Nonlinear control and aerospace applications

Attitude dynamics

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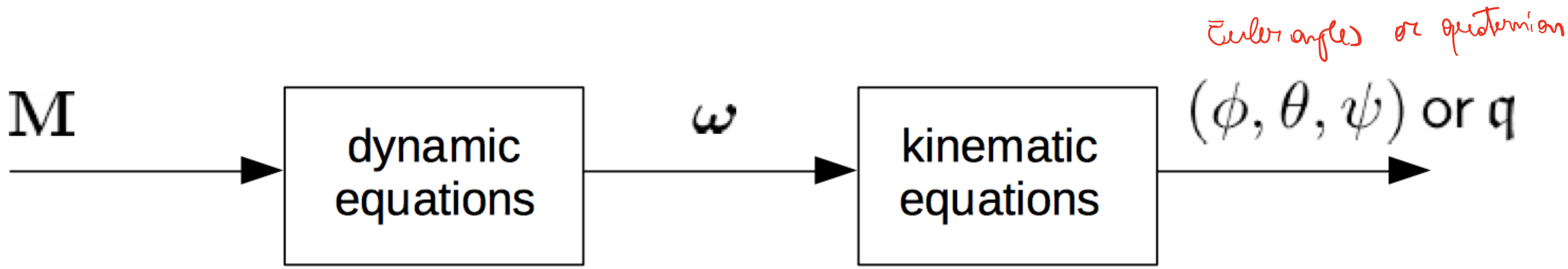
Outline

Introduction

- Controlling the spacecraft **orientation** (or **attitude**) in a continuous and autonomous manner is fundamental.
  - Spacecrafts and space stations orbiting around a planet, or during interplanetary navigation,
    - must capture the solar energy through panels,
    - need a communication link between on-board antennas and Earth stations/receivers or relay satellites.
  - Scientific satellites and space vehicles carry payloads to be pointed toward either celestial objects or Earth targets (e.g., the Hubble).
- The **objective** is to derive the *attitude dynamic equations* for a rigid body in rotational motion.
  - These equations, together with the kinematic equations, are fundamental for spacecraft attitude control.

Introduction

- The dynamic and kinematic equations can be seen as the series connection of two nonlinear systems:
  - the dynamic equations define a system from $\mathbf{M}$ to $\boldsymbol{\omega}$, where $\mathbf{M}$ is the moment applied to the body;
  - the kinematic equations define a system from $\boldsymbol{\omega}$ to (ϕ, θ, ψ) or $\mathfrak{q}$.



- Overall, it is a nonlinear system with:
  - input $\mathbf{M}$;
  - output (ϕ, θ, ψ), DCM or $\mathfrak{q}$. $\leftarrow$ output to control.

Introduction

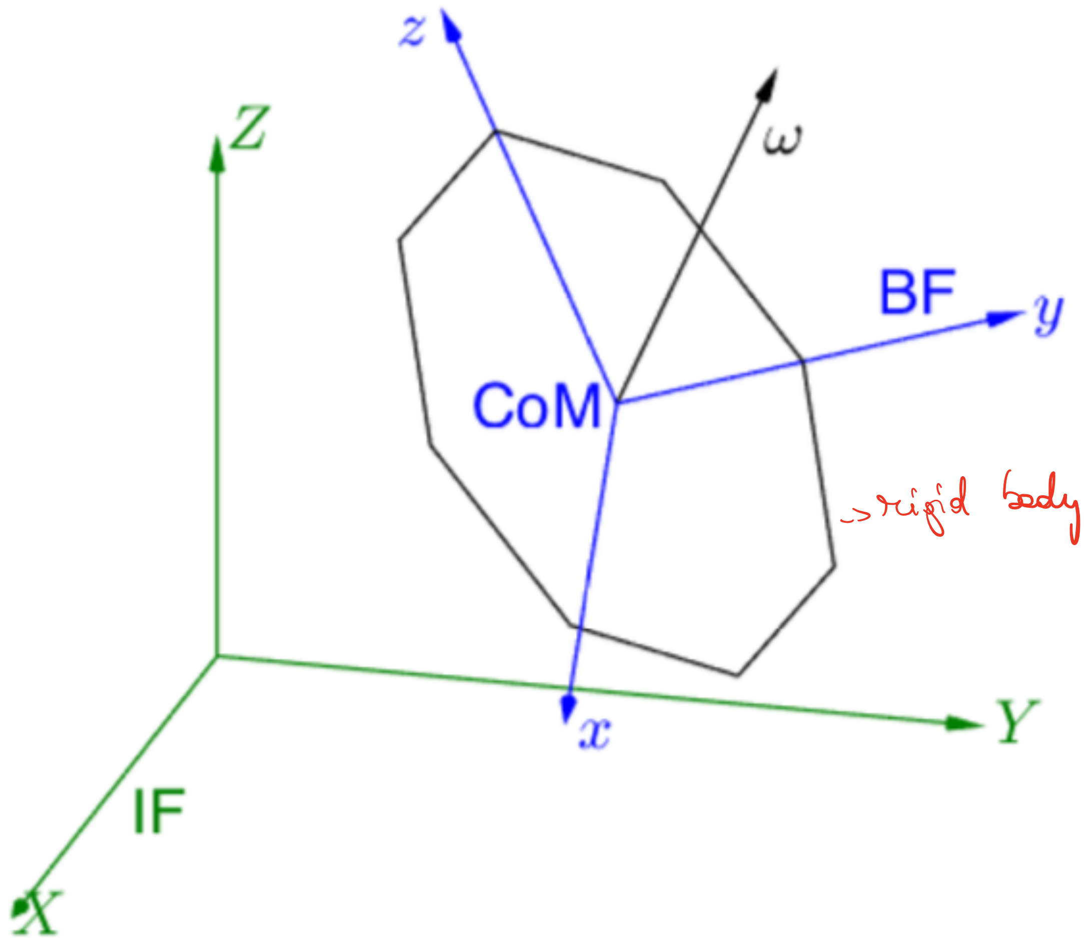
- Consider a rigid body rotating wrt an inertial reference frame with angular velocity $\boldsymbol{\omega} = \omega_1 \mathbf{b}_1 + \omega_2 \mathbf{b}_2 + \omega_3 \mathbf{b}_3$.

Inertial frame (IF):

- moving with a constant velocity
- origin: somewhere
- unit vectors: $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$
- axes: X, Y, Z.

Body frame (BF):

- moving with the body
- origin: body CoM
- unit vectors: $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$
- axes: x, y, z.



