Nonlinear control and aerospace applications

Rotations

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Outline

Introduction

more technical

- Controlling the **orientation** (or **attitude**) of a spacecraft (or aircraft) is fundamental.
  - Spacecrafts and space stations orbiting around a planet, or during interplanetary navigation,
    - must capture the solar energy through panels,
    - need a communication link between on-board antennas and Earth stations/receivers or relay satellites.
  - Scientific satellites and space vehicles carry payloads to be pointed toward either celestial objects or Earth targets (e.g., the Hubble).
- Depending on the mission objectives, the orientation must be known and controlled with respect to
  - a region on Earth (Earth pointing satellites) or
  - a celestial frame (inertial pointing satellites).

Introduction

- A spacecraft can be described as a rigid body, which moves with respect to some inertial frame.
- The body movement is given by a combination of a *translation* and a *rotation*.
- The **objective** here is to build the *mathematical tools* for properly describing the *rotational motion* of a rigid body.
  - This is fundamental to obtain the equations of motion of a spacecraft, which in turn are fundamental for spacecraft attitude control.
- Four rotation representations will be considered:
  1. Direction cosine matrices
  2. Euler angles
  3. Angle-axis
  4. Quaternions (Euler parameters).

Introduction

Notation

- Scalars: $a, b, A, B \in \mathbb{R}$.
- Column vectors:

$$\mathbf{r} = (r_1, \ldots, r_n) = [r_1 \ \ldots \ r_n]^T = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \in \mathbb{R}^{n \times 1}.$$

- Row vectors: $\mathbf{r}^T = [r_1 \ \ldots \ r_n] \in \mathbb{R}^{1 \times n}$.
- Matrices: $\mathbf{M} \in \mathbb{R}^{n \times m}$.
- Products:

result is a scalar

$$\mathbf{r} \cdot \mathbf{p} = \mathbf{r}^T \mathbf{p} = \textstyle\sum_{i=1}^{n} r_i p_i \qquad \text{dot product}$$

$$\mathbf{r} \times \mathbf{p} = \begin{bmatrix} r_2 p_3 - r_3 p_2 \\ r_3 p_1 - r_1 p_3 \\ r_1 p_2 - r_2 p_1 \end{bmatrix} \qquad \text{cross product.}$$

→ result is a vector

- Vector ℓ_2 (Euclidean) norm:

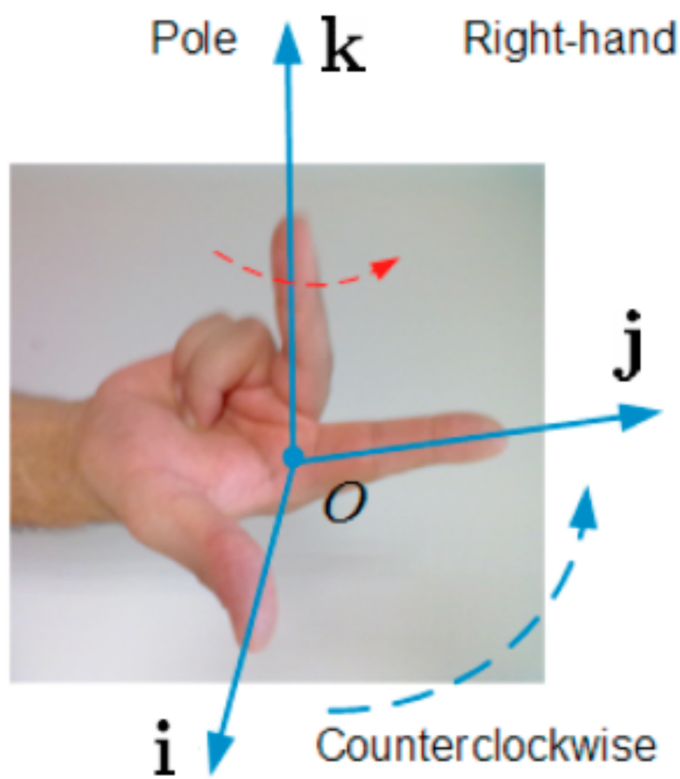
$$|\mathbf{r}| = \|\mathbf{r}\| = \|\mathbf{r}\|_2 = \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{\mathbf{r}^T \mathbf{r}} = \sqrt{\textstyle\sum_{i=1}^{n} r_i^2} = r.$$

→ not in bold

Reference frames

Definition

An orthogonal *frame of reference* $\mathscr{R} = \{O, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ (or *Cartesian coordinate system*) is formed by an origin O and a set of three unit vectors $\underbrace{\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}}_{\text{with unitary norm}}$ with origin in O, that are mutually orthogonal.



Reference frames

- Three main kinds of RFs (reference frames) can be distinguished:

  1. *Body frames*:
     - Origin and axes are defined by points of a rigid body, either a spacecraft or a planet.
     - Typically, the body Center of Mass (CoM) is taken as the origin.
  2. *Trajectory frames*: move with the body
     - A trajectory is the path of CoM of a body through the space (typically, an orbit).
     - The body CoM is taken as the origin.
     - The axes are aligned with three instantaneous directions of the trajectory.
  3. *Celestial frames*:
     - Origin and axes are defined by points and directions in the solar system or in the universe.

Reference frames

Vector representations

- Consider a reference frame F=$\{O, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$.
- A vector $\mathbf{r} \in \mathbb{R}^3$ can be written as a linear combination of the unit vectors of F as

$$\boxed{\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}} \leftarrow \text{physical vector}$$

where x, y, z are the *coordinates*.

- The vector can also be represented as

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leftarrow \text{coordinate vector.}$$

column vector

Reference frames

Vector representations

$$\mathbf{r} = \vec{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \;\leftarrow \text{physical vector}$$

defined in an absolute reference frame

the coordinate vector doesn't depend on the RF

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\leftarrow \text{coordinate vector.}$$

representation in a system

- The physical vector is an "abstract" object. We can think about it as a vector defined in an "absolute" reference frame.
- The coordinate vector is the representation of the physical vector in a given RF $\Rightarrow$ the coordinate vector depends on the RF.
- The physical vector is often denoted with $\vec{r}$. For simplicity, we use the same symbol $\mathbf{r}$ for both the physical and the coordinate vector.

Reference frames

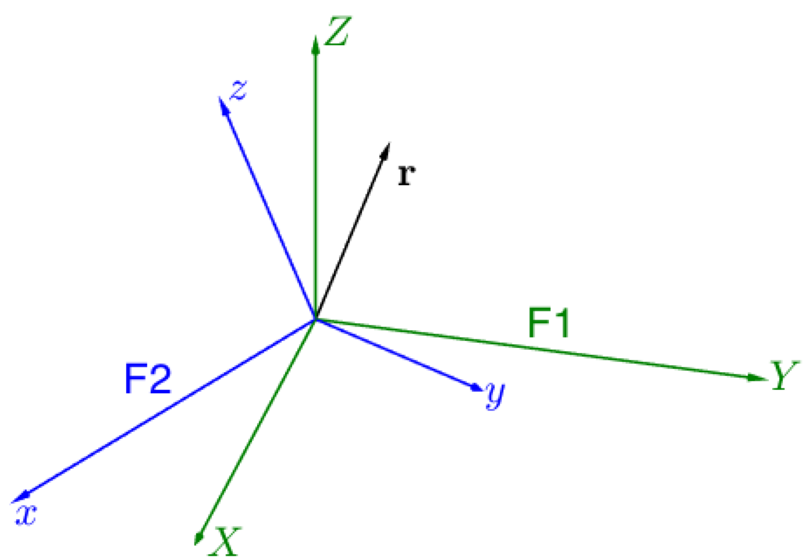
Vector representations: example

- Consider two RFs:
  - F1=$\{O, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$, with axes X, Y, Z.
  - F2=$\{O, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, with axes x, y, z.
- The physical vector

$$\mathbf{r} = -0.45\mathbf{I} + 0.04\mathbf{J} + 0.19\mathbf{K} = -0.4\mathbf{i} - 0.24\mathbf{j} + 0.16\mathbf{k}$$

has two different representations (coordinate vectors):

