Nonlinear control and aerospace applications

Attitude kinematics

Carlo Novara

Politecnico di Torino
Dip. Elettronica e Telecomunicazioni

Outline

1. Introduction

2. Kinematic equations
   - Vector derivative
   - Direction cosine matrix kinematics
   - Euler angle kinematics
   - Quaternion kinematics

3. Discussion

4. Appendix: Proofs

1 Introduction

2 Kinematic equations
- Vector derivative
- Direction cosine matrix kinematics
- Euler angle kinematics
- Quaternion kinematics

3 Discussion

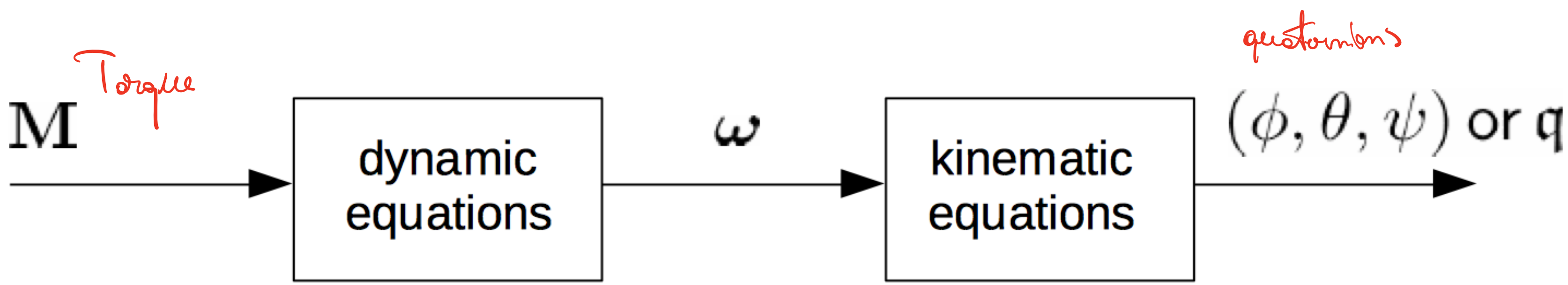
4 Appendix: Proofs

Introduction

- A spacecraft can be (approximately) described as a rigid body, which moves with respect to some inertial frame.
- The body movement is given by a combination of
  - a translation of the body center of mass (CoM);
  - a rotation of the body about an axis passing through the CoM.
- The study of both the translational and rotational motion is of paramount importance for spacecraft design and control.
- The **objective** here is to derive the *attitude kinematic equations* for a rigid body in rotational motion.
  - These equations, together with the dynamic equations, are fundamental for spacecraft attitude control.

Introduction

- The dynamic and kinematic equations can be seen as the series connection of two nonlinear systems:
  - the dynamic equations define a system from $\mathbf{M}$ to $\boldsymbol{\omega}$, where $\mathbf{M}$ is the moment applied to the body;
  - the kinematic equations define a system from $\boldsymbol{\omega}$ to (ϕ, θ, ψ) or $\mathfrak{q}$.



- Overall, it is a nonlinear system with:
  - input $\mathbf{M}$, → moment
  - output (ϕ, θ, ψ), DCM or $\mathfrak{q}$. ← output to control.

