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The median of the poisson distribution

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Abstract. The purpose of this paper is twofold: first, to provide a closed form expression for the median of the Poisson distribution and, second, to improve the known estimates of the difference between the median and the mean of the Poisson distribution. We use elementary techniques based on the monotonicity of certain sequences involving tail probabilities of the Poisson distribution and the Central Limit Theorem.

Key words: Poisson distribution, median, mean, central limit theorem, Poisson–gamma relation, Poisson mixture

1 Introduction

Let X be a random variable with distribution function $F(x) := P(X \leq x)$, $x \in \mathbb{R}$. Usual textbooks in probability and mathematical statistics (see, for instance, Bartoszyński and Niewiadomska–Bugaj (1996, p. 165), or Resnick (1999, p. 164)) define a median of X (or F) as any solution x of the simultaneous inequalities $F(x_-) \leq 1/2 \leq F(x)$, where $F(x_-)$ stands for the left limit of F at x. The point x does not need to be unique. Indeed, considering the set $A := \{x \in \mathbb{R} : \ F(x) = 1/2\}$ and taking into account the right–continuity of F, we have the following: either $A = \emptyset$ and, in such a case, x is unique, or A is an interval containing its left–end point and, in such a case, A is the set of all of the medians of X.

Recent developments on robust statistical modelling and inference have given the median of a distribution some renewed interest, as the median is a highly robust measure of location. A survey of important results related to the mean and median absolute deviations of a distribution can be found in Pham–Gia and Hung (2001). A major drawback, however, is that the median cannot usually be defined in closed form, even in the cases of well–known distribution functions. Many research papers are concerned with estimates of the difference between the median and the mean of certain distributions, such as Poisson (Chen and Rubin (1986), Choi (1994) and Hamza (1995)), gamma (Chen and Rubin (1986) and Choi (1994)), binomial (Kaas and Buhrman (1980), Göb (1994) and Hamza (1995)), beta (van de Ven and Weber (1993)) and negative binomial distributions (van de Ven and Weber (1993) and Göb (1994)), among others. In this context, it is useful to have an unambiguous definition of the median of a random variable X. The most commonly accepted is, perhaps, the following

$$\mathrm{Med}(X) := \inf\{z \in \mathbb{R} : \ F(z) \geq 1/2\}. \tag{1}$$

This definition implies, in particular, that the median of an integer–valued random variable X is also an integer. Apart from (1), there are other alternative definitions of the median (cf. Landers and Rogge (1981)).

The main motivation of this paper comes from the problem of best Poisson approximation of Poisson mixtures with respect to the Fortet–Mourier distance, considered in Adell and Lekuona (2003b). Such a problem may be posed in a more general setting with regard to an arbitrary metric (see, for instance, Serfling (1978), Deheuvels and Pfeifer (1986a, 1986b), Deheuvels et al. (1989) and Adell and Lekuona (2003a)). To be more precise, recall that a mixing random variable T is a nonnegative random variable independent of the standard Poisson process $(N_t, t \geq 0)$. The random variable N_T is called a Poisson mixture with mixing random variable T. On the other hand, denote by $\mathbb{N}$ the set of nonnegative integers and let X and Y be two $\mathbb{N}$–valued random variables. The Fortet–Mourier distance between X and Y is defined by

$$d(X, Y) := \sum_{n=0}^{\infty} \left| P(X \leq n) - P(Y \leq n) \right|.$$

Suppose that $(T_n, n \in \mathbb{N})$ is a sequence of mixing random variables converging to some $\lambda > 0$ in such a way that $ET_n = \lambda$, $n \in \mathbb{N}$, and $\mu_2(n) := E(T_n - \lambda)^2 \to 0$ as $n \to \infty$. If $E|T_n - \lambda|^3 = o\big(\mu_2(n)\big)$ as $n \to \infty$, then it is shown in Adell and Lekuona (2003b, Theorem 3.3) that

$$d\Big(N_{T_n}, N_{\lambda - \frac{a}{2\lambda}\mu_2(n)}\Big) = \frac{\mu_2(n)}{2\lambda} E\big|N_\lambda - (\lambda - a)\big| + o\big(\mu_2(n)\big), \ a \leq \frac{2\lambda^2}{\mu_2(n)}. \tag{2}$$

