Notes model theory

OZ

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Intro

Ehrenfeucht-Fraïssé games characterise the expressive power of logical languages [1]. Every Ehrenfeucht-Fraïssé game is an ultraproduct [2], a back-and-forth method for showing isomorphism between countably infinite structures, but only defined for finite structures in finite model theory. There are techniques in AI for model-checking algorithmically properties on generic objects like finite structures of some specified vocabulary. There is however a deeper relationship that can be described using ultrafilters and ultralimits.

We want a way to study discrete objects with continuous methods. This can be achieved via an ultraproduct construction.

Suppose α is a non-principal ultrafilter on $I \in 2^{\mathbb{N}}$ and A_i, B_i structures of a countable vocabulary τ. An ultrafilter is a filter that is maximal w.r.t. the partial ordering of any Boolean algebra in which the filter has been defined, in particular of the lattice $\mathbf{2} := \langle 2^{\mathbb{N}}, \subseteq \rangle$ when thinking of ultrafilters as subsets of the natural numbers $\mathbb{N}$ with the finite intersection property (i.e. $\inf\{\alpha \subseteq \mathbb{N} : |\alpha| < \aleph_0\} \neq 0$). An ultrafilter α is non-principal if it contains no finite set.

Assuming that α is a filter containing an infinite descending sequence with empty intersection, we have the following result.

Theorem. *If $A_i \equiv B_i$ for all $i \in I$, then player B has a winning strategy in $EF_\omega(\prod_i A_i/\alpha, \prod_i B_i/\alpha)$.*

Ultrafilters

Given a nonempty set L define a binary relation $\succ$ on it by forcing the following formulas true:

1. Reflexivity: $\forall x(x \succ x)$
2. Transitivity: $\forall xyz(x \succ y) \wedge (y \succ z) \to (x \succ z)$
3. Antisymmetry: $\forall xy(x \succ y) \wedge (y \succ x) \to (x = y)$

The pair $\langle L, \succ \rangle$ is called a *poset* and $\succ$ a *partial order* on L. For every two-element subset $\{x, y\}$ of L we define its *meet* and *join* as $x \sqcap y := \inf_\succ\{x, y\}$ and $x \sqcup y := \sup_\succ\{x, y\}$, respectively. A *lattice* is a poset where every two-element subset has meet and join (i.e. $\forall\{x, y\} \in 2^L(\exists(x \sqcap y) \in L \wedge \exists(x \sqcup y) \in L)$ is satisfied).

A *filter* is a subset F of a lattice L which contains all the *successors* (if we name "$\succ$" the "successor" relation) of any member of F (i.e. $\forall xy(y \in F) \wedge (x \succ y) \to (x \in F)$ holds). An *ultrafilter* α is a maximal filter with respect to the usual partial order relation that one can always define in any Boolean algebra (particularly in 2^L). Ultrafilters are characterised by the next equivalence.

Lemma. *If α is a filter in a Boolean algebra B, α is an ultrafilter if and only if for each $x \in B$ either $x \in \alpha$ or $1 - x \in \alpha$, but not both.*

Łoś lemma

Lemma (Łoś Lemma)**.** *If α is an ultrafilter and φ a first-order formula, then the ultraproduct of models of φ indexed by any index set $I \in \alpha$ is a model of φ, i.e.*

$$\prod_i A_i/\alpha \models \varphi \Leftrightarrow \{i \in I : A_i \models \varphi\} \in \alpha.$$

Ultraproducts

An ultraproduct is a mathematical construction that permits us to take the limit of any discrete object in any discrete space and actually build a limit object which exists in the ultraproduct of the spaces which is for instance a limit space. The ultraproduct construction is a universal way to go from discrete to continuous, back and forth, carrying along the axioms and the operations.

References

[1] N. Immerman. Descriptive complexity: a logician's approach to computation. *Notices of the AMS*, 1995.

[2] J. Väänänen. *Models and Games*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011.