Nonlinear control and aerospace applications

Attitude control

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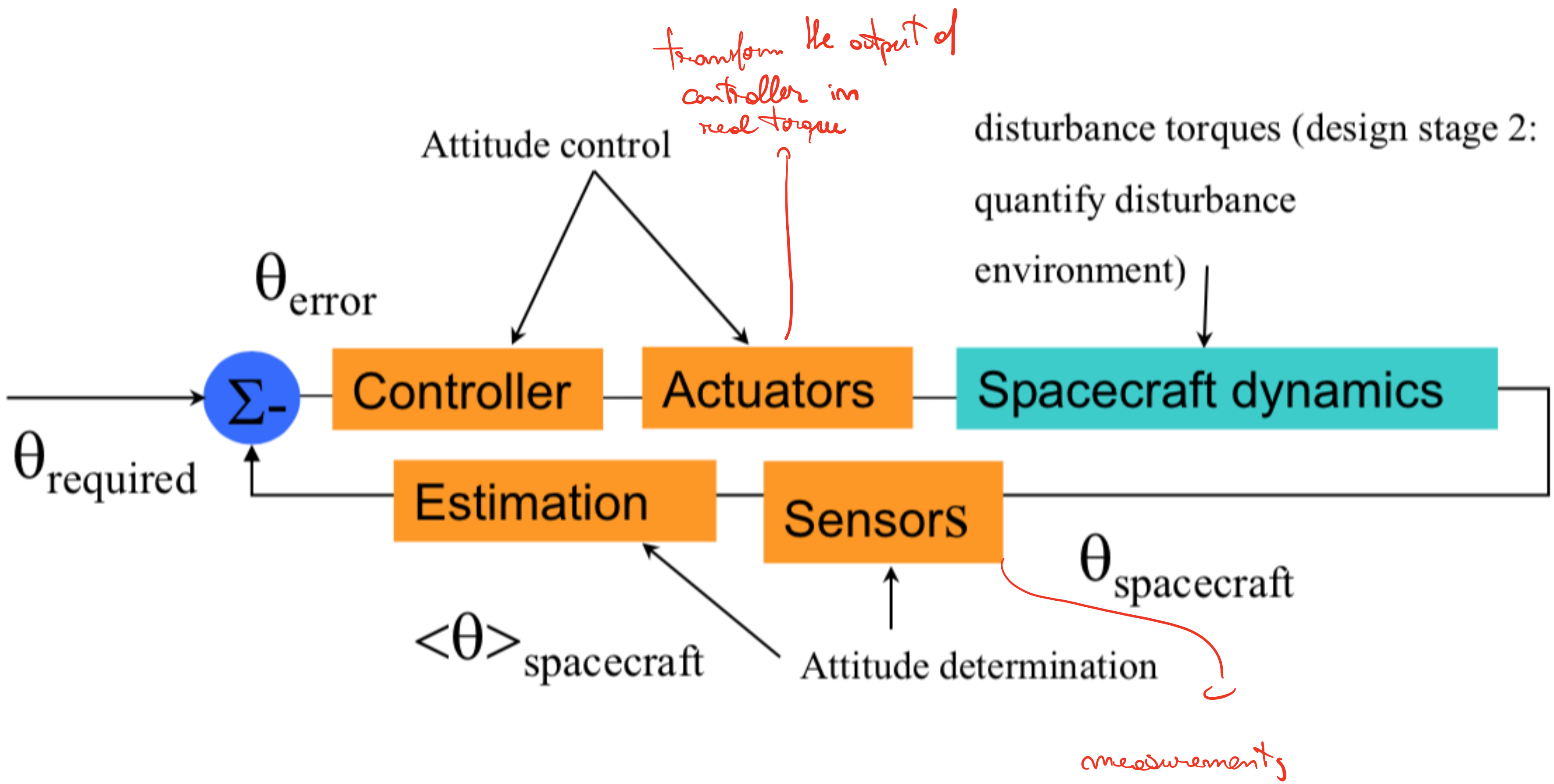
Outline

Introduction

- Accurate orientation is fundamental for most spacecraft missions:
  - telecommunications,
  - scientific missions,
  - capture the solar energy through panels.
- A fundamental component of a spacecraft (S/C) is the Attitude Control System (ACS), which is in charge of the following tasks:
  1. S/C orientation during the mission;
  2. S/C stabilization about a reference attitude in the presence of perturbing torques (aerodynamic, gravity gradient, solar radiation and wind, magnetic field).
- The ACS includes
  - sensors for attitude determination;
  - actuators capable of exerting the necessary command torques;
  - control algorithms.

The goal is having an accurate orientation

Introduction
Control loop



Sensors

- Sensors must be capable of measuring in real-time the satellite attitude, that is fundamental for control.
- Two main types of sensors can be distinguished:
  1. *Absolute attitude sensors*. They determine the attitude wrt the direction between the spacecraft and celestial bodies like the Sun, the Earth and the stars.
     - ★ They require the observation of some celestial body in a field of view.
     - ★ They include horizon sensors, orbital gyrocompasses, Sun sensors, Earth sensors, star trackers, magnetometers.
  2. *Relative attitude sensors*. They typically measure the angular rate and obtain the attitude by integration.

     angular velocity
     - ★ They do not require any observation instrument but they cannot directly measure the attitude and are thus subject to larger errors.

       the error can diverge

       the system is not asymptotically stable
     - ★ They are usually based on gyroscopes.

Actuators

- Controlling the attitude requires the use of suitable actuators.
- Keeping a predefined attitude and angular momentum requires the ACS to contrast the perturbation torques acting on the satellite.
- Three main types of actuators are employed to this scope:
  1. Actuators capable of delivering external torques.
     - ⋆ They include thrusters, momentum/reaction wheels, control moment gyros (CMGs). rockets exchange forces between spacecraft and inner wheels

more flexible
consume energy

  2. Actuators using environmental forces.
     - ⋆ They include gravity gradient, magnetic systems, aerodynamic systems, solar sails.

small amount of energy

  3. Dampers. Typically used to reduce nutation effects and torque disturbances. They also provide asymptotic stability properties.

Actuators

Actuators are usually broken into four classes: mass expulsion, momentum exchange, environmental and dissipative. An ACS may have actuators from any or all the classes.

