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ES1:

3.1 Let's assume that we have performed a 20-point DFT on a sequence of real-valued time-domain samples, and we want to send our $X(m)$ DFT results to a colleague using e-mail. What is the absolute minimum number of (complex) frequency-domain sample values we will need to type in our e-mail so that our colleague has complete information regarding our DFT results?

Answer:

As the signal is real, we need to consider that the resulting frequency components of a DFT, exhibit Hermitian symmetry. This means that the complex conjugate of any frequency component X(k) is found at X*(N-k), where N is the total number of points in the DFT, which in this case is 20.

Given this property, the DFT result is symmetric around the midpoint of the frequency array. This implies that the DFR result from:

k=1 to k=N/2 -1=9

completely determine the results for k=N/2 +1= 11 to k=N-1=19, due to the conjugate symmetry.

Thus, for a 20-point DFT of a real values sequence, the first N/2 +1 = 11 DFT results must be communicate.

My colleague therefore, can use the symmetry properties to infer the rest of the values.

ES2:

3.2 Assume a systems engineer directs you to start designing a system that performs spectrum analysis using DFTs. The systems engineer states that the spectrum analysis system's input data sample rate, f_s, is 1000 Hz and specifies that the DFT's frequency-domain sample spacing must be exactly 45 Hz.

(a) What is the number of necessary input time samples, N, for a single DFT operation?

(b) What do you tell the systems engineer regarding the spectrum analysis system's specifications?

Answer:

The sample rate (fs) = 1000 Hz.

The required frequency-domain sample spacing delat_f=45 Hz.

The relationship between the DFT size N, the sample rate fs and the frequency resolution delta_f is:

Delta_f = fs/N;

N = fs/delta_f = 1000Hz/45Hz = 22.22

Feedback to the system Engineer:

Since N must be an integer, the closest practical value for N is obtained by N=22 or 23, but this results in a frequency resolution of about 45.45 Hz and 43.48 Hz respectively which do not meet the exact requirement.

Achieving exactly 45 Hz resolution with a sample rate of 1000 Hz would require a non-integer N, which is impractical for DFT, as N must be an integer.

To address this issue, we sould either accept a slightly different frequency resolution as mentioned above, or adjust the sample rate.

For example, adjusting the sample rate fs to 990 Hz would allow for an N of 22, giving exactly 45 Hz resolution:

Fs=N*delta_f = 22*45 Hz = 990 Hz.

This adjustment to 990 Hz for the sample rate might be a minor change but will satisfy the exact requirements for the frequency resolution.

Of course, analysis and simulation are important to verify how these changes would impact the overall performance and accuracy of the spectrum analysis system.

ES3:

3.3 We want to compute an N-point DFT of a one-second-duration compact disc (CD) audio signal $x(n)$, whose sample rate is $f_s = 44.1$ kHz, with a DFT sample spacing of 1 Hz.

(a) What is the number of necessary $x(n)$ time samples, N?

(b) What is the time duration of the $x(n)$ sequence measured in seconds?

Hint: This Part (b) of the problem is trickier than it first appears. Think carefully.

Answer:

a) The relationship between the DFT size N, the sample rate fs, and the frequency resolution delta_f is given by:
Delta_f = fs/N

N=fs/delta_f = 44100 Hz/1 Hz = 44100 samples.

b) The duration of the sequences, T, for N samples collected at a sampling rate fs, is calculated by:
T=N/fs = 44100/44100 = 1 sec.

The DFT sample spacing of 1Hz specifically requires the sequence length in time, T, to be such that the frequency resolution delta_f is exactly 1 Hz.
When sampling an analog signal for a duration of T seconds at a sampling frequency fs, the total number of samples N is fsxT. If T needs to strictly ve 1 second to achieve a 1 Hz resolution in a DFT of N points, the given conditions perfectly align to make T exactly 1 second.

Anyway there is a second reasoning about the solution.
If the time duration is the time difference between the first and the last samples of the sequence, rather than N times the time interval ts=1/fs between the samples the time duration should be (N-1)*(1/fs) = 0.999 sec.