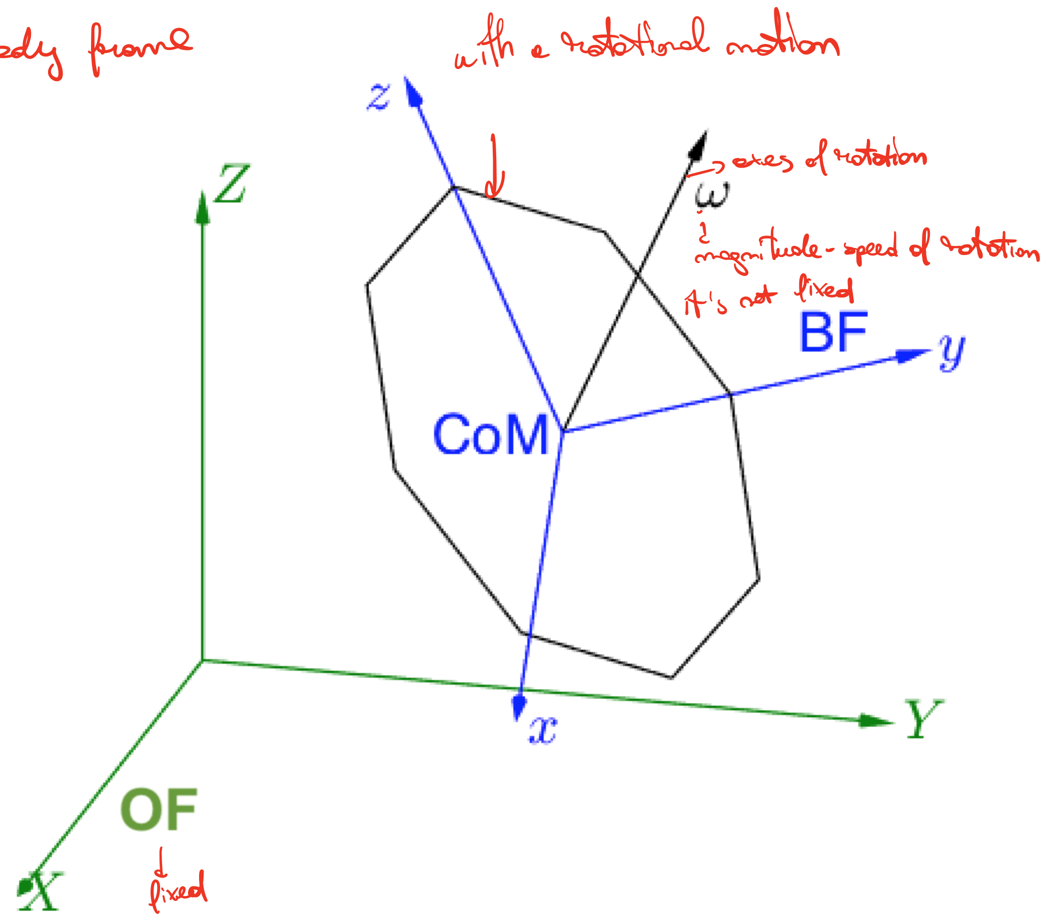
Introduction

- Consider a rigid body rotating wrt some observer reference frame, with angular velocity $\boldsymbol{\omega} = \omega_1 \mathbf{b}_1 + \omega_2 \mathbf{b}_2 + \omega_3 \mathbf{b}_3$.

vector based on body frame

Observer frame (OF):
- origin: somewhere
- unit vectors: $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$
- axes: X, Y, Z.

Body frame (BF): attached to the body
- rotating with the body
- origin: body CoM
- unit vectors: $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$
- axes: x, y, z.



Introduction

- The representations of rotations we consider are:
  1. Direction cosine matrix (DCM).
  2. Euler angles.
  3. Quaternions.
- In the following, after a brief note about the derivative of a vector, we will present the kinematic equations for these three cases.
- The proofs/derivations of these equations are given in the Appendix.

1 Introduction

2 Kinematic equations
- Vector derivative
- Direction cosine matrix kinematics
- Euler angle kinematics
- Quaternion kinematics

3 Discussion

4 Appendix: Proofs

Vector derivative

- For any physical vector $\mathbf{r} = x\mathbf{b}_1 + y\mathbf{b}_2 + z\mathbf{b}_3$: Combination of the vector in the body frame

$$\dot{\mathbf{r}} = \dot{x}\mathbf{b}_1 + \dot{y}\mathbf{b}_2 + \dot{z}\mathbf{b}_3 + x\dot{\mathbf{b}}_1 + y\dot{\mathbf{b}}_2 + z\dot{\mathbf{b}}_3.$$

→ derivative respect to time

- Consider the rotation

$$\delta\mathbf{b}_i = \delta\boldsymbol{\theta} \times \mathbf{b}_i$$

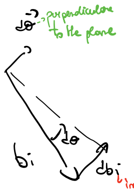
where $\delta\boldsymbol{\theta} = \boldsymbol{\omega}\delta t$ and $\delta t \to 0$. Then, $\dot{\mathbf{b}}_i = \boldsymbol{\omega} \times \mathbf{b}_i$.

- This implies the following relation:

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}_B + \boldsymbol{\omega} \times \mathbf{r}. \tag{1}$$

→ due the fact that the body frame is rotating

  - $\dot{\mathbf{r}}$ is the derivative in the observer frame
  - $\dot{\mathbf{r}}_B \doteq \dot{x}\mathbf{b}_1 + \dot{y}\mathbf{b}_2 + \dot{z}\mathbf{b}_3$ is the derivative in the body frame.