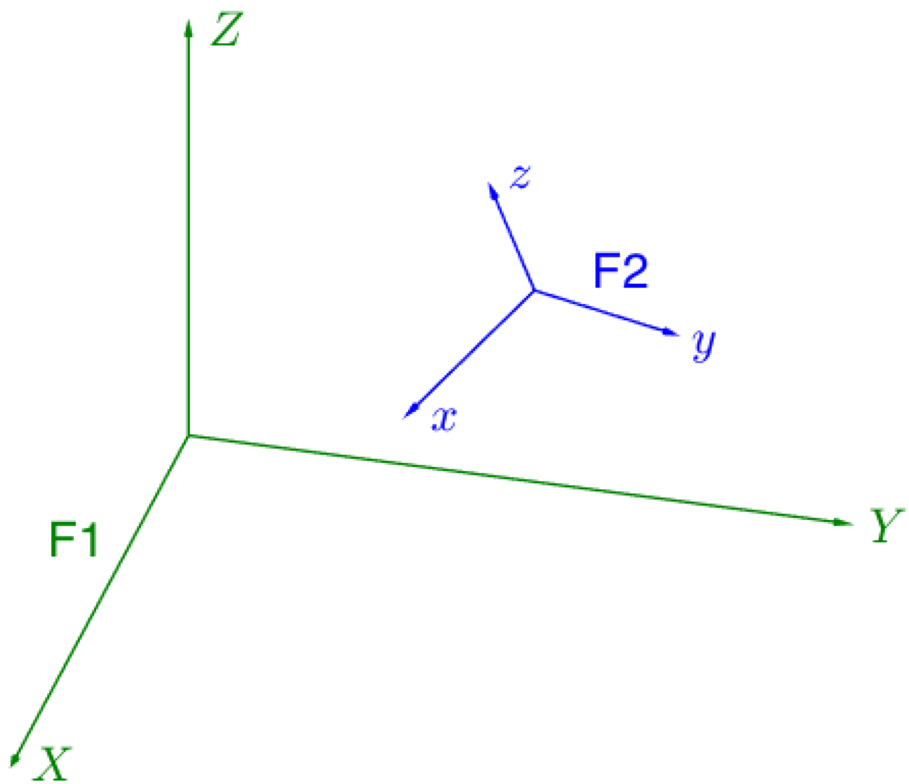
$$[-0.45\ 0.04\ 0.19]^T \quad \text{and} \quad [-0.4\ -0.24\ 0.16]^T.$$



Direction cosine matrices

- Consider two RFs:
  - F1=$\{O_1, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$, with axes X, Y, Z.
  - F2=$\{O_2, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, with axes x, y, z.

reference frames



Direction cosine matrices

- Consider a particle with position $\mathbf{R} = \mathbf{R}_o + \mathbf{r}$

position with respect to F1

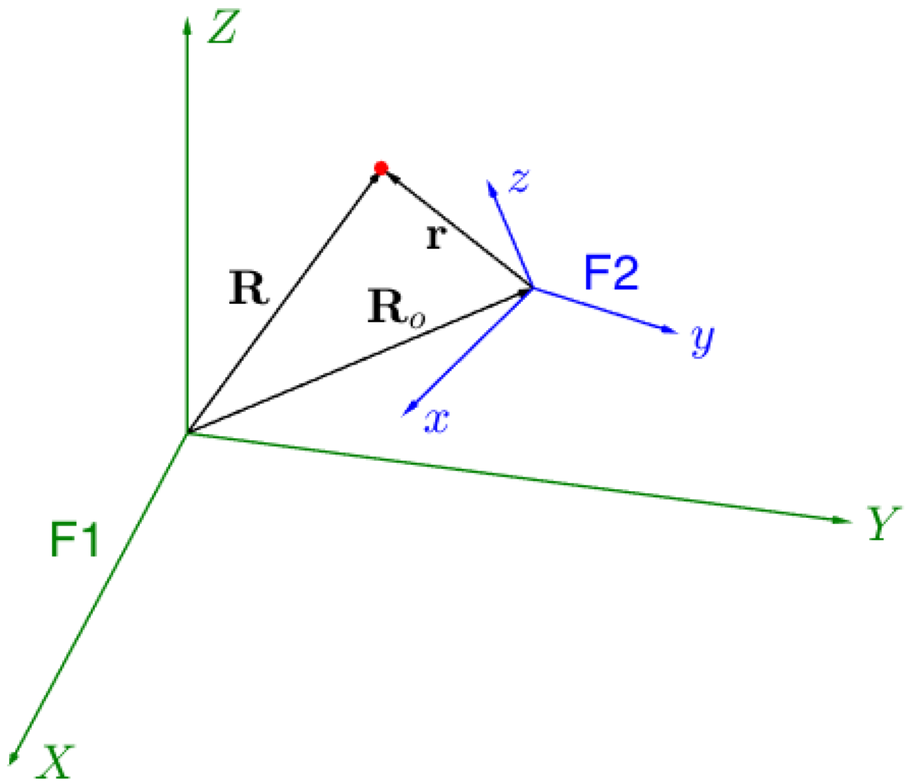
position in the second reference frame

origin of F_2

$$\mathbf{R} = X\mathbf{I} + Y\mathbf{J} + Z\mathbf{K}$$ position of the particle in F1

$$\mathbf{R}_o = X_o\mathbf{I} + Y_o\mathbf{J} + Z_o\mathbf{K}$$ position of the origin of F2

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$ position of the particle in F2.



Direction cosine matrices

- What is the relation between the coordinates X, Y an Z and the coordinates x, y and z? To answer, consider that:

because the products I·J and I·K are 0 because of orthogonality

$\mathbf{R}_o \cdot \mathbf{I}$

$\mathbf{r} \cdot \mathbf{I}$

$$
\begin{aligned}
X &= \mathbf{R} \cdot \mathbf{I} = (\mathbf{R}_o + \mathbf{r}) \cdot \mathbf{I} = X_o + \boxed{x\mathbf{I} \cdot \mathbf{i} + y\mathbf{I} \cdot \mathbf{j} + z\mathbf{I} \cdot \mathbf{k}} \\
Y &= \mathbf{R} \cdot \mathbf{J} = (\mathbf{R}_o + \mathbf{r}) \cdot \mathbf{J} = Y_o + x\mathbf{J} \cdot \mathbf{i} + y\mathbf{J} \cdot \mathbf{j} + z\mathbf{J} \cdot \mathbf{k} \\
Z &= \mathbf{R} \cdot \mathbf{K} = (\mathbf{R}_o + \mathbf{r}) \cdot \mathbf{K} = Z_o + x\mathbf{K} \cdot \mathbf{i} + y\mathbf{K} \cdot \mathbf{j} + z\mathbf{K} \cdot \mathbf{k}.
\end{aligned}
$$

- In matrix form:

Products of the unit vectors of the frames

9 scalar numbers

$$
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} + \mathbf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{T} \doteq \begin{bmatrix} \mathbf{I} \cdot \mathbf{i} & \mathbf{I} \cdot \mathbf{j} & \mathbf{I} \cdot \mathbf{k} \\ \mathbf{J} \cdot \mathbf{i} & \mathbf{J} \cdot \mathbf{j} & \mathbf{J} \cdot \mathbf{k} \\ \mathbf{K} \cdot \mathbf{i} & \mathbf{K} \cdot \mathbf{j} & \mathbf{K} \cdot \mathbf{k} \end{bmatrix}.
$$

The dot products $\mathbf{I} \cdot \mathbf{i}$, $\mathbf{I} \cdot \mathbf{j}$, ..., are the 9 *direction cosines* representing the orientation of each axis of one frame wrt each axis of the other.

$\mathbf{T}$ is the *direction cosine matrix* (DCM).

Direction cosine matrices

- The DCM can be expressed as

$$\mathbf{T} \doteq \begin{bmatrix} \mathbf{I}\cdot\mathbf{i} & \mathbf{I}\cdot\mathbf{j} & \mathbf{I}\cdot\mathbf{k} \\ \mathbf{J}\cdot\mathbf{i} & \mathbf{J}\cdot\mathbf{j} & \mathbf{J}\cdot\mathbf{k} \\ \mathbf{K}\cdot\mathbf{i} & \mathbf{K}\cdot\mathbf{j} & \mathbf{K}\cdot\mathbf{k} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = [T_{ij}]\,.$$

- Translations and rotations can be treated independently. Thus, in the following, we assume

$$\mathbf{R}_o = 0.$$

→ the systems have the same origin

= the vector $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ goes to ϕ

- Then,

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

relationship between the coordinates of the two system

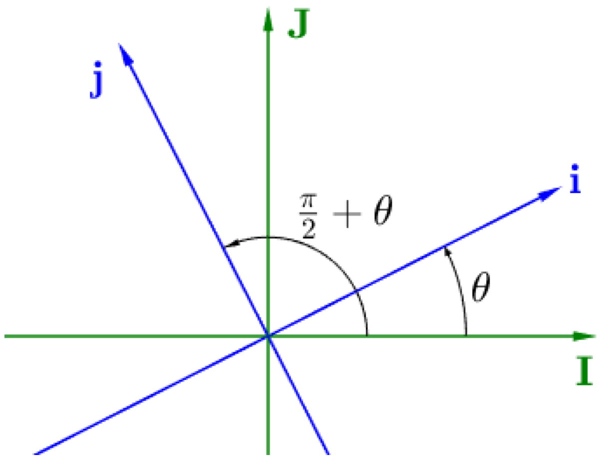
This shows that $\mathbf{T}$ is a **transformation**, giving the coordinates in F1 from the coordinates in F2.

Direction cosine matrices

- In two dimensions (e.g., with $z = Z = 0$) the direction cosines are

$$\boxed{\begin{aligned} &\mathbf{I}\cdot\mathbf{i} = \cos\theta = \mathbf{J}\cdot\mathbf{j} \\ &\mathbf{I}\cdot\mathbf{j} = \cos(\tfrac{\pi}{2}+\theta) = -\sin\theta = -\mathbf{J}\cdot\mathbf{i} \end{aligned}}$$

  where θ is the rotation angle.



