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MATHEMATICAL FINANCE

"TERM STRUCTURE ESTIMATION: A PARAMETRIC APPROACH BASED ON NELSON SIEGEL MODELS"

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Introduction

Interest rates are central to the functioning of financial markets and monetary policy. For this reason, the accurate modeling of the term structure of interest rates, called the yield curve, is of fundamental importance in many financial applications: pricing of government securities, assessing credit risk, business valuation and the formulation of monetary policy decisions.

This thesis focuses on two of the most influential and widely applied parametric models for yield curve estimation: the Nelson-Siegel (NS) model and its extension, the Nelson-Siegel-Svensson (NSS) model.

The Nelson-Siegel model has been originally introduced by Charles Nelson and Andrew Siegel in 1987. This model is able to represent yields as a function of three factors: level, slope and curvature. Each of these factors is associated with an intuitive economic and mathematical interpretation. Lee O. Svensson further extended the NS model in 1994 (NSS), adding a fourth parameter. The aim has been an increase in flexibility of the original model in fitting more complex curve shapes, for example in cases of economic distress or changing conditions.

The motivation for choosing this topic lies in the continuous relevance of yield curve modeling techniques in financial banks and institutions, combined with the opportunity to investigate the italian fixed income market through a practical application of the models. By applying and comparing the NS and NSS models to Italian Buoni del Tesoro Poliennali (BTPs), this thesis aims to demonstrate the models' differences and performances in a real context.

The thesis is centered around the comparative analysis of the two models. It wants to answer to a set of research questions:

1. Do the NS and NSS models accurately fit the Italian yield curve on a certain reference day?
2. Are the results of the two models stable when applied over a one month period?
3. Is the increased complexity of the NSS model justified by superior results or does it risk overfitting?

The structure of the thesis is organized as follows. The first chapter presents the theoretical formulation of the Nelson-Siegel and Nelson-Siegel-Svensson models, discussing and their mathematical structures and strenghts. In the second chapter the datasets used for the models' calibrations are introduced, and the optimization methodology, performed through the implementation of somee MATLAB codes, is explained. In the third chapter, that is the first empirical one, a cross-sectional analysis is performed by estimating both NS and NSS models on a single day, and compares the resulting yield curves among each other and against the official yield curve published by the European Central Bank (ECB). The fourth chapter expands the analysis to a dynamic framework, examining the evolution of model parameters and fit quality over a one month period.

The objective of this thesis is, on one hand to analyze the theory behind the models in a practical way and to apply it within the Italian financial context, on the other hand to develop a replicable framework that can serve as a basis for future research or practical applications.

1. Theoretical Frameworks

1.1 The Nelson Siegel Model

1.1.1 Theoretical formulation

The Nelson-Siegel (NS) model provides a formulation for fitting yield curves that is both parsimonious and flexible. This parametric model is able to capture the three primary yield curve shapes observed: monotonic, humped, and S-shaped curves. In particular, the NS model is derived from the solutions to second order differential equations.
As stated by Nelson and Siegel in the *Journal of Business 60*[1], according to the theory of the term structure of interest rates, if spot rates are generated by a differential equation, then forward rates, being forecasts, will be the solution to the equations.

It is possible to define an instantaneous forward rate function *f(m)*, where m is the time to maturity, specified as:

$$f(m) = \beta_0 + \beta_1 e^{-\frac{m}{\tau}} + \beta_2 \left(\frac{m}{\tau} * e^{-\frac{m}{\tau}}\right) \tag{1}$$

This function is the solution to a second-order linear differential equation with equal real roots. Such differential equations are used to model behaviors that exhibit exponential decay over time, and the solution yields a smooth and continuous function composed of exponential terms.

This equation generates a set of forward rate curves with monotonic, humped, or S shapes depending on the values of the parameters β_1 and β_2 and that also have asymptote β_0.[2]

From the forward rate function, we can compute the corresponding yield to maturity function *R(m)* as the average of the instantaneous forward rates over the maturity period, which is obtained by taking the average of the integrated instantaneous forward rate function $f(m)$:

[1] Nelson, Charles R., and Andrew F. Siegel. 1987. "Parsimonious Modeling of Yield Curves." *The Journal of Business* 60 (4): pp 474.
[2] Nelson, Charles R., and Andrew F. Siegel. 1987. "Parsimonious Modeling of Yield Curves." *The Journal of Business* 60 (4): pp 475.

(2) $$R(m) = \frac{1}{m}\int_0^m (f(x)dx)$$

That provides as a result the following function:

(3) $$R(m) = \beta_0 + \beta_1\left(\frac{1-e^{-\frac{m}{\tau}}}{\frac{m}{\tau}}\right) + \beta_2\left(\frac{1-e^{-\frac{m}{\tau}}}{\frac{m}{\tau}} - e^{-\frac{m}{\tau}}\right)$$

1.1.2 Interpretation of parameters (β_0, β_1, β_2, λ)

Following Diebold and Li (2006), I decided to adopt the reparameterization $\lambda = \frac{1}{\tau}$ to simplify interpretation and estimation.[3] In particular this reparameterization will simplify the computations and the analysis of results in the next chapters.

Thus, we can rewrite the yield function as:

(4) $$R(m) = \beta_0 + \beta_1\left(\frac{1-e^{-\lambda m}}{\lambda m}\right) + \beta_2\left(\frac{1-e^{-\lambda m}}{\lambda m} - e^{-\lambda m}\right)$$

Where:

- β_0 represents the long-term effect.
  Mathematically, this term is not affected by *m*, and remains constant across all maturities, as showed in figure 1.
  Economically, β_0 captures the asymptotic level of the yield curve, that is the long-term interest rate.
- β_1 represents the short-term effect, and it measures the steepness of the curve.
  This term multiplies $\left(\frac{1-e^{-\lambda m}}{\lambda m}\right)$ and decays exponentially with maturity: approximating using a Taylor expansion of the exponential term around $m = 0$ we can show that:

[3] Diebold, F. X., & Li, C. (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics*, 130(2), 337–364.

(5) $$e^{-\lambda m} = 1 - \lambda m + \frac{(\lambda m)^2}{2} \ldots. \;\; => \;\; 1 - e^{-\lambda m} \approx \lambda m - \frac{(\lambda m)^2}{2}$$

Thus:

(6) $$\frac{1 - e^{-\lambda m}}{\lambda m} \approx 1 - \frac{\lambda m}{2}$$

Therefore at maturity $m = 0$, the effect of the β_1 term is maximal. But as $m \rightarrow \infty$, its influence vanishes.

This evidentiate that β_1 captures the short-term component of the yield curve: its influence is strong at short maturities but fades rapidly, this can be easily seen and interpreted in the figure 1 below, with the short term line starting from 1 and fading after the first 5 years.

- β_2 represents the medium-term effect, and it captures the hump in the curve.

