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ES1:

11.1 Assume we have a four-sample $x(n)$ sequence, where index n is $1 \le n \le 4$, whose samples are

$$x(1) = 1,\ x(2) = 2,\ x(3) = 3,\ x(4) = 4.$$

(a) What is the average of $x(n)$?
(b) What is the variance of $x(n)$?
(c) What is the standard deviation of $x(n)$?

Answer:

a)

The average of a sequence is calculated as:

Average = $\left(\frac{1}{N}\right)\sum_{n=1}^{N} x(n) = \left(\frac{1}{4}\right)\left(x(1) + x(2) + x(3) + x(4)\right) = \left(\frac{1}{4}\right)(1 + 2 + 3 + 4) = 2.5$

b)

The variance is a measure of the dispersion of the sequence and is calculated as:

Variance = $\left(\frac{1}{N}\right)\sum_{n=1}^{N}(x(n) - \mu)^2 = 1/4[(1 - 2.5)^2 + (2 - 2.5)^2 + (3 - 2.5)^2 + (4 - 2.5)^2] = 1/4[2.25 + 0.25 + 0.25 + 2.25] = 1.25$

c)

The standard deviation is simply the square root of the variance:
Std = sqrt(Variance) = 1.118.

ES2:

11.2 This problem illustrates an important characteristic of the quantity known as the *average* (mean value) of a sequence of numbers. Suppose we have a six-sample $x(n)$ sequence, where index n is $1 \le n \le 6$, defined by

$$x(1) = 1, x(2) = -2, x(3) = 3, x(4) = -4, x(5) = 6, x(6) = \text{unspecified},$$

and the average of $x(n)$ is $x_{\text{ave}} = 4$. (Note that the sixth sample in $x(n)$ is not explicitly defined.) The difference between $x(n)$ and x_{ave} is the sequence $d_{\text{iff}}(n) = x(n) - x_{\text{ave}}$, given as

$$d_{\text{iff}}(1) = -3, d_{\text{iff}}(2) = -6, d_{\text{iff}}(3) = -1, d_{\text{iff}}(4) = -8, d_{\text{iff}}(5) = 2, d_{\text{iff}}(6) = \text{unspecified}.$$

(a) What is the value of $d_{\text{iff}}(6)$? Justify your answer.

Hint: The discussion of sequence averages in Appendix D's Section D.1 will be helpful here.

(b) What is the value of $x(6)$?

Answer:

a)

The key point is that the sum of all diff(n) values should equal zero because the average is the value around which the sequence values cluster.

$$\sum_{n=1}^{6} diff(n) = 0$$

Summing the given data:

-3 + (-6) + (-1) + (-8) + 2 + diff(6) = 0

Diff(6) = 16

b)

Using the relationship diff(n) = x(n) – x_ave:

Diff(6) = x(6) – x_ave

16 = x(6) – 4

X(6) = 16 + 4 = 20

ES3:

11.3 Let's look at an important topic regarding averaging. Assume we have two N-point discrete sequences, $x(n)$ and $y(n)$, where index n is $1 \le n \le N$, and the N-point averages of the two sequences are

$$x_{\text{ave}} = \frac{1}{N}\sum_{n=1}^{N} x(n) \ , \quad \text{and} \quad y_{\text{ave}} = \frac{1}{N}\sum_{n=1}^{N} y(n) \ .$$

Next, let's add the two sequences, element for element, to obtain a new N-point sequence $z(n) = x(n) + y(n)$. Is it correct to say that the average of $z(n)$, defined as

$$z_{\text{ave}} = \frac{1}{N}\sum_{n=1}^{N} z(n) \ ,$$

is equal to the sum of x_{ave} and y_{ave}? (In different words, we're asking, "Is the average of sums equal to the sum of averages?") Explain how you arrived at your answer.

Note: This problem is *not* "busy work." If the above statement $z_{\text{ave}} = x_{\text{ave}} + y_{\text{ave}}$ is true, it tells us that the average of a noisy signal is equal to the average of the noise-free signal plus the average of the noise.

