ranging from $x = 3$ to $x = 5$ quarters. We start at $k = 12$ to ensure that the conditional probability does not capture the same recession.

[Figure 1 about here.]

As can be seen in panel (a) of Figure 1, regardless of the window width, a rather clear pattern emerges. The probability that the economy is in a recession k quarters out given that it is in recession currently increases almost monotonically as k goes from 12 up to around 36-40 quarters, then decreases until around 56-60 quarters, at which point it starts increasing again. This pattern, which can be roughly approximated by a sine wave with a period of around 38 quarters, suggests to us that business cycles may reflect cyclical elements at a periodicity of around 9 to 10 years. Particularly interesting is the fall in the probability of a recession after 9-10 years, and the subsequent increase after reaching a minimum at

¹²Sources for all data series are provided in Appendix A.

around 14-15 years. Panel (b) of the Figure plots confidence intervals for the $x = 5$ case, which confirms that the general pattern we highlighted in panel (a)—namely, a local peak in the probability for $k \approx 36\text{-}40$ quarters, followed by a local trough around 56-60 quarters—is unlikely to have occurred simply by chance.¹³ In Appendix B we present confidence intervals for the $x = 3$ and $x = 4$ cases, which yield similar conclusions.

We take the above observations as the basis for formulating the hypothesis that post-war business cycles may contain cyclical elements that express themselves at a periodicity roughly between 36 and 40 quarters. It should be noted that this periodicity is slightly longer than commonly associated with business cycle phenomena as first proposed by Burns and Mitchell [1946].¹⁴ This may reflect the fact that Burns and Mitchell [1946] were focusing on pre-war data, where recessions were more frequent. The pattern in Figure 1 suggests that it may not be appropriate to focus business cycle analysis on fluctuations lasting less than 8 years, as is often taken as a benchmark, but should instead include a wider window of frequencies that would include periodicities of at least 10-12 years. In Appendix D, we document that a similar pattern to that observed in Figure 1 is also present in most other G7 countries. This may not be too surprising, as this set of countries often share recessions and expansions.

1.2 Looking for a Peak in Spectral Densities

We now examine the spectral properties of a set of non-trending U.S. macroeconomic variables. As noted previously, one potential way of describing the cyclical properties of stationary data is to focus on their spectral densities, which depicts the importance of cycles of different frequencies in explaining the data. If the spectral density of a time series displays a substantial peak at a given frequency, this is an indication of recurrent cyclical phenomena at that frequency. The traditional view, as expressed for example in Granger [1966] and Sargent [1987], is that most macroeconomic time series do not exhibit peaks in their spectral densities at business cycle frequencies. This view accordingly suggests that business cycle

¹³See Appendix B for details of how these confidence intervals were constructed.

¹⁴The traditional definition of the business cycle focuses on movements in macroeconomic variables at periodicities between 6 and 32 quarters. According to Baxter and King [1999] and Stock and Watson [1999], the reason was that the NBER chronology lists thirty complete cycles since 1858, with the shortest full cycle (peak to peak) being six quarters, and the longest thirty-nine quarters and 90% of these cycles being no longer than 32 quarters. While this definition may have seemed appropriate 30 years ago, it appears overly restrictive now given the more recent NBER cycle dates. For example, the cycle in the 1990s lasted 43 quarters from the peak in July 1990 to the subsequent peak in March 2001. Similarly, the cycle that started from the peak in 2007 has lasted more than 40 quarters so far, having apparently not yet reached another peak. For this reason—and this will be supported by our spectral evidence below—we argue that the definition of the business cycle should include fluctuations up to periodicities of at least 40 quarters, and maybe even up to 50 quarters.

theory should not seek to explain macroeconomic fluctuations as cyclical phenomena.¹⁵ In this section, we re-examine the validity of this consensus view relative to the alternative view in which business cycles may reflect cyclical forces that express themselves at a periodicity of around 36-40 quarters, where this timing is motivated by our above analysis of the conditional recession probabilities.

The main challenge we face when using spectral methods relates to the long-run properties of the data. In particular, spectral densities are only defined for stationary data, but many macroeconomic variables are non-stationary. It has therefore been common to substantially transform (i.e., de-trend) non-stationary variables before looking at their spectral properties. However, if a variable is thought to be the sum of a stationary cyclical component and a non-stationary trend component, there is no theory-free way of isolating the cyclical component. Moreover, it is well known that de-trending procedures can create spurious cycles. For this reason, the use of spectral methods is most informative if it can be applied to macroeconomic variables that can plausibly be argued to be stationary prior to any transformation.¹⁶ For example, this may be the case for labor market variables such as the employment and unemployment rates. For this reason, we begin by focusing on labor market variables and examine their spectral densities. Even if we accept that such labor market indicators can be thought of as stationary, we nevertheless believe that such measures most likely reflect both business cycle and longer-run phenomena. For example, demographic factors, which are typically thought of as being quite distinct from business cycle movements, tend to produce low-frequency movements in employment patterns. For this reason, we will focus on the shapes of spectral densities at periodicities shorter than 60 quarters, with the rationale that fluctuations at periodicities longer than 60 quarters are most likely reflecting factors distinct from business cycle phenomena.

We begin by examining the properties of the (log) of U.S. non-farm business (NFB) hours worked per capita from 1948Q1-2015Q2. This series is plotted in panel (a) of Figure 2. As the Figure shows, over the sample period hours exhibited substantial fluctuations, but with limited evidence of any long-run trend. For this reason, it seems plausible to treat this series as stationary, as is also confirmed formally by Dickey-Fuller tests.¹⁷ Accordingly, we begin

¹⁵In light of this, it is generally agreed upon (see e.g. Sargent [1987]) that business cycle research should focus mainly on explaining the co-movement properties of macro variables, as there is substantial co-movement across variables at business cycle frequencies. It is worth emphasizing that we do not question here the view that most macroeconomic aggregates co-move substantially over the business cycle, only the view that cyclical phenomena are not also present.

¹⁶If a series is non-stationary but known to be I(1), then in principle one can look at the spectrum of the first difference of the series. However, since the first-difference filter heavily emphasizes movements at the highest frequencies and de-emphasizes those at lower frequencies ones, doing so may substantially mask the properties of any cyclical component that may be present at somewhat lower frequencies.

¹⁷Specifically, we perform an augmented Dickey-Fuller test on the series. Akaike, Bayesian, and Hannan-

by looking directly at the spectral density of this series without any prior transformation (except de-meaning). The dark line in panel (b) of Figure 2 plots this spectral density over the range of periodicities from 4 to 60 quarters.¹⁸ Since it is common in macroeconomics to try to remove very low-frequency movements—that is, movements at frequencies much lower than business cycle frequencies—for comparison we also plot in the Figure the spectra obtained after first passing the series through a high-pass filter. In particular, each gray line in the Figure represents the spectral density of the series after it has been transformed using a high-pass filter that removes fluctuations with periodicities greater than P quarters in length, where P ranges from 100 to 200. The results interestingly suggest that the spectral properties of hours at periodicities around 36-40 quarters—the range of periodicities that interest us most—are very robust to whether or not one first removes very low-frequency movements from this series.

[Figure 2 about here.]

What does panel (b) of Figure 2 reveal about the cyclical properties of hours? To us, the dominant feature is the distinct hump in the spectral density surrounding the local peak at around 38 quarters, the bulk of which is contained in the 32- to 50-quarter range. This hump is much more pronounced than anything found at periodicities less than 32 quarters, which suggests that a significant proportion of the fluctuations in hours may come from some cyclical force with a periodicity of about 9-10 years, precisely in line with our earlier analysis using conditional recession probabilities. To make it clear that this is not simply a coincidence, and that the fluctuations in hours in this range of frequencies should indeed be thought of as business cycle phenomena, in panel (c) of Figure 2 we plot the hours series after having removed fluctuations longer than 60 quarters using the high-pass filter. In panels (a) and (c), we have also highlighted NBER recessions in gray. Unlike in panel (a), where there are clearly significant long-run fluctuations in hours that are unrelated to the NBER recessions, the fluctuations remaining in the de-trended hours series of panel (c) correspond very closely the standard narrative of the business cycle.¹⁹ In fact, to the extent that there

Quinn information criteria all suggest using a single lag. With this specification, the presence of a unit root is rejected at 5% significance.

¹⁸We obtain non-parametric power spectral density estimates by computing the discrete Fourier transform (DFT) of the series using a fast Fourier transform algorithm, and then smoothing it with a Hamming kernel. One key element is the number of points in the DFT, which determines the graphical resolution. In order to be able to clearly observe the spectral density between periodicities of 32 to 50 quarters, we use zero-padding to interpolate the DFT (see Section C in the Appendix for more details).

¹⁹We have checked that for the typical “RBC” moments (output, consumption, investment and hours standard deviations, standard deviations relative to the output one, autocorrelations, correlations with output and hours-average labor productivity correlation), we obtain the same business cycles pattern with Band-Pass(6,32) or (6,50) filtered data.

are fluctuations in panel (c) that do not correspond to NBER recessions, they are evidently due to unrelated movements much *shorter*, not longer, than the 32-50 quarter range in which the spectral hump appears.

1.2.1 Testing For a Local Peak

The local peak in the spectral density of hours worked observed in panel (b) of Figure 2 suggests that the labor market may be subject to cyclical forces that play out at periodicities around 38 quarters. However, as illustrated by the point-wise confidence bands shown in panel (d) of Figure 2,²⁰ there is substantial uncertainty surrounding our estimates of the spectral density. For this reason, it is desirable to examine the statistical significance of this observed local peak. Note that the point-wise confidence bands in panel (d) of Figure 2 are insufficient for this task, since “peaked-ness” is inherently a property of the size of the spectrum at one group of frequencies relative to another, making point-wise measures of sampling variability fundamentally inappropriate.²¹ To this end, we build on work by Canova [1996] and Reiter and Woitek [1999] to provide a set of statistical tests. Details of the tests and the bootstrap methods used to perform inference are presented in Appendix E. The tests we present that are built on the work of Canova [1996] exploit only the estimates of the spectral densities at the Fourier frequencies,²² while the tests built on Reiter and Woitek’s [1999] methodology use estimates over the entire continuum of frequencies.²³ The idea behind these tests is to compare the average estimated spectral density over some window around 38 quarters, which we refer to as the local peak and denote D_1 , to the average over a window around some slightly longer periodicity (e.g, around 50 quarters), which we refer to as the local trough and denote D_2 . The test statistic is then the ratio $D \equiv D_1/D_2$. A larger value of this test statistic indicates that the spectrum near 38 quarters is larger relative to its value

²⁰We use a bootstrap procedure to compute these confidence intervals. See Appendix C.4 for details.

²¹Reporting simultaneous confidence bands would be one way to jointly evaluate the sampling variability of the entire spectral density. However, even though such bands are computed using the full joint distribution of the density estimates, they nonetheless still only convey “point-wise” information, in the sense that they only show a likely range for the spectral density at each point individually. No information is conveyed, for example, about whether a large value at one frequency would tend to be associated with a large or small value at some other frequency. Since this is precisely the type of information that is relevant here, simultaneous bands are inappropriate for our purposes.

²²For a data series of length T , the Fourier frequencies are those given by j/T cycles/period for $j = 0, 1, \dots, T - 1$.

²³When presenting spectral estimates, it is common to smooth them first (as we did in Figure 2), since smoothing reduces sampling variability. However, the natural consequence of such smoothing is that it tends to obscure the presence of any local peaks (or troughs). Since our goal is to detect such peaks, we follow the literature and use the raw spectrum estimates in our statistical tests, and report p -values that account for the resulting larger sampling variability.

near the longer periodicity.²⁴

For each type of test, we report results under two different null hypotheses. The first null hypothesis is that the spectral density is flat over the region in question; that is, that $D = 1$. While we do not believe this to be a very pertinent null hypothesis, we provide results for it for the sake of completeness. We refer to this as the locally flat null. The second and, in our view, more relevant null hypothesis that we consider is one motivated by the influential work of Granger [1966], which argued that the spectral densities of most macroeconomic variables have shapes similar to those of persistent AR(1) processes, an argument that has since become the consensus view. Accordingly, the second null hypothesis is that the spectral density is that of the AR(1) process that best fits the data.²⁵ We refer to this case as the AR(1) null.

In Table 1 we present results for the tests proposed in Canova [1996] which use only the estimates of the spectral density at the Fourier frequencies. The question arises as to which Fourier frequencies to include in the local peak, and which to include in the local trough. It should be noted that, given the length of our data set, there are only five Fourier frequencies associated with periodicities between 30 and 60 quarters when we do not do zero-padding: those at 30, 33.8, 38.6, 45, and 54 quarters. The Fourier frequency at 38.6 quarters should clearly enter the local peak, while the one at 54 quarters should clearly enter the local trough. The first two entries in the first row of Table 1 report, under the locally flat and AR(1) nulls, respectively, bootstrapped p -values²⁶ for the case where the local peak only contains 38.6 quarters and the local trough only 54 quarters. As can be seen, the p -values are below 1% in both cases, indicating a strong rejection of both nulls. The remaining entries in the table examine results when we include different combinations of Fourier frequencies in the local peak and local trough. For example, in the last entry of the second row, we have a p -value of 3.8% for the case where there are two Fourier frequencies in the local peak (periodicities 38.6 and 33.8), two in the local trough (periodicities 45 and 54), and the relevant null hypothesis is the AR null. Note that rejection of a given null under any one of these peak/trough combinations is a rejection of the entire null, and in particular the null need not be rejected anew for each combination. For example, under the locally flat null,

²⁴Since our goal is to establish that there is a *local* peak in the spectrum near 38 quarters, rather than a global peak at that point (which, given the important movements in the data at very low frequencies, is almost certainly not the case), we do not compare the spectrum near 38 quarters with the, say, its *maximum* value at all longer periodicities.

²⁵Note that, since the spectral density of such a process is monotonically increasing in the periodicity (assuming a positive autoregressive parameter), we would expect to have $D < 1$ under the null hypothesis.

²⁶Canova [1996] derived an asymptotic distribution for the test statistic we use. For the sample sizes we encounter here, however, our simulations indicated that this asymptotic distribution is inappropriate, producing p -values that are much too small. As such, we use bootstrapped p -values throughout this paper.

the null hypothesis implies that $D = 1$ for every combination of peak/trough frequencies simultaneously, so that rejection of $D = 1$ for a single combination is in fact a rejection of the entire locally flat null. Thus, the results in Table 1 clearly reject both the locally flat and AR(1) nulls, implying that the local peak in the hours spectrum near 38 quarters is statistically significant. Among other things, this provides considerable evidence against the consensus view that the spectrum is well approximated by that of a persistent AR(1) process.

[Table 1 about here.]

In Table 2, we report an alternative set of tests, based on Reiter and Woitek [1999], which averages the estimated spectral density for hours over its continuous domain, as opposed to averaging only over the estimates at certain combinations of Fourier frequencies as in the Canova [1996] tests.²⁷ To implement these tests, we specify mid-points q_P and q_T for the local peak and local trough ranges, respectively, as well as a window width w , so that the peak range is given by the range of periodicities $[q_P - w, q_P + w]$, and the trough range by $[q_T - w, q_T + w]$. We set these mid-points to the periodicities associated with the local maximum and minimum of the (raw) estimated spectrum (in particular, $q_P = 38$ and $q_T = 54.1$), and then explore the consequences of varying the window widths. Table 2 presents the resulting p -values. For example, the first row of the table indicates that, with the local peak and trough ranges given by a ± 1 -quarter band around the respective mid-point, the locally flat null p -value is less than 2% and the AR(1) null p -value is less than 1%, establishing a strong rejection of both nulls for this case. Note that, similar to the Canova [1996] tests, rejection of a given null for any single window size is a rejection of the entire null. Thus, as can be seen from the p -values, we again find considerable evidence of a statistically significant local peak near 38 quarters for both the locally flat and AR(1) null hypotheses.

[Table 2 about here.]

1.2.2 Other Related Variables

We have above provided evidence that aggregate hours worked per capita appears to contain an important cyclical component, as captured by the peak in its spectrum in a window near 38 quarters. It is immediately relevant to ask whether this property is representative of

²⁷As argued in Reiter and Woitek [1999], the outcome of a Canova [1996] test “depends sensitively on the exact location of the Fourier frequencies, which in turn depends on the sample size” (p. 8). They provide simulation-based evidence that, as a result, the statistical power of the Canova [1996] test can be significantly improved by averaging the spectrum over the continuum of frequencies in the relevant peak or trough range, instead of just at the Fourier frequencies.

other measures of labor market activity. To this end, in Figure 3 we present estimates of the spectral density for two other closely related labor market measures: the employment rate and the unemployment rate. The employment rate covers the same post-war period as our hours worked series, while for the unemployment rate we take advantage of a longer series constructed by Ramey and Zubairy [2018], which is for the period 1890Q1-2015Q2, in hope of getting more precise estimates.²⁸ In both cases, we again see a hump in the spectral density in the same general range as observed for hours. The employment rate exhibits a shape similar to that of the hours worked series, with a peak near 38-39 quarters, while the longer unemployment series has its peak at a slightly shorter periodicity of around 35-36 quarters.²⁹

[Figure 3 about here.]

In Tables 3 and 4 we report the same set of test statistics that we reported for hours worked. Table 3 shows results for the Canova [1996] tests for both series. For the employment series, we use similar peak and trough combinations as we did for hours worked in Table 1.³⁰ For the unemployment series, which is of a different length and therefore has a different set of Fourier frequencies, this is not possible. We thus choose peak and trough combinations guided by the estimated spectrum in a manner similar to that for hours.³¹ As can be seen in the Table, for each series there are several peak/trough combinations that produce a rejection of each null hypothesis, indicating an overall rejection of both nulls for both series. In Table 4 we report results for the Reiter and Woitek [1999] tests. In both cases we fix the peak mid-point to $q_P = 38$ quarters (as with hours), however we allow the trough mid-point q_T to be dictated by the location of the local minimum for that series (54.9 quarters for employment, 43.5 quarters for unemployment). We again see rejections at 5% for each series/null combination. Combined, the results from these Tables indicate that labor market outcomes may embed cyclical forces.

[Table 3 about here.]

²⁸ We later also report results for the unemployment rate for the post war period.

²⁹The observation that the cyclical component of labor market outcomes may have a shorter periodicity in the pre-war period is not too surprising, given that it is well known that business cycles tended to be shorter on average in the pre-war period.

³⁰To economize on space, for the Canova [1996] tests we henceforth only consider peak/trough ranges containing either one or two Fourier frequencies each, while for the Reiter and Woitek [1999] tests we only consider window widths of $w = 1, 2$ quarters. Including extra Fourier frequencies for the Canova [1996] tests or considering wider windows for the Reiter and Woitek [1999] tests would not alter any of the conclusions.

³¹In particular, the estimated unemployment spectrum has a local peak near 36 quarters and a local trough near 42 quarters, and has Fourier frequencies in this general range of 35.9, 38.6, 41.8, and 45.6 quarters. We thus always include 35.9 quarters in the peak range and 41.8 quarters in the trough, and then explore the consequences of including various combinations of other frequencies.

[Table 4 about here.]

In addition to the above series, we display in Figure 4 the spectral densities for the following series: the post-war unemployment rate, capacity (i.e., capital) utilization, and a measure of overall factor usage that combines both labor utilization and capacity utilization. Note that for each of these series we tested whether they could plausibly be treated as stationary to justify examining their spectral densities without any de-trending. In Appendix E.2, we report a set of peak test results for those series. For each of these variables we again find strong evidence against both of our null hypotheses.³² In Appendix D, we also document that such cyclical features also appear to be present for labor and capital usage in the other countries of the G7.

[Figure 4 about here.]

1.3 Spectral Implications for Hours In Standard Models

In the previous sections we have shown that several measures of labor activity exhibit significant spectral peaks at a periodicity of around 36-40 quarters. In this subsection, we argue that this observation conflicts with the spectra implied by many modern business cycle models. To illustrate this, we report in Figure 5 the spectral density of hours implied by six models that we think span the range of quantitative models: two real models, a New Keynesian one, and three models with financial frictions. The first model is the simplest Real Business Cycle model described in Cooley and Prescott [1995]. In that stripped-down model, fluctuations only come from persistent technology shocks. The second model is a RBC model augmented to include variable capital utilization, as well as investment-specific technology shocks as a second source of fluctuations.³³ The third model is the rich New Keynesian model of Smets and Wouters [2007], which features seven shocks and a variety of real and nominal frictions. The three last models incorporate financial frictions. The fourth model is the financial frictions model of Carlstrom and Fuerst [1997], and the fifth is Bernanke, Gertler, and Gilchrist [1999], as estimated on U.S. data by Christensen and Dib [2008]. The

³²One may also be interested in knowing whether quantity variables such as output also exhibit a hump in their spectral densities in a similar range to that observed with the labor market variables. The difficulty with such variables, however, is that they are clearly non-stationary, so that some type of transformation is needed before one can examine the spectral density. For several different measures of output per capita, we have explored using the (controversial) method of passing the series through a high-pass filter to remove the non-stationary component. For the cases we examined, we did not find much evidence of a peak in filtered output. As we discuss in Appendix G, we provide evidence in support of the idea that this is likely the result of certain properties of productivity (as measured by utilization-adjusted TFP), rather than an absence of underlying cyclical forces altogether.

³³This augmented model is calibrated following Fernandez-Villaverde [2016].

sixth model, as proposed by Christiano, Motto, and Rostagno [2014], incorporates the microeconomics of the debt-contracting framework of Bernanke, Gertler, and Gilchrist [1999] into an otherwise standard monetary model of the business cycle and features a large number of shocks, including news and risk shocks.³⁴ In all cases we see a similar pattern: the spectra do not exhibit any peaks, being instead very similar to what one would obtain from a simple AR(1) process. This observation should not be too surprising: as is well known, the internal propagation of these models is rather weak, and therefore the endogenous variables largely inherit the properties of the exogenous driving forces, which in these cases are either equal to or close to AR(1) processes. Although these examples are only illustrative, in our exploration of different models we have not yet found an estimated model that produces a peak in the spectral density of hours similar to the one in the data.

[Figure 5 about here.]

1.4 Is There a Related Financial Cycle?

As the financial crisis of 2007 has revived the interest in connecting business cycles to financial conditions, we now turn to looking at the spectral properties of a set of financial market indicators. In parallel to our exploration of labor market variables, we again restrict attention to variables that appear to be stationary based on Dickey-Fuller tests, allowing us to directly examine their spectral densities with prior transformation.³⁵ In particular, we want to examine whether indicators of financial market conditions also exhibit peaks in their spectral densities in a range of frequencies similar to those observed for hours worked. Knowing whether this is the case will be important in helping to identify the type of model that may best explain the employment cycle we emphasized previously. To this end, we report spectrum estimates for four financial market indicators in Figure 6. Our main series of interest, which we will later also use in model estimation, is the spread between the Federal Funds (FF) rate and the rate on BAA bonds. This series runs from 1954Q3 to 2015Q2. This interest spread is a common measure of financial market risk discussed throughout the macro-financial linkage literature, and is typically interpreted as a measure of the market risk premium. The estimated spectral density for this measure of the risk premium is presented in panel (a) of Figure 6. As is clear from the Figure, we again find evidence of a hump in the spectral density in the range just above the standard business cycle range, though the

³⁴To simulate that model, we use the DYNARE code provided in the replication package of the Macroeconomic Model Data Base (see Wieland, Afanasyeva, Kuete, and Yoo [2016]).

³⁵We nevertheless again also show that the results are robust to first filtering the data with a high-pass filter in order to remove very low-frequency movements.

actual peak is located slightly below that found for our labor market measures (around 34 quarters).³⁶

[Figure 6 about here.]

To verify this visual impression, in Table 5 (Canova [1996] tests) and Table 6 (Reiter and Woitek [1999] tests) we report results from the same set of test statistics used for evaluating the significance of the peak for the labor market variables above.³⁷ In all cases, the data clearly reject that the spectral density is flat or similar to an AR(1) in the relevant range, in favor of the alternative that there is a local peak centered somewhere between 36 and 40 quarters. In Appendix F, we also examine the co-movement between this risk premium measure and our previous hours worked measure. We find the correlation between the two to be highest (in absolute value) at precisely the frequencies emphasized by the peaks in their spectral densities. Computing the spectral coherence between the two series, we find a maximum at 38 quarters over the 4 to 60 quarters periodicities. Together these observations suggest that the labor market and the financial market may share cyclical forces that express themselves at a periodicity of around nine years. In a later section our goal will be to explore a class of explanation to these shared cyclical patterns.

[Table 5 about here.]

[Table 6 about here.]

The other variables presented in Figure 6 are as follows. In panel (b) we show the spectral density for the Chicago Fed's National Financial Conditions Index (NFCI), which begins in 1971Q1. As described in Brave and Butters [2011], the NFCI, which is computed from a large sample of financial indicators, is a synthetic index between -1 and 1 that attempts to

³⁶For this interest rate spread measure there is also a second, smaller, peak near 21 quarters. This second peak, which also appears for two other spread series we consider (including the one in panel (c) of the Figure) but not in any of the other series, could be indicative of a second source of cyclicity relevant only for interest rate spreads. Alternatively, as is well known, a peak at periodicity n with a second peak at periodicity n/k for some integer k (e.g., at half the periodicity of the original) is indicative of a recurrent cyclical pattern with periodicity n , but where that pattern is not well captured by a simple sine wave. Even though 21 quarters is not exactly half of 34 quarters, one would not be able to reject the null hypothesis that, in the population spectrum, the smaller peak is at exactly half the periodicity of the larger one (i.e., that the smaller peak is simply a reflection of cyclical behavior at the larger periodicity). While this issue bears further study, for now we concentrate on the peak near 38 quarters.

³⁷Similar to the case with the unemployment series above, the length of our spread series differs from our hours series, and therefore so do the Fourier frequencies. There are four Fourier frequencies in the relevant range: 34.9, 40.7, 48.8, and 61. Thus, for the Canova [1996] tests we always include the first and last of these in the peak and trough range, respectively, and report results for including or not including the other two. For the Reiter and Woitek [1999] tests, as above we fix the peak mid-point at 38 quarters, and let the trough mid-point be dictated by the location of the local minimum, which in this case is at 60.6 quarters.

summarize financial conditions. In panel (c), we show the spectrum of the spread between the 3-month T-bill rate and the rate on AAA bonds. Relative to the series shown in panel (a), the advantage of this series is that it goes back to 1920Q1, potentially allowing for more precise estimates. In panel (d), we show the spectrum of the NFCI Risk Subindex (which captures volatility and funding risk in the financial sector). All of these series exhibit humps in their spectral densities in the range slightly beyond the standard business cycle range. The AAA risk premium series has a shape quite similar to our baseline spread series in panel (a), while the synthetic NFCI series and its risk sub-component have a hump that is slightly more spread out than the others. In Appendix E.2, we report the results of our formal peak tests for those three variables, as well as for a BAA/T-bill spread series also beginning in 1920Q1. In all cases, the data quite strongly reject the null hypothesis that there is no peak in the spectrum near 38 quarters. Thus, the overall picture that emerges from our visual and statistical evidence suggests a close link between the cycle in employment and the cycle in financial market conditions. This suggests to us that, when looking to explain such cyclical patterns, one may want to look toward models that feature real-financial linkages.

2 Explaining Spectral Peaks: A Class of Models

In the previous section we documented the presence of a significant peak in the spectral densities of several non-trending macroeconomic variables at a periodicity near 38 quarters. In particular, we emphasized the presence of this peak in variables related to both employment and financial market conditions. The presence of such peaks suggests that the macroeconomy may embed cyclical forces that manifest themselves at a frequency slightly lower than what has traditionally been associated with business cycle fluctuations. However, as we have discussed, we believe that such forces should nevertheless be viewed as a part of the business cycle—rather than something distinct from it—since they appear intimately linked to the occurrence of recessions. The object of the next two sections is to explore mechanisms that may lie behind such phenomena.

There are at least three classes of explanations for such cyclical behavior: explanations based mainly on properties of the exogenous driving forces, explanations based primarily of the properties of individual level behavior, and, finally, explanations based on equilibrium interactions—that is, explanations where cyclical outcomes arise as the result of market interactions between individuals who, in the absence of such interactions, would not tend to make cyclical choices. Since we want to better understand the mechanisms that may give rise to this latter possibility, the focus of this section will be to highlight certain properties that may be key for aggregate cyclical behavior to potentially emerge as the outcome of equi-

librium forces. To this end, we begin by examining how a hump-shaped spectral density can arise in a mechanical model where agent interactions are captured by a market-determined variable that each agent takes as given. The framework will allow us to isolate simple conditions under which endogenous cyclical outcomes emerge as a result of the equilibrium forces. The framework will also allow us to clarify potential challenges that arise when attempting to explain hump-shaped spectral densities in an equilibrium framework, especially when local instability and limit cycles cannot be ruled out. In Section 3 below, we will explore the empirical relevance of these notions using an optimization-based forward-looking model.