ES4:

3.4 Assume we have a discrete $x(n)$ time-domain sequence of samples obtained from *lowpass sampling* of an analog signal, $x(t)$. If $x(n)$ contains $N = 500$ samples, and it was obtained at a sample rate of $f_s = 3000$ Hz:

(a) What is the frequency spacing of $x(n)$'s DFT samples, $X(m)$, measured in Hz?

(b) What is the highest-frequency spectral component that can be present in the analog $x(t)$ signal where no aliasing errors occur in $x(n)$?

(c) If you drew the full $X(m)$ spectrum and several of its spectral replications, what is the spacing between the spectral replications measured in Hz?

Answer:

a) The frequency spacing delta_f between successive DFT samples is determined by:
Delta_f = fs/N = 3000 Hz / 500 = 6 Hz.

b) To avoid aliasinng in the samples signal x(n), the highest frequency component present in the analog signal x(t) must not exceed the Nyquist frequency, which is half of the sampling rate fs.
The Nyquist frequency f_N=fs/2 = 3000 Hz / 2 = 1500 Hz.

c) The spectral replications in the frequency domain occur at intervals equal to the sampling rate fs. This is because the spectrum of a sampled signal is periodic with a period equal to the sampling rate. Thus, the spacing between the spectral replications in X(m) is:

Fs = 3000 Hz.

ES5:

3.5 What are the magnitudes of the 8-point DFT samples of

(a) the $x_1(n) = 9, 9, 9, 9, 9, 9, 9, 9$ sequence (explain how you arrived at your solution)?

(b) the $x_2(n) = 1, 0, 0, 0, 0, 0, 0, 0$ sequence?

(c) the $x_3(n) = 0, 1, 0, 0, 0, 0, 0, 0$ sequence?

Answer:

a) The first sequence is a costant sequence, meaning each sample value is the same. The DFT of a costant sequence is zero at all frequencies except at the zero frequency (k=0), where it sums all the input samples.

This is given by the DFT definition_

X(k) = $\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$

X(0) = $\sum_{n=0}^{N-1} 9 = 72$ and X(k) = 0 for k = 1,2,...,7 because the sum of a fully cycle of complex exponentials is zero.

The magnitude of X(k) is:

- |X(0)|=72.
- |X(k)| = 0 for k=1,2,...,7

b) The second sequence is an impulse at n=0. The DFT of a single impulse is a constant across all frequency components because the transform of a single impulse is 1 at all frequencies.
Since the DFT formula involves a sum over the product of x(n) and a complex exponential, when x(n) is non-zero only at n=0, the sum becomes simply x(0) * 1.
|X(k)| = 1 for all k=0,1,2,...,7.

All frequency component will have the magnitude of the non-zero sample, which is 1.

c) The third sequence is an impulse at n=1. The DFT of an impulse at n=1 rotates through all the complex roots of unity as k varies, due to the phase term e^$^{j2\pi kn/N}$ in the DFT.

The DFT can be written as:

X(k)= $\sum_{n=0}^{N-1} x_3(n)e^{-j2\pi kn/8} = 1 * e^{-\frac{j2\pi k1}{8}} = e^{-j\pi k/4}$

This results in a complex number for each k, whose magnitude is 1, distributed uniformly around the unit circle in the complex plane.

The magnitude is:

|X(k)| = |$e^{-j\pi k/4}$| = 1 for all k =0,1,2,...,7.

In the complex plane, these points would lie on the unit circle, representing a phase shift that depends on kk, but their magnitudes remain consistent at 1. The phases would rotate through 0,−π/4,−π/2,−3π/4,−π,−5π/4,−3π/2,−7π/4.

Cooemnt about time-shifted version:

When a sequence x(n) is time-shifted by m samples, the DFT of the new sequence y(n) = x(n-m), is related to the original DFT by:

Y8k) = X(k) $e^{-j2\pi kn/N}$

This relationship shows that the DFT of a time-shifted sequence is the original DFT multiplied by a phase shift factor $e^{-j2\pi kn/N}$.

Since the magnitude of a complex number is invariant under multiplication by a phase factor, the magnitudes of the DFT coefficients of x2(n) and x3(n) remain the same.