- It follows that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

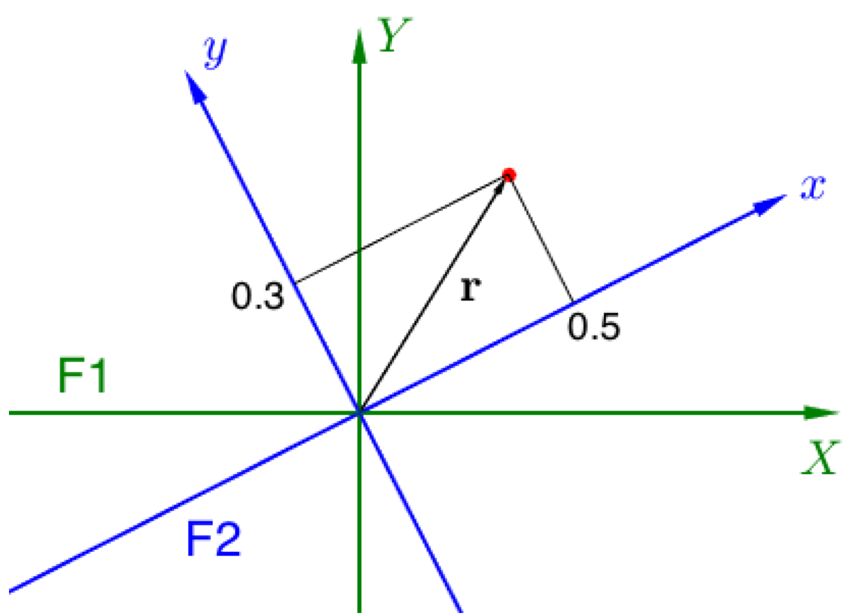
Direction cosine matrices

Example: 2D coordinate transformation

- Suppose that F2 is rotated wrt F1 of an angle $\theta = 0.15\,\pi$ rad.
- Consider a particle with position in F2 given by

$$\mathbf{r} = 0.5\,\mathbf{i} + 0.3\,\mathbf{j} = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}.$$

in F2



Direction cosine matrices

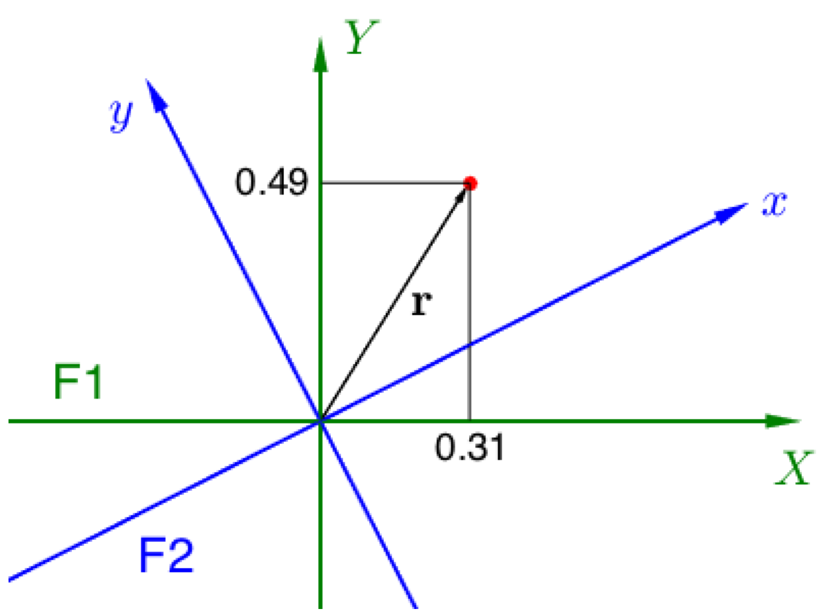
Example: 2D coordinate transformation

- The particle position in F1 is obtained through the following transformation:

in F1

$$\mathbf{R} = \begin{bmatrix} X \\ Y \end{bmatrix} = \mathbf{Tr} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.891 & -0.454 \\ 0.454 & 0.891 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.3093 \\ 0.4943 \end{bmatrix}$$



Direction cosine matrices

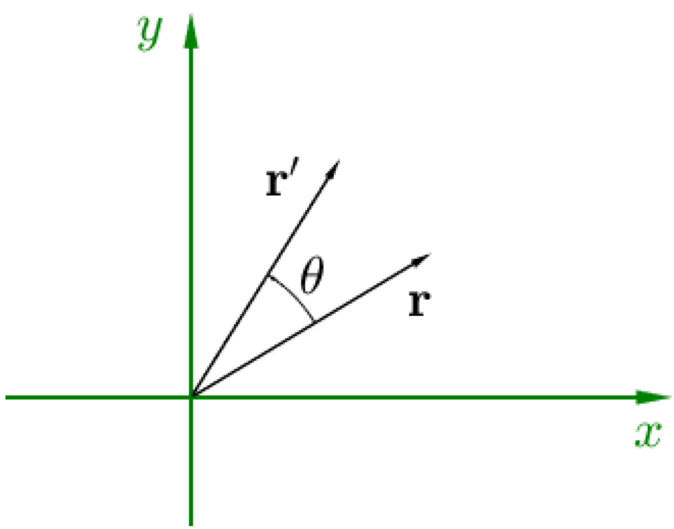
Example: 2D vector rotation

- Another interpretation of the DCM is the following:
  Consider a vector $\mathbf{r}$ in a reference frame with axes x, y:

$$\mathbf{r} = 0.5\,\mathbf{i} + 0.3\,\mathbf{j} = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}.$$

- Applying the DCM, we obtain a vector $\mathbf{r}'$ rotated of an angle θ in the same reference frame:

$$\mathbf{r}' = \mathbf{T}\mathbf{r} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.3093 \\ 0.4943 \end{bmatrix}.$$



Direction cosine matrices

Example: 2D rotation of a square

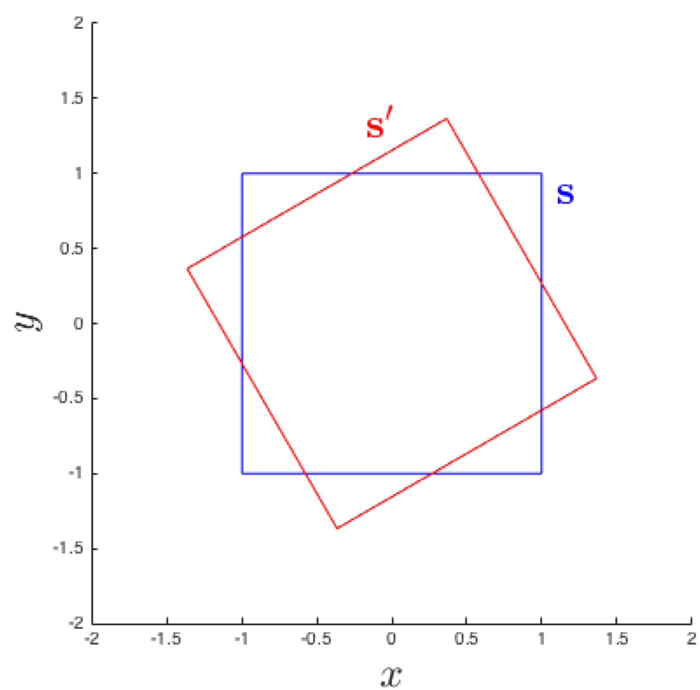
- A square can be represented by the following matrix:

coordinates of a point

$$\mathbf{S} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \leftarrow \begin{array}{c} \text{the columns} \\ \text{are the verices.} \end{array}$$

- Applying the DCM with $\theta = \pi/6$ rad:

$$\mathbf{S}' = \mathbf{TS} = \begin{bmatrix} 0.366 & -1.366 & -0.366 & 1.366 \\ 1.366 & 0.366 & -1.366 & -0.366 \end{bmatrix} \leftarrow \begin{array}{c} \text{rotated} \\ \text{square.} \end{array}$$



Direction cosine matrices

Discussion

- A DCM $\mathbf{T}$ has 2 interpretations:
  1. **Transformation** of coordinates F2 $\longrightarrow$ F1.
  2. **Rotation** of vectors in a given fixed frame.
     In the case of vectors that are frame axes, F1 $\longrightarrow$ F2.
- The following terminology is often used:
  - transformation $\leftrightarrow$ alias
  - rotation $\leftrightarrow$ alibi.
- Both interpretations are important.
- Any time that an DCM is used, it is necessary to understand which interpretation is being used.

Euler angles

c -> cos   s -> sin

- In three dimensions, we define the *elementary rotation matrices*:

coordinate that doesn't change

taken from the previous 2-D matrices

$$\mathbf{T}_1(\phi) \doteq \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

rotation about X (or x) of an angle ϕ

$$\mathbf{T}_2(\theta) \doteq \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

rotation about Y (or y) of an angle θ

$$\mathbf{T}_3(\psi) \doteq \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotation about Z (or z) of an angle ψ.

