Probability Theory and Statistics

Exercise 2

09.15-09.19.

Independence of events, conditional probability, law of total probability, Bayes' formula

1. We roll a fair die, let the outcome be x. Define the events

$$A = \{x \text{ is prime}\}, \qquad B = \{x \text{ is even}\}, \qquad C = \{x \leq 4\}.$$

Are A and C independent? If we know that x is even, what is the probability that it is prime? That is, what is $\mathbb{P}(A \mid B)$?

2. Let $\Omega = \{a, b, c, d, e\}$, with probabilities $\mathbb{P}(a) = \frac{1}{27}$, $\mathbb{P}(b) = \frac{14}{27}$, $\mathbb{P}(c) = \mathbb{P}(d) = \mathbb{P}(e) = \frac{4}{27}$, and

$$A_1 = \{d, e, a\}, \qquad A_2 = \{c, e, a\}, \qquad A_3 = \{c, d, a\}.$$

1. Show that $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$.
2. Show that A_1, A_2, A_3 are not independent.

3. What is the probability that out of two lottery tickets filled in independently, at least one of them has exactly four matches? (We choose 5 numbers from 90.)

4. At least one of the events A and B always occurs. If $\mathbb{P}(A \mid B) = 0.2$ and $\mathbb{P}(B \mid A) = 0.5$, compute $\mathbb{P}(A)$, $\mathbb{P}(B)$ and $\mathbb{P}(A \mid \overline{B})$. Are A and B independent?

5. Out of the events A, B, C, always an odd number occurs simultaneously. If $\mathbb{P}(A) = 3\mathbb{P}(C)$, $\mathbb{P}(B) = 2\mathbb{P}(C)$ and $\mathbb{P}(A \cap B \cap C) = \frac{1}{10}$, compute the conditional probabilities $\mathbb{P}(A \mid B)$, $\mathbb{P}(B \mid C)$ and $\mathbb{P}(C \mid A)$.

6. Compute the conditional probability that both values are even when rolling two dice, given that their sum is at least ten.

7. We flip a fair coin three times. Let A be the event that both heads and tails appear, and B the event that at most one tail appears. Are A and B independent?

8. Suppose A, B, C are mutually independent events with $\mathbb{P}(A) = 0.3$, $\mathbb{P}(B) = 0.4$, $\mathbb{P}(C) = 0.8$. Compute the probabilities of the following events:

1. all three events occur,
2. at least one of the three events occurs,
3. none of the events occurs.

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9. A proxy distributes incoming requests to 6 servers, each chosen independently and uniformly at random. With 4 concurrent requests, what is the probability that they are routed to different servers?

10. First, we draw a card from a standard 52-card deck. If it is a spade, we roll a fair die once; otherwise, we roll it twice. What is the probability that a six is rolled?

11. A box contains 15 tennis balls, of which 9 are unused. For three games, we randomly take out three balls for each game, then put them back afterwards (if an unused ball was taken, it becomes used during the game). What is the probability that in all three draws we end up with one new and two used balls?

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12. A hypothetical artificial intelligence is asked whether a certain statement is true or false. We know that the statement is true. The AI searches online, and responds „true" with probability equal to the proportion of „true" answers it encounters. There are exactly two sources online, each of which independently gives the answer „true" or „false" with probability 1/2. Given that the AI answered correctly, what is the probability that both sources contained the correct answer?

13. A test question has three possible answers. A hypothetical student knows the correct answer with probability p, otherwise guesses uniformly at random. Given that the student answered correctly, what is the probability that they actually knew the answer? What if $p = \frac{1}{4}$?

We roll a fair die, then flip a fair coin as many times as the die shows.

1. What is the probability that no heads occur?
2. Given that no heads occurred, what is the probability that the die showed 6?