Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Economic Dynamics & Control 29 (2005) 931–952
JOURNAL OF Economic Dynamics & Control
www.elsevier.com/locate/jedc

Eductive expectations coordination on deterministic cycles in an economy with heterogeneous agents ☆

Giorgio Negroni*

Università Cattolica, via Necchi 5, 20123 Milano, Italy

Received 31 October 2002; accepted 28 April 2004
Available online 30 October 2004

Abstract

In this paper we study expectations coordination on a deterministic 2-cycle by an 'eductive' approach in which agents are concerned with forecasting the forecasts of others. This approach is grounded in some Common Knowledge assumptions. We consider an overlapping generations economy in which agents have heterogeneous utility functions. We show that locally eductively stable 2-cycles can exist when agents have stationary beliefs. We also show that heterogeneity may hinder eductive coordination.

JEL classification: D64; D90; E10; E32; E37

Keywords: Coordination; Rational expectations; Deterministic cycles; Strong rationality; Heterogeneity

1. Introduction

An important piece of the Keynesian paradigm is that a significant part of the observed economic fluctuations may be endogenously generated by volatile,

☆ An earlier version of this paper circulated as Delta working paper 2000/17. I would like to thank R. Guesnerie for helpful discussions. I also thank A. Agliari and G. Weinrich. I am the only responsible for errors and imprecisions. Financial support from MURST gratefully acknowledged.

*Tel.: +39 02 7234 2919; fax: +39 02 7234 2923.
E-mail address: giorgio.negroni@unicatt.it (G. Negroni).

0165-1889/$ - see front matter
doi:10.1016/j.jedc.2004.04.004

self-fulfilling expectations. In recent years, this view has received a significant attention, see Woodford (1990a,b), Boldrin and Woodford (1990) and Guesnerie and Woodford (1992), among others. However, at this stage the theory is incomplete since expectations coordination on an endogenous cycle is postulated rather than proved to be a possible outcome.[1] This problem has already been studied by several authors; examples of learning cycles through different kinds of adaptive algorithms include Grandmont and Laroque (1986), Guesnerie and Woodford (1991) and Hommes and Sorger (1998); the E-stability criterion has been studied by Evans and Honkapohja (1995a,b), while a genetic algorithm was used by Bullard and Duffy (1995) and Dawid (1996).

In this paper we take a different viewpoint and consider an 'eductive' learning process. From a methodological point of view, our approach is motivated by the observation that the logical status of the rational expectations equilibrium is that of a Nash equilibrium, a point made by Townsend (1978), Evans (1983), Guesnerie (1993) and Bryant (1995), among others. This implies that the accuracy of an individual forecast depends on the set of actions taken by other agents, actions which ultimately depend on other's forecasts. Hence, in order to form his own forecast, each agent has to be concerned with forecasting the forecasts of others. We take an eductive view of this process. This is intended to describe agents' attempt to simulate other agents' reasoning and occurs in virtual time. This approach, in turn, needs to be grounded in some Common Knowledge (hereafter CK) assumptions. To be specific, suppose the following assumptions are satisfied:[2] (i) agents' Bayesian rationality is CK; (ii) the temporary equilibrium map, transforming individual forecasts into an aggregate outcome, is CK too as well as (iii) the existence of some restriction ensuring that the relevant variable agents have to forecast (in our case, prices) will take values in a neighborhood of the equilibrium under scrutiny (in our case, a given cycle of period 2). We ask whether, starting from these initial conditions, successive iterations of the CK operator may lead agents to educe that the equilibrium actions are CK. When this happens, borrowing the terminology suggested by Guesnerie (1992), we say that the equilibrium under scrutiny is locally strongly rational (or, equivalently, locally eductively stable).

Eductive stability is related to other learning processes. In particular, when fundamentals are identical, the conditions for local strong rationality and those for iterative E-stability coincide. Iterative E-stability corresponds to the appropriate temporary equilibrium map (with common and point expectations) being locally contracting. Notice, however, that eductive stability, being based on heterogeneous expectations, is a priori a more restrictive stability concept than iterative E-stability. When instead fundamentals are heterogeneous (as in the present paper[3]), the conditions for local strong rationality are, in general, more demanding that those for iterative E-stability; see Evans and Guesnerie (1993) and Guesnerie (2002).

[1] This criticism applies to all equilibrium concepts embedding the rational expectations hypothesis.

[2] See Guesnerie (1992, 2002) and Evans and Guesnerie (1993).

[3] We consider heterogeneous utility functions.

We assume throughout that agents have stationary beliefs. We say that agents have stationary beliefs if they expect that prices in all even (respectively, odd) periods are the same. Our purpose is to investigate the possibility of eductive coordination on a cycle of period 2 (hereafter 2-cycle) with both prices belonging to the decreasing arm of the backward perfect foresight (b.p.f.) map. We take a local approach and study coordination problems arising in a small neighborhood of a given 2-cycle. We show that, under the stationary beliefs assumption, locally strongly rational 2-cycles can exist. However, the necessary and sufficient conditions for eductive stability depend on the type of heterogeneity considered at the 2-cycle: we show that too much structural heterogeneity hinders coordination.

The paper is organized as follow. In Section 2 we present the economy and derive the temporary equilibrium map. In Section 3 we describe the eductive process and give a definition of local strong rationality. In Section 4 first the necessary and sufficient conditions for eductive coordination are derived, then the existence of locally strongly rational 2-cycles is discussed. In Section 5 we discuss some related works and conclude.

2. Model and equilibria

We consider a one-dimensional, one-step forward-looking economy populated by overlapping generations of agents. Agents live two periods: they work in the first and consume in the second. There is only one good. Using a constant return to scale technology, each young agent transforms one unit of labor supply into one unit of output. Both labor and good markets are competitive. Hence, the current nominal wage is equal to the current price of consumption good, p_t. All wage earnings are saved by keeping money balances which allows agents to consume when old. We make the following assumption:

Assumption 1. The economy is populated by two types (v) of agents: type 1, in proportion α, and type 2, in proportion $1-\alpha$. Agents of the same type have the same utility functions and the same expectations; however, utility functions and expectations differ across types.

In all our discussion we shall keep α fixed. Let $u_v(c_v)$ and $m_v(L_v)$ denote, respectively, the type v agent's utility from consuming when old and the disutility from working when young. These utility functions satisfy the standard assumptions. Let $R_{t+1} = p_t/p_{t+1}$ denote the real interest rate. The optimal labor supply of a generic type v agent is denoted L_v and it is the solution of the program

$$\max_{Lv} \mathrm{E}[u_v(\widetilde{R}^v_{t+1} L_v)] - m_v(L_v), \tag{1}$$

where E denotes the mean operator. Here, we denote by the stochastic variable $\widetilde{R}^v_{t+1}$ the generic v-agent's subjective expectation (i.e. his prior) of the real interest rate. We also denote by R^v_{t+1} the v-agent's subjective point expectation of the real interest rate.[4] The deterministic labor supply of a generic type v agent with a point expectation R^v_{t+1} is $L_v(R^v_{t+1})$ and it is uniquely defined by the first-order condition

$$R^v_{t+1}u'_v(R^v_{t+1}L_v) = m'_v(L_v). \tag{2}$$

B.p.f. dynamics are described by the b.p.f. map $p_t = F(p_{t+1};\alpha)$. Normalizing the stock of money, this map is well defined and it is derived (by the implicit function theorem) from the market clearing condition

$$\frac{1}{p_t} = \alpha L_1\left(\frac{p_t}{p_{t+1}}\right) + (1-\alpha)L_2\left(\frac{p_t}{p_{t+1}}\right). \tag{3}$$

We denote by $e_v(R_{t+1}) \stackrel{\text{def}}{=} R_{t+1}L'_v(R_{t+1})/L_v(R_{t+1})$ the elasticity of labor supply of a generic agent of type v; we also denote by $\varepsilon_v(R_{t+1}) \stackrel{\text{def}}{=} e_v(R_{t+1})\theta_v(R_{t+1})$ the weighted elasticity of labor supply where $\theta_v(R_{t+1}) \stackrel{\text{def}}{=} \alpha_v L_v(R_{t+1})/L(R_{t+1})$ is the share of type v agents' production on total production (here $\alpha_1 = \alpha$ and $\alpha_2 = 1-\alpha$) and $L(R_{t+1}) \stackrel{\text{def}}{=} \alpha L_1(R_{t+1}) + (1-\alpha)L_2(R_{t+1})$ is the aggregate labor supply.

The equilibrium concept we are concerned with is that of a 2-cycle.

Definition 1. A 2-cycle is a couple of prices (p_l, p_h) such that $p_l \neq p_h$, $p_l = F(p_h;\alpha)$ and $p_h = F(p_l;\alpha)$.

We assume that p_l (resp. p_h) is the equilibrium value taken by prices in all even (resp. odd) periods; hence, $R_{lh} = p_l/p_h$ (resp. $R_{hl} = p_h/p_h$) is the equilibrium value of the real interest rate arising when moving from an even (resp. odd) to the subsequent odd (resp. even) period. We further assume, without loss of generality, that $p_l < p^* < p_h$, where p^* denotes the unique (monetary) stationary state. It thus follows that $R_{lh} < 1 < R_{hl}$.

It is a standard result that a sufficient condition for the existence of 2-cycles is $F'(p^*;\alpha) < -1$; see Grandmont (1985). Notice that the slope of the b.p.f. map, evaluated at a generic equilibrium price p, is $F'(p;\alpha) = F'_1(p;\alpha) + F'_2(p;\alpha)$, where $F'_v(p;\alpha)$ is the partial derivatives of the b.p.f. map with respect to type v agents' expectations, evaluated at the equilibrium price p (the expression of $F'_v(p;\alpha)$ is given in Lemma 2 in Appendix A).

We limit our analysis to 2-cycles with both prices belonging to the decreasing arm of the b.p.f. map, that is 2-cycles such that $F'(p_l;\alpha) < 0$ and $F'(p_h;\alpha) < 0$. This, in turn, is possible with different signs of $F'_v(p_{t+1};\alpha)$. We shall consider the following configurations:

- S1: $F'_1(p_l;\alpha) < 0$, $F'_2(p_l;\alpha) < 0$, $F'_1(p_h;\alpha) < 0$ and $F'_2(p_h;\alpha) < 0$. This corresponds to a situation in which the offer curves of all agents of both types are negatively sloped at the 2-cycle,[5] i.e. $\varepsilon_v(R_{t+1}) < 0, v = 1,2$, at both R_{lh} and R_{hl}.
- S2: $F'_1(p_l;\alpha) < 0, F'_2(p_l;\alpha) < 0, F'_1(p_h;\alpha) > 0$ and $F'_2(p_h;\alpha) < 0$. This corresponds to a situation in which the offer curve of type 2 agents are negatively sloped at the 2-cycle while the offer curves of type 1 agents are negatively sloped at R_{hl}

[4] When $\widetilde{R}^v_{t+1}$ is used, agents are maximizing expected utility; when R^v_{t+1} is used, agents behave as if under certainty equivalent.