Answer:

z(n) is expressed as the sum between x(n) and y(n):

z(n) = x(n) + y(n)

Substituting into the average formula:

$$z(n)_{ave} = 1/N\sum_{n=1}^{N} z(n) = 1/N\sum_{n=1}^{N}[x(n) + y(n)] = \frac{1}{N}(\sum_{n=1}^{N} x(n) + \sum_{n=1}^{N} y(n))$$

The two terms above are the definition of $x_{ave}$and y_{ave}:

$$z(n)_{ave} = x_{ave} + y_{ave}$$

Therefore, it is correct to say that the average of z(n) is equal to the sum of x_{ave} and y_{ave}.

ES4:

11.4 Suppose we had three unity-magnitude complex numbers whose phase angles are $\pi/4$ radians, $-3\pi/4$ radians, and $-\pi/4$ radians. What is the average phase angle, measured in degrees, of the three phase angles? Show your work.

Answer:

The solution to this dilemma is to treat the two phase angles as the arguments of two complex numbers, add the two complex numbers, and determine the sum's argument (angle) to obtain the desired average phase angle.

Indeed, the resultant vector is the correct representation of the average phase.

Average phase angle = $\arg[e^{j\pi/4} + e^{-j3\pi/4} + e^{-j\pi/4}]$

The complex addition is performed in rectangular form.

Avg phase angle = arg[(cos(π/4) + jsin(π/4)) + (cos(-3 π/4) + sin(-3 π/4)) + (cos(-π/4) + jsin(-π/4))] =

= arg[(0.707 + j0.707) + (-0.707 -j0.707) + (0.707 -j0.707)] = arg[0.707 – j0.707] = - π /4.

Angle(degree) = - π /4 x(180/ π) = -45°.

ES5:

11.5 Assume we're averaging magnitude samples from multiple FFTs (fast Fourier transforms) and we want the variance of the *averaged FFT magnitudes* to be reduced below the variance of single-FFT magnitudes by a factor of 20. That is, we want

$$\sigma^2_{k\ \text{FFTs}} = \left(\frac{1}{20}\right)\cdot(\sigma^2_{\text{single FFT}}).$$

How many FFTs, k, must we compute and then average their magnitude samples?

Answer:

The variance of the average of k independent FFT magnitudes is reduced by a factor of k compared to the variance of a single FFT magnitude. This is a property of the variance of the mean of k independent and identically distributed random variables.

Thus, if σ^2_{kFFTs} is the variance of a single FFT magnitude, the variance of the average of k FFT magnitudes σ^2_{kFFTs} is:

$\sigma^2_{kFFTs} = \sigma^2_{singleFFT}/k$

Equating the two expressions:

$\sigma^2_{singleFFT}/k = (1/20)\ \sigma^2_{singleFFT}$

Therefore, 1/k=1/20 and finally:

K=20.

We must compute and then average the magnitude samples from 20 FFTs to achieve the desired reduction in variance by a factor of 20.

ES6:

11.6 Concerning the moving averager filters in the text's Figure 11–9, we stated that their transfer functions are equal. Prove that $H_{ma}(z) = H_{rma}(z)$.
Hint: $H_{ma}(z)$ is a geometric series that we'd like to represent as a closed-form equation. To obtain a closed-form equation for a geometric series, start by looking up *geometric series* in the Index.

Answer:

To prove that the transfer functions of the non-recursive moving averager $H_{ma}(z)$ and the recursive moving averager $H_{rma}(z)$ are equal, it is possible to derive their transfer functions and show that they are the same.

Non-recursive Moving Averager $H_{ma}(z)$:

The non-recursive moving average filter of length N is defined by:

$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(n-k)$

In the z-domain, this can be written as:

$Y(z) = \frac{1}{N}\ (X(z) + X(z)z^{-1} + X(z)z^{-2} + \ldots + X(z)z^{-(N-1)})$

$Y(z) = \frac{1}{N} X(z)(1 + z^{-1} + z^{-2} + z^{(N-1)})$

The transfer function $H_{ma}(z)$ is then:

$H_{ma}(z) = Y(z)/X(z) = \frac{1}{N}(1 + z^{-1} + z^{-2} + \ldots + z^{-(N-1)})$

This is a geometric series with a common ratio of z^{-1} and N terms. The sum of this series can be expressed in closed form as:

$H_{ma}(z) = \frac{1}{N} (1\text{-}z^{-N}/1\text{-}z^{-1})$

Recursive Moving Averager $H_{rma}(z)$:

The recursive moving averager filter is defined by:

$y(n) = \frac{1}{N} [x(n) - x(n-N)] + y(n-1)$

Taking the z-transform of both sides:

$Y(z) = \frac{1}{N} X(z) - \frac{1}{N} X(z)\, z^{-N} + Y(z)z^{-1}$

Solving for Y(z):

$Y(z)[1 - z^{-1}] = \frac{1}{N} (X(z) - X(z)\, z^{-N})$

$Y(z)[1 - z^{-1}] = \frac{1}{N} X(z)\, (1 - z^{-N})$

The transfer function $H_{rma}(z)$ is then:

$H_{rma}(z) = Y(z)/X(z) = \frac{1}{N} (1 - z^{-N}) /(1 - z^{-1})$

Equivalence of $H_{ma}(z)$ and $H_{rma}(z)$:

$H_{ma}(z) = \frac{1}{N} (1\text{-}z^{-N}/1\text{-}z^{-1})$

$H_{rma}(z) = \frac{1}{N} (1 - z^{-N}) /(1 - z^{-1})$

Thus, both $H_{ma}(z)$ and $H_{rma}(z)$ are equal, proving that the transfer functions for both the non-recursive and recursive moving average filters are the same.

ES7:

11.7 If we remove the $1/N$ multiplier from the recursive moving averager in the text's Figure 11–9(b), the remaining structure is called a *recursive running sum.* To exercise your digital network analysis skills, plot the frequency magnitude responses of a recursive running sum system for $N = 4$, 8, and 16 as we did in Figure 11–10.

Hint: The frequency response of a recursive running sum network is, of course, the discrete Fourier transform (DFT) of the network's rectangular impulse response. Note that the recursive running sum network's magnitude response curves will be similar, but not equal, to the curves in Figure 11–10.

Answer:

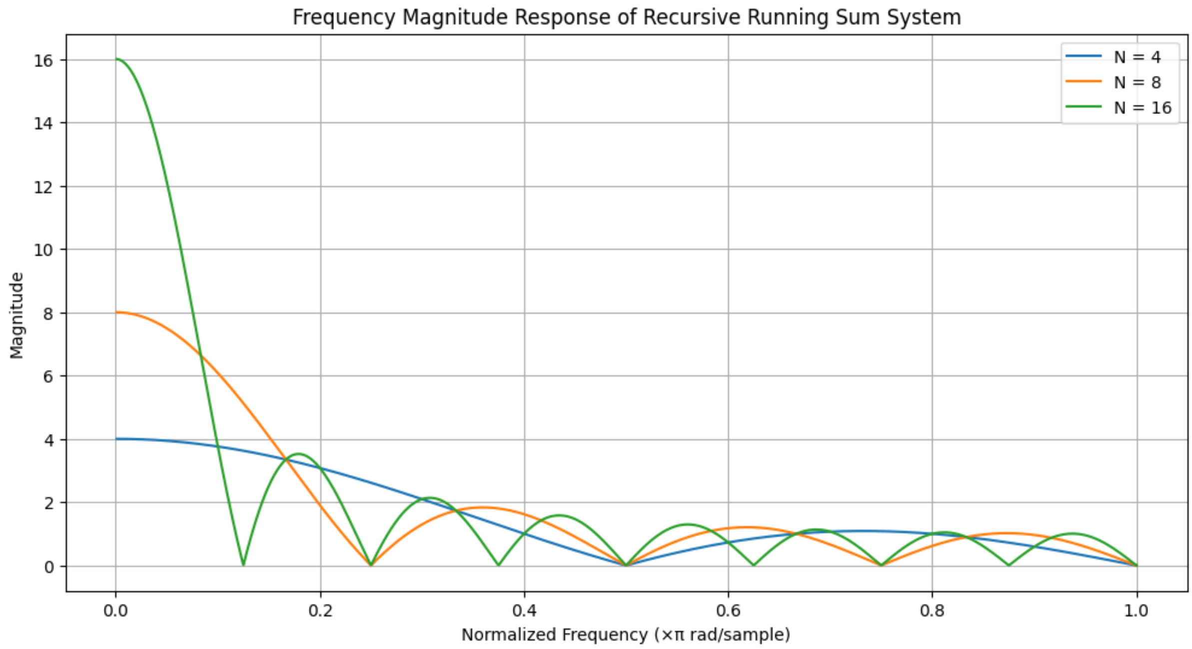
The difference equation of a recursive running sum without the 1/N factor is:

Y[n] = y[n-1] + x[n] – x[n-N]

The impulse response of the recursive running sum system is a rectangular pulse of width N.