Mass Expulsion	**Momentum Exc.**	**Environmental**	**Dissipative**
	Reaction wheel	Gravity gradient	Nutation damper
	Momentum wheel	Magnetic	GG viscous damper
	CMG	Aerodynamic	

There are two types of magnetic torquers. Those used for momentum dumping or control in momentum bias systems, and the eddy current dampers used in gravity gradient systems.

Range of torques available from some of these actuators (From Chobotov)

Actuator Type	**Torque Range (N-m)**
Reaction Control (RCS)	10^{-2} - 10
Magnetic Torquer	10^{-2} - 10^{-1}
Gravity Gradient	10^{-6} - 10^{-3}
Aerodynamic	10^{-5} - 10^{-3}
Reaction Wheel	10^{-1} - 1
Control Moment Gyro	10^{-2} - 10^{3}

From this table we see that RWs and CMGs are used when precision pointing and/or high torque is required. Satellites that perform rapid attitude maneuvers or have articulating payloads which create large disturbance torques require CMGs

Actuators

- The control laws that we will study can be actuated by means of thrusters, control moment gyros, reaction wheels or combinations of these actuators.

- In the case of thrusters, a pulse-width pulse-frequency (PWPF) modulation technique is commonly used.
  - PWPF translates the continuous-time command to an on/off signal.
  - The resulting signal is a sequence of square waves with given widths (in time) and frequencies.
  - Usually, the width is very short and the frequency is significantly faster than the rigid body dynamics.

Control System Approaches

- The ACS objectives can be
  1. vehicle attitude stabilization about a reference attitude;
  2. reference tracking in attitude manoeuvres.
- Attitude control can be
  - *Passive*: based on the body dynamic properties and/or environmental forces (gravity gradient and aerodynamic torques).
  - *Semi-active* (*semi-passive*): based on reaction wheels and/or the Earth magnetic field.
  - *Active*: based on thrusters.
- Complex manoeuvres can only be performed by means of active (or semi-active) control systems.

Control System Approaches

- *Passive control* achieves attitude stabilization by exploiting the body dynamic properties and/or the interactions with the surrounding environment.
  - Spinners:
    - Without power dissipation, the rotation about the minor and major axes is stable.
    - With power dissipation, only the rotation about the major axis is stable.
  - Environment perturbations (for instance the gravity gradient torques, aerodynamic torques) can lead to stable attitude and rate equilibriums.
  - Passive approaches are often used in combination with dampers.
  - In the case of large perturbations and/or perturbations with nonzero mean, passive control needs the assistance of active control.

Control System Approaches

- *Semi-active* (*semi-passive*) control. Much more common than passive control (this latter used in the early spacecraft missions).
  - Reaction wheels, control moment gyroscope (CMG):
    - Based on conservation of angular momentum.
  - Magnetic systems:
    - They create magnetic fields that interact with the Earth magnetic field, producing useful torques.
- *Active control* applies suitable and explicit torques to guide and keep the attitude and angular rate close to suitable reference values.
  - Typically based on thrusters.
  - It can both stabilize and manoeuvre the spacecraft attitude.
- This classification (passive, semi-active, active) is debatable. It is difficult to individuate the separations between the three classes.

Control System Approaches

- Another possible classification is as follows.
- *Spin stabilization*; made by setting the spacecraft spinning.
  - The gyroscopic action of the rotating spacecraft mass is the stabilizing mechanism.
  - Propulsion system thrusters are fired only occasionally to make desired changes in spin rate, or in the spin-stabilized attitude.
- *3-axis-stabilization*; possible methods:
  1. Small thrusters used to bring the spacecraft within a deadband of allowed attitude error.
  2. Electrically powered reaction wheels (or momentum wheels).
     - Mounted in three orthogonal axes.
     - They exchange angular momentum between spacecraft and wheels.
  3. Control moment gyroscope (CMG): a spinning rotor and one or more motorized gimbals that tilt the rotor's angular momentum. Two CMGs are need for 3-axis-stabilization.

Control System Approaches

- Reaction wheels and CMGs can be used in combination with thrusters or magnetics (magnetic torquers) for momentum damping.

Passive	Semi-passive	Active
Gravity gradient	Momentum bias with magnetics	Propellant
Spinner with nutation damper	Reaction wheels with magnetics for momentum dumping	Reaction wheels with propellant for momentum dumping
Dual spinner with nutation damper	CMGs with magnetics for momentum dumping	CMGs with propellant for momentum dumping

Control System Approaches

Attitude Control Methods and their Capabilities

Type	Pointing Options	Manueverability: Translation	Manueverability: Rotation	Accuracy	Lifetime Limits
Passive					
• Gravity gradient	• Earth local vertical only	• Minor adjustments with thrusters	• Very limited	• ±5° (two axes)	• None
• Gravity gradient & momentum bias wheel	• Earth local vertical only	• Minor adjustments with thrusters	• Very limited	• ±5° (three axes)	• Life of wheel bearings
Spinners					
• Pure spinner	• Inertially fixed any direction • Repoint with precession maneuvers	• Large _V along spin axis, minor adjust in other two axes with thrusters	• High propellant usage to move stiff momentum vector • Momentum vector same as above	• ±0.3017° to ±1° in two axes (proportional to spin rate)	• Thruster propellant
• Dual spin	• Limited only by articulation on despun platform	• Large _V along spin axis, minor adjust in other two axes with thrusters	• Despun platform constrained by its own geometry	• Same as above for spin section • Despun dictated by payload reference & pointing	• Thruster propellant (if applies) • Despin bearings
3-axis stabilized					
• Zero momentum (3 wheels &thrusters)	• No constraints	• Any direction,any level depending on size of thruster and main engine	• No constraints (1)	• 0.001° to 1° depending on sensor & actuator selection	• Propellant • Life of sensor bearings
• Bias momentum (1wheel & roll thrusters)	• Best suited for local vertical pointing	• Same as zero momentum with full set of thrusters; otherwise, not suited to translation	• Momentum vector of the bias wheel prefers to stay normal to orbit plane, constraining yaw maneuver	• Depends on sensors but generally less accurate than zero momentum; e.g., 0.1° to 1°	• Propellant • Life of sensor bearings

(1) High rates with thrusters or control moment gyros; low rate, accurate control with reaction wheels.