2.1 Demand Complementarities as a Source of Cyclical

2.1.1 The Environment

To understand the economic forces that may generate cyclicity, let us consider an environment with a large number N of agents indexed by j , where agent j makes a decision e_{jt} . Here, e_{jt} could represent a decision regarding an expenditure, a level of output or a level of investment. The decision rule of the individual is assumed to take the form³⁸

$$e_{jt} = \alpha_0 + \alpha_1 X_{jt} + \alpha_2 e_{jt-1} + \alpha_3 q_t + \mu_t , \quad (1)$$

where the variable X_{jt} is a stock variable which could represent capital, net worth, or a habit stock, which satisfies an accumulation equation of the form

$$X_{jt+1} = (1 - \delta)X_{jt} + \psi e_{jt} , \quad 0 < \delta < 1 . \quad (2)$$

The term μ_t in equation (1) represents an exogenous driving force,³⁹ and q_t represents some market-determined variable, such as a price or a matching rate. Since we allow α_1 to be positive or negative (or zero), for simplicity and without loss of generality we henceforth normalize $\psi = 1$.

A key aspect of this setup is the interaction between individuals as captured by the market-determined variable q_t . In particular, to capture equilibrium forces, we allow q_t to be an arbitrary function of the average behavior of others, i.e.,

$$q_t = Q \left(\frac{1}{N} \sum_{j=1}^N e_{jt} \right) , \quad (3)$$

where, following Cooper and John [1988], we will refer to the actions of agents as being (strategic) complements if $Q'(\cdot) > 0$ and (strategic) substitutes if $Q'(\cdot) < 0$. In comparison

³⁸In Beaudry, Galizia, and Portier [2016b], we show that most of the properties that we derive here are also present if the model includes a forward-looking component; that is, if we specify behavior as $e_{jt} = \alpha_0 + \alpha_1 X_{jt} + \alpha_2 e_{jt-1} + \alpha_5 \mathbb{E}_t e_{jt+1} + \alpha_3 q_t + \mu_t$.

³⁹The analysis can be easily extended to include idiosyncratic shocks as well.

with Cooper and John [1988], our framework embeds their static game within each period, with dynamic elements added through both the inertial forces captured by $\alpha_2 e_{jt-1}$ and the accumulation forces captured by $\alpha_1 X_{jt}$. We restrict attention to cases where $\alpha_2 \in [0, 1)$ since it is meant to reflect inertia. For now we will consider Q to be a linear function of aggregate behavior, i.e., we take $q_t = \alpha_4 e_t$, where $e_t = N^{-1} \sum_j e_{jt}$. Later we will discuss the effect of allowing the function $Q(\cdot)$ to be non-linear. However, we will always restrict attention to situations where $\alpha_3 Q'(e) < 1$ so as to rule out the possibility of static multiple equilibria. Throughout we will assume that μ_t is a stationary stochastic process of the form $\mu_t = D(L)\epsilon_t$ where $D(L)$ is a polynomial in the lag operator L , and ϵ_t are *i.i.d.* shocks with unit variance. Furthermore, as we are interested here in understanding the role of agent interactions in creating cyclical aggregate outcomes, we will exclude the possibility of exogenously driven cyclical by making the following assumption:

Assumption 1. *The spectral density $s_\mu(\omega)$ of μ_t is monotonic on frequencies $\omega \in [0, \pi]$.*

Note that Assumption 1 would hold if, for example, μ followed an AR(1) process. We now want to derive conditions under which aggregate behavior in this setup could feature cyclical behavior as captured by a hump-shaped spectral density. To see the forces at play explicitly, it is useful to write out the spectral density of e_t . Focusing on the symmetric outcome where $e_{jt} = e_t$, the evolution of e_t can be written as

$$e_t = \left(\frac{\alpha_1 + \alpha_2}{1 - \alpha_3 \alpha_4} + 1 - \delta \right) e_{t-1} - \frac{\alpha_2(1 - \delta)}{1 - \alpha_3 \alpha_4} e_{t-2} + \frac{1 - (1 - \delta)L}{1 - \alpha_3 \alpha_4} \mu_t ,$$

and accordingly the spectrum of e_t can be written as

$$s_e(\omega) = s_\mu(\omega) \times \frac{[1 - (1 - \delta) \exp\{i\omega\}] [1 - (1 - \delta) \exp\{-i\omega\}]}{(1 - \alpha_3 \alpha_4)^2} \times g(\omega) , \quad (4)$$

where $i = \sqrt{-1}$, $g(\omega) \equiv [B(\exp\{i\omega\})B(\exp\{-i\omega\})]^{-1}$, and $B(L) \equiv 1 - \left(\frac{\alpha_1 + \alpha_2}{1 - \alpha_3 \alpha_4} + 1 - \delta \right) L + \frac{\alpha_2(1 - \delta)}{1 - \alpha_3 \alpha_4} L^2$.

From equation (4), we see the different forces that can cause the spectrum to have a local peak. The first term represents the properties of the exogenous driving force. Assumption 1 implies that this term is monotonic, and so rules out that it can, by itself, be the source of a hump-shaped spectral density for e . It can be verified that the second term is also monotonic on $[0, \pi]$, and therefore also cannot directly create a hump-shaped spectral density for e . This leaves the last term, $g(\omega)$, which captures the transitional dynamic forces generated by the interaction of individual decision rules and the market-determined value of q .⁴⁰ The question

⁴⁰It should be noted that, even if each of these components were monotonic, their product could nevertheless exhibit a local peak. Such an interaction could be an alternative avenue to explain the observed pattern in the data. In our model of Section 3 that we take to the data, such a possibility will be allowed, but in the end it is not favored by the data, and so we do not focus on this possibility here.

we now ask is: under what conditions could agent interaction (through the endogenous determination of q) be the main reason for a humped-shaped spectral density? For this reason, we focus on situations where the following assumption is also satisfied.

Assumption 2. α_1, α_2 and δ are such that, when $\alpha_3\alpha_4 = 0$, the eigenvalues of the system⁴¹ represented by equations (1)-(3) are real, positive, and smaller than 1.

Assumption 2 implies that, if there were no forces linking agent decisions, as would be the case if either $\alpha_3 = 0$ or $\alpha_4 = 0$, then $g(\omega)$ would be monotonic and accordingly would not by itself introduce a local peak in the spectral density of e . In other words, Assumption 2 rules out the possibility that individual-level choices are cyclical when q is constant. This brings us to the main proposition for this section, which indicates necessary conditions for agent interactions to create a hump-shaped spectral density.⁴²

Proposition 1. Under Assumptions 1 and 2, for agent interactions to cause a hump-shaped spectral density—as captured by the introduction of a local peak in $g(\omega)$ —it is necessary that $\alpha_3\alpha_4$ be positive and that α_1 be negative.

Proposition 1 highlights two forces that are necessary to cause aggregate behavior to exhibit cyclicity even when individual-level behavior on its own would not be cyclical. The first of these two conditions is that the decisions of agents act as strategic complements. This is captured by the condition $\alpha_3\alpha_4 > 0$; that is, α_3 and α_4 need to be such that the decision by one agent to increase e_{jt} favors others to do the same. In addition to this complementarity property, it is also necessary that agents be accumulating a stock that tends to dampen their decision on e_{jt} . The intuition for this result is most easily understood in the case where e_{jt} is interpreted as an investment decision and X is interpreted as a stock of capital. Complementarity implies that agents would like to invest at the same time, but as they invest together, this leads to an increase in X_t which, due to decreasing returns, eventually leads them to want to dis-invest. These countervailing forces can thereby lead to cyclical behavior. That the conditions highlighted in Proposition 1 may lead to cyclicity in this simple setup may not be too surprising. However, what to us is more interesting is that these conditions are in fact *necessary* here. In particular, as we discuss later, few of the commonly estimated models in the literature embed these two forces simultaneously, and therefore many models in the literature appear by design limited in their capacity to explain aggregate cyclicity endogenously.

⁴¹By the eigenvalues of the system, we always mean in particular the eigenvalues of the deterministic version of the system (i.e., the eigenvalues of the system when μ_t does not appear in (1)). Equivalently, these are the eigenvalues of the AR(2) process for e_t defined by $B(L)e_t = \nu_t$ with ν_t i.i.d.

⁴²Proofs of all propositions are presented in Appendix H.

2.2 Pushing the Complementarities Further: Local Instability and Limit cycles

The second issue we want to address with the model represented by equations (1)-(3) relates to how such a system behaves if the complementarity forces, as captured by $\alpha_3\alpha_4$, become quite strong. In particular, Proposition 2 indicates that, even if we restrict our focus to the case where $\alpha_3\alpha_4 < 1$, as the complementarity forces become sufficient strong, this system will become unstable.

Proposition 2. *There exist an $\alpha^* \in (0, 1)$ such that if $\alpha^* < \alpha_3\alpha_4 < 1$ then the system represented by (1)-(3) is unstable. Moreover, at the point where it loses stability as $\alpha_3\alpha_4$ increases from zero, the eigenvalues of the system are complex if $\delta^2 < -\frac{\alpha_1}{\alpha_2} < (2 - \delta)^2$*

Proposition 1 and Proposition 2 together suggest that one needs to be aware of certain difficulties if one builds on a structure similar to equations (1)-(3). In particular, Proposition 1 suggests that allowing for complementarities across agents may be necessary to understand endogenously generated hump shaped spectral densities, while Proposition 2 suggests that it is precisely in such a case that instability can also arise. At first pass, one may want to disregard the possibility of instability since market economies do not appear to be explosive. However, this may be too dismissive. In particular, in our opinion, Proposition 2 should be taken as an indication that, when building on a framework that embeds complementarities and dynamics, one should take the possibility of local instability seriously. Since equations (1) and (2) are linear, and since we have assumed thus far that Q is also linear, any form of local instability in this setup implies explosiveness, and therefore could be ruled out by the observation that the economy is not explosive. But this equivalence between instability and explosiveness is a knife-edge feature that depends on not having any non-linearity in the model. In contrast, if we were to allow for some non-linearities in the model, for example by allowing Q to be non linear, then Proposition 2 suggests that as complementarities increase (e.g., by increasing α_3) the system can experience a Hopf bifurcation⁴³ resulting in the emergence of a locally unstable steady state surrounded by a stochastic limit cycle.⁴⁴ A stochastic limit cycle refers to the property of a stochastic dynamic system where the internal dynamics of the system (i.e., when the stochastic elements are set to zero) can support a perpetual cycle. As discussed in the introduction, the idea that macroeconomic fluctuations may reflect limit cycle forces has a long history in the literature, even though it has played a

⁴³Since the system under consideration is in discrete time, the bifurcation that occurs is more accurately referred to as a Neimark-Sacker bifurcation (see Kuznetsov [1998]).

⁴⁴See Beaudry, Galizia, and Portier [2015], [2016b] for further details about this statement including an extension of this statement to models with a forward looking component in which a saddle path limit cycle can arise.

minor role in modern macroeconomics. The value of Propositions 1 and 2 in this respect is to emphasize that the equilibrium and behavioral features which may allow a linear model to produce a hump-shaped spectral density may simultaneously be features that allow a non-linear version of the same type of model to produce the same pattern as the result of a limit cycle. Hence, when attempting to explain hump-shaped spectral densities, it appears warranted to explore whether such a pattern may reflect limit cycle forces, or whether it may be well captured by a stable linear structure. We explore this issue further in the next section.

2.3 Discussion

While the reduced-form model represented by equations (1)-(3) is largely mechanical, we believe it nevertheless offers a framework to help discuss the type of macroeconomic model that may offer promise in explaining the spectral density features we documented in Section 1. In particular, Proposition 1 emphasizes that, to produce a hump-shaped spectral density in activity as the result of equilibrium interactions, one likely needs agents' decisions to positively affect the decisions of others; that is, the decisions should be strategic complements. However, in many macroeconomic models, individual-level decisions tend to act as strategic substitutes (as would be captured in our model by having $\alpha_3\alpha_4 < 0$) due to standard price effects. This property remains true even in most New Keynesian models, since central bank behavior implies that interest rates are raised if one group increases their purchases, thereby causing others to decrease their purchases. Hence, to produce a hump-shaped spectral density, one likely needs to turn towards a model that departs sufficiently from standard neo-classical principles—which favor substitutability—to a model in which actions can act as strategic complements. Our empirical observations regarding the behavior of the risk premium on borrowing suggests that looking at models where financial frictions create such a complementarity appears promising, as the data indicate that the risk premium on borrowing tends to fall when activity is high, which may represent a source of complementarity.

It should be noted that many macroeconomic models that feature financial frictions do not in fact embed such complementarities. For example, relative to, say, a simple neo-classical benchmark, workhorse models of financial frictions such as Carlstrom and Fuerst [1997] and Bernanke, Gertler, and Gilchrist [1999] do not embed complementary forces but instead introduce an additional state variable in the form of net worth. While this addition tends to help amplify and prolong the effects of shocks, the framework presented in this section suggests that it is not a promising route to generate a hump-shaped spectral density, since



this net worth channels act like a stock variable X_{it} that favors—rather than dampens—activity (i.e., it features $\alpha_1 > 0$).⁴⁵ Accordingly, in the next section, we explore a model that includes financial frictions in a manner that allows for agents’ actions to potentially be complementary, and we allow for the accumulation of household capital which, when high, has a dampening effect on new purchases.⁴⁶ Our goal will be to use this model to explore a set of issues, namely, (i) how allowing for complementarities across agents can change one’s inference regarding the role played by exogenous forces in explaining the data, (ii) the sensitivity of such inferences with respect to the data frequencies used in estimation, and (iii) the implications of allowing for limit cycles in the estimation of the model.

2.4 Stochastic Limit Cycles and Predictability (Or Lack Thereof)

As we have seen, in dynamic environments with accumulation, strategic complementarities between agents actions may create cyclical aggregate outcomes which, if these forces are sufficiently strong, can involve local instability and the emergence of stochastic limit cycles. We discussed how this can arise in environments even where individual-level behavior is not in and of itself cyclical (see Assumption 2) and when complementarities remain modest (in the sense that the implied best-response elasticity is below one). Accordingly, in our empirical exploration we will explore whether allowing for stochastic limit cycle forces may help explain hump-shaped spectral densities. Since stochastic limit cycles are rarely allowed in the estimation of macroeconomic models, in this section we will briefly discuss some of their empirical properties. In particular, we aim to clarify the difference between stochastic and non-stochastic limit cycles in terms of regularity and predictability.

If agents’ decision rules in equation (1) were deterministic (i.e., $\mu_t = 0 \forall t$), complementarities across agents could also give rise to local instability and limit cycles as long as some non-linearity is present. However, the resulting cyclical dynamics would be far too regular to match the patterns observed in macroeconomic data. This is a well known criticism of limit cycle models. However, the addition of an exogenous stochastic process μ_t that affects behavior changes this implication considerably. In the presence of limit cycle forces, it is important to emphasize that the addition of a stochastic process influencing decisions does not simply add noise around an otherwise-deterministic cycle. Instead, the exogenous forces not only produce random “amplitude” shifts that temporarily perturb the system from the

⁴⁵Recall that Proposition 1 emphasizes that an accumulation variable should affect activity negatively if it is to generate spectral peaks.

⁴⁶One alternative possibility would be to pursue mechanisms more akin to those emphasized in Kiyotaki and Moore [1997]. We choose not to follow this route as we want to place financial frictions that affect households, as opposed to firms, central to the mechanism. See Mian and Sufi [2018] for an exposition of the importance of financial frictions on the household side in the business cycle.



limit cycle, but also create random “phase” shifts that accelerate or delay the cycle itself. Furthermore, even though μ_t is stationary, and while the amplitude displacement caused by a shock is temporary, the phase displacement will have a permanent component. Thus, following a shock the system will eventually converge back to the limit cycle, but the state of the system will be either permanently ahead or behind where it would have been in the absence of the shock. As a result, data simulated from a stochastic limit cycle model can easily be shown to have time paths similar to those found in actual economic data, including exhibiting significantly irregular business cycles (see Beaudry, Galizia, and Portier [2016b] for more details).

A similar property obtains for the *predictability* of stochastic limit cycle models. In fact, the implication of allowing for stochastic elements is even more stark for predictability than it is for regularity. A deterministic cycle is both perfectly regular and perfectly predictable arbitrarily far into the future. However, while introducing a small amount of stochastic variability in behavior would tend to make the cycle only slightly less regular, the degree of unpredictability (as measured, for example, by the forecast-error variance) arbitrarily far into the future will jump up discontinuously.⁴⁷ The reason for this discontinuity follows directly from the fact that the “phase” component of the system follows a random walk. As is well known, as long as the variance of the innovations to a random walk process is positive (even if it is arbitrarily small), the forecast-error variance of the process becomes infinite as the forecast horizon increases. In our context, this means that, as you go far enough into the future, no matter how small the variance of the exogenous shock process is, as long as it is not zero the phase of the cycle implied by the limit cycle becomes completely unpredictable. In other words, no matter what the current state is, many periods into the future the system is just as likely to be at the bottom of the cycle as it is at the top. Accordingly, a stochastic limit cycle should not be thought of as necessarily involving substantial predictability. Its predictability depends on the variance of shocks and the nature of the limit cycle itself, and in general it is possible to have a large amount of predictability, a small amount of predictability, or anything in between.

3 A New Keynesian Model

In this section we study an environment with real-financial linkages that focuses primarily on the household, and emphasizes how labor market risk (in particular, unemployment risk) can

⁴⁷Formally, let $V_t(\sigma_\mu^2) \equiv \lim_{k \rightarrow \infty} V_t(e_{t+k} - E_t[e_{t+k}])$ denote the limit of the forecast-error variance of e , conditional on information available as of date t , when the forecast horizon k extends infinitely far into the future, and where $V(\mu_t) = \sigma_\mu^2$ is the variance of the shock. Since the deterministic model is perfectly predictable we have $V_t(0) = 0$. However, it can be verified that $\lim_{\sigma_\mu^2 \rightarrow 0} V_t(\sigma_\mu^2) > 0$.

affect financial conditions through its effect on default risk, which in turn can affect consumer demand, thereby feeding back to the labor market. Our focus on the household is motivated in part by the work of Mian and Sufi [2018]. The model builds on the workhorse three-equation New Keynesian model, with the most important difference being our specification of the household problem and of the functioning of the financial sector. Following the insights gained in Section 2, we purposely build a model that allows for both complementarities across agents and for an accumulation variable that dampens current purchases. Our goal will be to use this model to explore how, depending on the mechanisms allowed, the frequency range targeted in the estimation, and the estimation method, inferences regarding the relative importance of endogenous and exogenous forces in explaining the data can differ.

3.1 The Model

We begin by presenting the household setup and the determination of lending rates in the banking sector. In Appendix I, we present all the details of the model.

3.1.1 The Determination of Household Consumption Decisions and Risk Premiums on Loans

Consider an environment with a mass one continuum of identical households, each composed of a mass one continuum of identical members (workers) and a household head. Each household, through its members, purchases consumption services on the market at nominal price P_t . Letting variables with h subscripts denote variables for household h , and those without denote aggregate variables, household h 's preferences are given by

$$\mathbb{E}_0 \sum_t \beta^t \xi_{t-1} [U(C_{ht} - \gamma C_{t-1}) + \nu(1 - e_{ht})]$$

where \mathbb{E}_t is the expectation operator, C_{ht} is the consumption services purchased by the household at time t , C_t is aggregate consumption services (so that there is external habit formation), e_{ht} is the fraction of employed household members, $U(\cdot)$ and $\nu(\cdot)$ are standard concave utility functions, and ξ_t represents an exogenous preference shifter. The worker-members of the household look for jobs and are ready to accept employment as long as the real wage $\frac{W_t}{P_t}$ is no smaller than the reservation value of their time to the household. In addition to purchasing consumption services on the market, the household also invests in durable goods. These durable goods could represent, for example, clothes, furniture, cars or houses. To avoid issues of indivisibility, we assume that households do not directly consume the services from their durable goods, but instead rent their durable goods to firms, who use them to produce and sell consumption services back to the households. This is why we specify

utility over consumption services instead of consumption goods. A households holding of durable goods is denoted by X_t ,⁴⁸ the nominal rental rate on durable goods is denoted by R_t^X , and the nominal price of durables is P_t^X . Durable goods accumulate according to

$$X_t = (1 - \delta)X_{t-1} + I_t \quad (5)$$

where I_t is the total amount of durable goods purchased by the household at time t and δ is the depreciation rate.

In order for the financial market to play a role, we assume that members of the household need to place orders with firms at the beginning of each period before they have received any wage or rental payments. For this reason, household members take out loans at the beginning of each period, with the plan to pay them back at the beginning of next period after they have received their income payments. The key market imperfection we introduce is that the financial link between a household and its members is imperfect, in the sense that if a household member cannot pay back its loan, it may be costly for banks to recover the loan amount from the household. In particular, if a household member is unable to pay back a loan—which will be the case when she cannot find a job—then with exogenous probability ϕ the bank can pay a cost $\Phi < 1$ (per unit of the loan) to recover the funds from the household, while with probability $1 - \phi$ it is prohibitively costly to pursue the household, in which case the bank is forced to accept a default. The variables ϕ and Φ will therefore control the degree of financial market imperfection, with $\phi = 1$ and $\Phi = 0$ creating a frictionless credit market. As we show in Appendix I, this financial market imperfection yields a budget constraint for household h of the form

$$D_{ht+1} = [\bar{e}_t + (1 - \bar{e}_t)\phi](1 + r_t)(D_{ht} + P_t C_{ht} + P_t^X I_{ht}^X) - (1 + i_t)Y_{ht}, \quad (6)$$

where D_{ht} is debt owed by the household when entering period t , r_t is the nominal interest rate charged on one-period loans by the banking sector, i_t is the risk-free interest rate banks pay on deposits, \bar{e}_t is the aggregate employment rate, and $Y_{ht} \equiv e_t W_t + R_t^X X_{ht} + \Pi_t$ is total nominal household income, where Π is total firm profits (of which each household receives an equal share). Note that the effective borrowing rate for the household is $[\bar{e}_t + (1 - \bar{e}_t)\phi](1 + r_t)$, which is the loan rate times the probability that the household will have to pay the loan back. The household has an Euler equation associated with the optimal choice of consumption services given by

$$U'(C_t - \gamma C_{t-1}) = \beta \frac{\xi_t}{\xi_{t-1}} [\bar{e}_t + (1 - \bar{e}_t)\phi](1 + r_t) \mathbb{E}_t \left[\frac{U'(C_{t+1} - \gamma C_t)}{1 + \pi_{t+1}} \right], \quad (7)$$

⁴⁸Since all households will be identical, for notational simplicity we will often drop h subscripts where no confusion should arise.

where $\pi_{t+1} \equiv P_{t+1}/P_t - 1$ is the inflation rate from period t to $t + 1$. If $\phi = 1$ then we have a standard Euler equation where the marginal rate of substitution in consumption across periods is set equal to the real rate of interest faced by households. When $\phi < 1$, Equation (7) reflects the fact that the household knows that it may default on some fraction of its loans. The household will also have an Euler equation associated with the purchase of durables given by⁴⁹

$$U'(C_t - \gamma C_{t-1}) = \beta \frac{\xi_t}{\xi_{t-1}} \times \mathbb{E}_t \left[\frac{U'(C_{t+1} - \gamma C_t)}{(1 + \pi_{t+1}) P_t^X} \left\{ \frac{R_{t+1}^X (1 + i_{t+1})}{[\bar{e}_{t+1} + (1 - \bar{e}_{t+1}) \phi] (1 + r_{t+1})} + (1 - \delta) P_{t+1}^X \right\} \right]. \quad (8)$$

Equation (8), when combined with (7), can be interpreted as an arbitrage condition that the return to holding a durable good must satisfy.⁵⁰

The central bank sets the nominal interest rate for safe debt i_t , which is also the bank deposit rate, and competition will lead r_t to be such that banks make zero profits. The increased probability of loan defaults when unemployment is high will cause banks to compensate by increasing their margins over the risk-free interest rate. In particular, as shown in Appendix I, in a zero profit equilibrium we will have

$$1 + r_t = (1 + i_t) \frac{1 + (1 - \bar{e}_t) \phi \Phi}{\bar{e}_t + (1 - \bar{e}_t) \phi}. \quad (9)$$

We refer to $r_t^p \equiv \frac{1+r_t}{1+i_t} - 1$ as the risk premium. Note here that the risk premium does not represent an excess return but simply corresponds to the amount needed to compensate for expected default. Using (9) to replace r_t in (7) we get

$$U'(C_t - \gamma C_{t-1}) = \beta \frac{\xi_t}{\xi_{t-1}} [1 + (1 - \bar{e}_t) \phi \Phi] (1 + i_t) \mathbb{E}_t \left[\frac{U'(C_{t+1} - \gamma C_t)}{1 + \pi_{t+1}} \right]. \quad (10)$$

From Equations (7) and (9) we see how unemployment risk, financial conditions, and purchasing decisions all become interrelated due to the fact that loans to households occasionally involve default and the rate of default $(1 - \bar{e}_t) \phi$ is endogenously determined. Equation (9) implies that as unemployment increases so does the risk premium on loans, while equation (7) indicates that a higher risk premium on loans will lead households to delay their purchases. Equation (10) gathers these two forces together indicating that higher unemployment (i.e.,

⁴⁹Note that the household treats the purchase of durable goods as it would any other asset.

⁵⁰In the case where $\phi = 1$ and $i_t = r_t$, equation (8) reduces to the standard asset-pricing condition

$$U'(C_t - \gamma C_{t-1}) = \beta \frac{\xi_t}{\xi_{t-1}} \mathbb{E}_t \left[\frac{U'(C_{t+1} - \gamma C_t)}{(1 + \pi_{t+1}) P_t^X} \{R_{t+1}^X + (1 - \delta) P_{t+1}^X\} \right],$$

a fall in \bar{e}_t) will lead to a delay of consumption. This is the source of strategic complementarity in this model: if an agent decides to purchase more goods, this will tend to lower the unemployment rate, which in turn allows banks to charge a lower borrowing rate, thereby stimulating other agents to purchase more. Note that this effect runs through \bar{e} , so it is external to the household, as was the case in Section 2.

The household can dictate the reservation wage to household members which implies that workers accept all jobs for which the real wage satisfies

$$\frac{W_t}{P_t} \geq \frac{v'(1-e_t)}{U'(C_t - \gamma C_{t-1})} \frac{[e_t + (1-e_t)\phi](1+r_t)}{(1+i_t)} + \frac{1+r_t}{1+i_t}(1-\phi) \left(C_t + \frac{P_t^X}{P_t} I_t^X \right). \quad (11)$$

In equilibrium, firms will offer wages that satisfy (11) with equality. Note that if $\phi = 1$ and $r_t = i_t$, then (11) implies accepting all wage offers where the real wage is higher than the marginal rate of substitution between leisure and consumption.⁵¹

3.1.2 Firms

There are two types of firms in the model: final good firms, and intermediate service firms. The final good sector is competitive and provides consumption services to households by buying a set of differentiated intermediate services, denoted C_{kt} , from the unit mass of intermediate service firms, and combining them in a standard way according to a Dixit-Stiglitz aggregator.⁵² Intermediate service producers, meanwhile, are monopolistically competitive in the supply of differentiated consumption services, and take the demand for such services from final good firms as given. These intermediate firms produce consumption services using durable goods, which can either be rented from households or produced anew according to the production technology $F(e_{kt}, \theta_t)$, where e_{kt} is labor hired by firm k , and θ_t is exogenous productivity. We assume that newly produced durable goods can immediately produce consumption services, and that the total output of consumption services is simply linear in the total quantity of durables; that is, we assume $C_{kt} = s[X_{kt} + F(e_{kt}, \theta_t)]$, $s > 0$, where X_{kt} is the amount of durable goods rented by firm k from households.⁵³ Moreover, after using newly produced durables to produce consumption services, the firm can sell the remaining

⁵¹Note that the extra term on the right-hand side of (11) reflects the fact that, by accepting a job, the household loses the possibility of being allowed to default on that worker's loan.

⁵²See Appendix I for further details.

⁵³Note that in the production of consumption services, the capital stock variable X_t and the employment level e_t enter in a separable manner as opposed to entering in the more common form of complements. We adopt this formulation for two reasons. First, from a theoretical point of view, this formulation implies that when X_t is high it necessarily tends to depress the desired level of employment since utility is concave in services. In contrast, if e_t and X_t were assumed to be complements, then when X_t is high it becomes ambiguous whether it depresses or favors current employment. Since from section 2, we know that having an accumulation variable that has a negative effect of new purchases is important for allowing a model to

un-depreciated amount to households at the market price P_t^X .⁵⁴ To increase generality, we will assume that the depreciation of new durable goods is given by $1 - \psi \geq \delta$, which allows new durable goods to potentially depreciate faster in the first period than in subsequent periods. This extension allow for the possibility of interpreting that a fraction of the new goods depreciate fully within the first period (i.e., are non-durable) and the remaining fraction depreciates at the standard durables rate δ .

Since intermediate service producers have a choice between two ways to obtain durables for use in production (i.e., renting existing durables and producing new ones using labor), if both ways are to be used in equilibrium then the net marginal cost of an additional unit of durables must be equalized across the two methods. For a new unit of durables, this marginal cost is equal to the wage cost per additional unit produced, less the value of the un-depreciated portion sold to households. For rented durables, this marginal cost is simply the rental rate. Thus, we must have

$$R_t^X = \frac{W_t}{F_e(e_{kt}, \theta_t)} - \psi P_t^X \quad (12)$$

Following the New Keynesian literature, we assume that the market for intermediate services is subject to sticky prices *à la* Calvo [1983]. This yields a standard New Keynesian Phillips curve, though as will become clear shortly, for our purposes we will not need to derive it.