The frequency response $H(e^{jw})$ of the recursive running sum is:

$H(e^{jw}) = \frac{1-e^{-jwN}}{1-e^{-jw}}$



As seen, the responses are similar to the moving averager frequency magnitude responses but differ due to the lack of the 1/N normalization factor. The key differences are in the scaling and the null points of the responses.

ES8:

11.8 In the text we said that the phase responses of both nonrecursive and recursive N-point moving averagers are linear. Why is it valid to make that statement?

Answer:

The statement that the phase responses of both nonrecursive and recursive N-point moving averagers are linear is valid because the filters are finite impulse response (FIR) filters with symmetric impulse responses.

The nonrecursive N-point moving averager (MA) filter is essentially a finite impulse response (FIR) filters. FIR filters are known for their ability to have a linear phase response, which means that all frequency components of the input signal are delayed by the same amount of time.

The recursive moving averager (RMA) filter is a infinite impulse response (IIR) filters. Although IIR filters generally do not have a linear phase, the specific structure of the recursive moving averager leads to a linear phase response over the passband. This structure is designed to mimic the behavior of the FIR moving averager, hence inheriting its linear phase characteristics in the frequency range of interest.

Both types of filters are implementations of the moving average process, which inherently produces a linear phase response due to the equal weighting of the input samples over the averaging period.

Nonrecursive MA filter:

The transfer function Hma(z) of a nonrecursive N-point moving averager has an impulse response:

Hma(z) = $\frac{1}{N}\sum_{k=0}^{N-1} z^{-k}$

The frequency response is :

Hma(e^{jw}) = $\frac{1}{N}\sum_{k=0}^{N-1} e^{-jwk}$

This sum is a geometric series with the sum formula:

$$\sum_{k=0}^{N-1} e^{-jwk} = \frac{1-e^{-jwk}}{1-e^{-jw}}$$

This can be simplified using Euler's formula:

Hma(e^{jw}) = $\frac{1}{N}\frac{e^{-\frac{jw(N-1)}{2}}(e^{\frac{jwN}{2}}-e^{-\frac{jwN}{2}})}{e^{-\frac{jw1}{2}}(e^{\frac{jw1}{2}}-e^{-\frac{jw1}{2}})} = \frac{e^{-\frac{jw(N-1)}{2}}2jsin\left(\frac{wN}{2}\right)}{2jsin\left(\frac{w}{2}\right)N} = \frac{e^{-\frac{jw(N-1)}{2}}\sin\left(\frac{wN}{2}\right)}{Nsin\left(\frac{w}{2}\right)}$

The phase response is:

Phase = arg($e^{-jw(N-1)/2}$) = -w(N-1/2)

This is a linear phase response with a constant group delay of N-1/2 samples.

Recursive RMA Filter:

The frequency response Hrma(e^{jw}) is :

Hrma(e^{jw}) = $\frac{1}{N}\frac{1-e^{-jwN}}{(1-e^{-jw})}$

Using the same simplification as before the result is:

Hrma(e^{jw}) = $\frac{e^{-\frac{jw(N-1)}{2}}\sin\left(\frac{wN}{2}\right)}{Nsin\left(\frac{w}{2}\right)}$

The phase response is again:

Phase = arg($e^{-jw(N-1)/2}$) = -w(N-1/2)

his is also a linear phase.

oth the nonrecursive and recursive N-point moving averagers have linear phase responses because their frequency responses are of the form:

$\frac{e^{-\frac{jw(N-1)}{2}}\sin\left(\frac{wN}{2}\right)}{Nsin\left(\frac{w}{2}\right)}$, where the exponential term introduces a linear phase shift.

ES9:

11.9 Draw a rough sketch of the frequency magnitude response, over the positive-frequency range, of a three-point moving averager. Clearly show the frequency magnitude response at $f_s/2$ Hz.

Note: The locations of the frequency response nulls are defined by the locations of the averager's transfer function zeros on its z-plane unit circle.

Answer:

A three-point moving averager (MA) has an impulse response h[n] given by:

H[n] = 1/3 , 0<=n<3.