Therefore, the best asymptotic choice of the Poisson parameter a, in the sense of minimizing the leading coefficient on the right–hand side in (2), is $a = \lambda - \mathrm{Med}(N_\lambda)$. In such a case, the leading coefficient becomes $E|N_\lambda - \mathrm{Med}(N_\lambda)|/(2\lambda)$.

The problem of best Poisson approximation of Poisson mixtures outlined above poses in a natural way the following questions: (a) to give a closed form expression for $\mathrm{Med}(N_\lambda)$ (see Theorem 1 in Section 2 and Theorem 2 in Section 3), (b) to give sharp upper and lower bounds for $\mathrm{Med}(N_\lambda) - \lambda$ (see Corollary 2 in Section3), and, finally, (c) to give a closed form expression for the mean absolute deviation $E|N_\lambda - \mathrm{Med}(N_\lambda)|$ (see Corollary 1 in Section 2).

As implicitly asserted before, the relevance of the median comes from the fact that it minimizes the mean absolute deviation of a random variable X with respect to an arbitrary point a. The proof of this result is omitted or left as an exercise in many textbooks. Here, we give a short simple proof in the case of nonnegative random variables (for other related proofs, see Muñoz–Pérez and Sanchez–Gomez (1990), Schwertman et al. (1990) and Bartoszyński and Niewiadomska–Bugaj (1996, p. 305)).

Proposition 1. *Let X be a nonnegative random variable with distribution function F such that $EX < \infty$. Then,*

$$f(a) := E|X-a| = EX + 2\int_0^a \left(F(z) - \frac{1}{2}\right) dz, \qquad a \geq 0. \tag{3}$$

As a consequence, f is convex and attains its minimum at $a = \mathrm{Med}(X)$.

Proof. For fixed $x \geq 0$, the function $\phi(a) := |x-a|$, $a \geq 0$, is absolutely continuous and satisfies

$$\begin{aligned} |x-a| - x = \phi(a) - \phi(0) &= \int_0^a \left(1_{[x,\infty)}(z) - 1_{[0,x)}(z)\right) dz \\ &= 2\int_0^a \left(1_{[0,z]}(x) - \frac{1}{2}\right) dz, \end{aligned}$$

where 1_B denotes the indicator function of the set B. Replacing x by X in the preceding formula, integrating and applying Fubini's theorem, we obtain (3). Since the integrand in (3) is right–continuous and nondecreasing, f is convex. From (1) and (3), we see that f attains its minimum at $a = \mathrm{Med}(X)$, thus completing the proof of Proposition 1.

2 The Poissonian median

Let N_λ be a random variable having the Poisson distribution with mean $\lambda \geq 0$. We define the median of N_λ as in (1) and denote by $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Our approach to find a closed form expression for the median of N_λ is simple. Instead of posing the question: what is the median of N_λ for a fixed $\lambda \geq 0$?, we pose the reciprocal question: which Poisson random variables N_λ have median equal to n for a fixed $n \in \mathbb{N}$? This point of view is implicitly considered in Chen and Rubin (1986) and Choi (1994). Our proofs use elementary techniques, namely, the monotonicity of certain sequences involving tail probabilities of the Poisson distribution and the Central Limit Theorem (CLT).