Introduction

- For a particle of the body with mass m_i:

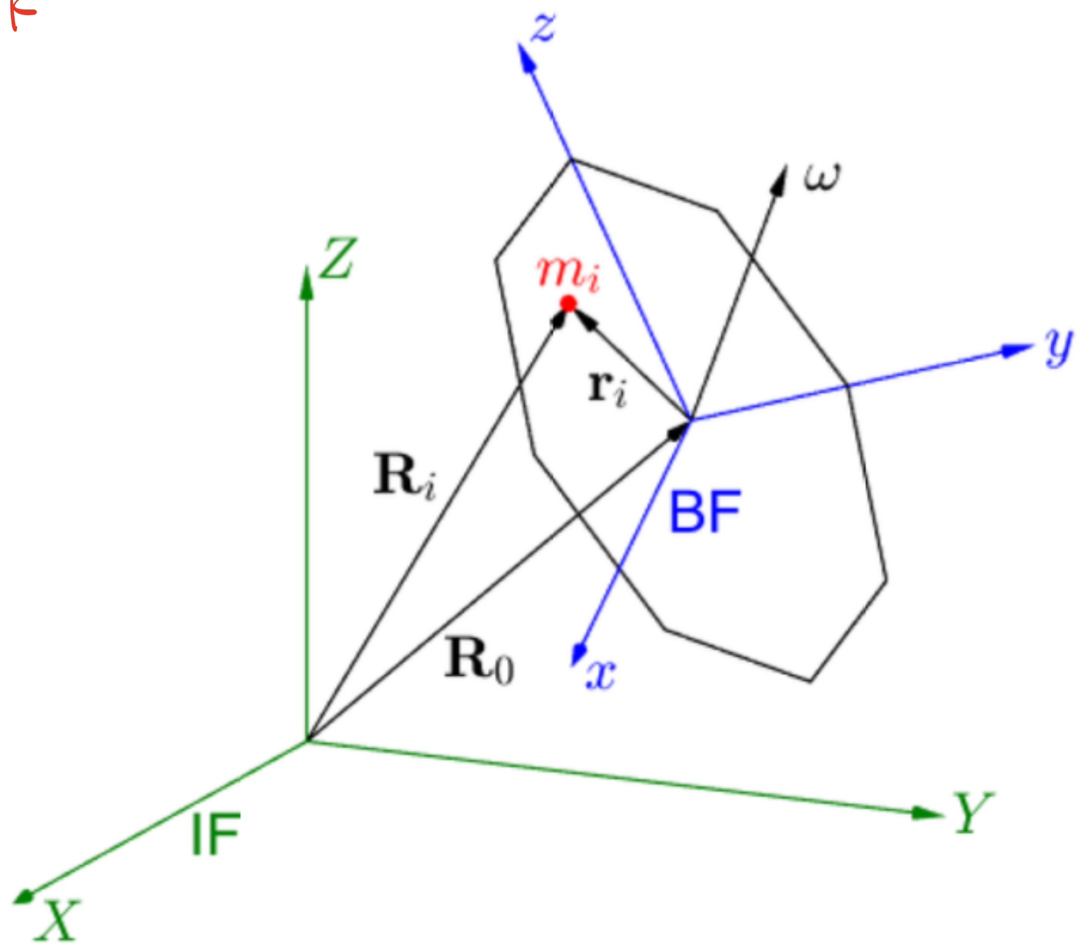
$$\mathbf{R}_i = X\mathbf{i}_1 + Y\mathbf{i}_2 + Z\mathbf{i}_3$$
$$\mathbf{R}_o = X_o\mathbf{i}_1 + Y_o\mathbf{i}_2 + Z_o\mathbf{i}_3$$

IF

position of the center

$$\mathbf{R}_i = \mathbf{R}_o + \mathbf{r}_i$$
$$\dot{\mathbf{R}}_i = \dot{\mathbf{R}}_o + \dot{\mathbf{r}}_{iB} + \boldsymbol{\omega} \times \mathbf{r}_i$$

$$\mathbf{r}_i = x\mathbf{b}_1 + y\mathbf{b}_2 + z\mathbf{b}_3$$
$$\dot{\mathbf{r}}_{iB} = \dot{x}\mathbf{b}_1 + \dot{y}\mathbf{b}_2 + \dot{z}\mathbf{b}_3$$
$$\boldsymbol{\omega} = \omega_1\mathbf{b}_1 + \omega_2\mathbf{b}_2 + \omega_3\mathbf{b}_3.$$



Angular momentum

- Angular momentum (moment of momentum) of the particle:

Position Vector

$$\mathbf{H}_i \doteq \mathbf{r}_i \times m_i \dot{\mathbf{R}}_i = \mathbf{r}_i \times m_i \left(\dot{\mathbf{R}}_o + \dot{\mathbf{r}}_{iB} + \boldsymbol{\omega} \times \mathbf{r}_i \right).$$

- Being $\dot{\mathbf{r}}_{iB} = \mathbf{0}$ (rigid body), m_i is fixed because part of the body

$$\begin{aligned} \mathbf{H}_i &= \mathbf{r}_i \times m_i \left(\dot{\mathbf{R}}_o + \boldsymbol{\omega} \times \mathbf{r}_i \right) \\ &= -\dot{\mathbf{R}}_o \times m_i \mathbf{r}_i + \mathbf{r}_i \times m_i \left(\boldsymbol{\omega} \times \mathbf{r}_i \right). \end{aligned}$$

Angular momentum

- The angular momentum of the entire body is

$$\begin{aligned}\mathbf{H} &= -\sum_i \dot{\mathbf{R}}_o \times m_i \mathbf{r}_i + \sum_i \mathbf{r}_i \times m_i (\boldsymbol{\omega} \times \mathbf{r}_i) \\ &= -\dot{\mathbf{R}}_o \times \sum_i m_i \mathbf{r}_i + \sum_i \mathbf{r}_i \times m_i (\boldsymbol{\omega} \times \mathbf{r}_i)\,.\end{aligned}$$

- By definition of CoM, $\sum_i m_i \mathbf{r}_i = \mathbf{0}$. Hence,

$$\mathbf{H} = \sum_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) m_i$$ → finite number of small masses

continuous body

$$\mathbf{H} = \int_B \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm \quad \text{for } m_i \to dm.$$

No translating quantities appear → the rotational motion is independent of the translational motion.

Inertia matrix

- Performing the products and using a matrix notation, in body coordinates we obtain

$$\mathbf{H} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \mathbf{J}\boldsymbol{\omega}$$

$$\left.\begin{array}{l} J_{11} = \int_B \left(y^2 + z^2\right) dm \\ J_{22} = \int_B \left(x^2 + z^2\right) dm \\ J_{33} = \int_B \left(x^2 + y^2\right) dm \end{array}\right\} \text{moments of inertia}$$

$$\left.\begin{array}{l} J_{12} = J_{21} = -\int_B xy\, dm \\ J_{13} = J_{31} = -\int_B xz\, dm \\ J_{23} = J_{32} = -\int_B yz\, dm \end{array}\right\} \text{products of inertia.}$$

$\mathbf{J}$ is the *inertia matrix* (or *inertia tensor*).

