Probability Theory and Statistics
Lecture 5

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Remainder Random Variables

Definition

A (one-dimensional) random variable is a function $X : \Omega \to \mathbb{R}$ that assigns to each outcome $\omega \in \Omega$ a real number $X(\omega) \in \mathbb{R}$.

Binomial distribution

Definition

A random variable X has the binomial distribution with parameters $n \in$ and $p \in [0,1]$ if

$$p_X(k) = \mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad \forall k \in \{0, 1, \ldots, n\}.$$

Notation: $X \sim \mathrm{B}(n; p)$.

Geometric distribution

Definition

A random variable X has the geometric distribution with parameter $p \in (0, 1)$ if

$$p_X(k) = \mathbb{P}(X = k) = (1 - p)^{k-1} p, \qquad \forall k \in \{1, 2, \ldots\}. \tag{1}$$

Notation: $X \sim \mathrm{Geo}(p)$.

Poisson Distribution

The binomial distribution models precisely the situation where we know how many trials we perform and the success probability in each.

There are situations where a certain random quantity X can also be viewed as the number of successes in many repeated and independent trials, but:

- the number of trials is large,
- the success probability in each trial is small,
- and neither the number of trials nor the success probability is necessarily known exactly.

Examples:

- the number of people older than 100 living in Budapest,
- the number of phone calls in Hungary between 4 and 5 pm today,
- the number of shooting stars observed in an hour on an August evening,
- further (more or less good) examples $\rightarrow$ in the exercise class.

In such cases X is approximately Poisson distributed.

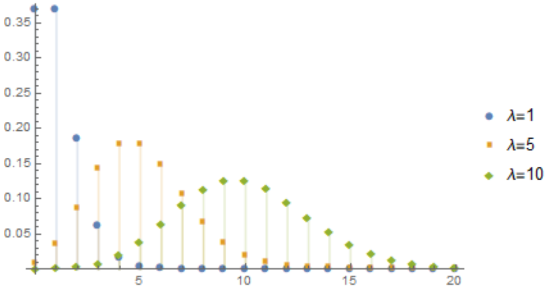
Poisson Distribution

Definition

A random variable X has the Poisson distribution with parameter $\lambda > 0$ if

$$p_X(k) = \mathbb{P}(X = k) = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, \qquad \forall k \in \{0, 1, 2, \ldots\}.$$

Notation: $X \sim \mathrm{Pois}(\lambda)$.



Property of Poisson

Theorem

The Poisson distribution is indeed a pmf.

(Proof)

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Intuition (see previous slide): the Poisson distribution is like a binomial distribution with „large n and small p".
This is not just a heuristic; it can be made mathematically precise (including how to choose p as a function of n), for which it helps to already know the notion of expectation. We will return to this soon.

Poisson Approximation to the Binomial Distribution

Recall: for many independent trials with identical success probability, if there are n trials and each succeeds with probability p, then the number of successes $\sim \mathrm{B}(n; p)$.

Questions:

- For large n and a suitably small, n-dependent $p = p_n$, can the distribution $\mathrm{B}(n; p_n)$ indeed be approximated by some $\mathrm{Pois}(\lambda)$? This would greatly simplify calculations, since we would not need to evaluate huge binomial coefficients (even when n is known exactly).
- If yes: how should we choose p_n; how should p_n tend to 0 as $n \to \infty$?
- What will the value of λ be (as a function of the p_n's)?

We have seen: if $X_n \sim \mathrm{B}(n; p_n)$, then $\mathbb{E}(X_n) = np_n$.
If $X \sim \mathrm{Pois}(\lambda)$, then $\mathbb{E}(X) = \lambda$.
Intuition: choosing $p_n = \lambda/n$ should work (this keeps the expectation unchanged).

Poisson Approximation to the Binomial Distribution

Theorem

Let n be a positive integer, $\lambda \in (0, \infty)$, and set $p_n = \lambda/n$. Then

$$\lim_{n\to\infty} \binom{n}{k} p_n^k (1 - p_n)^{n-k} = \frac{\lambda^k}{k!} \mathrm{e}^{-\lambda}, \qquad k \in \{0, 1, 2, \dots\}.$$

(Proof)

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(Proof)

The same proof works when p_n is not exactly λ/n but satisfies $np_n \to \lambda$. For example, the statement also holds for $p_n = \frac{\lambda}{n-42}$ or $p_n = \frac{\lambda}{n+\ln n}$. We may add to or subtract from n any quantity whose ratio to n tends to 0 as $n \to \infty$.

Expected Value — Discrete Random Variables

The expected value of a simple random variable is

$$\mathbb{E}(X) \stackrel{\mathrm{def}}{=} \sum_{k \in \mathrm{Ran}(X)} k\mathbb{P}(X = k) = \sum_{k \in \mathrm{Ran}(X)} k p_X(k). \tag{2}$$

Question: does the same formula work for an *arbitrary* discrete random variable X?

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Question: does the same formula work for an *arbitrary* discrete random variable X? Answer: yes, provided the expected value exists! **Example:**

Roll a fair die and let $X := \{\text{rolled number}\}$.

$$\mathbb{E}(X) = 3.5$$

Expected Value — Discrete Random Variables

Definition

Let X be a discrete random variable such that

$$\sum_{k \in \mathrm{Ran}(X)} |k| \, p_X(k) < \infty.$$

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In most cases $\mathrm{Ran}(X) \subseteq \mathbb{N}$, hence the absolute value does not matter.

Expected Value — Geometric and Poisson Distributions

Theorem

Let $X \sim \mathrm{Bin}(n; p)$ with $p \in (0, 1)$. Then $\mathbb{E}(X) = np$.

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(Proof)

Theorem

Let $X \sim \mathrm{Pois}(\lambda)$ *with* $\lambda > 0$. *Then* $\mathbb{E}(X) = \lambda$.

(Proof)

Further Properties of the Expected Value

Theorem

The linearity of expectation ($\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$, $\mathbb{E}(cX) = c\,\mathbb{E}(X)$, cf. Claim 3.2.3) holds for arbitrary (not necessarily discrete) random variables X, Y whose expected values exist.

(Proof)

For continuous random variables $\rightarrow$ the method of computing expectation will differ from the discrete case (see later), but linearity still holds.

Further Properties of the Expected Value

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(Proof)

For continuous random variables $\rightarrow$ the method of computing expectation will differ from the discrete case (see later), but linearity still holds.

Other easy properties: for a (not necessarily discrete) random variable X and real numbers $-\infty < a < b < \infty$,

- if $\mathbb{P}(a \leq X \leq b) = 1$, then $a \leq \mathbb{E}(X) \leq b$,
- if $\mathbb{P}(X \geq a) = 1$, then $\mathbb{E}(X)$ is definable (in the worst case with value $+\infty$) and $\mathbb{E}(X) \geq a$.
  Special case: if X is nonnegative, then $\mathbb{E}(X) \in [0, \infty]$,