Probability Theory and Statistics

Exercise 2 solutions

09.15. – 09.19.

Event algebra, classical and geometric probability spaces, Poincaré formula

1. A and C are independent iff $\mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C)$. Let us compute the probabilities. $\Omega = \{1, 2, 3, 4, 5, 6\}$, each elementary outcome has probability $\frac{1}{6}$, so we work in the classical model and compute probabilities as *favourable / total*.

$A = \{x \text{ is prime}\} = \{2, 3, 5\} \Longrightarrow \mathbb{P}(A) = \frac{3}{6} = \frac{1}{2}$

$C = \{x \leq 4\} = \{1, 2, 3, 4\} \Longrightarrow \mathbb{P}(C) = \frac{4}{6} = \frac{2}{3}$

$A \cap C = \{x \text{ is prime and } x \leq 4\} = \{2, 3\} \Longrightarrow \mathbb{P}(A \cap C) = \frac{2}{6} = \frac{1}{3}$

Since $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$, the events A and C are independent.

By definition, $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$. Proceeding as above,

$B = \{x \text{ is even}\} = \{2, 4, 6\} \Longrightarrow \mathbb{P}(B) = \frac{3}{6} = \frac{1}{2}$

$A \cap B = \{x \text{ is prime and even}\} = \{2\} \Longrightarrow \mathbb{P}(A \cap B) = \frac{1}{6} \Longrightarrow \mathbb{P}(A \mid B) = \frac{\frac{1}{6}}{\frac{1}{2}} = \dfrac{1}{3}$

Remark. For the second question one can also argue that knowing x is even restricts the sample space to B: $\Omega' = B = \{2, 4, 6\}$. On this new space, compute $\mathbb{P}(A)$ again as favourable/total. There is 1 favourable outcome (2) out of 3 total, hence $\frac{1}{3}$.

2. (a) By the statement, $A_1 \cap A_2 \cap A_3 = \{a\}$, so $\mathbb{P}(\{a\}) = \frac{1}{27}$, and moreover $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = \frac{1}{3}$.

(b) Counterexample: $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(\{a, e\}) = \frac{5}{27} \neq \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) = \frac{81}{729} = \frac{1}{9}$.

3. Let p be the probability that a given ticket has exactly four matches. A ticket can be filled in $\binom{90}{5}$ ways; the number of four-match tickets is $\binom{5}{4}\binom{85}{1}$ (choose which four of the five winning numbers are hit, and which is the fifth number from the 85 non-winning numbers). Thus, in the classical model

$$p = \frac{\binom{5}{4}\binom{85}{1}}{\binom{90}{5}} = \frac{5 \cdot 85}{\binom{90}{5}}.$$

Distinguish the two tickets. Let $A = \{\text{the first ticket has exactly four matches}\}$ and $B = \{\text{the other ticket has exactly four matches}\}$. Then $\mathbb{P}(A) = \mathbb{P}(B) = p$.

Since they are filled in independently, A and B are independent. The required probability is $\mathbb{P}(A \cup B)$.
Two approaches:

- By the inclusion–exclusion (Poincaré) formula,
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = p + p - p^2 = 2p - p^2 = 0.00001934.$$
- Or, $\mathbb{P}(A \cup B) = 1 - \mathbb{P}(\overline{A \cup B})$, and by De Morgan $\overline{A \cup B} = \overline{A} \cap \overline{B}$. Independence implies $\overline{A}$ and $\overline{B}$ are independent, so
$$\mathbb{P}(\overline{A} \cap \overline{B}) = \mathbb{P}(\overline{A}) \cdot \mathbb{P}(\overline{B}) = (1-p)(1-p).$$
Hence $\mathbb{P}(A \cup B) = 1 - (1-p)^2 = 0.00001934$.

Note: Of course $1 - (1-p)^2 = 1 - (1 - 2p + p^2) = 2p - p^2$.