3.1.3 The Central Bank and Equilibrium Outcomes

To close the model, we still need to specify how the central bank determines the risk-free interest rate. In order to keep the model tractable, we restrict attention to a monetary policy rule governed by only one parameter φ_e , which allows the central bank to only imperfectly control its objective of stabilizing inflation and employment. To this end, we assume that the central bank sets the nominal interest rate to induce an expected real interest rate that rises and falls with expected employment. By allowing the central bank to adjust only to expected variables, it can only imperfectly stabilize the economy. To be more precise, we assume that the central bank sets the nominal interest rate according to a rule of the form

$$1 + i_t \approx \Theta \mathbb{E}_t [e_{t+1}^{\varphi_e} (1 + \pi_{t+1})] \quad (13)$$

endogenously produce hump shaped spectral densities, this formulation is as easy way of permitting such a possibility. Second, from a conceptual point of view, when thinking of X_t as household capital, it appears more appropriate to assume that it does not directly increase the marginal product of labor in the market sector.

⁵⁴Note that in the market for durable goods the intermediate firms are price takers, while they are price setters in the consumption service market.

where φ_e controls the extent to which the central bank tries to stabilize inflation and employment, and Θ controls the steady state level of i . As we show shortly, the attractive feature of this monetary policy rule is that it gives the equilibrium equations the block recursive structure.⁵⁵

The equilibrium outcomes for this model are given by a set of nine equations determining the two aggregate quantities $\{C_t, I_t\}$, the employment level e_t , the relative prices $\left\{i_t, r_t, \frac{R_t^X}{P_t}, \frac{P_t^X}{P_t}, \frac{W_t}{P_t}\right\}$ and the inflation rate π_{t+1} .⁵⁶ Using the monetary policy rule (13), as shown in Appendix I the equilibrium equations have a convenient block-recursive structure whereby the variables e_t , X_{t+1} and r_t^p can be solved for first using the equations⁵⁷

$$1 + r_t^p = \frac{1 + (1 - e_t) \phi \Phi}{e_t + (1 - e_t) \phi} \quad (14)$$

$$X_{t+1} = (1 - \delta) X_t + \psi F(e_t, \theta_t), \quad (15)$$

$$\begin{aligned} U' (s(X_t + F(e_t, \theta_t)) - \gamma s(X_{t-1} + F(e_{t-1}, \theta_{t-1}))) &= \beta \Theta \frac{\xi_t}{\xi_{t-1}} [e_t + (1 - e_t) \phi] (1 + r_t^p) \\ &\times \mathbb{E}_t [U' (s(X_{t+1} + F(e_{t+1}, \theta_{t+1})) - \gamma s(X_t + F(e_t, \theta_t))) e_{t+1}^{\varphi_e}] . \end{aligned} \quad (16)$$

These three equations will provide our basis for exploring whether such a model can capture the spectral properties for hours and the risk premium that we documented in Section 1.

3.1.4 Shocks

There are two exogenous forces in the model that affect the determination of hours and the risk premium: the preference shifter ξ_t and the level of technology θ_t . Note that the preference shifter could alternatively be interpreted as a monetary shock, since allowing for a monetary shock gives rise to the exact same equations for the determination of hours and the risk premium. We use the more abstract interpretation of it as a preference shifter, since it allows for several different interpretations. In our estimation, we will focus on the

⁵⁵The precise form of the Taylor rule we use to obtain the block recursive property is

$$1 + i_t = \Theta \mathbb{E}_t \left[e_{t+1}^{\varphi_e} \frac{U' (C_{t+1} - \gamma C_t)}{\mathbb{E}_t \left[\frac{U' (C_{t+1} - \gamma C_t)}{1 + \pi_{t+1}} \right]} \right].$$

This deviates slightly from (13) due to Jensen's inequality, which is why (13) is expressed with an \approx symbol.

⁵⁶The relevant equilibrium equations correspond to (7), (8), (9), (11) with equality, (12), the aggregate consumption equation $C_t = s[X_t + F(e_t, \theta_t)]$, the accumulation equation (5), the Phillips curve, and the Taylor rule.

⁵⁷Given the values of e_t , X_{t+1} and r_t^p obtained from this system, the remaining equations simultaneously determine the remaining variables $\left\{C_t, \frac{R_t^X}{P_t}, \frac{P_t^X}{P_t}, \frac{W_t}{P_t}, \pi_t\right\}$. Note that, as we do not consider the implications of the model for inflation, we will not need to explicitly derive the optimal pricing behavior of firms.

case where technology is constant and the only stochastic driving force is the preference shifter, so as to see whether such a minimalist exogenous structure, once embedded in an environment with a potentially rich endogenous propagation mechanism, can capture the spectral properties of the data.

3.1.5 Limit Cycles and Arbitrage

As will become clear when we present our estimation results below, for certain parameterizations our model will feature limit cycles. In addition to the criticism about predictability noted in Section 2, another common criticism of earlier models featuring limit cycles is that these cycles would be subject to arbitrage forces that would tend to erase them. We wish to emphasize that, while this criticism may have been valid for models that did not feature rationally optimizing forward-looking agents (e.g., Hicks [1950] and Goodwin [1951]), it does not apply to our model, since agents are rationally optimizing and forward-looking. For example, even though there may be predictable cycles in the price of durable goods, so that agents could potentially borrow to purchase durables when the price is low and then sell them for a profit in the future when the price is high, the return they would earn from this strategy in equilibrium is necessarily less than the cost of servicing the associated debt, and would therefore not be optimal.

3.1.6 Equilibrium narrative

Before proceeding to the estimation of this model, it is useful to briefly discuss how agents in this environment would perceive macroeconomic fluctuations and why their behavior could generate endogenous cyclical outcomes. To this end, suppose the economy has been in recession for a while. Banks in this environment are reluctant to lend to people because unemployment is high. This leads them to charge a high premium on loans so as to cover expected defaults. However, if the recession has been going on for long enough, household capital will have depreciated significantly, causing the marginal utility of new purchases to be high. In fact, the marginal utility will eventually become sufficiently high that an agent will be ready to increase their borrowing in order to make new purchases even if the risk premium on borrowing remains high. At that point the tide starts to turn. As some individuals start purchasing more, this increases demand and reduces unemployment. Banks respond to this reduced unemployment risk by decreasing the risk premium they charge on loans which favors more purchasing by households. As a result a boom period emerges. The boom will continue until at some point households will have accumulated sufficient household capital such that the marginal utility of a new purchase becomes low. The low marginal utility induces household to reduce their borrowing even if the risk premium on loans charged by

bank is low. This is how a recession would endogenously start. Now, agents in this economy understand this boom-and-bust phenomenon, but this does not stop them from taking part, even if, individually, they would prefer stable consumption. In fact, the knowledge of an endogenous boom-bust cycle further pushes agents to go along with the cycle, knowing that in a boom it is a good time to accumulate—even over-accumulate—since risk premia on borrowing are currently low and a recession is known to be coming. Similarly, during a downturn, agents prefer to hold back from making purchases, since they know that financial conditions are temporarily bad. Note that this mechanism could be self sustaining, which would be the case if it reflected a limit cycle. Alternatively, the mechanism could dampen over time, requiring exogenous shocks in order to keep it going.

The addition of shocks to this model eliminates the perfect predictability of the endogenous cyclical forces, but does not change the endogenous mechanisms. Agents would no longer know exactly when a boom or a bust would finish, but they would know that the system is more susceptible to a bust if it has been booming for a while, and would know that the economy is more susceptible to a boom if it has been in a prolonged downturn.⁵⁸

It is important to emphasize that the above description of the economy does not involve indeterminacy or self full-filling beliefs.⁵⁹ A boom does not arise in this set up simply because people believe it will arise. It arises because the main capital stock has been depleted sufficiently, and this pushes people to take risks and borrow more. Changes in expectations are not the driving force; rather, they act more like an amplification mechanism. It should also be emphasized that the narrative behind this model is not especially novel as it is meant to capture many elements common to descriptions of the financial cycle as, for example, that proposed by Minsky [1986].

3.2 Functional Forms and Estimation

To bring our model to the data, it remains to specify functional forms and the stochastic process. We assume that period utility is CRRA and given by $U(c) = (c^{1-\omega} - 1)/(1 - \omega)$, while the production function is given by $F(e, \theta) = \theta e^\alpha$. As noted above, we take technology θ as constant, and for the purposes of estimation we normalize it to 1.⁶⁰ We also normalize $s = 1$. As shown above, we may reduce our system to three equations in the variables X , e , and r^p . Linearizing the two dynamic equations (15)-(16) with respect to $\log(X)$, $\log(e)$, r^p ,

⁵⁸Note that these implied patterns are consistent with the properties of the conditional probability of recession described in Section 1.

⁵⁹In Beaudry, Galizia, and Portier [2016b], it is shown that this type of model does not in general allow for self-fulfilling fluctuations unless one focuses on an environment where the complementarities are sufficiently strong to allow for multiple steady states.

⁶⁰Allowing for deterministic growth in the model does not change any results.

and $\mu_t \equiv -\Delta \log(\xi_t)$, we obtain equations of the form⁶¹

$$\widehat{X}_{t+1}^* = (1 - \delta) \widehat{X}_t^* + \psi \widehat{e}_t, \quad (17)$$

$$\widehat{e}_t = \alpha_1 \widehat{X}_t^* + \alpha_2 \widehat{e}_{t-1} + \alpha_3 \widehat{e}_{t+1} - \alpha_4 \widehat{r}_t^p + \alpha_4 \mu_t, \quad (18)$$

$$\widehat{r}_t^p = \varrho_1 \widehat{e}_t, \quad (19)$$

where \widehat{X} and \widehat{e} are log-deviations from steady state, \widehat{r}^p is the deviation in levels, $\widehat{X}^* \equiv \psi \widehat{X}/(\alpha\delta)$, and the α_j 's are functions of the structural parameters with $\alpha_1 < 0$ and $\alpha_2, \alpha_3, \alpha_4 > 0$. Note that, since the risk premium is a negative function of employment ($\varrho_1 < 0$), the above system has a structure similar to the model of Section 2. We assume that μ_t follows a stationary AR(1) process $\mu_t = \rho \mu_{t-1} + \epsilon_t$, where ϵ_t is a Gaussian white noise with variance σ^2 .

We estimate four different versions of our three-equation model. In all versions we use the dynamic equations in their linearized forms (17) and (18). In one version, which we refer to as the linear risk premium (RP) model, we use as the third equation the static risk premium equation (14) in its (log-)linearized form as given by (19). In another version, which we refer to as the no friction model, we assume that all household members receive the backing of the household (i.e., $\phi = 1$) and recovery from the household is costless (i.e., $\Phi = 0$), so that the risk premium is always zero, and thus there is no complementarity. This estimation will help to illustrate the importance of the complementarity in allowing our model to match the key features of the data. In a third version, we shut down both the complementarity channel (by setting $\phi = 1, \Phi = 0$) and the accumulation channel (by setting $\psi = 0$, so that $X_t = 0$ for all t). We refer to this as the canonical model, since it corresponds closely to the canonical New Keynesian model with habit.

In the final version of the model, which we refer to as the non-linear RP model, we allow the risk premium to be a non-linear function of (log-)employment.⁶² As discussed in Section 2, allowing for non-linearity in the strength of the complementarity will allow us to expand the parameter space to include situations where there may be local instability and limit cycles. In particular, to allow for a greater set of possibilities, for this case we let the debt-backing probability be a function of the employment rate, i.e., $\phi_t = \phi(e_t)$, and approximate (14) to the third order as

$$\widehat{r}_t^p = \varrho_1 \widehat{e}_t + \varrho_2 \widehat{e}_t^2 + \varrho_3 \widehat{e}_t^3, \quad (20)$$

where the coefficients ϱ_1, ϱ_2 and ϱ_3 are functions of the model parameters and of the three first

⁶¹See Appendix J for details.

⁶²We could also allow for non-linearities in both (17) and (18). However, we chose to allow for non-linearities in the risk premium so as to make the analysis more transparent, since it allows us to refer to results from Section 2.

derivatives of ϕ (ϕ' , ϕ'' and ϕ''') evaluated at the steady state.⁶³ To facilitate comparison with the linear RP model and to avoid issues of identification, we will fix $\phi' = 0$ in the estimation. Note that we take a third-order approximation since, for the model to be capable of producing (attractive) limit cycles, we will typically need the third-order coefficient ϱ_3 to be sufficiently positive (see Kuznetsov [1998] for details).

We estimate this model using the indirect inference method of Gourieroux, Monfort, and Renault [1993], where for each parameterization the model is solved by a first-order (linear RP, canonical, and no friction models) or third-order (non-linear RP model) perturbation method.⁶⁴ In the non-linear RP model, the solution and estimation are somewhat involved, as it allows for the possibility of a locally unstable steady state and limit cycles in a stochastic model with forward-looking agents. To our knowledge, such an exercise is novel.⁶⁵ For each version of the model, the parameters are chosen so as to minimize its distance to a set of features of the data that we have already emphasized. We focus on three sets of observations. The first set, which is used for all four models, corresponds to the spectral density of hours worked per capita (as shown in panel (b) of Figure 2).⁶⁶ The second set, which is used for both the linear and non-linear RP models, adds the spectral density of the risk premium (as shown in panel (c) of Figure 6).⁶⁷ For these first two sets, we aim to fit the point estimates of the spectral densities (using the non-detrended data) at periodicities between 2 and 50 quarters.⁶⁸ The last set of observations, which is used only for the non-linear RP model, is a set of five additional moments of the data: the correlation between hours and the risk premium, as well as the skewness and kurtosis of each of these two variables. Each of the data moments in this last set are obtained after first detrending the data series using a high-pass filter that removes fluctuations longer than 50 quarters. This is in line with our objective of using the current model to explain macroeconomic fluctuations arising at periodicities ranging from 2-50 quarters.

We calibrate three parameters for all four models: the depreciation rate is set to $\delta = 0.05$

⁶³See Appendix J.1.

⁶⁴Details of the solution and estimation are given in Section J of the Appendix.

⁶⁵For details about the solution method, see Galizia [2018].

⁶⁶Note that, in our model, since employed workers each work the same number of hours, hours is simply proportional to employment.

⁶⁷We use the BAA Corporate Bond spread series, rather than a series that directly measures interest rates faced by households, as the former is available going back to 1954, while we have only found quarterly measures of the latter that go back to the early 1970s. The coherences between the bond spread and those consumer spread series over the 32-50 quarter range—the range straddling the spectral peaks—are around 0.8, suggesting that fluctuations in the bond spread may be a reasonable proxy for fluctuations in consumer spreads. As a check on our results, we re-estimated the model also including a measurement error process on the risk premium, and found little change in the results (available upon request).

⁶⁸See Christiano and Vigfusson [2003], Qu and Tkachenko [2012] and Sala [2015] for previous work estimating DSGE type models in the frequency domain.

in order to match the average depreciation of houses and durable goods, the elasticity of the production function with respect to employment is set to $\alpha = 2/3$, and the monetary policy scale variable Θ is set so as to yield a steady state unemployment rate of 0.0583 (the average over our sample period). Depending on the particular model, we then estimate as many as ten parameters. For the two risk premium models, we estimate ω , γ , ψ , φ_e , ϕ , Φ , ρ , and σ , plus for the non-linear RP model, ϱ_2 and ϱ_3 (from which we can back out ϱ_1 , ϕ'' and ϕ'''). Because of the structure of the no friction and canonical models, and the fact that they are estimated only on the hours spectrum, to avoid identification issues we fix a number of the parameters in estimation. In particular, in addition to the parameters restricted by definition (ϕ and Φ , plus ψ in the canonical model), we fix ω and φ_e , plus ψ in the no friction model and γ in the canonical model, at the corresponding estimated values from the linear RP model.

[Figure 7 about here.]

Figures 7-9 illustrate the fit of the estimated model along the targeted dimensions for the four different versions of the model. Consider first panel (b) of Figures 7 and 8, which show the estimated spectral densities of hours and the risk premium, respectively, for the linear RP model. While our parsimonious model does not capture all the bumps and wiggles in the spectrum of hours in Figure 7, it nonetheless fits the overall pattern nicely, most importantly the peak in the spectrum near 40 quarters, though that peak is noticeably flatter than the one observed in the data. Meanwhile, while the model does not capture the smaller peak in the spectral density of the risk premium in Figure 8 observed around 21 quarters, it again fits reasonably well the overall hump-shaped pattern with a peak close to 40 quarters, though again that peak is flatter than the one observed in the data. Consider next panel (c) of Figure 7, which shows the fit of the hours spectrum for the no friction model, obtained by re-estimating the model after shutting down the complementarity channel. Despite the fact that for this model the estimation is no longer constrained to simultaneously match the risk premium spectrum, the fit of the hours spectrum is significantly worse than in the linear RP model, and in particular the no friction model is unable to replicate the peak in the spectrum of hours near 40 quarters. Thus, evidently the “complementarities” part of our accumulation-with-complementarities mechanism is of fundamental importance in allowing our model to capture the salient business cycle features of the data. As shown in panel (d) of Figure 7, the canonical model, which is obtained by re-estimating the model with both the complementarity and accumulation channels shut down, tells a similar story. Lastly, consider panel (a) of Figures 7 and 8, which show the results for the non-linear RP model. Relative to the linear RP model, the fits of both the spectra are noticeably better,

and in particular while both models generate peaks near 40 quarters, unlike in the linear RP model the peaks in the non-linear RP model are almost as pronounced as they are in the data.

[Figure 8 about here.]

[Figure 9 about here.]

The parameter estimates for the four models are presented in Table 7, along with bootstrap standard error estimates in parentheses.⁶⁹ Comparing columns (a) and (b) of the Table, we see that the non-shock parameter estimates are broadly similar between our two preferred models (the linear and non-linear RP models). Further, the estimated habit parameter (γ) of 0.53-0.59 is well in line with the values commonly found in the literature. The two parameter estimates that may be considered somewhat low relative to the literature are our estimates of the CRRA parameter (ω) of 0.24-0.3, implying a relatively high intertemporal elasticity of substitution of around 3-4, and of the Taylor rule elasticity (φ_e) of 0.042-0.047, which implies that a one-percentage-point increase in expected employment is associated with an increase in the annualized policy rate of about 17-19 basis points. The first of these parameters implies a strong response of consumption to the interest rate faced by households. Since the household interest rate is the sum of the pro-cyclical policy rate and the counter-cyclical risk premium, the second of these parameters tends to favor counter-cyclicity in the overall household interest rate. Taken together, these parameters help to increase the effect of the complementarity in the model—which, as we have shown, is helpful in matching key features of the data—by both increasing the degree of household interest rate counter-cyclicity and increasing the household’s consumption response to that counter-cyclicity.

[Table 7 about here.]

The most interesting finding regarding our parameter estimates revolves around the shock process parameters ρ and σ . In particular, as one moves leftward beginning from the canonical model (column (d) of Table 7) to the no friction model, the linear RP model, and finally to the non-linear RP model, the estimated persistence and standard deviation of the shock process both monotonically decrease. As a result, the unconditional standard deviation of the shock process (reported in the bottom row of the table) also monotonically decreases. Thus, sequentially allowing for accumulation, complementarity, and then non-linearity in the complementarity lets the model not only better fit the data (as discussed above), but do so

⁶⁹Standard errors were obtained by simulating $N = 200$ data sets from the corresponding model (using the point estimates for the parameters), and then re-estimating the model on each simulated data set.

with less reliance on exogenous stochastic forces. The upshot is that in the non-linear RP model the autocorrelation of the shock is effectively zero, so that almost all of the model's dynamics are due to endogenous forces. Further, the largest factor in reducing reliance on exogenous shocks comes from the introduction of the complementarity (i.e., moving from (c) to (b)), which is associated with a fall of more than 90% in the unconditional standard deviation of the estimated shock process. These results suggest that our accumulation-with-complementarities mechanism may be a promising avenue for those seeking to introduce stronger internal propagation and a lower reliance on exogenous shocks into business cycle models.

The strength and form of the internal propagation in each model can be seen more clearly in Table 8, which reports the eigenvalues of the first-order approximation to the solved model around the non-stochastic steady state, along with their moduli. In the canonical and no friction models, these eigenvalues are real, positive, and stable, with values given by 0.5617 and 0.5202, respectively. Thus, these models are characterized by monotonic convergence—which may explain their inability to capture the hump in the spectral densities near 40 quarters—and a relatively low degree of endogenous persistence. For example, in the canonical model, deviations from steady state have an endogenous half-life of slightly more than one quarter. For the no friction model, the larger eigenvalue is close to 0.95, but it can be verified that the smaller eigenvalue is the one driving most of the variance in hours, and it too is associated with an endogenous half-life of just over one quarter. In contrast, the linear RP model has a pair of complex eigenvalues, which allows it to generate spectral peaks, and these eigenvalues both have a modulus of 0.93, which is suggestive of a relatively larger degree of endogenous persistence (endogenous half-life of nine quarters). Finally, the non-linear RP model also has a pair of complex eigenvalues, but with a modulus exceeding one (1.12).⁷⁰ That is, when given the option, the data appears to favor a configuration featuring local instability and limit cycles, which generates significant internal propagation and a correspondingly lower reliance on exogenous processes to drive fluctuations. This suggests a more general point, which is that by ruling out parameterizations that produce local instability and limit cycles—as is implicitly done by standard solution methods (e.g., standard perturbation methods implemented in DYNARE.)—one may be significantly biasing the results of estimation.

[Table 8 about here.]

⁷⁰Note that we constrained the parameter space to allow only for parameterizations producing a determinate solution; that is, where the third eigenvalue (not shown in Table 8) of the unsolved system is unstable. However, we found no indication that this constraint was binding for the linear or non-linear RP models, suggesting that the data do not favor a configuration yielding indeterminacy.

To illustrate the deterministic mechanisms implied by the parameter estimates in the non-linear RP model, Figure 10 reports results obtained when feeding in a constant value of $\mu_t = 0$ for the exogenous process.⁷¹ Panel (a) of Figure 10 plots a simulated 270-quarter sample⁷² of hours generated from this deterministic version of the model. Two key properties should be noted. First, the estimated parameters produce endogenous cyclical behavior, with cycles of a reasonable length (around 38 quarters). This is consistent with Table 8, which indicates that the steady state is unstable and features two complex eigenvalues. The difficulty that some earlier models had in generating cycles of quantitatively reasonable lengths may have been one of the factors leading to limited interest in using a limit cycle framework to understand business cycles. However, as this exercise demonstrates, reasonable-length endogenous cycles can be generated in our framework relatively easily, precisely because the model possesses the two key features we highlighted in the previous section: complementarities, and an accumulation variable that affects current demand negatively. Second, notwithstanding the reasonable cycle length, it is clear when comparing the simulated data in panel (a) of Figure 10 to actual economic data that the fluctuations in the deterministic model are far too regular. These two properties of the deterministic model—i.e., a highly regular 38-quarter cycle—can also be seen clearly in the frequency domain. Panel (b) of Figure 10 plots the spectral density of hours for the deterministic model (gray line), along with the spectral density for the data (black line) for comparison. This spectral density exhibits an extremely large peak—characteristic of a highly regular cycle—at the 38-quarter periodicity,⁷³ while the spectral density of the data is much flatter.

[Figure 10 about here.]

[Figure 11 about here.]

Re-introducing the estimated shocks into the non-linear RP model, we see a markedly different picture in both the time and frequency domains. Figure 11 plots an arbitrary 270-quarter sample of log-hours generated from the full stochastic model. While clear cyclical patterns are evident, it is immediately obvious that the inclusion of shocks—even the

⁷¹Note that, as typically done when computing transitional dynamics, the model was solved (and simulated) using the estimated parameters presented in column (a) of Table 7, and in particular we did not first re-solve the model with $\sigma = 0$. Thus, agents in this deterministic simulation implicitly behave as though they live in the stochastic world. As a result, any differences between the deterministic and stochastic results are due exclusively to differences in the realized sequence of shocks, rather than differences in, say, agents' beliefs about the underlying data-generating process.

⁷²This is equal to the length of the sample period of the data.

⁷³The deterministic model spectral density also contains smaller peaks at integer multiples of the frequency of the main cycle (i.e., at around $19 = 38/2$ quarters, $12.67 = 38/3$ quarters, etc.). Such secondary peaks arise when the data exhibits a regular but not perfectly sinusoidal cycle, as is clearly the case in panel (a) of the Figure.

(essentially) i.i.d. shocks that are present in our model—results in fluctuations that are significantly less regular than those generated in the deterministic model, appearing qualitatively quite similar to the fluctuations found in actual data. This is confirmed by the hours spectral density (panel (a) of Figure 7), which matches the data quite well. In particular, the spectral density of the stochastic model includes a distinct peak close to 40 quarters, suggesting some degree of regularity at that periodicity, but without the exaggerated peak observed at this point in the deterministic model.

It should be emphasized that the exogenous shock process in the non-linear RP model primarily acts to accelerate and decelerate the endogenous cyclical dynamics, causing significant random fluctuations in the length of the cycle, while only modestly affecting its amplitude. In fact, somewhat counter-intuitively, when the shock is shut down (as in Figure 10), the variance of log-hours actually *increases* relative to the full stochastic case (the variance of log-hours in the stochastic case is 6.88, while in the deterministic case it is 7.48).⁷⁴ The role of complementarities in the model, however, is extremely important: if we shut down the endogenous risk premium (i.e, set $\phi = 1$ and $\Phi = \varrho_2 = \varrho_3 = 0$), but keep all other parameters at their estimated levels, the variance of log-hours in the model is less than 0.1 (compared with 6.88 with the complementarity). Thus, without the complementarities to amplify them, the small *i.i.d.* disturbances can only generate a tiny amount of volatility in hours.

3.3 Sensitivity of Results to Target Frequency Range

As we noted in Section 1, the spectral densities of several key macroeconomic variables exhibit peaks around 38 quarters, declining from there before reaching a local minimum at around 50 quarters, then increasing again beyond that point. This was our motivation for choosing to target the 2-50 quarter range in our estimation. We turn now to evaluating how this particular choice affects our results. In particular, since the lower end of the standard business cycle range in the literature is 6 quarters, we consider what happens if we estimate the parameters of the model restricting attention to periodicities from 6 to 50 quarters. We also consider what happens if we restrict attention further to the range beyond the traditional upper bound of 32 quarters (i.e., restricting to periodicities between 32 and 50 quarters),

⁷⁴Note that, since we have simply fed a constant sequence $\mu_t = 0$ of shocks into our model without first resolving it under the assumption that $\sigma = 0$, this phenomenon is not due in any way to rational-expectations effects. The fact that a fall in shock volatility can lead to a rise in the volatility of endogenous variables in a limit cycle model was pointed out in Beaudry, Galizia, and Portier [2016a]. Roughly speaking, because of the non-linear forces at play, shocks that push the system “inside” the limit cycle have more persistent effects than those that push it “outside”. For relatively small shocks, this leads to a decrease in outcome volatility when the shock volatility increases. See Beaudry, Galizia, and Portier [2016a] for a more detailed discussion of these mechanisms in the context of an estimated reduced-form univariate equation.

and we repeat these same exercises using 60 quarters as the upper bound instead of 50. We do these exercises for the non-linear RP model, though the results are of a similar nature for the linear RP model (available upon request). As we will see, the above choices make little difference to our results. As an additional exercise, we also report results when estimating over the 2-100 quarter range, and show that in this case the results are fundamentally changed.

We have also explored the effect of estimating the model only on frequencies between 2-32 quarters. This tends to favor inferring—as found in much of the literature—that persistent exogenous shocks drive business cycles. For example, if we estimate the linear version of the model to match both hours worked and the risk premium (as before), the resulting shock process has an autoregressive parameter of 0.99. In comparison, when estimating this same model over frequencies between 2-50 quarters we have an autoregressive parameter of only 0.14.

The top row of plots in Figure 12 shows the the hours and risk premium spectral densities for period ranges of the form $(x, 50)$, $x = 2, 6, 32$,⁷⁵ while the second row shows results for ranges of the form $(x, 60)$. The corresponding data spectral densities are also plotted (solid black lines) for comparison. In all cases, the effect of changing the lower bound to 6 or 32 quarters, or the upper bound to 60 quarters, has minimal effect on the parameter values, and this translates into a minimal change in the overall fit of the spectral densities: in all cases the non-linear RP model continues to match well the spectral peak near 40 quarters. Further, as shown in the first six rows of Table 9, the eigenvalues associated with the solved system remain complex (indicating cyclicity) and outside the unit circle (indicating the presence of a limit cycle) for all six of these different period ranges.

[Figure 12 about here.]

While increasing the upper bound of the target periodicities from 50 to 60 quarters has little effect on the results, the same is not true if we include even more low-frequency fluctuations. The last row of plots in Figure 12 shows the fit if we extend the upper bound to 100 quarters (note the scale change in the horizontal axis), while the last row of Table 9 reports the associated eigenvalues. The model no longer captures the peak in the hours spectrum near 40 quarters, since this is no longer the dominant feature of the data. Instead, the dominant feature is now the steep increase that occurs beyond 60 quarters, which reflects large but slow-moving forces (such as demographic changes) unrelated to the business cycle. The estimated autocorrelation of the exogenous driving force is now above 0.8, with an unconditional standard deviation of 0.0017 (an order of magnitude greater than

⁷⁵Note that the $(2, 50)$ results simply reproduce the information in Figures 7 and 8.

in the baseline case), suggesting that the dynamics are mainly driven by exogenous forces. This highlights the more general point that if one attempts to simultaneously explain fluctuations in hours data at all frequencies, not just those related to the business cycle, one may likely miss important business cycle features unless one explicitly includes in the model mechanisms to explain the lower frequencies movements. This may help to explain why few modern business cycle models—which are typically implicitly estimated to simultaneously fit all frequencies—generate a peak in the spectrum near 40 quarters.