To start with, we consider the sequence of functions $(f_n,\ n \in \mathbb{N})$ defined by

$$f_n(\lambda) := P(N_\lambda \leq n) = \sum_{k=0}^{n} \frac{e^{-\lambda}\lambda^k}{k!} = \frac{1}{n!}\int_\lambda^\infty e^{-u}u^n du, \qquad \lambda \geq 0, \tag{4}$$

where the last equality follows from the Poisson–Gamma relation (see, for instance, Johnson et al. (1992, p. 164)). Let $n \in \mathbb{N}$. Since $f_n(0) = 1$ and f_n strictly decreases to 0 as $\lambda \to \infty$, it is clear that there is a unique solution λ_n to the equation $f_n(\lambda) = 1/2$. Moreover, since $f_n(\lambda) < f_{n+1}(\lambda)$, $\lambda > 0$, we see that $\lambda_n < \lambda_{n+1}$. In fact, λ_n is the median of the gamma distribution $\Gamma(n+1, 1)$. The preceding properties are illustrated in Figure 1 below.

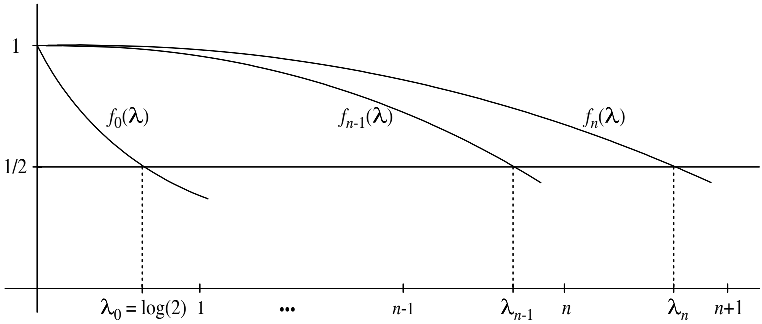

Fig. 1. Solutions of $f_n(\lambda) = 1/2$

For any $n \in \mathbb{N}$, denote by

$$g_n(u) := e^{-u}u^n, \ u \geq 0. \tag{5}$$

Although elementary, the following auxiliary result is crucial. A slightly different proof of it can be found in Teicher (1955).

Lemma 1. *The sequences* $\big(P(N_n \leq n),\ n \in \mathbb{N}\big)$ *and* $\big(P(N_n \geq n),\ n \in \mathbb{N}\big)$ *strictly decrease to* $1/2$.

Proof. Let $n \in \mathbb{N}$. By (4), we have

$$\begin{aligned} &P(N_n \leq n) - P(N_{n+1} \leq n+1) \\ &\quad = P(N_n \leq n+1) - P(N_{n+1} \leq n+1) - P(N_n = n+1) \\ &\quad = \frac{1}{(n+1)!}\int_n^{n+1} \big(g_{n+1}(u) - g_{n+1}(n)\big)du > 0, \end{aligned}$$

because $g_{n+1}(u)$ is strictly increasing in $[0, n+1]$. Similarly,

$$P(N_n \geq n) - P(N_{n+1} \geq n+1) = \frac{1}{n!}\int_n^{n+1} \big(g_n(n) - g_n(u)\big)du > 0.$$

By the CLT for the Poisson process, the sequences under consideration converge to $1/2$. This completes the proof of Lemma 1.

For notational reasons, we set $\lambda_{-1} = -1$. The following result gives us a closed form expression for the median of N_λ.

Theorem 1. *For any* $n \in \mathbb{N}$, *we have:* (*i*) $\lambda_n \in (n, n+1)$, *and (ii)* $\mathrm{Med}(N_\lambda) = n$, *whenever* $\lambda \in (\lambda_{n-1}, \lambda_n]$. *In particular*, $\mathrm{Med}(N_n) = n$.

Proof. By Lemma 1, $f_n(n) > 1/2$ and $f_n(n+1) = 1 - P(N_{n+1} \geq n+1) < 1/2$, thus showing (i). On the other hand, if $\lambda \in (\lambda_{n-1}, \lambda_n]$, we have the inequality $f_n(\lambda) = P(N_\lambda \leq n) \geq f_n(\lambda_n) = 1/2$, as well as $f_{n-1}(\lambda) = P(N_\lambda \leq n-1) < f_{n-1}(\lambda_{n-1}) = 1/2$, which, according to definition (1), implies (ii). Since, by part (i), $n \in (\lambda_{n-1}, \lambda_n)$, we see that $\mathrm{Med}(N_n) = n$, thus completing the proof of Theorem 1.