Control System Approaches

Type	Advantages	Disadvantages
Spin-stabilised (~1^o accuracy)	Simple, passive, long-life, provides scan motion, gyroscopic stability for large burns	Poor manoeuvrability, low solar cell efficiency (cover entire drum), no fixed pointing
3-axis stabilised (~0.001^o accuracy)	High pointing accuracy, rapid attitude slews possible, generate large power (Sun-facing flat solar arrays)	Expensive (~2 x spinner), complex, requires active closed-loop control, actuators for each body axis
Dual-spin stabilised (~0.1^o accuracy)	Provides both fixed pointing (on de-spun platform) and scanning motion, gyroscopic stability for large burns	Require de-spin mechanism, low solar cell efficiency (cover entire drum), cost can approach 3-axis if high accuracy
Gravity-gradient (~5^o accuracy)	Simple, low cost totally passive, long-life, provides simple passive Earth pointing mode	Low accuracy, almost no manoeuvrability, poor yaw stability, require deployment mechanism
Magnetic (~1^o accuracy)	Simple, low cost, can be passive with use of permanent magnet or active with use of electromagnets	Poor accuracy (uncertainty in Earth's magnetic field), magnetic interference with science payload

- In the following, we focus on active three-axis control.

Attitude control: a general setting

can be any, not only diagonal

- Consider a rigid body with inertia matrix $\mathbf{J}$. For this body, define the following variables:
  - $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$: angular velocity (body frame);
  - $\mathfrak{q} = (q_0, \mathbf{q}) = (q_0, q_1, q_2, q_3)$: attitude quaternion;
  - $\mathbf{x} = (\mathfrak{q}, \boldsymbol{\omega})$: state; state vector
  - $\mathbf{u} = (u_1, u_2, u_3)$: external moment (body frame).

- The state equations are given by the quaternion kinematic equation and the Euler dynamic equation (the latter holds in an inertial frame):

$$\dot{\mathfrak{q}} = \frac{1}{2}\mathbf{Q}\,\boldsymbol{\omega}$$

$$\dot{\boldsymbol{\omega}} = -\mathbf{J}^{-1}\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \mathbf{J}^{-1}\mathbf{u}$$

$$\mathbf{Q} \doteq \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}, \quad \boldsymbol{\omega}\times \doteq \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

Attitude control: a general setting

rotational motion of the body

- The general goal of *control* is to make the state vector $(\mathfrak{q}, \boldsymbol{\omega})$ track some (possibly time-varying) reference vector $(\mathfrak{q}_r, \boldsymbol{\omega}_r)$. → it can be a signal that changes in time
- It is thus important to quantify the distance between the reference and the actual state. To this aim, we define:
  - the *angular velocity tracking error*

ref. angular velocity

error

$$\tilde{\boldsymbol{\omega}} \doteq \boldsymbol{\omega}_r - \boldsymbol{\omega};$$

angular velocity of the body

  - the *quaternion tracking error* can't be defined as the difference

$$\tilde{\mathfrak{q}} \equiv \begin{bmatrix} \tilde{q}_0 \\ \tilde{\mathbf{q}} \end{bmatrix} \doteq \mathfrak{q}^{-1} \otimes \mathfrak{q}_r = \mathfrak{q}^* \otimes \mathfrak{q}_r.$$

we have to use the quaternion product

Motivation: $\tilde{\mathfrak{q}}$ is the quaternion that, starting from $\mathfrak{q}$, gives $\mathfrak{q}_r$ (intrinsic rotation): $\mathfrak{q} \otimes \tilde{\mathfrak{q}} = \mathfrak{q}_r$.

reference quaternion

Regulation

- The goal of *regulation* is

Classic goal of regulation

$$\mathfrak{q} \to \mathfrak{q}_r = \text{const} \quad \text{and} \quad \boldsymbol{\omega} \to \mathbf{0}$$

vector part

$$\tilde{\mathfrak{q}} \to \mathfrak{I} \doteq (1, \mathbf{0}) \quad \text{and} \quad \boldsymbol{\omega} \to \mathbf{0}.$$

↳ identity quaternion

- A simple linear Proportional-Derivative (PD) control law:

$$\mathbf{u} = k_p\, \tilde{\mathbf{q}} - k_d\, \boldsymbol{\omega} \tag{R1}$$

→ equal to $\tilde{\omega}$

simple → 2 parameters to choose

where $k_p > 0$ and $k_d > 0$ are parameters to tune.

- With this law, the state equations of the closed-loop system are autonomous and are written as

$$\boxed{\begin{aligned} \dot{\tilde{\mathfrak{q}}} &= -\tfrac{1}{2}\boldsymbol{\omega}^q \otimes \tilde{\mathfrak{q}} \\ \dot{\boldsymbol{\omega}} &= \mathbf{J}^{-1}\left(-\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + k_p\, \tilde{\mathbf{q}} - k_d\, \boldsymbol{\omega}\right). \end{aligned}}$$

See the appendix for a proof of the kinematic equation.

Regulation

- This system has two equilibrium points: $(\tilde{q}_0, \tilde{\mathbf{q}}, \boldsymbol{\omega}) = (\pm 1, \mathbf{0}, \mathbf{0})$. Both signs of $\tilde{q}_0$ correspond to the same attitude $(\beta \to 2\pi + \beta)$.

+ -

Theorem

The equilibria $(\pm 1, \mathbf{0}, \mathbf{0})$ *of the closed-loop system are loc. asymptotically stable. Moreover, for any initial condition* $(\tilde{q}_0(0), \tilde{\mathbf{q}}(0), \boldsymbol{\omega}(0))$,

$$\lim_{t \to \infty} (\tilde{q}_0(t), \tilde{\mathbf{q}}(t), \boldsymbol{\omega}(t)) = (\pm 1, \mathbf{0}, \mathbf{0}).$$

whatever the starting point it is

- The proof is based on the Lyapunov function

$$V = \frac{1}{4}\boldsymbol{\omega}^T J \boldsymbol{\omega} + \frac{1}{2} k_p \tilde{\mathbf{q}}^T \tilde{\mathbf{q}} + \frac{1}{2} k_p (1 \mp \tilde{q}_0)^2.$$