[5] The link between offer curve and b.p.f. map is discussed in Grandmont (1985).

(i.e. $\varepsilon_1(R_{hl})<0$) and positively sloped at R_{lh} (i.e. $\varepsilon_1(R_{lh})>0$). This is compatible with both prices belonging to the decreasing arm of the b.p.f. map if $F_1'(p_h;\alpha)+F_2'(p_h;\alpha)<0$, i.e. $0<\varepsilon_1(R_{lh})<-\varepsilon_2(R_{lh})$; see Lemma 2 in Appendix A.

- S3: $F_1'(p_l;\alpha)>0, F_2'(p_l;\alpha)<0, F_1'(p_h;\alpha)>0$ and $F_2'(p_h;\alpha)<0$. This corresponds to a situation in which the offer curves of type 2 agents are negatively sloped at the 2-cycle while the offer curves of type 1 agents are positively sloped at the 2-cycle (i.e. $\varepsilon_2(R_{t+1})<0$ and $\varepsilon_1(R_{t+1})>0$ at both R_{lh} and R_{hl}). This is compatible with both prices belonging to the decreasing arm of the b.p.f. map if $F_1'(p_l;\alpha)+F_2'(p_l;\alpha)<0$ and $F_1'(p_h;\alpha)+F_2'(p_h;\alpha)<0$, i.e. $0<\varepsilon_1(R_{lh})<-\varepsilon_2(R_{lh})$ and $0<\varepsilon_1(R_{hl})<-\varepsilon_2(R_{hl})$; see Lemma 2 in Appendix A.

2.1. Temporary equilibrium map

We assume throughout that agents have stationary beliefs.

Definition 2. Agents have stationary beliefs if they expect that the price in all even (resp. odd) periods is constant.

In terms of coordination of agents' expectations, this is a strong assumption. It means that agents believe in stationary cycles. This belief is CK among present and future generations. Formally, this assumption implies that $R_{t+1}^v = 1/R_{t+2}^v$. Notice, however, that it does neither imply that agents have common expectations nor it excludes that agents have stochastic expectations.

One may question whether the stationary beliefs assumption makes our approach consistent with the observation of past prices (although the eductive approach does not require any observation of past prices). In fact, it would not be satisfactory to assume that agents hold stationary beliefs while the prices they observe contradict their beliefs. The eductive approach does not suffer this criticism: either the equilibrium is eductively stable (and the observed prices confirm agents beliefs) or it is unstable, and the eductive argument does not intend to make any prediction. Moreover, this criticism does not apply if the eductive argument is applied as of the first period. However, one may nevertheless claim that the observation of past prices 'solves' the coordination problem: after all, if a 2-cycle was observed in the past it seems plausible to suppose that, if agents believe in stationary cycles, they will expect the observed cycle also for the current and future periods. According to this argument, the coordination problem will thus be trivially solved by history, giving to the observed cycle the status of natural focal point. Notice, however, that at the heart of our approach is the fact that agents, in order to form their own forecast, cannot avoid to ask themselves what other agents are expecting. Therefore, even if we suppose that in the past the economy followed a 2-cycle, a rational agent living in the present period and not knowing others' expectations (but knowing that the current outcome depends on others' expectations) has to find whether the rates of return corresponding to the observed cycle are necessarily expected by other agents for current and future periods.

Suppose that the optimal choice of labor supply can *not* be made conditionally on the current nominal wage; this corresponds to a situation in which the current nominal wage will be known only at the end of the period. Agents have thus to form an expectation on next period price and on the current nominal wage as well.[6] In this context, the stationary beliefs assumption seems quite natural; see Woodford (1990a), Dawid (1996) and Desgranges and Negroni (2003). Imposing this assumption implies that, in a generic period t, with t even, the type v agent's labor supply is $L_v(\widetilde{R}^v_{lh})$, where $\widetilde{R}^v_{t+1} = \widetilde{R}^v_{lh} = \widetilde{p}^v_l/\widetilde{p}^v_h$ and where $\widetilde{p}^v_t = \widetilde{p}^v_l$ and $\widetilde{p}^v_{t+1} = \widetilde{p}^v_h$. Here, we denote by the stochastic variables $\widetilde{p}^v_l$ (resp. $\widetilde{p}^v_h$) the v-agent's subjective expectation of the market clearing price in all even (resp. odd) periods. Analogously, in a generic period t, with t odd, the generic type v agent's labor supply is $L_v(1/\widetilde{R}^v_{lh})$.

Normalizing to 1 the stock of money, the market clearing condition requires

$$\frac{1}{p_t} = \alpha L_1\left(\frac{\widetilde{p}^1_l}{\widetilde{p}^1_h}\right) + (1-\alpha)L_2\left(\frac{\widetilde{p}^2_l}{\widetilde{p}^2_h}\right). \tag{4}$$

This determines a unique price for period t as a function of the individuals' price expectations. We write

$$p_t = G_s((\widetilde{p}^v_l, \widetilde{p}^v_h)_{v=1,2}; \alpha) \stackrel{\text{def}}{=} \frac{1}{\alpha L_1(\widetilde{p}^1_l/\widetilde{p}^1_h) + (1-\alpha)L_2(\widetilde{p}^2_l/\widetilde{p}^2_h)} \tag{5}$$

and call this the temporary equilibrium map. When instead agents have point expectations (common to agents of the same type) we have the 'simplified' temporary equilibrium map[7]

$$p_t = G_p((p^v_l, p^v_h)_{v=1,2}; \alpha) \stackrel{\text{def}}{=} \frac{1}{\alpha L_1(p^1_l/p^1_h) + (1-\alpha)L_2(p^2_l/p^2_h)}. \tag{6}$$

3. Eductive coordination

In this section we briefly describe our eductive approach to expectations coordination on 2-cycles. From the viewpoint of agents living at some period t, with t even, the eductive process is based on the following CK assumption:

Assumption 2. It is CK among agents living at t that:

(i) for every $t' \geqslant t$, agents living at period t' are rational and know that period t' prices are determined by agents' expectations at t' according to the temporary equilibrium map $G_s((\widetilde{p}^v_l, \widetilde{p}^v_h)_{v=1,2}; \alpha)$;
(ii) for every $t' \geqslant t$, Point (i) holds true for agents living at period t'.

[6] A different institutional context arises when the current nominal wage is known at the time agents effectively supply their labor services. In this context, agents have to form an expectation on next period price only. See Desgranges and Negroni (2003) and Negroni (2004) for more detailed analysis.

[7] Notice that $G_s((\widetilde{x}^v, \widetilde{y}^v)_{v=1,2}; \alpha) = G_s((\widetilde{x}^v/\widetilde{y}^v)_{v=1,2}; \alpha)$ and $G_p((x^v, y^v)_{v=1,2}; \alpha) = G_p((x^v/y^v)_{v=1,2}; \alpha)$.

Notice that CK of the temporary equilibrium map implies CK of stationary beliefs. We take a local viewpoint and analyze a process of expectations coordination arising in a small neighborhood of a given 2-cycle. In order to study local eductive stability, we need some information concerning the way in which agents form their beliefs. This is provided by next assumption which introduces a local restriction on the set of possible prices.[8]

Assumption 3. For every t even (resp. odd) and for every t' and t'' odd (resp. even), with $t' > t$ and $t'' > t'$, it is CK among agents living in t that it will be CK among agents living at t' that the price at t'' will belong to $\Omega_h^0 \equiv [p_h - \omega_h^0, p_h + \omega_h^0]$ (resp. $\Omega_l^0 \equiv [p_l - \omega_l^0, p_l + \omega_l^0]$).

Since $p_l < p^* < p_h$, to ensure consistency we need to assume that ω_h^0 and ω_l^0 satisfy the condition $p_l + \omega_l^0 < p^* < p_h - \omega_h^0$. This ensures that the sets Ω_h^0 and Ω_l^0 do not have any intersection and that they do not include the stationary state itself.

Since agents are concerned with a periodical equilibrium, agents living in a generic period t have to educe the prices in all even and odd periods. Suppose t is even. Assumptions 2 and 3 imply that, after the first step of the process, it is CK that prices in the even period t and in the odd period $t+1$ will be in the new sets

$$\begin{aligned} \Omega_l^1 &= G_s((\Delta(\Omega_l^0 \times \Omega_h^0))_{v=1,2}) \cap \Omega_l^0, \\ \Omega_h^1 &= G_s((\Delta(\Omega_h^0 \times \Omega_l^0))_{v=1,2}) \cap \Omega_h^0, \end{aligned} \tag{7}$$

where $\Delta(\Omega_l^0 \times \Omega_h^0)$ denotes the set of agent's subjective probability distributions over $\Omega_l^0 \times \Omega_h^0$. Every further step is analogous. Hence, after n steps it is CK for agents living in t that prices in the even period t and in the odd period $t+1$ belong to the sets

$$\begin{aligned} \Omega_l^n &= G_s((\Delta(\Omega_l^{n-1} \times \Omega_h^{n-1}))_{v=1,2}) \cap \Omega_l^{n-1}, \\ \Omega_h^n &= G_s((\Delta(\Omega_h^{n-1} \times \Omega_l^{n-1}))_{v=1,2}) \cap \Omega_h^{n-1}. \end{aligned} \tag{8}$$

Eductive expectations coordination among agents born in period t obtains if, starting from arbitrary initial beliefs in Ω_h^0 and Ω_l^0, the sequences of sets of admissible expectations, Ω_h^n and Ω_l^n, converge, respectively, to $\{p_h\}$ and $\{p_l\}$ as $n \to \infty$. When this happens we say that the 2-cycle is locally eductively stable or locally strongly rational. This leads to the following formal definition of eductive stability:

Definition 3. A 2-cycle (p_l, p_h) is locally strongly rational if there exists local CK restrictions Ω_l^0 and Ω_h^0 of the 2-cycle such that, $\lim_{n\to+\infty} \Omega_s^n = \{p_s\}$ with $p_s = p_l$ (resp. $p_s = p_h$) if period t is even (resp. odd).

In order to proceed in our analysis we need to know how, in a small neighborhood of the equilibrium cycles, the system reacts to changes in expectations, which are heterogeneous and stochastic. This information is provided by the following lemma:

[8] Concerning the logical status of this restriction, see Guesnerie (2002).

