Probability Theory and Statistics

Exercise 3 solutions

09.22. – 09.26.

Combinatorics, Discrete Random Variables, Binomial-, Poisson-, Geometric distribution

1.

1. **How many 5-person committees can be formed?**

   Solution. Choose any 5 of the 12:
$$\binom{12}{5} = 792.$$

2. **How many 5-person committees contain at least two seniors?**

   Solution. Sum over the possible numbers of seniors (2, 3, or 4):
$$\binom{4}{2}\binom{8}{3} + \binom{4}{3}\binom{8}{2} + \binom{4}{4}\binom{8}{1} = 6 \cdot 56 + 4 \cdot 28 + 1 \cdot 8 = 336 + 112 + 8 = 456.$$
   (Equivalently, $\binom{12}{5} - \binom{8}{5} - \binom{4}{1}\binom{8}{4} = 792 - 56 - 280 = 456$.)

3. **Two particular students, Alice and Bob, refuse to serve together. How many 5-person committees can be formed?**

   Solution. Subtract those containing both Alice and Bob:
$$\binom{12}{5} - \binom{10}{3} = 792 - 120 = 672.$$

2. An ice-cream shop sells 7 flavors. You order a cup with 10 scoops (scoops indistinguishable except for flavor; order does not matter).

1. **How many different cups can you order?**

   Solution. Stars and bars: number of nonnegative solutions to $x_1 + \cdots + x_7 = 10$:
$$\binom{10+7-1}{7-1} = \binom{16}{6} = 8008.$$

2. **How many different cups if the cup must contain at least one vanilla scoop?**

   Solution. Require $x_{\text{vanilla}} \geq 1$. Let $y_{\text{vanilla}} = x_{\text{vanilla}} - 1$, then $y_{\text{vanilla}} + x_2 + \cdots + x_7 = 9$:
$$\binom{9+7-1}{7-1} = \binom{15}{6} = 5005.$$

(Equivalently, $8008 - \binom{15}{5} = 8008 - 3003 = 5005$.)

3. **How many different cups if no single flavor is used more than 3 times?**

   Solution. Count solutions to $x_1 + \cdots + x_7 = 10$ with $0 \le x_i \le 3$ by inclusion–exclusion.
   Total without upper bounds: $\binom{16}{6} = 8008$.
   Let A_i be $x_i \ge 4$. Then
   $$|A_i| = \binom{10-4+7-1}{7-1} = \binom{12}{6} = 924, \quad \sum |A_i| = 7 \cdot 924 = 6468.$$
   For pairs $A_i \cap A_j$ ($x_i, x_j \ge 4$): subtract 8, solve sum = 2:
   $$\binom{2+7-1}{7-1} = \binom{8}{6} = 28, \quad \sum |A_i \cap A_j| = \binom{7}{2} \cdot 28 = 21 \cdot 28 = 588.$$
   Triples would require ≥ 12 scoops, impossible for total 10.
   Thus
   $$8008 - 6468 + 588 = 2128.$$

3. **(a)** The sample space has size $2^3 = 8$, and the probability space is classical (uniform). Hence probabilities can be computed in "favorable/total" form.
$\{X = 1\} = \{\text{the first symbol is „tails”}\} = \{IIF, III, IFI, IFF\}$, so $\mathbb{P}(X = 1) = 4/8 = 1/2$. One can also argue that the first symbol is tails with the same probability as heads, and if it is tails, then $\{X = 1\}$ occurs (otherwise it does not).
$\{X = 3\} = \{\text{the first two symbols are not „tails”, but the third is}\} = \{FFI\}$, hence $\mathbb{P}(X = 3) = 1/8$.
Since $\{X \text{ is odd}\} = \{X = 1\} \cup \{X = 3\}$ and the events $\{X = 1\}$ and $\{X = 3\}$ are disjoint,
$$\mathbb{P}(X \text{ is odd}) = \mathbb{P}(X = 1) + \mathbb{P}(X = 3) = \frac{5}{8}.$$

(b) Y is *not* a random variable, because if it were, then $Y\colon \Omega \to \mathbb{R}$ would be a function, i.e., $Y(FFF)$ would be a fixed real number (not depending on any further randomness).