[Table 9 about here.]

3.4 Is Allowing for Non-Linearities and Local Instability Important?

As noted above, the fit of the estimated linear and non-linear RP models are quite similar, as are a number of the estimated structural parameters. One may then naturally ask: in what ways (if any) are the non-linearities important? We highlighted one way above, which is that allowing for non-linearities expands the parameter space to include parameterizations that produce limit cycles. If the dynamics of the economy do indeed feature limit cycles, by considering only a linear approximation (and the parameterizations that yield a valid rational expectations solution for this approximation), the estimation will necessarily be biased. For example, if the estimated non-linear RP model were the true model, but one employed the linear approximation to it (i.e., the linear RP model) in estimation, one would incorrectly conclude that the steady state is locally stable.

Allowing for non-linearities also has implications for (a)symmetry in the model. For example, in the estimated non-linear RP model, the business cycle is asymmetric, with booms lasting longer on average than recessions. This can be seen most clearly by looking at the deterministic component of the business cycle (i.e., the limit cycle illustrated in panel (a) of Figure 10), which features booms that last 23 quarters (trough to peak) and downturns in employment that last 15.5 quarters (peak to trough). There are also implications for the symmetry of the response of the economy to shocks. To illustrate this, panel (a) of Figure 13 plots the response of hours in the non-linear RP model to a one-standard-deviation positive shock, along with minus the response to a one-standard-deviation negative shock, conditional on initially being at the peak of the cycle. Panel (b) of the Figure plots the same except beginning from the trough of the cycle.⁷⁶ The responses are clearly different depending both on whether the shock is positive or negative and on whether the economy is initially in a boom or a bust. One particularly interesting implication of the Figure is that when

⁷⁶See notes to Figure for further details.

the economy is at the peak of a boom period and is hit by a negative (i.e., contractionary) shock, the magnitude of the response is substantially larger than if it is initially in a bust. This suggests that peak times could be periods where the economy is particularly sensitive to negative shocks (e.g., a financial disruption).

[Figure 13 about here.]

A third important way in which the non-linear and linear RP models differ is in their implications for the relationship between the variance of the shock process and the resulting variance of the endogenous variables. As we already noted above, the standard deviation (s.d.) of hours in the non-linear RP model is in fact smaller than the s.d. of hours obtained when we feed in a constant stream of zeros for the shocks but do not re-solve the model under the assumption that the s.d. of the shocks is zero. This property in fact holds more generally. To illustrate this, in Figure 14 we plot the elasticity of the s.d. of hours with respect to σ (the s.d. of the shock innovation) for each of the linear and non-linear RP models over a range of σ 's. The horizontal axis in the Figure is $\sigma/\hat{\sigma}$, where $\hat{\sigma}$ is the estimated value of the innovation s.d. for the corresponding model (so that $\sigma/\hat{\sigma} = 1$ corresponds to the elasticity at the estimated value of σ). Note that, for the non-linear RP model, we re-solve the model for each different value of the s.d. of the shock innovation, so that the s.d. of hours is obtained from the appropriate rational expectations solution.⁷⁷ As the Figure shows, the elasticity for the linear RP model (light gray) is constant and equal to one, indicating that a one percent increase in σ always leads to a one percent increase in the s.d. of hours. The same is not true for the non-linear RP model (dark gray), in which the elasticity is negative over the range of standard deviations shown (and actually declining over most of that range). For example, near the estimated value of σ , a one percent increase in σ is associated with a 0.45 percent fall in the s.d. of hours. To the extent that this sort of relationship—which is typical of models featuring limit cycles—is present in the real world, it has the intriguing implication that reducing the volatility of economic shocks may not help stabilize the economy.

[Figure 14 about here.]

Conclusion

Why do market economies experience business cycles? There are at least two broad classes of explanations. On the one hand, it could be that market economies are inherently stable

⁷⁷This is unnecessary for the linear RP model, since the rational expectations solution is invariant to the s.d. of the shock in that case.

and that observed booms and busts are mainly due to persistent outside disturbances. On the other hand, it could be that the economy is locally unstable—or close to unstable—in that there are not strong forces that tend to push it towards a stable resting position. Instead, the economy's internal forces may endogenously favor cyclical outcomes, where booms tend to cause busts, and vice versa. The contribution of this paper has been to provide theory and evidence in support of this second view, while simultaneously highlighting the key elements that influence inference on this dimension.

We have emphasized three elements that have led us to infer that business cycles may more likely be generated by strong endogenous forces as opposed to persistent exogenous shocks. However, in concluding, we would like to emphasize the one element we view as most important for this debate, which is the question of what business cycle theory should aim to explain. If one adopts the conventional consensus that business cycle theory should be mainly concerned with movements in macroeconomic aggregates that arise at periodicities between 6 and 32 quarters, then standard models with weak internal propagation mechanisms can offer a reasonable explanation to the data. In contrast, if one agrees that business cycle theory should extend its focus to include slightly lower frequency movements, such as those associated with fluctuations of up to 50 quarters,⁷⁸ then the need to consider strong internal propagation mechanisms becomes much more relevant. In particular, we have documented that many macroeconomic aggregates appear to exhibit a peak in their spectral densities at periodicities between 32 and 50 quarters, and that the implied movements coincide with NBER cycle dating. Moreover, we have emphasized that such a pattern is very unlikely to have been a spurious draw from a AR(1) process. Given that cyclically sensitive variables such as unemployment, capacity utilization, and risk premia all exhibit such a peak, we believe that explaining this cyclical pattern should be a priority in business cycle analysis. If one accepts this point, then inferring that the economy has a strong internal propagation mechanism will, in our opinion, becomes more likely. While we have made the explicit case for this inference using our model, we conjecture that if one uses a different model and successfully explains these observed humps in the spectral density, then most likely the estimated model will require complementarities that generate a strong internal cyclical mechanism.⁷⁹ Whether the resulting model delivers the more extreme form of endogenous

⁷⁸ A popular approach in the estimation of macroeconomic models is to include (almost) all frequencies. For example, this is the case when using likelihood-based methods to fit unfiltered (or, at most, first-differenced) data. In principle this is fine if the model is built to explain both business cycle fluctuations and the relatively large lower-frequency fluctuations associated with, for example, demographic changes. However, if the model is built to understand business cycles but is estimated using all frequencies, then the estimation may favor parameters that help to explain the large lower-frequency movements at the detriment of explaining business cycle movements. This point was illustrated in Section 3.

⁷⁹ An alternative class of explanations would be one were the exogenous force itself is highly cyclical.

propagation as implied by limit cycles, or if instead it favors more damped fluctuations (as we observed when estimating a linear version of our model) will likely depend on model details. Nonetheless, in either case, we conjecture that one's inference regarding the role of shocks in driving business cycles is likely to be greatly diminished if one tries to explain features of the data we emphasized such as the hump shaped spectral density for hours worked and risk premia.

While this is certainly possible, we have not been able to find evidence of cyclicalities in variables commonly considered as drivers of business cycles such as technological change or oil prices.

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Appendix to “Putting the Cycle Back into Business Cycle Analysis”

Note: Much of this material can be placed in an online appendix

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A Data

- U.S. Population: Total Population: All Ages including Armed Forces Overseas, obtained from the FRED database (**POP**) from 1952Q1 to 2015Q2. Quarters from 1947Q1 to 1952Q1 are obtained from linear interpolation of the annual series of National Population obtained from U.S. Census, where the levels have been adjusted so that the two series match in 1952Q1.
- U.S. Total GDP is obtained from the Bureau of Economic Analysis National Income and Product Accounts. Real quantities are computed as nominal quantities (Table 1.1.5) over prices (Table 1.1.4.). Sample is 1947Q1-2015Q2.
- U.S. Non-Farm Business Hours, Total Hours, Unemployment Rate (16 years and over), and the employment rate (employment-population ratio, 16 years and over) are obtained from the Bureau of Labor Statistics. Sample is 1948Q1-2015Q2.
- U.S. Capacity utilization: Manufacturing (SIC), Percent of Capacity, Quarterly, Seasonally Adjusted, obtained from the FRED database, (**CUMFNS**). Sample is 1948Q1-2015Q2.
- U.S. TFP: utilization-Adjusted quarterly-TFP series for the U.S. Business Sector, produced by John Fernald, series ID: **dtpf_util**, 1947Q1-2015Q2.
- U.S. Spread: Moody’s Seasoned Baa Corporate Bond Minus Federal Funds Rate, quarterly average, obtained from the FRED database, (**BAAFFM**). Sample is 1954Q3-2015Q2.
- U.S. AAA and BAA rates are from FRED (series names AAA and BAA). Samples are from 1920Q1-2015Q2. Spreads are computed by subtracting the T-bill rate, which is taken from Ramey and Zubairy [2018].
- U.S. National Financial Conditions Index: Chicago Fed National Financial Conditions Index, Index, Quarterly, Not Seasonally Adjusted, obtained from the FRED database, (**NFCI**). Sample is 1973Q1-2015Q2.
- U.S. National Financial Conditions Risk Subindex: Chicago Fed National Financial Conditions Risk Subindex, Index, Quarterly, Not Seasonally Adjusted, obtained from the FRED database, (**NFCIRISK**). Sample is 1973Q1-2015Q2.
- U.S. Policy rate : computed from the FRED database as Moody’s Seasoned Baa Corporate Bond Yield (**BAAFFM**) minus the spread (**BAAFFM**). Sample is 1954Q3-2015Q2. The real policy rate is obtained by subtracting realized inflation, computed using the “Consumer Price Index for All Urban Consumers: All Items” (**CPIAUCSL** from the FRED database).

- Business cycles reference dates: NBER for the U.S.A. (<https://goo.gl/n7tS9s>), Owyang, Ramey, and Zubairy [2013] for Canada, Economic Cycle Research Institute (<https://goo.gl/RycWU>) for the other countries of the G7.
- Capacity utilization rates (Manufacturing) for the other countries of the G7: obtained from the FRED database: **BSCURT02DEQ160S** for Germany (sample: 1960Q1-2017Q4), **BSCURT02GBQ160S** for the U.K. (sample: 1985Q1-2017Q4), **BSCURT02ITQ160S** for Italy (sample: 1968Q4-2017Q4), **BSCURT02FRQ160S** for France (sample: 1976Q1-2017Q4), **BSCURT02JPQ160S** for Japan (sample: 1974Q1-2017Q4); obtained from the Bank of Canada (<https://goo.gl/MqzkaE>) for Canada (sample: 1987Q1-2017Q4).
- Unemployment rates for the other countries of the G7: obtained from the FRED database: **LRUNTTTDEQ156S** for Germany (sample: 1962Q1-2017Q3), **AURUKM** for the U.K. (sample: 1948Q1-2067Q4), **ITAURHARMQDSMEI** and **LRHUTTTJPQ156S** for Italy (sample: 1979Q4-2017Q3), **LRUN74TTFRQ156N** for France (sample: 1967Q4-2017Q3), **LRHUTTTJPQ156S** for Japan (sample: 1960Q1-2017Q4), **LRHUTTTCAQ156S** for Canada (sample: 1960Q1-2017Q4);
- Factors usage series for the G7 countries are constructed as $\frac{1}{3} \times \log(\text{capacity utilization rate}/100) + \frac{2}{3} \times \log(1 - \text{unemployment rate}/100)$.

B Conditional Probability of a Recession

In this Appendix, we discuss how to compute confidence intervals for the conditional probabilities of recession (see panel (b) of Figure 1), and present confidence intervals for the 3- and 4-quarter window cases. The intervals were constructed as follows. Fix a time horizon k and a window size x . Let r_t be an indicator for whether the economy is in recession at date t , and let $y_t^{k,x}$ be an indicator for whether the economy is in recession at some point in an x -quarter window around date $t+k$. We may then run the regression

$$y_t^{k,x} = \beta_0^{k,x} + \beta_1^{k,x} r_t + \epsilon_t^{k,x}.$$

Letting $\tilde{\beta}_0^{k,x}$ and $\tilde{\beta}_1^{k,x}$ denote the true values of the regression parameters, the quantity $\tilde{p}^{k,x} \equiv \tilde{\beta}_0^{k,x} + \tilde{\beta}_1^{k,x}$ is the true probability of being in recession some time in the x -quarter window around date $t+k$, conditional on being in recession at date t . Letting hats denote estimates of these parameters, $\hat{p}^{k,x} \equiv \hat{\beta}_0^{k,x} + \hat{\beta}_1^{k,x}$ is a point estimate of that probability which, it can be verified, coincides asymptotically with the point estimates shown in Figure 1.ⁱ Letting $\hat{\Omega}$ denote an estimate of the variance-covariance matrix of $(\hat{\beta}_0^{k,x}, \hat{\beta}_1^{k,x})'$, and $\eta \equiv (1, 1)'$, then $\hat{p}^{k,x}$ is approximately distributed as $N(\tilde{p}^{k,x}, \eta' \hat{\Omega} \eta)$. We can then construct approximate confidence intervals from this distribution in the usual way. In practice, we use the Newey and West [1994] HAC estimator for $\hat{\Omega}$.

In Figure 15, we present 66%, 80%, and 90% confidence intervals for the $x = 3$ and $x = 4$ quarter windows, point estimates for which are presented in panel (a) of Figure 1. In both cases, the confidence intervals indicate that the general pattern we highlighted in Figure 1—namely, a local peak in the probability for $k \approx 36$ -40 quarters, followed by a local trough around 56-60 quarters—is unlikely to have occurred simply by chance. Furthermore, in conjunction with panel (b) of Figure 1 we see that as the window size increases, the estimates become less uncertain and the aforementioned pattern becomes more starkly apparent.

[Figure 15 about here.]

ⁱIndeed, even in our finite sample the two different ways of estimating this probability produce nearly identical results.

C Spectral Density Estimation

C.1 Schuster's Periodogram

We estimate the spectral density of series $\{x_t\}_{t=0}^{T-1}$ of finite length T by first computing the Discrete Fourier Transform (DFT) X_k , which results from sampling the Discrete Time Fourier Transform (DTFT) at frequency intervals $\Delta\omega = \frac{2\pi}{T}$ in $[-\pi, \pi)$:

$$X_k = X\left(e^{i\frac{2\pi}{T}k}\right) = \sum_{t=0}^{T-1} x_t e^{-i\frac{2\pi}{T}kt}, \quad (\text{C.1})$$

for $k = 1, \dots, T - 1$. We then can compute samples of the Sample Spectral Density (SSD) S_k from samples of Schuster's periodogram I_k ⁱⁱ according to

$$S_k = I_k = \frac{1}{T} |X_k|^2 \quad (\text{C.2})$$

Taking advantage of the fact that X is even, this amounts to evaluating the spectral density at T frequencies equally spaced between 0 and π .ⁱⁱⁱ

C.2 Zero-Padding to Increase the Graphic Resolution of the Spectrum

As we have computed only T samples of the DTFT $X(e^{i\omega})$, we might not have a detailed enough picture of the shape of the underlying function $X(e^{i\omega})$, and therefore of the spectral density $|X(e^{i\omega})|^2$. This problem is particularly acute if one is interested in the behavior of the spectrum at longer periodicities (i.e., lower frequencies). Specifically, since we uniformly sample frequencies, and since the periodicity p corresponding to frequency ω is given by $p = \frac{2\pi}{\omega}$, the spectrum is sparser at longer periodicities (and denser at shorter ones). While the degree of accuracy with which the samples of X_k describe the shape of $X(e^{i\omega})$ is dictated and limited by the length T of the data set, we can nonetheless increase the number of points at which we sample the DTFT in order to increase the graphic resolution of the spectrum. One common (and numerically efficient) way to do this is to add a number of zeros to the end of the sequence x_t before computing the DFT. This operation is referred to as zero-padding. As an example, suppose that we add exactly T zeros to the end of the length- T sequence $\{x_t\}$. One can easily check that this has no effect on the DFT computed at the original T sampled frequencies, instead simply adding another set of T sampled frequencies at the midpoints between each successive pair of original frequencies.^{iv}

If one is interested in the behavior of the spectral density at long enough periodicities, zero-padding in this way is useful. We will denote by N the number of samples at which the DTFT

ⁱⁱAnother approach for obtaining the spectral density is to take a Fourier transform of the sequence of autocovariances of x . We show below that this method gives essentially the same result when applied to our hours series.

ⁱⁱⁱSee Priestley [1981] for a detailed exposition of spectral analysis, Alessio [2016] for practical implementation and Cochrane [2005] for a quick introduction.

^{iv}This is true when the number of zeros added to the end of the sample is an integer multiple of T . When instead a non-integer multiple is added, the set of frequencies at which the padded DFT is computed no longer contains the original set of points, so that the two cannot be directly compared in this way. Nonetheless, the overall pattern of the sampled spectrum is in general unaffected by zero-padding.

(and thus the SSD) is sampled, meaning that $T' = N - T$ zeros will be added to the sequence $\{x_t\}$ before computing the DFT. In the main text, we have set $N = 1,024$.^v

C.3 Smoothed Periodogram Estimates

We obtain the raw spectrum estimate of a series non-parametrically as the squared modulus of the DFT of the (zero-padded) data sequence, divided by the length of the data set.^{vi} This estimate is called Schuster's periodogram, or simply the periodogram. It turns out that the periodogram is asymptotically unbiased, but is not a consistent estimate of the spectrum, and in particular the estimate of the spectrum at a given frequency ω_k is generally quite unstable (i.e., it has a high standard error). Notwithstanding this fact, the over all pattern of the spectrum is much more stable, in the sense that the average value of the estimated spectrum within a given frequency band *surrounding* ω_k is in fact consistent. In order to obtain a stable and consistent estimate of the spectrum, we exploit this fact by performing frequency-averaged smoothing. In particular, we obtain our estimate of the SSD $S(\omega)$ by kernel-smoothing the periodogram $I(\omega)$ over an interval of frequencies centered at ω . Since the errors in adjacent values of $I(\omega)$ are uncorrelated in large samples, this operation reduces the standard errors of the estimates without adding too much bias. In our estimations, we use a Hamming window of length $W = 13$ as the smoothing kernel.^{vii}

C.4 Confidence Intervals

We compute bootstrap confidence intervals for our spectral density estimates as follows. Let $X = (x_1, \dots, x_T)'$ be a sample from a (mean-zero) data series with (true) spectrum $S(\omega)$, and let $\hat{S}(\omega_k; X)$, be our consistent estimator of this spectrum at the discrete frequencies $\omega_1, \dots, \omega_K$ computed from the sample X . We would like to know the sampling distribution of the ratio of the estimated spectrum to the true smoothed spectrum, i.e., of

$$\frac{\hat{S}(\omega_k; X)}{S(\omega_k)}. \quad (\text{C.3})$$

To estimate this distribution, suppose that $\hat{S}(\cdot; X)$ is the “true” spectrum, and then simulate N data series of length T from a process that has this “true” spectrum. Letting X_j denote the j -th simulated series, we can then use the empirical distribution of

$$\frac{\hat{S}(\omega_k; X_j)}{\hat{S}(\omega_k; X)}$$

as an approximation to the sampling distribution of (C.3).

To simulate data from a series with “true” spectrum $\hat{S}(\cdot; X)$, we first take the inverse Fourier transform of $\hat{S}(\cdot; X)$ to recover the autocovariance function (ACF), $\hat{R}_k \equiv \mathbb{E}[x_t, x_{t+k}]$, that is associated with \hat{S} .^{viii} Note that a series drawn from this estimated ACF will necessarily have precisely

^vAs is well known, standard numerical routines for computing the DFT (i.e., those based on the Fast Fourier Transform algorithm) are computationally more efficient when N is a power of 2, which is why we set $N = 1,024$ rather than, say, $N = 1,000$.

^{vi}Note that we divide by the original length of the series (i.e., T), rather than by the length of the zero-padded series (i.e., N).

^{vii}Using alternative kernel functions makes little difference to the results.

^{viii}It can be verified that, since in our case $K > 2T - 1$, given the way we compute the estimated spectrum the inverse Fourier transform of the spectrum does indeed recover the associated ACF.

the spectrum $\widehat{S}(\cdot; X)$. For a series of length T , this estimated ACF implies a covariance matrix given by

$$\widehat{\Sigma} = \begin{pmatrix} \widehat{R}_0 & \widehat{R}_1 & \cdots & \widehat{R}_{T-1} \\ \widehat{R}_1 & \widehat{R}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \widehat{R}_1 \\ \widehat{R}_{T-1} & \cdots & \widehat{R}_1 & \widehat{R}_0 \end{pmatrix}.$$

Letting B denote the Cholesky decomposition of $\widehat{\Sigma}$, and ϵ_j a vector drawn from the $N(0, I_T)$ distribution, the vector $X_j \equiv B\epsilon_j$ is drawn from the above ACF function.

D Other G7 Countries

In this appendix we examine whether the motivating evidence reported in the main text for the U.S. also appears in other countries. In Figure 16, we report the probability of a recession at time $t + i$ conditional (plus or minus a window) on a recession at time t . The data on business cycles reference dates are taken from Owyang, Ramey, and Zubairy [2013] for Canada and from the Economic Cycle Research Institute for the other countries. As can be seen, for most of these countries the conditional probability of a recession initially increases, peaks around 40 quarters and then starts to decrease, which is quite close to that observed for the U.S..

[Figure 16 about here.]

We also examine whether the spectral densities of cyclical sensitive variables in other G7 countries echo the observations observed for the U.S. based on hours worked per capita. However, getting measures of hours worked per capita over a sufficiently long period is difficult in many of these countries. Two series of interest which are readily available over the post war period are the unemployment rate and the rate of capital utilization. We therefore create a composite of these two variables that we denote “factors usage”. It is constructed as $\frac{1}{3} \times \log(\text{capacity utilization rate}/100) + \frac{2}{3} \times \log(1 - \text{unemployment rate}/100)$. The spectral density for this composite factors usage variable is plotted in Figure 17 for the other than the U.S. G7 countries. In four out of six of these countries, there is again a sizeable peak in the spectral density around 9-10 years. The exceptions are France which exhibits a peak somewhere around 13 year instead and Italy for which the peak is quite modest.

[Figure 17 about here.]

E Testing for a Peak in Spectral Density: Methodology and Additional Results

In this Appendix, we present the methodology used for our formal peak tests presented in Section 1. We also report results of these tests for a number of additional series not presented in the main text.

E.1 Methodology

For a mean-zero series X_t of length T , let $I(\omega)$ be the raw estimated periodogram at (angular) frequency ω .^{ix} The test statistic is then $D = D_1/D_2$, where we obtain D_j as follows. Let Ω_1 be the set of frequencies corresponding to the local peak range, and Ω_2 be that for the trough range. For the Canova [1996] tests, Ω_j is finite,^x and we may let n_j denote the number of elements it contains. For the Canova [1996] tests, the average size of the periodogram over the relevant range is then

$$D_j \equiv \frac{1}{n_j} \sum_{\omega \in \Omega_j} I(\omega).$$

For our Reiter and Woitek [1999] tests, meanwhile, we may write $\Omega_j = [\underline{\omega}_j, \bar{\omega}_j]$, where $\underline{\omega}_j$ and $\bar{\omega}_j$ are the lower and upper bounds of the relevant frequency interval, respectively. Thus, for the Reiter and Woitek [1999] tests, the average size of the periodogram over the relevant range is

$$\begin{aligned} D_j &\equiv \frac{1}{\bar{\omega}_j - \underline{\omega}_j} \int_{\underline{\omega}_j}^{\bar{\omega}_j} I(\omega) d\omega \\ &= \frac{1}{2\pi} \hat{\gamma}_0 + \frac{1}{\bar{\omega}_j - \underline{\omega}_j} \sum_{k=1}^{T-1} \frac{\sin(k\bar{\omega}_j) - \sin(k\underline{\omega}_j)}{\pi k} \hat{\gamma}_k, \end{aligned}$$

where

$$\hat{\gamma}_k \equiv \frac{1}{T} \sum_{t=1}^{T-k} X_t X_{t+k}, \quad k = 0, 1, \dots, T-1,$$

is the (biased) estimate of the series' auto-covariance function.

To obtain a distribution for D under the relevant null hypothesis (locally flat or AR(1)), we simulate $N = 100,000$ data sets, each of length T , from the relevant null distribution and compute D for each one. To draw from the AR distribution, we first estimate the autocorrelation parameter from the data series (via regression), and then simulate data sets using this parameter estimate. 

E.2 Additional Peak Test Results

In Table 10, we report results of Reiter and Woitek [1999] peak tests for the four additional U.S. variables that we plot in Figure 4: the post-war unemployment rate, the rate of capacity utilization and our measure of overall factor usage. We also report the results for the financial series shown in panels (b)-(d) of Figure 6, plus for the BAA/T-bill spread.^{xi} As seen in the Table, we observe a strong rejection of both null hypotheses for each variable.

[Table 10 about here.]

In Table 11, meanwhile, we report results of Reiter and Woitek [1999] tests for our measure of overall factors usage for the other G7 countries. With the exception of France and Germany, both nulls are rejected at 5% for all countries, and with the exception of France alone, both nulls

^{ix}That is, $I(\omega) = (2\pi T)^{-1} |\sum_{t=0}^T X_t e^{-i\omega t}|^2$.

^xIn particular, if $\omega \in \Omega_j$ in this case, then we can write $\omega = 2\pi k/T$ for some integer k , $0 \leq k \leq T-1$.

^{xi}Since the length of these series vary, the Fourier frequencies vary. This makes a simple systematic implementation of test based only on Fourier frequencies difficult. For this reason we focus here on reporting Reiter and Woitek [1999] type tests.

are rejected at 10%. Despite these exceptions for our factors usage measure, if one looks at the unemployment rate (results not shown), both nulls are rejected for France and Germany at 5%, with the exception of the locally flat null for France, which would be rejected at 6%.

[Table 11 about here.]

F Counter-Cyclical of the AAA/FFR Spread at Peak Frequencies

In this appendix, we look at the co-movement between the BAA/FF spread and our hours worked measured at various frequencies. To do so, we first filter the two series with a Band-Pass filter ($P, P + 26$), for P going from 6 quarters to 32, and compute the correlation between those two filtered series. The first correlation corresponds to the typical (6,32) “business cycle” filter, while the last one (32,58) corresponds to the peak in spectral density of hours and second peak in the spectral density of the spread. Panel (a) of Figure 18 displays the correlation between the two filtered series when varying the band of the filter. The spread is always counter-cyclical, even more so for the (32,58) band of periods that correspond to the peak in spectral densities. Panel (b) shows the coherence between the two series. The spectral coherence is a measure of the degree of relationship, as a function of frequency, between the two time series. As we can see, over the periodicities 4 to 60 quarters, coherence is maximum at 38 quarters.

[Figure 18 about here.]

G Spectral Properties of Detrended GDP

Our observations of a distinct peak in the spectral density of a set of macroeconomic variables may appear somewhat at odds with conventional wisdom. In particular, it is well known, at least since Granger [1966], that several macroeconomic variables do not exhibit such peaks, and for this reason the business cycle is often defined in terms of co-movement between variables instead of reflecting somewhat regular cyclical behavior. According to us, this perspective on business cycle dynamics may be biased by the fact that it often relies on examining the spectral properties of transformed non-stationary variables, such as detrended GDP. We instead have focused on variables—which we like to call cyclically sensitive variables—where business cycle fluctuations are large in relation to slower “trend” movements. For such variables, the breakdown between low-frequency trend and cycle is potentially less problematic if the series can still be considered stationary. In contrast, if one focuses on quantity variables, for example GDP, one needs to believe that the detrending procedure used to make it stationary is allowing one to isolate the relevant cyclical properties. This is certainly questionable as the detrending procedure most often changes the spectral properties dramatically. To see this, in Figure 19 we report the same information we reported before regarding the spectral density, but in this case the series is real per capita GDP. Here we see that the spectra of the non-detrended data and of the filtered data have very little in common with each other. Since it does not make much sense to report the spectral density of non-detrended GDP (it is clearly non-stationary), in Panel (a) of Figure 20 we focus on the spectral density of GDP after removing low-frequency movements using the various high-pass filters. These spectral densities are in line with conventional wisdom: even when we have removed very low-frequency movements, we do not detect any substantial peak in the spectral density of GDP around 40 quarters. How can this be?

What explains the different spectral properties of filter-output versus the level of hours worked? There are at least two possibilities. First, it could be that the filtering we implemented on GDP is simply not isolating cyclical properties. Alternatively, if one believes that such a filtering procedure is isolating cyclical properties, the answer to the puzzle may lie in the behavior of (detrended) TFP. Panel (b) of Figure 20 plots the spectral density of the (log of the) product of TFP by hours, after having removed low-frequency movements in the same way we have done for GDP and other variables. Note that the spectral density of this variable is very similar to the GDP one, as if, for periodicity below 80 quarters, GDP could be approximatively seen as being produced linearly with hours only, whose productivity would be scaled by TFP. If Hours and TFP were uncorrelated, then the spectral density of GDP (in logs) would be the sum of the ones of (log) Hours and TFP. This is approximatively what we have, as shown in panels (c) and (d). The spectral density of TFP shows a quick pick-up it just above periodicities of 40 quarters. As with GDP, we do not see any marked peaks in the spectral density of TFP. An interesting aspect to note is that if we add the spectral density of hours worked to that of TFP, we get almost exactly that of GDP. This suggests that looking at the spectral density of GDP may be a much less informative way to understand business cycle phenomena than looking at the behavior of cyclically sensitive variables such as hours worked and capacity utilization. Instead, GDP may be capturing two distinct processes: a business cycle process associated with factors usage and a lower-frequency process associated with movement in TFP. For this reason, we believe that business cycle analysis may gain by focusing more closely on explaining the behavior of cyclically sensitive variables at business cycle frequencies.