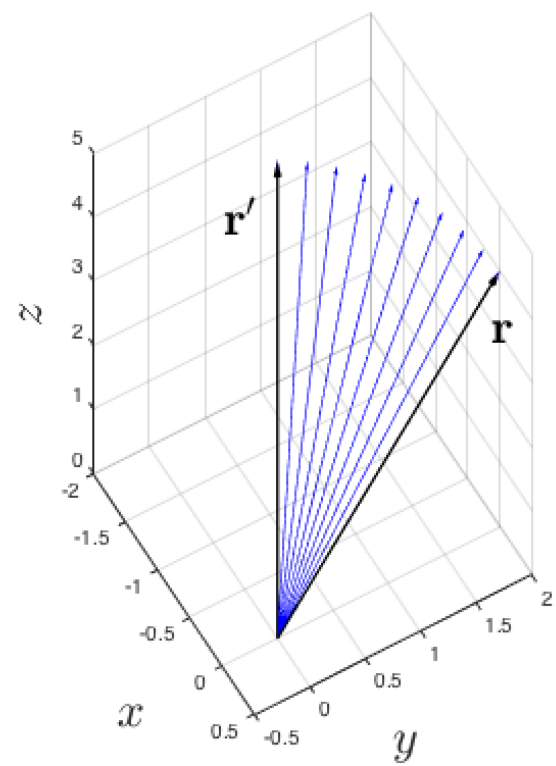
- Any rotation can be expressed as a product of $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$.
- ψ, θ and ϕ are called the *Euler angles*.

Euler angles

Example: 3D elementary rotation of a vector

- Consider the vector $\mathbf{r} = 0\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.
- The matrix $\mathbf{T}_3(\pi/3)$ rotates any vector about the z axis of an angle $\pi/3$ rad $= 60^o$.
- The rotated vector is $\mathbf{r}' = \mathbf{T}_3(\pi/3)\mathbf{r} = -1.73\mathbf{i} + 1\mathbf{j} + 4\mathbf{k}$.

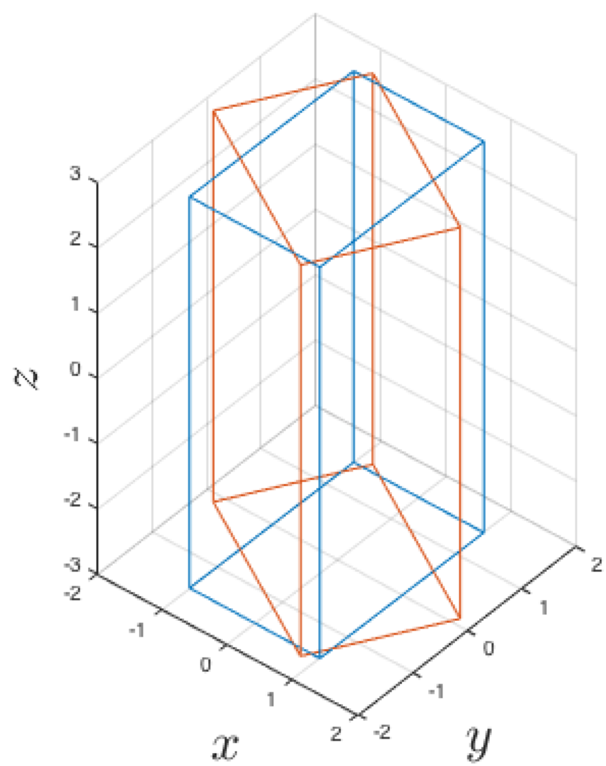
z component unchanged



Euler angles

Example: 3D elementary rotation of an hyper-rectangle

- An hyper-rectangle can be represented by a matrix whose columns are the vertices (blue).
- Applying $\mathbf{T}_3(\pi/3)$ to this matrix, we obtain an hyper-rectangle rotated about the z axis of an angle $\pi/3$ rad $= 60^o$ (red).



Euler angles

- There exist 12 possible combinations of the 3 elementary rotations (with non-sequentially repeated indexes), which can be grouped as follows:
  - 6 Tait–Bryan rotations: index are all different
    - $123 \rightarrow \mathbf{T}_1(\phi)\mathbf{T}_2(\theta)\mathbf{T}_3(\psi)$;
    - $321 \rightarrow \mathbf{T}_3(\psi)\mathbf{T}_2(\theta)\mathbf{T}_1(\phi)$;
    - ...
  - 6 proper Euler rotations: two indexes are repeated $\downarrow$ not sequentially
    - $313 \rightarrow \mathbf{T}_3(\phi)\mathbf{T}_1(\theta)\mathbf{T}_3(\psi)$;
    - $323 \rightarrow \mathbf{T}_3(\psi)\mathbf{T}_2(\theta)\mathbf{T}_3(\phi)$;
    - ...

- Commonly used: Tait–Bryan 123 and 321; proper Euler 313.

Euler angles

- Tait–Bryan 123:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{T}_{123} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where $\mathbf{T}_{123}(\phi, \theta, \psi) \doteq \mathbf{T}_1(\phi)\mathbf{T}_2(\theta)\mathbf{T}_3(\psi)$ is the *rotation matrix* (or *attitude matrix*) given by

$$\mathbf{T}_{123} = \begin{bmatrix} c\theta\, c\psi & -c\theta\, s\psi & s\theta \\ c\phi\, s\psi + s\phi\, s\theta\, c\psi & c\phi\, c\psi - s\phi\, s\theta\, s\psi & -s\phi\, c\theta \\ s\phi\, s\psi - c\phi\, s\theta\, c\psi & s\phi\, c\psi + c\phi\, s\theta\, s\psi & c\phi\, c\theta \end{bmatrix}.$$

Euler angles

dovrebbe essere la rotazione attorno all'originale θ del sist. di riferimento FISSO

- Tait–Bryan 321:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{T}_{321} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where $\mathbf{T}_{321}(\psi, \theta, \phi) \doteq \mathbf{T}_3(\psi)\mathbf{T}_2(\theta)\mathbf{T}_1(\phi)$ is the rotation matrix

$$\mathbf{T}_{321} = \begin{bmatrix} c\theta\, c\psi & -c\phi\, s\psi + s\phi\, s\theta\, c\psi & s\phi\, s\psi + c\phi\, s\theta\, c\psi \\ c\theta\, s\psi & c\phi\, c\psi + s\phi\, s\theta\, s\psi & -s\phi\, c\psi + c\phi\, s\theta\, s\psi \\ -s\theta & s\phi\, c\theta & c\phi\, c\theta \end{bmatrix}.$$

Quella usata da Solvo per calcolo di L
in sperimentale_2.m

Euler angles

- Proper Euler 313:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{T}_{313} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

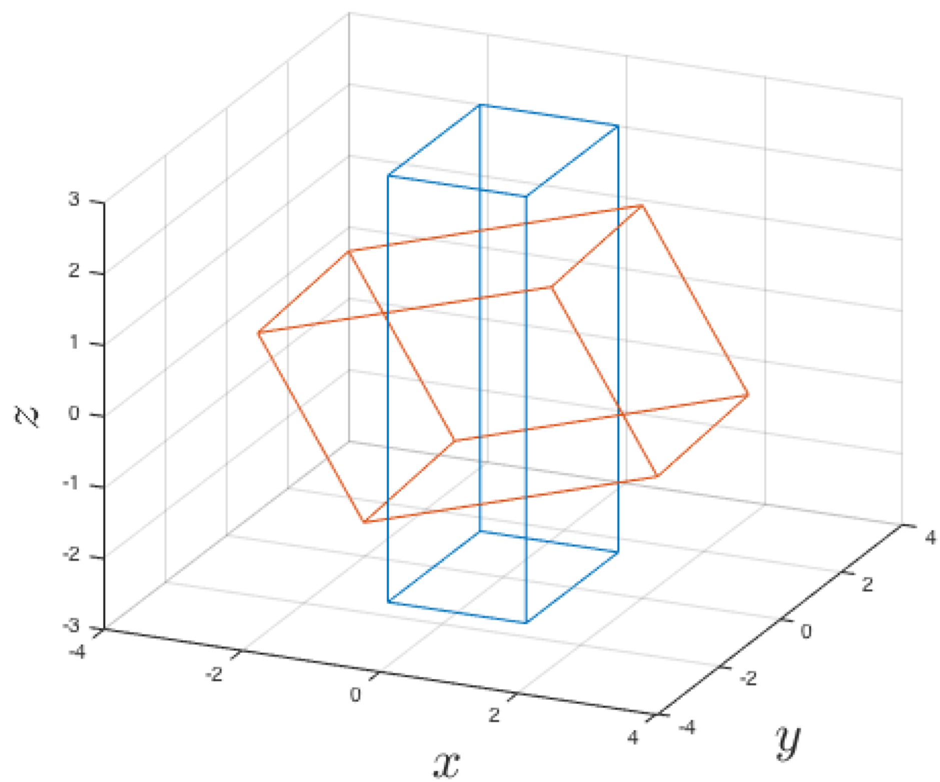
where $\mathbf{T}_{313}(\phi,\theta,\psi) \doteq \mathbf{T}_3(\phi)\mathbf{T}_1(\theta)\mathbf{T}_3(\psi)$ is the rotation matrix

$$\mathbf{T}_{313} = \begin{bmatrix} c\phi\, c\psi - s\phi\, c\theta\, s\psi & -c\phi\, s\psi - s\phi\, c\theta\, c\psi & s\phi\, s\theta \\ s\phi\, c\psi + c\phi\, c\theta\, s\psi & -s\phi\, s\psi + c\phi\, c\theta\, c\psi & -c\phi\, s\theta \\ s\theta\, s\psi & s\theta\, c\psi & c\theta \end{bmatrix}.$$

Euler angles

Example: 3D generic rotation of an hyper-rectangle

- An hyper-rectangle can be represented by a matrix whose columns are the vertices (blue).
- Applying $\mathbf{T}_1(\pi/4)\mathbf{T}_2(\pi/3)\mathbf{T}_3(\pi/3)$ to this matrix, we obtain the rotated hyper-rectangle shown in the figure (red).



