EXERCISE I.2 (HOMEWORK 1, ON LINEAR REGRESSION)

LET $E : \mathbb{R} \to \mathbb{R}$ SUCH THAT $E(x) = \max(0, 1-x)$

1) WHERE E IS DIFFERENTIABLE?

SINCE THE RESTRICTION OF E TO THE SETS $(-\infty, 1)$ AND $(1, +\infty)$ IS A POLINOMIAL (RESPECTIVELY $1-x$ AND 0), E IS DIFFERENTIABLE IN ALL THE POINTS IN THE SET $\mathbb{R}\setminus\{1\}$.

IN 1 INSTEAD IT IS NOT DIFFERENTIABLE BECAUSE THE LEFT DERIVATIVE IS -1 AND THE RIGHT DERIVATIVE IS 0.

SUMMING UP E IS DIFFERENTIABLE IN x IF AND ONLY IF $x \in \mathbb{R}\setminus\{1\}$.

2) SHOW THAT $\partial E(x) = \begin{cases} \{-1\} & \text{IF } x < 1 \\ [-1, 0] & \text{IF } x = 1 \\ \{0\} & \text{IF } x > 1 \end{cases}$

THE CONDITIONS FOR THE INTERVALS $(-\infty, 1)$ AND $(1, +\infty)$ FOLLOWS FROM THE DIFFERENTIABILITY OF E IN THOSE INTERVALS.

LET'S PROVE THE CONDITION $\partial E(1) = [-1, 0]$

WE HAVE TO PROVE THAT $\forall y \in \mathbb{R}\ \forall \lambda \in [-1, 0]$ HOLDS THAT

$$\max(0, 1-y) \geq \max(0, 0) + \langle \lambda, (y-1) \rangle$$

THAT IS EQUIVALENT TO

$$\max(0, 1-y) \geq \lambda y - \lambda. \qquad (*)$$

IF $y \in [1, +\infty)$ THE INEQUALITY $(*)$ IS EQUIVALENT TO $0 \geq \lambda(y-1)$. SINCE $y-1 \geq 0$ THEN THIS HOLDS IF AND ONLY IF $\lambda \in (-\infty, 0]$

IF $y \in (-\infty, 1)$ THE INEQUALITY $(*)$ IS EQUIVALENT TO $1-y \geq \lambda(y-1)$.

SINCE $1-y > 0$ THIS IS EQUIVALENT TO $1 \geq -\lambda$ THAT IS $\lambda \in [-1, +\infty)$

SUMMING UP, $\partial E(1) = [-1, 0]$.