[Figure 19 about here.]

[Figure 20 about here.]

H Proofs

Proof of Proposition 1: The proof requires showing that, under Assumption 2, $\alpha_3\alpha_4 > 0$ and $\alpha_1 < 0$ are necessary for $g(\omega)$ to be humped-shaped on $\omega \in [0, \pi]$. Sargent [1987] (pp. 262–65) shows that for such a hump shape to arise it is necessary that the roots of $B(z) = 0$ be complex, and that if we write $B(L)$ as $1 - t_1 L - t_2 L^2$, then it must also be the case that $4t_2 + t_1(1 - t_2) < 0$.

The proof will proceed in two steps. The first step will be to show that $\alpha_1 < 0$ is necessary for the roots of $B(z) = 0$ be to complex under Assumption 2. The second step will be to show that $\alpha_3\alpha_4 > 0$ is necessary for $4t_2 + t_1(1 - t_2) < 0$ under Assumption 2. Before proceeding with these two steps, it is helpful to make explicit some of the implications of Assumption 2. It is straightforward to verify that Assumption 2 requires that

$$\alpha_1 < \delta(1 - \alpha_2), \quad (\text{H.4})$$

$$(1 - \delta + \alpha_1 + \alpha_2)^2 > 4\alpha_2(1 - \delta), \quad (\text{H.5})$$

which in turn imply

$$\alpha_1 + \alpha_2 < 1, \quad (\text{H.6})$$

$$(1 - \delta + \alpha_1 + \alpha_2)[1 + \alpha_2(1 - \delta)] > 4\alpha_2(1 - \delta). \quad (\text{H.7})$$

First step: For the roots of $B(z) = 0$ to be complex, there must exist a real $\phi \equiv \frac{1}{1 - \alpha_3\alpha_4}$ such that

$$[1 - \delta + (\alpha_1 + \alpha_2)\phi]^2 < 4\alpha_2(1 - \delta)\phi,$$

which is equivalent to

$$(1 - \delta)^2 + 2\phi(\alpha_1 - \alpha_2)(1 - \delta)\phi + (\alpha_1 + \alpha_2)^2\phi^2 < 0.$$

Since the left-hand side of this expression is a convex quadratic function of ϕ , a necessary condition for it to hold for some range of ϕ is if the roots of that quadratic function are real. One can verify that this is only the case if $(\alpha_1 - \alpha_2)^2 > (\alpha_1 + \alpha_2)^2$. Since $\alpha_2 > 0$, this is not possible if $\alpha_1 \geq 0$. Thus, α_1 needs to be negative for $g(\omega)$ to be hump-shaped.

Second Step: To have $4t_2 + t_1(1 - t_2) < 0$, we need

$$[1 - \delta + (\alpha_1 + \alpha_2)\phi][1 + \alpha_2(1 - \delta)\phi] < 4\alpha_2(1 - \delta)\phi,$$

which is in turn equivalent to the condition

$$4\alpha_2(1 - \delta) - (\alpha_1 + \alpha_2) - \alpha_2(1 - \delta)^2 > (1 - \delta) \left[\frac{1}{\phi} + (\alpha_1 + \alpha_2)\alpha_2\phi \right]. \quad (\text{H.8})$$

Assumption 2 implies that the first of these two inequalities is reversed when $\phi = 1$ (see condition (H.7)), and therefore so is the second. Since only the right-hand side of (H.8) depends on ϕ , in order for (H.8) to hold for some $\phi < 1$ (i.e., some $\alpha_3\alpha_4 < 0$), the right-hand side of (H.8) must be increasing in ϕ over some range of $\phi < 1$. But this in turn requires $(\alpha_1 + \alpha_2) > 1/\phi^2$ for some $\phi < 1$, which is ruled out by Assumption 2 (see condition (H.7)). Hence, $\alpha_3\alpha_4 > 0$ ($\phi > 1$) is necessary for $g(\omega)$ to be hump-shaped. This completes the proof.

Proof Proposition 2: The eigenvalues of the system are given by the roots of $B(z^{-1}) = 0$, where $B(L) = 1 - [1 - \delta + (\alpha_1 + \alpha_2)\phi]L + \phi\alpha_2(1 - \delta)L^2$, and $\phi \equiv \frac{1}{1 - \alpha_3\alpha_4}$. A sufficient condition for the system to be unstable is that $\phi\alpha_2(1 - \delta) > 1$. Since $\alpha_2(1 - \delta) > 1$ by Assumption 2, there must exist a $\phi^* > 1$ for which the system is unstable, and accordingly there exists an $\alpha^* < 1$ as stated in the proposition.

If the system has complex roots when it loses stability as ϕ increases, this will happen at the point where $\phi = \frac{1}{\alpha_2(1 - \delta)}$. For the roots to be complex when $\phi = \frac{1}{\alpha_2(1 - \delta)}$, it must be the case that $\left[1 - \delta + \frac{\alpha_1 + \alpha_2}{\alpha_2(1 - \delta)}\right]^2 < 4$. This will happen if $-\alpha_1 > \delta^2\alpha_2$ and $\alpha_2 > -\frac{\alpha_1}{(2 - \delta)^2}$, that is, when α_1 is sufficiently negative and α_2 sufficiently positive.

I Steps in Deriving the Model of Section 3

The model can be seen as an extension of the canonical three-equation New Keynesian model. The extra elements we add are habit persistence, the accumulation of durable goods (or houses), and a credit market imperfection that generates a counter-cyclical risk premium.

I.1 Model Summary

The economy is populated with a continuum of households of mass one, final good and intermediate services firms, commercial banks and a central bank. Each household (or *family*) will be composed of a continuum of mass one of worker members and an atomless family head who coordinates family activities. Time is discrete, and a period of time is divided into four sub-periods.

Final good firms Final good firms buy consumption services from the set of intermediate firms, and bundle them into a final good according to a Dixit-Stiglitz aggregator.

Intermediate firms Intermediate firms produce a set of differentiated consumption services using durable goods as input. These durable goods will be composed of existing durable goods rented from the households and new ones produced by these intermediate firms using labor. After the production of consumption services occurs, the remaining undepreciated stock of new durable goods will be sold to households. We assume that firms are price takers on the market for durable goods, but that they are monopolistic competitors when selling their variety of consumption services. As standard in a New Keynesian framework, they will only be able to adjust their prices upon the arrival of price change opportunity, which occurs according to a Poisson distribution.

Commercial Banks Orders for final goods and new durable goods are made by each individual household member before going to the labor market, and these orders are financed by credit granted by banks at interest rate r . To finance their lending, banks take deposits on which they pay the risk-free nominal rate i . Because some workers will not find a job and may as a result default on their debt, the interest rate r charged by banks to borrowers will include a risk premium r^p over and above the risk-free rate. We assume free entry and perfect competition in the banking market, so that banks make no profits.

Central Bank The central bank sets the risk free nominal interest rate i according to a type of Taylor rule.

Households A family purchases consumption services and durable goods. At the beginning of each period, after the resolution of aggregate uncertainty and the clearing of the last period's debts and deposits, each worker in the family places orders for consumption services and newly produced durable goods, financed by borrowing from banks against their uncertain labor income, with the imperfect backing of their family. Because of a financial imperfection that we describe below, the cost of borrowing will be the risky rate $r = (1 + r^p)(1 + i) - 1$. The uncertainty at this stage is only idiosyncratic, and comes from the fact that jobs are indivisible. All the workers inelastically supply one unit of labor, but firms will demand only $e_t \leq 1$ jobs, so that a fraction $1 - e_t$ of workers will be unemployed, and will not be able to repay their debt the next period. We assume that each employed worker works a fixed number of hours normalized to one, so that e_t is also hours.

The financial friction We assume that for workers that were employed the last period, loan contracts are fully enforceable, so that their debt is always repaid. The financial friction takes the form of a costly enforcement of debt repayments by the family for the household members that do not find a job. In particular, not all non-performing loans can be recovered by going after the family. When a household member cannot pay back a loan—which will be the case when they were unable to find a job—then with exogenous probability ϕ the bank can pay a cost $\Phi < 1$ per unit of the loan to recover the funds from the household (which the bank will always choose to do), while with probability $1 - \phi$ it is too costly to pursue the household, in which case the bank is forced to accept a default. To fix ideas, we assume that, for unemployed workers who have their debt covered by the household, the household transfers to the unemployed worker the funds necessary to cover the debt.^{xii}

I.2 Timing

Borrowing is in one-period bonds. A family enters the period with outstanding debt and bank deposits for each of its workers, and a stock of durable goods that is managed by the family head.

In the first sub-period, interest on last-period deposits are paid, and past debts with banks are settled. Workers that were employed in the previous period pay back their loans, and return any remaining cash balances to the household.

In the second sub-period, workers first borrow from banks, and then use the proceeds to order consumption services and durable goods. Final good services firms receive demand (paid orders) and make orders to intermediate good firms, which also receive the orders for new durable goods from the households. A worker who wants to borrow can try to do it on his own, or can get backing from his family. If there is no backing by the family, a worker who borrows but does not find a job will not be able to repay loans. Backing by the family will therefore reduce the risk premium on loans, and in equilibrium will always be chosen by the workers. As we have explained above, backing by the family will be imperfect: with probability ϕ the bank will pursue the household, and with probability $1 - \phi$ it is too costly to pursue the household.

In the third sub-period, intermediate good firms rent durable goods from the household head and hire workers to produce any new durable goods. As noted above, there will always be a measure one of workers looking for a job, but because of job indivisibility, only a fraction e_t of workers will find a job. Within a match, the firm makes a take-it-or-leave-it wage offer to the worker, the equilibrium level of which will be at a reservation wage, which will be decided by the family head.

^{xii}An equivalent formulation would have the household simply absorbing that workers debt, to be repaid in a future period.

Finally, in the fourth sub-period, production takes place, wages are paid to workers, rental payments for durable goods are made to the household head, orders are fulfilled, and all consumption services and durable goods are brought back to the household and shared equally among family members. Firms' and banks' profits, if any, are returned to the household.

I.3 Households

There is a continuum of mass one of households (families) indexed by h who purchase consumption services from the market and accumulate durable goods. The preferences of family h are given by $U(h) = \mathbb{E}_0 \sum_t \beta^t \xi_{t-1} [U(C_{ht} - \gamma C_{t-1}) + \nu(1 - e_{ht})]$, with $U'(\cdot) > 0, U''(\cdot) < 0, \nu'(\cdot), \nu''(\cdot) < 0$ and $0 \leq \gamma < 1 - \delta$, where δ is the depreciation rate of the durable good. C_{ht} represents the consumption services purchased by household h in period t , $C_t \equiv \int_0^1 C_{ht} dh$ denotes the average level of consumption in the economy, β is the discount factor, and ξ_t denotes an exogenous shock to the discount factor at date t . Note that this preference structure assumes the presence of external habit. Each family owns a stock of durable goods X_{ht} that is rented to firms in order to produce intermediate services. This stock of durable goods is accumulated according to the equation

$$X_{ht+1} = (1 - \delta)X_{ht} + I_{ht}. \quad (\text{I.9})$$

A family is composed of a continuum of measure one of workers indexed by j . All the members of the family share the utility $U(h)$, where $C_{ht} = \int_0^1 C_{jht} dj$ is the total consumption services bought by the workers (C_{jht} for each worker) in household h , and total purchases of new durable goods is similarly given by $I_{ht} = \int_0^1 I_{jht} dj$.

We go into details of a family behavior starting with the second sub-period of period t , when family i has inherited from the past a net debt position given by D_{ht} ; that is, the family owes a bank the amount D_{ht} . It is useful to note at this stage that, because all families are identical, D_{ht} will be zero in equilibrium.

The outstanding debt of the family D_{ht} is equally shared among the unit measure of workers. Worker j is told by the family head to go to bank with its share of existing debt D_{jht} ($= D_{ht}$) to apply for a new loan $L_{jht} \geq D_{jht}$, where the first D_{jht} units of that loan are used to settle the past debt and $L_{jht} - D_{jht}$ is left and available for spending. When granting the loan, the banks open the worker a checking account (which cannot have a negative balance), where the initial amount in the deposit would be $L_{jht} - D_{jht}$. As noted above, a worker can apply for their loan either or without family backing. We only consider the case where all workers apply for loans with family backing. This is not restrictive, as the risk premia on backed and non-backed loans will be such that workers will never choose a non-backed loan.

Banks can transfer the balances in the checking account to other agents, and these deposits earn the safe (central bank-determined) interest rate i at the beginning of next period. The workers can then use this bank money to order consumption goods C_{jht} at nominal price P_t and new durable goods I_{jht} at nominal price P_t^X . When a firm receives an order, the bank money is transferred to that firm, which then uses it to similarly place orders with the intermediate good firms (in the case of a final good firm) or to pay workers and rent durables (in the case of an intermediate good firm). With these latter payments, much of the bank money will be transferred back to (employed) workers and the household head (who maintains a family bank account) at the end of the period. The remaining bank money (reflecting intermediate good firm profits) is also transferred to the household head. After paying off their own loans, employed workers are asked by the family head to transfer their remaining bank money back to the family account.

The head of the family coordinates the family activities by telling workers how much to borrow, how much to purchase and how much of the bank money to transfer to the family. On top of that,

the family head manages the stock of durable goods (or houses) X_{ht} of the family. She does so by renting it on the market at rate R_t^X .

In the third sub-period, firms wish to hire e_t workers from of the unit measure pool of workers, so that $1 - e_t$ workers will be left unemployed.^{xiii} Firms make take-it-or-leave-it offers to the workers in the form of a nominal wage W_t , who accept it as long as it is not below a reservation level set by the head of the family. In equilibrium, all offered jobs will be accepted. Because there is a unit measure of families with a unit measure of workers each, we will have $e_{ht} = e_t$, so that e_t is the probability of a worker in family h being employed.

Finally, as noted above, in the fourth sub-period, production takes place, all wage and rental payments are made, profits are transferred, an consumption and new durable goods are delivered to those who ordered them, which in turn are returned to the family to be split equally among its members.

In the first sub-period of the next period, debts need to be settled after interest on bank money is paid. Workers who were employed the last period ended it with a bank account balance of $W_t + L_{jht} - D_{jht} - P_t C_{jht} - P_t^X I_{jht}$. This balance receives an interest payment $1 + i_t$ at the beginning of period $t+1$. The worker, meanwhile, owes the bank $(1 + r_t)L_{jht}$ at this point. The bank will limit its lending such that $(1 + r_t)L_{jht} \leq (1 + i_t)W_t$ so as to ensure that employed workers can always pay back their loans. Further, since $r_t \geq i_t$ (reflecting the risk premium), it will never be optimal for workers to borrow more than they intend to spend, and thus we will have $L_{jht} = D_{jht} + P_t C_{jht} + P_t^X I_{jht}$. Thus, an amount $T_{jht}^e = (1 + i_t)W_t - (1 + r_t)(D_{jht} + P_t C_{jht} + P_t^X I_{jht}) > 0$ will be transferred back to the family account by each employed worker.

The bank account balance of an unemployed worker at the end of period t , meanwhile, is given by $L_{jht} - D_{jht} - P_t C_{jht} - P_t^X I_{jht} = 0$, which as noted above is optimally set to zero. Further, as with employed workers, unemployed workers owe an amount $(1 + r_t)L_{jht}$ to the bank at the beginning of period $t+1$. For these workers, the bank decides whether or not to pursue the family for repayment. If it is too costly (which which happens with probability ϕ), the loan is not repaid. With probability $1 - \phi$, pursuing the household is worthwhile for the bank, in which case the family transfers $T_{jht}^u = (1 + r_t)(D_{jht} + P_t C_{jht} + P_t^X I_{jht})$ to the worker so they can repay the loan.

Therefore, family h 's net debt at the beginning of period $t+1$ will be given by

$$\begin{aligned} D_{ht+1} &= (1 - e_t) \phi T_{jht}^u - e_t T_{jht}^e - (1 + i_t) (R_t^X X_{ht} + \Pi_t) \\ &= [e_t + (1 - e_t) \phi] (1 + r_t) (D_{ht} + P_t C_{ht} + P_t^X I_{ht}^X) - (1 + i_t) (e_t W_t + R_t^X X_{ht} + \Pi_t) , \end{aligned} \quad (\text{I.10})$$

where Π_t is total profits of all firms and banks.

I.4 Banks

Banks remunerate deposits at rate i_t , receive interest r_t on the fraction $e_t + (1 - e_t)\phi$ of loans that are repaid, and incur processing costs of Φ per unit of loans to bailed-out workers (i.e., those made to the fraction $(1 - e_t)\phi$ of workers who end up unemployed and are bailed out by the household). Thus, bank profits are

$$\Pi_t^{\text{banks}} = \{[e_t + (1 - e_t) \phi] (1 + r_t) - (1 + i_t) [1 + (1 - e_t) \phi \Phi]\} L_t ,$$

^{xiii}As the supply of workers is always inelastically one, equilibrium e_t must be less than or equal to one. To simplify notation, we will also let e_t denote the probability that a worker is offered a job, whereas it technically should be written $\min\{e_t, 1\}$ to allow for off-equilibrium labor demand.

where L_t is the total volume of loans. Free entry into banking implies zero profits, i.e.,

$$1 + r_t = (1 + i_t) \frac{1 + (1 - e_t) \phi \Phi}{e_t + (1 - e_t) \phi}. \quad (\text{I.11})$$

Defining the risk premium as $(1 + r_t) = (1 + i_t)(1 + r_t^p)$, the above equation implies

$$1 + r_t^p = \frac{1 + (1 - e_t) \phi \Phi}{e_t + (1 - e_t) \phi} \quad (\text{I.12})$$

If there is no unemployment risk ($e_t = 1$), or no default or recovery costs ($\phi = 1$ and $\Phi = 0$), one can check that there is no risk premium ($r_t^p = 0$).

I.5 Firms

The final good sector is competitive. This sector provides consumption services to households by buying a set of differentiated intermediate services, denoted C_{kt} , from intermediate service firms and combining them using a Dixit-Stiglitz aggregator. We assume a measure one of intermediate service firms, indexed by k . The objective of the final good firm is thus to solve

$$\max P_t C_t - \int_0^1 P_{kt} C_{kt} dj$$

subject to

$$C_t = \left(\int_0^1 C_{kt}^\eta dk \right)^{\frac{1}{\eta}}$$

with $\eta \in (0, 1)$ and where P_{kt} is the nominal price of intermediate service k . This gives rise to demand for intermediate service k given by

$$C_{kt} = \left(\frac{P_{kt}}{P_t} \right)^{-\frac{1}{1-\eta}} C_t.$$

Details of intermediate firms are presented in Section 3.1.2.

I.6 Equilibrium Outcomes

As noted in the text, the equilibrium outcome for this model is determined from a set of nine equations in the endogenous variables. The first two equilibrium conditions are the bank zero-profit condition (I.11) and the intermediate goods optimality condition (12). We derive the three next equilibrium conditions from the household's behavior. The head of family h maximizes

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \xi_{t-1} [U(C_{ht} - \gamma C_{t-1}) + v(1 - e_{ht})]$$

subject to the intertemporal budget constraint (I.10) and the durables accumulation equation (I.9). We can use the first-order conditions of this problem and the equilibrium conditions (the bank zero profit condition (I.11), $D_{ht} = 0$ and $C_{ht} = C_t$) to obtain

$$U'(C_t - \gamma C_{t-1}) = \beta \frac{\xi_t}{\xi_{t-1}} (1 + i_t) [1 + (1 - e_t) \phi \Phi] \mathbb{E}_t \left[\frac{U'(C_{t+1} - \gamma C_t)}{1 + \pi_{t+1}} \right], \quad (\text{I.13})$$

$$U'(C_t - \gamma C_{t-1}) = \beta \frac{\xi_t}{\xi_{t-1}} \mathbb{E}_t \left[\frac{U'(C_{t+1} - \gamma C_t)}{(1 + \pi_{t+1}) P_t^X} \left\{ \frac{R_{t+1}^X}{1 + (1 - e_{t+1}) \phi \Phi} + (1 - \delta) P_{t+1}^X \right\} \right] \quad (\text{I.14})$$

with an additional wage condition given by (11). Equation (I.13) is derived from the Euler equation for debt accumulation, (I.14) is derived from the Euler equation for the durable good accumulation, and the wage condition is the reservation wage of the household, which is in equilibrium the actual wage offered by firms. We also have the aggregate law of motion for the stock of durable goods (15) and the consumption services market-clearing condition

$$C_t = s [X_t + F(e_t, \theta_t)] . \quad (\text{I.15})$$

The last two equations will be given by the nominal interest rate policy rule of the central bank (13), and the optimal pricing decision of firms under Calvo adjustment costs which, as noted in the text, we do not need to explicitly obtain.

Block Recursive Structure

Using the monetary policy rule (13),^{xiv} the equilibrium equations have a block recursive structure whereby the variables C_t , e_t , X_{t+1} and r_t^p can be solved for first using equations (I.11), (I.12), (I.15) and the combination of (I.13) and (13), which is given by

$$U' (C_t - \gamma C_{t-1}) = \beta \frac{\xi_t}{\xi_{t-1}} \Theta [e_t + (1 - e_t) \phi] (1 + r_t^p) \mathbb{E}_t [U' (C_{t+1} - \gamma C_t) e_{t+1}^{\rho_e}] \quad (\text{I.16})$$

Given that the above equations determine the quantity variables, the rest of the system then simultaneously determines the remaining variables $\left\{ \frac{R_t^X}{P_t}, \frac{P_t^X}{P_t}, \frac{W_t}{P_t}, \pi_t \right\}$. In particular, as we do not consider the implications of the model for inflation, we do not need to explicitly derive the optimal pricing behavior of firms. Finally, using (I.15) to eliminate C from the system, we end up with the three equations (16)-(14)

J Solving and Estimating the Model

J.1 Deriving the Estimated Equations

Substituting the functional forms $U(c) = (c^{1-\omega} - 1)/(\omega - 1)$ and $F(e, \theta) = \theta e^\alpha$, as well as the normalizations $s = \theta = 1$ and the definition $X_t^* \equiv X_t^{\psi/(\alpha\delta)}$ into equations (15) and (16) from the main text, equations (17) and (18) can then be obtained by linearizing these two equations around the non-stochastic steady state with respect to $\log(e_t)$, $\log(X_t^*)$, r_t^p , and $\mu_t \equiv -\Delta \log(\xi_t)$. In particular, the coefficients in (18) are given by

$$\begin{aligned} \alpha_1 &\equiv -\frac{\alpha\delta^2(1-\delta-\gamma)}{(1-\delta)\kappa}, \\ \alpha_2 &\equiv \frac{\gamma\alpha\delta(1-\delta-\psi)}{(1-\delta)\kappa}, \\ \alpha_3 &\equiv \frac{\alpha\delta-\tau\varphi_e}{\kappa}, \\ \alpha_4 &\equiv \frac{\tau}{\kappa}, \end{aligned}$$

^{xiv}Specifically, we use the version in footnote 55.

where $\tau \equiv (1 - \gamma)(\psi + \delta)/\omega$, $\kappa \equiv (1 + \gamma - \psi)\alpha\delta + \tau\Xi$, $\Xi \equiv (1 - \phi)e_s/[e_s + (1 - e_s)\phi]$, and e_s is the steady state employment rate.

For the linear RP model, we may obtain the (log-)linearized version of equation (14) as $\hat{r}_t^p = \varrho_1 \hat{e}_t$, where $\varrho_1 = -(1 - \phi + \phi\Phi)e_s/[e_s + (1 - e_s)\phi]^2$. For the non-linear RP model, we have $\phi_t = \phi(e_t)$ and $r_t^p = f(e_t)$, where

$$f(e) = \frac{1 + (1 - e)\phi(e)\Phi}{e + (1 - e)\phi(e)} - 1.$$

Letting $\tilde{e} \equiv \log e$ and $g(\tilde{e}) \equiv f(\exp\{\tilde{e}\})$, and noting that $\phi'(e_s) = 0$ by assumption, the third-order approximation to (14) is given by

$$\hat{r}_t^p = \varrho_1 \hat{e}_t + \frac{1}{2} g'' \hat{e}_t^2 + \frac{1}{6} g''' \hat{e}_t^3,$$

where ϱ_1 is as in the linear RP model, and the derivatives of g are evaluated at the steady state and are implicitly dependent on the corresponding derivatives of ϕ of the same order or lower. In practice, we directly estimate ϱ_2 and ϱ_3 in the expression $\hat{r}_t^p = \varrho_1 \hat{e}_t + \varrho_2 \hat{e}_t^2 + \varrho_3 \hat{e}_t^3$, and then solve for the values of ϕ'' and ϕ''' such that $g'' = 2\varrho_2$ and $g''' = 6\varrho_3$.

J.2 Solution Method

For the three linear versions of the model (linear RP, no friction, and canonical), we use standard (linear) perturbation methods to solve the model (see, e.g., Fernández-Villaverde, Rubio-Ramírez, and Schorfheide [2016]). For the non-linear RP model, since we would like to allow for the possibility of local instability and limit cycles, standard non-linear perturbation methods are not appropriate. In particular, as a first step standard methods require obtaining a rational-expectations solution to the linearized (i.e., linear RP, in this case) model. If the model is locally unstable (as is necessarily the case if it features limit cycles), such a solution cannot exist. We therefore instead use the perturbation method discussed in Galizia [2018]. The method differs from standard methods in that it does not require that the linear approximation to the solution of the non-linear model also be a solution to the linearized model, an assumption implicit in standard methods. See Galizia [2018] for further details.

In practice, we solve the non-linear RP model in this manner to a third-order approximation. Given a parameterization and an associated solution in state-space form, as part of the Galizia [2018] method we need to verify that the system indeed remains bounded (in expectation). It is not possible to confirm this analytically, so we employ numerical methods instead. In particular, to minimize computational burden, we simply check whether, for a given initial (non-zero) state of the system,^{xv} the deterministic path for hours (i.e., the one obtained by feeding in a constant stream of zeros for the shock) explodes, where we define an explosion as a situation where, within the first 270 simulated quarters (the length of our data set), the absolute value of hours exceeds $10 \times \bar{L}$, where \bar{L} is the maximum absolute value of hours (in log-deviations from the mean) observed in our data set ($\bar{L} = 0.146$).

J.3 Estimation Procedure

To estimate the model, we use an indirect inference method as follows. Let $x_t \in \mathbb{R}^n$ denote a vector of date- t observations in our data set, $t = 1, \dots, T$, and let $\mathbf{x}_T \equiv (x'_1, \dots, x'_T)'$ denote the full data set in matrix form. Let $F : \mathbb{R}^{T \times n} \rightarrow \mathbb{R}^q$ be the function that generates the q -vector of features of

^{xv}We use the same initial point for every parameterization.

the data we wish to match (i.e., $F(\mathbf{x}_T)$ is a vector containing all relevant spectrum values, plus, for the non-linear RP model, the correlation, skewness and kurtosis for hours and the risk premium).

Suppose we simulate M data sets of length T from the model using the parameterization θ . Collect the m -th simulated simulated data set in the matrix $\tilde{\mathbf{x}}_T^m(\theta) \in \mathbb{R}^{T \times n}$, $m = 1, \dots, M$. The estimation strategy is to choose the parameter vector θ to minimize the Euclidean distance between $F(\mathbf{x}_T)$ and the average value of $F(\tilde{\mathbf{x}}_T^m(\theta))$, i.e., we seek the parameter vector

$$\hat{\theta} = \underset{\theta \in \Theta}{\operatorname{argmin}} \left[F(\mathbf{x}_T) - \frac{1}{M} \sum_{m=1}^M F(\tilde{\mathbf{x}}_T^m(\theta)) \right]' \left[F(\mathbf{x}_T) - \frac{1}{M} \sum_{m=1}^M F(\tilde{\mathbf{x}}_T^m(\theta)) \right],$$

where Θ is the parameter space. In practice, we simulate $M = 3,000$ data sets for each parameter draw, and estimate $\hat{\theta}$ using MATLAB's `fminsearch` optimization function. We explored several different weighting matrices in estimation which all gave similar results. In the main text, we report results based on using an identity matrix as the weighting matrix.

J.3.1 The Parameter Space

We estimate the parameters of the model imposing several restrictions on the parameter space Θ . First, we require that $0 \leq \gamma, \psi < 1$. Second, we require that the policy rate reacts positively to expected hours, but not so strongly as to cause current hours to fall in response to an increase in expected hours, i.e., $0 < \varphi_e < \alpha\delta/\tau$, where τ is defined above. Third, we impose that the degree of complementarity is never so strong as to generate temporary multiple equilibria.^{xvi} This latter property is ensured if the function $\hat{e}_t + \frac{\tau}{\kappa} R^p(\hat{e}_t)$ (see equation (22) in the main text) is strictly increasing in \hat{e}_t (so that it is invertible). This in turn requires $1 + \frac{\tau}{\kappa} R_1^p > 0$ and $R_3^p \geq R_2^{p^2}/[3(\kappa/\tau + R_1^p)]$. Fourth, we impose that the shock process is stationary, i.e., $|\rho| < 1$. Finally, we require that the parameters be such that a solution to the model exists and is unique (see Appendix J.2). Note that none of the estimated parameters in either the linear or non-linear RP models is on the boundary of the set of constraints we have imposed.