Euler angles

Extrinsic and intrinsic rotations

- Let $\mathbf{T}_\diamond$ denote a rotation matrix, where $\diamond$ stands for any combination of 1, 2 and 3 with non-sequentially repeated numbers.
- Given a rotation matrix $\mathbf{T}_\diamond$, two RFs can be defined:
  - Fixed frame, with axes (X, Y, Z).
  - Rotating frame, with axes (x, y, z).
- Consider for example the matrix $\mathbf{T}_{123} = \mathbf{T}_1(\phi)\mathbf{T}_2(\theta)\mathbf{T}_3(\psi)$. The rotating frame is constructed as follows:
  - Beginning: $(x, y, z) = (X, Y, Z)$
  - $\mathbf{T}_1$: rotation about the x-axis $\rightarrow (x, y', z')$
  - $\mathbf{T}_2$: rotation about the y'-axis $\rightarrow (x', y', z'')$
  - $\mathbf{T}_3$: rotation about the z''-axis $\rightarrow (x'', y'', z'')$.

Euler angles

Extrinsic and intrinsic rotations

- Any rotation has two interpretations:
  - **Extrinsic rotation**: about the axes of the fixed frame.
  - **Intrinsic rotation**: about the axes of the rotating frame. axis of the body frame
- An extrinsic rotation corresponds to an intrinsic rotation of the same angles but with inverted order of the elementary rotations.
- For example, the matrix $\mathbf{T}_{123} = \mathbf{T}_1(\phi)\mathbf{T}_2(\theta)\mathbf{T}_3(\psi)$ corresponds to

extrinsic rotation: of original ref frame

1) a rotation by ψ about Z;
2) a rotation by θ about Y;
3) a rotation by ϕ about X.

intrinsic rotation: body frame

1) a rotation by ϕ about x;
2) a rotation by θ about y';
3) a rotation by ψ about z''.

the order of the rotations changes

- An extrinsic rotation is denoted with $Z - Y - X$, an intrinsic rotation with $x - y' - z''$, where y' and z'' are the rotated axes.

Rotation matrices

Properties

- $\mathbf{T}_\diamond$ are *linear transformations* ($\diamond$ stands for any combination of 1, 2 and 3 with non-sequentially repeated numbers).
- $\mathbf{T}_\diamond$ are *orthogonal matrices*, that is, square matrices with real entries whose columns are orthonormal vectors.
- Main property:

$$\mathbf{T}_\diamond^{-1} = \mathbf{T}_\diamond^T, \qquad \mathbf{T}_\diamond^T \mathbf{T}_\diamond = \mathbf{T}_\diamond \mathbf{T}_\diamond^T = \mathbf{I}.$$

- Orthogonal transformations preserve
  - the lengths of vectors;
  - the angles between vectors.

Rotation matrices

Properties

- Orthogonal matrices have all the eigenvalues (either complex or real) with absolute value equal to 1. 3D rotation matrices have one eigenvalue equal to 1.
- Orthogonal matrices have the determinant equal to 1 or -1. Rotation matrices have the determinant equal to 1.

  set of orthogonal matrices
- For null angles, $\mathbf{T}_\diamond$ become identity matrices.
  - It follows for example that $\mathbf{T}_{123} = \mathbf{T}_3(\psi)$ for $\theta = \phi = 0$.
- The matrix product is **non-commutative** $\Rightarrow$ the rotation composition depends on the order.
- $\mathbf{T}_\diamond$ has 9 elements (3×3 matrix), which depend on 3 angles only;
  - the minimum number of parameters required to describe a rotation is 3.

Rotation matrices

Singularities

- Consider for example the Tait–Bryan 123 rotation with $\theta = \frac{\pi}{2}$:

$$\mathbf{T}_{123}\left(\phi, \tfrac{\pi}{2}, \psi\right) = \begin{bmatrix} 0 & 0 & 1 \\ c\phi\, s\psi + c\psi\, s\phi & c\phi\, c\psi - s\psi\, s\phi & 0 \\ s\phi\, s\psi - c\psi\, c\phi & c\phi\, s\psi + c\psi\, s\phi & 0 \end{bmatrix}$$

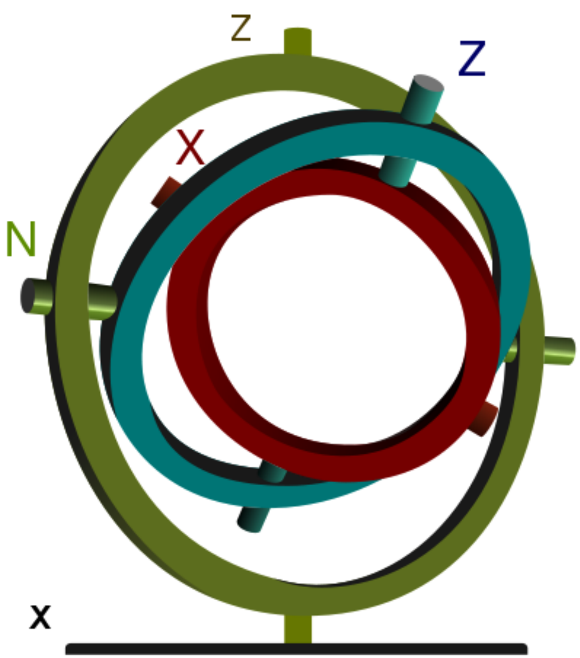
$$= \begin{bmatrix} 0 & 0 & 1 \\ \sin(\phi + \psi) & \cos(\phi + \psi) & 0 \\ -\cos(\phi + \psi) & \sin(\phi + \psi) & 0 \end{bmatrix}.$$

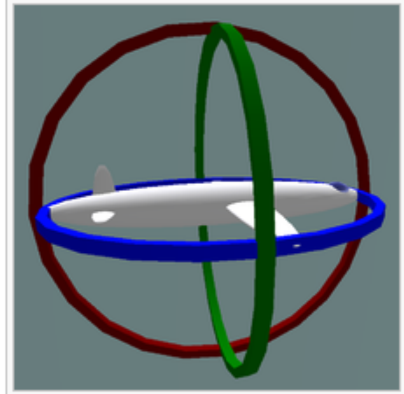
- Only the sum of ϕ and ψ can be determined from the DCM.
  - One angle is undermined.
- This phenomenon is known as the *gimbal lock*, and corresponds to a loss of a degree of freedom.

Rotation matrices

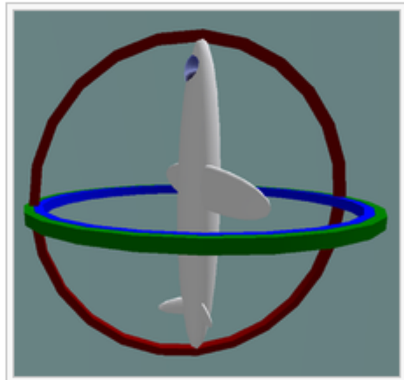
Singularities

Three-gimbal system: used on boats and inertial platforms.





Normal situation: the three gimbals are independent



Gimbal lock: two out of the three gimbals are in the same plane, one degree of freedom is lost

→ lose info about one angle

Rotation matrices

Singularities

- Critical situations:
  - Tait–Bryan rotations: $\cos\theta = 0$;
  - proper Euler rotations: $\sin\theta = 0$.
- In these situations, a **degeneracy at the poles** occurs:
  - it is not possible to determine ϕ and ψ from the DCM;
  - only the sum or the difference of ϕ and ψ can be determined.
- The singularity corresponds to a loss of a degree of freedom.
- We will see that
  - singularities appear in the kinematic equations;
  - gimbal lock can be overcome using non-minimal representations.

