Probability Theory and Statistics

Exercise 1 solutions

09.08-09.12.

Event algebra, classical and geometric probability spaces, Poincaré formula

1. As a sample space we may choose, e.g., the set of length-2 F–I sequences, where the first letter is the result of the first toss and the second letter the result of the second toss (F = heads, I = tails):

$$\Omega = \{FF, FI, IF, II\}.$$

So the sample space has 4 elements. On each toss heads and tails are equally likely; if we take every subset of Ω as an event, then each elementary (i.e., singleton) event has probability $1/4$, and in general a k-element event has probability $k/4$, where $k \in \{0, 1, 2, 3, 4\}$.
Thus this is a classical probability space. If, however, we chose $\Omega = \{0, 1, 2\}$ (again taking all subsets as events), where the outcome records the number of heads, then we would *not* get a classical probability space, because $\mathbb{P}(\{0\}) = \mathbb{P}(\{2\}) = 1/4$ but $\mathbb{P}(\{1\}) = 1/2$, i.e., not all elementary events are equally likely.

2. *(Final answer only.)*

a) yes, Ω b) yes, $\emptyset$

c) yes, e.g.: $A \backslash B$ d) impossible

(Justification for d): it is easy to see that the event in d) is a proper, nonempty subset of $\overline{A} \cap \overline{B}$. However, among the proper subsets of $\overline{A} \cap \overline{B}$ the only one expressible from A and B by the usual set operations is $\emptyset$.)

3. *(Final answer only.)*

a) e.g.: $B_1 \cap A_7 \cap Ka$ b) e.g.: $B_1 \cup B_2 \cup B_3$

c) e.g.: $\overline{Ka \cup Ko}$ d) e.g.: $B_3 \cap \overline{\bigcup_{i=2}^{6} A_i} \cap \overline{\bigcup_{i=8}^{10} A_i}$

e) e.g.: $B_8 \cap \overline{\bigcup_{i=2}^{6} A_i} \cap \overline{\bigcup_{i=8}^{9} A_i}$ (there are 4+4 in total) f) not possible

4. a) $A \subseteq B$ (because if an outcome is in A, then by $A = A \cap B$ it is also in $A \cap B$, hence in both A and B, thus in B).
b) $B \subseteq A$ (because if an outcome is in $A \cup B$, then it is in A as well. The set $A \cup B$ contains exactly those outcomes that are in A or in B. Since all of these are in A, we get $B \subseteq A$).
c) $A \cap B = \emptyset$, because intersecting A with $\overline{B}$ does not remove any elements; hence

every element of A lies in $\overline{B}$, i.e., A has no element in B.
d) $A = B$. Indeed, if A had an element not in B, it would lie in $A \cup B$ but not in $A \cap B$. Similarly, B cannot have an element outside A. Thus A and B have the same elements, i.e., $A = B$.

5. Since $B \subseteq \overline{C}$, we have

$$\mathbb{P}\Big(A \cap (B \cup \overline{C})\Big) = \mathbb{P}(A \cap \overline{C}) = \mathbb{P}(\text{the sum is 7 and none is a three}).$$

If we choose $\Omega = \{1,2,3,4,5,6\}^3 = \{(a,b,c) \mid a,b,c \in \{1,\dots,6\}\}$, then (in this classical model)

$$\mathbb{P}(A \cap \overline{C}) = \frac{\big|\{(1,1,5),(1,5,1),(5,1,1),(4,2,1),(4,1,2),(2,4,1),(2,1,4),(1,4,2),(1,2,4)\}\big|}{216} = \frac{1}{24}.$$

Moreover,

$$\mathbb{P}\big((A \cup C) \cap \overline{B}\big) = \mathbb{P}(A \cup C) = \mathbb{P}(C) + \mathbb{P}(A \setminus C) = 1 - \mathbb{P}(\overline{C}) + \mathbb{P}(A \cap \overline{C}) = \frac{(216-125)+9}{216} = \frac{25}{54},$$

since $\mathbb{P}(\overline{C}) = 125/216$ (when $\overline{C}$ occurs, each of the three dice has 5 possible values).

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6. $\frac{1}{2}$.
Let the two numbers be $x, y \in [0,1]$, so $(x,y) \in [0,1]^2 =: \Omega$, and use geometric probability. Let

$$A = \{x > 2y\}, \qquad B = \{y > 2x\}.$$

$$\mathbb{P}(A \cup B) = \frac{T_{\text{useful}}}{T_{\text{total}}} = \frac{\frac{1}{4} + \frac{1}{4}}{1} = \frac{1}{2}.$$

7. $\frac{1}{2}$.
Let $x \in [0,2]$ and $y \in [0,3]$, so $(x,y) \in [0,2] \times [0,3] =: \Omega$, and use geometric probability. Let A, B, C be the events

$$\begin{cases} A = \{x + y > 1\} & \Longrightarrow \; A = \{y > 1 - x\}, \\ B = \{x + 1 > y\} & \Longrightarrow \; B = \{y < 1 + x\}, \\ C = \{y + 1 > x\} & \Longrightarrow \; C = \{y > x - 1\}. \end{cases}$$

$$\mathbb{P}(A \cap B \cap C) = \frac{T_{\text{useful}}}{T_{\text{total}}} = \frac{3}{2 \cdot 3} = \frac{1}{2}.$$

8. $\frac{2}{\pi}$.