^{xvi}By temporary multiple equilibria, we mean a situation where, for a given X_t , e_{t-1} and expectation about e_{t+1} , there are multiple values of e_t consistent with the dynamic equilibrium condition.

K More on Spectral Density Estimation

K.1 Smoothing and Zero-Padding with a Multi-Peaked Spectral Density

To illustrate the effects of smoothing and zero-padding, in this section we compare the estimated spectral density with the known theoretical one for a process that exhibits peaks in the spectral density at periods 20, 40 and 100 quarters. We think this is a good description of the factor variables we are studying (i.e., hours worked, unemployment, capacity utilization), that display both business cycle movements and lower-frequency movements unrelated to the business cycle. We construct our theoretical series as the sum of three independent stationary AR(2) processes, denoted x_1 , x_2 and x_3 .

Each of the x_i follows an $AR(2)$ process

$$x_{it} = \rho_{i1}x_{it-1} + \rho_{i2}x_{it-2} + \varepsilon_{it},$$

where ε_i is i.i.d. $N(0, \sigma_i^2)$. The spectral density of this process can be shown to be given by

$$S(\omega) = \sigma_i^2 \left\{ 2\pi[1 + \rho_{i1}^2 + \rho_{i2}^2 + 2(\rho_{i1}\rho_{i2} - \rho_{i1})\cos(\omega) - 2\rho_{i2}\cos(2\omega)] \right\}^{-1}$$

It can also be shown (see, e.g., Sargent [1987]) that for a given ρ_{i2} , the spectral density has a peak at frequency $\bar{\omega}_i$ if

$$\rho_{i1} = -\frac{4\rho_{i2}\cos(2\pi/\bar{\omega}_i)}{1 - \rho_{i2}}$$

We set $\bar{\omega}_i$ equal to 20, 40, and 100 quarters, respectively, for the three processes, and ρ_{i2} equal to -0.9, -0.95, and -0.95. The corresponding values for ρ_{i1} are 1.802, 1.9247, and 1.9449. We set σ_i equal to 6, 2, and 1. We then construct $x_t = x_{1t} + x_{2t} + x_{3t}$. The theoretical spectral density of x is shown in Figure 21. As in the factors usage series we are using in the main text, the spectral density shows long-run fluctuations, but the bulk of the business cycle movements is explained by movements at the 40-quarter periodicity, although we observe another peak at periodicity 20 quarters.

[Figure 21 about here.]

We simulate this process 1,000,000 times, with $T = 270$ for each simulation, which is the length of our observed macroeconomic series. We estimate the spectral density for various values of N (zero-padding) and W (length of the Hamming window). Higher N corresponds to higher resolution, and higher W to more smoothing. On each panel of Figure 22, we report the mean of the estimated spectrum over the 1,000,000 simulations (solid grey line), the mean \pm one standard deviation (dashed lines), and the theoretical spectrum (solid black line). As we can see moving down the figure (i.e., for increasing W), more smoothing tends to reduce the error variance, but at the cost of increasing bias. Effectively, the additional smoothing “blurs out” the humps in the true spectrum. For example, with no zero-padding ($N = 270$), the peak in the spectral density at 40 quarters is (on average) hardly detected once we have any smoothing at all. Meanwhile, moving rightward across the figure (i.e., for increasing N), we see that more zero-padding tends to reduce the bias (and in particular, allows for the humps surrounding the peaks to be better picked up on average), but typically increases the error variance. As these properties suggest, by appropriately choosing the combination of zero-padding and smoothing, one can minimize the error variance while maintaining the ability to pick up the key features of the true spectrum (e.g., the peaks at 20 and 40 quarters).

[Figure 22 about here.]

K.2 Smoothing and Zero-Padding with Non-Farm Business Hours per Capita

Figure 23 presents estimates of the spectral density of U.S. Non Farm Business Hours per Capita for different choices of the zero-padding parameter (N) and different lengths of the Hamming window (W). The results indicate that, as long as the amount of zero-padding is not too small (i.e., N larger), we systematically observe the peak at around 40 quarters in the spectral density. In fact, it is only with minimal zero-padding (N low) and a wide smoothing window (W high) that the peak is entirely washed out. We take this as evidence of the robustness of that peak.

[Figure 23 about here.]

K.3 Detrending with a Polynomial Trend

In this section, we check that detrending our hours series with a polynomial trend of degrees 1 to 5 does not affect our main finding; namely, the existence of a peak in the spectrum at a periodicity around 40 quarters. Plots confirming that our finding is robust to polynomial detrending are shown in Figure 24.

[Figure 24 about here.]

K.4 Alternative Estimators

As another robustness test, we estimate the spectrum using the SPECTRAN package (for MATLAB), which is described in Marczak and Gómez [2012]. The spectrum is computed in this case as the Fourier transform of the covariogram (rather the periodogram as we have done thus far). Smoothing is achieved by applying a window function of length M to the covariogram before taking its Fourier transform.^{xvii} Three different window shapes are proposed: the Blackman-Tukey window, Parzen window, and Tukey-Hanning window. The width of the window used in estimation is set as a function of the number of samples of the spectrum. In the case where no zero-padding is done ($N = 270$), these “optimal” widths correspond to lengths of, respectively, $M = 68, 89$, and 89 quarters for the three methods.^{xviii} Figure 25 shows the estimated spectrum of Non Farm Business hours for the three windows and with or without zero-padding ($N = 270, 512$, or 1024). Results again confirm the existence of a peak at a periodicity around 40 quarters, as long as there is enough zero-padding.

[Figure 25 about here.]

L Non Normality

Here we also examined whether the business cycle fluctuations which we have been focusing upon are approximately normally distributed. To this end, we perform Bai and Ng [2005] test, which combines skewness and kurtosis into a single statistic for serially dependant data. This is done

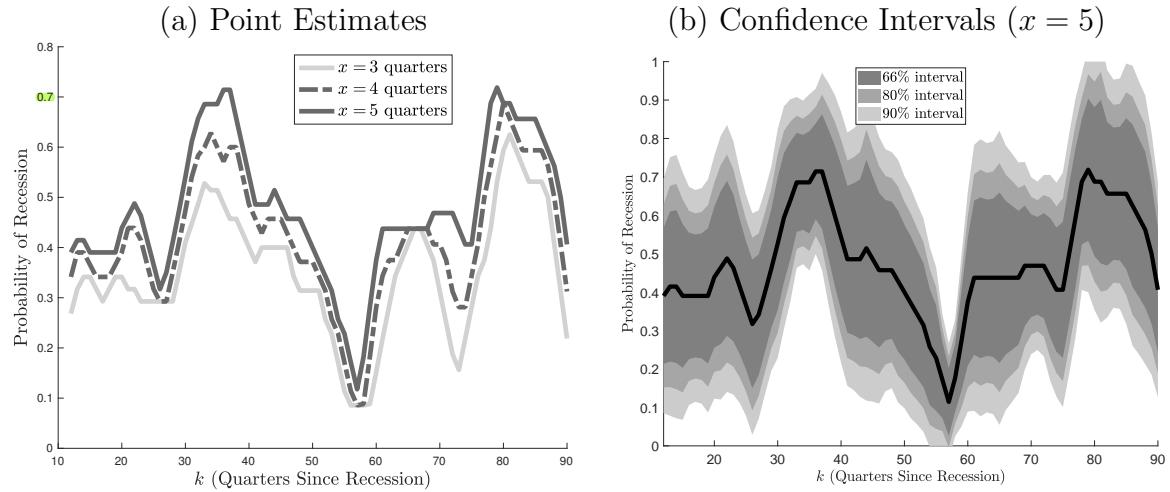
^{xvii}Specifically, the k -th-order sample autocovariance is first multiplied by $w(|k|)$, where the window function w is an even function with the property that $\max_k w(k) = w(0) = 1$, and the window length $M > 0$ is such that $w(|k|) \neq 0$ for $|k| = M - 1$ and $w(|k|) = 0$ for $|k| \geq M$.

^{xviii}Note that, in contrast to the kernel-smoothing case, in this case a wider window corresponds to less smoothing.

on hours and spread after using a high-pass linear filter to remove periodicities greater than 60 quarters, which allows us to retain all the variation that we have argued is relevant for the business cycle, while removing more medium- and long-run fluctuations. The null hypothesis for the test is normality. For non-farm business hours and spread, the p-values we find are, respectively, 1.2% and .02%. This indicates that linear-Gaussian models might not be an appropriate way to describe these data, and that one may need to allow for some type of non-linearity in order to explain these movements. Accordingly, we explore a class of explanation that allows for such non-linearities. Moreover, when we estimate our model, we use the skewness and kurtosis properties of the data to help identify parameters.

Figures

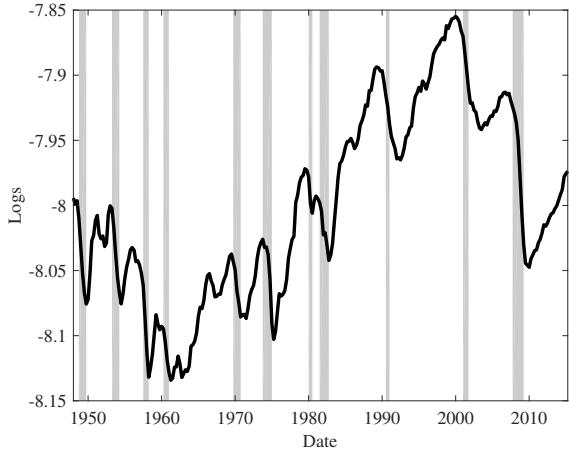
Figure 1: Conditional Probability of Being in a Recession



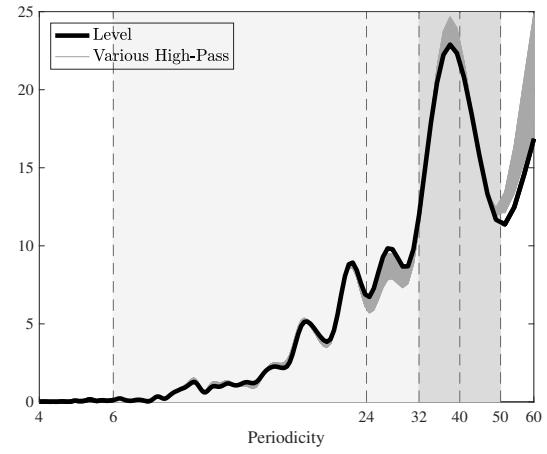
Notes: Panel (a) displays the fraction of time the economy was in a recession within an x -quarter window around time $t + k$, conditional on being in a recession at time t , where x is allowed to vary between 3 and 5 quarters. Panel (b) shows confidence intervals for the $x = 5$ case. See Appendix B for the $x = 3$ and $x = 4$ cases, as well as for details of how these confidence intervals were constructed. The figure was constructed using NBER recession dates over the period 1946Q1-2017Q2.

Figure 2: Properties of Hours Worked per Capita

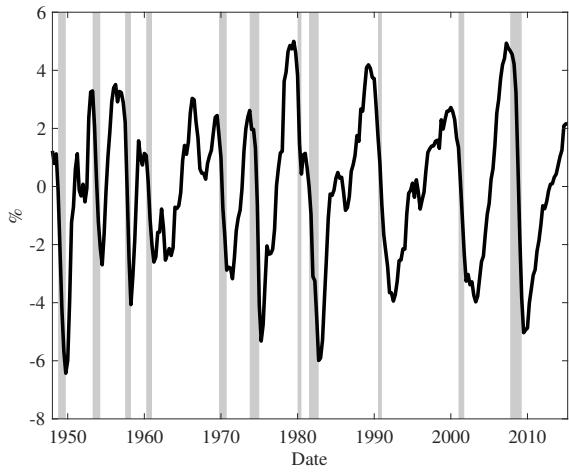
(a) Data



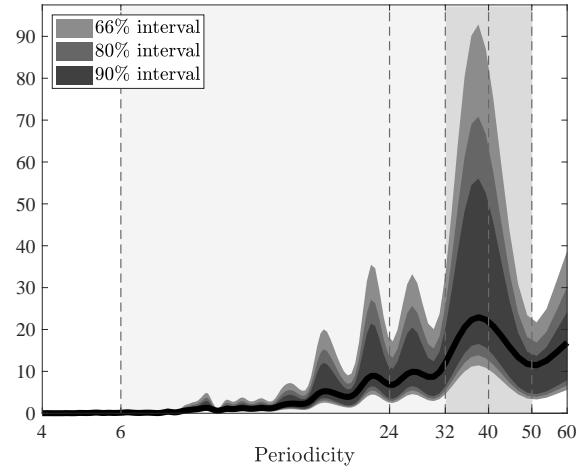
(b) Spectral Density



(c) High-Pass (60) Filtered Hours

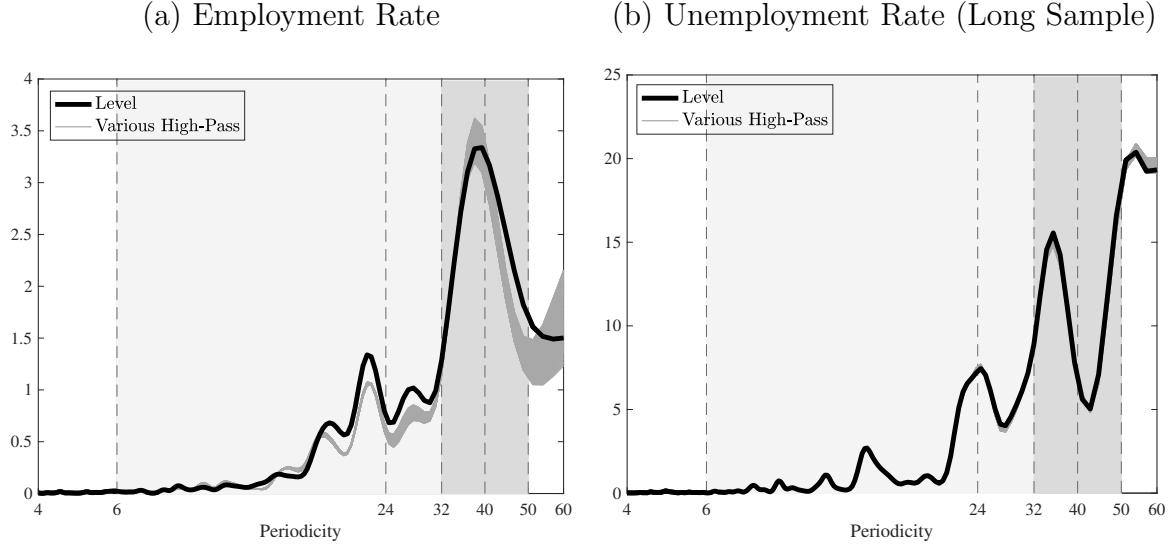


(d) Spectral Density: Confidence Bands



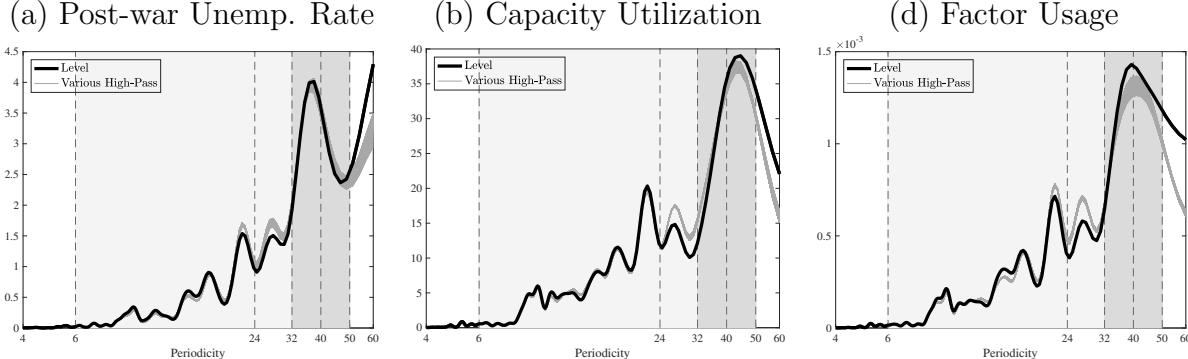
Notes: Panel (a) plots the log of Non-Farm Business Hours divided by Total Population. Panel (b) is an estimate of the spectral density of hours in levels (black line) and for 101 series that are high-pass (P) filtered version of the levels series, with P between 100 and 200 (gray lines). A high-pass (P) filter removes all fluctuations of period greater than P . Panel (c) displays high-pass (60) filtered hours. Panel (d) shows bootstrapped 66%, 80% and 90% point-wise confidence bands for the spectral density.

Figure 3: Spectral Density for U.S. Employment Rate and Unemployment Rate



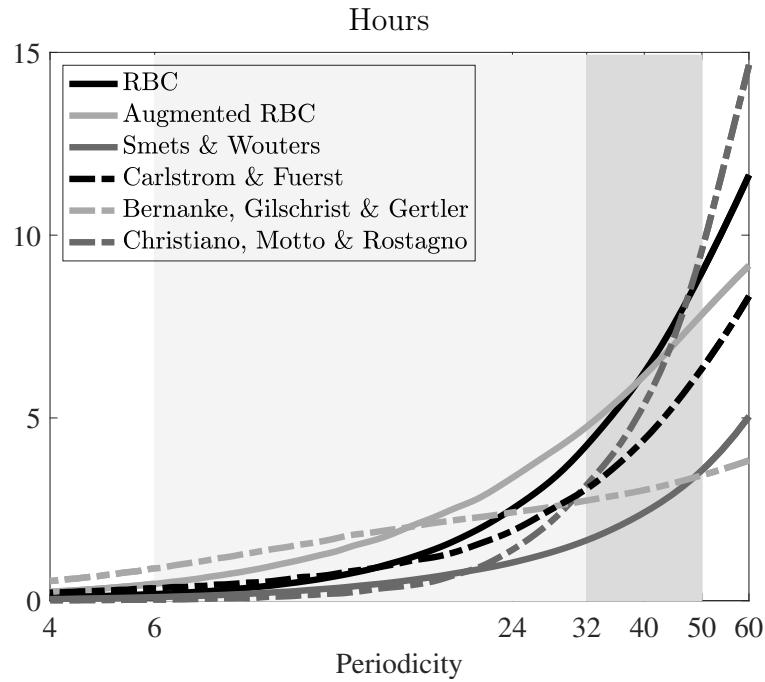
Notes: These panels display estimates of the spectral density of the employment rate (panel (a)) and unemployment rate (panel (b)) in levels (black line) and for 101 series that are high-pass (P) filtered version of the levels series, with P between 100 and 200 (gray lines). A high-pass (P) filter removes all fluctuations of period greater than P . Employment rate sample is 1948Q1-2015Q2 and unemployment rate sample is 1890Q1-2015Q2. For unemployment, the smoothing window width has been reduced from 13 to 9 quarters as we have roughly twice more observations.

Figure 4: Spectral Density of Other U.S. Series



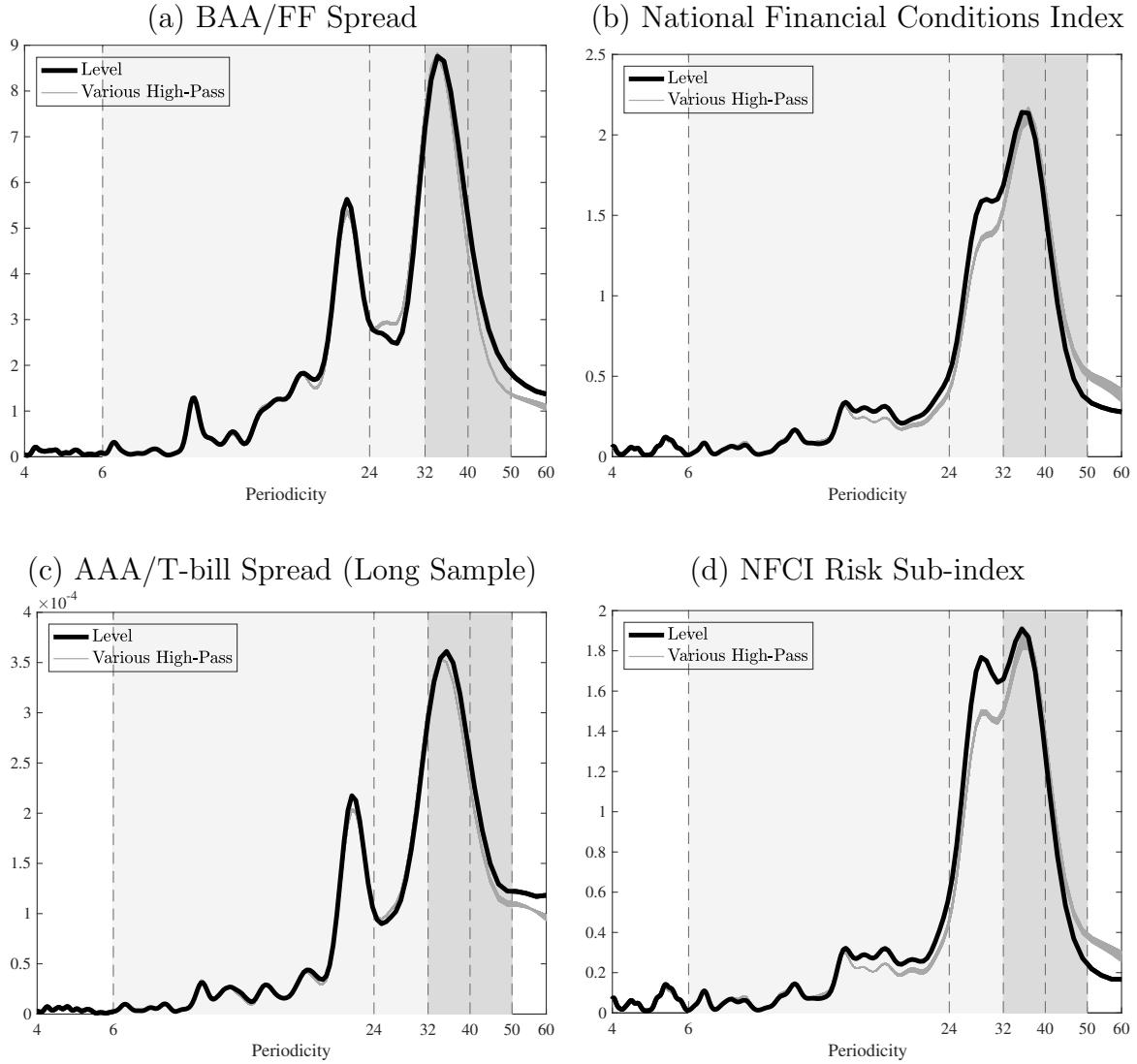
Notes: Panels (a) to (c) show estimates of the spectral density for the series in levels (black line) and for 101 series that are high-pass (P) filtered version of the levels series, with P between 100 and 200 (gray lines). A high-pass (P) filter aims to remove all fluctuations of period greater than P . For the three panels, the sample is 1948Q1-2015Q2. Factors usage is defined as $\frac{1}{3} \times \log(\text{capacity utilization rate}/100) + \frac{2}{3} \times \log(1 - \text{unemployment rate}/100)$.

Figure 5: Spectral Densities of Hours in Some Standard Models



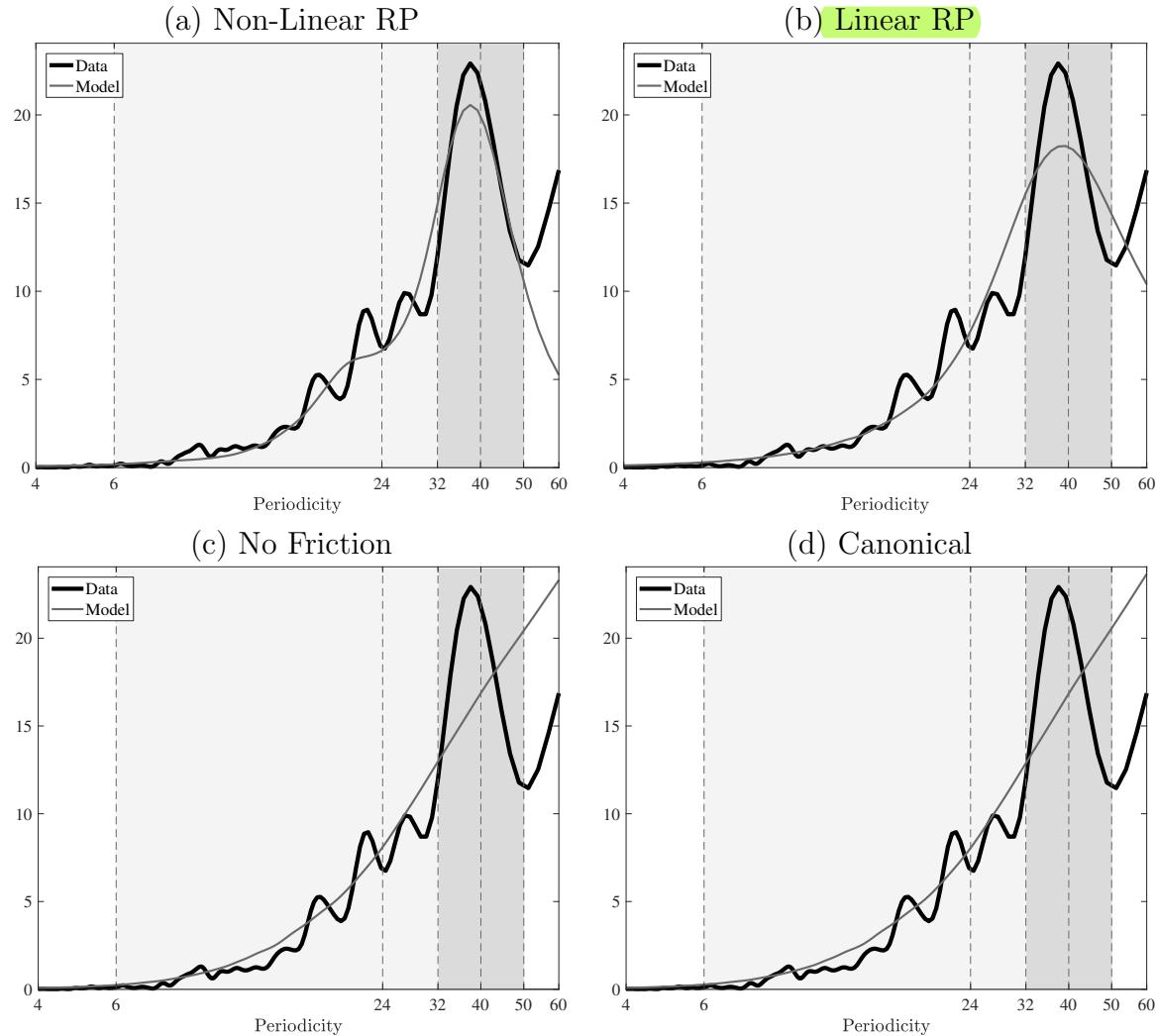
Notes: The figure displays the mean spectral density of hours over 1000 simulations of length 270. Models and parameters values are Cooley and Prescott [1995] for the standard RBC model, Fernandez-Villaverde [2016] for the augmented RBC (with variable capital utilization and investment specific technology shocks), Smets and Wouters [2007], Carlstrom and Fuerst [1997] and Christensen and Dib [2008] for a version of Bernanke, Gertler, and Gilchrist [1999] estimated on U.S. data and Christiano, Motto, and Rostagno [2014]. For better visual display, all the series have been standardized to have unit variance.

Figure 6: Spectral Density of Some Financial Series



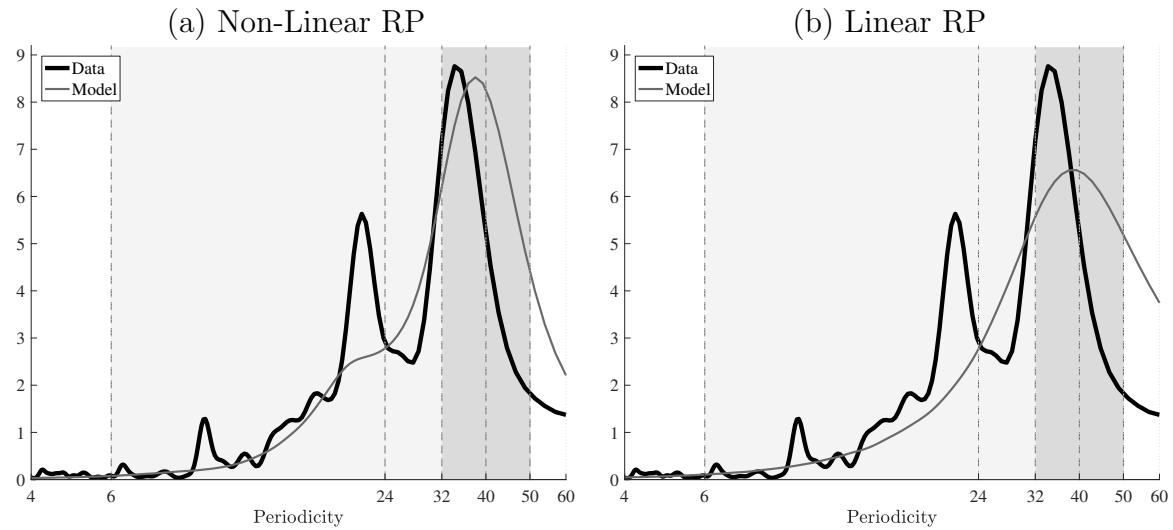
Notes: Panels (a) to (d) show estimates of the spectral density for the series in levels (black line) and for 101 series that are high-pass (P) filtered version of the levels series, with P between 100 and 200 (gray lines). A high-pass (P) filter aims to remove all fluctuations of period greater than P . All series end in 2015Q2, and start, respectively, in 1954Q3 for (a), 1973Q1 for (b), 1920Q1 for (c) and 1973Q1 for (d).