$|d\vec{\theta}| = d\theta$

$d\vec{\theta} = \vec{\omega}\, dt$

$db_i = d\vec{\theta} \times b_i$

db_i → infinitesimal increment due to the rotation in dt → $d\theta$

We can write this relation in terms of coordinatete vectors of a Reference frame RF

$r \to r_B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad \dot{r}_B = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$ $\quad \dot{r} \to [\dot{r}]_B = T \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$

$[\dot{r}]_B$ → Vector expressed in BF coordinates

x, y, z → r coordinates in RF

T → rotation RF→BF

DCM kinematics

- Define the matrix

$$\boldsymbol{\omega}\times \doteq \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

- The DCM kinematic equation is

$$\dot{\mathbf{T}} = \mathbf{T}\boldsymbol{\omega}\times .$$

- Also useful is the transpose kinematic equation

$$\dot{\mathbf{T}}^\top = -\boldsymbol{\omega}\times \mathbf{T}^\top.$$

A physical vector $r = x b_1 + y b_2 + z b_3 = X i_1 + Y i_2 + Z i_3$ has coordinate vectors and derivatives given by:

$$BF: \; r_B = (x, y, z) \qquad OF: \; r_O = (X, Y, Z)$$
$$\dot{r}_B = (\dot{x}, \dot{y}, \dot{z}) \qquad \dot{r}_O = (\dot{X}, \dot{Y}, \dot{Z})$$

T is the rotation matrix OF -> BF (transformation BF -> OF). So:

$$r_O = T r_B$$
$$\dot{r}_O = \dot{T} r_B + T \dot{r}_B$$

We suppose that r_B is an arbitrary constant vector. Then:

$$\dot{r} = \dot{r}_B + w \times r = w \times r$$

that in terms of coordinate vectors becomes: $(\dot{r})_B = w \times r_B$

$\dot{r}$ expressed in BF coordinates

Attention: $(\dot{r})_B \neq \dot{r}_B = 0$. So $(\dot{r})_B = T^T \dot{r}_O$

By substitution we obtain: $T^T \dot{r}_O = w \times r_B \Rightarrow \dot{r}_O = T w \times r_B$. But we know that $\dot{r}_O = \dot{T} r_B + T \dot{r}_B = \dot{T} r_B$. So

$$(\dot{T} - T w \times) r_B = 0$$

arbitrary $\Rightarrow$ $\dot{T} = T w \times$

$$\dot{T}^T = - w \times T^T$$

Euler angle kinematics

Tait-Bryan 321

- Tait-Bryan 321 rotation with angles (ψ, θ, ϕ):

components of angular speed

derivatives of angles

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi\, c\theta \\ 0 & -s\phi & c\phi\, c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}.$$

- Inverting the matrix, we obtain the Tait-Bryan 321 kinematic equation:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \frac{1}{c\theta} \begin{bmatrix} c\theta & s\phi\, s\theta & c\phi\, s\theta \\ 0 & c\phi\, c\theta & -s\phi\, c\theta \\ 0 & s\phi & c\phi \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$$

singularity

- Singularity for $c\theta = 0$ (gimbal lock).

The goal is to describe the time evolution of the rotation Euler angles in function of $\omega_1, \omega_2, \omega_3$.

The overall rotation from OF to BF can be seen as a sequence of 3 intrinsic elementary rotations:

1. OF -> F1;
2. F1 -> F2; |-> intermediate frames
3. F2 -> BF

For example in Tait-Bryan 321 intrinsic rotation, the axes of F1 are x' y' z while those of F2 are x'', y', z' with $x'' = x$

Euler angle kinematics
Tait-Bryan 123

- Tait-Bryan 123 rotation with angles (ϕ, θ, ψ):

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} c\theta\, c\psi & s\psi & 0 \\ -c\theta\, s\psi & c\psi & 0 \\ s\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \frac{1}{c\theta} \begin{bmatrix} c\psi & -s\psi & 0 \\ c\theta\, s\psi & c\theta\, c\psi & 0 \\ -s\theta\, c\psi & s\theta\, s\psi & c\theta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$$

- Singularity for $c\theta = 0$ (gimbal lock).