Euler's rotation theorem

Theorem

Any movement of a rigid body where one point is fixed is equivalent to a rotation about an axis passing through the fixed point. The axis of rotation is the eigenvector $\mathbf{u}$ *(eigenaxis) corresponding to the eigenvalue* 1 *of the rotation matrix.*

Proof. From linear algebra, a rotation matrix $\mathbf{T}$ has one eigenvalue equal to 1. The corresponding eigenvector $\mathbf{u}$ is unchanged by the rotation:

$$\mathbf{T}\mathbf{u} = \mathbf{u}.$$

u is not changed by the transformation
u→axis of the rotation

This means that $\mathbf{u}$ has the same components in the original and rotated reference frames. It follows that $\mathbf{u}$ is the axis of rotation. □

- From this theorem, it follows that any rotation can be described by 2 quantities, involving 4 variables:
  - a rotation angle (1 variable)
  - a rotation axis (3 variables) ← *angle-axis representation.*

: three dimensions

Euler's rotation theorem

Example

- Consider the rotation matrix

$$\mathbf{T}_{123}\left(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}\right) = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{3}}{2} \\ \frac{3\sqrt{2}\sqrt{3}}{8} & \frac{\sqrt{2}\sqrt{3}}{8} & -\frac{1}{4} \\ -\frac{\sqrt{2}}{8} & \frac{5\sqrt{2}}{8} & \frac{\sqrt{3}}{4} \end{bmatrix}.$$

- The eigenvalues are $\{0.0464 + 0.9989i,\ 0.0464 - 0.9989i, 1\}$. The corresponding eigenvectors are

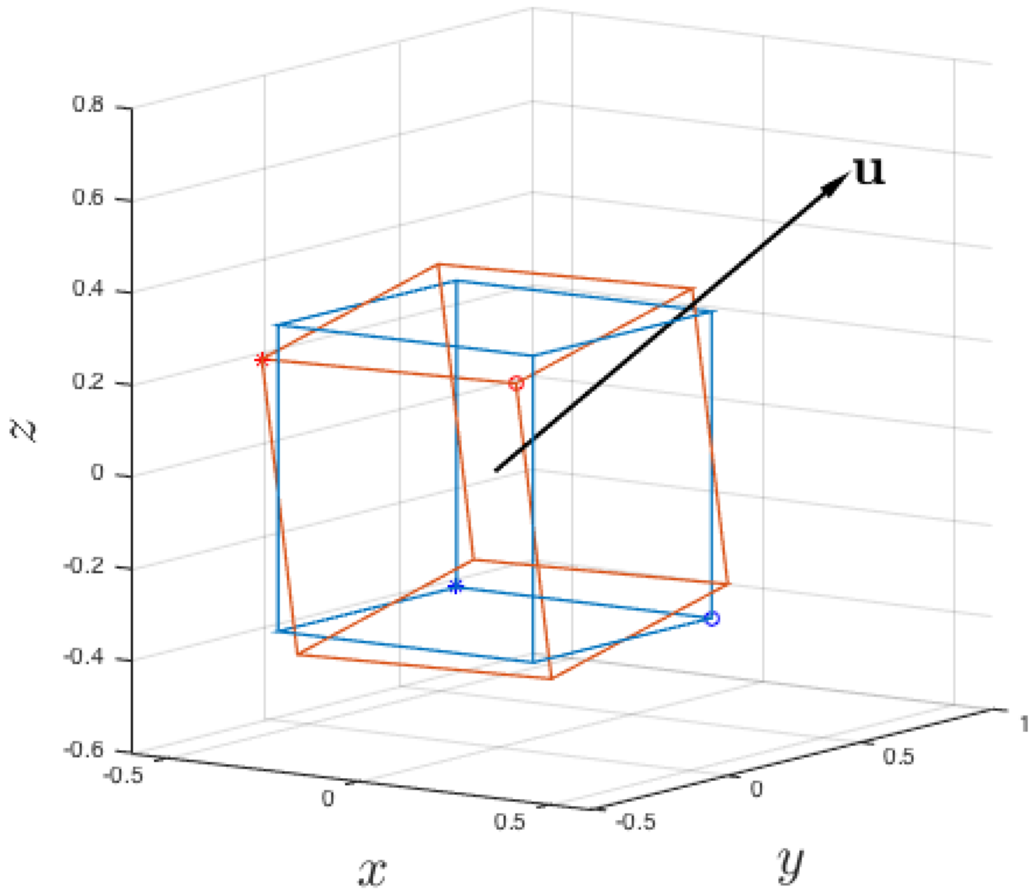
$$\begin{bmatrix} -0.25 + 0.53\,\mathrm{i} & -0.25 - 0.53\,\mathrm{i} & 0.57 \\ 0.60 & 0.60 & 0.52 \\ -0.27 - 0.47\,\mathrm{i} & -0.27 + 0.47\,\mathrm{i} & 0.64 \end{bmatrix}.$$

- The last column $\mathbf{u} = [0.57\ 0.52\ 0.64]^T$ is a vector representing the axis of rotation (eigenaxis).

Euler's rotation theorem

Example

- The result of $\mathbf{T}_{123}\left(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}\right)$ applied to a cube is shown in the figure. The axis of rotation is $\mathbf{u} = [0.57\ 0.52\ 0.64]^T$.



Euler parameters

- Based on the Euler's theorem, the following 4 variables (called the *Euler parameters*) can be used to describe a rotation of an angle β about an axis defined by a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$:

$$
\begin{aligned}
q_0 &\doteq \cos\frac{\beta}{2} \\
q_1 &\doteq u_1 \sin\frac{\beta}{2} \\
q_2 &\doteq u_2 \sin\frac{\beta}{2} \\
q_3 &\doteq u_3 \sin\frac{\beta}{2}.
\end{aligned}
$$

components of the rotation axis

- Note that only 3 of these 4 parameters are independent:

$$\sqrt{\sum_{i=0}^{3} q_i^2} = 1.$$

- As we'll see next, the vector $\mathbf{q} = (q_0, q_1, q_2, q_3)$ is a *quaternion*.

Quaternions

- Quaternions are mathematical objects introduced by the Irish mathematician Hamilton that are
  - a generalization of complex numbers to a 3D-space;
  - efficient rotation operators.
- The Euler parameters form a *quaternion*.
- Advantages wrt Euler angles/rotation matrices:
  - more efficient from a computational point of view;
  - less sensitive to rounding errors;
  - gimbal lock avoided.

set of elements of the space such that any element is a linear combination of basis

Definition

Quaternions are elements of a 4D linear vector space with basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, where the basis vectors satisfy the following multiplication rules:

$$
\begin{aligned}
&\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i} \otimes \mathbf{j} \otimes \mathbf{k} = -1 \\
&\mathbf{i} \otimes \mathbf{j} = -\mathbf{j} \otimes \mathbf{i} = \mathbf{k} \\
&\mathbf{j} \otimes \mathbf{k} = -\mathbf{k} \otimes \mathbf{j} = \mathbf{i} \\
&\mathbf{k} \otimes \mathbf{i} = -\mathbf{i} \otimes \mathbf{k} = \mathbf{j}.
\end{aligned}
$$

Quaternions

- The following notations are equivalent to indicate a quaternion $\mathfrak{q}$:

first part → real part

$$
\begin{aligned}
\mathfrak{q} &= q_0 + \mathbf{q} \quad \text{→ vector part → imaginary part} \\
&= q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \\
&= \cos\frac{\beta}{2} + \mathbf{u}\sin\frac{\beta}{2} \quad \text{from Euler parameters} \\
&= e^{\mathbf{u}\frac{\beta}{2}} \\
&= \left(\cos\frac{\beta}{2}, u_1\sin\frac{\beta}{2}, u_2\sin\frac{\beta}{2}, u_3\sin\frac{\beta}{2}\right) \\
&= (q_0, q_1, q_2, q_3) \\
&= (q_0, \mathbf{q}) = \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \cos\frac{\beta}{2} \\ \mathbf{u}\sin\frac{\beta}{2} \end{bmatrix}
\end{aligned}
$$

  - q_0 is the *real part*,
  - $\mathbf{q}$ is the *imaginary* (or *vector*) *part*.
  - A quaternion with null real part is said *pure*.