Figure 7: Fit of Hours Spectral Density



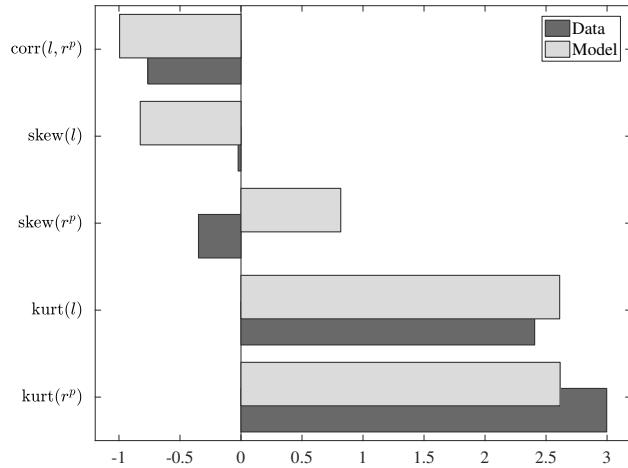
Notes: This figure compares the estimate of the spectral density of U.S. Non-Farm Business hours per capita with the ones obtained from our four estimated models.

Figure 8: Fit of Risk Premium Spectral Density



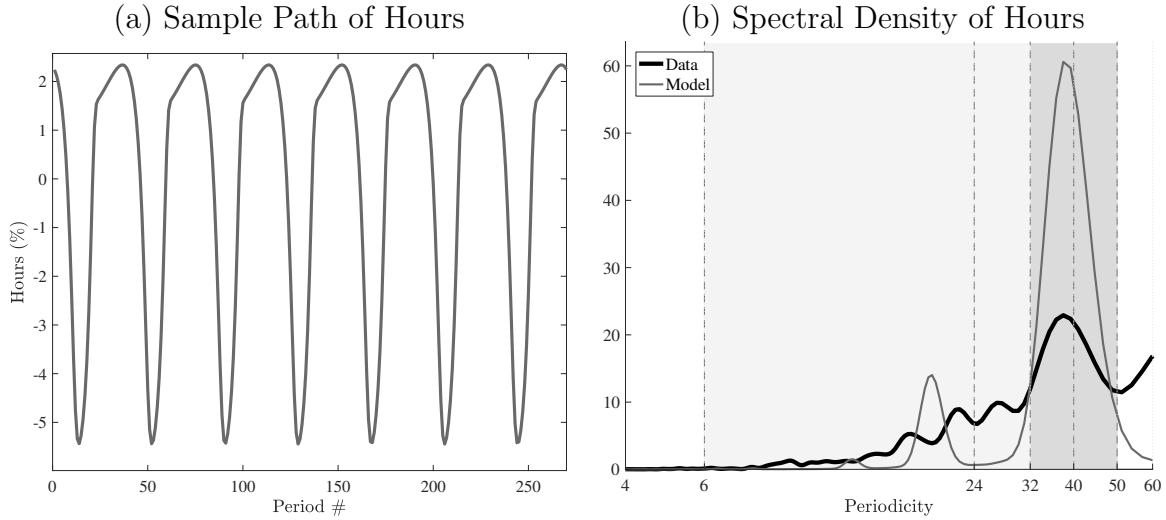
Notes: This figure compares the estimate of the spectral density of U.S. risk premium with the one obtained from our estimated non-linear and linear RP models.

Figure 9: Fit of Other Moments (Non-Linear RP)



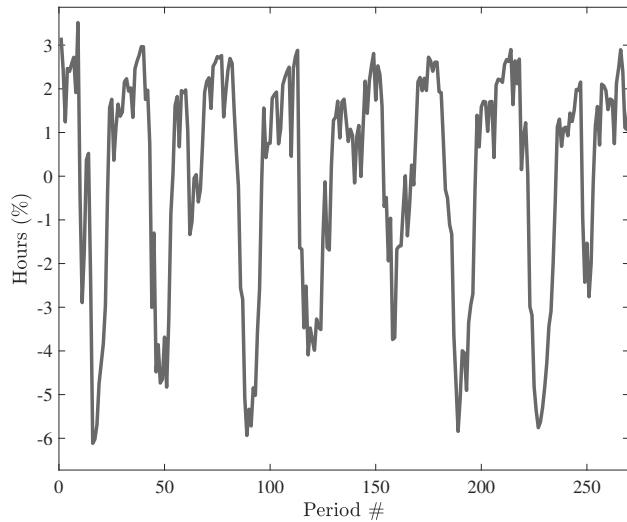
Notes: This figure compares various moments estimated using U.S. data with the ones obtained from the estimated non-linear RP model. All series have been high-pass (50) filtered, in order to remove all fluctuations of period greater than 50 quarters.

Figure 10: Non-Linear RP Model When Shocks Are Turned Off



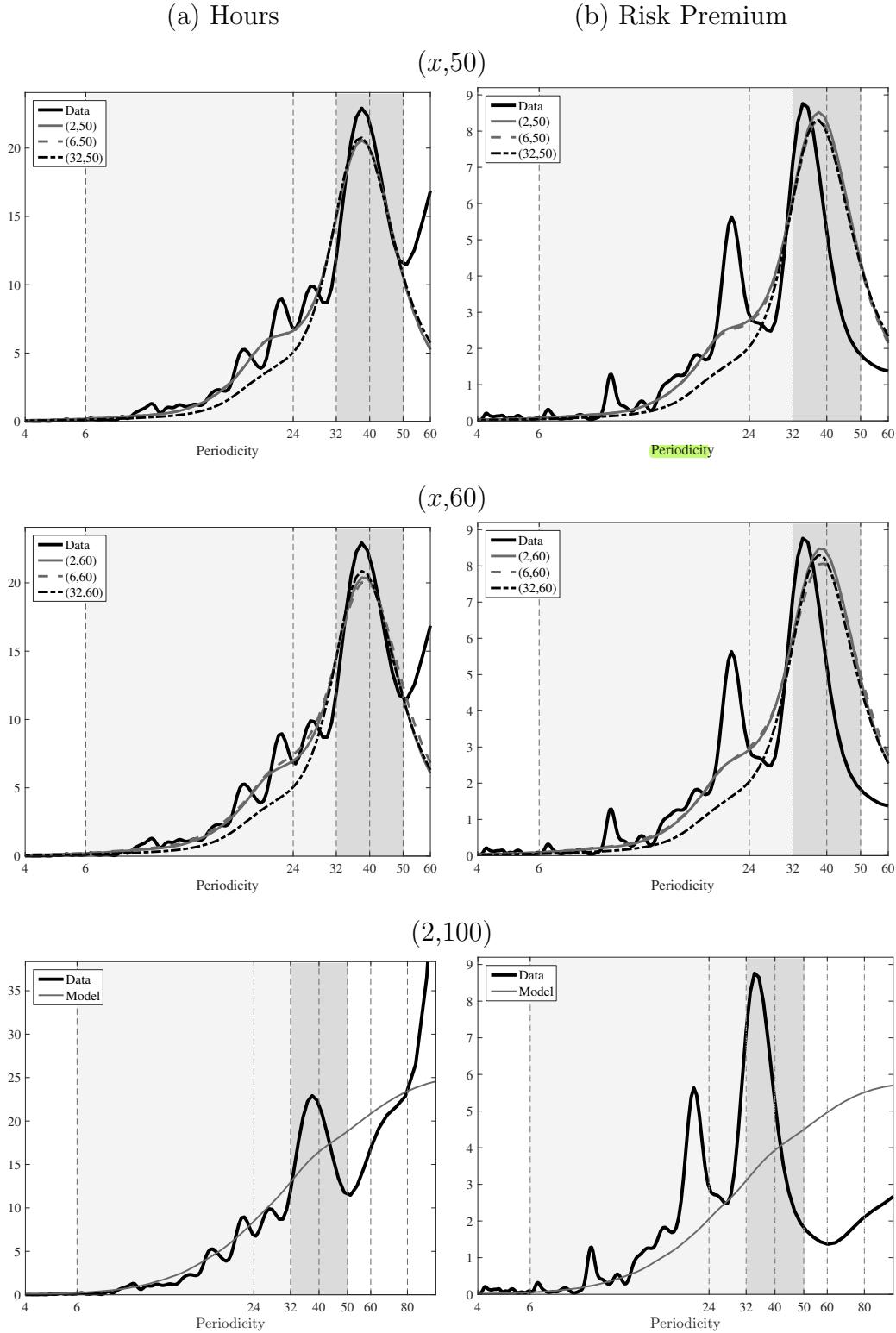
Notes: This figure corresponds to the counterfactual simulation of the estimated non-linear RP model when shocks are turned off, and initial conditions place the system on the limit cycle. Panel (a) shows the evolution of hours along the limit cycle. Panel (b) compares the spectral density of hours in the data (black line) and in the estimated model when simulated without shocks (gray line).

Figure 11: Sample Path of Hours in the Non-Linear RP Model with Shocks



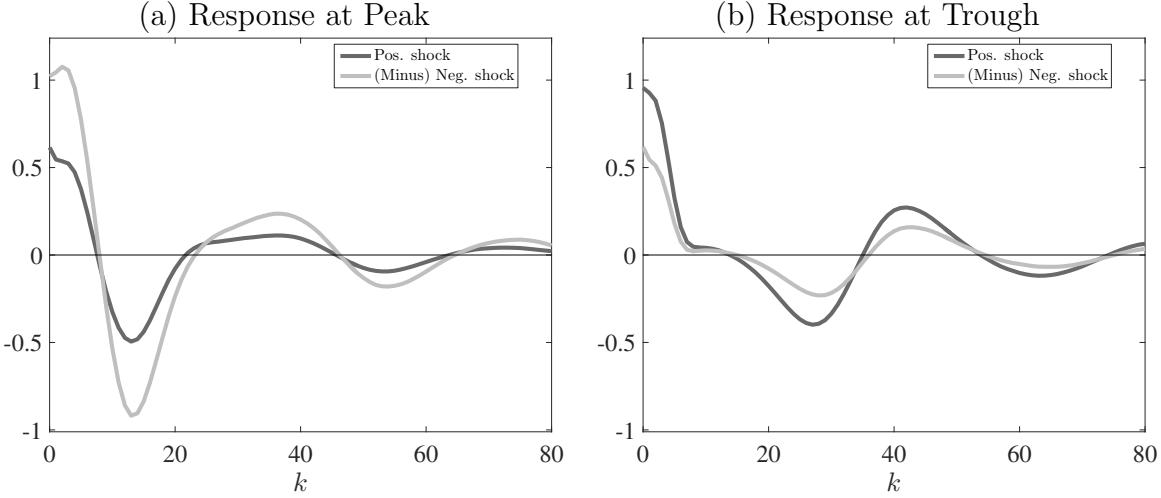
Notes: This figure shows one particular 270-quarter simulation of the estimated non-linear RP model.

Figure 12: Estimated and Data Spectral Densities for Various Estimations



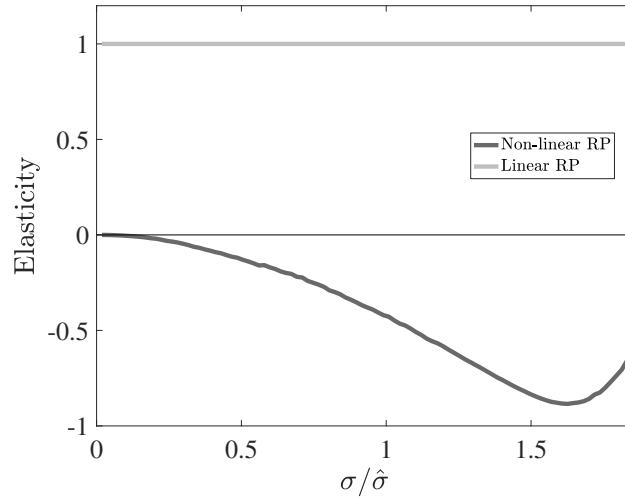
This figure shows estimated and actual spectral densities of hours (column (a)) and the risk premium (column (b)) when we target different period ranges.

Figure 13: Hours Impulse Response at Cycle Peak (Non-Linear RP)



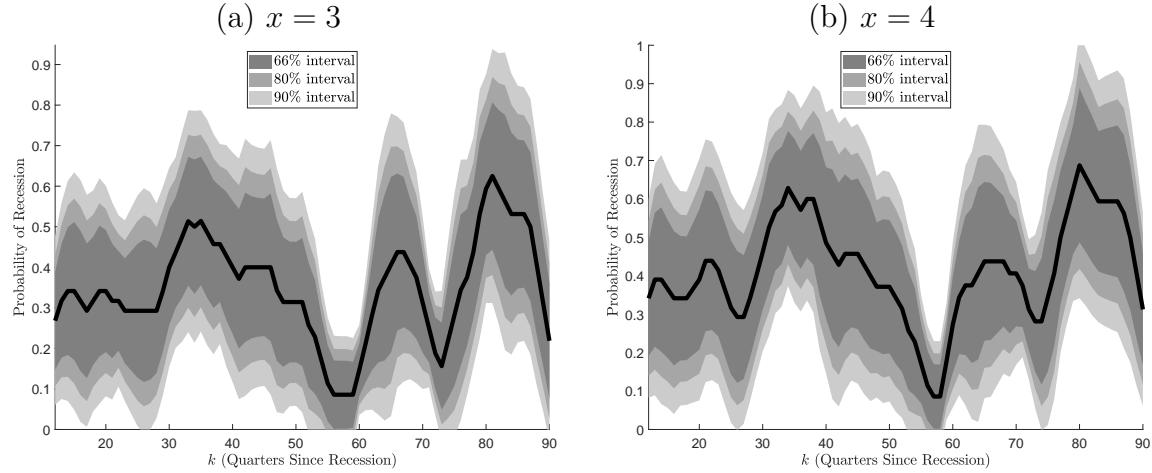
Notes: Figure shows the average response of hours in the non-linear RP model to a one-standard-deviation shock to ϵ_t at date $t + k$, for $k = 0, \dots, 80$, conditional on initially being at a peak (panel (a)) or trough (panel (b)) of the business cycle. Dark gray line shows response to a positive shock, light gray shows minus the response to a negative shock. Responses obtained as an average across 100,000 simulations, where for each simulation we compute the difference between a random simulation for the path of hours and the path that would occur if the first simulated shock were one standard deviation higher and the rest unchanged. For panel (a) we set the initial state as $(X_t, I_{t-1}, \mu_{t-1}) = (0.127, 1.731, 0)$, while for panel (b) we set it as $(-8.210, -6.030, 0)$, which correspond respectively to the peak and trough of the deterministic cycle.

Figure 14: Elasticity of s.d.(e) w.r.t. σ (Linear and Non-Linear RP)



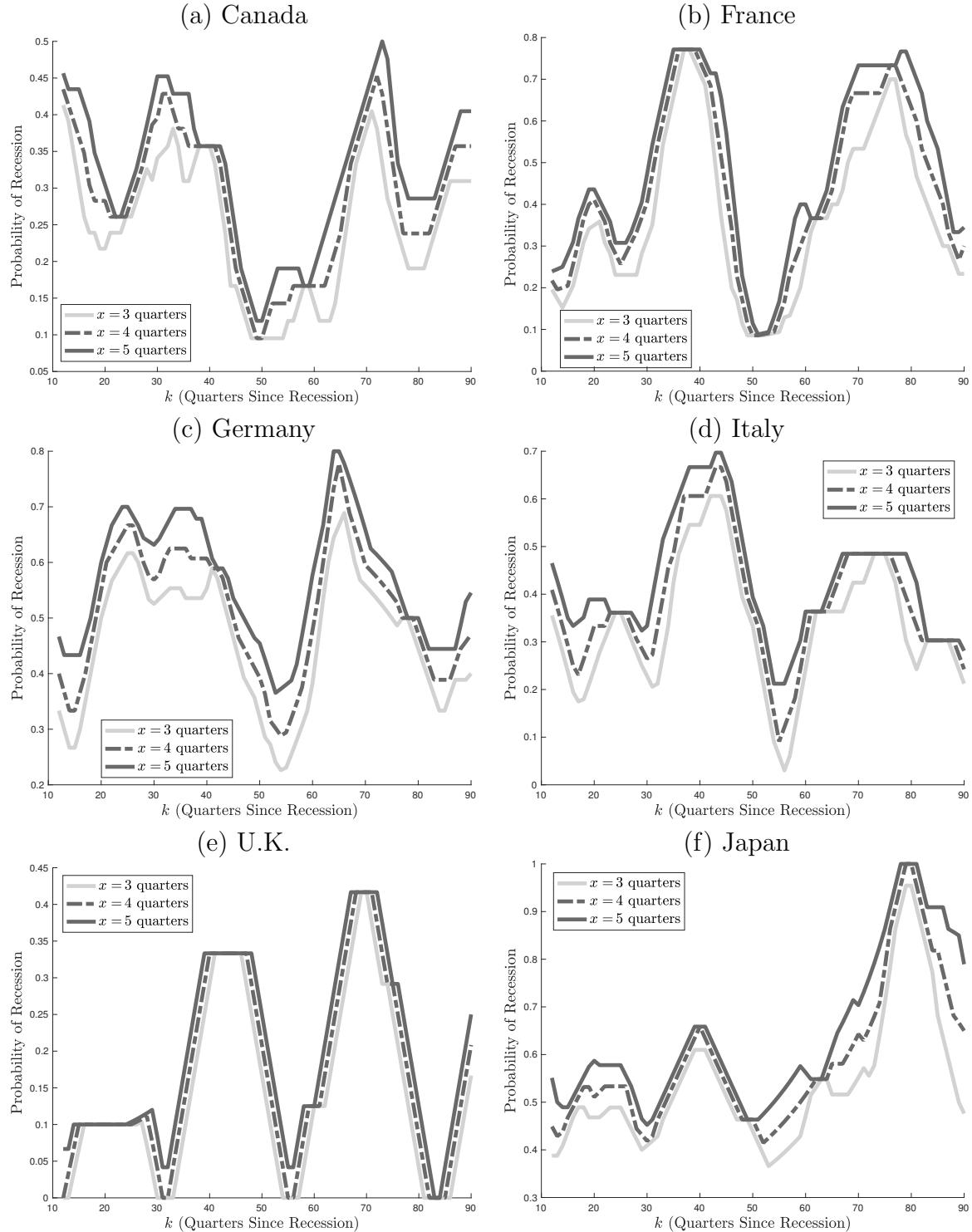
Notes: Figure plots the elasticity of the s.d. of hours with respect to σ for the linear RP (light gray line) and non-linear RP (dark gray line) models. The elasticity for the linear RP model can be shown analytically to equal 1 at all times. To compute the elasticity for the non-linear RP model, for each value of σ we re-solved the model, and then computed the sample s.d. of hours from 10,000,000 simulated periods of data.

Figure 15: Confidence Intervals for Recession Probability Plots



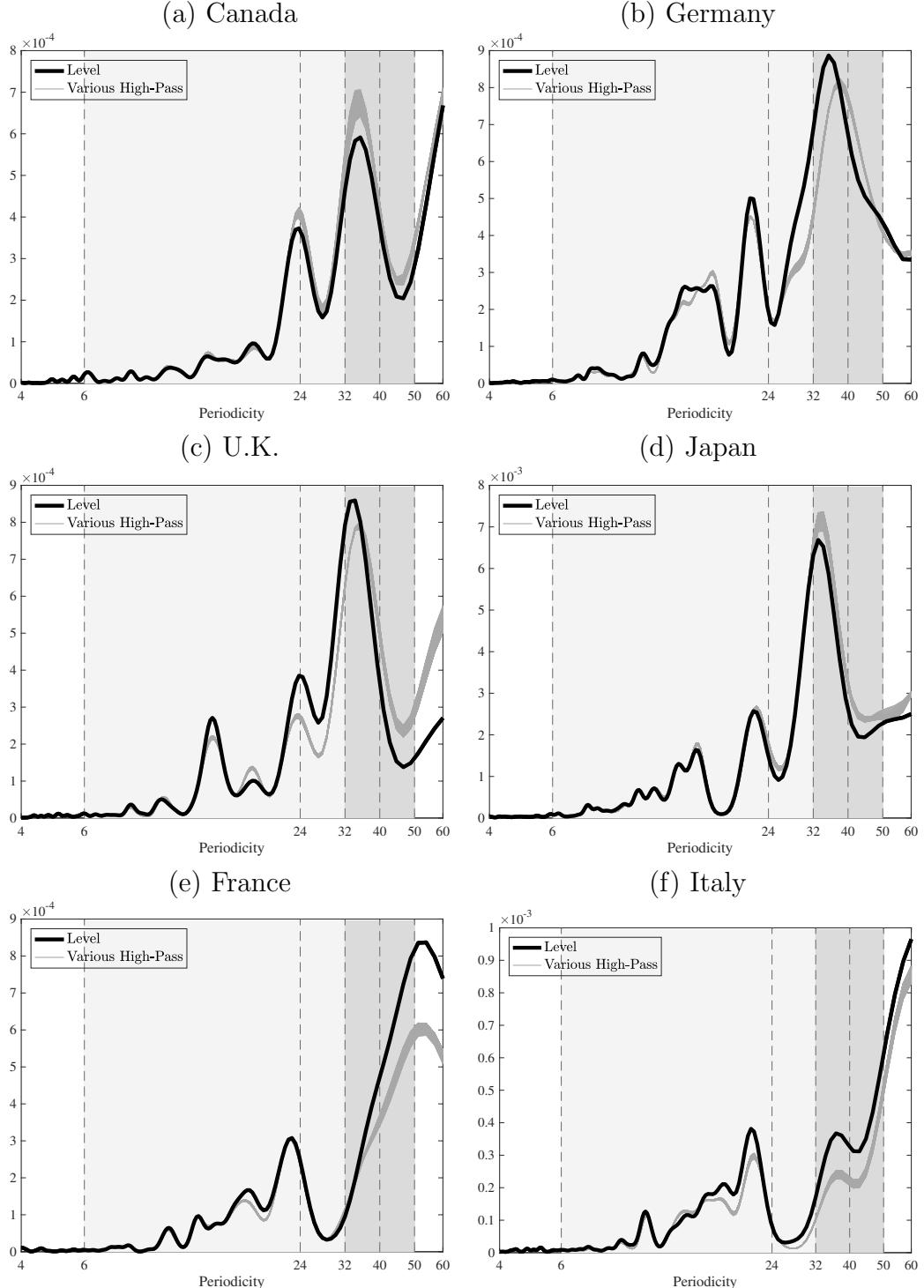
Notes: Panels (a) and (b) shows 66%, 80%, and 90% confidence intervals (gray shaded areas) for window widths $x = 3$ and $x = 4$, respectively. The solid black line in each panel gives the point estimates.

Figure 16: Conditional Probability of Being in a Recession: Other G7 countries



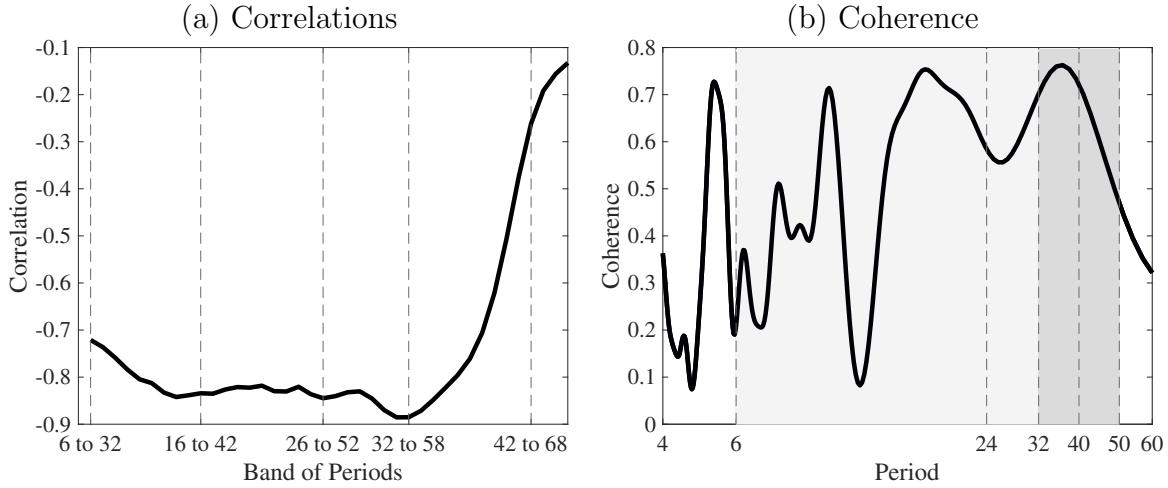
Notes: Each panel displays the fraction of time the economy was in a recession within an x -quarter window around time $t + k$, conditional on being in a recession at time t , where x is allowed to vary between 3 and 5 quarters.

Figure 17: Spectral Density of Factors Usage in the Other G7 Countries



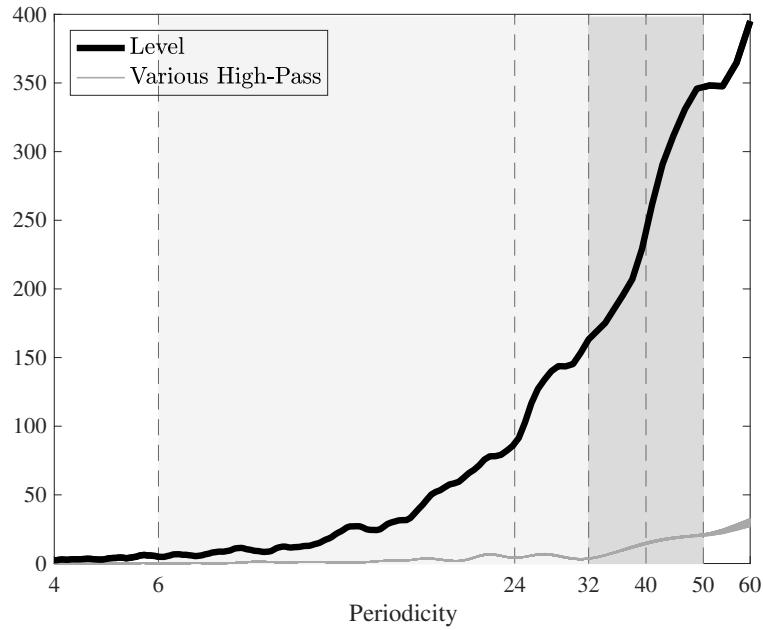
Notes: Each panel displays an estimate of the spectral density of factors usage in levels (black line) and for 101 series that are high-pass (P) filtered version of the levels series, with P between 100 and 200 (gray lines). A high-pass (P) filter removes all fluctuations of period greater than P . Factors usage is defined as $\frac{1}{3} \times \log(\text{capacity utilization rate}/100) + \frac{2}{3} \times \log(1 - \text{unemployment rate}/100)$.