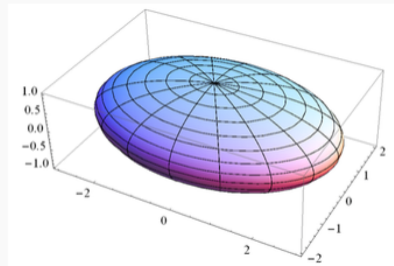
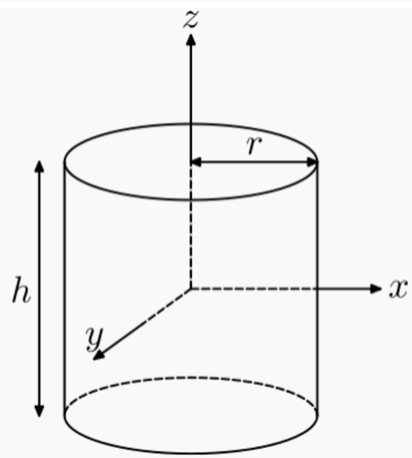
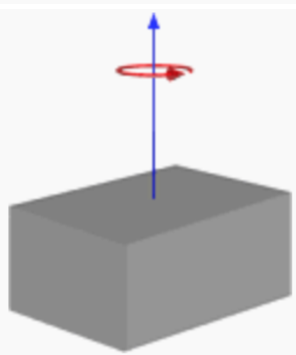
Principal axes of inertia

- A body frame where the inertia matrix $\mathbf{J}$ is *diagonal* can always be found by means of a *rotation*:

$$\mathbf{J} = \mathbf{T}^T \mathbf{J}' \mathbf{T}$$

  - $\mathbf{J}' = \left[J'_{ij} \right]$: non-diagonal inertia matrix;
  - $\mathbf{J} = \mathrm{diag}(J_1, J_2, J_3) = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$: diagonal inertia matrix;
  - $\mathbf{T} = [\mathbf{e}_1 \; \mathbf{e}_2 \; \mathbf{e}_3]$
    - $\mathbf{e}_i$: eigenvectors of $\mathbf{J}'$;
    - $\mathbf{T}$ is a rotation matrix, since $\mathbf{J}'$ is real and symmetric.
- With this transformation, $\mathbf{J}$ is a diagonal matrix with entries equal to the *eigenvalues* of $\mathbf{J}'$.
  - The eigenvalues of $\mathbf{J}'$ are said the *principal moments of inertia*.
  - The eigenvectors of $\mathbf{J}'$ are said the *principal axes of inertia*.

Examples of principal moments/axes

Ellipsoid (solid) of semiaxes *a*, *b*, and *c* with mass *m*		$I_a = \frac{m(b^2+c^2)}{5}$ $I_b = \frac{m(a^2+c^2)}{5}$ $I_c = \frac{m(a^2+b^2)}{5}$
Solid cylinder of radius *r*, height *h* and mass *m*. This is a special case of the thick-walled cylindrical tube, with $r_1 = 0$. (Note: X-Y axis should be swapped for a standard right handed frame).		$I_z = \frac{mr^2}{2}$ [1] $I_x = I_y = \frac{1}{12}m\left(3r^2+h^2\right)$
Solid cuboid of height *h*, width *w*, and depth *d*, and mass *m*. For a similarly oriented cube with sides of length s, $I_{CM} = \frac{ms^2}{6}$		$I_h = \frac{1}{12}m\left(w^2+d^2\right)$ $I_w = \frac{1}{12}m\left(h^2+d^2\right)$ $I_d = \frac{1}{12}m\left(h^2+w^2\right)$

Euler moment equations

- Suppose that a moment $\mathbf{M} = M_1\mathbf{b}_1 + M_2\mathbf{b}_2 + M_3\mathbf{b}_3$ is acting on the body B. The II law of dynamics for a rotating body is

$$\dot{\mathbf{H}} = \mathbf{M}.$$

- Since $\dot{\mathbf{H}} = \dot{\mathbf{H}}_B + \boldsymbol{\omega} \times \mathbf{H}$, the equation becomes

$$\dot{\mathbf{H}}_B + \boldsymbol{\omega} \times \mathbf{H} = \mathbf{M}.$$

- Being $\mathbf{H} = \mathbf{J}\boldsymbol{\omega}$ and $\dot{\mathbf{H}}_B = \mathbf{J}\dot{\boldsymbol{\omega}}$, we obtain the *Euler moment equation*:

diagonal

inertia matrix

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} = \mathbf{M}.$$

moment applied to the body

angular speed vector

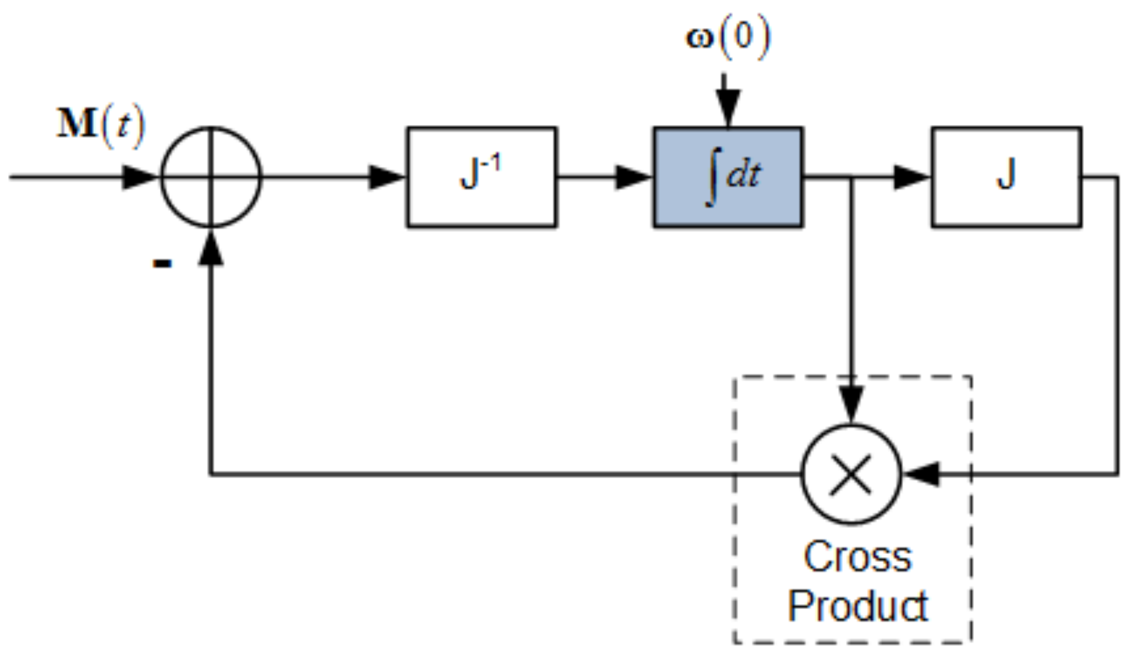
- This equation is nonlinear. In general, no analytical solution is available.

