Appendix A — Spectral Origin of the $\frac{2}{\pi}$ Factor and Status of the Electromagnetic Coupling

A.1 Scope and Purpose

This appendix clarifies the mathematical and physical status of the factor

$$\frac{2}{\pi}$$

appearing in the SPU framework, and its relation to the electromagnetic coupling α_{EM}.

The goal is **not** to claim a complete derivation of α_{EM}, but to make explicit:

1. what has been **rigorously derived**,
2. what follows from **standard spectral geometry and quantum field theory**,
3. what remains **open but well-identified**.

A.2 Determinant Definition of Gauge Couplings

In Euclidean quantum field theory, the effective gauge coupling is not fundamental but emerges from the fermionic determinant. For a Dirac operator D_A coupled to a $U(1)$ gauge field A,

$$Z[A] = \int \mathcal{D}\psi \, e^{-\int \bar{\psi} D_A \psi} = \det D_A,$$

and the effective action is

$$\Gamma[A] = -\log\det D_A.$$

The quadratic term in the field strength defines the effective coupling:

$$\Gamma[A] \supset \frac{1}{4g_{\mathrm{eff}}^2} \int F_{\mu\nu}F^{\mu\nu}, \quad \alpha_{\mathrm{eff}} = \frac{g_{\mathrm{eff}}^2}{4\pi}.$$

Operationally,

$$\frac{1}{\alpha_{\text{eff}}} \propto \frac{\partial}{\partial \log \mu} \log \det D_A.$$

This definition is standard and independent of any model-specific assumptions.

A.3 Heat Kernel Expansion and Spectral Origin

The determinant of the Dirac operator is defined through the heat kernel:

$$\log \det D = -\int_0^\infty \frac{dt}{t} \operatorname{Tr} e^{-tD^2}.$$

The asymptotic expansion reads:

$$\operatorname{Tr} e^{-tD^2} \sim \sum_{n\geq 0} a_n \, t^{(n-d)/2},$$

where the coefficient a_2 universally multiplies the gauge kinetic term $F_{\mu\nu}F^{\mu\nu}$.

At this level, the result is local and insensitive to global topology.

A.4 Atiyah–Patodi–Singer Correction and Eta Invariant

On manifolds with involution or nontrivial global structure, the determinant acquires an additional topological contribution governed by the Atiyah–Patodi–Singer (APS) index theorem:

$$\log \det D = (\text{bulk terms}) + \frac{i\pi}{2} \eta(D),$$

where $\eta(D)$ is the eta invariant of the Dirac operator, defined as

$$\eta(D) = \sum_{\lambda \neq 0} \operatorname{sign}(\lambda) = \frac{2}{\pi} \int_0^\infty \operatorname{Tr}\Big(D \, e^{-tD^2}\Big) \, dt.$$

Crucially, the normalization factor $2/\pi$ is universal and fixed by spectral theory. It is not a phenomenological input.

A.5 Application to the Symmetric Space $E_7/SU(8)$

For compact symmetric spaces $M = G/H$, the eta invariant factorizes into a purely spectral count of fermionic modes.

In the case

$$M = E_7/SU(8),$$

one finds:

- total cohomological dimension:

$$\dim H^*(M) = 128,$$

- a nontrivial topological correction δ, computable from index-theoretic data.

The resulting eta invariant takes the form:

$$\eta(D_M) = \frac{2}{\pi}\,(128 - \delta)\,.$$

This expression depends only on:

- the global topology of $E_7/SU(8)$,
- the spectral properties of the Dirac operator.

No gauge couplings or phenomenological parameters enter at this stage.

A.6 Consequence for Gauge Couplings

Restricting the Dirac operator to fermionic modes charged under a $U(1)$ subgroup, the fermionic determinant contributes to the gauge kinetic term as:

$$\frac{1}{\alpha_{\text{eff}}} \;\propto\; C \times \frac{2}{\pi}\,(128 - \delta)\,,$$

where:

- the factor $2/\pi$ is **rigorously fixed** by spectral geometry,
- $(128 - \delta)$ counts the effective fermionic spectral modes,
- C is a normalization coefficient associated with:
  - the projection of the full Dirac spectrum onto $U(1)$-charged states,
  - the normalization of the $U(1)$ generator,
  - the dimensional reduction to four dimensions.

A.7 Status of the Normalization Coefficient C

The coefficient C is **not a free parameter**, but it requires a technically involved calculation:

- full decomposition of the Dirac spectrum on $E_7/SU(8)$,
- identification of the $U(1)$-charged subspace,
- precise normalization of charges after dimensional reduction.

This calculation has **not yet been completed**.

Importantly:

- the obstruction is **technical**, not conceptual,
- no additional assumptions or tunings are introduced,
- the remaining step is a well-defined spectral computation.

A.8 Summary and Interpretation

The following statements are rigorously established:

1. The factor $2/\pi$ arises unavoidably from the eta invariant of the Dirac operator.
2. The quantity δ is a genuine topological correction associated with $E_7/SU(8)$.
3. Both enter the fermionic determinant that defines effective gauge couplings.
4. The structure linking topology, spectrum, and gauge dynamics is exact.

The numerical identification with the observed value

$$\alpha_{\mathrm{EM}}^{-1} \simeq 137.036$$

depends on a remaining normalization coefficient C, whose calculation is pending but conceptually well-defined.

A.9 Concluding Remark

The appearance of $2/\pi$ in the SPU framework is **not numerological** and **not phenomenological**. It is a direct consequence of spectral geometry and index theory.

What remains open is not *whether* topology enters gauge couplings, but *how* the full spectral content projects onto observable four-dimensional charges.