Regulation

- Other effective control laws are the following:

$$\mathbf{u} = k_p \operatorname{sign}(\tilde{q}_0)\, \tilde{\mathbf{q}} - k_d\, \boldsymbol{\omega} \tag{R2}$$

$$\mathbf{u} = k_p\, \tilde{\mathbf{q}} - k_d \left(1 + \tilde{\mathbf{q}}^T \tilde{\mathbf{q}}\right) \boldsymbol{\omega} \tag{R3}$$

$$\mathbf{u} = k_p \operatorname{sign}(\tilde{q}_0)\, \tilde{\mathbf{q}} - k_d \left(1 + \tilde{\mathbf{q}}^T \tilde{\mathbf{q}}\right) \boldsymbol{\omega}. \tag{R4}$$

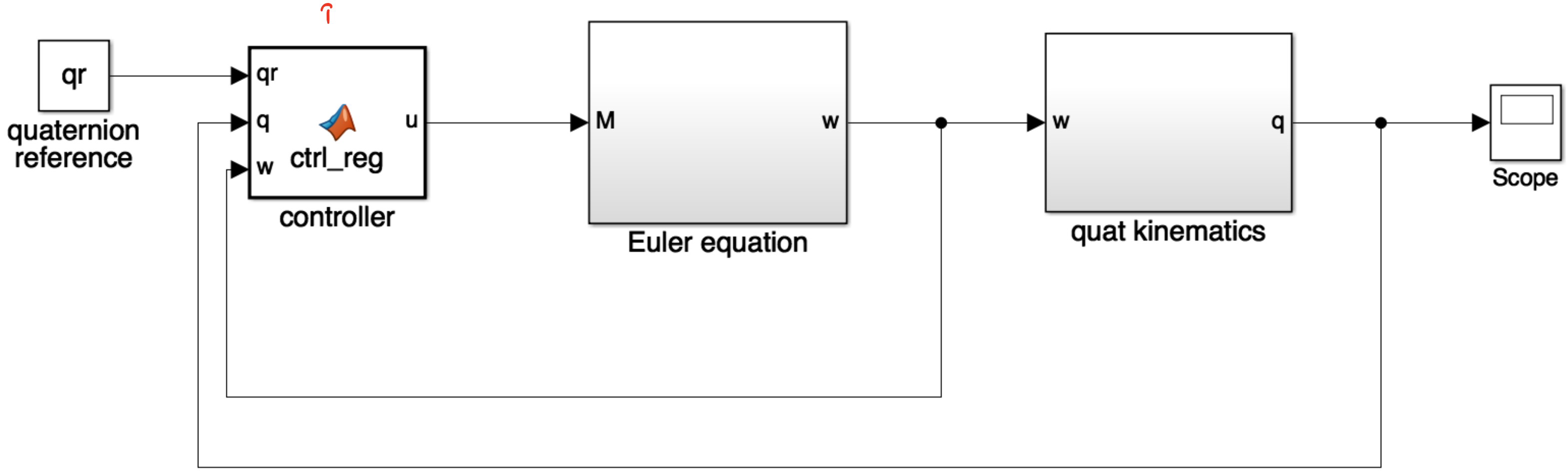
  where $k_p > 0$ and $k_d > 0$ are parameters to tune.
  - (R2) is similar to (R1) but guarantees the shortest path to the final orientation.
  - $(1+\tilde{q}^T\tilde{q})$
  - (R3) and (R4) are nonlinear, possibly allowing a better performance in terms of response time and command activity.
  - (R4) is similar to (R3) but guarantees the shortest path.
- The closed-loop equilibrium points using (R2), (R3) and (R4) are the same as those using (R1).
- Stability results similar to the one shown above hold also for these modified laws.

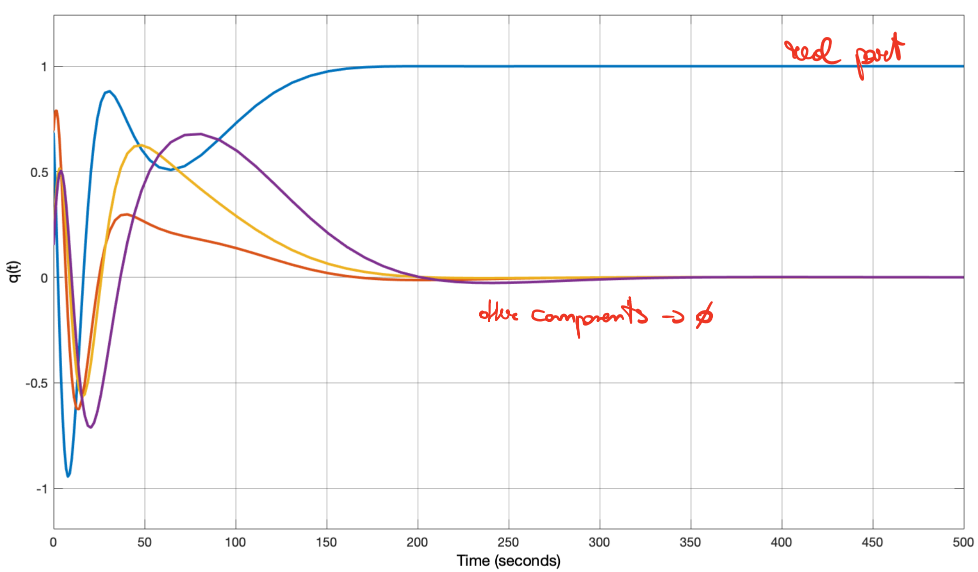
Application: satellite attitude regulation

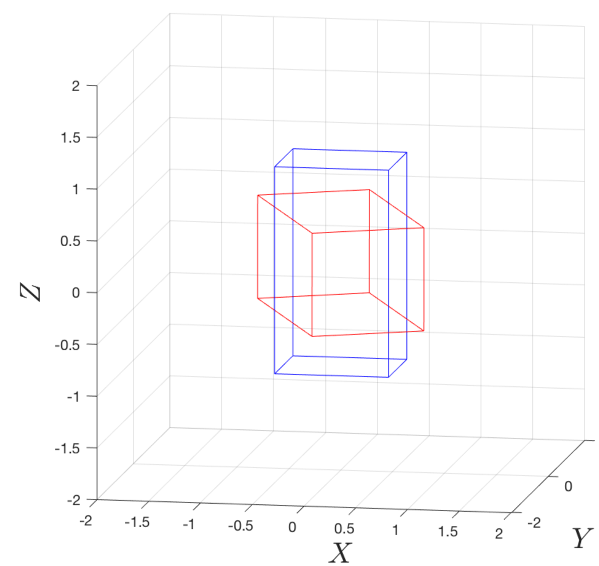
- A satellite on an Earth orbit is considered, with inertia matrix $\mathbf{J} = \mathrm{diag}(10\,000, 9\,000, 12\,000)\,\mathrm{kg\,m^2}$.
- Two main reference frames: orbital frame (OF) and satellite (body) frame (BF).
  origin centre on the centre of the orbital
  - The satellite attitude is described by the rotation OF$\rightarrow$BF.
- Non-inertial effects, gravity gradient moment, third body gravity, atmosphere drag, solar radiation are considered negligible.
- The following initial quaternion and angular velocity are assumed: $\mathbf{q}(0) = (0.6853, 0.6953, 0.1531, 0.1531)$, $\boldsymbol{\omega}(0) = (0.53, 0.53, 0.053)\,\mathrm{rad/s}$.
- The control task is to bring the satellite attitude to the identity quaternion.
- The control laws (R1)-(R4) are applied to the satellite system to accomplish this task.
- Satisfactory regulation performance obtained for all these control laws.

