Nonlinear control and aerospace applications

Kalman Filters

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Outline

Introduction

- In many real-world situations:
  - Not all the state variables of a system are measured: sensors may be expensive; the available space may be not sufficient.
  - For safety/fault tolerance reasons, it can be useful to use multiple sensors for the same variable (redundancy).
  - Even though a variable is measured, it may be subject to relevant disturbances/noises/errors.
- Observers/filters are algorithms that estimate the state of a system from input/output measurements. They can be useful to
  - estimate variables that are not measured (thus saving money/space);
  - increase safety: redundancy and fault tolerance;
  - improve the accuracy of sensors via a suitable data fusion, attenuating the disturbance/noise/error effects.
- Widely used observers:
  - linear systems: Kalman Filter (KF);
  - nonlinear systems: Extended Kalman Filter (EKF).

Kalman Filter

- The formulation in discrete time is relatively simpler and more suitable for direct on-line implementation.
- Consider a discrete-time Linear Time Varying (LTV) system

$$x_{k+1} = F_k x_k + G_k u_k + d_k$$
$$y_k = H_k x_k + d_k^y$$

where $k \in \mathbb{Z}$ is the time index, x_k is the state, u_k is the input, y_k is the output, d_k is a process disturbance and d_k^y is a measurement noise.

- Suppose that:
  - x_k, d_k, d_k^y: not measured/unknown
  - y_k, u_k: measured.
- The **goal** is to obtain a (possibly accurate) estimate $\hat{x}_k$ of x_k, from current and past measurements of y_k and u_k.

Kalman Filter

- The Kalman Filter (KF), like many other state-of-the art observers, is based on the following operations:

  1. **Prediction.** At time $k-1$, compute a prediction x_k^p of the state x_k, using the system model:

  $$x_k^p = F_{k-1}\hat{x}_{k-1} + G_{k-1}u_{k-1}.$$

  2. **Update.** At time k, the prediction x_k^p is corrected using the output measurement y_k, giving the "more accurate" estimate

  $$\begin{aligned} \hat{x}_k &= x_k^p + K_k \delta y_k \\ \delta y_k &= y_k - H_k x_k^p. \end{aligned}$$

- K_k: gain matrix, chosen to minimize the variance of the estimation error norm $\mathcal{E}\left[\|x_k - \hat{x}_k\|_2^2\right]$.
  - $\mathcal{E}[f(v)] \doteq \int_V f(x)p_v(x)dx$: expected value;
  - $p_v(\cdot)$: probability density function (typically assumed Gaussian).

Kalman Filter

- **Notation/definitions:**
  - x_k^p: prediction of x_k (computed at step $k-1$)
  - $\hat{x}_k$: estimate of x_k (computed at step k)
  - $\delta x_k \doteq x_k - x_k^p$: state prediction error
  - $\tilde{x}_k \doteq x_k - \hat{x}_k$: state estimation error
  - $P_k^p \doteq E\left[\delta x_k \delta x_k^\top\right]$: covariance matrix of the prediction error
  - $P_k \doteq E\left[\tilde{x}_k \tilde{x}_k^\top\right]$: covariance matrix of the estimation error
  - $Q_d \doteq E\left[d_k d_k^\top\right]$: covariance matrix of d_k
  - $R_d \doteq E\left[d_k^y d_k^{y\top}\right]$: covariance matrix of d_k^y.

Kalman Filter

Preliminary operations

- **KF design**: Choice of Q_d and R_d. These matrices can be defined from the available information on d_k and d_k^y.
  - Typically, they are chosen as diagonal matrices with the variances of d_k and d_k^y on the diagonal.
  - A trial and error tuning may be required (especially if a poor information is available on d_k and d_k^y).
- Initialization:
  - $\hat{x}_0 =$ estimated initial state (typically 0),
  - $P_0 =$ estimated initial covariance matrix (typically $\sim I$).
- The KF algorithm is now presented. Its properties are discussed afterwards.