Lemma 1. *Consider an economy populated by heterogeneous agents satisfying Assumption* 1. *Given stochastic expectations* $(\widetilde{p}_l^v, \widetilde{p}_h^v)_{v=1,2}$ *there exist point expectations* $(p_l^v, p_h^v)_{v=1,2}$ *such that, in a sufficiently small neighborhood of the equilibrium cycle, the behavior of the temporary equilibrium map* $p_t = G_s((\widetilde{p}_l^v, \widetilde{p}_h^v)_{v=1,2}; \alpha)$ *can be approximated by*

$$p_t - p_j \simeq \left.\frac{\partial G_p((p_l^v, p_h^v)_{v=1,2}; \alpha)}{\partial p_l^1}\right|_{p_l, p_h} (p_l^1 - p_l) + \left.\frac{\partial G_p((p_l^v, p_h^v)_{v=1,2}; \alpha)}{\partial p_l^2}\right|_{p_l, p_h} (p_l^2 - p_l)$$
$$+ \left.\frac{\partial G_p((p_l^v, p_h^v)_{v=1,2}; \alpha)}{\partial p_h^1}\right|_{p_l, p_h} (p_h^1 - p_h) + \left.\frac{\partial G_p((p_l^v, p_h^v)_{v=1,2}; \alpha)}{\partial p_h^2}\right|_{p_l, p_h} (p_h^2 - p_h),$$

where $p_j = p_l$ (*resp.* $p_j = p_h$) *if t is an even* (*resp. odd*) *period.*

Proof. see Appendix A.

This lemma says that, in a sufficiently small neighborhood of the equilibrium cycle, the temporary equilibrium map with stochastic and heterogeneous expectations, $G_s((\widetilde{p}_l^v, \widetilde{p}_h^v)_{v=1,2}; \alpha)$, can be approximated by the temporary equilibrium map with point (but still heterogeneous) expectations, $G_p((p_l^v, p_h^v)_{v=1,2}; \alpha)$. This means that relaxing the assumption of point expectations does not involve any additional eductive difficulty. Precisely, the effect of individual expectations on the determination of current period equilibrium price is additive across agents' types. For each type, it depends both on the sensitivity of the temporary equilibrium map with respect to agents' expectations and on the discrepancy between expectations and the equilibrium price. The remaining of the paper builds upon this Lemma which shows that Axiom 2 of Guesnerie (2002) actually holds true in the economy we are considering.

4. Eductive stability with stationary beliefs

We are now in a position to give our first result stating the necessary and sufficient conditions for local eductive stability of 2-cycles. These conditions are derived in terms of the weighted elasticities of labor supply $\varepsilon_v(R_{t+1})$.

Proposition 1. *Consider an economy with heterogeneous agents and let assumption* 1 *be satisfied. Assume that agents have stationary beliefs.*

(i) *A* 2-*cycle* (p_l, p_h) *satisfying the conditions* $\varepsilon_1(R_{lh}) < 0$, $\varepsilon_2(R_{lh}) < 0$, $\varepsilon_1(R_{hl}) < 0$ *and* $\varepsilon_2(R_{hl}) < 0$, *is locally strongly rational if and only if*

$$|\varepsilon_1(R_{lh}) + \varepsilon_2(R_{lh}) + \varepsilon_1(R_{hl}) + \varepsilon_2(R_{hl})| < 1;$$

(ii) *a* 2-*cycle* (p_l, p_h) *satisfying the conditions* $\varepsilon_1(R_{lh}) > 0$, $\varepsilon_2(R_{lh}) < 0$, $\varepsilon_1(R_{hl}) < 0$ *and* $\varepsilon_2(R_{hl}) < 0$ *is locally strongly rational if and only if*

$$|\varepsilon_1(R_{lh})| + |\varepsilon_2(R_{lh}) + \varepsilon_1(R_{hl}) + \varepsilon_2(R_{hl})| < 1;$$

(iii) *a* 2*-cycle* (p_l, p_h) *satisfying the conditions* $\varepsilon_1(R_{lh})>0$, $\varepsilon_1(R_{hl})>0$, $\varepsilon_2(R_{lh})<0$ *and* $\varepsilon_2(R_{hl})<0$ *is locally strongly rational if and only if*

$$|\varepsilon_1(R_{lh})+\varepsilon_1(R_{hl})|+|\varepsilon_2(R_{lh})+\varepsilon_2(R_{hl})|<1.$$

Proof. see Appendix A.

Observe that the conditions for local eductive stability depend on the specific type of heterogeneity considered at the 2-cycle.[9] This is a consequence of the fact that, as shown in Section 3, eductive coordination is based on the elimination of the largest sets of not admissible expectations. As shown in the proof of the above proposition, these sets, in turn, are derived under particular patterns of point expectations, patterns which depend on the heterogeneity configuration considered.

The conditions stated in the above proposition mean that eductive coordination on a specific equilibrium cycle occurs if and only if, at that cycle, agents are not too reactive to their expectations. This is in accordance with the general intuition developed in Guesnerie (2002). Since determinacy can be interpreted as absence of agents' overreaction to a change of expectations, Proposition 1 can be used to discuss the link between successful local eductive stability and determinacy of the cycle.[10] Since we are concerned with 2-cycles with both prices on the decreasing arm of the b.p.f. map, it turns out that determinacy is a necessary and sufficient condition for local strong rationality when the sign of the partial derivatives of the temporary equilibrium maps are identical across agents.[11] In fact, when the elasticities of labor supplies, evaluated at the 2-cycle, are negative, Point (i) of Proposition 1 becomes $\varepsilon_1(R_{lh})+\varepsilon_2(R_{lh})+\varepsilon_1(R_{hl})+\varepsilon_2(R_{hl})>-1$, which is exactly the condition for determinacy (see Lemma 2 in Appendix A). Hence when these signs are identical, heterogeneity (of expectations and of utility functions) does not create eductive difficulties; we are in the case in which local strong rationality and iterative E-stability still coincide. On the contrary, when the signs of the partial derivatives of the temporary equilibrium maps are different across agents, determinacy is no longer sufficient for local eductive stability. More precisely, while local eductive stability implies determinacy, the converse is no longer true. Local strong rationality is now more demanding than iterative E-stability. This is in accordance with Evans and Guesnerie (1993) and Guesnerie (2002). As argued by these authors, this is due to the fact that the effect on actual prices of individual reactions, induced by a change of

[9] A careful reading of the proposition proof shows that the determinants of the Jacobian matrices J_q, with $q=1,2,3$, are all zero. This implies that the relevant eductive process is one-dimensional, meaning that agents are here concerned with educing a single real interest rate rather than educing two distinct price levels. This is a direct consequence of the stationarity beliefs assumption.

[10] A link between stability of the learning process and determinacy has emerged also in the literature on adaptive learning of cycles; see Grandmont (1985), Grandmont and Laroque (1986), Guesnerie and Woodford (1991), Evans and Honkapohja (1995a,b) and Hommes and Sorger (1998). However, all these papers are concerned with representative agents economies in which agents have to forecast price levels (rather than real interest rates) and do not have stationary beliefs.

[11] This obtains also when agents have identical utility functions, see Negroni (2004).

individual expectations, is amplified when agents' expectations and utility functions are heterogeneous.

At this point (after a careful reading of the proposition proof) it should be clear that, since what matters is the sign of the partial derivatives of the temporary equilibrium map (evaluated at the cycle), limiting our analysis to two types of agents is by no means restrictive. Our results extend to an economy populated by an arbitrary number of agents types, v, provided this number is finite.[12] To see this, let us consider the following sets: $V^+_{lh} \equiv \{v \mid \varepsilon_v(R_{lh})>0\}$, $V^-_{lh} \equiv \{v \mid \varepsilon_v(R_{lh})<0\}$, $V^+_{hl} \equiv \{v \mid \varepsilon_v(R_{hl})>0\}$ and $V^-_{hl} \equiv \{v \mid \varepsilon_v(R_{hl})<0\}$.

Moreover, let

$$\eta^+(R_{lh}) \equiv \sum_{v\in V^+_{lh}} \varepsilon_v(R_{lh}), \eta^-(R_{lh}) \equiv \sum_{v\in V^-_{lh}} \varepsilon_v(R_{lh});$$

$\eta^+(R_{hl})$ and $\eta^-(R_{hl})$ are analogously defined. Point (i) of Proposition 1 then becomes $|\eta^-(R_{lh})+\eta^-(R_{hl})|<1$; Point (ii) becomes $|\eta^+(R_{lh})|+|\eta^-(R_{lh})+\eta^-(R_{hl})|<1$ and Point (iii) becomes $|\eta^+(R_{lh})+\eta^+(R_{hl})|+|\eta^-(R_{lh})+\eta^-(R_{hl})|<1$.

Having found the necessary and sufficient conditions for local eductive stability, a question naturally arising is the existence of locally strongly rational 2-cycles, that is the existence of 2-cycles for which the postulate of coordinated beliefs is justified according to the eductive approach. In order to derive our results, we compare the condition for eductive stability with the condition for the existence of 2-cycles derived from the Poincaré–Hopf index theorem.[13] Consider thus the mapping $p \to \psi_F(p)$, where[14]

$$\psi_F(p) = p - F(F(p)), \tag{9}$$

defined for $p \in B \overset{\text{def}}{=} [p_m, p_M]$. Assume that $\psi_F(p)$ points outward at the boundaries of B. In the class of model we are considering this obtains if both leisure and consumption are normal goods; see Azariadis and Guesnerie (1986), Woodford (1990a) and Guesnerie and Woodford (1992). The zeroes of $\psi_F(p)$ are the stationary state (p^*) and the 2-cycle (p_l, p_h). Given the behavior of $\psi_F(p)$ at the boundaries, we know from Poincaré–Hopf theorem that the sum of the indices of the zeroes of ψ_F is $+1$. Let $\psi'_F(p) = 1 - F'(F(p))F'(p)$. The index of a particular zero is $+1$ (resp. -1) if $\psi'_F(p)>0$ (resp. $\psi'_F(p)<0$) at that zero. The stationary state is the unique zero if $\psi'_F(p^*)>0$. When instead $\psi'_F(p^*)<0$, there must exist at least two other zeroes (i.e. at least one 2-cycle); moreover, since the sum of the indices is $+1$, there must exists (at least) one 2-cycle with index $+1$, that is

$$\psi'_F(p_l, p_h) = 1 - [F'_1(p_h) + F'_2(p_h)][F'_1(p_l) + F'_2(p_l)] > 0. \tag{10}$$

Taking into account Lemma 2 in Appendix A, this condition becomes

$$\varepsilon_1(R_{hl}) > -1 - \varepsilon_2(R_{hl}) - \varepsilon_2(R_{lh}) - \varepsilon_1(R_{lh}). \tag{11}$$

We are now in a position to give our final result.