4. Since X is the number of sixes in two rolls, $\mathrm{Ran}(X) = \{0, 1, 2\}$. Thus
$$\mathbb{P}(X \text{ is even}) = \mathbb{P}(\{X=0\} \cup \{X=2\}) = \mathbb{P}(X=0) + \mathbb{P}(X=2) = \mathbb{P}(\text{no six}) + \mathbb{P}(\text{two sixes}) = \frac{25}{36} + \frac{1}{36} =$$

(Here we used that $\mathbb{P}$(no six) and $\mathbb{P}$(two sixes) were already computed earlier.)

5. The event $\{0 < Y < 3\}$ means that among A, B, C at least one occurs but not all, i.e., we seek the probability of $A \cup B \cup C \setminus (A \cap B \cap C)$:

$$\mathbb{P}(A \cup B \cup C \setminus A \cap B \cap C) = \mathbb{P}(A \cup B \cup C) - \mathbb{P}(A \cap B \cap C).$$

By the inclusion–exclusion (Poincaré) formula for three events,

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(C \cap A) + \mathbb{P}(A \cap B \cap C) \\ &= 0.5 + 0.4 + 0.3 - 0.3 - 0.2 - 0.1 + 0.1 = 0.7.\end{aligned}$$

Hence $\mathbb{P}(0 < Y < 3) = \mathbb{P}(A \cup B \cup C) - \mathbb{P}(A \cap B \cap C) = 0.7 - 0.1 = 0.6$.

$\{Y = 0\} = \overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$.

Therefore $p_Y(0) = \mathbb{P}(Y = 0) = \mathbb{P}(\overline{A \cup B \cup C}) = 1 - \mathbb{P}(A \cup B \cup C) = 1 - 0.7 = 0.3$ (using the union computed above).

$\{Y = 3\} = A \cap B \cap C \implies p_Y(3) = \mathbb{P}(Y = 3) = \mathbb{P}(A \cap B \cap C) = 0.1$.

$\{Y = 2\} = A \cap B \cap \overline{C} \ \cup \ A \cap \overline{B} \cap C \ \cup \ \overline{A} \cap B \cap C$. Since these are disjoint, the probability of the union is the sum:

$$\mathbb{P}(Y = 2) = \mathbb{P}(A \cap B \cap \overline{C}) + \mathbb{P}(A \cap \overline{B} \cap C) + \mathbb{P}(\overline{A} \cap B \cap C).$$

Each term is obtained by subtracting the triple intersection from the corresponding pairwise intersection:

$$\mathbb{P}(A \cap B \cap \overline{C}) = \mathbb{P}(A \cap B) - \mathbb{P}(A \cap B \cap C) = 0.3 - 0.1 = 0.2,$$

$$\mathbb{P}(A \cap \overline{B} \cap C) = \mathbb{P}(C \cap A) - \mathbb{P}(A \cap B \cap C) = 0.1 - 0.1 = 0,$$

$$\mathbb{P}(\overline{A} \cap B \cap C) = \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B \cap C) = 0.2 - 0.1 = 0.1.$$

Thus $p_Y(2) = \mathbb{P}(Y = 2) = 0.2 + 0 + 0.1 = 0.3$.

Since the possible values of Y are $0, 1, 2, 3$ and exactly one of these occurs,

$$p_Y(1) = 1 - p_Y(0) - p_Y(2) - p_Y(3) = 0.3.$$

6. Model the sample space as $\Omega := [10]^2 = \{(x, y) \in \mathbb{R}^2 \mid x, y \in \{1, 2, \ldots, 10\}\}$ (and take $\mathcal{F} = \mathcal{P}(\Omega)$). For each $(x, y) \in \Omega$, set $X(x, y) = x$ and $Y(x, y) = y$. Then

$$\{X \leq Y\} = \{(x, y) \in \Omega \mid X(x, y) \leq Y(x, y)\} = \{(x, y) \in \Omega \colon x \leq y\}.$$