Euler angle kinematics

Proper Euler 313

- Proper Euler 313 rotation with angles (ϕ, θ, ψ):

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} s\theta\, s\psi & c\psi & 0 \\ s\theta\, c\psi & -s\psi & 0 \\ c\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \frac{1}{s\theta} \begin{bmatrix} s\psi & c\psi & 0 \\ s\theta\, c\psi & -s\theta\, s\psi & 0 \\ -c\theta\, s\psi & -c\theta\, c\psi & s\theta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$$

- Singularity for $s\theta = 0$ (gimbal lock).

Quaternion kinematics

- The quaternion kinematic equations can be written in different equivalent forms:

$$\dot{\mathfrak{q}} = \frac{1}{2}\,\mathfrak{q} \otimes (0, \boldsymbol{\omega})$$
$$\dot{\mathfrak{q}} = \frac{1}{2}\,\boldsymbol{\Omega}\,\mathfrak{q}$$
$$\dot{\mathfrak{q}} = \frac{1}{2}\,\mathbf{Q}\,\boldsymbol{\omega}$$

 where

$$\boldsymbol{\Omega} \doteq \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}, \quad \mathbf{Q} \doteq \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}.$$

- No singularities occur.

1 Introduction

2 Kinematic equations
- Vector derivative
- Direction cosine matrix kinematics
- Euler angle kinematics
- Quaternion kinematics

3 Discussion

4 Appendix: Proofs

Discussion

- We obtained the kinematics equations in the general form

$$\dot{\mathbf{x}} = f(\mathbf{x}, \boldsymbol{\omega})$$ state equations

  where $\mathbf{x} = \mathbf{T}$, $\mathbf{x} = (\phi, \theta, \psi)$ or $\mathbf{x} = \mathfrak{q}$.

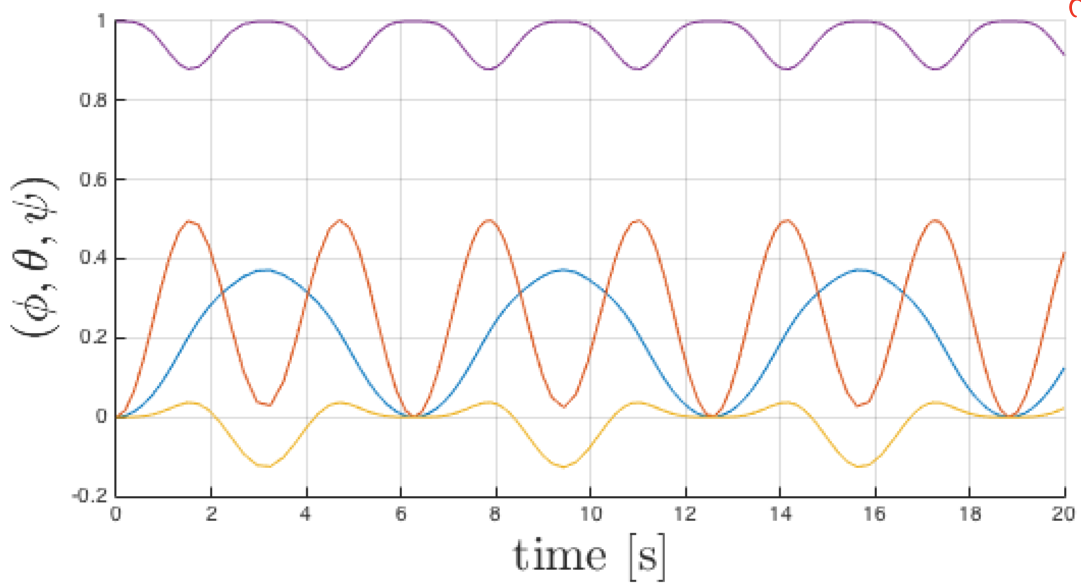
- This is a *state equation*, with state $\mathbf{x}$ and input $\boldsymbol{\omega}$.

- State equations allow us to predict the time evolution of a system:
  - Given an initial state $\mathbf{x}(0) = \mathbf{x}_0$ and an input $\boldsymbol{\omega}(t)$, $t \in [0, \infty)$, $\mathbf{x}(t)$ can be computed for any future time $t \in [0, \infty)$.
  - This computation is done by integration:
    - analytical (possible only in very particular cases);
    - numerical (always possible).

