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Star-shapedness of Richter–Aumann integral on a measure space with atoms: theory and economic applications ☆

Antonio D'Agata*

Dipartimento di Analisi dei Processi Politici, Sociali ed Istituzionali, University of Catania, Via Vittorio Emanuele 8, Catania 95131, Italy

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Abstract

This paper provides sufficient (as well as necessary) conditions for the integral of a correspondence defined on a measure space with atoms to exhibit star-shaped values. This result is used to analyze the existence of a Nash equilibrium in games with a measure space of agents with atoms and of a competitive equilibrium in economies with mixed markets. In either case it is shown that an exact equilibrium exists whenever atoms are "small enough".

JEL classification: C62; C72; D43; D50

Keywords: Existence of equilibrium; Non-cooperative games; Imperfect competition; Richter–Aumann integral

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*Fax: +95-746-41-93.
E-mail address: adagata@unict.it.

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1. Introduction

According to Richter [16] and Aumann [3], the integral of a bounded and integrable correspondence defined on an atomless measure space is convex. This result has been applied in economics to obtain theorems on existence of equilibrium based on fixed-point arguments; for example, it has been applied to prove the existence of a Walrasian equilibrium in economies with an atomless measure space of agents [4,18] and the existence of a Nash equilibrium in games with an atomless measure space of players [15,19]. To be more specific, for example in games with a nonatomic measure space of players, a Nash equilibrium is a fixed point of the best-reply correspondence, and this fixed point is known to exist because, by Richter–Aumann's result, the integrals of the best-reply correspondence and of the strategy sets are convex valued. However, if the measure exhibits atoms, then the convexity of the integral cannot be ensured, and consequently, existence results of this kind cannot be provided.

Convexity is not necessary for applying fixed-point arguments; in fact, for example, acyclicity would be sufficient as well (see, e.g. [7]). Therefore, intuition suggests that whenever we deal with atomic measures, if it is possible to ensure that the integral of the relevant correspondence satisfies conditions that are weaker than convexity but still sufficient for the existence of a fixed point, then the desired existence results can be recovered.

The main aim of this paper is to provide conditions ensuring that the integral of a correspondence defined on a measure space with atoms is star shaped (notice that star-shapedness implies contractibility and, therefore, acyclicity). It will be shown that in case where the integral of the atomic part is neither convex nor star shaped, the integral of a correspondence exhibits star-shaped values whenever the atoms are "small enough" if and only if the integrals of the atomic and of the non-atomic parts of the correspondence satisfy a particular spanning condition.[1] The sufficiency result will be used in two applications: first, we prove the existence of a Nash equilibrium in games with an atomic measure space of players and second, we prove the existence of a general equilibrium in an economy with an atomic measure space of agents. In either case we show that an equilibrium exists if atoms are "small enough". However, although our argument is similar to the limiting argument used by the literature on oligopolistic equilibria in "large economies", it is worth emphasizing that, with the exception of Roberts [17],[2] our

[1] The spanning condition that we introduce is quite general and does not impose any algebraic or topological condition upon the atomic component of the integral. By contrast, it is quite easy to show that the convexity, star-shapedness of even the contractibility of the integral can be obtained by imposing the corresponding conditions on the atomic part of the integral. For the sake of completeness, we shall provide also these results, although the examples in Section 2 show that this approach is not fruitful in applications.

[2] Roberts [17] assumes convexity of preferences and works within a differentiable context.

approach yields the existence of an *exact* equilibrium rather than an *approximate* one (see, for example, [11]).

The next section provides two examples illustrating intuitively why the integral of a correspondence is star shaped if atoms are "small enough". This intuition will be used in Section 3 to provide a technical preliminary result, which will find applications in Sections 4 and 5. Section 6 contains some final remarks concerning the possibility of weakening the spanning condition.