Application: satellite attitude regulation

contains one
of the control laws







Matlab code min 37

Tracking

- The goal of *tracking* is

$$\mathfrak{q}(t) \to \mathfrak{q}_r(t) \quad \text{and} \quad \boldsymbol{\omega}(t) \to \boldsymbol{\omega}_r(t)$$

$$\tilde{\mathfrak{q}}(t) \to \mathfrak{I} \doteq (1, \mathbf{0}) \quad \text{and} \quad \tilde{\boldsymbol{\omega}}(t) \to \mathbf{0}.$$

proportional derivative controller

Poor performances and poor robustness

- The above regulation controllers may be used. However, for challenging tracking tasks, their performance may be not satisfactory.
- An effective method for nonlinear systems is represented by *sliding-mode control*.
  - This method may allow for high performance and robustness levels.
- Sliding-mode control design develops in two main steps:
  1. definition of a *sliding surface*
  2. design of a feedback law.

Tracking

- The system to control is

similar to normal form

instead of identity matrix

$$\begin{cases} \dot{\mathbf{q}} = \frac{1}{2}\mathbf{Q}\,\boldsymbol{\omega} \\ \dot{\boldsymbol{\omega}} = -\mathbf{J}^{-1}\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} + \mathbf{J}^{-1}\mathbf{u} \end{cases}$$

external moment

  - $\mathbf{y} = \mathbf{q}$ (vector part of q): output to control we want to control the entire quaternion
  - the system is MIMO: $\mathbf{u} \in \mathbb{R}^3$, $\mathbf{y} \in \mathbb{R}^3$
  - it is a "generalized normal form"
  - relative degree $\gamma = 2$.

- The sliding surface function is defined as

$$\mathbf{s}(\mathbf{q}, \boldsymbol{\omega}, t) \doteq \tilde{\boldsymbol{\omega}} + k_2\,\tilde{\mathbf{q}}.$$

On the sliding surface, *the tracking error converges to* 0.

Tracking

- The derivative is

$$\begin{aligned}\dot{\boldsymbol{s}} &= \overbrace{\dot{\boldsymbol{\omega}}_r - \dot{\boldsymbol{\omega}}}^{\dot{\tilde{\boldsymbol{\omega}}}} + k_2\,\dot{\tilde{\mathbf{q}}} \\ &= \dot{\boldsymbol{\omega}}_r + \mathbf{J}^{-1}\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} - \mathbf{J}^{-1}\mathbf{u} + \frac{k_2}{2}\left(\tilde{q}_0\tilde{\boldsymbol{\omega}} + \tilde{\mathbf{q}} \times (\boldsymbol{\omega}_r + \boldsymbol{\omega})\right).\end{aligned}$$

  See the appendix for the expression of $\dot{\tilde{\mathbf{q}}}$.

- With $\dot{\boldsymbol{s}} = 0$, the sliding surface is *invariant* (invariant set). Imposing $\dot{\boldsymbol{s}} = 0$ and inverting wrt $\mathbf{u}$ the above expression, we obtain

$$\mathbf{u}_s = \mathbf{J}\left(\dot{\boldsymbol{\omega}}_r + \frac{k_2}{2}\left(\tilde{q}_0\tilde{\boldsymbol{\omega}} + \tilde{\mathbf{q}} \times (\boldsymbol{\omega}_r + \boldsymbol{\omega})\right)\right) + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega}. \qquad \text{(T1)}$$

- A further term is added, to make the sliding surface *attractive*. The complete control law is

$$\mathbf{u} = \mathbf{u}_s + k_1\,\mathbf{J}\,\tanh(\eta\,\boldsymbol{s}).$$

Tracking

- Another sliding mode control law, guaranteeing the shortest reorientation, is the following:

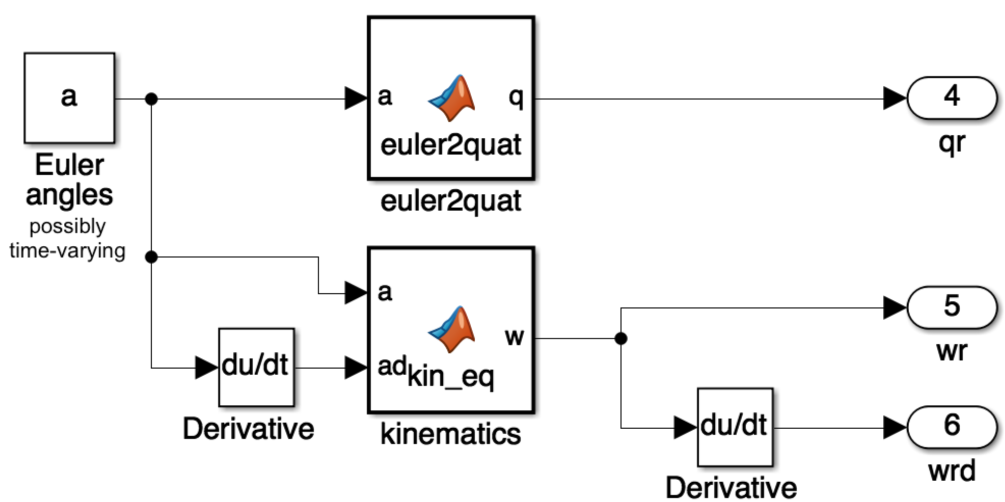
$$\mathbf{s}(\mathbf{q},\boldsymbol{\omega}) = \tilde{\boldsymbol{\omega}} + k_2\,\mathrm{sign}(\tilde{q}_0)\,\tilde{\mathbf{q}}$$