[12] The expression in Lemma 1 has to be appropriately modified.

[13] See Chiappori and Davila (1996) and Guesnerie and Woodford (1991).

[14] To simplify the notation we omit to show the dependency of the b.p.f. map on α.

Proposition 2. *Consider an economy with heterogeneous agents and let Assumption* 1 *be satisfied. Assume that agents have stationary beliefs. Let* $\psi'_F(p^*)<0$, *then there exists* 2*-cycles satisfying condition* (11).

(i) *If one of these* 2*-cycle is such that* $\varepsilon_2(R_{lh})<0, \varepsilon_2(R_{hl})<0, \varepsilon_1(R_{lh})<0, \varepsilon_1(R_{hl})<0$ *and* $1+\varepsilon_2(R_{lh})+\varepsilon_2(R_{hl})>0$, *then it is locally strongly rational*;

(ii) *if one of these* 2*-cycle is such that* $\varepsilon_2(R_{lh})<0, \varepsilon_2(R_{hl})<0$, $1+\varepsilon_2(R_{lh})+\varepsilon_2(R_{hl})>0$, $0<\varepsilon_1(R_{lh})<\min[-\varepsilon_2(R_{lh}), 1+\varepsilon_2(R_{lh})+\varepsilon_2(R_{hl})]$ *and* $-1-\varepsilon_2(R_{lh})-\varepsilon_2(R_{hl})+\varepsilon_1(R_{lh})<\varepsilon_1(R_{hl})<0$, *then it is locally strongly rational*;

(iii) *if one of these* 2*-cycle is such that* $\varepsilon_2(R_{lh})<0, \varepsilon_2(R_{hl})<0$, $1+\varepsilon_2(R_{lh})+\varepsilon_2(R_{hl})>0$, $0<\varepsilon_1(R_{lh})<\min[-\varepsilon_2(R_{lh}), 1+\varepsilon_2(R_{lh})+\varepsilon_2(R_{hl})]$ *and* $0<\varepsilon_1(R_{hl})<\min[-\varepsilon_2(R_{hl}), 1+\varepsilon_2(R_{hl})+\varepsilon_2(R_{lh})-\varepsilon_1(R_{lh})]$, *then it is locally strongly rational.*

Proof. see Appendix A.

Point (i) refers to 2-cycles belonging to configuration S1; Points (ii) and (iii) refer to 2-cycles belonging to configurations S2 and S3, respectively. This Proposition says that, in an economy with heterogeneous agents, locally strongly rational 2-cycles (with both prices on the decreasing arm of the b.p.f. map) can effectively exist. It does not say that locally strongly rational 2-cycles always exist. In particular, while in configuration S1 locally strongly rational 2-cycle always exist (provided $1+\varepsilon_2(R_{lh})+\varepsilon_2(R_{hl})>0$), this is no longer the case for the other two configurations. This is not due to fact that the 2-cycles (with both prices on the decreasing arm of the b.p.f. map) detected by Poincaré–Hopf are incompatible the conditions defining configurations S2 and S3; these cycles always exist.[15] Rather, it is due to the fact that the detected cycles may not satisfy the conditions for local strong rationality of Proposition 1. We are here in the case in which heterogeneity hinders eductive coordination.

Proposition 2 is illustrated in Fig. 1 (this figure is drawn for fixed ε_2). Locally strongly rational 2-cycles exist in region OAB (configuration S1), in region OBC (configuration S2) and in region OCD (configuration S3). However, while 2-cycles belonging to configuration S2 (resp. S3) also exist in region BCHL (resp. DEFHC), they are not locally strongly rational.

Although Proposition 2 complements previous Proposition 1, the intuition behind them is the same: locally strongly rational 2-cycles can exist provided agents are not too reactive to their expectations.

An example: We now give a numerical example of the existence of locally strongly rational cycles in configurations S1 and S2.

[15] When $1+\varepsilon_2(R_{hl})+\varepsilon_2(R_{lh})>0$, if the detected 2-cycle has both prices on the decreasing arm of the b.p.f. map, it must be in one of the three configurations considered. It belongs to configuration S2 if $0<\varepsilon_1(R_{lh})<-\varepsilon_2(R_{lh})$ and to configuration S3 if $0<\varepsilon_1(R_{lh})<-\varepsilon_2(R_{lh})$ and $0<\varepsilon_1(R_{hl})<-\varepsilon_2(R_{hl})$. On the contrary, when $1+\varepsilon_2(R_{hl})+\varepsilon_2(R_{lh})<0$, the detected 2-cycle cannot be in configuration S1: it must be either in S2 or in S3. However, in this case no locally strongly rational 2-cycle can exist in these configurations.

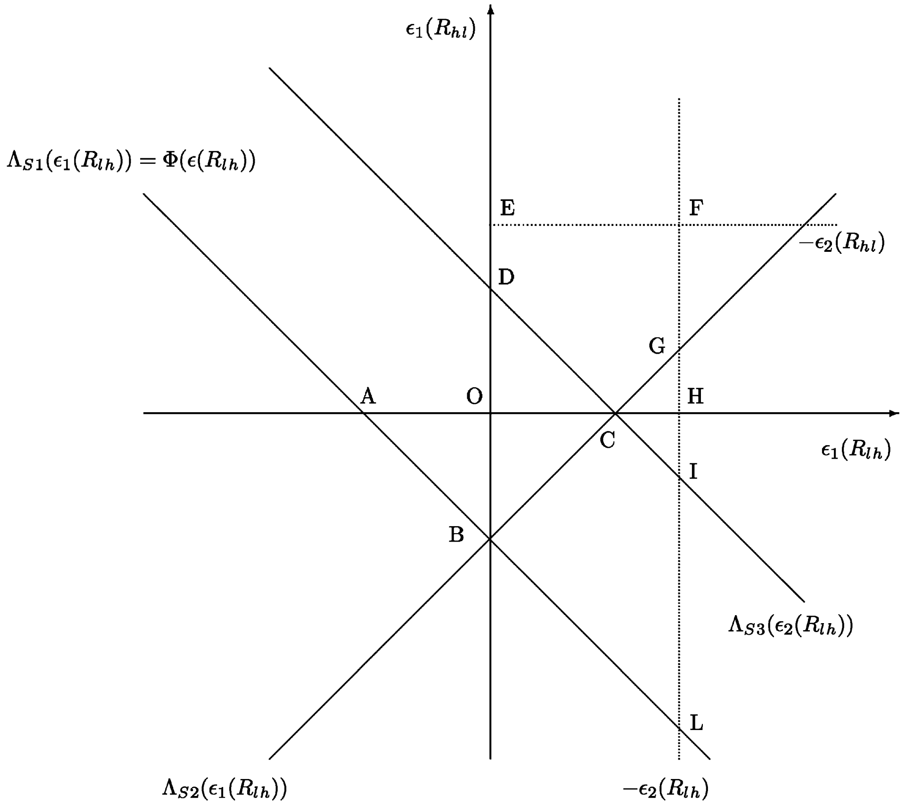

Fig. 1. The figure illustrates Proposition 2. It is drawn for fixed ε_2 satisfying the conditions $1+\varepsilon_2(R_{lh})+\varepsilon_2(R_{hl})<-\varepsilon_2(R_{lh})$ and $1+\varepsilon_2(R_{lh})+\varepsilon_2(R_{hl})<-\varepsilon_2(R_{hl})$. The line going through AB is $\varepsilon_1(R_{hl})=\Lambda_{S1}(\varepsilon_1(R_{lh}))=\Phi(\varepsilon_1(R_{lh}))$; the line going through BC is $\varepsilon_1(R_{hl})=\Lambda_{S2}(\varepsilon_1(R_{lh}))$ and the line going through CD is $\varepsilon_1(R_{hl})=\Lambda_{S3}(\varepsilon_1(R_{lh}))$; see the proposition proof. When 2-cycles exist, then $\varepsilon_1(R_{hl})>\Phi(\varepsilon_1(R_{lh}))$. A 2-cycle arising in configuration S1 satisfies Point (i) of Proposition 1 if $\varepsilon_1(R_{hl})>\Lambda_{S1}(\varepsilon_1(R_{lh}))$; a 2-cycle arising in configuration S2 satisfies Point (ii) of Proposition 1 if $\varepsilon_1(R_{hl})>\Lambda_{S2}(\varepsilon_1(R_{lh}))$; lastly, a 2-cycle arising in configuration S3 satisfies Point (iii) of Proposition 1 if $\varepsilon_1(R_{hl})<\Lambda_{S3}(\varepsilon_1(R_{lh}))$.

Let $u_v(c_{t+1})=a_v c_{v,t+1}-\frac{1}{2}c_{v,t+1}^2$ and $m_v(L_{v,t})=L_{v,t}$ be the utility functions. The labor supply of a generic agent of type v with point expectations R_{t+1}^v is $L_v=(a_v R_{t+1}^v-1)/(R_{t+1}^v)^2$. The b.p.f. map is $F(p_{t+1};\alpha)\overset{\text{def}}{=}(p_{t+1})^2/[p_{t+1}(\alpha a_1+(1-\alpha)a_2)-1]$.

- Let $a_1=3.1, a_2=3.2, \alpha=0.1$. In this case the stationary state is $p^*=0.456\,62$ and the 2-cycle is $(p_l,p_h)=(0.393\,53, 0.606\,47)$. Since $\varepsilon_1(R_{lh})=-1.080\,3\times10^{-3}$, $\varepsilon_1(R_{hl})=-7.092\,3\times10^{-2}$, $\varepsilon_2(R_{lh})=-6.430\,5\times10^{-2}$ and $\varepsilon_2(R_{hl})=-0.673\,72$, we are in configuration S1. This cycle is determinate being $|\varepsilon_1(R_{lh}) + \varepsilon_2(R_{lh}) + \varepsilon_1(R_{hl}) + \varepsilon_2(R_{hl})| = 0.810\,03 < 1$. Since $1 + \varepsilon_2(R_{lh}) + \varepsilon_2(R_{hl}) = 0.26198 > 0$, all the conditions of Point (i) of Proposition 2 are satisfied.