4. The statement gives $\mathbb{P}(A \cup B) = 1$. Using inclusion–exclusion,
$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$
The three probabilities are related via the given conditionals:
$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\,\mathbb{P}(B) = 0.2\,\mathbb{P}(B), \qquad \mathbb{P}(A \cap B) = \mathbb{P}(B \mid A)\,\mathbb{P}(A) = 0.5\,\mathbb{P}(A).$$
Thus $0.2\,\mathbb{P}(B) = 0.5\,\mathbb{P}(A)$, i.e. $\mathbb{P}(B) = \frac{0.5}{0.2}\,\mathbb{P}(A) = \frac{5}{2}\,\mathbb{P}(A)$. Substitute into inclusion–exclusion:
$$1 = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(A) + \tfrac{5}{2}\mathbb{P}(A) - \tfrac{1}{2}\mathbb{P}(A) = 3\,\mathbb{P}(A).$$
Hence $\mathbb{P}(A) = \frac{1}{3}$ and $\mathbb{P}(B) = \frac{5}{2} \cdot \frac{1}{3} = \frac{5}{6}$.
$$\mathbb{P}(A \mid \overline{B}) = \frac{\mathbb{P}(A \cap \overline{B})}{\mathbb{P}(\overline{B})} = \frac{\mathbb{P}(\overline{B} \mid A)\,\mathbb{P}(A)}{\mathbb{P}(\overline{B})} = \frac{\left[1 - \mathbb{P}(B \mid A)\right]\mathbb{P}(A)}{1 - \mathbb{P}(B)} = \frac{(1-0.5)\cdot\frac{1}{3}}{1 - \frac{5}{6}} = \frac{\frac{1}{6}}{\frac{1}{6}} = 1.$$
A and B are independent iff $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. The RHS is $\frac{1}{3} \cdot \frac{5}{6} = \frac{5}{18}$, while from above $\mathbb{P}(A \cap B) = 0.5 \cdot \mathbb{P}(A) = \frac{1}{6} \neq \frac{5}{18}$. Thus they are not independent.

5. "An odd number of A, B, C occurs" means that the only possible cases are
$$A\bar{B}\bar{C}, \quad \bar{A}B\bar{C}, \quad \bar{A}\bar{B}C, \quad ABC,$$
while the events "none" and "exactly two" have probability 0. In particular,
$$\mathbb{P}(A \cap B \cap \bar{C}) = \mathbb{P}(A \cap C \cap \bar{B}) = \mathbb{P}(B \cap C \cap \bar{A}) = 0,$$

so

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \cap C) = \mathbb{P}(B \cap C) = \mathbb{P}(A \cap B \cap C) = \tfrac{1}{10}.$$

Let $x = \mathbb{P}(C)$. Then $\mathbb{P}(A) = 3x$, $\mathbb{P}(B) = 2x$, and since $\mathbb{P}(\bar{A} \cap \bar{B} \cap \bar{C}) = 0$, we have $\mathbb{P}(A \cup B \cup C) = 1$. By inclusion–exclusion,

$$1 = \mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(AB) - \mathbb{P}(AC) - \mathbb{P}(BC) + \mathbb{P}(ABC) = (3x + 2x + x) - 3 \cdot \tfrac{1}{10} + \tfrac{1}{10}.$$

Hence $1 = 6x - \frac{2}{10}$, so $6x = \frac{12}{10}$ and therefore $x = \mathbb{P}(C) = \frac{1}{5}$. Consequently

$$\mathbb{P}(A) = \tfrac{3}{5}, \qquad \mathbb{P}(B) = \tfrac{2}{5}, \qquad \mathbb{P}(A \cap B \cap C) = \tfrac{1}{10}.$$

Now the required conditional probabilities are

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\frac{1}{10}}{\frac{2}{5}} = \frac{1}{4}, \qquad \mathbb{P}(B \mid C) = \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)} = \frac{\frac{1}{10}}{\frac{1}{5}} = \frac{1}{2}, \qquad \mathbb{P}(C \mid A) = \frac{\mathbb{P}(C \cap A)}{\mathbb{P}(A)} =$$