The favorable outcomes are those (x, y) with $x \le y$. If $x = 1$ there are 10 possibilities $(1, 1), \dots, (1, 10)$; if $x = 2$ there are 9; $\dots$; if $x = 10$ there is 1, namely $(10, 10)$. Hence there are

$$10 + 9 + \cdots + 1 = \frac{11 \cdot 10}{2} = 55$$

favorable cases. The total number of cases is $10^2 = 100$, and the probability space is classical, so the desired probability is

$$\frac{55}{100} = \frac{11}{20}.$$

Remark. Since X and Y are outcomes of two independent die rolls, their roles are symmetric; thus $\mathbb{P}(X \le Y)$ and $\mathbb{P}(Y \le X)$ must be equal (independence and equal pmfs are both crucial). By inclusion–exclusion,

$$\begin{aligned} 1 = \mathbb{P}(\Omega) &= \mathbb{P}(\{X \le Y\} \cup \{Y \le X\}) = \mathbb{P}(X \le Y) + \mathbb{P}(Y \le X) - \mathbb{P}(\{X \le Y\} \cap \{Y \le X\}) \\ &= 2\mathbb{P}(X \le Y) - \mathbb{P}(X = Y). \end{aligned}$$

Since $\mathbb{P}(X = Y) = \mathbb{P}(\{(1, 1), (2, 2), \dots, (10, 10)\}) = \frac{10}{100} = \frac{1}{10}$, we get $\frac{11}{10} = 2\mathbb{P}(X \le Y)$, hence $\mathbb{P}(X \le Y) = \frac{11}{20}$. This approach only requires counting $\{X = Y\}$ (easy) instead of $\{X \le Y\}$.
Equivalently, partition Ω into $\{X < Y\}$, $\{X = Y\}$, and $\{Y < X\}$, note that $\mathbb{P}(X < Y) = \mathbb{P}(Y < X)$ by symmetry, and use $\{X \le Y\} = \{X < Y\} \cup \{X = Y\}$.

7. (a) Assume bulbs are (approximately) independent and identically defective with probability $p = 0.01$. Then with $n = 100$, the number of defectives $X \sim \mathrm{Bin}(100, 0.01)$. Hence

$$\begin{aligned} \mathbb{P}(X \le 3) &= \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) \\ &= 0.99^{100} + 100 \cdot 0.01 \cdot 0.99^{99} + \binom{100}{2} 0.01^2 0.99^{98} + \binom{100}{3} 0.01^3 0.99^{97} \approx 0.9816. \end{aligned}$$

(b*) It is easy to check that $\mathbb{P}(X = 1) > \mathbb{P}(X = 0)$. Since on average 1 out of 100 bulbs is defective, a natural guess is that "1 defective" is most probable. We now prove it.

1. (Simple, specific to the present numbers.) $\mathbb{P}(X = 0) = 0.99^{100} \approx 0.3660$, $\mathbb{P}(X = 1) = 100 \cdot 0.01 \cdot 0.99^{99} \approx 0.3697$, $\mathbb{P}(X = 2) = \binom{100}{2} 0.01^2 0.99^{98} \approx 0.1849$.

Among these, $\mathbb{P}(X=1)$ is the largest, and their sum exceeds 0.9, so $\mathbb{P}(X=k) < 0.1$ for all $k > 2$. Thus $\mathbb{P}(X=1)$ is the largest over all $k = 0, 1, \ldots, 100$.

2. (General method, works for general n and p.) Check for which $k \in \{0, 1, \ldots, 99\}$ we have $\mathbb{P}(X = k+1) \geq \mathbb{P}(X = k)$. This holds iff

$$1 \leq \frac{\mathbb{P}(X=k+1)}{\mathbb{P}(X=k)} = \frac{\binom{100}{k+1} 0.01^{k+1} 0.99^{100-k-1}}{\binom{100}{k} 0.01^{k} 0.99^{100-k}} = \frac{0.01}{0.99} \cdot \frac{100-k}{k+1} = \frac{1}{99} \cdot \frac{100-k}{k+1},$$

i.e.,

$$99(k+1) \leq 100 - k,$$

which yields $k \leq 1/100$. Therefore for $k \geq 1$ the sequence decreases, so $k = 1$ is the most probable value.