→ for the shortest path

$$\mathbf{u}_s = \mathbf{J}\left(\dot{\boldsymbol{\omega}}_r + \frac{k_2}{2}\left(|\tilde{q}_0|\,\tilde{\boldsymbol{\omega}} + \mathrm{sign}(\tilde{q}_0)\,\tilde{\mathbf{q}} \times (\boldsymbol{\omega}_r + \boldsymbol{\omega})\right)\right) + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega}. \qquad \text{(T2)}$$

$$\mathbf{u} = \mathbf{u}_s + k_1\,\mathbf{J}\,\tanh(\eta\,\boldsymbol{s}).$$

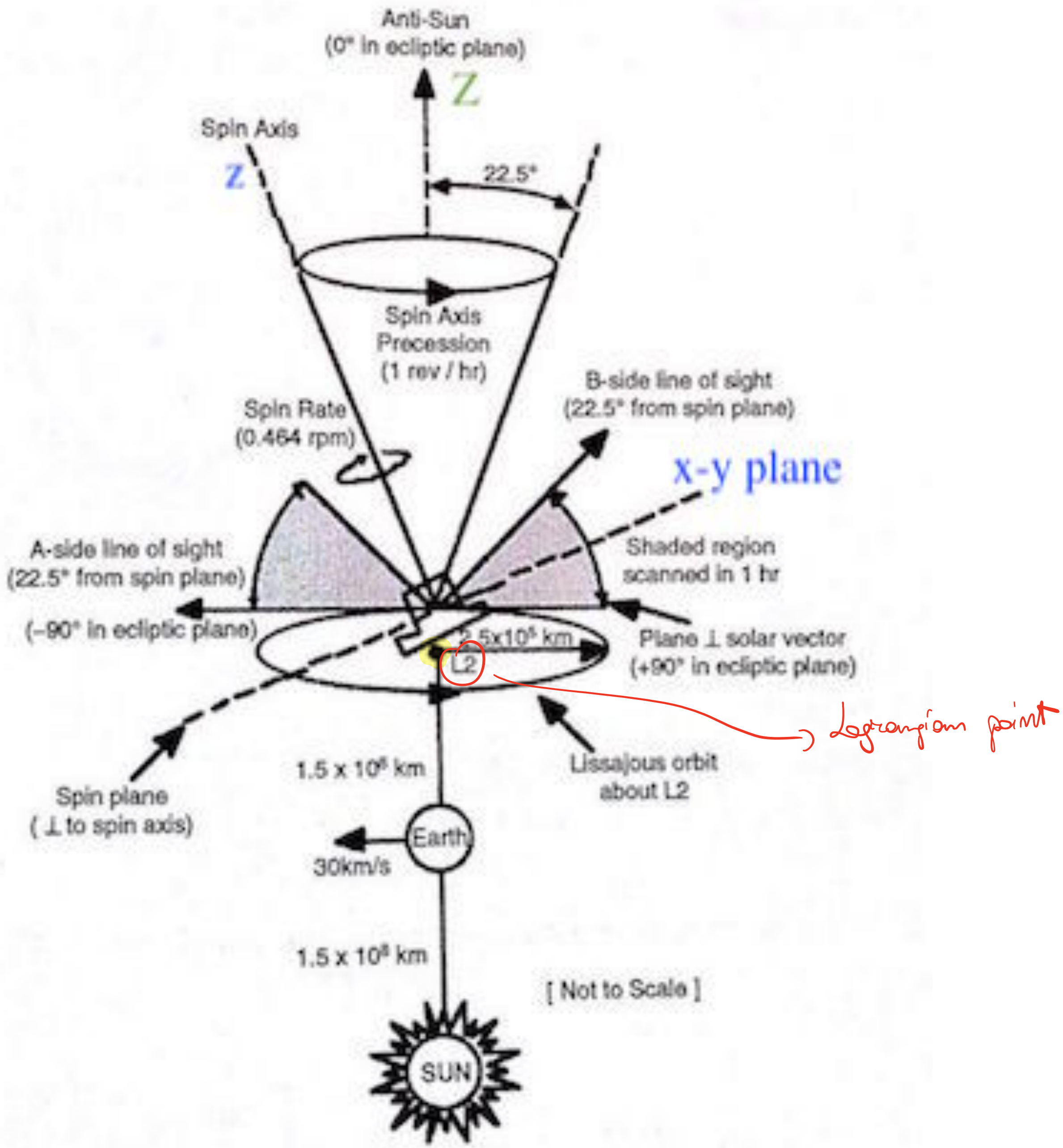
- To impose the desired closed-loop system behavior, a *reference generator (guidance)* has to be used → $\mathbf{q}_r$, $\boldsymbol{\omega}_r$ and $\dot{\boldsymbol{\omega}}_r$ must be consistent with each other, according to the kinematic equations. Example:



Application: WMAP attitude control

- The WMAP mission was to create a full-sky map of the cosmic micro-wave background and to measure its anisotropy with 0.3^o angular resolution.
  - Scientific goals: measuring the Hubble constant, estimating the age of the universe, checking the existence of the dark matter.
- The WMAP spacecraft (S/C) is in an orbit about the Sun-Earth L_2 Lagrange point (180 day period, $1.5e^6\,\mathrm{km}$ far from the Earth).
- The universe is scanned as the Earth revolves around the Sun.
- To ensure scanning of a "large" space region:
  - the S/C must spin about its z-axis at $0.04859\,\mathrm{rad/s}$;
  - the S/C z-axis must spin about the Sun direction at $0.001745\,\mathrm{rad/s}$;
  - A $22.5^o \pm 0.25^o$ angle between the z-axis and the Sun direction must be maintained for power stability and science quality.