  This term multiplies:

(7) $$\left(\frac{1 - e^{-\lambda m}}{\lambda m} - e^{-\lambda m}\right)$$

Using again the Taylor expansion, we find:

(8) $$\frac{1 - e^{-\lambda m}}{\lambda m} \approx 1 - \frac{\lambda m}{2} \; ; \; e^{-\lambda m} \approx 1 - \lambda m$$

So:

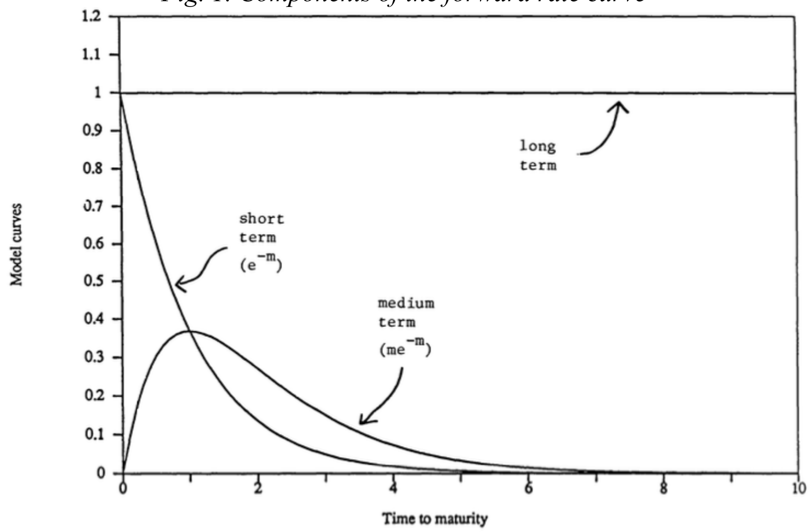
(9) $$\left(\frac{1 - e^{-\lambda m}}{\lambda m} - e^{-\lambda m}\right) \approx \; 1 - \frac{\lambda m}{2} - 1 + \lambda m$$

Therefore, the effect of this term will be 0 at maturity $m = 0$, it will then increase until it reaches a maximum and will decay to 0 as $m \rightarrow \infty$.

It has no effect at the short and long ends, it contributes only in the middle.

- As emphasized by Diebold and Li[4], the parameter λ explains how quickly the influence of β_1 and β_2 decay over time, where low values of λ produce slow decay and allow the curvature to extend into longer maturities.

Fig. 1. Components of the forward rate curve



Nelson, Charles R., and Andrew F. Siegel. 1987. "Parsimonious Modeling of Yield Curves." The Journal of Business 60 (4): pp 477.

1.1.3 Strengths and applications of the model

Nelson and Siegel (1987) analyze and demonstrate that their model provides an excellent in sample fit to U.S. Treasury data, with a median R^2 of 95.9% across 37 yield curves (as shown in the last row of the fourth column in table 1). They also show that the model is robust with respect to the parameter τ: even when τ is fixed across maturities, the fit (captured by the Standard Deviation) deteriorates by only 0.57 basis points on average, as shown by the difference between the median value of Standard Deviation with the best τ and of Standard Deviation with fixed τ at 50 in table 1[5].

[4] Diebold, F. X., & Li, C. (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics*, 130(2), 337–364.

[5] Nelson, Charles R., and Andrew F. Siegel. 1987. "Parsimonious Modeling of Yield Curves." *The Journal of Business* 60 (4): pp 480-481.

Table 1. Measures of Model Fit

Set No.	τ	Second-Order Model: SD at Best τ	R^2	SD, τ = 50	First Term Only: SD at Best τ
1	50	16.09	92.4	16.09	46.71
2	40	13.00	88.9	13.67	36.42
3	30	11.22	72.3	12.45	13.46
4	60	6.01	86.7	6.12	9.00
5	40	12.92	87.8	14.52	30.97
6	40	13.47	93.3	13.52	13.32
7	80	15.61	49.7	15.90	17.11
8	10*	10.43	81.7	22.42	23.00
9	20	19.85	88.8	20.34	19.56
10	50	18.33	95.2	18.33	18.10
11	30	4.88	98.8	6.11	6.95
12	300	12.28	93.8	12.43	12.16
13	50	7.76	99.4	7.76	7.67
14	30	11.08	98.0	11.32	11.22
15	60	10.51	95.7	10.75	15.20
16	10*	6.28	97.3	7.30	7.55
17	110	5.11	98.3	5.71	5.74
18	20	7.51	86.4	10.12	11.10
19	170	4.12	98.8	4.46	4.05
20	20	5.79	98.8	9.26	9.98
21	20	20.04	96.7	25.17	25.55
22	365*	15.08	98.3	15.84	15.41
23	40	10.01	99.1	11.65	14.78
24	30	2.91	99.6	5.13	6.17
25	20	7.25	97.4	7.45	7.34
26	100	5.18	93.9	5.33	5.09
27	300	3.71	97.3	4.03	3.65
28	50	5.38	95.5	5.38	5.28
29	110	6.72	85.6	6.90	6.59
30	70	1.95	98.0	2.10	2.21
31	365*	3.74	91.6	4.02	3.68
32	20	4.89	96.1	5.80	4.83
33	40	3.16	99.1	3.22	3.19
34	120	7.24	96.1	7.82	7.11
35	90	15.34	86.3	15.51	15.07
36	365*	5.53	95.9	6.17	5.43
37	180	3.01	99.0	4.25	2.97
Median	50	7.25	95.9	7.82	9.00

NOTE.—Standard deviations are in basis points.
* Best fit realized at boundary of range of search.

Nelson, Charles R., and Andrew F. Siegel. 1987. "Parsimonious Modeling of Yield Curves." The Journal of Business *60 (4): pp 481.*

In their 2006 paper, Diebold and Li[6] extend the static Nelson-Siegel model into a dynamic fraamework by computing the parameters over time according to simple autoregressive processes. After estimating the Nelson-Siegel parameters β_0, β_1 and β_2 from monthly U.S. Treasury zero-coupon yields, they model each series using an *AR(1)* process.

[6] Diebold, F. X., & Li, C. (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics*, 130(2), 337–364.

This means that each coefficient at time *t+1* is forecasted as a linear function of its value at time *t*. Then forecasts of one step ahead for each β are plugged into the Nelson-Siegel function in order to compute a forecast of the entire yield curve at different time horizons.

In particular, Table 2 reports the out of sample forecast accuracy of the model at 6 months ahead horizon, while table 3 at 12 months ahead, across different maturities (3 months, 1 year, 3 years, 5 years, and 10 years). For each maturity, the tables present statistics such standard deviation and the root mean squared error (RMSE). These allow a direct comparison between the Nelson-Siegel forecasts and those from a random walk benchmark.

Table 2.
Out-of-sample 6-month-ahead forecaste results

Maturity (τ)	Mean	Std. Dev.	RMSE
Nelson–Siegel with AR(1) factor dynamics			
3 months	0.083	0.510	0.517
1 year	0.131	0.656	0.669
3 years	−0.052	0.748	0.750
5 years	−0.173	0.758	0.777
10 years	−0.251	0.676	0.721
Random walk			
3 months	0.220	0.564	0.605
1 year	0.181	0.758	0.779
3 years	0.099	0.873	0.879
5 years	0.048	0.860	0.861
10 years	−0.020	0.758	0.758

Diebold, F. X., & Li, C. (2006). Forecasting the term structure of government bond yields. Journal of Econometrics, 130(2), pp. 355.