Figure 18: Correlations and Coherence Between (Filtered for Correlations, Level for Coherence) Hours and BAA/FFR Spread



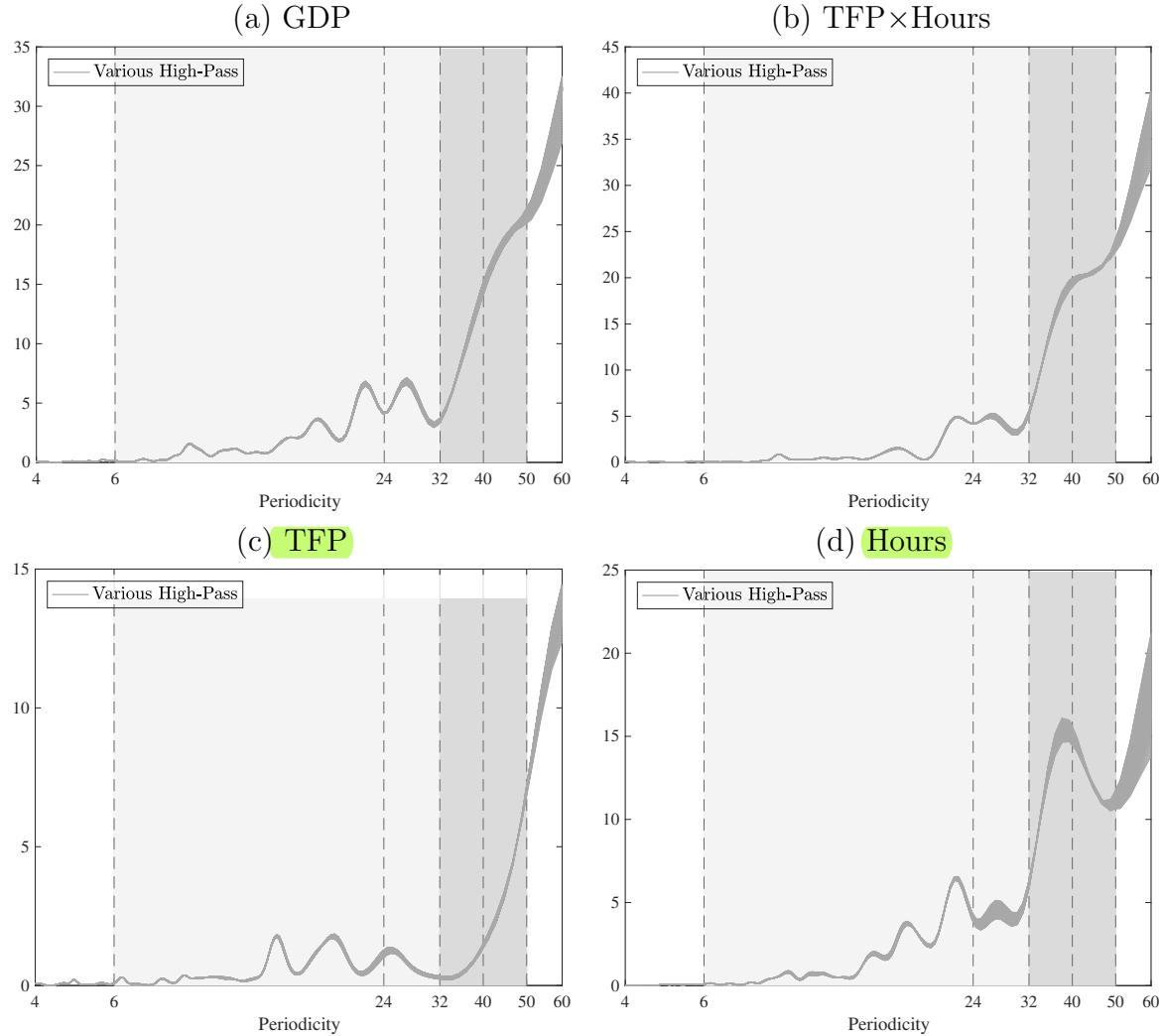
Panel (a) of This Figure displays the correlation between filtered Hours and BAA/FFR Spread, when the filter is a Band-Pass filter ($P, P + 26$), for P going from 6 quarters to 42. Panel (b) displays the coherence between the two series in levels. Sample is 1954Q3-2015Q2.

Figure 19: Spectral Density of U.S. GDP per Capita, Levels and Various High-Pass Filters



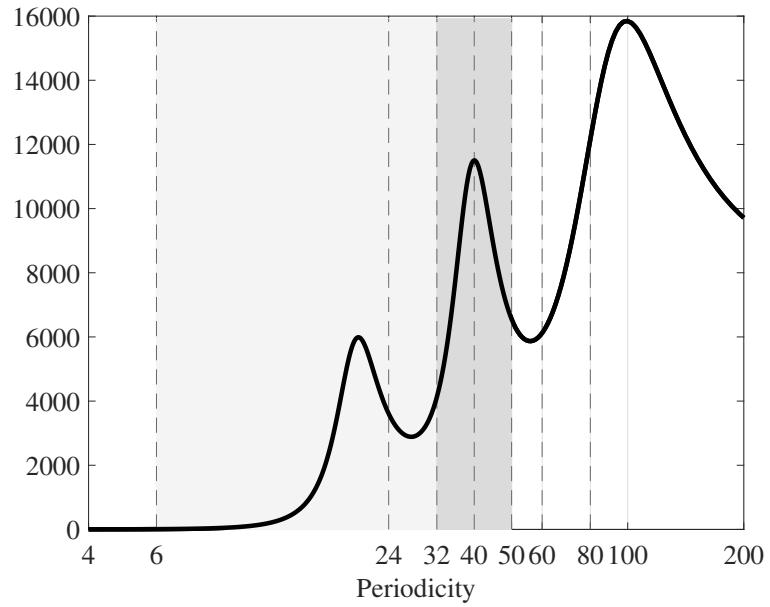
Notes: This figure shows an estimate of the spectral density of U.S. GDP per capita in levels (black line) and for 101 series that are high-pass (P) filtered version of the levels series, with P between 100 and 200 (grey lines). A high-pass (P) filter removes all fluctuations of period greater than P .

Figure 20: Decomposing the Spectral Density of GDP



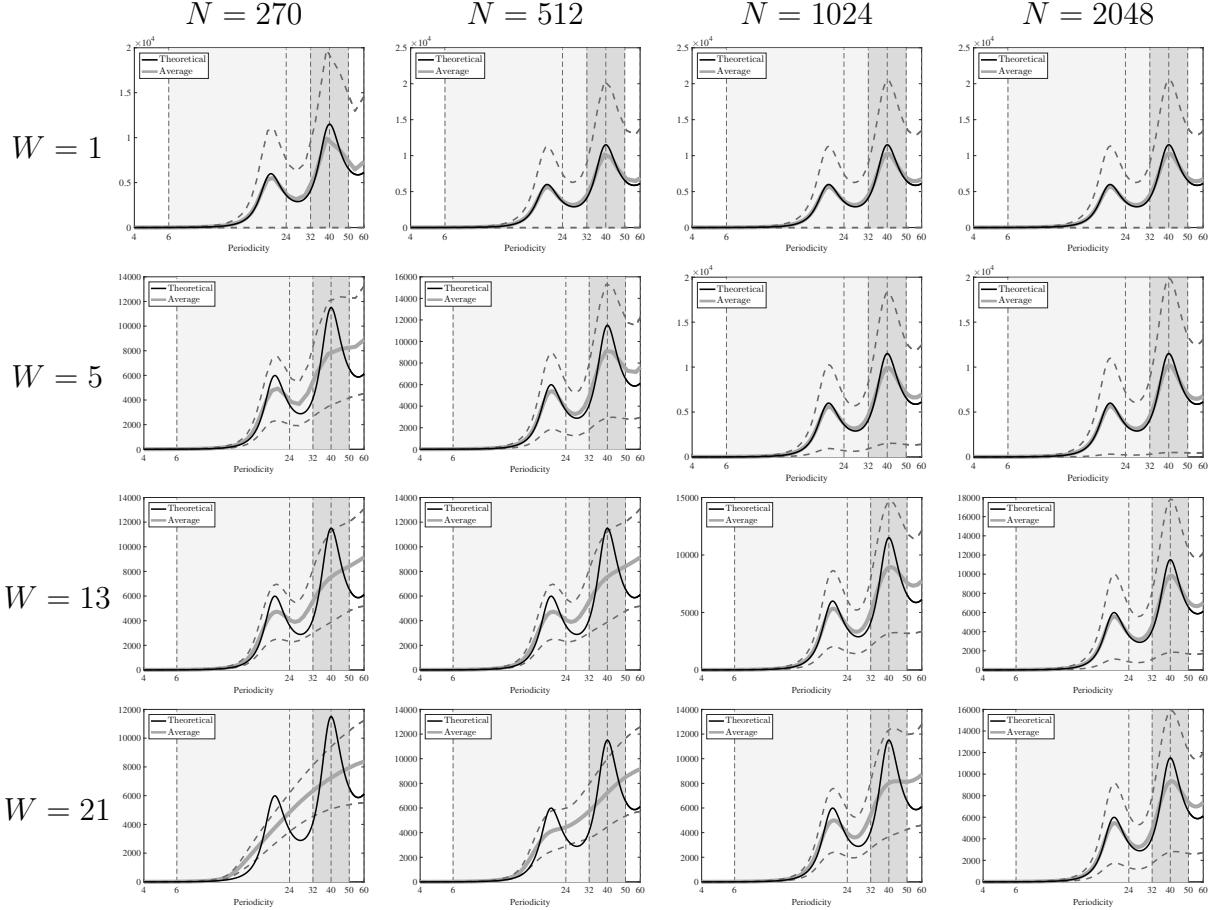
Notes: This figure shows an estimate of the spectral density of U.S. GDP per capita (panel (a)) and Total Factor Productivity (panel (b)) for 101 series that are high-pass (P) filtered version of the levels series, with P between 100 and 200 (grey lines). A high-pass (P) filter removes all fluctuations of period greater than P . TFP is the corrected quarterly TFP series of Fernald [2014].

Figure 21: Theoretical Spectral Density (Sum of Three AR(2))



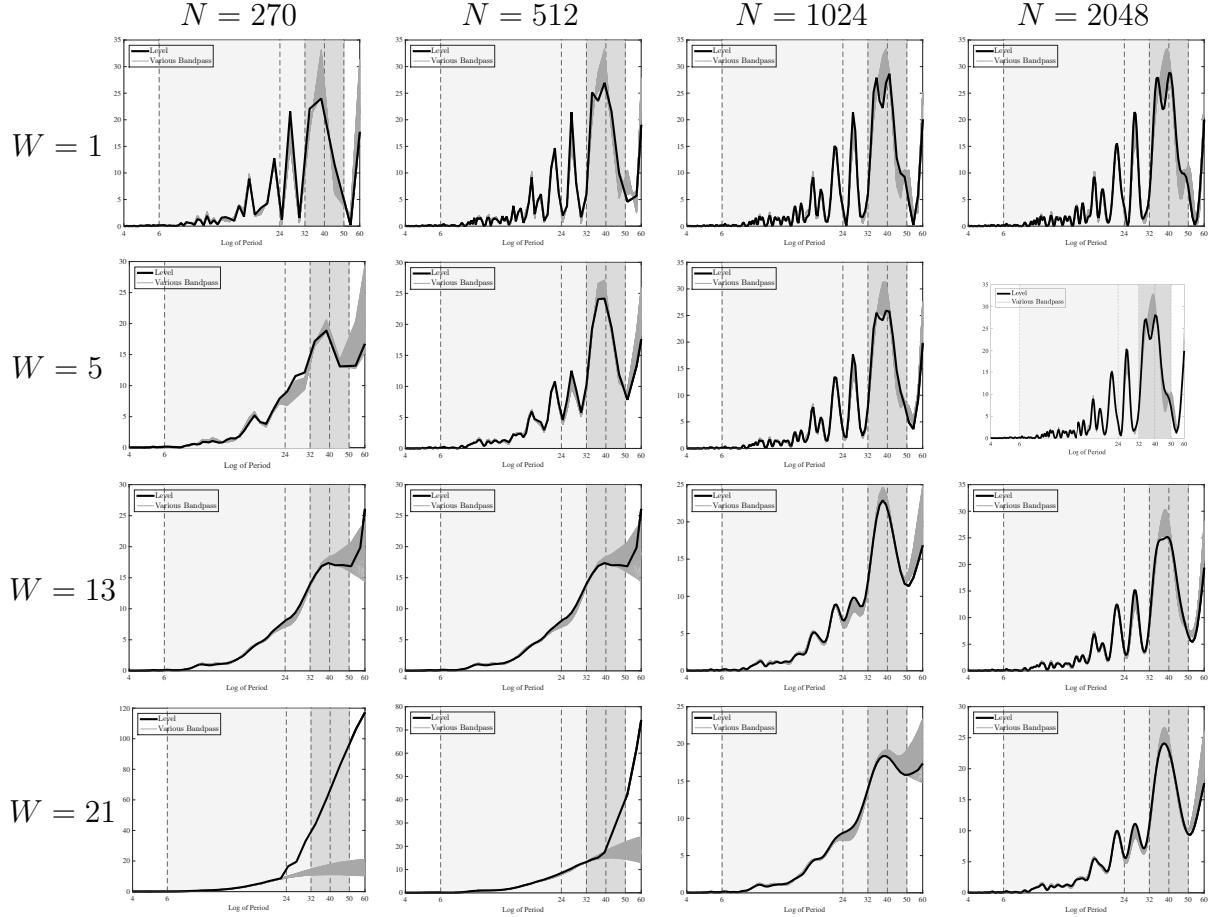
Notes: Figure shows the theoretical spectral density of the sum of three independent AR(2) processes, which have peaks in their spectral densities at, respectively, 20, 40 and 100 quarters.

Figure 22: Effects of Smoothing and Zero-Padding (Sum of Three AR(2))



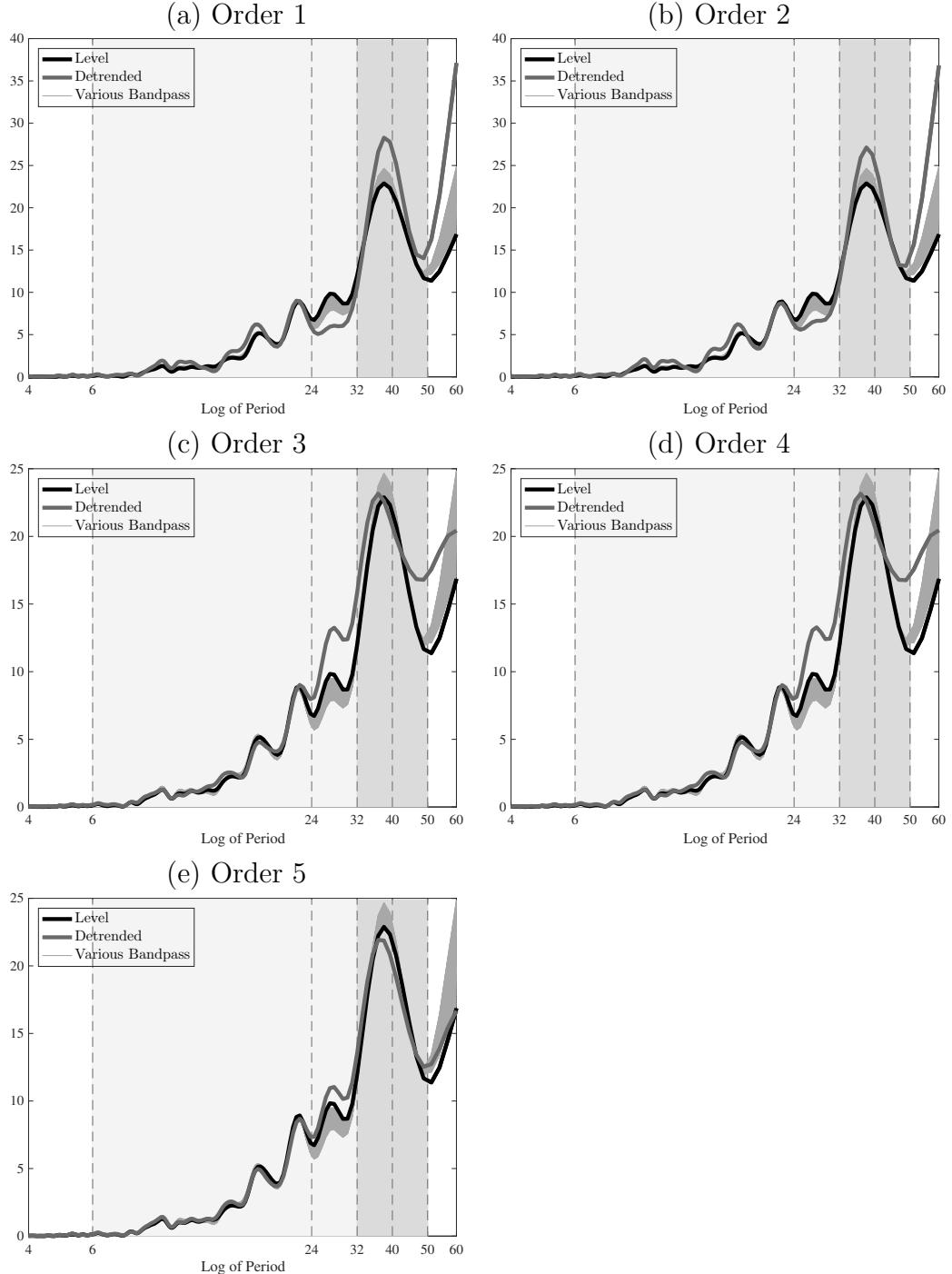
Notes: Figure shows estimates of the spectral density using simulations of the sum of three independent AR(2), which have peaks in their spectral densities at, respectively, 20, 40 and 100 quarters. The black line is the theoretical spectrum, the solid grey line is the average estimated spectrum over 1,000,000 simulations, and the dotted grey lines corresponds to that average \pm one standard deviation, and bounded below by zero. W is the length of the Hamming window (smoothing parameter) and N is the number of points at which the spectral density is evaluated (zero-padding parameter).

Figure 23: Changing Smoothing and Zero-Padding



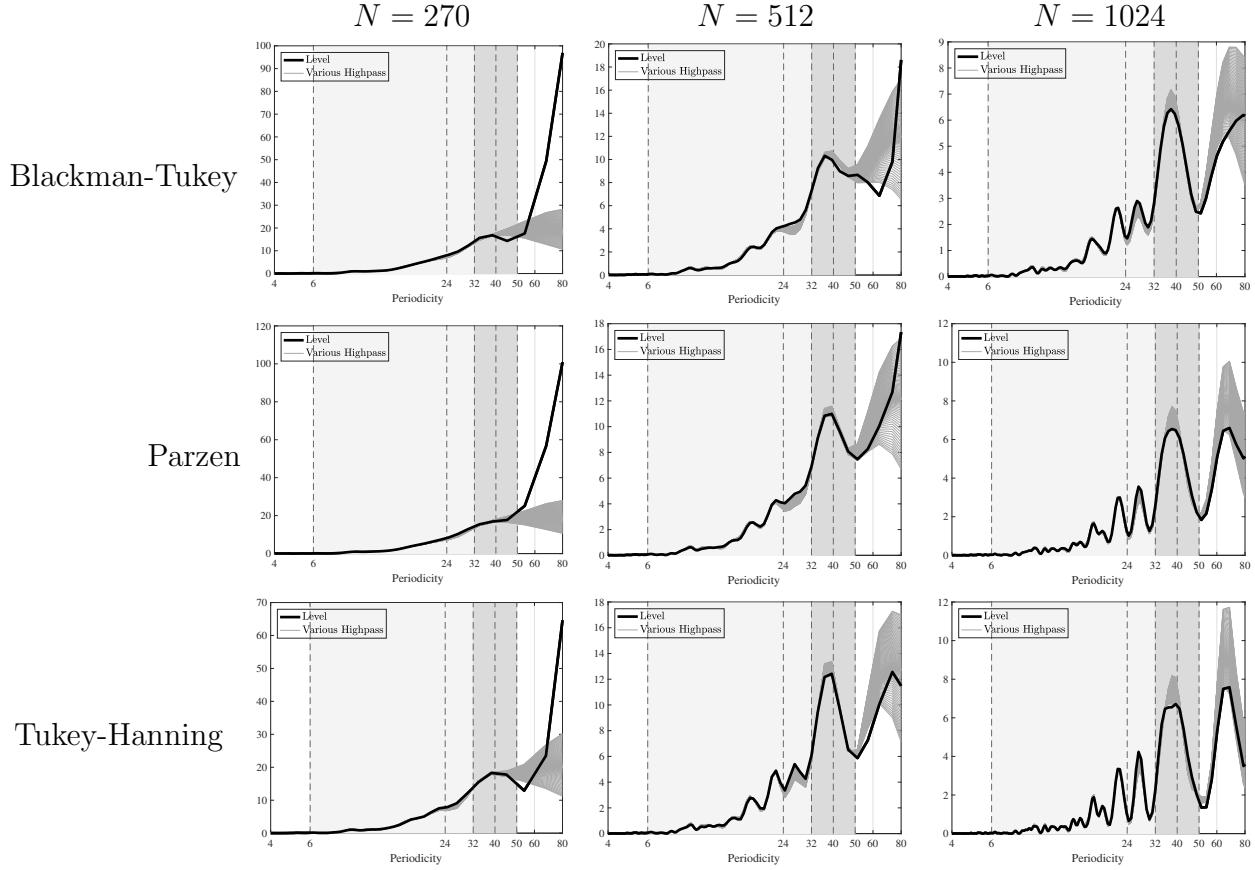
Notes: Figure shows estimates of the spectral density of U.S. Non-Farm Business Hours per Capita over the sample 1947Q1-2015Q2. The different lines correspond to estimates of the spectral density of hours in levels (black line) and of 101 series that are high-pass (P) filtered version of the levels series, with P between 100 and 200 (thin grey lines). W is the length of the Hamming window (smoothing parameter) and N is the number of points at which the spectral density is evaluated (zero-padding parameter).

Figure 24: Using a Polynomial Trend of Various Orders for Benchmark Smoothing ($W = 13$) and Zero-Padding ($N = 1024$)



Notes: Figure shows estimates of the spectral density of U.S. Non-Farm Business Hours per Capita over the sample 1947Q1-2015Q2, when polynomial trends of order 1 to 5 have been removed from the data. The different lines correspond to estimates of the spectral density of hours in levels (black line), of hours detrended with a polynomial trend (thick grey line) and of 101 series that are high-pass (P) filtered version of the levels series, with P between 100 and 200 (thin grey lines).

Figure 25: Non-Farm Business Hours with Various Windows and Estimation Using the Covariogram



Notes: This Figure shows estimates of the spectral density of U.S. Non Farm Business Hours per Capita over the sample 1947Q1-2015Q2, as computed from the covariogram using the SPECTRAN package. The different lines correspond to estimate of the spectral density of hours in levels (black line) and of 101 series that are high-pass (P) filtered version of the levels series, with P between 100 and 200 (thin grey lines). N is the number of points at which is evaluated the spectral density (zero padding).

Tables

Table 1: Canova [1996] Peak Tests: Non-Farm Business Hours

Trough period:	54		45, 54	
Peak period	locally flat	AR(1)	locally flat	AR(1)
38.6	1.0%	0.5%	11%	4.9%
33.8, 38.6	1.0%	0.4%	11%	3.8%
30, 33.8, 38.6	1.5%	0.5%	18%	5.6%

Notes: Table displays bootstrapped p-values for the test statistic $D = D_1/D_2$ under the locally flat and AR(1) null hypotheses for different combinations of periodicities contained in the peak and in the trough.

Table 2: Reiter and Woitek [1999] Peak Tests: Non-Farm Business Hours

	locally flat	AR(1)
$w = 1$	1.4%	0.3%
$w = 2$	2.5%	0.5%
$w = 3$	3.9%	0.7%

Notes: Table displays bootstrapped p-values for the test statistic $D = D_1/D_2$ under the locally flat and AR(1) null hypotheses for different window sizes around the local peak and trough mid-points.

Table 3: Canova [1996] Peak Tests: Employment and Unemployment

Employment					
Trough period:	54		45, 54		
Peak period	locally flat	AR(1)	locally flat	AR(1)	
38.6	3.9%	1.9%	10.5%	4.6%	
33.8, 38.6	5.1%	2.1%	14.4%	5.2%	

Unemployment					
Trough period:	41.8		41.8, 45.6		
Peak period	locally flat	AR(1)	locally flat	AR(1)	
35.9	4.1%	3.1%	1.3%	0.6%	
35.9, 38.6	5.7%	4.6%	1.8%	1.0%	

Notes: Table displays bootstrapped p -values for the test statistic $D = D_1/D_2$ under the locally flat and AR(1) null hypotheses for different combinations of periodicities contained in the peak and in the trough.

Table 4: Reiter and Woitek [1999] Peak Tests: Employment and Unemployment

	$w = 1$		$w = 2$	
	locally flat	AR(1)	locally flat	AR(1)
Employment	1.3%	0.3%	3.2%	0.6%
Unemployment	0.8%	0.4%	1.0%	0.5%

Notes: Table displays bootstrapped p -values for the test statistic $D = D_1/D_2$ under the locally flat and AR null hypotheses for different window sizes around the local peak and trough mid-points. The local peak mid-point is fixed at 38 quarters for both series, while the local trough mid-point is 54.9 quarters for employment and 43.5 quarters for unemployment.

Table 5: Canova [1996] Peak Tests: BAA/FF Spread

Trough period:	61		48.8, 61	
Peak period	locally flat	AR(1)	locally flat	AR(1)
34.9	0.4%	0.2%	3.2%	1.1%
34.9, 40.7	0.6%	0.3%	4.8%	2.0%

Notes: Table displays bootstrapped p -values for the test statistic $D = D_1/D_2$ under the locally flat and AR(1) null hypotheses for different combinations of periodicities contained in the peak and in the trough.

Table 6: Reiter and Woitek [1999] Peak Tests: BAA/FF Spread

	locally flat	AR(1)
$w = 1$	0.6%	0.2%
$w = 2$	0.6%	0.2%
$w = 3$	0.7%	0.2%

Notes: Table displays bootstrapped p -values for the test statistic $D = D_1/D_2$ under the locally flat and AR(1) null hypotheses for different window sizes around the local peak and trough mid-points.

Table 7: Estimated Parameter Values

		(a) Non-Linear RP	(b) Linear RP	(c) No Friction	(d) Canonical
ω	CRRA parameter	0.2997 (0.0200)	0.2408 (0.0423)	0.2408*	0.2408*
γ	Habit	0.5335 (0.0031)	0.5876 (0.0512)	0.9405 (0.0836)	0.5876*
ψ	One minus initial dep.	0.4000 (0.0028)	0.2994 (0.0644)	0.2994* —	0* —
φ_e	Taylor rule	0.0421 (0.0028)	0.0467 (0.0057)	0.0467*	0.0467*
ϕ	Debt backing	0.8668 (0.0067)	0.8827 (0.0074)	1*	1*
Φ	Recovery cost	0.0421 (0.0029)	0.0458 (0.0067)	0*	0*
ϱ_2	Risk prem. (2nd order)	0.0167 (0.0008)	—	—	—
ϱ_3	Risk prem. (3rd order)	0.5929 (0.0575)	—	—	—
ρ	Autocorrelation	-0.0000 (0.0000)	0.1387 (0.0799)	0.8541 (0.1118)	0.8609 (0.1077)
σ	Innovation s.d.	0.00016 (0.00003)	0.00027 (0.00010)	0.00148 (0.00076)	0.00148 (0.00075)
<i>s.d. (μ)</i>	Implied uncond. s.d.	0.00016	0.00027	0.00285	0.00292

Notes: Table displays the estimated parameters of the model for each of the four estimation scenarios with standard errors in parentheses. * indicates calibrated values. Estimates for the non-linear RP model imply $\varrho_1 = -0.1624$, $\phi'' = -3.1$ and $\phi''' = -227.1$, while for the linear RP model we have $\varrho_1 = -0.1506$. In the bottom row, we report the unconditional standard deviation of the shock process μ_t implied by the point estimates for ρ and σ .

Table 8: Eigenvalues at the Steady State

	Non-Linear RP	Linear RP	No Friction	Canonical
$\lambda_{11}, \lambda_{12}$	$1.1032 \pm 0.2164i$	$0.9258 \pm 0.1372i$	0.5617, 0.9488	0.5202
$ \lambda_{11} $	1.1242	0.9359	0.5617	0.5202
$ \lambda_{12} $	1.1242	0.9359	0.9488	—

Notes: Table reports the eigenvalues of the first-order approximation to the solved model around the non-stochastic steady state. Note that the solved canonical model has only one dimension (X is no longer a relevant state variable) and therefore has only one eigenvalue.

Table 9: Eigenvalues at the Steady State for Various Estimations

Est. Range	$\lambda_{11}, \lambda_{12}$	$ \lambda_{1j} $
(2,50)	$1.1032 \pm 0.2164i$	1.1242
(6,50)	$1.1048 \pm 0.2175i$	1.1260
(32,50)	$1.0626 \pm 0.2258i$	1.0864
(2,60)	$1.1182 \pm 0.1987i$	1.1358
(6,60)	$1.1212 \pm 0.1959i$	1.1382
(32,60)	$1.0671 \pm 0.2230i$	1.0902
(2,100)	0.6055, 0.9453	0.6055, 0.9453

Notes: Each line of this table corresponds to a different estimation of the non-linear RP model. In each estimation we target a different range of periods.

Table 10: Reiter and Woitek [1999] Peak Tests: Other U.S. Variables

	$w = 1$		$w = 2$	
	locally flat	AR	locally flat	AR
Unemployment (Post-War)	1.5%	0.5%	4.4%	1.4%
Capacity Utilization	5.9%	3.2%	6.8%	3.3%
Factor Usage	4.3%	1.7%	7.2%	2.7%
NFCI	0.9 %	0.4%	0.7%	0.3%
AAA/T-bill Spread	0.0%	0.0%	0.2%	0.0%
NFCI Risk Sub-index	0.0%	0.0%	0.1%	0.0%
BAA/T-bill Spread	3.7%	1.5%	3.0%	0.9%

Notes: Table displays bootstrapped p -values for the test statistic $D = D_1/D_2$ under the locally flat and AR(1) null hypotheses for different window sizes around the local peak and trough mid-points. The local peak mid-point is fixed at 38 quarters for all series, with the exception of capacity utilization which is fixed at 42 quarters, while the local trough mid-point is dictated by the location of the local minimum.

Table 11: Reiter and Woitek [1999] Peak Tests: Factors Usage (G7 Countries)

	$w = 1$		$w = 2$	
	locally flat	AR	locally flat	AR
U.K.	0.2%	0.1%	0.1%	0.0%
Germany	7.8%	3.2%	7.4%	2.5%
France	26%	12%	28%	13%
Japan	0.2%	0.2%	1.4%	0.9%
Canada	1.1%	0.9%	1.3%	0.7%
Italy	0.7%	0.5%	1.9%	1.3%

Notes: Table displays bootstrapped p -values for the test statistic $D = D_1/D_2$ under the locally flat and AR null hypotheses for different window sizes around the local peak and trough mid-points. The local peak mid-point is fixed at 38 quarters for all series, while the local trough mid-point is dictated by the location of the local minimum.