Euler moment equations

- The Euler moment equation

$$\mathbf{J}\dot{\boldsymbol{\omega}} = \mathbf{M} - \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega}.$$

can be implemented using the following block diagram:



Euler moment equations

- Assuming a body frame in which $\mathbf{J} = \mathrm{diag}(J_1, J_2, J_3)$ and performing the product, the Euler moment equation becomes

$$J_1\,\dot{\omega}_1 + (J_3 - J_2)\,\omega_2\,\omega_3 = M_1$$
$$J_2\,\dot{\omega}_2 + (J_1 - J_3)\,\omega_1\,\omega_3 = M_2$$
$$J_3\,\dot{\omega}_3 + (J_2 - J_1)\,\omega_1\,\omega_2 = M_3.$$

- This can be written in matrix form as

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} 0 & \sigma_1\omega_3 & 0 \\ \sigma_2\omega_3 & 0 & 0 \\ \sigma_3\omega_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} M_1/J_1 \\ M_2/J_2 \\ M_3/J_3 \end{bmatrix}$$

$$\sigma_1 = \frac{J_2 - J_3}{J_1},\ \sigma_2 = \frac{J_3 - J_1}{J_2},\ \sigma_3 = \frac{J_1 - J_2}{J_3}.$$

- These equations are nonlinear. In general, no analytical solution is available.

Free motion of a symmetric body

- Suppose that no external input
  - $\mathbf{M} = \mathbf{0}$ (free motion = free response = homogeneous equation);
  - $J_1 = J_2 \neq J_3$ (axisymmetry, with symmetry axis $\mathbf{b}_3$).
- The Euler equations become

$$\begin{aligned} J_1\dot{\omega}_1 + \omega_2\omega_3(J_3 - J_1) &= 0 \\ J_2\dot{\omega}_2 + \omega_1\omega_3(J_1 - J_3) &= 0 \\ J_3\dot{\omega}_3 &= 0. \end{aligned}$$

- The third one implies $\omega_3 = \text{const.}$
- It follows that:
  1. The body rotates with constant velocity about the symmetry axis z.
  2. The equations become *linear*, allowing a complete theoretical analysis and the calculation of the analytical solution.

Free motion of a symmetric body

constant

- Defining $\eta \doteq \omega_3(J_3 - J_1)/J_1$, we obtain

$$\dot{\omega}_1 + \eta\omega_2 = 0 \quad (1)$$
$$\dot{\omega}_2 - \eta\omega_1 = 0. \quad (2)$$

- Multiplying (1) by ω_1 and (2) by ω_2, and adding the two equations, we obtain

$$\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 = 0$$

simplify dt

$$\omega_1\, d\omega_1 + \omega_2\, d\omega_2 = 0$$

integrating

$$\omega_1^2 + \omega_2^2 = \text{const}.$$

norm of w

- Since $\omega_3 = \text{const}$, we have that $|\boldsymbol{\omega}| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \text{const}$, showing that the norm of the body angular velocity is constant.

Free motion of a symmetric body

- Differentiating (1) and using (2), we obtain

$$\ddot{\omega}_1 + \eta^2 \omega_1 = 0$$

  that is the *harmonic oscillator equation*.

- An harmonic oscillator is a *marginally stable* (*or simply stable*) system (with two complex conjugate poles having null real parts).

- It follows that:
  1. ω_1 always remains bounded (without converging to the origin).
  2. In particular, ω_1 is an harmonic signal.

- Analogous results hold for ω_2.

Free motion of a symmetric body

- Applying the Laplace transform to the oscillator equation:

$$s^2\Omega_1(s) - s\omega_1(0) - \dot{\omega}(0) + \eta^2\Omega_1(s) = 0$$

$$\Omega_1(s) = \frac{s\omega_1(0)}{s^2+\eta^2} + \frac{\dot{\omega}_1(0)}{s^2+\eta^2}.$$

- Applying the inverse Laplace transform:

$$\omega_1(t) = \omega_1(0)\cos(\eta t) + \frac{\dot{\omega}_1(0)}{\eta}\sin(\eta t).$$

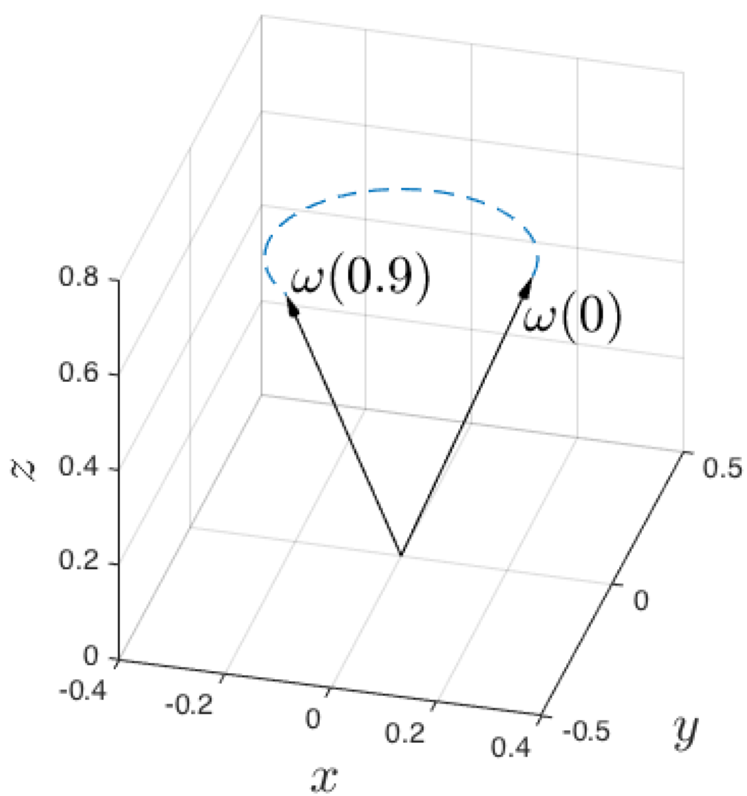
- A similar result holds for $\omega_2(t)$:

$$\omega_2(t) = \omega_1(0)\sin(\eta t) - \frac{\dot{\omega}_1(0)}{\eta}\cos(\eta t).$$

Free motion of a symmetric body

Example

- A body with the following inertia matrix is considered: $\mathrm{diag}(J_1, J_2, J_3) = \mathrm{diag}(50, 50, 400)\ \mathrm{kg\,m^2}$.
- The following initial conditions are assumed: $\omega(0) = (0.25, 0, 0.63)\ \mathrm{rad/s}$.



ω_3 is constant;

ω_1 and ω_2 are harmonic signals;

trajectory of $\boldsymbol{\omega}$ close to the z axis;

trajectory of (ω_1, ω_2) = combination of harmonic curves $\rightarrow$ ellipse;

$\boldsymbol{\omega}$ draws the body cone.

Free motion of an asymmetric body

- Suppose that
  - $\mathbf{M} = \mathbf{0}$ (free motion = free response = homogeneous equation);
  - $J_1 \neq J_2 \neq J_3 \neq J_1$ (asymmetry).
- Suppose also that $\omega_3 = \omega_o + \epsilon$, where
  - ω_o: constant angular speed;
  - ϵ: perturbation.
- For $\epsilon \to 0$, the moment equations become

$$\left.\begin{aligned} J_1\dot{\omega}_1 + (J_3 - J_2)\omega_2\omega_o &= 0 \\ J_2\dot{\omega}_2 + (J_1 - J_3)\omega_1\omega_o &= 0 \end{aligned}\right] \text{LTI}$$

$$J_3\dot{\epsilon} + (J_2 - J_1)\omega_1\omega_2 = 0. \to \text{non lineare}$$

- The first two equations are *linear time invariant*.