Example

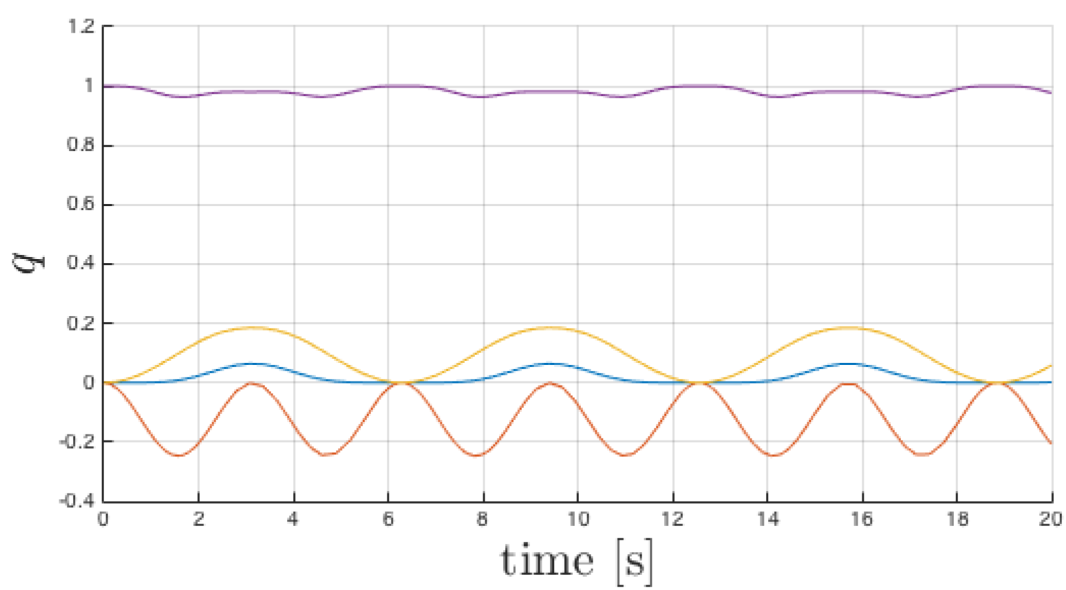
- Consider the kinematic equation for the Tait-Bryan 321 rotation with angles (ψ, θ, ϕ).
- Suppose that:
  - $\mathbf{x}(0) = (\phi(0), \theta(0), \psi(0)) = (0, 0, 0)$,
  - $\boldsymbol{\omega}(t) = (0.2\sin(t), 0.5\sin(2t), 0)$.
- The corresponding time evolution, obtained by integration of the kinematic equation is shown in the figure (magenta: $\cos(\theta)$).

no singularity occurs



Example

- Consider the quaternion kinematic equation.
- Suppose that:
  - $\mathbf{x}(0) = \mathfrak{q}(0) = (0, 0, 0, 1)$,
  - $\boldsymbol{\omega}(t) = (0.2\sin(t), 0.5\sin(2t), 0)$.
- The corresponding time evolution, obtained by integration of the kinematic equation is shown in the figure.



1 Introduction

2 Kinematic equations
- Vector derivative
- Direction cosine matrix kinematics
- Euler angle kinematics
- Quaternion kinematics

3 Discussion

4 Appendix: Proofs

DCM kinematics

Preliminary considerations

- Equation (1) is a relation between physical vectors but it can be written in terms of coordinate vectors of a RF.
  - All vectors in an equation must be expressed in the same RF.
- In BF coordinates, the vectors are given by

$$\mathbf{r} \to \mathbf{r}_B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \dot{\mathbf{r}}_B = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \quad \dot{\mathbf{r}} \to (\dot{\mathbf{r}})_B = \mathbf{T}^\top \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix}$$

where $(\dot{\mathbf{r}})_B$ is the vector $\dot{\mathbf{r}}$ expressed in BF coordinates, T is the rotation OF $\to$ BF, and X, Y, Z are the $\mathbf{r}$ coordinates in OF.