If, instead of (1), we use the definition of a median given at the beginning of Section 1, from Theorem 1 we conclude the following. If $\lambda \neq \lambda_n$ for any $n \in \mathbb{N}$, then N_λ has a unique median given by Theorem 1. If $\lambda = \lambda_n$ for some $n \in \mathbb{N}$, then the interval $[n, n+1)$ is the set of all of the medians of N_λ.

The following result provides a closed form expression for the mean deviation of N_λ with respect to its median.

Corollary 1. *For any $\lambda \geq 0$, we have*

$$E\left|N_\lambda - \mathrm{Med}(N_\lambda)\right| = 2\lambda P\left(N_\lambda = \mathrm{Med}(N_\lambda)\right) + 2\left(\mathrm{Med}(N_\lambda) - \lambda\right)\left(P\left(N_\lambda \leq \mathrm{Med}(N_\lambda)\right) - \frac{1}{2}\right).$$

In particular, for any $n \in \mathbb{N}$, we have $E\left|N_n - \mathrm{Med}(N_n)\right| = 2nP(N_n = n)$ and $E\left|N_{\lambda_n} - \mathrm{Med}(N_{\lambda_n})\right| = 2\lambda_n P(N_{\lambda_n} = n)$.

Proof. Let $\lambda \geq 0$ and $a \geq 0$. We claim that

$$E\left|N_\lambda - a\right| = 2\lambda P\left(N_\lambda = [a]\right) + 2(a - \lambda)\left(P\left(N_\lambda \leq [a]\right) - \frac{1}{2}\right), \tag{6}$$

where $[a]$ stands for the integer part of a. Indeed, denote by $x_+ = \max(0, x)$. Taking into account that $|x| = 2x_+ - x$, we obtain

$$\begin{aligned} E\left|N_\lambda - a\right| &= a - \lambda + 2E\left(N_\lambda - a\right)_+ = a - \lambda + 2\sum_{k=[a]+1}^{\infty} (k - a)e^{-\lambda}\frac{\lambda^k}{k!} \\ &= a - \lambda + 2\lambda P\left(N_\lambda \geq [a]\right) - 2aP\left(N_\lambda \geq [a] + 1\right), \end{aligned}$$

Which shows claim (6). Corollary 1 is an immediate consequence of (6) and Theorem 1.

3 Refinements

From Theorem 1, it is readily seen that $-1 < \mathrm{Med}(N_\lambda) - \lambda < 1$. Sharper estimates of this kind are available in the literature. For instance, Chen and Rubin (1986) obtain $-1 < \mathrm{Med}(N_\lambda) - \lambda < 1/3$, while Hamza (1995) gives the bound $\left|\mathrm{Med}(N_\lambda) - \lambda\right| \leq \log 2$. Choi (1994) has obtained the bounds

$$-\lambda_0 = -\log 2 \leq \mathrm{Med}(N_\lambda) - \lambda < 1/3, \qquad \lambda > 0, \tag{7}$$

thus giving a positive answer to conjecture 1 of Chen and Rubin (1986). Choi (1994) derives (7) from the estimates

$$n + \frac{2}{3} < \lambda_n \leq \min\left(n + \log 2, n + \frac{2}{3} + \frac{1}{2n+2}\right), \qquad n \in \mathbb{N}. \tag{8}$$

The bounds in (7) are the best possible absolute bounds. This can be seen as follows. Since $\lambda_0 = \log 2$, it follows from Theorem 1 (ii) that $\mathrm{Med}(N_{\lambda_0}) - \lambda_0 = -\log 2$. On the other hand, let $(\varepsilon_n, n \in \mathbb{N}^*)$ be a sequence converging to 0 such that $0 < \varepsilon_n < \lambda_n - \lambda_{n-1}$, $n \in \mathbb{N}^*$. It follows from Theorem 1 (ii) and (8) that

$$\mathrm{Med}(N_{\lambda_{n-1}+\varepsilon_n}) - (\lambda_{n-1} + \varepsilon_n) = n - \lambda_{n-1} - \varepsilon_n \in \left(\frac{1}{3} - \varepsilon_n - \frac{1}{2n}, \frac{1}{3} - \varepsilon_n\right). \quad (9)$$