Kalman Filter

KF algorithm

1. Prediction:

$$x_k^p = F_{k-1}\hat{x}_{k-1} + G_{k-1}u_{k-1}$$

$$P_k^p = F_{k-1}P_{k-1}F_{k-1}^\top + Q_d.$$

2. Update:

$$S_k = H_k P_k^p H_k^\top + R_d$$

$$K_k = P_k^p H_k^\top S_k^{-1}$$

$$\delta y_k = y_k - H_k x_k^p$$

$$\hat{x}_k = x_k^p + K_k \delta y_k$$

$$P_k = (I - K_k H_k) P_k^p.$$

Assumptions ($\forall i, k$):

- The noises are i.i.d. and white:
  - Zero-mean: $\mathcal{E}[d_k] = 0$, $\mathcal{E}[d_k^y] = 0$.
  - Bounded variance: $\mathrm{var}(d_k) < \infty$, $\mathrm{var}(d_k^y) < \infty$.
  - Auto-uncorrelation[1]: $\mathcal{E}[d_k d_i^\top] = \delta_{ki} Q_d$, $\mathcal{E}[d_k^y (d_i^y)^\top] = \delta_{ki} R_d$.
- Noise cross-uncorrelation: $\mathcal{E}[d_k (d_i^y)^\top] = 0$.
- Input-noise cross-uncorrelation: $\mathcal{E}[d_k u_i^\top] = 0$, $\mathcal{E}[d_k^y u_i^\top] = 0$.
- The system is globally observable.

Theorem

The linear KF guarantees an estimation error with zero mean and minimum variance:

$$K_k = \arg\min_{\mathcal{K}} \mathcal{E}\left[\|x_k - \hat{x}_k\|_2^2 \right], \ \forall k.$$

[1] Kronecker delta: $\delta_{kk} = 1$, $\delta_{ki} = 0 \, (i \neq k)$.

LTI Kalman Filter

- Consider the discrete-time Linear Time Invariant (LTI) system

$$x_{k+1} = Fx_k + Gu_k + d_k$$
$$y_k = Hx_k + d_k^y$$

where $k \in \mathbb{Z}$ is the time index, x_k is the state, u_k is the input, y_k is the output, d_k is a process disturbance and d_k^y is a measurement noise.

- It can be proven that, if $Q_d, R_d \succ 0$ and (F, H) is observable, then

$$\lim_{k\to\infty} P_k^p \to \bar{P}.$$

- This implies that also P_k, S_k and K_k converge to given values P, S and K as $k \to \infty$.

LTI Kalman Filter

- Asymptotic equations:

$$\begin{aligned}\bar{P} &= FPF^\top + Q_d \\ S &= H\bar{P}H^\top + R_d \\ K &= \bar{P}H^\top S^{-1} \\ P &= (I - KH)\,\bar{P}.\end{aligned}$$

- Through simple operations, we obtain

$$\bar{P} = F\left(\bar{P} - \bar{P}H^\top(H\bar{P}H^\top + R_d)^{-1}H\bar{P}\right)F^\top + Q_d$$

$$K = \bar{P}H^\top(H\bar{P}H^\top + R_d)^{-1}.$$

The first of these equations is called *discrete algebraic Riccati equation*. The second one gives the filter gain matrix.

LTI Kalman Filter

- Predicted and estimated state equations:

$$
\begin{aligned}
x_k^p &= F\hat{x}_{k-1} + Gu_{k-1} \\
\hat{x}_k &= x_k^p + K(y_k - Hx_k^p).
\end{aligned}
$$

- Defining $L \doteq FK$, taking $\hat{x}_{k-1}$ from the second one and replacing it in the first one, we obtain

$$
\begin{aligned}
x_k^p &= (F - LH)x_{k-1}^p + Ly_{k-1} + Gu_{k-1} \\
\hat{x}_k &= x_k^p + K(y_k - Hx_k^p).
\end{aligned}
$$

  These are the LTI KF equations: x_k^p is the filter state, $\hat{x}_k$ is the output, and y_k,u_k are the inputs.

- LTI is a sub-case of LTV $\Rightarrow$ the above theorem holds $\Rightarrow$ the LTI KF guarantees a minimum variance estimation error.
- Matlab: the command `kalman` gives L and K.