ES6:

3.6 Consider sampling exactly three cycles of a continuous $x(t)$ sinusoid resulting in an 8-point $x(n)$ time sequence whose 8-point DFT is the $X(m)$ shown in Figure P3–6. If the sample rate used to obtain $x(n)$ was 4000 Hz, write the time-domain equation for the discrete $x(n)$ sinusoid in trigonometric form. Show how you arrived at your answer.

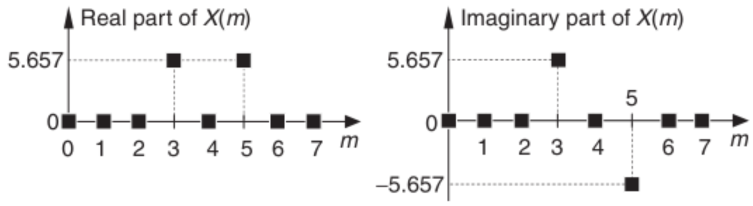

Figure P3–6

Answer:

The real part of X(m) shows significant values only at m=3 and m=5.

The imaginary part of X(m) complements this, with significant values also at m=3 and m=5 but with opposite signs. This indicates sinusoidal components at these indices.

The indices m=3 and m=5 are the key frequencies in the DFT.

Since m=3 and m=5 are symmetric around N/2 in an 8-point DFT, they represent the same frequency but are complex conjugates of each other, implying a real sinusoidal component in x(n).

The periodicity of the sinusoid in x(n) matches these DFT indices, confirming that the sinusoid completes three cycles in eight samples.

With a sample rate fs=4000 Hz, the frequency fm (for m=3) of the sinusoid is given by the fraction of the sample rate represented by three cycles over eight samples:

Fm = 3*4000 Hz/8 = 1500 Hz, for m=3.

Given that three complete cycles occur in eight samples, the frequency in radian/sample ω0 is:

ω0 = 2π*3/8

The time-domain sequence x(n) can be formulated as:

x(n) = A cos(2π*3n/8= = A cos (3πn/4)

ES7:

3.7 In the text's Section 3.1 we discussed the computations necessary to compute the $X(0)$ sample of an N-point DFT. That $X(0)$ output sample represents the zero Hz (DC) spectral component of an $x(n)$ input sequence. Because it is the DC component, $X(0)$ is real-only and we're free to say that an $X(0)$ sample *always* has zero phase. With that said, here are two interesting DFT problems:

(a) Given that an N-point DFT's input sequence $x(n)$ is real-only, and N is an even number, is there any value for m (other than $m = 0$) for which an $X(m)$ DFT output sample is always real-only?

(b) Given that N is an odd number, is there any value for m (other than $m = 0$) where an $X(m)$ DFT output sample is always real-only?

Answer:

For a sequence x(n) of length N, the DFT X(m) is given by:

X(m) = $\sum_{n=0}^{N-1} x(n)e^{-j2\pi mn/N}$

For x(n) real-only, the complex exponential can be expanded into its cosine and sine components:

X(m) = $\sum_{n=0}^{N-1} x(n)[\cos\left(\frac{2\pi mn}{N}\right) - jsin(2\,\pi mn/N)]$

a) N is an even number
   X(m) can be real-only when m=0 and m=N/2.

   Fro m= N/2, X(N/2) = $\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi(\frac{N}{2})n/N}$ = $\sum_{n=0}^{N-1} x(n)\, e^{-j\pi n}$ = $\sum_{n=0}^{N-1} x(n)\,(-1)^n$

   The expression (-1)^n will alternate between +1 nad -1 for successive n.
   This expression is purely real as the imaginary part of the exponential term is zero for any integer n.

b) N is an odd number

   Analysing the general form, when N is odd, the frequencies m!= 0 do not lead to the sine component uniformly cancelling out across the sum, as there isn't a straightforward repeating pattern that covers all n such that sin($2\pi mn$) always equals zero.

   Thus, for N odd, there is no m!=0 where X(m) is always real-only.