Application: WMAP attitude control



Application: WMAP attitude control

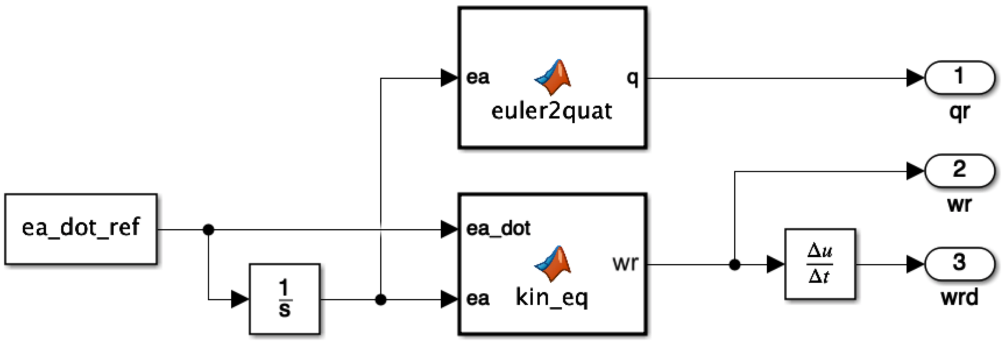
- Two main reference frames: Sun reference frame (SRF) and S/C frame (or body frame, BF).
  - The S/C attitude is described by a 313 rotation SRF→BF.
- Non-inertial effects of SRF rotation, gravity gradient moment, third body gravity, solar radiation are treated as disturbances.
- The reference angular velocities and angles of the rotation are the following:

$$\begin{aligned} \dot{\phi}_r &= 0.001745\,\mathrm{rad/s} \\ \dot{\theta}_r &= 0\,\mathrm{rad/s} \\ \dot{\psi}_r &= 0.04859\,\mathrm{rad/s} \\ \theta_r &= 22.5^o = 0.3927\,\mathrm{rad}. \end{aligned}$$

ϕ_r and ψ_r are obtained by integration of $\dot{\phi}_r$ and $\dot{\psi}_r$.

Application: WMAP attitude control

- Reference generator (guidance):



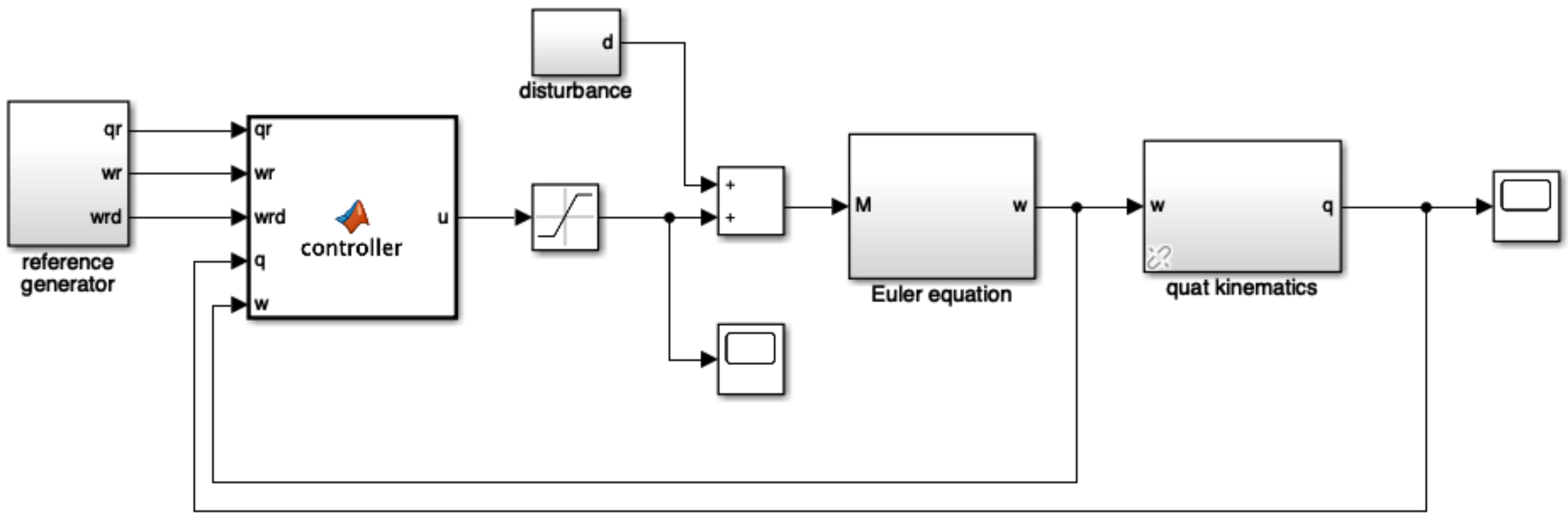
- The S/C inertia matrix is

$$\mathbf{J} = \begin{bmatrix} 399 & -2.81 & -1.31 \\ -2.81 & 377 & 2.54 \\ -1.31 & 2.54 & 377 \end{bmatrix} \mathrm{kg\,m^2}.$$

- The following initial quaternion and angular velocity are assumed: $\mathfrak{q}(0) = (0, 0, 1, 0)$, $\boldsymbol{\omega}(0) = (0.1, -0.9, 0.12)\,\mathrm{rad/s}$.
- The control task for the satellite is to track the reference signals.