- Let $a_1 = 2.8, a_2 = 3.2, \alpha = 0.1$. In this case the stationary state is $p^* = 0.46296$ and the 2-cycle is $(p_l, p_h) = (0.40194, 0.59806)$. Since $\varepsilon_1(R_{lh}) = 1.051\,9 \times 10^{-2}$, $\varepsilon_1(R_{hl}) = -5.851\,6 \times 10^{-2}$, $\varepsilon_2(R_{lh}) = -0.120\,63$ and $\varepsilon_2(R_{hl}) = -0.671\,34$, we are in configuration S2 being $\varepsilon_1(R_{lh}) < -\varepsilon_2(R_{lh})$. This cycle is determinate and $|\varepsilon_1(R_{lh})| + |\varepsilon_2(R_{lh}) + \varepsilon_1(R_{hl}) + \varepsilon_2(R_{hl})| = 0.86101 < 1$. Notice that $-1 - \varepsilon_2(R_{lh}) - \varepsilon_2(R_{hl}) + \varepsilon_1(R_{lh}) = -0.197\,51 < \varepsilon_1(R_{hl}) < 0$; $1 + \varepsilon_2(R_{lh}) + \varepsilon_2(R_{hl}) = 0.208\,03 > 0$ and $\min[-\varepsilon_2(R_{lh}), 1 + \varepsilon_2(R_{lh}) + \varepsilon_2(R_{hl})] = 0.120\,63$ so that $0 < \varepsilon_1(R_{lh}) < 0.120\,63$. All the conditions of Point (ii)of Proposition 2 are satisfied.

5. Concluding remarks

In this paper we have studied the possibility of eductive expectations coordination on 2-cycles in a one-dimensional, one-step forward-looking economy. This type of learning describes a virtual process, anchored in CK of rationality, by means of which agents attempt to simulate the reasoning used by other agents in order to select their own forecast. We have studied expectations coordination problems arising in a small neighborhood of cycles of period 2.

Besides the different approach, our paper differs from most of the existing literature on expectations coordination on endogenous cycles since we focus on the role played by agents' heterogeneity. Two sources of heterogeneity have been considered. The first (considered in some works on adaptive learning[16]) is represented by heterogeneity of expectations; the second (almost neglected in the adaptive learning literature) by structural heterogeneity (i.e. heterogeneity of utility functions).

We have proved that locally strongly rational 2-cycles can exist when agents have stationary beliefs. However, heterogeneity of utility functions and expectations may hinder the process of eductive coordination. We have shown that the crucial factor determining when heterogeneity is at hindrance is represented by the sign of the partial derivatives of the temporary equilibrium map, evaluated at the equilibrium cycle. When these signs are the same for all agents (i.e. when agents' utility functions are not too different), then locally strongly rational 2-cycles exist; when instead these signs are different, the existence of locally strongly rational 2-cycles is more demanding: too much structural heterogeneity hinders eductive coordination.

Heterogeneity of expectations and of structural parameters has also been discussed in Honkapohja and Mitra (2003) and Negroni (2003), both concerned with adaptive learning of the stationary state. The first paper shows how different forms of heterogeneity in structure, forecasts and adaptive learning rules affect the

[16]See Grandmont (1994, Remark 2.3), Brock and Hommes (1997, 1998), Goeree and Hommes (2000), Hommes (1998, Section 6), Evans and Honkapohja (1996, 1999), Evans et al. (1996) and Slok and Sorensen (1997).

conditions for convergence of dynamics with learning. In particular, it shows that mild forms of heterogeneity in learning rules do not affect the stability conditions of the stationary state even in presence of structural heterogeneity; however, substantial differences in learning rules can hinder the stability of learning dynamics. The second paper only considers heterogeneous learning within a class of adaptive learning rules (with constant gain term). It shows that the crucial factor determining when heterogeneity is at hindrance is represented by the same condition on the sign of the partial derivatives of the temporary equilibrium map that we have detected in the present paper.

So far there has been no systematic attempt to study eductive stability of cycles; see, however, Guesnerie (1992) and Desgranges and Negroni (2003) for a first assessment of the problem, and Negroni (2004) for a more detailed analysis limited to an economy with identical utility functions. This last paper shows that the eductive approach can also be used to address the issue of equilibrium selection and focuses on the selection problem arising in a small neighborhood of the bifurcation giving rise to the 2-cycle. By exploiting the equivalence between determinacy and local strong rationality which holds in an economy with identical agents, the above paper shows that the 2-cycle is selected provided the bifurcation is supercritical.[17] Although this argument can be applied to an economy with heterogeneous agents, we expect that the 2-cycle is selected only if agents are not too heterogeneous. This because, as we have seen in the present paper, when utility functions are quite heterogeneous, determinacy no longer implies local strong rationality.

So far we have considered 2-cycles with both prices belonging to the decreasing arm of the b.p.f. map; we now briefly show that our approach can be immediately applied to 2-cycles with one price (p_l) on the decreasing arm of the b.p.f. map, and the other price (p_h) on the increasing one. When 2-cycles of this type are considered, two new configurations may be defined. These are:

- $L2: \; F_1'(p_l;\alpha)<0, \quad F_2'(p_l;\alpha)<0, \quad F_1'(p_h;\alpha)>0, \quad F_2'(p_h;\alpha)<0$ but $F_1'(p_h;\alpha)+F_2'(p_h;\alpha)>0$. This obtains when $\varepsilon_1(R_{lh})>-\varepsilon_2(R_{lh})$.
- $L3: \; F_1'(p_l;\alpha)>0, \quad F_2'(p_l;\alpha)<0, \quad F_1'(p_h;\alpha)>0, \quad F_2'(p_h;\alpha)<0$ but $F_1'(p_l;\alpha)+F_2'(p_l;\alpha)<0$ and $F_1'(p_h;\alpha)+F_2'(p_h;\alpha)>0$. This obtains when $0<\varepsilon_1(R_{hl})<-\varepsilon_2(R_{hl})$ and $\varepsilon_1(R_{lh})>-\varepsilon_2(R_{lh})$.

It follows that the necessary and sufficient conditions for local strong rationality stated at Points (ii) and (iii) of Proposition 1 still apply.[18] Moreover in configurations L2 and L3 locally strongly rational 2-cycles exist under the conditions stated, respectively, at Points (ii) and (iii) of Proposition 2, except that now it must be $-\varepsilon_2(R_{lh})<1+\varepsilon_2(R_{lh})+\varepsilon_2(R_{hl})$ and $-\varepsilon_2(R_{lh})<\varepsilon_1(R_{lh})<1+\varepsilon_2(R_{lh})+\varepsilon_2(R_{hl})$.

[17] This follows from the fact that, close to the bifurcation, since the 2-cycle must belong to an arbitrarily small neighborhood of the indeterminate stationary state, necessarily the 2-cycle must have both prices on the decreasing arm of the b.p.f. map. In an economy with identical utility functions, this 2-cycle is determinate and locally strong rationality.

[18] However, the relationship between determinacy and local strong rationality of 2-cycles satisfying configurations L2 and L3 is much more complex. In particular, determinacy is no longer implied by local strong rationality. We left the analysis to the interested reader.

In principle this approach can also be applied to cycles of higher order. Our conjecture is that, although the conditions for local strong rationality can be derived, the proof of the existence of locally eductively stable cycles can be a quite difficult task. We conjecture that the conditions for their existence become increasingly demanding with the periodicity of the cycle, since this increases the dimensions of heterogeneity that have to be taken into account.

Appendix A

Lemma 2. *Consider an economy populated by heterogeneous agents satisfying Assumption* 1. *Let α be given and let $p_t = F(p_{t+1};\alpha)$ be the backward perfect foresight map.*

(a) *The derivative of this b.p.f. map, evaluated at the equilibrium price, is*

$$\begin{aligned} F'(p_{t+1};\alpha) &= F_1'(p_{t+1};\alpha) + F_2'(p_{t+1};\alpha) \\ &= \frac{\varepsilon_1(R_{t+1}) + \varepsilon_2(R_{t+1})}{1 + \varepsilon_1(R_{t+1}) + \varepsilon_2(R_{t+1})} R_{t+1}, \end{aligned} \tag{A.1}$$

where

$$F_v'(p_{t+1};\alpha) = \frac{R_{t+1}\varepsilon_v(R_{t+1})}{1 + \varepsilon_1(R_{t+1}) + \varepsilon_2(R_{t+1})}, \tag{A.2}$$

for $v = (1,2)$, *and where*

$$\begin{aligned} \varepsilon_1(R_{t+1}) &\stackrel{\text{def}}{=} \frac{R_{t+1}L_1'(R_{t+1})}{L_1(R_{t+1})}\frac{\alpha L_1(R_{t+1})}{L(R_{t+1})} \stackrel{\text{def}}{=} e_1(R_{t+1})\theta_1, \\ \varepsilon_2(R_{t+1}) &\stackrel{\text{def}}{=} \frac{R_{t+1}L_2'(R_{t+1})}{L_2(R_{t+1})}\frac{(1-\alpha)L_2(R_{t+1})}{L(R_{t+1})} \stackrel{\text{def}}{=} e_2(R_{t+1})\theta_2. \end{aligned}$$

(b) $F_v'(p_{t+1};\alpha) < 0$ *if* $\varepsilon_v(R_{t+1}) < 0$.

Proof. Under Assumption 1, the market clearing condition is

$$\frac{1}{p_t} = \alpha L_1\left(\frac{p_t}{p_{t+1}}\right) + (1-\alpha)L_2\left(\frac{p_t}{p_{t+1}}\right) \stackrel{\text{def}}{=} L(R_{t+1}).$$

This implicitly defines the b.p.f. map, $p_t = F(p_{t+1};\alpha)$. The slope of this b.p.f. map is $F'(p_{t+1};\alpha) = F_1'(p_{t+1};\alpha) + F_2'(p_{t+1};\alpha)$ where, after few computations,

$$F'(p_{t+1};\alpha) = \frac{R^2\left[\frac{\alpha L_1'(R)}{L(R)} + \frac{(1-\alpha)L_2'(R)}{L(R)}\right]}{R\frac{\alpha L_1'(R)}{L(R)} + R\frac{(1-\alpha)L_2'(R)}{L(R)} + 1}$$

and

$$F_1'(p_{t+1};\alpha) = \frac{R^{2\,\frac{\alpha L_1'(R)}{L(R)}}}{R\frac{\alpha L_1'(R)}{L(R)} + R\frac{(1-\alpha)L_2'(R)}{L(R)} + 1},$$

$$F_2'(p_{t+1};\alpha) = \frac{R^{2\,\frac{(1-\alpha)L_2'(R)}{L(R)}}}{R\frac{\alpha L_1'(R)}{L(R)} + R\frac{(1-\alpha)L_2'(R)}{L(R)} + 1}.$$