Table 3.
Out-of-sample 12-month-ahead forecaste results

Maturity (τ)	Mean	Std. Dev.	RMSE
Nelson–Siegel with AR(1) factor dynamics			
3 months	0.150	0.724	0.739
1 year	0.173	0.823	0.841
3 years	−0.123	0.910	0.918
5 years	−0.337	0.918	0.978
10 years	−0.531	0.825	0.981
Random walk			
3 months	0.416	0.930	1.019
1 year	0.388	1.132	1.197
3 years	0.236	1.214	1.237
5 years	0.130	1.184	1.191
10 years	−0.033	1.051	1.052

Diebold, F. X., & Li, C. (2006). Forecasting the term structure of government bond yields. Journal of Econometrics, 130(2), pp. 356-357.

Their empirical results show that this dynamic approach based on the NS model outperforms a random walk benchmark in out of sampling forecasting, particularly for medium term horizons such as 6 and 12 months. Table 2 and Table 3 illustrate this forecasting advantage, showing lower root mean squared forecast errors (RMSFE) across all maturities.

1.2 Extension to Nelson-Siegel-Svensson

1.2.1 Introduction of second curvature term (β_3, λ_2)

The Nelson-Siegel (NS) model provides a parsimonious and flexible representation of the yield curve. However, some studies, such as the work of Gürkaynak[7] for the Federal Reserve Board

[7] Gürkaynak, R. S., Sack, B., & Wright, J. H. (2006). *The U.S. Treasury Yield Curve: 1961 to the Present*. Finance and Economics Discussion Series 2006-28, Board of Governors of the Federal Reserve System (pag14).

to make public the Treasury yield curve estimates, have shown that in some cases it lacks sufficient flexibility to fit the entire term structure, especially for securities with long maturities.

In response to these limitations, Lars E.O. Svensson[8] proposed an extension to the NS model by introducing an additional curvature component. The Nelson-Siegel-Svensson (NSS) model, adds a fourth parameter β_3 and an associated second decay parameter λ_2, enabling the model to fit yield curves with more than one hump more accurately. Svensson was driven by the need to better capture the complexity of the Swedish government bond market between 1992 and 1994, as it was under stressed economic conditions caused by a banking crisis.

Mathematically, the instantaneous forward rate function in the NSS model is defined as:

$$(10) \qquad f(m) = \beta_0 + \beta_1 e^{-\frac{m}{\tau_1}} + \beta_2 \left(\frac{m}{\tau_1} * e^{-\frac{m}{\tau_1}} \right) + \beta_3 \left(\frac{m}{\tau_2} * e^{-\frac{m}{\tau_2}} \right)$$

The associated yield function, as in the NS model, is derived by integrating the forward rate over the maturity horizon:

$$(11) \qquad R(m) = \frac{1}{m} \int_0^m f(x) dx$$

This leads to the following expression for the yield to maturity, where I replaced τ with λ as described above and as proposed by Diebold and Li:

$$(12) \quad R(m) = \beta_0 + \beta_1 \left(\frac{1 - e^{-m\lambda_1}}{m\lambda_1} \right) + \beta_2 \left(\frac{1 - e^{-m\lambda_1}}{m\lambda_1} - e^{-m\lambda_1} \right) + \beta_3 \left(\frac{1 - e^{-m\lambda_2}}{m\lambda_2} - e^{-m\lambda_2} \right)$$

[8] Svensson, L. E. O. (1994). Estimating and interpreting forward interest rates: Sweden 1992–1994. *NBER Working Paper No. 4871*. National Bureau of Economic Research.

The two terms β_2 and β_3 both represent a hump (or a dip) in the curve, each with a maximum at a different maturity and its own rate of decay depending on the corresponding decay parameters λ_1 or λ_2. These parameters allow the model to fit the term structure even in situations of changing behavior of the curve, allowing for a second hump.

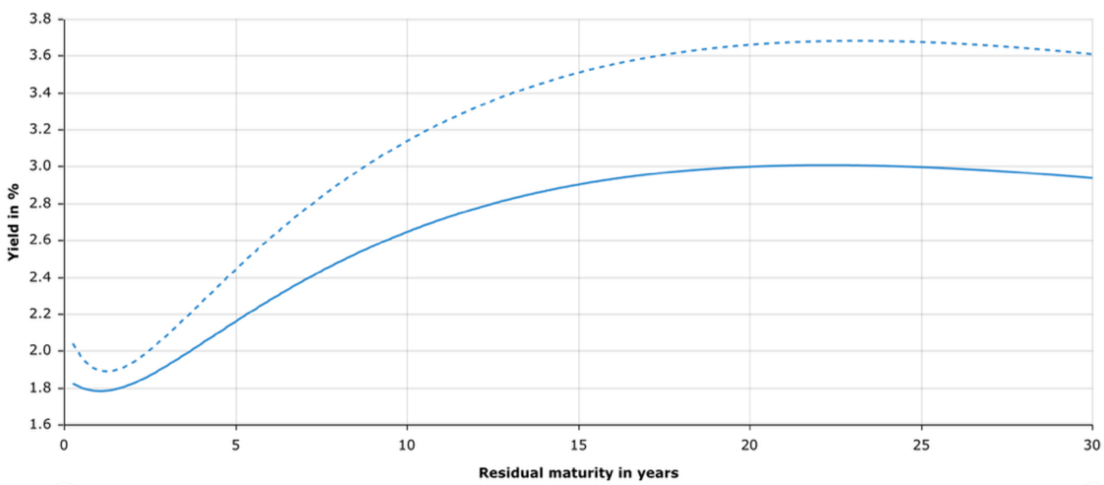
From an economic perspective, the parameters maintain intuitive interpretations:

- The effects of β_2 and β_3 determine togheter the positions of the first and second humps,
- The decay parameters $\lambda_1$and λ_2, as in the NS model, control how quickly the influence of β_1 , β_2 and β_3 decay over time.

1.2.2 Application in central banks

The Nelson-Siegel-Svensson (NSS) model has been adopted by central banks, most notably the European Central Bank (ECB), to estimate the zero-coupon yield curve. The ECB applies the NSS model on a daily basis to construct and publish the official euro area yield curves based on AAA-rated government bonds, as shown by the continuous line in figure 2, and a yield curve based on all government bonds (dotted line). These yield curves are publicly available on the ECB's data portal[9].

