Probability Theory and Statistics

Exercise 4 solutions

09.29. – 10.03.

Expected value, Expected value of transformed random variables, deviation

1. Let X be the maximum of the two dice (a random variable). We can compute the expected value of X as follows:

$$\mathbb{E}(X) = \sum_{k=1}^{6} k \cdot \mathbb{P}(X = k).$$

The maximum is $X = 1$ if both dice show 1 – there is only one such outcome, so $\mathbb{P}(X = 1) = 1/36$ (there are 36 possible outcomes in total). $\mathbb{P}(X = 2) = 3/36$, since there are three favorable cases: one die shows 1 and the other shows 2 (in either order), or both dice show 2. Proceeding similarly for each $k \in \{1, 2, \ldots, 6\}$, we obtain

$$\mathbb{E}(X) = 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{9}{36} + 6 \cdot \frac{11}{36} = \frac{1}{36} \sum_{i=1}^{6} i(2i - 1) = 4.4722.$$

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2. Let X be the number of flips needed. Note that $\mathrm{Ran}(X) = \{2, 3, 4, \ldots\}$ (since to get two equal consecutive outcomes, at least two flips are needed, but any larger number is also possible). Define $Y := X - 1$. Then Y is geometrically distributed with parameter $1/2$.

Indeed, for $k \geq 1$, if the experiment has not ended after the k-th flip, then it ends at the $(k + 1)$-th flip exactly when the $(k + 1)$-th outcome equals the k-th, which has probability $1/2$, independent of the previous outcomes. Thus for all $k \geq 1$,

$$\mathbb{P}(Y > k + 1 | Y > k) = 1/2.$$

Since $\mathbb{P}(Y > 0) = 1$, the multiplication rule gives

$$\mathbb{P}(Y > k + 1) = (1/2)^{k+1},$$

so

$$\mathbb{P}(Y = k) = \mathbb{P}(Y > k-1) - \mathbb{P}(Y > k) = (1/2)^{k-1} - (1/2)^k = (1/2)^k = (1/2)^{k-1} \cdot (1-1/2).$$

Thus Y is indeed Geo(1/2). Therefore, by linearity of expectation,

$$\mathbb{E}(X) = \mathbb{E}(Y+1) = \mathbb{E}(Y)+1 = 2+1 = 3, \quad \mathbb{D}^2(X) = \mathbb{D}^2(Y) = \frac{1-p}{p^2} = 2, \quad \mathbb{D}(X) = \sqrt{2}.$$

Remark: On practice class at least one student suggested the following alternative solution. The sample space may be taken as the set of finite F–H sequences ending with two identical outcomes but containing no two consecutive identical ones before, e.g. $FF, HH, FHH, HFF, FHFF, HFFH, \ldots$ For each $k = 2, 3, 4, \ldots$ there are exactly 2 such sequences of length k. The value X is precisely the length of the outcome. A sequence of length k has probability $(1/2)^k$, hence $\mathbb{P}(X = k) = 2(1/2)^k = (1/2)^{k-1}$ for $k = 2, 3, 4, \ldots$. From this, $\mathbb{E}(X)$ can be computed directly as in the case of the geometric distribution, and similarly for $\mathbb{D}(X)$.

3. As we have already seen earlier, the distribution of the number of defective bulbs is binomial with parameters $n = 100$ and $p = 0.01$. Thus $\mathbb{E}(X) = np = 100 \cdot 0.01 = 1$.

4. Let X denote the number of points. Then X is geometrically distributed, since we repeat until the first success (a point inside the unit circle). So $X \sim \mathrm{Geo}(p)$, where

$$p = \frac{\text{area of circle}}{\text{area of square}} = \frac{\pi \cdot 1^2}{4} = \frac{\pi}{4} \approx 0.7854.$$

Therefore

$$\mathbb{E}(X) = \frac{1}{p} = \frac{4}{\pi} \approx 1.2732.$$

5. Let X be the number of broken telephones on a given day. Since the number of telephones is unbounded and each fails independently with small probability, X has a Poisson distribution: $X \sim \mathrm{Poi}(\lambda)$. The problem states

$$\mathbb{P}(X = 0) = \frac{12}{360} = \frac{1}{30}.$$

But for a Poisson variable $\mathbb{P}(X = 0) = e^{-\lambda}$, hence $\lambda = -\ln(1/30) \approx 3.401$. Thus $\mathbb{E}(X) = \lambda = 3.401$.