Quaternions

Algebra

- There exists the *null element*, that is $\mathfrak{O} = (0, \mathbf{0})$.
- The *complex conjugate* of a quaternion $\mathfrak{q} = q_0 + \mathbf{q}$ is

$$\mathfrak{q}^* \doteq q_0 - \mathbf{q} = (q_0, -\mathbf{q}) = \cos\frac{\beta}{2} - \mathbf{u}\sin\frac{\beta}{2} = e^{-\mathbf{u}\frac{\beta}{2}}.$$

- The *norm* of a quaternion is

$$|\mathfrak{q}| = \|\mathfrak{q}\| = \|\mathfrak{q}\|_2 = |\mathfrak{q}^*| = \sqrt{\mathfrak{q} \cdot \mathfrak{q}^*} = \sqrt{\sum_{i=0}^{3} q_i^2}.$$

- The *reciprocal* of a quaternion $\mathfrak{q}$ is

$$\mathfrak{q}^{-1} = \mathfrak{q}^* / |\mathfrak{q}|$$

$$\boxed{\mathfrak{q}^{-1} = \mathfrak{q}^*} \qquad \text{for a unit quaternion.}$$

Quaternions

Algebra

- Sum: $\mathfrak{q} + \mathfrak{p} = q_0 + p_0 + \mathbf{q} + \mathbf{p}$.
- Dot product: $\mathfrak{q} \cdot \mathfrak{p} = \sum_{i=0}^{3} q_i p_i$.
- Quaternion product (Hamilton product):

$$\begin{aligned} \mathfrak{q} \otimes \mathfrak{p} &= (q_0 + \mathbf{q}) \otimes (p_0 + \mathbf{p}) = \dots \\ &= (q_0 p_0 - \mathbf{q} \cdot \mathbf{p}) + (q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{q} \times \mathbf{p}) \end{aligned}$$

of vector parts

$$\mathbf{q} \cdot \mathbf{p} = \sum_{i=1}^{3} q_i p_i \qquad \text{dot product}$$

$$\mathbf{q} \times \mathbf{p} = \begin{bmatrix} q_2 p_3 - q_3 p_2 \\ q_3 p_1 - q_1 p_3 \\ q_1 p_2 - q_2 p_1 \end{bmatrix} \qquad \text{cross product.}$$

- The *identity element* is $\mathfrak{I} \doteq (1, \mathbf{0})$: $\mathfrak{q} \otimes \mathfrak{I} = \mathfrak{q}$, $\mathfrak{I} \otimes \mathfrak{q} = \mathfrak{q}$.
- The quaternion product is associative, non-commutative.
- The cross product is non-associative, anti-commutative. → add a minus

Quaternions

Algebra

- The cross product can also be written as

elements of q — elements of p — matrix by definition

$$\mathbf{q}\times\mathbf{p} = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad \mathbf{q}\times \doteq \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}.$$

- The quaternion product can also be computed as

$$\begin{aligned} \mathfrak{q}\otimes\mathfrak{p} &= (q_0 p_0 - \mathbf{q}\cdot\mathbf{p}) + (q_0\mathbf{p} + p_0\mathbf{q} + \mathbf{q}\times\mathbf{p}) \\ &= \begin{bmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & q_0\mathbf{I} + \mathbf{q}\times \end{bmatrix} \begin{bmatrix} p_0 \\ \mathbf{p} \end{bmatrix} \\ &= \begin{bmatrix} p_0 & -\mathbf{p}^T \\ \mathbf{p} & p_0\mathbf{I} - \mathbf{p}\times \end{bmatrix} \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix}. \end{aligned}$$

Quaternions

Rotations

- Let a 3D vector $\mathbf{r} = (x, y, z)$ be given.
- Consider a rotation of $\mathbf{r}$ about an axis $\mathbf{u} = (u_1, u_2, u_3)$ of an angle β:

rotation matrix

$$\mathbf{r}' = \mathbf{T}(\beta, \mathbf{u})\mathbf{r}.$$

original vector

rotation axis

- Both $\mathbf{r}$ and $\mathbf{r}'$ can be seen as the vector parts of quaternions with null real part, given by $(0, \mathbf{r})$ and $(0, \mathbf{r}')$.

Theorem

Define the unit quaternion

$$\mathfrak{q} \doteq \left(\cos\frac{\beta}{2}, u_1 \sin\frac{\beta}{2}, u_2 \sin\frac{\beta}{2}, u_3 \sin\frac{\beta}{2}\right).$$

The rotated vector $\mathbf{r}'$ can be computed as follows:

$$(0, \mathbf{r}') = \mathfrak{q} \otimes (0, \mathbf{r}) \otimes \mathfrak{q}^*.$$

- **Rotation composition:** Given a rotation composition

$$\mathbf{T} = \mathbf{T}_1 \mathbf{T}_2 \dots \mathbf{T}_n,$$ matrix product

the quaternion corresponding to the rotation $\mathbf{T}$ is

$$\mathfrak{q} = \mathfrak{q}_1 \otimes \mathfrak{q}_2 \otimes \cdots \otimes \mathfrak{q}_n$$

where $\mathfrak{q}_i$ is the quaternion corresponding to the rotation $\mathbf{T}_i$.

- **Inverse rotation:** Given a rotation defined by a unit quaternion $\mathfrak{q}$, the inverse rotation is defined by the quaternion

$$\mathfrak{q}^{-1} = \mathfrak{q}^*.$$

Quaternions

Rotations

- Elementary rotations:

$$\mathbf{T}_1(\phi) \quad \leftrightarrow \quad \mathfrak{q}_1(\phi) = \left(\cos\tfrac{\phi}{2}, \sin\tfrac{\phi}{2}, 0, 0\right)$$

$$\mathbf{T}_2(\theta) \quad \leftrightarrow \quad \mathfrak{q}_2(\theta) = \left(\cos\tfrac{\theta}{2}, 0, \sin\tfrac{\theta}{2}, 0\right)$$

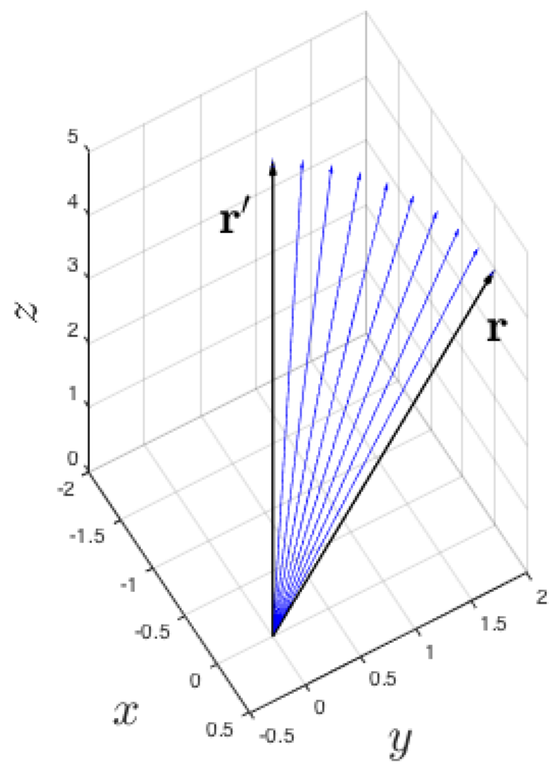
$$\mathbf{T}_3(\psi) \quad \leftrightarrow \quad \mathfrak{q}_3(\psi) = \left(\cos\tfrac{\psi}{2}, 0, 0, \sin\tfrac{\psi}{2}\right).$$

These are called the *elementary quaternions*.

Quaternions

Example: 3D elementary rotation of a vector

- Consider the vector $\mathbf{r} = 0\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.
- The quaternion $\mathfrak{q}_3(\pi/3)$ rotates any vector about the z axis of an angle $\pi/3$ rad $= 60^o$.
- The rotated vector is computed as $(0, \mathbf{r}') = \mathfrak{q}_3 \otimes (0, \mathbf{r}) \otimes \mathfrak{q}_3^*$, giving $\mathbf{r}' = -1.73\mathbf{i} + 1\mathbf{j} + 4\mathbf{k}$.

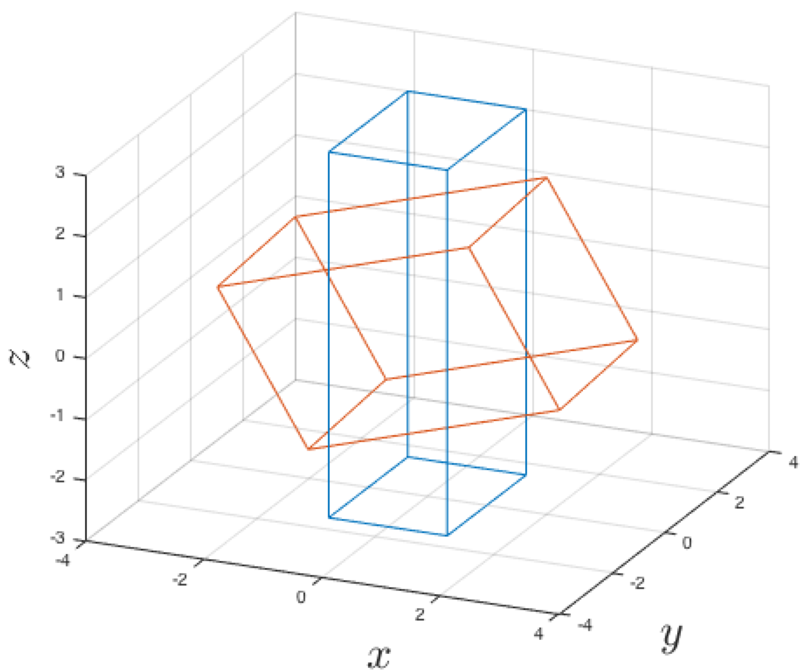


Same result as that obtained with the rotation matrix $\mathbf{T}_3(\pi/3)$.

Quaternions

Example: 3D generic rotation of an hyper-rectangle

- An hyper-rectangle can be represented by a matrix whose columns are the vertices (blue).
- Applying $\mathfrak{q}_1(\pi/4) \otimes \mathfrak{q}_2(\pi/3) \otimes \mathfrak{q}_3(\pi/3)$ to each column of this matrix, we obtain the rotated hyper-rectangle shown in the figure (red).