ES8:

3.8 Using the following rectangular form for the DFT equation:

$$X(m) = \sum_{n=0}^{N-1} x(n) \cdot [\cos(2\pi nm / N) - j\sin(2\pi nm / N)]$$

(a) Prove that the $f_s/2$ spectral sample is $X(N/2) = N \cdot \sin(\theta)$ when the $x(n)$ input is a sinusoidal sequence defined by

$$x(n) = \sin[2\pi(f_s/2)nt_s + \theta].$$

N is an even number, frequency f_s is the $x(n)$ sequence's sample rate in Hz, time index $n = 0, 1, 2, ..., N-1$, and θ is an initial phase angle measured in radians.

Hint: Recall the trigonometric identity $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.

(b) What is $X(N/2)$ when $x(n) = \sin[2\pi(f_s/2)nt_s]$?

(c) What is $X(N/2)$ when $x(n) = \cos[2\pi(f_s/2)nt_s]$?

Answer:

a) X(m) = $\sum_{n=0}^{N-1} \sin\left(2\pi\left(\frac{fs}{2}\right)n\left(\frac{1}{fs}\right) + \theta\right)[\cos\left(\frac{2\pi nN}{2N}\right) - jsin(\frac{2\pi nN}{2N})] =$

= $\sum_{n=0}^{N-1} \sin(\pi n + \theta)[\cos(\pi n) - jsin(\pi n)] = \sum_{n=0}^{N-1} \sin(\pi n + \theta)[(-1)^n - j0]$

= $\sum_{n=0}^{N-1} \sin(\pi n + \theta)(-1)^n$

Considering the trigonometric identity:

Sin(α+β) = sin(α)cos(β) + cos(α)sin(β)

= $\sum_{n=0}^{N-1}[\sin(\pi n)\cos(\theta) + \cos(\pi n)\sin(\theta)](-1)^n$ = $\sum_{n=0}^{N-1}[0\cos(\theta) + (-1)^n \sin(\theta)](-1)^n$ = $\sum_{n=0}^{N-1}(-1)^n \sin(\theta)(-1)^n$=$\sum_{n=0}^{N-1}(-1)^{2n}\sin(\theta)$ = $\sum_{n=0}^{N-1}(1)^n \sin(\theta) = N\sin(\theta)$

b) X(m) = $\sum_{n=0}^{N-1} \sin\left(2\pi\left(\frac{fs}{2}\right)n\left(\frac{1}{fs}\right)\right)[\cos\left(\frac{2\pi nN}{2N}\right) - jsin(\frac{2\pi nN}{2N})] =$

= $\sum_{n=0}^{N-1} \sin(\pi n)[\cos(\pi n) - jsin(\pi n)] = \sum_{n=0}^{N-1} 0[(-1)^n - j0] = 0$

The same result can be derived from the result of a), indeed:

If $\theta = 0$, then $\sin(\theta) = 0$ and this implies that:

$X(N/2) = N\sin(\theta) = N\sin(0) = 0$

c) $X(m) = \sum_{n=0}^{N-1} \cos\left(2\pi\left(\frac{fs}{2}\right)n\left(\frac{1}{fs}\right)\right)[\cos\left(\frac{2\pi nN}{2N}\right) - jsin(\frac{2\pi nN}{2N})] =$

$=\sum_{n=0}^{N-1} \cos(\pi n)\,[\cos(\pi n) - jsin(\pi n)] = \sum_{n=0}^{N-1}(-1)^n[(-1)^n - j0] = \sum_{n=0}^{N-1}(-1)^{2n} = \sum_{n=0}^{N-1}(1)^n = N$

The same result can be derived from the result of a), indeed:

Cos(x) = sin(x + π/2)

X(n) = cos[2π(fs/2)nTs] = sin[2π(fs/2)nTs + $\pi/2$]

X(N/2) = $N\sin(\theta) = N\sin(\pi/2) = N$

ES9:

3.9 To gain some practice in using the algebra of discrete signals and the geometric series identities in Appendix B, and to reinforce our understanding of the output magnitude properties of a DFT when its input is an exact integer number of sinusoidal cycles:

(a) Prove that when a DFT's input is a complex sinusoid of magnitude A_o (i.e., $x(n)=A_o e^{j2\pi fnt_s}$) with exactly three cycles over N samples, the output magnitude of the DFT's $m = 3$ bin will be $|X(3)| = A_o N$.
Hint: The first step is to redefine $x(n)$'s f and t_s variables in terms of a sample rate f_s and N so that $x(n)$ has exactly three cycles over N samples. The redefined $x(n)$ is then applied to the standard DFT equation.

