Probability Theory and Statistics

Exercise 1

09.08-09.12.

Event algebra, classical and geometric probability spaces, Poincaré formula

1. We toss two coins. How many elements does the sample space have? What are the probabilities of the individual events?

2. We roll two dice. Let A and B be the events where A is the set of outcomes with a two-digit sum, and B is the set where the sum is even. Express the following in terms of A and B, where possible:

a) the sum is an integer b) the sum is irrational

c) the sum is 11 d) the sum is 7

3. From a deck of French cards we remove the face cards, then draw several cards. Let A_i denote the event that a card of value i is drawn, S, D, C, H that we drew a spade, diamond, club, or heart respectively, and B_i that i cards are drawn. Express the following, where possible:

a) we draw the seven of diamonds (and nothing else)

b) fewer than 4 cards are drawn

c) every drawn card is a spade or club

d) exactly three sevens are drawn (and nothing else)

e) four sevens and four tens are drawn (and nothing else)

f) three sevens and one other card are drawn (*)

4. For which events A and B are the following true?

a) $A = A \cap B$ b) $A = A \cup B$ c) $A = A \cap \overline{B}$ d) $A \cup B = A \cap B$

5. We roll three dice. Let

$$A = \{\text{sum } = 7\} \qquad B = \{\text{all even}\} \qquad C = \{\text{at least one 3}\}$$

Compute $\mathbb{P}(A \cap (B \cup \overline{C}))$ and $\mathbb{P}((A \cup C) \cap \overline{B})$.

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6. Choose two numbers uniformly at random from $[0, 1]$. What is the probability that one is more than twice the other?

7. Choose a random point x on $(0, 2)$ and y on $(0, 3)$. What is the probability that the segments of length x, y, and 1 can form a triangle?

8. Choose a random point from the unit disk and fix a square inscribed in the unit circle. What is the probability that the point lies inside the square?

9. Choose a random point $P = (a, b)$ from the unit square. What is the probability that the polynomial $p(x) = ax^2 - 2bx + 1$ has no real roots?

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10. In the Hungarian lottery "5 out of 90", 5 distinct numbers are drawn from 1 to 90. Let A be the event that all drawn numbers are at most 50; B that all drawn numbers are even; and C that all drawn numbers are at least 20. Compute $\mathbb{P}(A)$, $\mathbb{P}(B)$, $\mathbb{P}(A \cap B)$, $\mathbb{P}(A \cup B)$, $\mathbb{P}(B \cap C)$, and $\mathbb{P}(A \cup B \cup C)$.

11. Two fair coins are tossed n times. What is the probability that both "two heads" and "two tails" outcomes appear during the sequence?

12. Verify that for $\mathbb{P}(A) = 0.7$, $\mathbb{P}(B) = 0.6$, $\mathbb{P}(C) = 0.9$ the following hold:

a) $\mathbb{P}(A \cap B) \geq 0.3$ b) $\mathbb{P}(A \cap B \cap C) \geq 0.2$