Denoting by

$$c_n := \frac{1}{e}\left(1 + \frac{1}{n}\right)^{n-1}, \ n \in \mathbb{N}^*, \quad (10)$$

we state the following results, the proof of which is differed to Section 4.

Theorem 2. *For any $n \in \mathbb{N}$, we have*

$$n + \frac{2}{3} < \lambda_n < n + \frac{2}{3} + \frac{8(1 - c_{n+1})}{81 c_{n+1}}. \quad (11)$$

Corollary 2. *Let $\lambda \geq 0$ and $k \in \mathbb{N}$. Then,*

$$\mathrm{Med}(N_\lambda) - \lambda < 1/3 \quad (12)$$

and

$$-\frac{2}{3} - \frac{8(1 - c_{k+1})}{81 c_{k+1}} < \mathrm{Med}(N_\lambda) - \lambda, \qquad \lambda \geq k. \quad (13)$$

It turns out that the last term in the upper bound in (11) is of order n^{-1}. Indeed, using the inequalities $1 - e^{-x} \leq x$ and $\log(1+x) \geq x - x^2/2$, $x \geq 0$, we have

$$1 - c_{n+1} = 1 - e^{-\left(1 - n\log\left(1 + \frac{1}{n+1}\right)\right)} \leq 1 - n\left(\frac{1}{n+1} - \frac{1}{2(n+1)^2}\right) \leq \frac{3}{2(n+1)}.$$

We therefore obtain from (11) that

$$\lambda_n < n + \frac{2}{3} + \frac{4}{27(n+1)c_{n+1}}.$$

Actually, as follows by calculus, the upper bound for λ_n given in (11) is better than the corresponding upper bound in (8), provided that $n \geq 5$. In the same way, the lower bound in (13) is better than $-\log 2$, whenever $\lambda > 5$. On the other hand, the best possible asymptotic lower and upper bounds for $\mathrm{Med}(N_\lambda) - \lambda$ are $-2/3$ and $1/3$, respectively. In other words, we have

$$-\frac{2}{3} = \liminf_{\lambda \to \infty}\left(\mathrm{Med}(N_\lambda) - \lambda\right) \leq \limsup_{\lambda \to \infty}\left(\mathrm{Med}(N_\lambda) - \lambda\right) = \frac{1}{3}. \quad (14)$$

In fact, the first equality in (14) follows from Theorem 1 (ii), Theorem 2 and (13), by noting that $c_n \to 1$ as $n \to \infty$ and that $\mathrm{Med}(N_{\lambda_n}) - \lambda_n = n - \lambda_n$. The second equality in (14) follows from (9) and (12).

Finally, in a recent paper, Alm (2003) has shown conjecture 2 of Chen and Rubin (1986), by proving that the sequence $(\lambda_n - n, n \in \mathbb{N})$ decreases from $\log 2$ to $2/3$ as n increases from 0 to ∞. This result complements the bounds given in Theorem 2, as well as the first equality in (14), since we have

$$\lim_{n\to\infty} \uparrow \left(\mathrm{Med}(N_{\lambda_n}) - \lambda_n\right) = \lim_{n\to\infty} \uparrow (n - \lambda_n) = -\frac{2}{3}.$$

4 The proofs

To show Theorem 2, we need to improve Lemma 1. This is done in the following two auxiliary results.

Lemma 2. *The sequence* $\left(P(N_{n+a} \le n),\ n \in \mathbb{N}\right)$ *strictly decreases to* $1/2$, *for any* $a \in [0, 2/3]$.