Fig. 2. ECB Yield Curves for the 08-06-2025

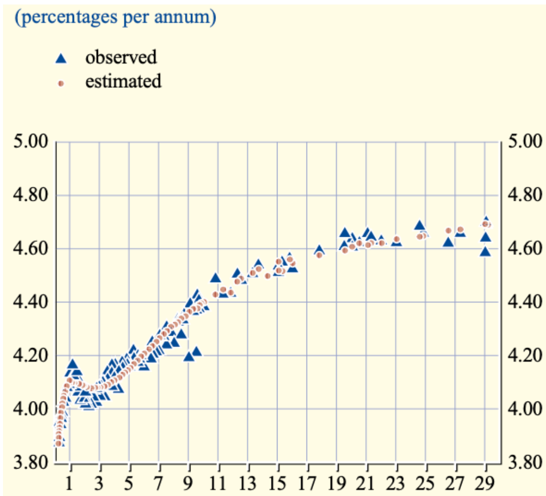


ECB official website: https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html

[9] ECB official website: https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html

The methodology behind the application of the model is explained in the ECB Monthly Bulletin of February 2008, where it is stated that *"This model strikes a good balance between different criteria, including goodness of fit, smoothness and stability of the curve".*[10]

Fig.3. Observed yields versus estimated yields using the Svensson model



Sources: ECB calculations and EuroMTS; ratings provided by Fitch Ratings.
Note: Data for 14 December 2007. The horizontal axis refers to years to maturity. Due to the close fit of the estimated values to the observed values, the 160 observed points are not clearly visible.

ECB (2008), "Estimating and Interpreting the Yield Curve," Monthly Bulletin, Feb. 2008, pp. 99

As illustrated in Figure 3, provided by the Monthly Bulletin of February 2008, the model provides an excellent fit to observed bond yields, even in the presence of data gaps, particularly for longer maturities, where fewer bonds are available. So it explains that the model has been chosen since, even relying on only a small number of parameters, the NSS model can capture the general shape of the yield curve with high precision.[11]

Also Deutsche Bundesbank discussed the usefulness of estimating the term structure of interest rate, with a focus on the fit of the NSS model for this purpose. In particular with their October 1997 Monthly Report[12], the Bundesbank explained its parametric approach, including both Nelson-Siegel and its Svensson extension, to model yield curves across maturities, even though an official yield curve is not currently provided.

[10] ECB (2008), "Estimating and Interpreting the Yield Curve," Monthly Bulletin, Feb. 2008, pp. 98
[11] ECB (2008), "Estimating and Interpreting the Yield Curve," Monthly Bulletin, Feb. 2008, pp. 99
[12] Deutsche Bundesbank. (1997, October). *Estimating the term structure of interest rates*. Monthly Report, pp. 60–65

2. Data and Methodoloy

2.1 The selection of BTPs

To compute the estimations and comparisons of the Nelson-Siegel and Nelson-Siegel-Svensson models in the next sections, I constructed two datasets made of 109 Italian BTPs (Buoni del Tesoro Poliennali). I selected BTPs given their liquidity, standardized payment structure (semi-annual coupons or zero-coupon) and wide range of maturities, which make them particularly suitable for yield curve estimation.

2.1.1 First Dataset: Specific Reference Date

In the first dataset the 109 BTPs are observed on a single reference day: May 20, 2024**.** The dataset is composed by the ISIN, the Price, the Coupon Rate and the Maturity Date of each bond[13]. This static cross-sectional dataset allows for the calibration of the yield curve models under stable market conditions and the comparison between the models fitting power. The selected day does not coincide with any major monetary policy announcements, helping to ensure that pricing is driven mainly by expectations and term premium.

2.1.2 Second Dataset: One-Month period

The second dataset includes a similar set, made by the same 109 BTPs, with daily price observations from April 15 to May 15, 2024, creating a panel dataset. This broader sample enables a dynamic analysis of the models' performance and parameter stability over time. By observing how estimated parameters evolve and how closely the models fit market prices over a one-month horizon.

[13] Official data source: https://www.exactnetwork.net/autenticazione/ssl?id=0

2.2 Objective function and optimization

Starting from the BTPs datasets, the objective is to evaluate a set of parameters that will allow the NS and NSS models to extrapolate a family of spot rates, that associated to a vector of different maturities will create a yield curve that is able to replicate as closely as possible the observed one. The estimated yield curves are compared to the yield curves published by the European Central Bank (ECB) to evaluate the actual model fit of the results.

The optimization aims to minimize the pricing errors on the set of selected bonds, in order to be able to evaluate and plot a yield curve by implementing the corresponding parameters into the models. For each bond, the price estimated by the model is derived by discounting future cash flows using the spot rates obtained from the parametric yield curve.

The objective function is defined as the sum of squared differences between the model-implied prices and the actual market prices:

$$Mispricing = \sum_{i=1}^{n} \left(P_i^{Model} - P_i^{Market}\right)^2 \tag{13}$$

where P_i^{Model} is the price of bond i obtained using the estimated curve, and P_i^{Market} is the observed market price that can be found in the market. The calibration minimizes this quantity.

All the parameters of the model, including the betas and the decay parameters (λ for Nelson-Siegel and λ_1, λ_2 for Nelson-Siegel-Svensson), are optimized simultaneously using the fminunc function in MATLAB. This function searches for the parameter vector that minimizes the mispricing function, starting from an initial guess.

This method differs from earlier approaches, such as those mentioned by Nelson and Siegel in their original work (1987)[14] and by Diebold and Li (2006)[15], where the decay parameter λ was

[14] Nelson, C. R., & Siegel, A. F. (1987). *Parsimonious modeling of yield curves*. The Journal of Business, 60(4), 478-479.
[15] Diebold, F. X., & Li, C. (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics*, 130(2), 349–351.

fixed or chosen by a grid search across a predefined set of values, and then the other parameters were estimated conditionally. In contrast, the procedure implemented in the optimization process estimates λ together with the β parameters, without fixing it in advance. This joint estimation provides on one hand a more precise and flexible fit of the term structure, but on the other hand it requires a careful selection of the initial values, and it will in addition increase computational time and complexity.

3. Cross-Sectional Analysis: a Single-Day Comparison

3.1 Model Calibration Results

From the estimation of the Nelson-Siegel (NS) and Nelson-Siegel-Svensson (NSS) models, calibrated on the cross-sectional Italian bond market dataset for May 20th, it is possible to analyze the differences in both quantitative results and in the shape of the yield curves.

As shown in Table 5, which summarizes all the optimized parameters for both the models, we can see by considering the obtained mispricing errors that the NSS model offers a better fit than the simpler NS model. In particular, the minimum mispricing obtained with NSS is €1320.76, compared to €1392.72 for NS, highlighting NSS's better ability to replicate observed bond prices.