2. Two illustrative examples

Example 1. Consider a two good general equilibrium model with mixed markets (see, for example, [6,8,20]) in which the index set of agents is $I = N \cup A = [0,1] \cup \{2\}$, the atom being the point 2. We assume that on I a measure μ is defined, where μ is Lebesgue on [0,1] and $\mu(\{2\}) = 1$. Indicating by (x_{i1}, x_{i2}) the consumption bundle of agent i, we assume that preferences of agents are represented by the following utility functions:

$$u(x_{i1}, x_{i2}) = \begin{cases} 17x_{i1}x_{i2}/4 & \text{if } x_{i1} \leqslant x_{i2}/4, \\ x_{i1}^2 + x_{i2}^2 & \text{if } x_{i2}/4 \leqslant x_{i1} \leqslant 4x_{i2}, \\ 17x_{i1}x_{i2}/4 & \text{if } x_{i1} \geqslant x_{i2}/4 \end{cases}$$

for $i \in [0,1]$ and

$$u(x_{i1}, x_{i2}) = \begin{cases} 10x_{i1}x_{i2}/3 & \text{if } x_{i1} \leqslant x_{i2}/3, \\ x_{i1}^2 + x_{i2}^2 & \text{if } x_{i2}/3 \leqslant x_{i1} \leqslant 3x_{i2}, \\ 10x_{i1}x_{i2}/3 & \text{if } x_{i1} \geqslant x_{i2}/3 \end{cases}$$

for $i = 2$.

Suppose, moreover, that the initial endowment of agent i is $\omega(i) = (1,3)$ for $i \in [0,1]$ and that $\omega(i) = (1,1)$ for $i = 2$. Finally, set $p_2 = 1$. Given a non-negative real number α, consider now the economy $\mathscr{E}_\alpha$ which is obtained from the preceding economy by considering on set I the measure $\mu_\alpha = \mu\chi_{I\setminus A} + \alpha\mu(1 - \chi_{I\setminus A})$, and where $\chi_{I\setminus A}$ denotes the characteristic function with respect to set $I\setminus A$. We use number α to "shrink" atom A. Fig. 1 (Fig. 2) illustrates some indifference curves and the initial endowment of an agent in set N (in set A).

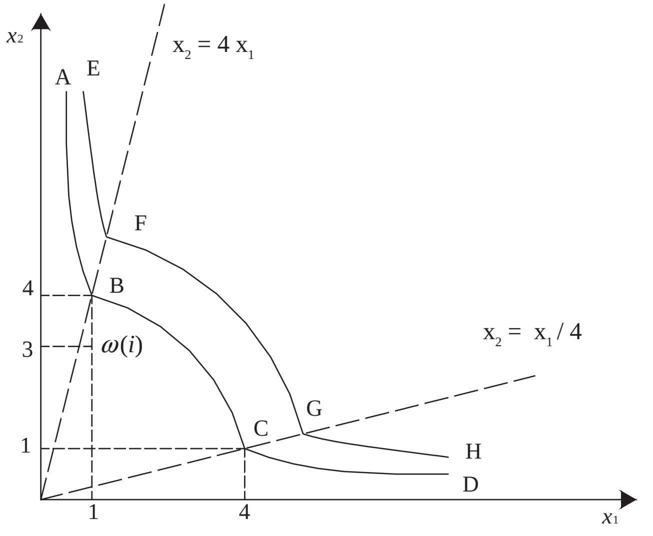

Fig. 1. Curves *ABCD* and *EFGH* indicates two indifference curves of an agent belonging to the nonatomic part of the economy. Point $\omega(i)$ indicates the initial endowment.

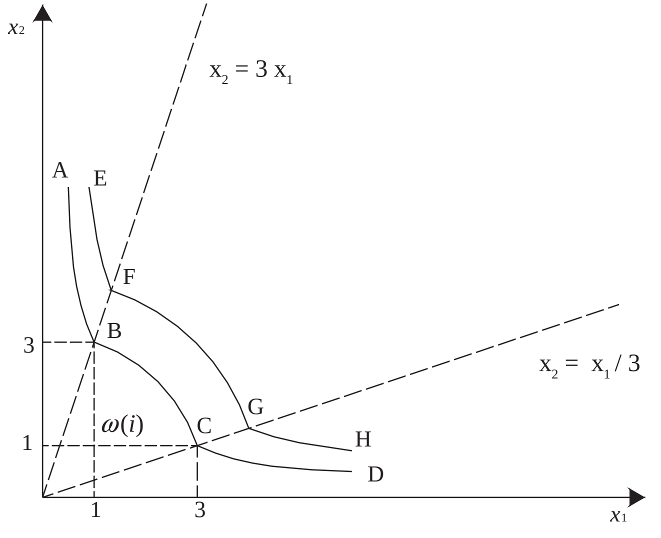

Fig. 2. Curves *ABCD* and *EFGH* indicates two indifference curves of the atom. Point $\omega(i)$ indicates the atom's initial endowment.