Now let Y be the number of days (out of 360) with at least 2 failures. Then

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X \leq 1) = 1 - (e^{-\lambda} + \lambda e^{-\lambda}) \approx 0.8533.$$

So $Y \sim \mathrm{Bin}(360; 0.8533)$ and $\mathbb{E}(Y) = 360 \cdot 0.8533 \approx 307.18$.

6. (a) Let X be the number of defective bulbs on that day. Then $X \sim \text{Bin}(n = 10, p = 1/20)$. Thus $\mathbb{E}(X) = np = 1/2$, $\mathbb{D}^2(X) = np(1-p) = 10 \cdot 1/20 \cdot 19/20 = 19/40 = 0.475$.

(b) The required probability is

$$\mathbb{P}(4 < X < 8) = \mathbb{P}(X = 5) + \mathbb{P}(X = 6) + \mathbb{P}(X = 7),$$

which equals

$$\binom{10}{5}(1/20)^5(19/20)^5 + \binom{10}{6}(1/20)^6(19/20)^4 + \binom{10}{7}(1/20)^7(19/20)^3 \approx 0.0000637.$$

(c) For large n, the number of defective bulbs can be approximated by a Poisson random variable Y with parameter $\lambda = \lim_{n\to\infty} n(1/2n) = 1/2$. Hence

$$\mathbb{P}(Y = 3) = \frac{\lambda^3}{3!}e^{-\lambda} = \frac{(1/2)^3}{6}e^{-1/2} \approx 0.0126.$$

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7. $\mathbb{E}((X-3)^2) = \mathbb{E}(X^2 - 6X + 9) = \mathbb{E}(X^2) - 6\mathbb{E}(X) + 9$. We know $\mathbb{E}(X) = 3.5 = 21/6$ and $\mathbb{E}(X^2) = 91/6$, so

$$\mathbb{E}((X-3)^2) = \frac{91}{6} - \frac{126}{6} + \frac{54}{6} = \frac{19}{6} = 3\tfrac{1}{6}.$$

8. a) $\mathbb{P}(Y = 1) = 5/9$, $\mathbb{P}(Y = -1) = 4/9$. b) $\mathbb{E}(Y) = 1/9$. c) $\mathbb{E}((-1)^X) = 1/9$.

9. $X \sim \text{Bin}(n = 3, p = 1/4)$. Then

$$\mathbb{E}(Y) = \mathbb{E}(X^2) = \sum_{k=0}^{3} k^2 \mathbb{P}(X = k) = 9/8.$$

So $\mathbb{E}(Z) = \mathbb{E}(X^2 + X + 1) = \mathbb{E}(X^2) + \mathbb{E}(X) + 1 = 9/8 + 3/4 + 1 = 23/8$. Also $\mathbb{D}^2(X) = np(1-p) = 9/16$, hence $\mathbb{D}(X) = 3/4$.

10. The pmf is

$$\mathbb{P}(X = k) = \binom{k+2}{2} p^3 (1-p)^k, \quad k = 0, 1, 2, \ldots$$

Then

$$\mathbb{E}\left(\frac{1}{(X+1)(X+2)}\right) = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} \binom{k+2}{2} p^3 (1-p)^k = \frac{p^2}{2}.$$

11. Sample space size 27. The distribution of X (number of surviving deer):

$$\mathbb{P}(X=0)=2/9, \quad \mathbb{P}(X=1)=2/3, \quad \mathbb{P}(X=2)=1/9.$$

Thus $\mathbb{E}(X)=8/9$, $\mathbb{D}^2(X)=26/81$, $\mathbb{D}(X)\approx 0.567$.

12. $\mathbb{D}^2(2X+1)=4\mathbb{D}^2(X)=4\lambda$.

13. (a) $\mathbb{E}((2+X)^2)=14$. (b) $\mathbb{D}^2(4+3X)=45$.

14. With $p=1/3$, $\mathbb{E}(X)=3$, $\mathbb{D}^2(X)=6$. Then

$$\mathbb{E}((3-X)^2)=6, \quad \mathbb{D}(5-2X)=\sqrt{24}\approx 4.899.$$