Same result as that obtained with the rotation matrix $\mathbf{T}_1(\pi/4)\mathbf{T}_2(\pi/3)\mathbf{T}_3(\pi/3)$.

Changes of representation

- Four different rotation representations have been introduced:
  1. Direction cosine matrix (DCM)

$$\mathbf{T} = [T_{ij}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}.$$

  2. Euler angles (ϕ, θ, ψ):
     - Tayt-Brian angles 321;
     - proper Euler angles 313.
  3. Angle-axis $(\beta, \mathbf{u})$.
  4. Quaternions (Euler parameters) $\mathfrak{q} = (q_0, \mathbf{q})$.

- In the following, we will see how to change representation.

Changes of representation

DCM $\leftrightarrow$ Euler angles

- Euler angles $\to$ DCM: trivial.
- DCM $\to$ Tayt-Brian angles 321 $(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$:

$$\begin{aligned} \phi &= \operatorname{atan2}\left(T_{32}, T_{33}\right) \\ \theta &= \operatorname{atan2}\left(-T_{31}, s\phi\, T_{32} + c\phi\, T_{33}\right) \\ \psi &= \operatorname{atan2}\left(-c\phi\, T_{12} + s\phi\, T_{13}, c\phi\, T_{22} - s\phi\, T_{23}\right). \end{aligned}$$

- DCM $\to$ Proper Euler angles 313 $(\theta \neq 0)$:

$$\begin{aligned} \phi &= \operatorname{atan2}\left(T_{13}, -T_{23}\right) \\ \theta &= \operatorname{atan2}\left(s\phi\, T_{13} - c\phi\, T_{23}, T_{33}\right) \\ \psi &= \operatorname{atan2}\left(-c\phi\, T_{12} - s\phi\, T_{22}, c\phi\, T_{11} + s\phi\, T_{21}\right). \end{aligned}$$

- $\operatorname{atan2}(x, y)$ = four quadrant inverse tangent of y/x (see the next slide).

Changes of representation

Note

- Function $\operatorname{atan2}(x, y)$:

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

- In Matlab, $\operatorname{atan2}(0, 0) = 0$.

Changes of representation

DCM$\leftrightarrow$ angle-axis

- Angle-axis $\rightarrow$ DCM:

$$\mathbf{T} = \left[\begin{array}{ccc} u_1^2(1-c\beta)+c\beta & u_1u_2(1-c\beta)-u_3s\beta & u_1u_3(1-c\beta)+u_2s\beta \\ u_1u_2(1-c\beta)+u_3s\beta & u_2^2(1-c\beta)+c\beta & u_2u_3(1-c\beta)-u_1s\beta \\ u_1u_3(1-c\beta)-u_2s\beta & u_2u_3(1-c\beta)+u_1s\beta & u_3^2(1-c\beta)+c\beta \end{array}\right].$$

- DCM $\rightarrow$ angle-axis ($s\beta \neq 0$):

$$\beta = \arccos\left(\tfrac{T_{11}+T_{22}+T_{33}-1}{2}\right)$$

$$\mathbf{u} = \tfrac{1}{2s\beta}\left[\begin{array}{c} T_{32}-T_{23} \\ T_{13}-T_{31} \\ T_{21}-T_{12} \end{array}\right].$$

Changes of representation

DCM $\leftrightarrow$ quaternions

- Quaternions $\rightarrow$ DCM:

$$\mathbf{T} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}.$$

- DCM $\rightarrow$ quaternions ($q_0 \neq 0$):

$$q_0 = \tfrac{1}{2}\sqrt{T_{11} + T_{22} + T_{33} + 1}$$

$$\mathbf{q} = \tfrac{1}{4q_0} \begin{bmatrix} T_{32} - T_{23} \\ T_{13} - T_{31} \\ T_{21} - T_{12} \end{bmatrix}.$$

  If $q_0 = 0$, more complicated formulas can be used.

- No singularities occur: $\mathbf{q}$ can always be computed from $\mathbf{T}$ (gimbal lock avoided).

Changes of representation

Quaternions $\leftrightarrow$ Euler angles

- Euler angles $\to$ quaternions: elementary quaternions.
- Quaternions $\to$ Euler angles: pass through the DCM.

Quaternions $\leftrightarrow$ angle-axis

- Angle-axis $\to$ quaternions: trivial.
- Quaternion $\to$ angle-axis $(\beta \neq 2k\pi,\ k = 0, 1, \ldots)$:

$$\beta = 2\ \arccos q_0$$
$$u_i = q_i / \sin \frac{\beta}{2}.$$

Changes of representation

Examples

- Euler angles $313 \rightarrow$ DCM. Consider the 313 Euler angles $\left(\frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{3}\right)$. The corresponding DCM is

$$\mathbf{T} = \mathbf{T}_3\left(\frac{\pi}{8}\right)\mathbf{T}_1\left(\frac{\pi}{4}\right)\mathbf{T}_3\left(\frac{\pi}{3}\right) = \begin{bmatrix} 0.227 & -0.935 & 0.270 \\ 0.757 & -0.005 & -0.653 \\ 0.612 & 0.353 & 0.707 \end{bmatrix}.$$

  Note that numerical approximations may be critical.

- DCM $\rightarrow$ Euler angles 313:

$$\begin{aligned} \phi &= \operatorname{atan2}\left(T_{13}, -T_{23}\right) = \tfrac{\pi}{8} \\ \theta &= \operatorname{atan2}\left(s\phi\, T_{13} - c\phi\, T_{23}, T_{33}\right) = \tfrac{\pi}{4} \\ \psi &= \operatorname{atan2}\left(-c\phi\, T_{12} - s\phi\, T_{22}, c\phi\, T_{11} + s\phi\, T_{21}\right) = \tfrac{\pi}{3}. \end{aligned}$$

- Euler angles $\rightarrow$ quaternions. Consider the 313 Euler angles $\left(\frac{\pi}{8}, \frac{\pi}{4}, \frac{\pi}{3}\right)$. The corresponding elementary quaternions are

$$
\begin{aligned}
\mathfrak{q}_3\left(\tfrac{\pi}{8}\right) &= \left(\cos\tfrac{\pi}{16}, 0, 0, \sin\tfrac{\pi}{16}\right) = (0.981, 0, 0, 0.195) \\
\mathfrak{q}_1\left(\tfrac{\pi}{4}\right) &= \left(\cos\tfrac{\pi}{8}, \sin\tfrac{\pi}{8}, 0, 0\right) = (0.924, 0.383, 0, 0) \\
\mathfrak{q}_3\left(\tfrac{\pi}{3}\right) &= \left(\cos\tfrac{\pi}{6}, 0, 0, \sin\tfrac{\pi}{6}\right) = (0.866, 0, 0, 0.5)\,.
\end{aligned}
$$

The total rotation quaternion is

$$
\mathfrak{q} = \mathfrak{q}_3\left(\frac{\pi}{8}\right) \otimes \mathfrak{q}_1\left(\frac{\pi}{4}\right) \otimes \mathfrak{q}_3\left(\frac{\pi}{3}\right) = \left(\ 0.695, \quad 0.362, \quad -0.123, \quad 0.609\ \right).
$$

Changes of representation

Examples

- DCM$\rightarrow$ quaternions. Consider the DCM

$$\mathbf{T} = \begin{bmatrix} 0.227 & -0.935 & 0.270 \\ 0.757 & -0.005 & -0.653 \\ 0.612 & 0.353 & 0.707 \end{bmatrix}.$$

The corresponding quaternion is $\mathfrak{q} = (q_0, \mathbf{q})$, where

$$q_0 = \tfrac{1}{2}\sqrt{T_{11} + T_{22} + T_{33} + 1} = 0.695$$

$$\mathbf{q} = \frac{1}{4q_0}\begin{bmatrix} T_{32} - T_{23} \\ T_{13} - T_{31} \\ T_{21} - T_{12} \end{bmatrix} = \begin{bmatrix} 0.362 \\ -0.123 \\ 0.609 \end{bmatrix}.$$

Changes of representation

Examples

- Quaternions $\rightarrow$ DCM. Consider the quaternion

$$\mathfrak{q} = \left(\ 0.695, \quad 0.362, \quad -0.123, \quad 0.609\ \right).$$

The corresponding DCM is

$$\mathbf{T} = \begin{bmatrix} 0.227 & -0.935 & 0.270 \\ 0.757 & -0.005 & -0.653 \\ 0.612 & 0.353 & 0.707 \end{bmatrix}.$$

- Quaternions $\rightarrow$ DCM $\rightarrow$ Euler angles 313:

$$\begin{aligned}
\phi &= \operatorname{atan2}\left(T_{13}, -T_{23}\right) = \tfrac{\pi}{8} \\
\theta &= \operatorname{atan2}\left(s\phi\, T_{13} - c\phi\, T_{23}, T_{33}\right) = \tfrac{\pi}{4} \\
\psi &= \operatorname{atan2}\left(-c\phi\, T_{12} - s\phi\, T_{22}, c\phi\, T_{11} + s\phi\, T_{21}\right) = \tfrac{\pi}{3}.
\end{aligned}$$