Extended Kalman Filter

- Consider the discrete-time nonlinear system

$$x_{k+1} = f(x_k, u_k) + d_k$$
$$y_k = h(x_k) + d_k^y$$

where $k \in \mathbb{Z}$ is the time index, x_k is the state, u_k is the input, y_k is the output, d_k is a process disturbance and d_k^y is a measurement noise.

- Suppose that:
  - x_k, d_k, d_k^y: not measured
  - y_k, u_k: measured.

- The **goal** is to obtain a (possibly accurate) estimate $\hat{x}_k$ of x_k, from current and past measurements of y_k and u_k.

Extended Kalman Filter

- The matrices F_k and H_k are obtained linearizing the system along the estimated trajectory:
  - $F_k \doteq \frac{\partial f}{\partial x}(\hat{x}_k, u_k)$: Jacobian of f computed at $(\hat{x}_k, u_k)$.
  - $H_k \doteq \frac{\partial h}{\partial x}(\hat{x}_k)$: Jacobian of h computed at $\hat{x}_k$.

- Same as for the linear KF:
  - Notation/definitions.
  - EKF design (choice of Q_d and R_d).
  - Initialization.

Extended Kalman Filter

EKF algorithm

1. Prediction:

$$x_k^p = f\left(\hat{x}_{k-1}, u_{k-1}\right)$$
$$P_k^p = F_{k-1} P_{k-1} F_{k-1}^\top + Q_d.$$

2. Update:

$$S_k = H_k P_k^p H_k^\top + R_d$$
$$K_k = P_k^p H_k^\top S_k^{-1}$$

$$\delta y_k = y_k - h\left(x_k^p\right)$$
$$\hat{x}_k = x_k^p + K_k \delta y_k$$
$$P_k = \left(I - K_k H_k\right) P_k^p.$$

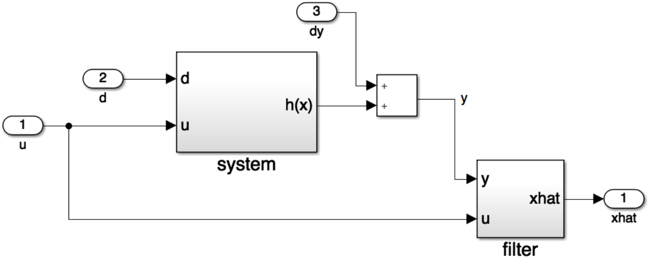
Advantages and drawbacks of EKF

- Advantages:
  - estimator for nonlinear systems;
  - relatively simple;
  - it works well in many applications.
- Drawbacks:
  - it is not optimal (the linear Kalman Filter is optimal, in the sense that it minimizes the estimation error variance);
  - in general, stability is not theoretically guaranteed;
  - its performance can be seriously affected by model uncertainties and errors in the initial condition choice.
- In many aerospace applications, the EKF works.

Discussion

Prediction and correction (KF and EKF)

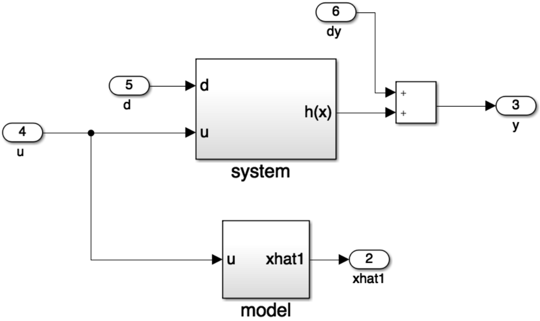
- In the prediction step, the filter/observer uses the model equations to obtain a preliminary estimate of the state.
  - In the update step, the filter corrects the preliminary estimate according to the current output measurement. This allows us to
    - increase the estimation accuracy;
    - enhance the filter stability properties;
    - filter the effects of disturbances/noises.



Discussion

Open-loop and closed-loop estimation (KF and EKF)

- A filter/observer is structurally different from a model.
- Using a model for estimation leads to worse results, since a model does not use the information given by the output measurement.
  - A model works in open-loop, while a filter works in closed-loop.