Application: WMAP attitude control

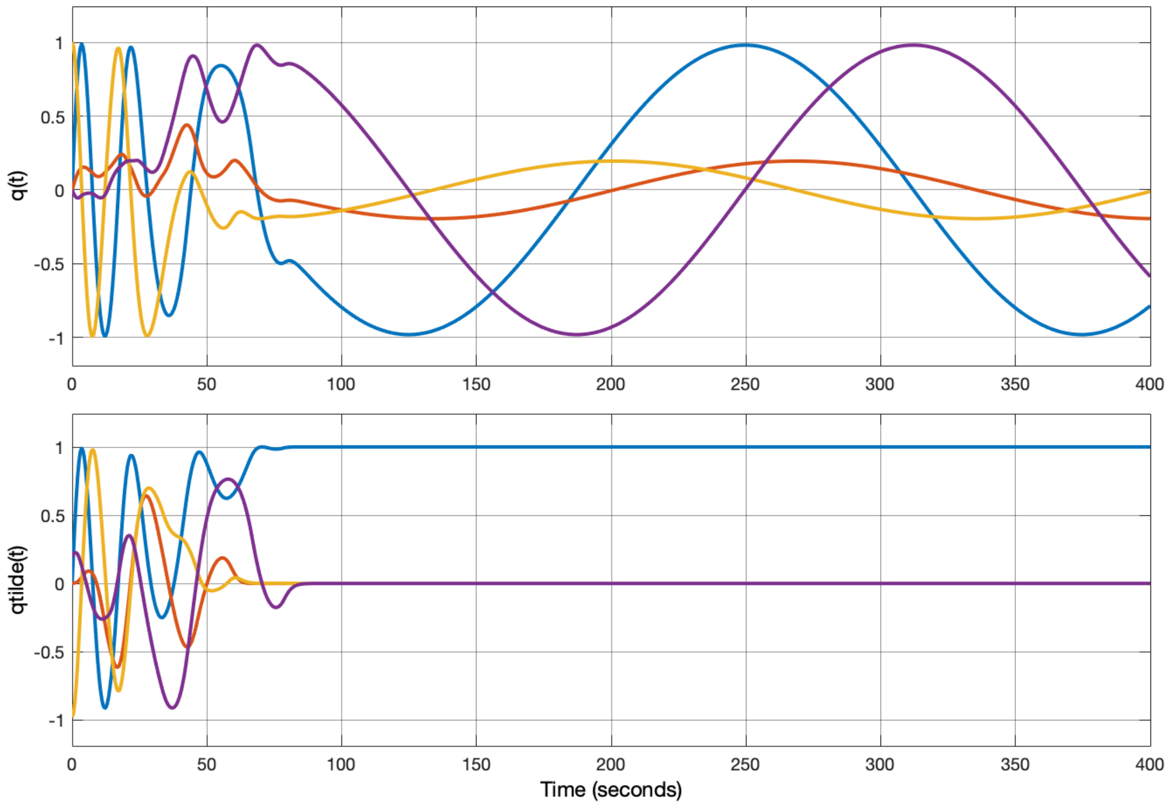
- A disturbance is added to the input. The controller is designed using an approximate inertia matrix $\hat{\mathbf{J}} \neq \mathbf{J}$, $\hat{\mathbf{J}} \simeq \mathbf{J}$.
- The control laws (T1)-(T2) are applied to the S/C system to accomplish this task.



Matlab code at 33.

Application: WMAP attitude control

- Tracking performance: steady-state tracking error $\left|\tilde{\theta}\right| < 0.1^o$ for all the considered control laws.



Appendix

Technicalities

- Consider the following relations:

$$\dot{\mathfrak{q}} = \tfrac{1}{2}\mathbf{Q}\boldsymbol{\omega} = \tfrac{1}{2}\mathfrak{q} \otimes \omega^q = -\mathbf{q} \cdot \boldsymbol{\omega} + q_0\boldsymbol{\omega} + \mathbf{q} \times \boldsymbol{\omega}$$

$$\begin{aligned} (\dot{\mathfrak{q}})^* &= -\mathbf{q} \cdot \boldsymbol{\omega} - q_0\boldsymbol{\omega} - \mathbf{q} \times \boldsymbol{\omega} \\ &= -\left(-\boldsymbol{\omega} \cdot (-\mathbf{q}) + q_0\boldsymbol{\omega} + \boldsymbol{\omega} \times (-\mathbf{q})\right) = -\tfrac{1}{2}\omega^q \otimes \mathfrak{q}^*. \end{aligned}$$

- Quaternion conjugation is a linear operation: $(\dot{\mathfrak{q}})^* = \frac{d}{dt}(\mathfrak{q}^*) = \dot{\mathfrak{q}}^*$. It follows that

$$\dot{\mathfrak{q}}^* = -\frac{1}{2}\omega^q \otimes \mathfrak{q}^*.$$

- The time derivative of $\tilde{\mathfrak{q}}$ is thus given by

$$\begin{aligned} \dot{\tilde{\mathfrak{q}}} &= \dot{\mathfrak{q}}^* \otimes \mathfrak{q}_r + \mathfrak{q}^* \otimes \dot{\mathfrak{q}}_r = -\tfrac{1}{2}\boldsymbol{\omega}^q \otimes \mathfrak{q}^* \otimes \mathfrak{q}_r \\ &+ \tfrac{1}{2}\mathfrak{q}^* \otimes \mathfrak{q}_r \otimes \boldsymbol{\omega}_r^q = -\tfrac{1}{2}\boldsymbol{\omega}^q \otimes \tilde{\mathfrak{q}} + \tfrac{1}{2}\tilde{\mathfrak{q}} \otimes \boldsymbol{\omega}_r^q. \end{aligned}$$

Appendix

Technicalities

- The two terms of $\dot{\tilde{q}}$ can be computed as

$$-\tfrac{1}{2}\boldsymbol{\omega}^q \otimes \tilde{\mathbf{q}} = \tfrac{1}{2}\begin{bmatrix} 0 & \boldsymbol{\omega}^T \\ -\boldsymbol{\omega} & -\boldsymbol{\omega}\times \end{bmatrix}\begin{bmatrix} \tilde{q}_0 \\ \tilde{\mathbf{q}} \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} \boldsymbol{\omega}^T\tilde{\mathbf{q}} \\ -\tilde{q}_0\,\boldsymbol{\omega} + \tilde{\mathbf{q}}\times\boldsymbol{\omega} \end{bmatrix}$$

$$\tfrac{1}{2}\tilde{\mathbf{q}} \otimes \boldsymbol{\omega}_r^q = \tfrac{1}{2}\begin{bmatrix} 0 & -\boldsymbol{\omega}_r^T \\ \boldsymbol{\omega}_r & -\boldsymbol{\omega}_r\times \end{bmatrix}\begin{bmatrix} \tilde{q}_0 \\ \tilde{\mathbf{q}} \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} -\boldsymbol{\omega}_r^T\tilde{\mathbf{q}} \\ \tilde{q}_0\,\boldsymbol{\omega}_r + \tilde{\mathbf{q}}\times\boldsymbol{\omega}_r \end{bmatrix}.$$

- The derivative $\dot{\tilde{q}}$ can thus be written as

$$\dot{\tilde{\mathbf{q}}} = \begin{bmatrix} \dot{\tilde{q}}_0 \\ \dot{\tilde{\mathbf{q}}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \tilde{\mathbf{q}}^T(\boldsymbol{\omega} - \boldsymbol{\omega}_r) \\ \tilde{q}_0(\boldsymbol{\omega}_r - \boldsymbol{\omega}) + \tilde{\mathbf{q}}\times(\boldsymbol{\omega}_r + \boldsymbol{\omega}) \end{bmatrix}.$$