

# Category Theory and Computational Complexity

Marco Larrea

Octavio Zapata

December 22, 2014

A first-order dependence logic  $D$  is a class which consists of all  $D$ -definable properties where  $D := (FO + \mu.\bar{t})$  and  $\mu.\bar{t}$  denotes that term  $t_{|\bar{t}|}$  is functionally dependent on  $t_i$  for all  $i \leq |\bar{t}|$ . The model class  $FO$  is as always defined as the class of models of all first-order sentences (i.e.  $FO := \{S : (\exists \tau)(\exists \varphi \in L(\tau)) S = Mod(\varphi)\}$  where  $L(\tau)$  is a first-order language of type  $\tau$ ) and  $\mu.\bar{t}$  is interpreted as a recursively generated tuple of terms which we naturally identify with the set  $[[\bar{t}]] := \{1, 2, \dots, |\bar{t}|\}$ .  $D$  sentences are capable to characterise variable dependence and in general they are proven to be as expressive as the sentences of the second order  $\Sigma_1^1$  fragment. The intuitionistic dependence version  $ID$  has the same expressive power as full  $SO$ . It is a fact that  $MID$ -model checking is  $PSPACE$ -complete where  $MID$  is the intuitionistic implication fragment of the modal dependence logic  $MD$  which contains at least two modifiers. Hence,  $(FO + \mu.\bar{t}) = NP$ ,  $ID = \Sigma_* P$  and  $MID = PSPACE$ . On the other hand,  $PSPACE = IP = QIP$ , and so  $MID = QIP$  which is the quantum version of the interactive polytime class  $IP$ .

We shall try to cook up a purely algebraic definition for the class of structures  $MID$  and extend such categorical logic in order to capture other quantum and classical complexity classes.

Ehrenfeucht-Fraïssé games characterise the expressive power of logical languages [?]. Every Ehrenfeucht-Fraïssé game is an ultraproduct [?]. A back-and-forth method for showing isomorphism between countably infinite structures. If  $F$  is an ultrafilter (i.e.  $F \subseteq 2^{\mathbb{N}}$  and  $\forall X \subseteq \mathbb{N} (X \notin F \Leftrightarrow \mathbb{N} \setminus X \in F)$  holds) then the reduced product  $\prod_i M_i / F$  is an ultraproduct of the sets  $M_i$ ,  $i \in I$ . Recall that

$$f \sim g \Leftrightarrow \{i \in I : f(i) = g(i)\} \in F$$

for all infinite sequences  $f, g \in \prod_i M_i$  and any index set  $I$  is the relation which induces the equivalence classes that conform the ultraproduct

$$\prod_i M_i / F = \{[f] : f \in \prod_i M_i\}.$$

**Lemma 1** (Łoś Lemma). *If  $F$  is an ultrafilter and  $\varphi$  a first-order formula, then the ultraproduct of models of  $\varphi$  indexed by any index set  $I \in F$  is a model of  $\varphi$ , i.e.*

$$(\prod_i A_i / F, \alpha) \models \varphi \Leftrightarrow \{i \in I : (A_i, \alpha_i) \models \varphi\} \in F.$$