Proof. Let $n \in \mathbb{N}$ and let g_n be the function defined in (5). As in the proof of Lemma 1, we have from (4) and Fubini's theorem that

$$\begin{aligned}
&P(N_{n+a} \le n) - P(N_{n+1+a} \le n+1) \\
&\quad = P(N_{n+a} \le n+1) - P(N_{n+1+a} \le n+1) - P(N_{n+a} = n+1) \\
&\quad = \frac{1}{(n+1)!} \int_{n+a}^{n+1+a} \left(g_{n+1}(u) - g_{n+1}(n+a)\right) du \\
&\quad = \frac{1}{(n+1)!} \int_{n+a}^{n+1+a} \int_{n+a}^{u} g'_{n+1}(z)\, dz\, du \\
&\quad = \frac{1}{(n+1)!} \int_{n+a}^{n+1+a} (n+1+a-z) g'_{n+1}(z)\, dz \\
&\quad = \frac{1}{(n+1)!} \int_{n+a}^{n+1+a} g_n(z) P_n(z)\, dz, \qquad (15)
\end{aligned}$$

where

$$P_n(z) := (n+1+a-z)(n+1-z).$$

By the CLT for the Poisson process, we have

$$\begin{aligned}
\lim_{n\to\infty} P(N_{n+a} \le n) &= \lim_{n\to\infty} \left(\frac{N_{n+a} - (n+a)}{\sqrt{n+a}} \le -\frac{a}{\sqrt{n+a}} \right) \\
&= P(Z \le 0) = \frac{1}{2}, \qquad (16)
\end{aligned}$$

where Z is a standard normal random variable. Therefore, in view of (15), Lemma 2 will follow as soon as we show that

$$\int_{n+a}^{n+1} g_n(z) P_n(z)\, dz > \int_{n+1}^{n+1+a} g_n(z) \left(-P_n(z)\right) dz. \qquad (17)$$

Since $a \in [0, 2/3]$, it follows by calculus that

$$\int_{n+a}^{n+1} P_n(z)\, dz = \frac{(1-a)^2}{2} - \frac{(1-a)^3}{6} \ge \frac{a^3}{6} = \int_{n+1}^{n+1+a} \left(-P_n(z)\right) dz. \qquad (18)$$

Hence, (17) follows from (18) and the fact that $g_n(z)$ strictly decreases in $[n, \infty)$, since

$$\int_{n+a}^{n+1} g_n(z)P_n(z)dz > g_n(n+1)\int_{n+a}^{n+1} P_n(z)dz$$
$$\geq g_n(n+1)\int_{n+1}^{n+1+a} \big(-P_n(z)\big)dz > \int_{n+1}^{n+1+a} g_n(z)\big(-P_n(z)\big)dz.$$

The proof of Lemma 2 is complete.

Lemma 3. *Let $n \in \mathbb{N}^*$ and let c_n be as in (10). Let $a \in (0, 1/3)$ be such that*

$$R_n(a) := 3a^2 - a^3 + (a-1)^3 c_n \leq 0. \tag{19}$$

Then,

$$P(N_{n-a} \geq n) > P(N_{n+1-a} \geq n+1).$$

Proof. Let $n \in \mathbb{N}^*$ and let g_n be as in (5). As in the proof of Lemma 2, we have

$$\begin{aligned}
&P(N_{n-a} \geq n) - P(N_{n+1-a} \geq n+1\big) \\
&\quad = -\big(P(N_{n-a} \leq n) - P(N_{n+1-a} \leq n) - P(N_{n-a} = n)\big) \\
&\quad = -\frac{1}{n!}\int_{n-a}^{n+1-a} \big(g_n(u) - g_n(n-a)\big)du \\
&\quad = -\frac{1}{n!}\int_{n-a}^{n+1-a}\int_{n-a}^{u} g_n'(z)dzdu \\
&\quad = -\frac{1}{n!}\int_{n-a}^{n+1-a} (n+1-a-z)g_n'(z)dz \\
&\quad = -\frac{1}{n!}\int_{n-a}^{n+1-a} g_{n-1}(z)Q_n(z)dz,
\end{aligned} \tag{20}$$

where

$$Q_n(z) := (n+1-a-z)(n-z).$$

Condition (19) implies, after some simple computations, that

$$\int_{n-a}^{n} Q_n(z)dz = \frac{a^2}{2} - \frac{a^3}{6} \leq -\frac{(a-1)^3}{6}c_n = c_n\int_{n}^{n+1-a}\big(-Q_n(z)\big)dz. \tag{21}$$