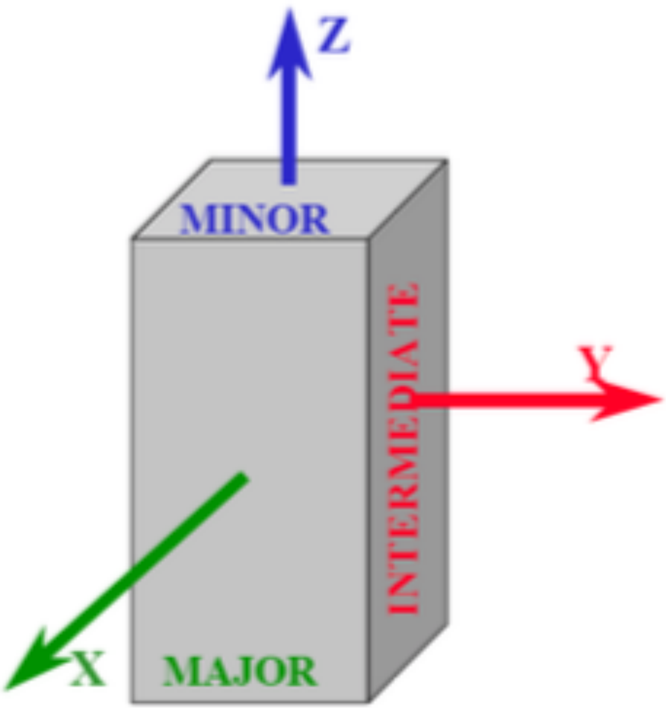
Free motion of an asymmetric body

- Differentiating the first, using the second and defining $\gamma \doteq \omega_o \sqrt{\left(1 - \frac{J_3}{J_1}\right)\left(1 - \frac{J_3}{J_2}\right)}$, we obtain $\dot{\omega}_2$ from the second equation

$$\ddot{\omega}_1 + \gamma^2 \omega_1 = 0.$$

- If γ is real ($J_3 > J_1, J_2$ or $J_3 < J_1, J_2$), we obtain again an harmonic oscillator, implying (marginal) stability.
- If γ is imaginary ($J_1 > J_3 > J_2$ or $J_2 > J_3 > J_1$), then we have a positive real pole, implying instability.
- It follows that:
  1. The motion about the minor and major principal axes is *stable*.
  2. The motion about the intermediate principal axis is *unstable*.

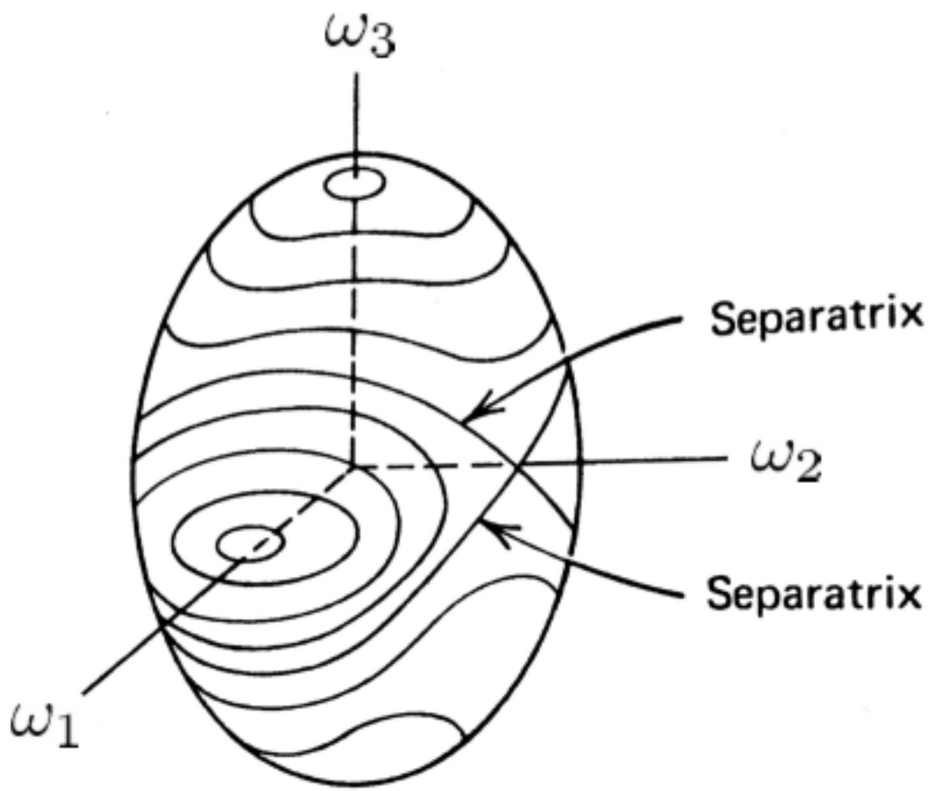
Free motion of an asymmetric body



- $I_{xx} > I_{yy} > I_{zz}$
- Major axis spin is stable
- Minor axis spin is stable
- Intermediate axis spin is unstable
- Energy dissipation changes these results
  $\rightarrow$ Minor axis spin becomes unstable
- This is called the Major-Axis Rule

Free motion of an asymmetric body

- In general, a trajectory of $\boldsymbol{\omega}$ is a *polhode*.
- The polhode curves can be represented on the *Poinsot ellipsoid*.



Free motion of an asymmetric body

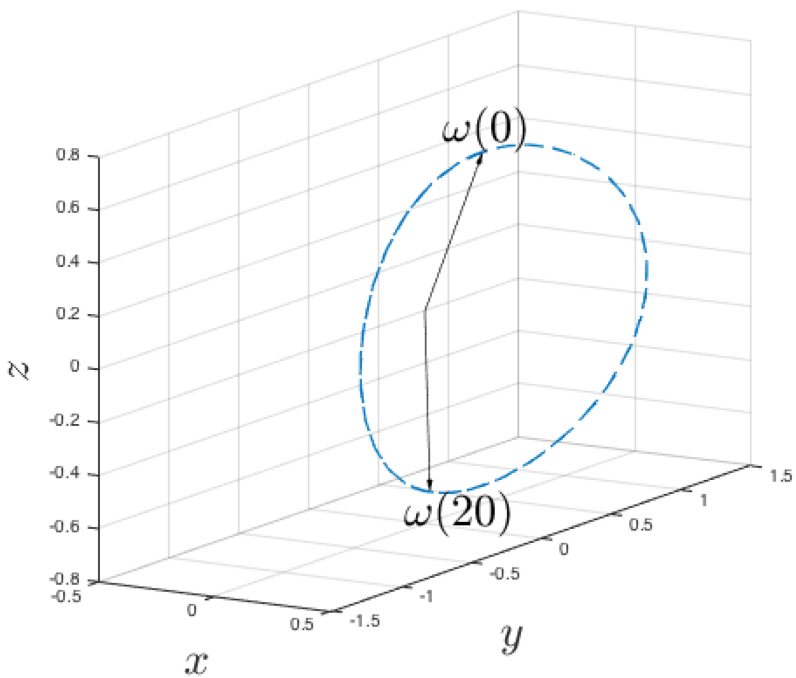
Example

- A body with the following inertia matrix is considered: $\mathrm{diag}(J_1, J_2, J_3) = \mathrm{diag}(500, 50, 200)\ \mathrm{kg\,m}^2$.