Table 4. Nelson-Siegel vs Nelson-Siegel_Svensson Optimization Results

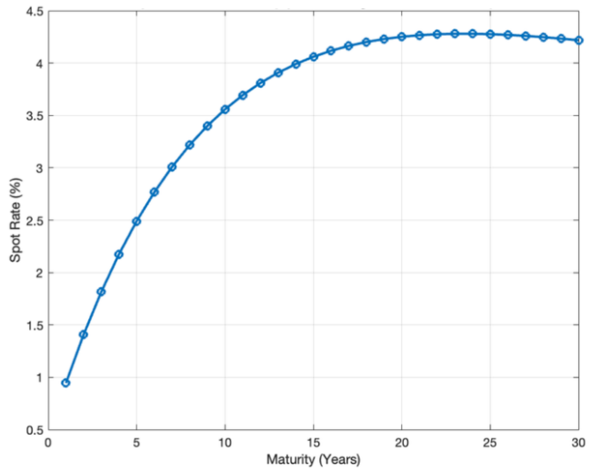
NS Parameter	Estimated Value	NSS Parameter	Estimated Value
Lambda (λ)	0.1022	Lambda1 (λ_1)	0.2717
		Lambda2 (λ_2)	0.2772
Beta 0 (β_0)	0.025715	Beta 0 (β_0)	0.035283
Beta 1 (β_1)	-0.021512	Beta 1 (β_1)	-0.005035
Beta 2 (β_2)	0.087578	Beta 2 (β_2)	3.698957
		Beta 3 (β_3)	-3.716906
Min Mispricing	1392.723772	Min Mispricing	1320.759514

MATLAB

Beyond this numerical comparison, the two models produce different yield curve shapes that are represented in graph 11 and graph 12. The NS model, generates a relatively smooth curve that rises in the short to medium term and then gradually flattens and declines across longer maturities. This is a classic hump-shaped profile that is coherent with the structure of the NS specification.

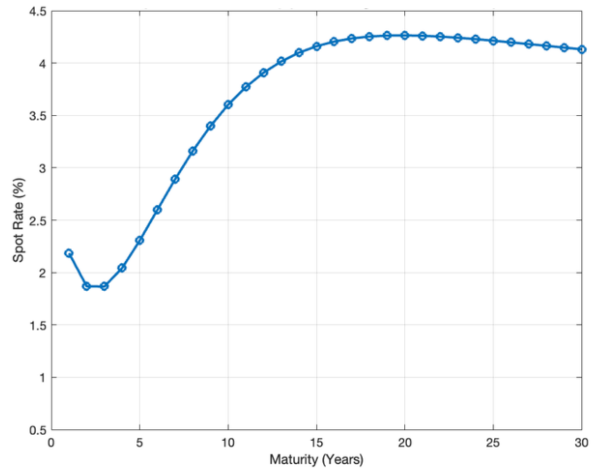
In contrast, the NSS model introduces a second exponential decay term through an additional λ parameter , enabling the model to represent a second curvature component. As a result, the NSS curve (Figure 5) presents a pronounced initial drop at very short maturities, followed by a moderate rise in the medium-long term and a slower decline in the very long term.

Fig. 4. Spot Rates vs. Maturity (Nelson-Siegel)



MATLAB

Fig. 5. Spot Rates vs. Maturity (Nelson-Siegel-Svensson)



MATLAB

3.2 Mathematical Interpretation

As carefully explained in the previous chapters of the thesis, β_0 represents the level component, and determines the asymptotic long-term yield toward which the curve tends as maturity increases. In the NS analysis the value found is of 0.0257, this indicates that the yield curve in the very long run will eventually approach the 2.57% level.

β_1, that is the slope parameter, is negative and modest in magnitude (-0.021512). Since its influence is mainly concentrated in the short-term part of the curve, this negative value implies that yields at shorter maturities start below the long-term level, as can be graphically visualized in Figure 4. However, β_2, the curvature parameter, reaches a relatively high value of 0.087578. This coefficient is responsible for the presence of the hump in the medium-term segment of the yield curve. The result is a curve that increases from short maturities to a mid-term peak and then slowly decreases for longer maturities. Mathematically, the NS model produces a curve that is smooth but unable to replicate more complex curve shapes.

In contrast, the Nelson-Siegel-Svensson model produces a more complex term structure, enabled by the introduction of a second exponential decay term and an additional beta coefficient. The estimated values include λ_1 equal to 0.2717 and λ_2 sligtlhy higher, equal to 0.2772, while the beta coefficients are $\beta_0 = 0.035283$, $\beta_1 = -0.005035$, $\beta_2 = 3.698957$, and $\beta_3 = -3.716906$. The level parameter β_0, at approximately 3.53%, is higher than its NS counterpart, indicating a higher long-term yield, even though it can be seen in the graphs that on a 30 years maturity the yield is actually on the same level. While β_1, which controls the slope component, is close to zero and only slightly negative (-0.005035), thus causing a minimal influence on the short end of the curve. Interestingly, while this value would suggest a mostly flat short-term curve, there is actually a dip, this suggests that the shape is dominated by the interaction of the curvature components β_2 and β_3.

An apparent anomaly emerges when examining the last two parameters. Since β_2 is strictly positive and β_3 is opposite and negative, we would expect a dip in the short-medium term and a successive hump in the long term. However, as presented in Figure 6, the curve performs the opposite behavior. This can be explained by analyzing the numerical interaction between the model's parameters.

In the very short-term, the effect of the two curvature terms is close to zero, but the negative value of β_3 provides a negative downward effect before the positive effect of β_2 starts to afflict the curve, since the β_3 parameter and the λ_2 are sligtly higher, in absolute values, than their counterparts. As a result, the yield curve exhibits a drop in short maturities, producing the initial dip.

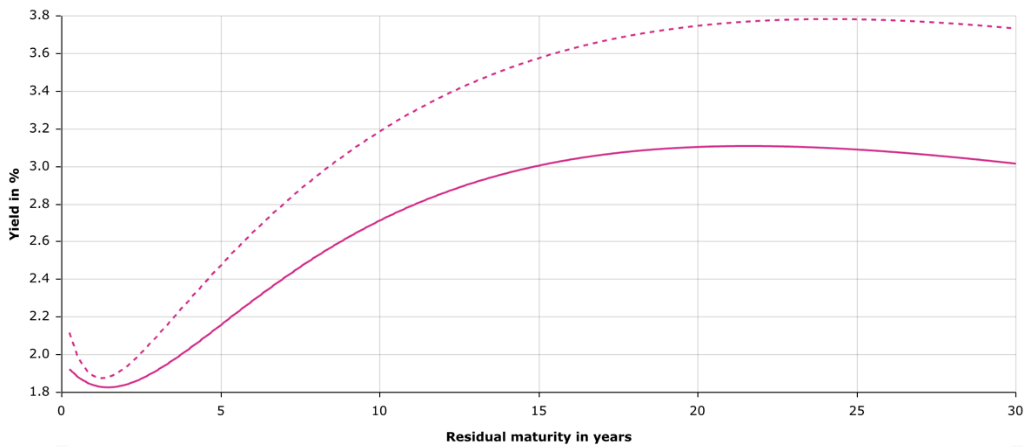
As maturity increases toward the 5–20 year range, both curvature terms reach their maximum contributions. However, since β_2 is positive and large, while β_3 is negative and nearly equal in size, the β_2 term begins to dominate, especially as the effect of β_3 starts decaying more rapidly with time due to a higher lambda. This creates the upward movement that is evident in the medium-long term**,** resulting in a hump peaking around 20 years**.**

Finally, in the very long term, both curvature terms decay toward zero, and the yield curve reflects the combined effect of β_0 and β_1. Since β_1 is close to zero, the curve dips gently, explaining the soft decline observed beyond 20 years.