DCM kinematics

- A physical vector $\mathbf{r} = x\mathbf{b}_1 + y\mathbf{b}_2 + z\mathbf{b}_3 = X\mathbf{i}_1 + Y\mathbf{i}_2 + Z\mathbf{i}_3$ has coordinate vectors and derivatives given by

$$\begin{aligned} \text{BF}: \quad & \mathbf{r}_B = (x, y, z) \\ & \dot{\mathbf{r}}_B = (\dot{x}, \dot{y}, \dot{z}) \\ \text{OF}: \quad & \mathbf{r}_O = (X, Y, Z) \\ & \dot{\mathbf{r}}_O = (\dot{X}, \dot{Y}, \dot{Z}). \end{aligned}$$

- Let $\mathbf{T}$ be the rotation matrix OF$\rightarrow$BF (transformation BF$\rightarrow$OF). The following relations hold:

$$\begin{aligned} \mathbf{r}_O &= \mathbf{T}\mathbf{r}_B \\ \dot{\mathbf{r}}_O &= \dot{\mathbf{T}}\mathbf{r}_B + \mathbf{T}\dot{\mathbf{r}}_B. \end{aligned}$$

DCM kinematics

- Suppose now that $\mathbf{r}_B$ is an arbitrary constant vector. Then,

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}_B + \boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}.$$

- This equation is a relation between physical vectors but it can be expressed in terms of coordinate vectors:

$$(\dot{\mathbf{r}})_B = \boldsymbol{\omega} \times \mathbf{r}_B$$

  where $(\dot{\mathbf{r}})_B$ is the vector $\dot{\mathbf{r}}$ expressed in BF coordinates.

- Note that $(\dot{\mathbf{r}})_B \neq \dot{\mathbf{r}}_B = 0$. The expression of $(\dot{\mathbf{r}})_B$ is

$$(\dot{\mathbf{r}})_B = \mathbf{T}^\top \dot{\mathbf{r}}_O.$$

DCM kinematics

- Hence,
$$\mathbf{T}^\top \dot{\mathbf{r}}_O = \boldsymbol{\omega} \times \mathbf{r}_B \quad \Rightarrow \quad \dot{\mathbf{r}}_O = \mathbf{T}\boldsymbol{\omega} \times \mathbf{r}_B.$$
- Since
$$\dot{\mathbf{r}}_O = \dot{\mathbf{T}}\mathbf{r}_B + \mathbf{T}\dot{\mathbf{r}}_B = \dot{\mathbf{T}}\mathbf{r}_B$$
we obtain
$$(\dot{\mathbf{T}} - \mathbf{T}\boldsymbol{\omega}\times)\mathbf{r}_B = 0.$$
- Considering that $\mathbf{r}_B$ is arbitrary, we obtain the DCM kinematic equation
$$\dot{\mathbf{T}} = \mathbf{T}\boldsymbol{\omega} \times .$$
- The transpose kinematic equation directly follows:
$$\dot{\mathbf{T}}^\top = -\boldsymbol{\omega} \times \mathbf{T}^\top.$$

Euler angle kinematics

- The goal is to describe the time evolution of the rotation Euler angles in function of ω_1, ω_2, ω_3.
- The overall rotation from OF to BF can be seen as a sequence of 3 intrinsic elementary rotations:
  1. OF $\rightarrow$ F1;
  2. F1 $\rightarrow$ F2;
  3. F2 $\rightarrow$ BF.

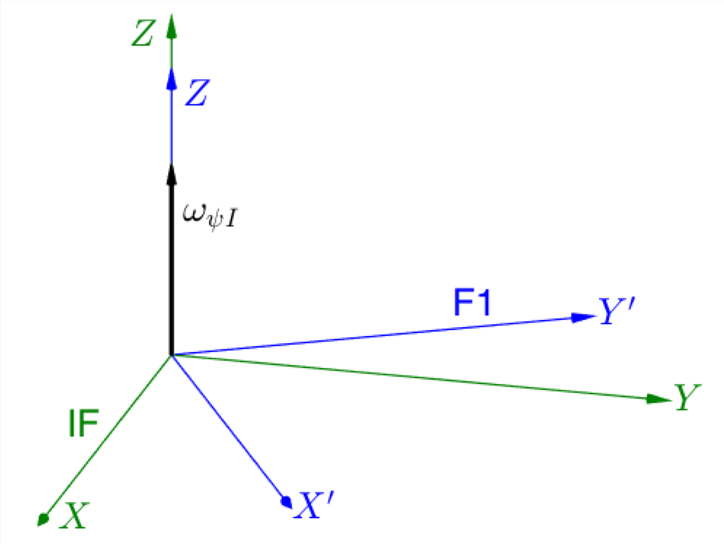
  where F1 and F2 are intermediate frames.
- We consider the Tait-Bryan 321 rotation. The proof for the other cases is similar.
- According to the intrinsic rotation interpretation, the axes of F1 are X', Y', Z, while the axes of F2 are X'', Y', Z', with $X'' = x$.