Discussion

Current/delayed versions (KF and EKF)

- For both the KF and EKF filters, two versions are possible:
  - **Current**. The estimator uses all measurements up to time k to estimate x_k. The estimate is thus given by

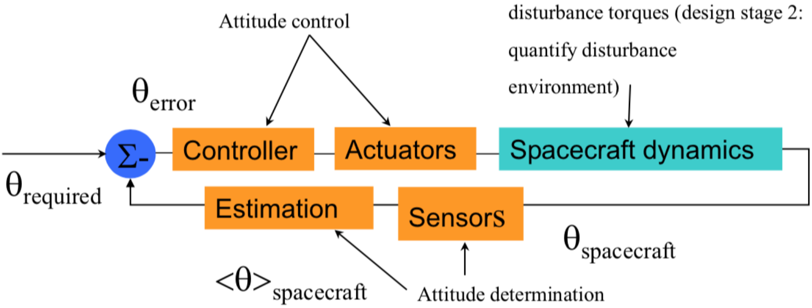
$$\hat{x}_k = x_k^p + K_k \delta y_k.$$

  - **Delayed**. The estimator uses only past measurements up to time $k-1$. The estimate is thus given by

$$x_k^p = f\left(\hat{x}_{k-1}, u_{k-1}\right).$$

- The "delayed" implementation is easier to embed in digital control loops. Depending on the application, the "current" implementation may be not suitable/possible.

Spacecraft attitude determination



Spacecraft attitude determination

- Consider a spacecraft (S/C) with inertia matrix $\mathbf{J}$, subject to a moment $\mathbf{u}$ and a disturbance $\mathbf{d}^u$. S/C attitude state equations:

$$\dot{\mathfrak{q}} = \frac{1}{2}\mathbf{Q}\,\boldsymbol{\omega}$$
$$\dot{\boldsymbol{\omega}} = \mathbf{J}^{-1}\left(-\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \mathbf{u} + \mathbf{d}^u\right)$$
$$\mathbf{Q} \doteq \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}, \quad \boldsymbol{\omega}\times \doteq \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

- Suppose the state $\mathbf{x} = (\mathfrak{q}, \boldsymbol{\omega})$ can only be partially measured.
- The measured output is

$$\mathbf{y} = h(\mathbf{x}) + \mathbf{d}^y$$

where $\mathbf{d}^y$ is a measurement noise and h depends on the sensors available on the S/C.

Spacecraft attitude determination

- The state equations can be written as follows:

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\,\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{B}\mathbf{d}^u$$
$$\mathbf{y} = h(\mathbf{x}) + \mathbf{d}^y$$

$$\mathbf{x} \doteq \begin{bmatrix} \mathfrak{q} \\ \boldsymbol{\omega} \end{bmatrix}, \quad \mathbf{A}(\mathbf{x}) \doteq \begin{bmatrix} \mathbf{0} & \frac{1}{2}\mathbf{Q} \\ \mathbf{0} & -\mathbf{J}^{-1}\boldsymbol{\omega} \times \mathbf{J} \end{bmatrix}, \quad \mathbf{B} \doteq \begin{bmatrix} \mathbf{0} \\ \mathbf{J}^{-1} \end{bmatrix}.$$

- These can be discretized via the forward Euler method:

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k)\,\mathbf{x}_k + \mathbf{G}\mathbf{u}_k + \mathbf{d}_k$$
$$\mathbf{y}_k = h(\mathbf{x}_k) + \mathbf{d}_k^y$$

where $\mathbf{F}(\mathbf{x}_k) \doteq \mathbf{I} + \tau\mathbf{A}(\mathbf{x}_k)$, $\mathbf{G} \doteq \tau\mathbf{B}$ and $\mathbf{d}_k \doteq \tau\mathbf{B}\mathbf{d}_k^u$ and τ is the sampling time.