Answer: $\boxed{\mathbb{P}(A \mid B) = \tfrac{1}{4},\ \mathbb{P}(B \mid C) = \tfrac{1}{2},\ \mathbb{P}(C \mid A) = \tfrac{1}{6}.}$

6. Introduce the notation:

$$A = \{\text{both dice show even}\}, \qquad B = \{\text{the sum is at least } 10\}.$$

We seek $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$. The sample space is $\Omega = \{(i,j) : i, j = 1, 2, 3, 4, 5, 6\}$ with $|\Omega| = 36$ and equal probabilities.

$B = \{(6,4), (4,6), (5,5), (5,6), (6,5), (6,6)\}$, so $|B| = 6$ and $\mathbb{P}(B) = \frac{6}{36} = \frac{1}{6}$.

$A \cap B = \{(6,4), (4,6), (6,6)\}$, so $|A \cap B| = 3$ and $\mathbb{P}(A \cap B) = \frac{3}{36} = \frac{1}{12}$.

Therefore $\mathbb{P}(A \mid B) = \dfrac{\frac{1}{12}}{\frac{1}{6}} = \dfrac{1}{2}$.

7. Choose the elementary outcomes so they are equally likely by distinguishing the order of flips:

$$\Omega = \{FFF, FFH, FHF, HFF, HHF, HFH, FHH, HHH\},$$

where H denotes heads and F tails. Then

$$A = \{FFH, FHF, HFF, HHF, HFH, FHH\}, \qquad B = \{FFF, FFH, FHF, HFF\}.$$

Thus $\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{6}{8} = \frac{3}{4}$, $\mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{4}{8} = \frac{1}{2}$.

Independence holds iff $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. Now $A \cap B = \{FFH, FHF, HFF\}$ so $\mathbb{P}(A \cap B) = \frac{3}{8}$, and indeed $\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$. Hence A and B are independent.

8. (a) The required probability is $\mathbb{P}(A \cap B \cap C)$, which by mutual independence equals $\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = 0.096$.
(b) *Solution 1.* We want $\mathbb{P}(A \cup B \cup C)$. By inclusion–exclusion and mutual independence,

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C) \\ &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(B)\mathbb{P}(C) + 0.096 \\ &= 0.916.\end{aligned}$$

Solution 2. $\mathbb{P}(A \cup B \cup C) = 1 - \mathbb{P}(\overline{A} \cap \overline{B} \cap \overline{C})$. Complements of mutually independent events are also mutually independent, so

$$\mathbb{P}(\overline{A} \cap \overline{B} \cap \overline{C}) = \mathbb{P}(\overline{A})\mathbb{P}(\overline{B})\mathbb{P}(\overline{C}) = (1-\mathbb{P}(A))(1-\mathbb{P}(B))(1-\mathbb{P}(C)) = 0.7 \cdot 0.6 \cdot 0.2 = 0.084,$$

hence $\mathbb{P}(A \cup B \cup C) = 0.916$.
(c) The probability is $\mathbb{P}(\overline{A} \cap \overline{B} \cap \overline{C}) = 0.084$, as just computed.

9. Let A_i ($1 \leq i \leq 4$) be the event that up to the i-th request, all requests have been assigned to distinct (still free) servers. Clearly, $\mathbb{P}(A_1) = \frac{6}{6} = 1$. For larger i, the conditionals are easy:

- $\mathbb{P}(A_2 \mid A_1) = \frac{5}{6}$ (five free servers remain),
- $\mathbb{P}(A_3 \mid A_1 \cap A_2) = \frac{4}{6}$,
- $\mathbb{P}(A_4 \mid A_1 \cap A_2 \cap A_3) = \frac{3}{6}$.