Let Ω be the unit disk; we use a geometric probability model. Its area is $T_{\text{total}} = T_\Omega = \pi$. The favorable region is the inscribed square. Its diagonal equals the circle's diameter 2, so one way to compute the square's area is via the kite formula: $T_{\text{useful}} = 2 \times 2/2 = 2$ (of course, other methods give the same). Therefore

$$\mathbb{P}(\text{the point falls inside the square}) = \frac{T_{\text{useful}}}{T_{\text{total}}} = \frac{2}{\pi}.$$

9. $\frac{2}{3}$.

With $(a,b) \in [0,1]^2 =: \Omega$ and geometric probability: a quadratic has no real root iff its discriminant

$$D = (-2b)^2 - 4a \cdot 1 = 4b^2 - 4a$$

is negative, i.e., $b^2 < a$. Since $0 < b$, this is equivalent to $b < \sqrt{a}$. Hence

$$\mathbb{P}(p \text{ has no real root}) = \frac{T_{\text{useful}}}{T_{\text{total}}} = \int_0^1 \sqrt{a}\, da = \left[\tfrac{2}{3} a^{3/2}\right]_0^1 = \frac{2}{3}.$$

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10. *Solution sketch*: $\mathbb{P}(A) = \frac{\binom{50}{5}}{\binom{90}{5}} = 0.0482$, $\mathbb{P}(B) = \frac{\binom{45}{5}}{\binom{90}{5}} = 0.0278$, $\mathbb{P}(A \cap B) = \frac{\binom{25}{5}}{\binom{90}{5}} = 0.0012$. By the inclusion–exclusion formula, $\mathbb{P}(A \cup B) = 0.0748$. Also $\mathbb{P}(B \cap C) = \frac{\binom{36}{5}}{\binom{90}{5}} = 0.0086$, and with $\mathbb{P}(C) = \frac{\binom{71}{5}}{\binom{90}{5}}$, $\mathbb{P}(A \cap C) = \frac{\binom{31}{5}}{\binom{90}{5}} = 0.0039$, and $\mathbb{P}(A \cap B \cap C) = \frac{\binom{16}{5}}{\binom{90}{5}} = 9.939 \cdot 10^{-5}$, inclusion–exclusion gives $\mathbb{P}(A \cup B \cup C) = 0.3587$ (or 0.3586 if using previously rounded values).

11. A suitable sample space is $\Omega = \{FF, FI, IF, II\}^n$; this yields a classical model (cf. Problem 1), with each elementary outcome of probability $1/4^n$. Let A be the event that during the n double-coin tosses we see both "two heads" and "two tails" at least once. Assume the coins are distinguishable. Then

$$\overline{A} = \{\text{we never see two heads}\} \cup \{\text{we never see two tails}\}.$$

To avoid "two heads" on every toss, each of the n tosses must be one of the three outcomes FF, FI, or IF, hence probability $3^n/4^n$. Similarly, avoiding "two tails" also has probability $3^n/4^n$. For their intersection (neither "two heads" nor "two tails" ever), every toss must be FI or IF, yielding probability $1/2^n$. Therefore, by inclusion–exclusion,

$$\mathbb{P}(A) = 1 - \mathbb{P}(\overline{A}) = 1 - 2 \cdot \frac{3^n}{4^n} + \frac{1}{2^n}.$$

12. a) Rearranging the two-set inclusion–exclusion formula gives $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$. Since $\mathbb{P}(A) + \mathbb{P}(B) = 1.3$ and $\mathbb{P}(A \cup B) \leq 1$, we get $\mathbb{P}(A \cap B) \geq 1.3 - 1 = 0.3$.

b) Apply the two-set inclusion–exclusion formula to $A \cap B$ and C:

$$\mathbb{P}((A \cap B) \cup C) = \mathbb{P}(A \cap B) + \mathbb{P}(C) - \mathbb{P}(A \cap B \cap C).$$

Using part a), $\mathbb{P}(A \cap B) + \mathbb{P}(C) \geq 1.2$. Thus

$$1 \geq \mathbb{P}((A \cap B) \cup C) \;\geq\; 1.2 - \mathbb{P}(A \cap B \cap C),$$

whence $\mathbb{P}(A \cap B \cap C) \geq 0.2$.