3.3 Economic Interpretation and Contextual Analysis

The NSS model represents more accurately the economic conditions of Italy, but also of the broader Euro Area on May 20th. This is supported by a visive comparison between the NSS yield curve and the official yield curve published by the European Central Bank (ECB), shown in Figure 7. The two curves show strong similarity in their shapes, while mantaining differences in yield levels for corresponding maturities, reflecting the distinct characteristics of Italian government securities structure compared to those of the broader Euro Area.

Fig. 7. ECB Yield Curves for the 20-05-2025



ECB official website: https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html

In mid-2025, the Italian bond market has been influenced by several macroeconomic and geopolitical factors. Of particular importance is the news that annual inflation in the euro zone dropped to 1.9% in May, falling below the ECB's 2% target for the first time since 2021 as can be consulted on the data published by the ECB on their Data Portal[16].

This decline, as stated by ECB policymaker and Finland's central bank governor Olli Rehn in an interview with *Reuters*[17], influences market expectations of further rate cuts, thereby contributing to a steep drop in short-term yields, as captured by the dip in the first 1-3 year

[16] ECB Data Portal (2025, May). Euro-area quarterly and monthly HICP inflation series. Retrieved from: https://data.ecb.europa.eu/main-figures/inflation

[17] Reuters. (2025, April 28). *ECB may cut rates below neutral, Rehn says.* Reuters. Retrieved from: https://www.reuters.com/business/finance/ecb-may-cut-rates-below-neutral-rehn-says-2025-04-28/

maturities of the NSS curve. However, the NS model, fails to capture this short-term dynamic and thus is not able to include the initial drop in short-term rates.

The mid-term rise observed in both models yield curves, reflects the expectation of the market about the risk and uncertainty for medium-term investments in government securities. Indeed, Italy faces a high public debt level, and investors demand a premium for holding medium-term debt due to economic risks.

Finally, in the long-term segment, both the Nelson-Siegel and Nelson-Siegel-Svensson models capture a slow decline in yields. According to an OECD report (Bouis et al., 2014[18]), a long-term downward trend can be explained by population aging, shifts in demand for safer investments, and slower economic growth. To confirm this view, a recent cross-country study performed by SOA Research Institute[19], found that a 1% increase in the proportion of the population aged 40–64 is associated with an average 1.29% reduction in real long-term bond yields**.** These dynamics are particularly relevant in the current context of Italy and the broader Euro Area, where demographic pressures and low productivity growth persist.

In summary, the economic environment of Italy in May 2025 provides a plausible rationale for the complex shape of the NSS-estimated curve.

[18] Bouis, R. et al. (2014), "Factors behind the Decline in Real Long-Term Government Bond Yields", OECD Economics Department Working Papers, No. 1167, OECD Publishing, Paris.
[19] SOA Research Institute. (2023). *Demographics and Productivity: Drivers of Economic Growth*. Society of Actuaries, pp 13.

4. Time-Series Analysis: Dynamics of the Nelson-Siegel Model

4.1 Methodology for Dynamic Analysis

In this chapter the Nelson-Siegel (NS) and the Nelson-Siegel-Svensson (NSS) models are evaluated over a 30-day period, only focusing on effective trading days, thereby excluding weekends and holidays. The objective is to analyze and discuss the temporal stability of the parameters and to assess the overall fit of the models over time. The calibrations for the single days are performed through the same optimization procedure explained in Chapter 3 of this thesis.

4.2 Descriptive Statistics of Estimated Parameters

Tables 7 and 8 present the full time-series of estimated parameters and resulting mispricing errors for the NS and NSS models, respectively.

Table 7. Time-series of results for Nelson-Siegel model over 30-days

Date	Lambda	Beta0	Beta1	Beta2	Misprice
15-Apr-2025	0.10497	0.024233	-0.019943	0.091345	1414.4
16-Apr-2025	0.10256	0.023636	-0.019518	0.092873	1407.1
17-Apr-2025	0.10204	0.023994	-0.020512	0.092412	1391.7
22-Apr-2025	0.10295	0.024141	-0.02087	0.091379	1345.9
23-Apr-2025	0.10142	0.023261	-0.019545	0.093066	1370.9
24-Apr-2025	0.10227	0.023772	-0.020928	0.09087	1381.5
25-Apr-2025	0.10222	0.023491	-0.020156	0.091836	1400.9
28-Apr-2025	0.10252	0.023768	-0.020659	0.092322	1369.5
29-Apr-2025	0.101	0.023577	-0.020182	0.092778	1346.1
30-Apr-2025	0.10102	0.023304	-0.020262	0.092762	1364.7
02-May-2025	0.10096	0.023579	-0.020146	0.093262	1381.1
05-May-2025	0.10136	0.024426	-0.020852	0.091123	1362.1
06-May-2025	0.1001	0.023749	-0.020402	0.093852	1353.1
07-May-2025	0.10169	0.024057	-0.021036	0.090917	1391.3
08-May-2025	0.10033	0.023851	-0.020388	0.091611	1412.2
09-May-2025	0.1002	0.023945	-0.020217	0.092162	1389.5
12-May-2025	0.10087	0.024245	-0.019243	0.090606	1425.2
13-May-2025	0.10005	0.023791	-0.019034	0.093164	1447.2
14-May-2025	0.10116	0.024289	-0.019401	0.091985	1452.7
15-May-2025	0.1021	0.024552	-0.020093	0.089578	1463.3

Source: MATLAB

Table 8. Time-series of results for Nelson-Siegel-Svensson model over 30-days

Date	Lambda1	Lambda2	Beta0	Beta1	Beta2	Beta3	Misprice
15-Apr-2025	0.26488	0.27412	0.03363	-0.0044118	2.2138	-2.2228	1345.6
16-Apr-2025	0.26874	0.26499	0.033616	-0.0044195	-5.4518	5.4422	1335.2
17-Apr-2025	0.27966	0.26874	0.034451	-0.0044141	-1.9463	1.9302	1318.1
22-Apr-2025	0.27659	0.26562	0.034044	-0.0052176	-1.8794	1.866	1274.7
23-Apr-2025	0.26972	0.26236	0.033437	-0.0044595	-2.7819	2.7712	1297.7
24-Apr-2025	0.28344	0.26895	0.033941	-0.0037952	-1.4843	1.4656	1303.9
25-Apr-2025	0.26568	0.28031	0.03365	-0.003528	1.4436	-1.4592	1324.2
28-Apr-2025	0.26724	0.27093	0.033659	-0.0047348	5.6052	-5.6178	1294.3
29-Apr-2025	0.26708	0.26193	0.03361	-0.0051017	-3.9334	3.9226	1271.3
30-Apr-2025	0.27014	0.2758	0.033869	-0.0037239	3.7641	-3.7811	1286.2
02-May-2025	0.27075	0.27428	0.034312	-0.0043203	5.9672	-5.983	1305.8
05-May-2025	0.26474	0.27107	0.034345	-0.0053625	3.1926	-3.2059	1288.5
06-May-2025	0.27294	0.26722	0.034633	-0.0052325	-3.6423	3.6273	1279.8
07-May-2025	0.27722	0.27046	0.034249	-0.0045548	-3.1133	3.0958	1315.8
08-May-2025	0.27135	0.26772	0.034204	-0.0046384	-5.6766	5.6611	1337.2
09-May-2025	0.27425	0.2625	0.034399	-0.005045	-1.7427	1.7284	1316.9
12-May-2025	0.27391	0.26276	0.034316	-0.0038643	-1.8276	1.8145	1353.4
13-May-2025	0.26155	0.27164	0.034341	-0.0040409	2.0278	-2.0398	1374.3
14-May-2025	0.27763	0.26357	0.034675	-0.0039124	-1.4823	1.4684	1381
15-May-2025	0.27914	0.26543	0.034373	-0.0038136	-1.5149	1.4993	1390.3