By the multiplication rule,

$$\mathbb{P}(A_4 \cap A_3 \cap A_2 \cap A_1) = \mathbb{P}(A_4 \mid A_1 \cap A_2 \cap A_3)\,\mathbb{P}(A_3 \mid A_1 \cap A_2)\,\mathbb{P}(A_2 \mid A_1)\,\mathbb{P}(A_1).$$

Since $A_j \subseteq A_i$ for $j > i$, we have $A_j \cap A_i = A_j$, hence $\mathbb{P}(A_4 \cap A_3 \cap A_2 \cap A_1) = \mathbb{P}(A_4)$. Therefore

$$\mathbb{P}(A_4) = \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} = \frac{5}{18}.$$

10. Computing the probability of rolling a six is easy once we know how many rolls were made, which depends on whether the drawn card was a spade. Thus it is natural to apply the *law of total probability* with the partition {Spade, Not spade}:

$$\mathbb{P}(\text{At least one six}) = \mathbb{P}(\text{At least one six} \mid \text{Spade})\mathbb{P}(\text{Spade}) + \mathbb{P}(\text{At least one six} \mid \text{Not spade})\mathbb{P}(\text{Not spa}$$

The needed probabilities are:

$$\mathbb{P}(\text{Spade}) = \frac{1}{4},$$
$$\mathbb{P}(\text{Not spade}) = 1 - \mathbb{P}(\text{Spade}) = \frac{3}{4},$$
$$\mathbb{P}(\text{At least one six} \mid \text{Spade}) = \frac{1}{6} \quad \text{(one roll)}.$$

For the last conditional (two rolls), it is easier via the complement:

$$\begin{aligned}\mathbb{P}(\text{At least one six} \mid \text{Not spade}) &= 1 - \mathbb{P}(\text{No six in two rolls}) \\ &= 1 - \mathbb{P}(\text{1st not six}) \cdot \mathbb{P}(\text{2nd not six}) = 1 - \frac{5}{6} \cdot \frac{5}{6} = \frac{11}{36}.\end{aligned}$$

Substituting,

$$\mathbb{P}(\text{At least one six}) = \frac{1}{6} \cdot \frac{1}{4} + \frac{11}{36} \cdot \frac{3}{4} = \frac{13}{48}.$$

11. Let A_i ($i = 1, 2, 3$) be the event that for game i we draw exactly 1 new and 2 used balls. Clearly,

$$\mathbb{P}(A_1) = \frac{\binom{9}{1}\binom{6}{2}}{\binom{15}{3}},$$

since favourable choices pick 1 of the 9 unused and 2 of the 6 used, out of all $\binom{15}{3}$ triples.

To speak about game 2 we need to know what happened in game 1. While $\mathbb{P}(A_2)$ is awkward, the conditional $\mathbb{P}(A_2 \mid A_1)$ is easy: after A_1, there are 8 unused and 7 used, so

$$\mathbb{P}(A_2 \mid A_1) = \frac{\binom{8}{1}\binom{7}{2}}{\binom{15}{3}}.$$

If A_1 and A_2 both occur, then for game 3 we have 7 unused and 8 used, hence

$$\mathbb{P}(A_3 \mid A_1 \cap A_2) = \frac{\binom{7}{1}\binom{8}{2}}{\binom{15}{3}}.$$

We need $\mathbb{P}(A_1 \cap A_2 \cap A_3)$, which by the multiplication rule equals

$$\mathbb{P}(A_3 \mid A_1 \cap A_2) \cdot \mathbb{P}(A_2 \mid A_1) \cdot \mathbb{P}(A_1) = \frac{\binom{7}{1}\binom{8}{2}}{\binom{15}{3}} \cdot \frac{\binom{8}{1}\binom{7}{2}}{\binom{15}{3}} \cdot \frac{\binom{9}{1}\binom{6}{2}}{\binom{15}{3}} = 0.0472.$$

12. We apply Bayes' theorem. Let A_i be the event that exactly i sources say the statement is true ($i = 0, 1, 2$). Then A_0, A_1, A_2 form a partition, and by independence

$$\mathbb{P}(A_2) = \tfrac{1}{4}, \qquad \mathbb{P}(A_1) = \tfrac{1}{2} \text{ (since TF has prob. } \tfrac{1}{4} \text{ and FT has prob. } \tfrac{1}{4}), \qquad \mathbb{P}(A_0) = \tfrac{1}{4}.$$