Spacecraft attitude determination

- The equations are quasi-LTV: the matrix $\mathbf{F}(\mathbf{x}_k)$ depends on $\mathbf{x}_k$.
  - In standard LTV systems, the matrices are known and independent of the state, input and output variables.
  - Here, this matrix is not exactly known but estimated as $\mathbf{F}(\hat{\mathbf{x}}_k)$.
- The EKF matrices can be thus taken as follows:

$$\mathbf{F}_k \doteq \mathbf{F}(\hat{\mathbf{x}}_k)$$

$$\mathbf{H}_k \doteq \tfrac{\partial h}{\partial \mathbf{x}}(\hat{\mathbf{x}}_k).$$

- In general, $\mathbf{F}(\hat{\mathbf{x}}_k) \neq \frac{\partial f}{\partial \mathbf{x}}(\hat{\mathbf{x}}_k, \mathbf{u}_k)$ but $\mathbf{F}(\hat{\mathbf{x}}_k) \cong \frac{\partial f}{\partial \mathbf{x}}(\hat{\mathbf{x}}_k, \mathbf{u}_k)$.
- Another possible choice of the $\mathbf{F}_k$ matrix is the following:

$$\mathbf{F}_k \doteq \frac{\partial f}{\partial \mathbf{x}}(\hat{\mathbf{x}}_k, \mathbf{u}_k).$$

Spacecraft attitude determination

EKF algorithm (attitude dynamics)

1. Prediction:

$$\mathbf{x}_k^p = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{G}\mathbf{u}_{k-1}$$

$$\mathbf{P}_k^p = \mathbf{F}_{k-1}\mathbf{P}_{k-1}\mathbf{F}_{k-1}^\top + \mathbf{Q}_d.$$

2. Update:

$$\mathbf{S}_k = \mathbf{H}_k\mathbf{P}_k^p\mathbf{H}_k^\top + \mathbf{R}_d$$

$$\mathbf{K}_k = \mathbf{P}_k^p\mathbf{H}_k^\top\mathbf{S}_k^{-1}$$

$$\delta\mathbf{y}_k = \mathbf{y}_k - h(\mathbf{x}_k^p)$$

$$\hat{\mathbf{x}}_k = \mathbf{x}_k^p + \mathbf{K}_k\delta\mathbf{y}_k$$

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\,\mathbf{P}_k^p.$$

Spacecraft attitude determination and regulation

- A spacecraft on an Earth orbit is considered, with inertia matrix $\mathbf{J} = \mathrm{diag}(1900, 2400, 1600)\,\mathrm{kg\,m}^2$.
- A saturation of $10\,\mathrm{Nm}$ is assumed for all the three inputs.
- The following initial quaternion and angular velocity are assumed: $\mathfrak{q}(0) = (0.6853, 0.6953, 0.1531, 0.1531)$, $\boldsymbol{\omega}(0) = (0.03, -0.05, 0.02)\,\mathrm{rad/s}$.
- Random disturbances $\mathbf{d}_k^u$ and $\mathbf{d}_k^y$ are supposed to act on the system, with zero-mean and $\mathrm{std}(\mathbf{d}^u) = 10^{-3}\,\mathrm{Nm}$, $\mathrm{std}(\mathbf{d}^y) = 10^{-5}$.
- All other disturbances/perturbations are assumed to be negligible.
- The control task is to bring the spacecraft attitude to the identity quaternion.

Spacecraft attitude determination and regulation

- The angular velocity sensor (gyroscope) is not available. The S/C attitude (quaternion) is measured: $h(\mathbf{x}) = \mathbf{C}\mathbf{x}$, $\mathbf{C} = [I_{4\times4}\ \mathbf{0}_{4\times3}]$, $\mathbf{H}_k = \partial h/\partial\mathbf{x} = \mathbf{C}$, $\forall k$.
- An EKF is designed to estimate the spacecraft state. Motivations:
  - cost reduction;
  - simpler system design;
  - redundancy, backup system, fault tolerance;
  - noise/disturbance attenuation.
- Choice of $\mathbf{Q}_d$ and $\mathbf{R}_d$: Covariance matrices of $\mathbf{d}_k = \tau\mathbf{G}\mathbf{d}_k^u$ and $\mathbf{d}_k^y$ (i.e., diagonal matrices with the variances on the diagonal). To increase robustness, null diagonal elements can be replaced with "small" values.
- The control laws R1-R4 are applied to the spacecraft system, using for feedback the EKF estimated state.
- A satisfactory regulation performance is obtained for each of these control laws.

Spacecraft attitude determination and regulation