Euler angle kinematics

Tait-Bryan 321

1 Rotation OF $\rightarrow$ F1:

- F1 is rotated wrt OF of an angle ψ about the Z axis.
- In OF, the angular velocity due to the ψ rotation is $\boldsymbol{\omega}_{\psi I} = [0\ 0\ \dot{\psi}]^\top$.



- Coordinate transformations:
  - ★ $\mathbf{T}_3(\psi)$: F1 $\rightarrow$ OF,
  - ★ $\mathbf{T}_3(\psi)^T = \mathbf{T}_3(-\psi)$: OF $\rightarrow$ F1.
- In F1, the angular velocity is $\boldsymbol{\omega}_{\psi 1} = \mathbf{T}_3(-\psi)\,\boldsymbol{\omega}_{\psi I} = \boldsymbol{\omega}_{\psi I}$.

Euler angle kinematics
Tait-Bryan 321

2 Rotation F1 $\rightarrow$ F2; angle θ about the Y' axis:

- In F1, the angular velocity due to the θ rotation is

$$\boldsymbol{\omega}_{\theta 1} = [0\ \dot{\theta}\ 0]^\top.$$

- In F2, the angular velocity due to the θ rotation is

$$\boldsymbol{\omega}_{\theta 2} = \mathbf{T}_2(-\theta)\,\boldsymbol{\omega}_{\theta 1} = \boldsymbol{\omega}_{\theta 1}.$$

- In F2, the angular velocity due to the ψ rotation is

$$\boldsymbol{\omega}_{\psi 2} = \mathbf{T}_2(-\theta)\,\boldsymbol{\omega}_{\psi 1} = \mathbf{T}_2(-\theta)\,\boldsymbol{\omega}_{\psi I}.$$

Euler angle kinematics

Tait-Bryan 321

3 Rotation F2 $\rightarrow$ BF; angle ϕ about the X'' axis:

- In F2, the angular velocity due to the ϕ rotation is

$$\boldsymbol{\omega}_{\phi 2} = [\dot{\phi}\ 0\ 0]^\top.$$

- In BF, the angular velocity due to the ϕ rotation is

$$\boldsymbol{\omega}_{\phi B} = \mathbf{T}_1(-\phi)\,\boldsymbol{\omega}_{\phi 2} = \boldsymbol{\omega}_{\phi 2}.$$

- In BF, the angular velocity due to the θ rotation is

$$\boldsymbol{\omega}_{\theta B} = \mathbf{T}_1(-\phi)\,\boldsymbol{\omega}_{\theta 2} = \mathbf{T}_1(-\phi)\,\boldsymbol{\omega}_{\theta 1}.$$

- In BF, the angular velocity due to the ψ rotation is

$$\boldsymbol{\omega}_{\psi B} = \mathbf{T}_1(-\phi)\,\boldsymbol{\omega}_{\psi 2} = \mathbf{T}_1(-\phi)\mathbf{T}_2(-\theta)\,\boldsymbol{\omega}_{\psi I}.$$

Euler angle kinematics

Tait-Bryan 321

- In BF, the total angular velocity is

$$
\begin{aligned}
\boldsymbol{\omega} &= \boldsymbol{\omega}_{\phi B} + \boldsymbol{\omega}_{\theta B} + \boldsymbol{\omega}_{\psi B} \\
&= \boldsymbol{\omega}_{\phi 2} + \mathbf{T}_1(-\phi)\,\boldsymbol{\omega}_{\theta 1} + \mathbf{T}_1(-\phi)\mathbf{T}_2(-\theta)\,\boldsymbol{\omega}_{\psi I} \\
&= \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{T}_1(-\phi)\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{T}_1(-\phi)\mathbf{T}_2(-\theta)\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \\
&= \mathbf{I}_1\dot{\phi} + \mathbf{A}_2\dot{\theta} + \mathbf{B}_3\dot{\psi} = \begin{bmatrix} \mathbf{I}_1 & \mathbf{A}_2 & \mathbf{B}_3 \end{bmatrix}\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}
\end{aligned}
$$

  - $\mathbf{I}_1$ is the first column of the identity matrix,
  - $\mathbf{A}_2$ is the second column of $\mathbf{A} \doteq \mathbf{T}_1(-\phi)$,
  - $\mathbf{B}_3$ is the third column of $\mathbf{B} \doteq \mathbf{T}_1(-\phi)\mathbf{T}_2(-\theta)$.