→ intermediate => intermediate axis
↓
motion here is unstable

divergence from the axis

- The following initial conditions are assumed: $\omega(0) = (0.25, 0, 0.63)\ \mathrm{rad/s}$.



$\boldsymbol{\omega}$ diverges from the z axis;

trajectory of $\boldsymbol{\omega}$ sometimes close to, other times far from the z axis.

Free motion of an asymmetric body

Energy dissipation effects

- The assumption of rigid body does not hold in a real-world system:
  - structural deflection; model not so correct
  - liquid slosh due to accelerations about the CoM.

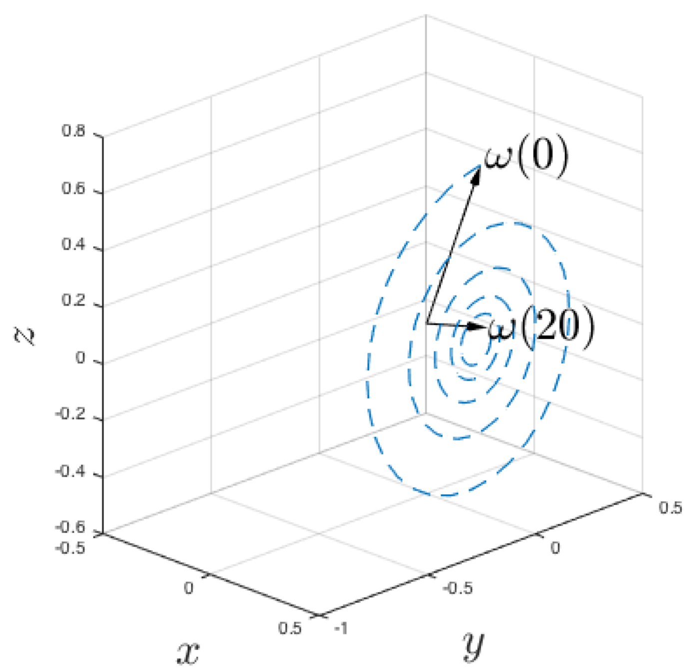
- The assumption of *semirigid body* is intermediate (between rigid and real body) and more realistic:
  - no moving parts;
  - the body dissipates energy.

- With energy dissipation:
  1. The motion about the major principal axis is *asymptotically stable*.
  2. The motion about the minor and intermediate principal axis is *unstable*.

Free motion of an asymmetric body

Internal energy dissipation effects: an example

- A body with the following inertia matrix is considered: $\mathrm{diag}(J_1, J_2, J_3) = \mathrm{diag}(400, 60, 50)\ \mathrm{kg\,m}^2$.
- The following initial conditions are assumed: $\omega(0) = (0.25, 0, 0.63)\ \mathrm{rad/s}$.
- Friction is included in the Euler equations.



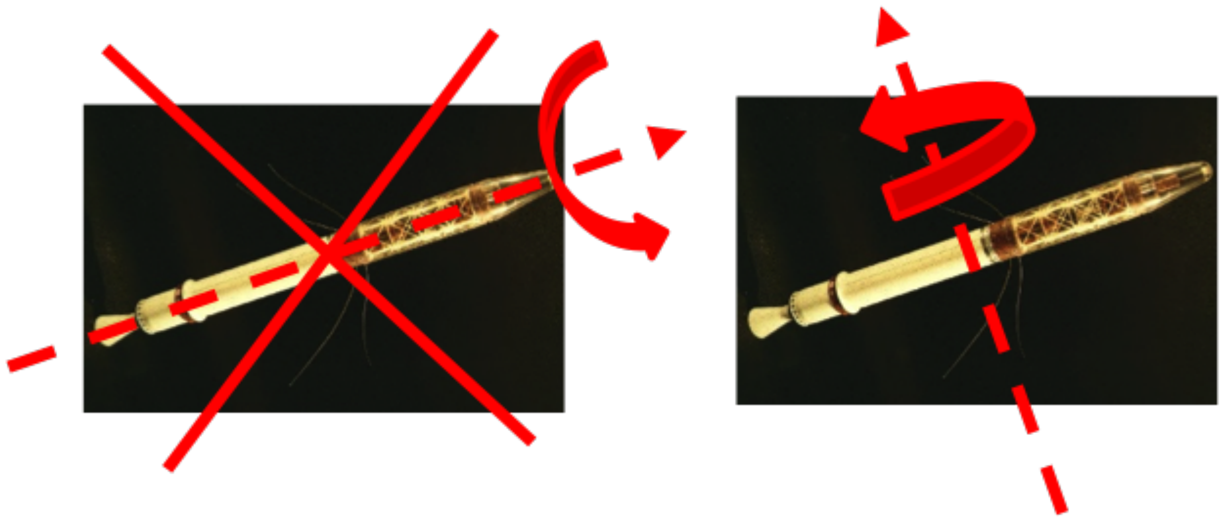
divergence

$\boldsymbol{\omega}$ diverges from the z axis;

it approaches the x axis (the major axis, in this case).

Convergence

Explorer 1 satellite (1958)



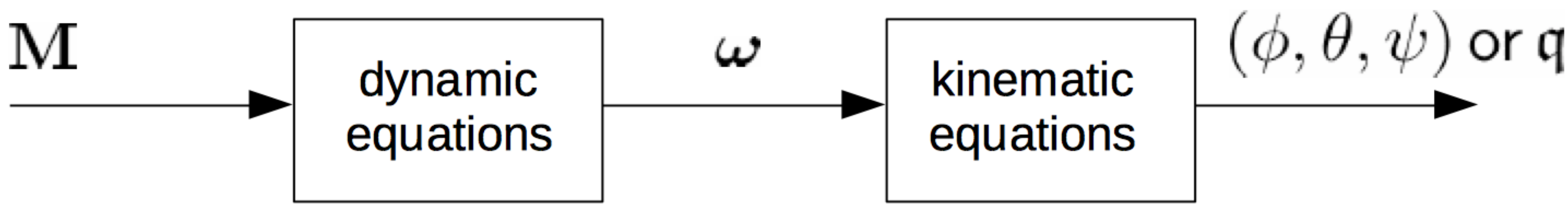
- Set to spin about an axis parallel to its length.
- However, this is the minor axis of the satellite.
- Before it had orbited the Earth once, the angular momentum vector had moved to the major axis.
- It spent the rest of its mission wheeling though space.
- Fortunately, its instruments and power supply were unaffected by the orientation of the satellite and its mission was a success:
  - discovery of the Van Allen radiation belts around the Earth.