8. Let X be the number of sixes. With n rolls, $X \sim \mathrm{Bin}\left(n, \frac{1}{6}\right)$. Choose n so that $\mathbb{P}(X \geq 2) \geq 0.5$:

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X < 2) = 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) = 1 - \left(\frac{5}{6}\right)^n - n\left(\frac{5}{6}\right)^{n-1} \frac{1}{6}.$$

Plugging in $n = 9$ still gives a probability below 0.5, while for $n = 10$ it exceeds 0.5, so the answer is 10.

Remark. $\mathbb{P}(X < 2)$ is strictly decreasing in n: if among the first $n+1$ rolls there are fewer than two sixes, then among the first n there are also fewer than two. One can also verify monotonicity analytically by differentiating the expression in n.

9. Let X be the number of services that are down. Since each service is independently down with probability 0.2, we have $X \sim \mathrm{Bin}(10, 0.2)$. The desired probability is

$$\mathbb{P}(X \leq 3) = 0.8^{10} + 10 \cdot 0.2 \cdot 0.8^9 + \binom{10}{2} 0.2^2 0.8^8 + \binom{10}{3} 0.2^3 0.8^7 = 0.8791.$$

(Alternatively, let $Y = 10 - X$ be the number of *working* services, so $Y \sim \mathrm{Bin}(10, 0.8)$ and we seek $\mathbb{P}(Y \geq 7)$. The computation is essentially the same, using $\binom{10}{k} = \binom{10}{10-k}$.)

10. We repeat a random experiment—here, choosing a point uniformly in the unit interval—until a desired event occurs—here, the point falls into the middle third. The required number of trials is geometrically distributed, with parameter equal to the probability of the desired event. That probability is 1/3, so $X \sim \text{Geo}(1/3)$. As X takes positive integer values,

$$\begin{aligned}\mathbb{P}(X < 5) &= \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \mathbb{P}(X = 4) \\ &= \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^3 \cdot \frac{1}{3} = \frac{27 + 18 + 12 + 8}{81} = \frac{65}{81} \approx 0.8025.\end{aligned}$$

11. X is geometrically distributed, since it counts the number of independent trials until a specified event occurs. Here a trial is a success if the fair die shows a number less than 3 (i.e., 1 or 2), which has probability 1/3, so $X \sim \text{Geo}(1/3)$.

$$\mathbb{P}(2 \le X \le 3) = \mathbb{P}(X = 2) + \mathbb{P}(X = 3) = \frac{2}{3} \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} = \frac{10}{27}.$$

$$\mathbb{P}(X \ge 3) = 1 - \mathbb{P}(X < 3) = 1 - \mathbb{P}(X = 1) - \mathbb{P}(X = 2) = 1 - \frac{1}{3} - \frac{2}{3} \cdot \frac{1}{3} = 1 - \frac{5}{9} = \frac{4}{9} = \frac{12}{27}.$$

Therefore,

$$\mathbb{P}(2 \le X \le 3) < \mathbb{P}(X \ge 3).$$

12. Let X be the number of flips needed to get the first head, and Y the number of flips *after the first head* needed to get the second head. We must compute $\mathbb{P}(X = Y)$. Both X and Y are Geo(1/2), since each counts the number of independent trials with success probability $p = 1/2$ until the first success. Note that if $\{X = Y\}$ occurs, then $\{X = Y = k\}$ occurs for some $k \in \{1, 2, \ldots\}$; the events $\{X = Y = k\}$ are pairwise disjoint and their union is $\{X = Y\}$. Hence

$$\mathbb{P}(X = Y) = \sum_{k=1}^{\infty} \mathbb{P}(X = Y = k) = \sum_{k=1}^{\infty} \mathbb{P}(\{X = k\} \cap \{Y = k\}).$$

For any k, the events $\{X = k\}$ and $\{Y = k\}$ are independent (the pre- and post-first-head phases are independent), so

$$\sum_{k=1}^{\infty} \mathbb{P}(\{X = k\} \cap \{Y = k\}) = \sum_{k=1}^{\infty} \mathbb{P}(X = k)\mathbb{P}(Y = k) = \sum_{k=1}^{\infty} \left(p(1-p)^{k-1}\right)^2 \Big|_{p=1/2} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{2k}.$$

Using the geometric series,

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{2k} = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^{k} = \frac{1/4}{1 - 1/4} = \frac{1}{3}.$$

Thus the desired probability is $\boxed{\frac{1}{3}}$.