The transfer function H(z) in the z-domain is:

H(z) = 11/3(1+z^{-1} + z^{-2})

The frequency response H(e^{jw}) is obtained by substituting z = e^{jw}:

H(e^{jw}) = 1/3 (1+ e^{-jw} + e^{j2w})

This can be simplπified using Euler's formula:

$H(e^{jw})$ = 1/3 (1+ cos(w) – jsin(w) + cos(2w) – jsin(2w))

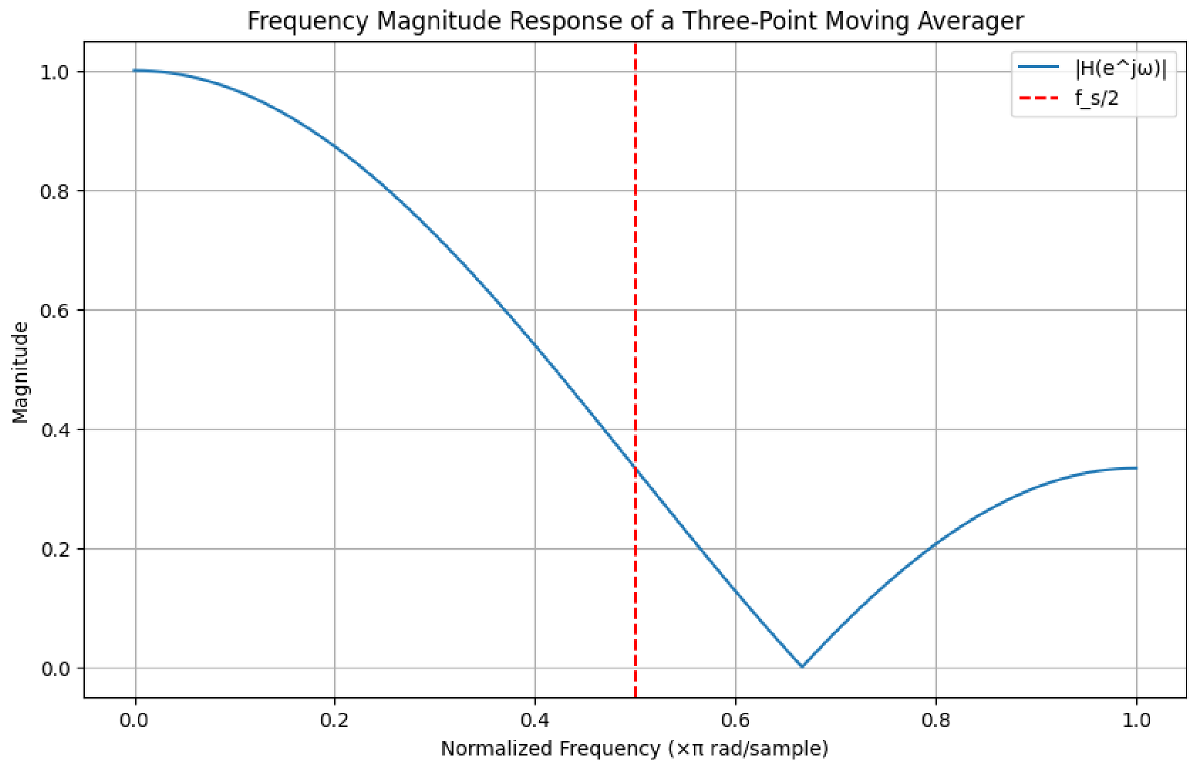
Grouping the real and imaginary parts:

$H(e^{jw})$ = 1/3 (1+ cos(w) + cos(2w)) -j1/3(sin(w) + sin(2w))

The magnitude of the frequency response is:

$|H(e^{jw})|$= 1/3sqrt((1+ cos(w) + cos(2w)) -j1/3(sin(w) + sin(2w))^2)

Evaluating the magnitude at some frequencies:



ES10:

11.10 Think about building a two-stage filter comprising a four-point moving averager in cascade (series) with a two-point moving averager.

(a) Draw a rough sketch of the frequency magnitude response of the two-stage filter.

(b) Does the cascaded filter have a linear phase response? Justify your answer.