Dynamic-kinematic equations

- The dynamic equations discussed so far describe the time evolution of the angular speed vector $\boldsymbol{\omega}$.
- However, a description of the body attitude in terms of angles (or quaternions) is of interest for control purposes.
  - The aim of control is to impose to the body desired values of the orientation angles (or quaternions).
- Such a description can be obtained putting together the dynamics equations with the kinematic equations:
  - the dynamic equations define a system from $\mathbf{M}$ to $\boldsymbol{\omega}$;
  - the kinematic equations define a system from $\boldsymbol{\omega}$ to the Euler angles or the quaternions.

Dynamic-kinematic equations

- The dynamic and kinematic equations can be seen as the series connection of two nonlinear systems:
  - the dynamic equations define a system from $\mathbf{M}$ to $\boldsymbol{\omega}$;
  - the kinematic equations define a system from $\boldsymbol{\omega}$ to (ϕ, θ, ψ) or $\mathfrak{q}$.



- Overall, it is a nonlinear system with:
  - input $\mathbf{M}$;
  - output (ϕ, θ, ψ), DCM or $\mathfrak{q}$. $\leftarrow$ output to control.

Dynamic-kinematic equations

Tait-Bryan 321

- State equations: $\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\,\mathbf{x} + \mathbf{B}\mathbf{u}$

$$\mathbf{x} = \begin{bmatrix} \phi \\ \theta \\ \psi \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \qquad \mathbf{A}(\mathbf{x}) = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & s\phi\, t\theta & c\phi\, t\theta \\ 0 & 0 & 0 & 0 & c\phi & -s\phi \\ 0 & 0 & 0 & 0 & s\phi/c\theta & c\phi/c\theta \\ \hline 0 & 0 & 0 & 0 & \sigma_1\omega_3 & 0 \\ 0 & 0 & 0 & \sigma_2\omega_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_3\omega_2 & 0 & 0 \end{array}\right]$$

$$\mathbf{u} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}, \quad \mathbf{B} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1/J_1 & 0 & 0 \\ 0 & 1/J_2 & 0 \\ 0 & 0 & 1/J_3 \end{array}\right].$$

Dynamic-kinematic equations

Tait-Bryan 123

- State equations: $\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\,\mathbf{x} + \mathbf{B}\mathbf{u}$

$$\mathbf{x} = \begin{bmatrix} \phi \\ \theta \\ \psi \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \qquad \mathbf{A}(\mathbf{x}) = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & c\psi/c\theta & -s\psi/c\theta & 0 \\ 0 & 0 & 0 & s\psi & c\psi & 0 \\ 0 & 0 & 0 & -t\theta\, c\psi & t\theta\, s\psi & 1 \\ \hline 0 & 0 & 0 & 0 & \sigma_1\omega_3 & 0 \\ 0 & 0 & 0 & \sigma_2\omega_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_3\omega_2 & 0 & 0 \end{array}\right]$$

$$\mathbf{u} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}, \quad \mathbf{B} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1/J_1 & 0 & 0 \\ 0 & 1/J_2 & 0 \\ 0 & 0 & 1/J_3 \end{array}\right].$$

Dynamic-kinematic equations

Proper Euler 313

- State equations: $\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\,\mathbf{x} + \mathbf{B}\mathbf{u}$

$$\mathbf{x} = \begin{bmatrix} \phi \\ \theta \\ \psi \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \quad \mathbf{A}(\mathbf{x}) = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & s\psi/s\theta & c\psi/s\theta & 0 \\ 0 & 0 & 0 & c\psi & -s\psi & 0 \\ 0 & 0 & 0 & -ct\theta\, s\psi & -ct\theta\, c\psi & 1 \\ \hline 0 & 0 & 0 & 0 & \sigma_1\omega_3 & 0 \\ 0 & 0 & 0 & \sigma_2\omega_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_3\omega_2 & 0 & 0 \end{array}\right]$$

$$\mathbf{u} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}, \quad \mathbf{B} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1/J_1 & 0 & 0 \\ 0 & 1/J_2 & 0 \\ 0 & 0 & 1/J_3 \end{array}\right].$$

Dynamic-kinematic equations

Quaternions

- State equations: $\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\,\mathbf{x} + \mathbf{B}\mathbf{u}$

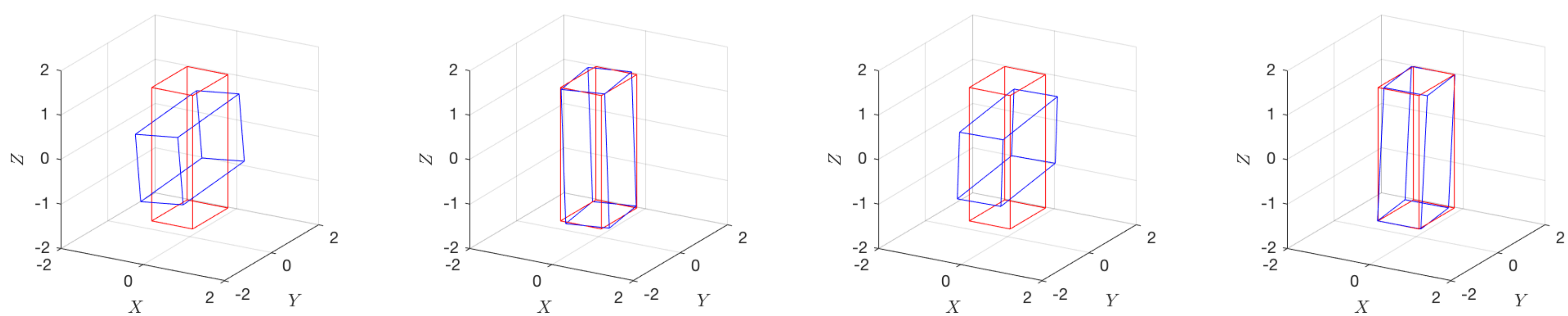
$$\mathbf{x} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \quad \mathbf{A}(\mathbf{x}) = \frac{1}{2}\left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & -q_1 & -q_2 & -q_3 \\ 0 & 0 & 0 & 0 & q_0 & -q_3 & q_2 \\ 0 & 0 & 0 & 0 & q_3 & q_0 & -q_1 \\ 0 & 0 & 0 & 0 & -q_2 & q_1 & q_0 \\ \hline 0 & 0 & 0 & 0 & 0 & 2\sigma_1\omega_3 & 0 \\ 0 & 0 & 0 & 0 & 2\sigma_2\omega_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sigma_3\omega_2 & 0 & 0 \end{array}\right]$$

$$\mathbf{u} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}, \quad \mathbf{B} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1/J_1 & 0 & 0 \\ 0 & 1/J_2 & 0 \\ 0 & 0 & 1/J_3 \end{array}\right].$$

Dynamic-kinematic equations

Example: free rotational motion of an hyper-rectangular body

- Consider a rigid body with
  - shape: hyper-rectangle with dimensions $1 \times 1.5 \times 3\ \mathrm{m}^3$;
  - mass: $1000\ \mathrm{kg}$;
  - inertia matrix: $\mathrm{diag}(937.5, 833.3, 270.8)\ \mathrm{kg\,m}^2$;
  - initial conditions: $\mathbf{x}(0) = (\mathfrak{q}(0), \boldsymbol{\omega}(0))$, $\mathfrak{q}(0) = (1, 0, 0, 0)$, $\boldsymbol{\omega}(0) = (1, 0.1, 0)\ \mathrm{rad/s}$.



