Nonlinear control and aerospace applications

Random variables and stochastic processes

Carlo Novara

Politecnico di Torino
Dip. Elettronica e Telecomunicazioni

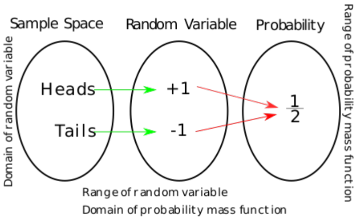
Outline

Random variables

Definitions

- An *outcome* is a possible result of an experiment.
- The *sample space* Ω is the set of all possible outcomes.
- *A random variable (or stochastic variable)* is a number $v \in V \subseteq \mathbb{R}$ associated with the outcomes of an experiment. More formally, a *random variable* is a function $v : \Omega \to V \subseteq \mathbb{R}$.
- The set V of all possible values of v is called the *range* of v.
- In general, v may take values in V with a certain probability.

Example: Coin toss.



Random variables

Definitions

- *A random vector (or stochastic vector)* $v \in V \subseteq \mathbb{R}^n$ is a collection of random variables $v_i \in \mathbb{R}$: $v = (v_1, \ldots, v_n)$.
- A value obtained from a specific outcome is called a *realization* of the random variable (vector).
- In the following:
  - The expression "random variable" will be used also to indicate a random vector.
  - A random variable (vector) will be abbreviated with RV.
  - RVs of arbitrary dimension n will be considered.

Random variables

Examples

- Coin toss:
  - $\Omega = \{\text{head}, \text{tail}\}$ (2 possible outcomes).
  - $V = \{1, -1\}$.
- Dice roll:
  - $\Omega = \{\text{face1}, ..., \text{face6}\}$ (6 possible outcomes).
  - $V = \{1, 2, \ldots, 6\}$.
- Measurement of a physical observable: $v =$ "true" value + error.
  - $\Omega =$ set of all possible "states" of the physical observable and errors (∞ possible outcomes).
  - $V =$ "true" value + error set.

Random variables

Probability functions - discrete RV

- The range V is the set of all possible values of v.
- If V is <u>countable</u> $\Rightarrow$ *discrete* RV.
- A discrete RV v is described by a *probability mass function (PMF)* $p_v(x)$.
- The PMF gives the probability of v to take a value x:

$$\mathrm{prob}(v = x) = p_v(x).$$

- Suppose $V = \{x_1, x_2, \ldots\}$. Clearly,

$$\mathrm{prob}(v \in V) = \sum_{\forall i} p_v(x_i) = 1.$$

Random variables

Probability functions - continuous RV

- The range V is the set of all possible values of v.
- If V is uncountable (e.g., a real interval) $\Rightarrow$ *continuous* RV.
- A continuous RV v is described by a *probability density function (PDF)* $p_v(x)$[1].
- The PDF multiplied by dx gives the probability that v falls between x and $x + dx$:
$$\mathrm{prob}(x \leq v < x + dx) = p_v(x)dx.$$
- The PDF integral over a set $A \subseteq V$ gives the probability of v to take a value in A:
$$\mathrm{prob}(v \in A) = \int_A p_v(x)dx.$$
- Clearly, $\mathrm{prob}(v \in V) = \int_V p_v(x)dx = 1$.

[1] Same symbol p_v for both continuous and discrete cases, different meaning.

Random variables

Definitions - continuous RV

- *Expected value* (expectation, mean, average) of a function $f(v)$ of a random variable v:

$$\mathcal{E}[f(v)] \doteq \int_V f(x) p_v(x) dx.$$

- *(Raw) moments* of the distribution[2]: $\mathcal{E}[v^k],\ k = 1, 2, \ldots$
- *Central moments* of the distribution: $\mathcal{E}[(v - \mathcal{E}[v])^k],\ k = 1, 2, \ldots$
- *Expected value* of $v = 1^{st}$ moment.
- *Variance* of v: $\sigma^2 \doteq \mathrm{var}(v) \doteq \mathcal{E}[(v - \mathcal{E}[v])^2] = 2^{nd}$ central moment.
- *Standard deviation*: $\sigma \doteq \mathrm{std}(v) = \sqrt{\mathrm{var}(v)}$.
- *Covariance matrix*: $P \doteq \mathrm{cov}(v) \doteq \mathcal{E}[(v - \mathcal{E}[v])(v - \mathcal{E}[v])^\top] \in \mathbb{R}^{n \times n}$.