(b) Prove that when a DFT's input is a real-only sinewave of peak amplitude A_o (i.e., $x(n) = A_o\sin(2\pi fnt_s)$) with exactly three cycles over N samples, the output magnitude of the DFT's m=3 bin will be $|X(3)| = A_oN/2$.
Hint: Once you redefine $x(n)$'s f and t_s variables in terms of a sample rate f_s and N so that $x(n)$ has exactly three cycles over N samples, you must convert that real sinewave to complex exponential form so that you can evaluate its DFT for $m = 3$.

The purpose of this problem is to remind us that DFT output magnitudes are proportional to the size, N, of the DFT. That fact is important in a great many DSP analysis activities and applications.

Answer:

a)

- X(n)=$A_0 e^{j2\pi f n T_s}$
- f is such that there are exactly three cycles in N samples.

To have three cycles in N samples, we set f=3/N cycles per sample, assuming fs=1/Ts:

- f=3fs/N
- X(n)=$A_0 e^{j2\pi 3nfs/(Nfs)}$ = $A_0 e^{j2\pi 3n/N}$

The DFT formula is:

X(k) = $\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi mn/N}$

X(m=3) = $\sum_{n=0}^{N-1} A_0 e^{j2\pi 3n/N}\, e^{-j2\pi 3n/N}$ = $\sum_{n=0}^{N-1} A_0$= $A_0 N$

b)

- X(n)=$A_0 \sin\,(2\pi f n T_s)$
- Again, f=3/N cycles per sample.

Using Euler's formula:

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

X(n) = $A_o \frac{e^{\frac{j2\pi 3n}{N}} - e^{-\frac{j2\pi 3n}{N}}}{2j}$

The DFT of x(n) for m=3:

X(m=3) = $\sum_{n=0}^{N-1} A_o \frac{e^{\frac{j2\pi 3n}{N}} - e^{-\frac{j2\pi 3n}{N}}}{2j}\; e^{j2\pi 3n/N} = A_0/2j(\sum_{n=0}^{N-1} e^{\frac{j2\pi 3n}{N}} - e^{-\frac{j2\pi 3n}{N}} - \sum_{n=0}^{N-1} e^{-j2\pi 3n/N}\, e^{-j2\pi 3n/N}\;)$

X(3) = $\frac{A_0(N-0)}{2j} = \frac{A_0 N}{2j}$

This summation evaluates to zero because it completes multiple full cycles around the unit circle in the complex plane, since -12π is an integer multiple of 2π.

Magnitude:

|X(3)| = $\frac{A_0 N}{2}$

The N/2 factor arises because only one side of the sinusoid contributes to the specific DFT bin at m=3, with the sinusoid's negative counterpart contributing to m=N−3.

ES10:

3.10 Consider performing the 5-point DFT on the following $x_1(n)$ time-domain samples

$$x_1(n) = [1, 2.2, -4, 17, 21],$$

and the DFT's first sample is $X_1(0) = 37.2$. Next, consider performing the 5-point DFT on the following $x_2(n)$ time samples

$$x_2(n) = [1, 2.2, -4, 17, Q],$$

and that DFT's first sample is $X_2(0) = 57.2$. What is the value of Q in the $x_2(n)$ time sequence? Justify your answer.

Answer:

X(0) represents the DC components and is simply the sum of all the samples in the time-domain sequence.

The DC component X1(0) is:

X1(0) = 1+2.2+(-4)+17+21=37.2

This result confirm what expected.

X2(0)=1+2.2+(-4)+17+Q=57.2

Q=57.2 – (1+2.2+(-4)+17)) = 57.2 – 16.2 = 41.