Category Theory and Computational Complexity

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A first-order dependence logic D is a class which consists of all D-definable properties where $\mathsf{D} := (\mathsf{FO} + \mu.\bar{t})$ and $\mu.\bar{t}$ denotes that term $t_{|\bar{t}|}$ is functionally dependent on t_i for all $i \leq |\bar{t}|$. Recall that a P-definable property is a closed under isomorphism class P of structures. The model class of a first-order sentence φ is the class of structures $\mathsf{Mod}(\varphi) := \{A : A \models \varphi\}$ where the notation $A \models \varphi$ refers to the *'the snow is white' iff the snow is white* type of Tarskian semantics, of φ being true when interpreted in A under all α assignment function ($A \models \varphi[\alpha]$).

The model class FO is, as always, defined as the class of models of all first-order sentences (i.e. $\mathsf{FO} := \{S : (\exists\tau)(\exists\varphi \in L(\tau))\ S = \mathsf{Mod}(\varphi)\}$ where $L(\tau)$ is a first-order language of type τ) and $\mu.\bar{t}$ is interpreted as a recursively generated tuple of terms which we naturally identify with the set $[|\bar{t}|] := \{1, 2, \ldots, |\bar{t}|\}$. D sentences are capable to characterise variable dependence and in general they are proven to be as expressive as the sentences of the second order Σ^1_1 fragment [**?**]. The intuitionistic dependence version ID has the same expressive power as full SO [**?**]. It is a fact that MID-model checking is PSPACE-complete [**?**] where MID is the intuitionistic implication fragment of the modal dependence logic MD which contains at least two modifiers. Hence, $(\mathsf{FO} + \mu.\bar{t}) = \mathsf{NP}, \mathsf{ID} = \Sigma_*\mathsf{P}$ and $\mathsf{MID} = \mathsf{PSPACE}$. On the other hand, $\mathsf{PSPACE} = \mathsf{IP} = \mathsf{QIP}$ [**?**], and so $\mathsf{MID} = \mathsf{QIP}$ which is the quantum version of the interactive polytime class IP.

We shall try to cook up a purely algebraic definition for the class of structures MID and extend such categorical logic in order to capture other quantum and classical complexity classes.

Ehrenfeucht-Fraïssé games characterise the expressive power of logical languages [**?**]. Every Ehrenfeucht-Fraïssé game is an ultraproduct [**?**]. A back-and-forth method for showing isomorphism between countably infinite structures. If F is an ultrafilter (i.e. $F \subseteq 2^{\mathbb{N}}$ and $\forall X \subseteq \mathbb{N}(X \notin F \Leftrightarrow \mathbb{N} \setminus X \in F)$ holds) then the reduced product $\prod_i M_i/F$ is an ultraproduct of the sets M_i, $i \in I$. Recall that

$$f \sim g \Leftrightarrow \{i \in I : f(i) = g(i)\} \in F$$

for all infinite sequences $f, g \in \prod_i M_i$ and any index set I is the relation which induces the equivalence classes that conform the ultraproduct

$$\prod_i M_i/F = \{[f] : f \in \prod_i M_i\}.$$

Lemma 0.1 (Łoś Lemma)**.** *If F is an ultrafilter and φ a first-order formula, then the ultraproduct of models of φ indexed by any index set $I \in F$ is a model of φ, i.e.*

$$(\prod_i A_i/F, \alpha) \models \varphi \Leftrightarrow \{i \in I : (A_i, \alpha_i) \models \varphi\} \in F.$$