Since $g_{n-1}(z)$ is strictly decreasing in $[n-1, \infty)$, we have from (21) that

$$\begin{aligned}
&\int_{n}^{n+1-a} g_{n-1}(z)\big(-Q_n(z)\big)dz \\
&\quad > \frac{g_{n-1}(n+1-a)}{g_{n-1}(n-a)}g_{n-1}(n-a)\int_{n}^{n+1-a}\big(-Q_n(z)\big)dz \\
&\quad \geq c_n g_{n-1}(n-a)\int_{n}^{n+1-a}\big(-Q_n(z)\big)dz \\
&\quad \geq g_{n-1}(n-a)\int_{n-a}^{n} Q_n(z)dz > \int_{n-a}^{n} g_{n-1}(z)Q_n(z)dz.
\end{aligned}$$

This, togehter with (20) shows Lemma 3.

Proof of Theorem 2. Since $\lambda_0 = \log 2$, statement (11) is trivial for $n = 0$. Assume that $n \in \mathbb{N}^*$. By Lemma 2, we have $f_n(n + 2/3) > 1/2$, thus showing the lower bound in (11). On the other hand, let $a \in (0, 1/3)$. Recalling (19) and taking into account that $0 \le c_n \le 1$, we have

$$R_n(a) = (c_n - 1)a^3 + 3(1 - c_n)a^2 + 3ac_n - c_n$$
$$\le \frac{c_n - 1}{27} + \frac{1 - c_n}{3} + 3ac_n - c_n = 3ac_n + \frac{8 - 35c_n}{27}.$$

Therefore, choosing

$$a(n) := (35c_n - 8)(81c_n)^{-1},$$

we see that $R_n\big(a(n)\big) \le 0$. Since c_n increases to 1 as $n \to \infty$, then $a(n)$ increases to $1/3$ as $n \to \infty$. Since $a(n) \in (0, 1/3)$, we use the CLT for the Poisson process as in (16) to conclude that the sequence $\big(P(N_{k-a(n)} \ge n), k \ge n\big)$ converges to $1/2$. By Lemma 3, this sequence strictly decreases to $1/2$. As a consequence,
$f_n\big(n + 1 - a(n+1)\big) = 1 - P\big(N_{n+1-a(n+1)} \ge n + 1\big) < 1/2$. This shows that

$$\lambda_n < n + 1 - a(n+1) = n + \frac{2}{3} + \frac{8(1 - c_{n+1})}{81c_{n+1}}.$$

The proof of Theorem 2 is complete.

Proof of Corollary 2. Let $k \in \mathbb{N}$ and $\lambda \ge k$. Then $\lambda \in (\lambda_{n-1}, \lambda_n]$, for some $n \ge k$. By Theorems 1 and 2, we have

$$\mathrm{Med}(N_\lambda) - \lambda = n - \lambda < n - \lambda_{n-1} < 1/3, \qquad n \in \mathbb{N}^*.$$

This inequality shows (12), since the case $n = 0$ is trivial. Again by Theorems 1 and 2, we have

$$\mathrm{Med}(N_\lambda) - \lambda \ge n - \lambda_n > -\frac{2}{3} - \frac{8(1 - c_{n+1})}{81c_{n+1}} \ge -\frac{2}{3} - \frac{8(1 - c_{k+1})}{81c_{k+1}},$$

the last inequality because c_n is increasing. The proof of Corollary 3 is complete.

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