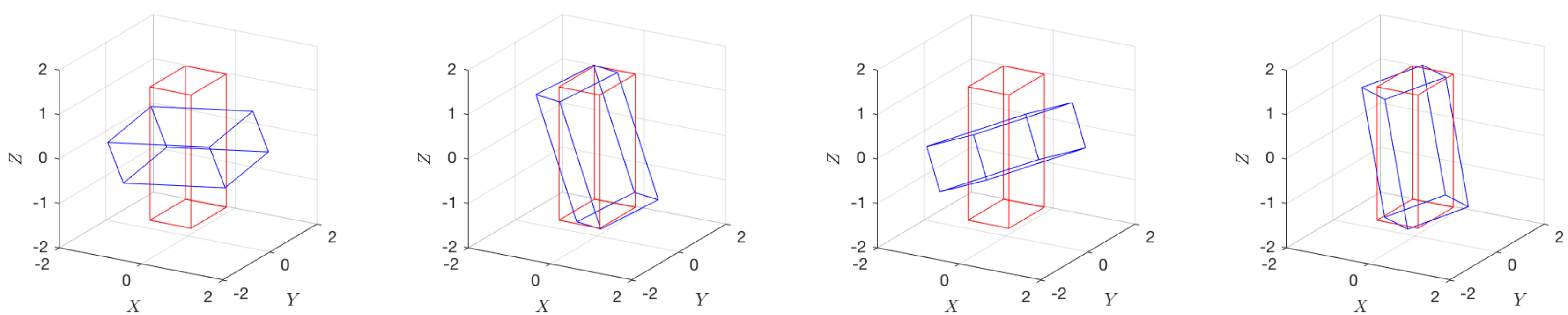
Red: initial position. Blue: position after 11.72, 18.76, 30.48, 37.52 s.

- The rotation is always about an axis close to the x axis.

Dynamic-kinematic equations

Example: free rotational motion of an hyper-rectangular body

- Same body with
  - initial conditions $\mathbf{x}(0) = (\mathfrak{q}(0), \boldsymbol{\omega}(0))$, $\mathfrak{q}(0) = (1, 0, 0, 0)$, $\boldsymbol{\omega}(0) = (0.1, 1, 0)$ rad/s.



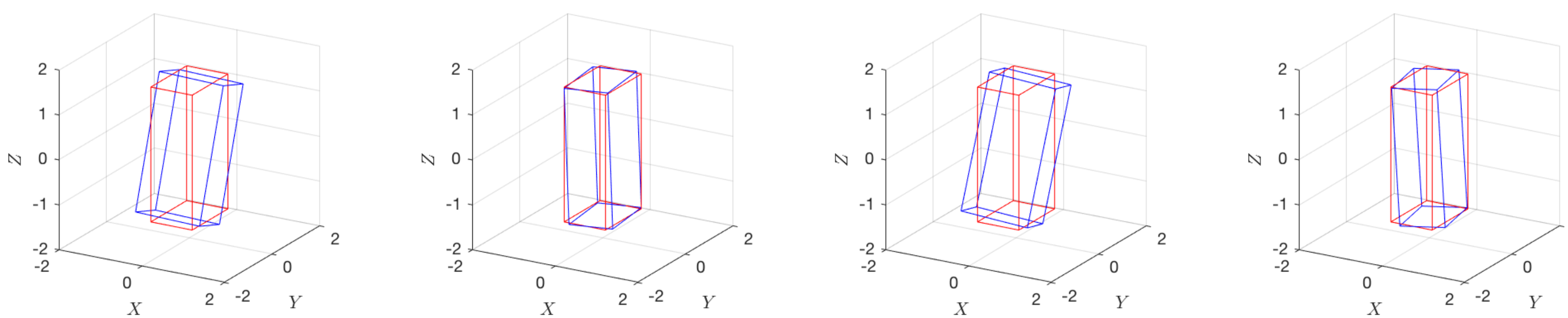
Red: initial position. Blue: position after 11.72, 18.76, 30.48, 37.52 s.

- After a certain time, the rotation axis tends to diverge from the y axis.

Dynamic-kinematic equations

Example: free rotational motion of an hyper-rectangular body

- Same body with
  - initial conditions $\mathbf{x}(0) = (\mathfrak{q}(0), \boldsymbol{\omega}(0))$, $\mathfrak{q}(0) = (1, 0, 0, 0)$, $\boldsymbol{\omega}(0) = (0, 0.1, 1)$ rad/s.



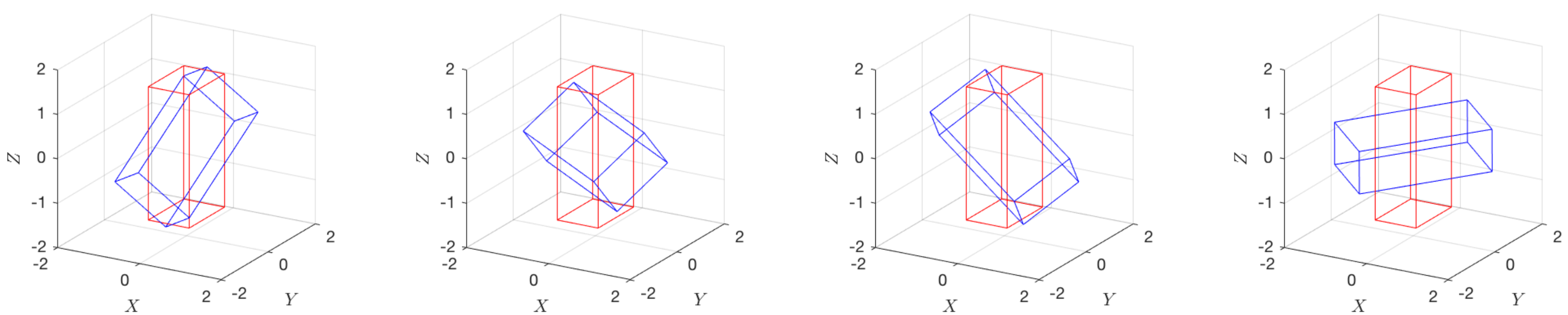
Red: initial position. Blue: position after 11.72, 18.76, 30.48, 37.52 s.

- The rotation is always about an axis close to the z axis.

Dynamic-kinematic equations

Example: free rotational motion of an hyper-rectangular body

- Same body with
  - energy dissipation;
  - initial conditions $\mathbf{x}(0) = (\mathfrak{q}(0), \boldsymbol{\omega}(0))$, $\mathfrak{q}(0) = (1, 0, 0, 0)$, $\boldsymbol{\omega}(0) = (0, 0.1, 1)$ rad/s.



Red: initial position. Blue: position after 11.72, 18.76, 30.48, 37.52 s.

- After a certain time, the rotation axis get close to the x axis.