Source: MATLAB

However, to facilitate the analysis and comparison of the two models, Tables 5 and 6 summarize key descriptive statistics, including mean, standard deviation (SD), minimum, maximum, and median, for each parameter across the 30-day period.

Table 5. Summary table for Nelson-Siegel model over 30-days

	Mean	SD	Median	Max	Min
Lambda	0.10159	0.0011777	0.10139	0.10497	0.10005
Beta0	0.023883	0.00035904	0.023821	0.024552	0.023261
Beta1	-0.020169	0.00057859	-0.020199	-0.019034	-0.021036
Beta2	0.091995	0.0010707	0.092073	0.093852	0.089578
Misprice	1393.5	34.531	1390.4	1463.3	1345.9

Source: MATLAB

Table 6. Summary table for Nelson-Siegel-Svensson model over 30-days

	Mean	SD	Median	Max	Min
Lambda1	0.27183	0.0058505	0.27105	0.28344	0.26155
Lambda2	0.26852	0.0050832	0.26823	0.28031	0.26193
Beta0	0.034088	0.00038084	0.034226	0.034675	0.033437
Beta1	-0.0044295	0.00056023	-0.0044168	-0.003528	-0.0053625
Beta2	-0.61313	3.4296	-1.6288	5.9672	-5.6766
Beta3	0.59914	3.4297	1.6139	5.6611	-5.983
Misprice	1319.7	35.355	1316.3	1390.3	1271.3

Source: MATLAB

In the case of the NS model (Table 5), the results indicate strong temporal stability. Indeed the Standard Deviation (SD) is extremely low for the decay factor λ, the level parameter β_0 and for the slope parameter β_1, at 0.00089, 0.00036, and 0.00058 respectively. While the β_2 parameter, which governs the curvature, shows slightly more variation (SD ≈ 0.0032), yet remains within acceptable bounds. Additionally, the comparison of the maxima of the minima over time confirms the conclusion of low volatility of the parameters.

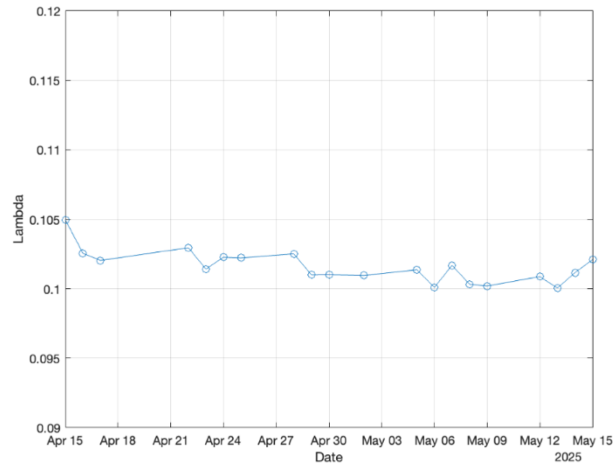
By contrast, while the NSS model (Table 6) also demonstrates stability in λ_1, λ_2, β_0 and β_1 that present low SD over time and close maxima and minima, if we continue the analysis to the last two curvature parameters β_2 and β_3 we can clearly see that the results become extremely unstable: the SD rises sharply. As seen in Table 6, their standard deviations are significantly higher, both above 3.4, and the range between their maxima and minima exceeds 10 points. This reflects the greater flexibility of the NSS model, but also indicates potential overfitting or sensitivity to outliers in the dataset.

Regarding model fit, the mispricing error is on average lower in the NSS compared to the NS, as can be implied by the lower mean (respectively 1319.7 and 1393.5) and median (respectively 1316.3 and 1390.4). Additionally, by performing an analysis on the maxima and minima mispricing errors in the two models, it appears that the mispricing errors in the NS and in the NSS models have on average the same stability, as the difference between the maximum and minimum mispricing errors of the 30-days is 117.4 in NS compared to 119 in NSS, while the SD is approximately really close in the two models (34.531 in NS and 35.355 in NSS).

4.3 Evolution of Parameters and Fit over Time

To better understand the dynamics of model behavior over time, Figures 8–10 track the evolution of the decay parameters and mispricing errors over time.

Fig. 8. Optimal Lambda over time in NS

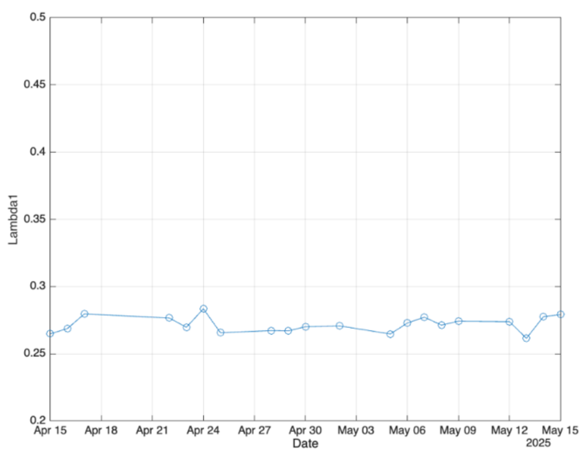


Source: MATLAB

In the NS model (Figure 8), the decay parameter λ remains extremely stable across the observation window, with minimal day-to-day fluctuations. This confirms the reliability of the NS model's structure in capturing the general shape of the yield curve during the period.

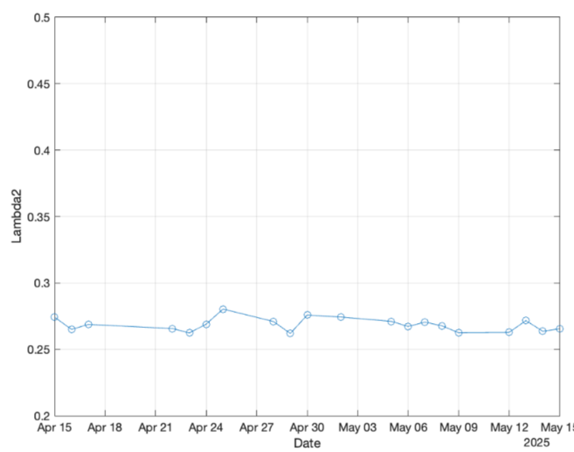
As shown in Figure 9 and in Figure 10, also for the NSS model the levels of respectively λ_1 and λ_2 maintain their stability, although small fluctuations are more visible than in the NS case.