Let $\varepsilon_v(R_{t+1}) \stackrel{\text{def}}{=} e_v(R_{t+1})\theta_v(R_{t+1})$ where $e_v(R_{t+1}) \stackrel{\text{def}}{=} R_{t+1}L_v'(R_{t+1})/L_v(R_{t+1})$ is the elasticity of labor supply of a generic agent of type v, $\theta_v(R_{t+1}) \stackrel{\text{def}}{=} \alpha_v L_v(R_{t+1})/L(R_{t+1})$ is the share of type v agents' production on total production (here $\alpha_1 = \alpha$ and $\alpha_2 = 1-\alpha$) and $L(R_{t+1})$ is the aggregate labor supply. After few manipulations, we get (A.1) and (A.2). This proves Point (a). Notice now that, from the first-order conditions defining the optimal labor supply of the generic agent of type v, the elasticity of labor supply of this generic agent (evaluated at the equilibrium real rate of interest, R) is

$$e_v(R) = \frac{1-\rho_{u_v}(RL_v(R))}{\rho_{u_v}(RL_v(R)) + \rho_{m_v}(L_v(R))}, \tag{A.3}$$

where $\rho_{u_v}(x_v) = -x_v u_v''(x_v)/u_v'(x_v) > 0$ and $\rho_{m_v}(y_v) = y_v m_v''(y_v)/m_v'(y_v) > 0$ denote, respectively, the coefficient of relative risk aversion associated with the functions u_v and m_v. The sign of $e_v(R)$ is thus determined by the sign of the numerator. Hence $e_v(R) < 0$ if $\rho_{u_v}(RL_v(R)) > 1$. Also notice that, when negative, $e_v(R)$ is always larger than -1. It thus follows that $1 + \theta_1 e_1(R) + \theta_2 e_2(R) > 0$, being $0 \leqslant \theta_1 = 1-\theta_2 \leqslant 1$. This implies that the sign of (A.2) is determined by the sign of the numerator. This proves Point (b). $\square$

Lemma 3. *Consider an economy with heterogeneous agents satisfying Assumption* 1. *Let* α *be given. Consider the 'simplified' temporary equilibrium map* $p_t = G_p((p_t^v, p_{t+1}^v)_{v=1,2};\alpha)$. *Let*

$$G_{v,1}(R_{t+1};\alpha) \stackrel{\text{def}}{=} G_{v,1}(p_t, p_{t+1};\alpha) \stackrel{\text{def}}{=} \left.\frac{\partial G_p((p_t^v, p_{t+1}^v)_{v=1,2};\alpha)}{\partial p_t^v}\right|_{p_t^v = p_t,\, p_{t+1}^v = p_{t+1}}$$

and

$$G_{v,2}(R_{t+1};\alpha) \stackrel{\text{def}}{=} G_{v,2}(p_t, p_{t+1};\alpha) \stackrel{\text{def}}{=} \left.\frac{\partial G_p((p_t^v, p_{t+1}^v)_{v=1,2};\alpha)}{\partial p_{t+1}^v}\right|_{p_t^v = p_t,\, p_{t+1}^v = p_{t+1}}$$

for $v = 1, 2$. *Then* $G_{v,2}(R_{t+1}) = -G_{v,1}(R_{t+1})R_{t+1}$ *where* $G_{v,1}(R_{t+1}) = -\varepsilon_v(R_{t+1})$.

Proof. Consider the 'simplified' temporary equilibrium map (derived under point expectations common to all agents of the same type)

$$p_t = G_p((p_t^v, p_{t+1}^v)_{v=1,2};\alpha) \stackrel{\text{def}}{=} \frac{1}{\alpha L_1\left(\frac{p_t^1}{p_{t+1}^1}\right) + (1-\alpha)L_2\left(\frac{p_t^2}{p_{t+1}^2}\right)}.$$

Under the stationary beliefs assumption the point expectations, common to all type v agents, are $p_t^v = p_l^v$ (resp. p_h^v) and $p_{t+1}^v = p_h^v$ (resp. p_l^v) if t is even (resp. odd). Differentiating with respect to the common expectations p_t^1 and p_{t+1}^1 held by agents of type 1, and evaluating at the equilibrium prices, we get, respectively,

$$G_{1,1}(R_{t+1};\alpha) \stackrel{\text{def}}{=} G_{1,1}(p_t, p_{t+1};\alpha) = -\frac{\partial L_1(R_{t+1})}{\partial R_{t+1}^i(1)} \frac{\alpha}{L(R_{t+1})} \frac{p_t}{p_{t+1}},$$

$$G_{1,2}(R_{t+1};\alpha) \stackrel{\text{def}}{=} G_{1,2}(p_t, p_{t+1};\alpha) = \frac{\partial L_1(R_{t+1})}{\partial R_{t+1}^i(v)} \frac{\alpha}{L(R_{t+1})} \left[\frac{p_t}{p_{t+1}}\right]^2.$$

The expressions for $G_{2,1}(p_t, p_{t+1};\alpha)$ and $G_{2,2}(p_t, p_{t+1};\alpha)$ are analogously derived by differentiating the market clearing condition with respect to the common expectations p_t^2 and p_{t+1}^2 held by agents of type 2. Notice that $G_{v,2}(R_{t+1};\alpha) = -G_{v,1}(R_{t+1};\alpha)R_{t+1}$. After few computations we obtain, respectively, $G_{v,1}(R_{t+1};\alpha) = -\varepsilon_v(R_{t+1})$ and $G_{v,2}(R_{t+1};\alpha) = \varepsilon_v(R_{t+1})R_{t+1}$, where $\varepsilon_v(R_{t+1})$ is the weighted elasticity of labor supply of type v agents (see previous Lemma 2). $\square$

We now give the proofs of Lemma 1 and of the propositions of the main text.

Proof of Lemma 1. Consider a connected neighborhood $\Omega_l \times \Omega_h$ of the 2-cycle (p_l, p_h). We first show that when expectations are in $\Omega_l \times \Omega_h$, the set of prices $G_s((\Delta(\Omega_l \times \Omega_h))_{v=1,2})$, derived under stochastic expectations, reduces to the set of prices $G_p((\Omega_l \times \Omega_h)_{v=1,2})$, derived under point expectations. Then we show that, when the supports of subjective expectations are small neighborhoods of the equilibrium cycle, the lemma follows from a direct application of a first-order Taylor expansion.

Let Ω_{lh} be the set of expected real interest rates R_{lh}^v when the expected prices (p_l^v, p_h^v) are, respectively, $p_l^v \in \Omega_l$ and $p_h^v \in \Omega_h$. The set Ω_{lh} is connected. Consider the generic agent of type v. The first-order condition defining $L_v(R_{t+1}^v)$ is $Z_v(R_{t+1}^v) = m_v'(L_v)$, with

$$Z_v(R_{t+1}^v) \stackrel{\text{def}}{=} R_{t+1}^v u_v'(R_{t+1}^v L_v).$$

Notice now that, for every given L_v, $Z_v(R_{t+1}^v)$ is continuous. As the set Ω_{lh} is connected, this implies that the set $Z_v(\Omega_{lh})$ is a connected part of $\mathscr{R}$, i.e. an interval. Therefore, for every distribution $\tilde{R}_{lh}^v$ in $\Delta(\Omega_{lh})$, there is a real value R_{lh}^v in Ω_{lh} such that

$$\mathrm{E}[Z_v(\tilde{R}_{lh}^v)] = Z_v(R_{lh}^v). \tag{A.4}$$

Now, for every $\tilde{R}_{lh}^v$ in $\Delta(\Omega_{lh})$, write $L_v = L_v(\tilde{R}_{lh}^v)$ for short and consider the value R_{lh}^v satisfying condition (A.4). The first-order condition defining L_v is

$$\mathrm{E}[Z_v(\tilde{R}_{lh}^v)] = m_v'(L_v).$$

We then have $Z_v(R_{lh}^v) = m_v'(L_v)$, meaning that L_v is also the optimal labor supply with point expectations R_{lh}^v. Therefore, $L_v[\Delta(\Omega_{lh})]$ is included in $L_v(\Omega_{lh})$. As the reciprocal inclusion holds true, we have just shown that $L_v(\Omega_{lh}) = L_v[\Delta(\Omega_{lh})]$ that is, for each agent of type v, the set of optimal labor supplies with point expectations and the set of optimal labor supply with stochastic expectations coincide. Since all agents of the same type have the same utility functions and the same expectations, this implies that $\alpha L_1(\Delta(\Omega_{lh})) = \alpha L_1(\Omega_{lh})$ and $(1-\alpha)L_2(\Delta(\Omega_{lh})) = (1-\alpha)L_2(\Omega_{lh})$. Therefore, $\alpha L_1(\Delta(\Omega_{lh})) + (1-\alpha)L_2(\Delta(\Omega_{lh}))$ and $\alpha L_1(\Omega_{lh}) + (1-\alpha)L_2(\Omega_{lh})$ coincide. This means that $G_s((\Delta(\Omega_l \times \Omega_h))_{v=1,2})$ and $G_p((\Omega_l \times \Omega_h)_{v=1,2})$ coincide too.

Consider now the optimal labor supply when agents have point expectations. We show that the lemma follows when the sets Ω_l and Ω_h are small neighborhoods of the equilibrium cycle. Market clearing in a generic even period implies that

$$\Omega_l = \frac{1}{\alpha L_1(\Omega_{lh}) + (1-\alpha)L_2(\Omega_{lh})}.$$

Notice that $L_v(\Omega_{lh})$ is the interval of individual optimal supplies when the generic agent of type v expect $R^v_{t+1} \in \Omega_{lh}$, that is when $p^v_h \in \Omega_h$ and $p^v_l \in \Omega_l$. Then, when the sets Ω_l and Ω_h are small neighborhood of the 2-cycle, taking into account previous Lemma 3, the lemma follows from a first-order Taylor expansion, being higher order terms negligible. By the same argument we can show that the same obtains when the expected real interest rate is in Ω_{hl}. This proves the lemma. □