Euler angle kinematics

Tait-Bryan 321

- For the Tait-Bryan 321 rotation with angles (ψ, θ, ϕ), simple calculations yield

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi\, c\theta \\ 0 & -s\phi & c\phi\, c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}.$$

- Inverting the matrix, we obtain the kinematic equation

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \frac{1}{c\theta} \begin{bmatrix} c\theta & s\phi\, s\theta & c\phi\, s\theta \\ 0 & c\phi\, c\theta & -s\phi\, c\theta \\ 0 & s\phi & c\phi \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$$

Quaternion kinematics

- The goal is to describe the time evolution of the rotation quaternion $\mathfrak{q}$ in function of ω_1, ω_2, ω_3.
- Note that both the quaternion and the angular velocity change in time:

$$\begin{aligned}\mathfrak{q} &\equiv \mathfrak{q}(t)\\ \boldsymbol{\omega} &= (\omega_1,\, \omega_2, \omega_3) \equiv \boldsymbol{\omega}(t).\end{aligned}$$

- At time $t + \Delta t$, we have the rotation $\mathfrak{q}(t)$ at time t composed with the rotation $\Delta\mathfrak{q}(t)$ occurred from time t to time $t + \Delta t$.
- The quaternion at time $t + \Delta t$ is thus given by

$$\mathfrak{q}(t + \Delta t) = \mathfrak{q}(t) \otimes \Delta\mathfrak{q}(t).$$

composition with quaternion multiplication

Quaternion kinematics

- The incremental quaternion $\Delta\mathfrak{q}(t)$ can be written as follows.
  - Let $\omega = |\boldsymbol{\omega}|$ be the angular speed magnitude;
    - for a small Δt, the rotation angle is $\omega\Delta t$.
  - Let $\mathbf{u}$ be the rotation axis, with $|\mathbf{u}| = 1$;
    - it follows that $\boldsymbol{\omega} = \omega\mathbf{u}$.

  Then, for small Δt,

  components of quaternion in standard form

  when $\Delta t \to 0$

$$\Delta\mathfrak{q} \cong \begin{bmatrix} \cos\frac{\omega\Delta t}{2} \\ \mathbf{u}\sin\frac{\omega\Delta t}{2} \end{bmatrix} \cong \begin{bmatrix} 1 \\ \mathbf{u}\frac{\omega\Delta t}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\boldsymbol{\omega}\Delta t}{2} \end{bmatrix}.$$

- The quaternion derivative is thus given by

$$\dot{\mathfrak{q}} = \lim_{\Delta t\to 0} \frac{\mathfrak{q}(t+\Delta t)-\mathfrak{q}(t)}{\Delta t} = \lim_{\Delta t\to 0} \frac{\mathfrak{q}\otimes\Delta\mathfrak{q}-\mathfrak{q}}{\Delta t}$$

$$= \lim_{\Delta t\to 0} \frac{\mathfrak{q}\otimes(\Delta\mathfrak{q}-(1,\mathbf{0}))}{\Delta t} = \lim_{\Delta t\to 0} \frac{\mathfrak{q}\otimes((1,\boldsymbol{\omega}\Delta t/2)-(1,\mathbf{0}))}{\Delta t}.$$

Quaternion kinematics

- Consider that $\left(1, \frac{\boldsymbol{\omega}\Delta t}{2}\right) - (1, \mathbf{0}) = \left(0, \frac{\boldsymbol{\omega}\Delta t}{2}\right)$. Then,

$$\dot{\mathfrak{q}} = \lim_{\Delta t \to 0} \frac{\mathfrak{q} \otimes \left(0, \frac{\boldsymbol{\omega}\Delta t}{2}\right)}{\Delta t} = \frac{1}{2}\mathfrak{q} \otimes (0, \boldsymbol{\omega}) \quad \Rightarrow \quad \dot{\mathfrak{q}} = \frac{1}{2}\mathfrak{q} \otimes (0, \boldsymbol{\omega}).$$

- Equivalent equations are the following:

same equation

$$\dot{\mathfrak{q}} = \frac{1}{2}\boldsymbol{\Omega}\,\mathfrak{q} \qquad \dot{\mathfrak{q}} = \frac{1}{2}\mathbf{Q}\,\omega$$

$$\boldsymbol{\Omega} \doteq \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}, \qquad \mathbf{Q} \doteq \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}.$$