Fig. 9. Optimal Lambda1 over time in NSS



Source: MATLAB

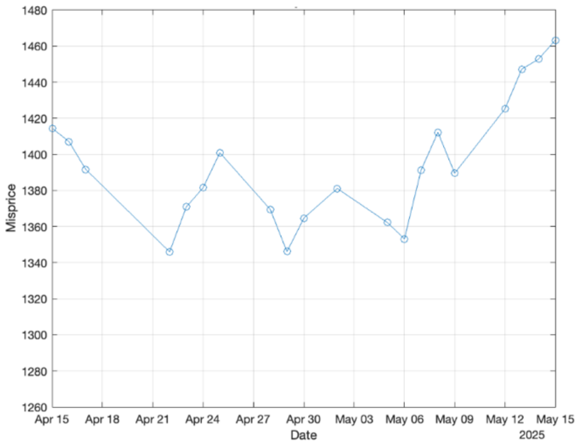
Fig. 10. Optimal Lambda2 over time in NSS



Source: MATLAB

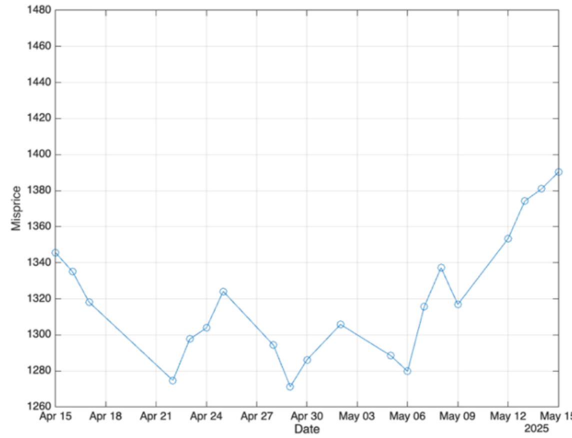
For what concerns the mispricicing error, Figure 11 and Figure 12 display the evolution of the mispricing error over time for the NS and NSS models, respectively. It is evident that the two curves follow a mostly similar pattern. For instance, peaks tend to occur on the same days, with both models showing a decline in mispricing during the initial days and an increase in the last days, in particular with the maximum located in both the models on the last trading day analyzed (15 of May). This indicates that, despite structural differences between the models, their performance, in terms of pricing accuracy, reacts similarly to day-to-day market conditions.

Fig. 11. Minimum Misprice over time in NS



Source: MATLAB

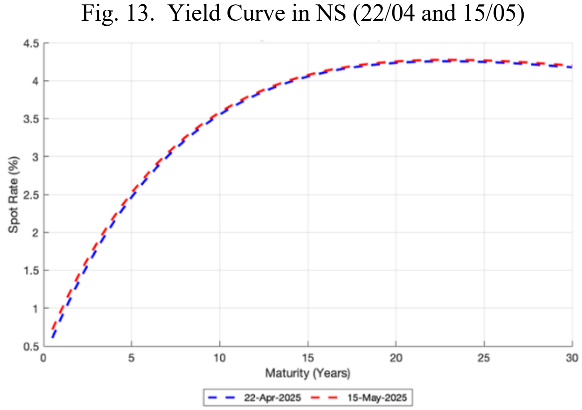
Fig. 12. Minimum Misprice over time in NSS



Source: MATLAB

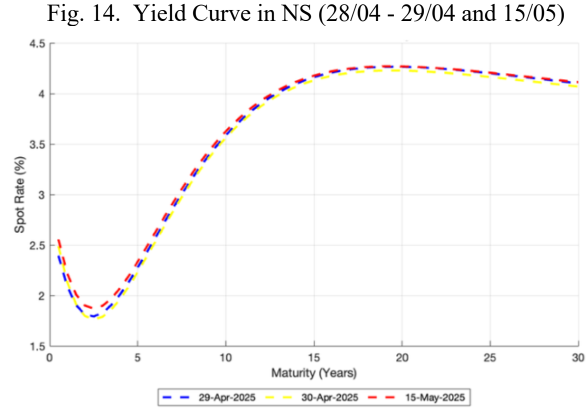
Figures 13 and 14 plot the estimated yield curves for selected days in both the NS and NSS models. Specifically, two sets of yield curves are presented: one corresponding to the day with the **highest mispricing error** (15 May, in red) and one to the **lowest mispricing day** (22 April for NS and 29 April for NSS, in blue). Furthermore, for the NS model a third curve is estimated: the 28 April has been chosen, since in this day β_2 and β_3 were opposite to the ones of the 29 April. From these graphical representations we can conclude that overall, despite differences in estimates and mispricing errors, the shape of the yield curves remains consistent across days. Indeed, even from the analysis of the best and worst calibration results, the curves closely overlap, suggesting that the structural form of the estimated term structure remains stable over time.

Fig. 13. Yield Curve in NS (22/04 and 15/05)



Source: MATLAB

Fig. 14. Yield Curve in NS (28/04 - 29/04 and 15/05)



Source: MATLAB

Conclusions

This thesis has analyzed the application of the Nelson-Siegel and Nelson-Siegel-Svensson models to the Italian government bond market, with focus on economic and mathematical interpretation. Through the single day and the one month empirical analysis, models' fit, parameters stability and ability to effectively capture the Italian yield curve are analyzed.

The results demonstrate that the NSS model provides a better fit when calibrated on a single day, with a superior ability to represent the real Italian yield curve, such as the dip in short maturities explained by the decreasing inflation and subsequent monetary policy expectations. The better fit is also reflected in lower mispricing errors and graphically confirmed by the curves comparison with the ECB benchmark.

However, through the one month time series analysis, it can be evinced that the simpler NS model shows advantages in terms of parameter stability. The NS parameters have extremely low fluctuations over the month, guaranting an easier and accurate interpretability. In contrast, the NSS parameters, in particular the two curvature terms, present a high volatility. The higher volatility in some parameters increases the difficulty of the mathematical interpretation of the terms, suggesting a certain degree of overfitting. However, despite the instability, the NSS maintains a better average fit as demonstrated by the lower mispicing error at any time.

From an economic point of view, both models estimated curves are aligned with the Italian macroeconomic context during the sample period. Indeed both models capture the lower yield rate in the short term due to expectations caused by low inflation and a medium term hump related to required risk premiums, even though the NSS model is able to better represent these dynamics.

In sum, the thesis demonstrates that while the Nelson-Siegel-Svensson model excels at fitting the yield curve, in absence of regulations and controls in the optimization process it can incur the risk of overfitting. This observation underscores the importance of balancing model complexity and flexibility in yield curve modeling.

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