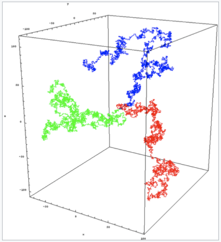
---

[2] For $n > 1$, $v^k = (v_1^k, \ldots, v_n^k)$, $\sqrt{v} = (\sqrt{v_1}, \ldots, \sqrt{v_n})$.

Stochastic processes

Definitions

- A *stochastic process* (or *random process*) is a collection of random variables $\{v(t) \in V, t \in T\}$, where $V \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}$.
- Usually, t represents the time and T the time range.
  - $T = [0, \infty) \implies$ continuous-time process.
  - $T = \{0, 1, 2, \ldots\} \implies$ discrete-time process (k often used instead of t).
- Examples: random walk, Wiener process, Poisson process.



Three random walks in three dimensions

Stochastic processes

Definitions

- A process v is *identically distributed* if $v(t)$ has the same PDF $\forall t$.
- A process v is *independent* if, for any collection of distinct times $t_1, t_2, \ldots, t_m$, the joint PDF of the RVs $v(t_1), v(t_2), \ldots, v(t_m)$ coincides with the product of their individual PDFs, i.e., if

$$p_{v(t_1),\ldots,v(t_m)}(x_1, \ldots, x_m) = \prod_{i=1}^{m} p_{v(t_i)}(x_i).$$

- Notation: *i.i.d.* = *iid* = *independent identically distributed*.
- A process v is *white* if it is independent and $\mathcal{E}[v(t)] = 0$, $\mathrm{var}(v(t)) < \infty$, $\forall t$.
- Two processes v and u are *independent* if, for any collection of distinct times $t_1, t_2, \ldots, t_m$, the random vectors $(v(t_1), v(t_2), \ldots, v(t_m))$ and $(u(t_1), u(t_2), \ldots, u(t_m))$ are independent.

Stochastic processes

Definitions

- Given two processes $v(t), u(t) \in \mathbb{R}^n$, their *covariance* is

$$\mathrm{cov}(v(t), u(\tau)) \doteq \mathcal{E}\left[v(t) - \mathcal{E}[v(t)])(u(\tau) - \mathcal{E}[u(\tau)])^\top\right].$$

  $u(t) \neq v(t)$: *cross-covariance*; $u(t) = v(t)$: *auto-covariance*.

- *Correlation*: $\mathrm{corr}(v(t), u(\tau)) \doteq \frac{\mathrm{cov}(v(t), u(\tau))}{\mathrm{std}(v(t))\,\mathrm{std}(u(\tau))}$.

  $u(t) \neq v(t)$: *cross-correlation*; $u(t) = v(t)$: *auto-correlation*.

- Two processes $v(t), u(t) \in \mathbb{R}^n$ are *uncorrelated* if

$$\mathrm{cov}(v(t), u(\tau)) = 0,\ \forall t, \tau.$$

- If two processes are independent, then they are uncorrelated. The inverse is in general not true.

Stochastic processes

Examples of stochastic processes

- Sequence of coin tosses: $\{v(k) \in V, k \in T\}$, $V = \{-1, 1\}$, $T = \{0, 1, 2, \ldots\}$. The stochastic process is $(v(0), v(1), v(2), \ldots, v(k), \ldots) = (1, -1, -1, \ldots, 1, \ldots)$.
- Discrete-time random walk: $v(k+1) = v(k) + d(k)$, where $V \in \mathbb{R}^n$, $k \in T = \{0, 1, 2, \ldots\}$ and $d(k) \in \mathbb{R}^n$ is a white noise.
- Continuous-time Wiener process (Brownian motion): $v(t + dt) = v(t) + d(t)$, where $V \in \mathbb{R}^n$, $t \in T = [0, \infty)$ and $d(t) \in \mathbb{R}^n$ is a Gaussian white noise with variance dt.
- The Wiener process can be discretized, giving a discrete-time random walk equation.