Let B be the event that the AI outputs "true". Conditional on both sources being true we certainly get "true", so $\mathbb{P}(B \mid A_2) = 1$. Conditional on exactly one source being true, the AI outputs "true" with probability 1/2, hence $\mathbb{P}(B \mid A_1) = \frac{1}{2}$. If neither source is true, the AI outputs "false", so $\mathbb{P}(B \mid A_0) = 0$.

Therefore, by Bayes' theorem,

$$\mathbb{P}(A_2 \mid B) = \frac{\mathbb{P}(B \mid A_2)\mathbb{P}(A_2)}{\mathbb{P}(B \mid A_2)\mathbb{P}(A_2) + \mathbb{P}(B \mid A_1)\mathbb{P}(A_1) + \mathbb{P}(B \mid A_0)\mathbb{P}(A_0)} = \frac{1 \cdot \frac{1}{4}}{1 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} + 0 \cdot \frac{1}{4}} = \frac{1}{2}.$$

(Of course, the term $\mathbb{P}(B \mid A_0)\mathbb{P}(A_0)$ can be omitted from the denominator since it equals 0.)

13. Let $H = \{\text{answers correctly}\}$ and $T = \{\text{knows the answer}\}$. By Bayes' theorem we can write $\mathbb{P}(T \mid H)$ as

$$\mathbb{P}(T \mid H) = \frac{\mathbb{P}(H \mid T)\mathbb{P}(T)}{\mathbb{P}(H \mid T)\mathbb{P}(T) + \mathbb{P}(H \mid \overline{T})\mathbb{P}(\overline{T})} = \frac{1 \cdot p}{1 \cdot p + \frac{1}{3} \cdot (1-p)} = \frac{3p}{2p+1}.$$

Here $\mathbb{P}(T) = p$, $\mathbb{P}(\overline{T}) = 1 - p$, $\mathbb{P}(H \mid T) = 1$, and $\mathbb{P}(H \mid \overline{T}) = \frac{1}{3}$ (uniform guessing among three choices). Substituting $p = \frac{1}{4}$ gives $\mathbb{P}(T \mid H) = \frac{1}{2}$.

(a) Let A_i be the event that the die shows i and let B be the event that we get 0 heads. We seek $\mathbb{P}(B)$. Since $A_1, \ldots, A_6$ form a partition of the sample space, we may use the law of total probability:

$$\mathbb{P}(B) = \mathbb{P}(B \mid A_1)\mathbb{P}(A_1) + \cdots + \mathbb{P}(B \mid A_6)\mathbb{P}(A_6).$$

For any $i = 1, \ldots, 6$ we have $\mathbb{P}(A_i) = \frac{1}{6}$ and $\mathbb{P}(B \mid A_i) = \frac{1}{2^i}$ (each coin flip is fair and independent). Hence

$$\mathbb{P}(B) = \sum_{i=1}^{6} \frac{1}{2^i} \cdot \frac{1}{6} = \frac{1}{6} \sum_{i=1}^{6} \frac{1}{2^i} = \frac{1}{6} \cdot \frac{\frac{1}{2}\left(1 - (\frac{1}{2})^6\right)}{1 - \frac{1}{2}} = \frac{21}{128}.$$

(b) The required probability is $\mathbb{P}(A_6 \mid B)$. By Bayes' theorem,

$$\mathbb{P}(A_6 \mid B) = \frac{\mathbb{P}(B \mid A_6)\mathbb{P}(A_6)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B \mid A_6)\mathbb{P}(A_6)}{\mathbb{P}(B \mid A_1)\mathbb{P}(A_1) + \cdots + \mathbb{P}(B \mid A_6)\mathbb{P}(A_6)} = \frac{\frac{1}{384}}{\frac{21}{128}} = \frac{1}{63}.$$