Answer:

a)

The impulse response h4[n] of a 4-point moving averager is:

$H4[n] = 1/4$, $0<=n<4$

The transfer function is:

$H4(z) = ¼(1 + z^{-1} + z^{-2} + z^{-3})$

The frequency response is:

$H4(e^{jw}) = ¼(1 + e^{-jw} + e^{-j2w} + e^{-j3w})$

The impulse response h2[n] of a 2-point moving averager is:

$H2[n] = ¼^2$, $0<=n<2$

The transfer function is:

$H2(z) = 1/2(1 + z^{-1})$

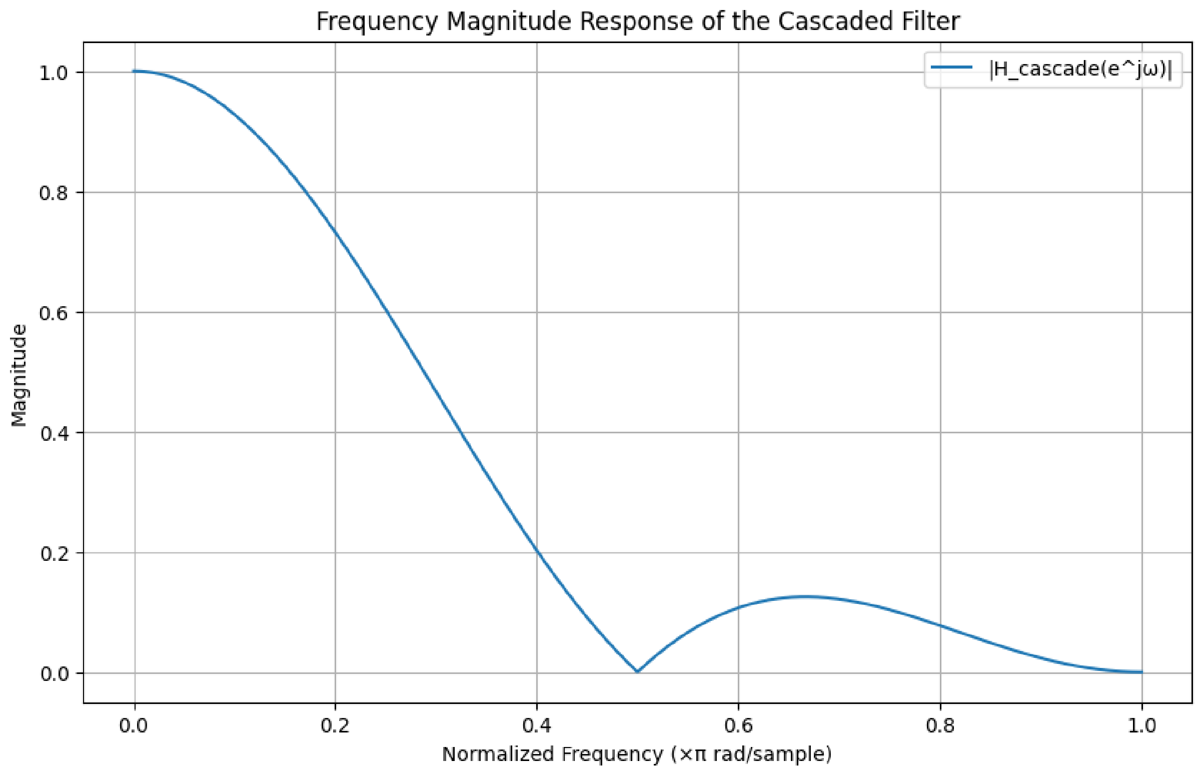
The frequency response is:

$H2(e^{jw}) = 1/2(1 + e^{-jw})$

Cascade frequency response:

The overall frequency response of the cascaded filter is the product of the individual frequency responses:

$Hcascade(e^{jw}) = H4(e^{jw})\ H2(e^{jw})$



b)

To determine if the cascaded filter has a linear phase response, it is needed to analyze the phase response of the individual filters and their combined effect.

The four-point moving averager is a FIR filter with symmetric coefficients [1/4, ¼, ¼, ¼] Symmetric FIR filters have a linear phase response.

The two-point moving averager is also a FIR filter with symmetric coefficients [1/2,1/2]. Symmetric FIR filters have a linear phase response.

Since both the 4-point and 2-point moving averagers are symmetric FIR filters with linear phase responses, their cascade will also have a linear phase response. The phase response of the combined filter will be the sum of the linear phases of the individual filters, which results in an overall linear phase response.