Proof of Proposition 1. Consider first the case S1 in which $\varepsilon_1(R_{lh})<0$, $\varepsilon_1(R_{hl})<0$, $\varepsilon_2(R_{hl})<0$ and $\varepsilon_2(R_{lh})<0$. Suppose that period t is even. Let $\Omega^0_h \equiv [p_h - \omega^0_h, p_h + \omega^0_h]$ and $\Omega^0_l \equiv [p_l - \omega^0_l, p_l + \omega^0_l]$ be the initial restrictions. Consider the first step of the eductive process. Since[19] $G_{v,1}(R_{lh})>0$ and $G_{v,2}(R_{lh})<0$, it follows that inf Ω^1_l (resp. sup Ω^1_l) is obtained when agents of both types expect $p^v_t = \inf \Omega^0_l$ (resp. $p^v_t = \sup \Omega^0_l$) and $p^v_{t+1} = \sup \Omega^0_h$ (resp. $p^v_{t+1} = \inf \Omega^0_h$). Analogously, since $G_{v,1}(R_{hl})>0$ and $G_{v,2}(R_{hl})<0$, it follows that inf Ω^1_h (resp. sup Ω^1_h) is obtained when agents of both types expect $p^v_{t+1} = \inf \Omega^0_h$ (resp. $p^v_{t+1} = \sup \Omega^0_h$) and $p^v_{t+2} = \sup \Omega^0_l$ (resp. $p^v_{t+2} = \inf \Omega^0_l$). Taking into account Lemma 1, we obtain

$$J_1 \begin{pmatrix} \omega^0_l \\ \omega^0_h \end{pmatrix} \leqslant \begin{pmatrix} p^1_{l,t} - p_l \\ p^1_{h,t+1} - p_h \end{pmatrix} \leqslant -J_1 \begin{pmatrix} \omega^0_l \\ \omega^0_h \end{pmatrix},$$

where $p^1_{l,t}$ and $p^1_{h,t+1}$ represent, respectively, the price educed after the first step of the eductive process in the even period t and in the odd period $t+1$, respectively. Notice that

$$J_1 = \begin{bmatrix} -G_{1,1}(R_{lh}) - G_{2,1}(R_{lh}) & G_{1,2}(R_{lh}) + G_{2,2}(R_{lh}) \\ G_{1,2}(R_{hl}) + G_{2,2}(R_{hl}) & -G_{1,1}(R_{hl}) - G_{2,1}(R_{hl}) \end{bmatrix}. \tag{A.5}$$

Proceeding inductively, after n steps it is CK that

$$J_1 \begin{pmatrix} \omega^{n-1}_l \\ \omega^{n-1}_h \end{pmatrix} \leqslant \begin{pmatrix} p^n_{l,t} - p_l \\ p^n_{h,t+1} - p_h \end{pmatrix} \leqslant -J_1 \begin{pmatrix} \omega^{n-1}_l \\ \omega^{n-1}_h \end{pmatrix}.$$

Local eductive stability obtains, as $n \to \infty$, if and only if J_1 is a contraction. Notice that, since $G_{v,2}(R_{lh}) = -G_{v,1}(R_{lh})R_{lh}$ and $G_{v,2}(R_{hl}) = -G_{v,1}(R_{hl})R_{hl}$, $\det[J_1] = 0$. Hence local strong rationality obtains if and only if $-1<\mathrm{tr}[J_1]<1$, condition

[19] To simplify the exposition we omit to show the dependency of $G_{v,j}(R)$ on α.

satisfied when

$$|G_{1,1}(R_{lh}) + G_{2,1}(R_{lh}) + G_{1,1}(R_{hl}) + G_{2,1}(R_{hl})| < 1.$$

Taking Lemma 3 into account proves the proposition when $\varepsilon_1(R_{lh}) < 0$, $\varepsilon_1(R_{hl}) < 0$, $\varepsilon_2(R_{hl}) < 0$ and $\varepsilon_2(R_{lh}) < 0$.

Consider now the case S2 in which $\varepsilon_1(R_{lh}) > 0$, $\varepsilon_1(R_{hl}) < 0$, $\varepsilon_2(R_{hl}) < 0$, $\varepsilon_2(R_{lh}) < 0$. Suppose that period t is even. Let Ω_h^0 and Ω_l^0 be the initial restrictions and consider the first step. The only difference with respect to the previous case is that now $G_{1,1}(R_{lh}) < 0$ (and hence $G_{1,2}(R_{lh}) > 0$). It follows that $\inf \Omega_l^1$ is obtained when type 1 (resp. type 2) agents expect $p_t^1 = \sup \Omega_l^0$ (resp. $p_t^2 = \inf \Omega_l^0$) and $p_{t+1}^1 = \sup \Omega_h^0$ (resp. $p_{t+1}^2 = \inf \Omega_h^0$). Instead, $\sup \Omega_l^1$ is obtained when type 1 (resp. type 2) agents expect $p_t^1 = \inf \Omega_l^0$ (resp. $p_t^2 = \sup \Omega_l^0$) and $p_{t+1}^1 = \sup \Omega_h^0$ (resp. $p_{t+1}^2 = \inf \Omega_h^0$). Since $G_{v,1}(R_{hl}) > 0$ (and hence $G_{v,2}(R_{hl}) < 0$), the largest set of admissible prices in the odd period $t+1$ is obtained under the same pattern of expectations discussed in previous case. Taking into account Lemma 1 we obtain

$$J_2 \begin{pmatrix} \omega_l^0 \\ \omega_h^0 \end{pmatrix} \leqslant \begin{pmatrix} p_{l,t}^1 - p_l \\ p_{h,t+1}^1 - p_h \end{pmatrix} \leqslant -J_2 \begin{pmatrix} \omega_l^0 \\ \omega_h^0 \end{pmatrix},$$

where

$$J_2 = \begin{bmatrix} G_{1,1}(R_{lh}) - G_{2,1}(R_{lh}) & -G_{1,2}(R_{lh}) + G_{2,2}(R_{lh}) \\ G_{1,2}(R_{hl}) + G_{2,2}(R_{hl}) & -G_{1,1}(R_{hl}) - G_{2,1}(R_{hl}) \end{bmatrix}. \tag{A.6}$$

Proceeding inductively, after n steps it is CK that

$$J_2 \begin{pmatrix} \omega_l^{n-1} \\ \omega_h^{n-1} \end{pmatrix} \leqslant \begin{pmatrix} p_{l,t}^n - p_l \\ p_{h,t+1}^n - p_h \end{pmatrix} \leqslant -J_2 \begin{pmatrix} \omega_l^{n-1} \\ \omega_h^{n-1} \end{pmatrix}.$$

Local eductive stability obtains, as $n \to \infty$, if and only if J_2 is a contraction. Since $G_{v,2}(R_{lh}) = -G_{v,1}(R_{lh})R_{lh}$ and $G_{v,2}(R_{hl}) = -G_{v,1}(R_{hl})R_{hl}$, it follows that $\det[J_2] = 0$. Hence, local strong rationality obtains if and only if $-1 < \mathrm{tr}[J_2] < 1$, condition satisfied when

$$|G_{1,1}(R_{lh})| + |G_{2,1}(R_{lh}) + G_{1,1}(R_{hl}) + G_{2,1}(R_{hl})| < 1.$$

Taking Lemma 3 into account proves the proposition when $\varepsilon_1(R_{lh}) > 0$, $\varepsilon_1(R_{hl}) < 0$, $\varepsilon_2(R_{hl}) < 0$, $\varepsilon_2(R_{lh}) < 0$.

Lastly, consider the case S3 in which $\varepsilon_1(R_{lh}) > 0$, $\varepsilon_1(R_{hl}) > 0$, $\varepsilon_2(R_{hl}) < 0$, $\varepsilon_2(R_{lh}) < 0$. Suppose that period t is even. Let Ω_h^0 and Ω_l^0 be the initial restrictions and consider the first step. Since the signs of $G_{v,1}(R_{lh})$ and $G_{v,2}(R_{lh})$ are as in S2, the largest set of admissible prices in the even period t is obtained as before. However, since $G_{v,1}(R_{hl}) < 0$ and $G_{v,2}(R_{hl}) > 0$, the largest set of admissible prices in the odd period $t+1$ is now obtained under different conditions with respect to the previous case S2. In fact, $\inf \Omega_h^1$ is obtained when type 1 (resp. type 2) agents expect $p_{t+1}^1 = \sup \Omega_h^0$ (resp. $p_{t+1}^2 = \inf \Omega_l^0$) and $p_{t+2}^1 = \inf \Omega_l^0$ (resp. $p_{t+2}^2 = \sup \Omega_l^0$). Instead, $\sup \Omega_h^1$ is obtained when type 1 (resp. type 2) agents expect $p_{t+1}^1 = \inf \Omega_h^0$ (resp. $p_{t+1}^2 = \sup \Omega_h^0$)

and $p^1_{t+2} = \sup \Omega^0_l$(resp. $p^2_{t+2} = \inf \Omega^0_l$). Taking into account Lemma 1 we obtain

$$J_3 \begin{pmatrix} \omega^0_l \\ \omega^0_h \end{pmatrix} \leqslant \begin{pmatrix} p^1_{l,t} - p_l \\ p^1_{h,t+1} - p_h \end{pmatrix} \leqslant -J_3 \begin{pmatrix} \omega^0_l \\ \omega^0_h \end{pmatrix},$$

where

$$J_3 = \begin{bmatrix} G_{1,1}(R_{lh}) - G_{2,1}(R_{lh}) & -G_{1,2}(R_{lh}) - G_{2,2}(R_{lh}) \\ -G_{1,2}(R_{hl}) - G_{2,2}(R_{hl}) & G_{1,1}(R_{hl}) - G_{2,1}(R_{hl}) \end{bmatrix}. \tag{A.7}$$

Proceeding inductively, after n steps it is CK that

$$J_3 \begin{pmatrix} \omega^{n-1}_l \\ \omega^{n-1}_h \end{pmatrix} \leqslant \begin{pmatrix} p^n_{l,t} - p_l \\ p^n_{h,t+1} - p_h \end{pmatrix} \leqslant -J_3 \begin{pmatrix} \omega^{n-1}_l \\ \omega^{n-1}_h \end{pmatrix}.$$

Local strong rationality obtains as $n \to \infty$ if and only if J_3 is a contraction. Since $G_{v,2}(R_{lh}) = -G_{v,1}(R_{lh})R_{lh}$ and $G_{v,2}(R_{hl}) = -G_{v,1}(R_{hl})R_{hl}$, $\det[J_3] = 0$. Hence, local strong rationality obtains if and only if $-1 < \mathrm{tr}[J_3] < 1$, condition satisfied when

$$|G_{1,1}(R_{lh}) + G_{1,1}(R_{hl})| + |G_{2,1}(R_{lh}) + G_{2,1}(R_{hl})| < 1.$$

Taking Lemma 3 into account proves the proposition when $\varepsilon_1(R_{lh}) > 0$, $\varepsilon_1(R_{hl}) > 0$, $\varepsilon_2(R_{hl}) < 0$, $\varepsilon_2(R_{lh}) < 0$. □

Proof of Proposition 2. We have to prove that, in each configuration considered, the Poincaré–Hopf conditions for the existence of cycles is compatible with the conditions for local strong rationality given in Proposition 1.

Let $\psi'_F(p^*) < 0$. Then there exists (at least) one 2-cycle such that $\varepsilon_1(R_{hl}) > \Phi(\varepsilon_1(R_{lh}))$ where $\Phi(\varepsilon_1(R_{lh}))$ denotes the r.h.s. of (11) as a function of $\varepsilon_1(R_{lh})$; $\Phi(\varepsilon_1(R_{lh}))$ is decreasing.

In configuration S1, $\varepsilon_1(R_{lh}) < 0$, $\varepsilon_1(R_{hl}) < 0$, $\varepsilon_2(R_{hl}) < 0$ and $\varepsilon_2(R_{lh}) < 0$. The condition for local strong rationality is given by Point (i) of Proposition 1 and it can be written as $\varepsilon_1(R_{hl}) > -1 - \varepsilon_2(R_{hl}) - \varepsilon_2(R_{lh}) - \varepsilon_1(R_{lh})$. Let $\Lambda_{S1}(\varepsilon_1(R_{lh}))$ denote the r.h.s. of this inequality, as a function of $\varepsilon_1(R_{lh})$. We have $\Lambda_{S1}(\varepsilon_1(R_{lh})) = \Phi(\varepsilon_1(R_{lh}))$. However, $\varepsilon_1(R_{hl}) > \Lambda_{S1}(\varepsilon_1(R_{lh}))$ is compatible with $\varepsilon_1(R_{lh}) < 0$ and $\varepsilon_1(R_{hl}) < 0$ only when $1 + \varepsilon_2(R_{hl}) + \varepsilon_2(R_{lh}) > 0$. This proves Point (i).

In configuration S2, $\varepsilon_1(R_{lh}) > 0$, $\varepsilon_1(R_{hl}) < 0$, $\varepsilon_2(R_{hl}) < 0$, $\varepsilon_2(R_{lh}) < 0$ and $0 < \varepsilon_1(R_{lh}) < -\varepsilon_2(R_{lh})$. The condition for local strong rationality is given by Point (ii) of Proposition 1 and can be written as $\varepsilon_1(R_{hl}) > -1 - \varepsilon_2(R_{hl}) - \varepsilon_2(R_{lh}) + \varepsilon_1(R_{lh})$. Let $\Lambda_{S2}(\varepsilon_1(R_{lh}))$ denote the r.h.s. of this inequality, as a function of $\varepsilon_1(R_{lh})$. $\Lambda_{S2}(\varepsilon_1(R_{lh}))$ is increasing. When $\varepsilon_1(R_{lh}) = 0$, $\Lambda_{S2}(0) = \Phi(0) = -1 - \varepsilon_2(R_{hl}) - \varepsilon_2(R_{lh})$. If $1 + \varepsilon_2(R_{hl}) + \varepsilon_2(R_{lh}) > 0$, we have $\Lambda_{S2}(\varepsilon_1(R_{lh})) > \Phi(\varepsilon_1(R_{lh}))$ for $\varepsilon_1(R_{lh}) > 0$ and the detected 2-cycle with $\varepsilon_1(R_{lh}) > 0$, $\varepsilon_1(R_{hl}) < 0$, $\varepsilon_2(R_{hl}) < 0$ and $\varepsilon_2(R_{lh}) < 0$ is locally strongly rational. This cycle is in configuration S2 if $0 < \varepsilon_1(R_{lh}) < \min[-\varepsilon_2(R_{lh}), 1 + \varepsilon_2(R_{hl}) + \varepsilon_2(R_{lh})]$. This proves Point (ii).

In configuration S3, $\varepsilon_1(R_{lh}) > 0$, $\varepsilon_1(R_{hl}) > 0$, $\varepsilon_2(R_{hl}) < 0$, $\varepsilon_2(R_{lh}) < 0$, $\varepsilon_1(R_{lh}) < -\varepsilon_2(R_{lh})$ and $\varepsilon_1(R_{hl}) < -\varepsilon_2(R_{hl})$. The condition for local strong rationality is given by Point (iii) of Proposition 1 and can be written as $\varepsilon_1(R_{hl}) < 1 + \varepsilon_2(R_{hl}) + \varepsilon_2(R_{lh}) - \varepsilon_1(R_{lh})$. Let $\Lambda_{S3}(\varepsilon_1(R_{lh}))$ denote the r.h.s. of this inequality, as a function of $\varepsilon_1(R_{lh})$. $\Lambda_{S3}(\varepsilon_1(R_{lh}))$ is decreasing. When $\varepsilon_1(R_{lh}) = 0$, $\Lambda_{S3}(0) = 1 + \varepsilon_2(R_{hl}) + \varepsilon_2(R_{lh})$, $\Phi(0) = -1 - \varepsilon_2(R_{hl}) - \varepsilon_2(R_{lh})$ and $\Lambda_{S3}(0) > \Phi(0)$ provided $1 + \varepsilon_2(R_{hl}) + \varepsilon_2(R_{lh}) > 0$. Since $\Lambda_{S3}(\varepsilon_1(R_{lh}))$ and $\Phi(\varepsilon_1(R_{lh}))$ have identical slope, we conclude that if $1 + \varepsilon_2(R_{hl}) + \varepsilon_2(R_{lh}) > 0$ the detected 2-cycle with $\varepsilon_1(R_{lh}) > 0$, $\varepsilon_1(R_{hl}) > 0$, $\varepsilon_2(R_{hl}) < 0$ and $\varepsilon_2(R_{lh}) < 0$ is locally strongly rational. This cycle is in configuration S3 if $0 < \varepsilon_1(R_{hl}) < [\Lambda_{S3}(\varepsilon_1(R_{lh})), 1 + \varepsilon_2(R_{hl}) + \varepsilon_2(R_{lh})]$ and $0 < \varepsilon_1(R_{hl}) < \min[-\varepsilon_2(R_{hl}), 1 + \varepsilon_2(R_{hl}) + \varepsilon_2(R_{lh})]$. This proves Point (iii). $\square$

References

Azariadis, C., Guesnerie, R., 1986. Sunspots and cycles. Review of Economic Studies 53, 725–806.

Boldrin, M., Woodford, M., 1990. Equilibrium models displaying endogenous fluctuations and chaos. Journal of Monetary Economy 25, 189–222.

Brock, W.A., Hommes, C., 1997. A rational route to randomness. Econometrica 65, 1059–1095.

Brock, W.A., Hommes, C., 1998. Heterogeneous beliefs and roads to chaos in a simple asset pricing model. Journal of Economic Dynamics and Control 22, 1235–1274.

Bryant, J., 1995. Does a constant money-growth rule helps stabilize inflation?: experimental evidence. A comment. Carnegie-Rochester Conference on Public Policy 43, 157–162.

Bullard, J., Duffy, J., 1995. On learning and the stability of cycles. Mimeo.

Chiappori, P.A., Davila, J., 1996. Cycles and sunspots: the Poincaré–Hopf approach. Journal of Mathematical Economics 26, 269–284.

Dawid, H., 1996. Learning of cycles and sunspots equilibria by genetic algorithms. Journal of Evolutionary Economics 6, 361–373.

Desgranges, G., Negroni, G., 2003. Expectations coordination on a sunspot equilibrium: an eductive approach. Macroeconomic Dynamics 7, 7–41.

Evans, G., 1983. The stability of rational expectations in macroeconomics models. In: Frydman, R., Phelps, E. (Eds.), Individual Forecasting and Aggregate Outcomes. Cambridge University Press, New York, pp. 69–94.

Evans, G., Guesnerie, R., 1993. Rationalizability, strong rationality and expectational stability. Games and Economic Behavior 5, 632–646.

Evans, G., Honkapohja, S., 1995a. Adaptive learning and expectational stability: an introduction. In: Kirman, A., Salmon, M. (Eds.), Learning and Rationality in Economics. Basil Blackwell, Oxford, pp. 102–126.

Evans, G., Honkapohja, S., 1995b. Local convergence of recursive learning to steady states and cycles in stochastic nonlinear models. Econometrica 63, 195–206.

Evans, G., Honkapohja, S., 1996. Least squares learning with heterogeneous agents. Economics Letters 53, 197–201.

Evans, G., Honkapohja, S., 1999. Learning dynamics. In: Taylor, J., Woodford, M. (Eds.), Handbook of Macroeconomics, vol. 1. North-Holland, Amsterdam, pp. 449–542.

Evans, G., Honkapohja, S., Marimon, R., 1996. Convergence in monetary inflation models with heterogeneous learning rules. Cepr working paper 1310.

Goeree, J.K., Hommes, C., 2000. Heterogeneous beliefs and the non-linear cobweb model. Journal of Economic Dynamics and Control 24, 761–798.

Grandmont, J.M., 1985. On endogenous competitive business cycles. Econometrica 53, 995–1046.

Grandmont, J.M., 1994. Expectations formation in large socioeconomic systems. Cepremap working paper 9424.

Grandmont, J.M., Laroque, Y., 1986. Stability of cycles and expectations. Journal of Economic Theory 40, 138–151.

Guesnerie, R., 1992. An exploration of the eductive justification of the rational expectations hypothesis. American Economic Review 82, 1254–1278.

Guesnerie, R., 1993. Theoretical tests of the rational expectations hypothesis in economic dynamical models. Journal of Economic Dynamics and Control 17, 847–864.

Guesnerie, R., 2002. Anchoring economic predictions in common knowledge. Econometrica 70, 439–480.

Guesnerie, R., Woodford, M., 1991. Stability of cycles with adaptive learning rules. In: Heller, W. (Ed.), Equilibrium Theory and Applications. Cambridge University Press, New York, pp. 111–133.

Guesnerie, R., Woodford, M., 1992. Endogenous fluctuations. In: Laffont, J.J. (Ed.), Recent Advances in Economic Theory and Econometrics. Cambridge University Press, New York, pp. 289–312.

Hommes, C., 1998. On the consistency of backward-looking expectations: the case of the cobweb. Journal of Economic Behavior and Organization 33, 333–362.

Hommes, C., Sorger, G., 1998. Consistent expectations equilibria. Macroeconomic Dynamics 2, 287–321.

Honkapohja, S., Mitra, K., 2003. Learning stability in economies with heterogenous agents. Mimeo, University of Helsinki.

Negroni, G., 2003. Adaptive expectations coordination in an economy with heterogeneous agents. Journal of Economic Dynamics and Control 28, 117–140.

Negroni, G., 2004. Eductive expectations coordination on deterministic cycles in an economy with identical fundamentals. Mimeo, Catholic University of Milan. Journal of Economic Behavior and Organization, forthcoming.

Slok, T., Sorensen, J.P., 1997. How small shocks and heterogeneous expectations can create swings in the exchange rate. Mimeo.

Townsend, M., 1978. Market anticipations, rational expectations and bayesian analysis. International Economic Review 19, 481–494.

Woodford, M., 1990a. Learning to believe in sunspots. Econometrica 58, 277–307.

Woodford, M., 1990b. Equilibrium models of endogenous fluctuations: an introduction. NBER working paper 3360.