Moreover, the aggregate excess demand correspondence is the following:

$$\zeta(p_1)=\begin{cases}\left(\dfrac{(3+\alpha)-(1+\alpha)p_1}{2p_1},\dfrac{-(3+\alpha)+(1+\alpha)p_1}{2}\right) & \text{if } p_1\geqslant 4,\\ \left(\dfrac{\alpha-(2+\alpha)5p_1}{2(4+p_1)p_1},\dfrac{\alpha+(1-\alpha)p_1}{2(4+p_1)}\right) & \text{if } 3\leqslant p_1\leqslant 4,\\ \left(\dfrac{-1}{4+p_1}-\dfrac{2\alpha}{3+p_1},\dfrac{p_1}{4+p_1}+\dfrac{2\alpha p_1}{3+p_1}\right) & \text{if } 1\leqslant p_1\leqslant 3,\\ \left(\left[\dfrac{-1}{4+p_1},\dfrac{11}{1+4p_1}\right]+\left\{\dfrac{-2\alpha}{3+p_1},\dfrac{2\alpha}{3+p_1}\right\},\left[\dfrac{-11}{1+4p_1},\dfrac{p_1}{4+p_1}\right]\right. & \\ \quad\left.+\left\{\dfrac{2\alpha p_1}{3p_1+1},\dfrac{-2\alpha p_1}{3p_1+1}\right\}\right) & \text{if } p_1=1,\\ \left(\dfrac{11}{1+4p_1}+\dfrac{2\alpha}{1+3p_1},\dfrac{-11}{1+4p_1}-\dfrac{2\alpha}{1+3p_1}\right) & \text{if } \dfrac{1}{3}\leqslant p_1\leqslant 1,\\ \left(\dfrac{11}{1+4p_1}+\dfrac{\alpha(1-p_1)}{2p_1},\dfrac{-11}{1+4p_1}-\dfrac{\alpha(1-p_1)}{2}\right) & \text{if } \dfrac{1}{4}\leqslant p_1\leqslant \dfrac{1}{3},\\ \left(\dfrac{(3-p_1)+\alpha(1-p_1)}{2p_1},\dfrac{-(4+\alpha)+p_1(1+\alpha)}{2}\right) & \text{if } 0<p_1\leqslant \dfrac{1}{4}.\end{cases}$$

It is easy to check that for $\alpha\in[0,\infty)$, the economy $\mathscr{E}_\alpha$ has an equilibrium only if $p_1=1$. In fact, if $p_1\neq 1$, then the aggregate excess demand mapping is actually a function which, as it is easy to check, never takes zero values; so, there cannot be an equilibrium for $p_1\neq 1$. If $p_1=1$, then the excess demand set of each agent is composed of two vectors; however, the existence of the non-atomic sector ensures that the aggregate excess demand of this part of the economy is convex and the aggregate excess demand is the union of convex sets. In fact, at $p_1=1$ the image of the excess demand correspondence is: $\left[-\frac{1}{5},\frac{11}{5}\right]+\{\alpha/2,-\alpha/2\}$, $\left[\frac{1}{5},-\frac{11}{5}\right]+\{-\alpha/2,\alpha/2\}$. It is also possible to check that $\mathscr{E}_\alpha$ has an equilibrium if $\alpha\in\left[0,\frac{22}{5}\right]$. Figs. 3 and 4 illustrate the excess demand set for $\alpha=5$ and 1, respectively. It can be seen that while for $\alpha=5$ the excess demand set is not convex, it is convex for $\alpha=1$. More precisely, the excess demand set is convex for $\alpha\in\left[0,\frac{12}{5}\right]$.

This example points out first that the aggregate excess demand can have any shape, but the atomic and the non-atomic part of it always lie on the same subspace of the commodity space. Second, that by decreasing appropriately the size of the atom, the aggregate excess demand correspondence becomes "well shaped" (in this case, convex) and an equilibrium could be ensured by the usual fixed-point theorems (notice, however, that for $\alpha\in\left[\frac{12}{5},\frac{22}{5}\right]$ the excess demand set is not convex but an equilibrium still exists). The next example makes it clear that shrinking the atom is the right general method to ensure the "well-shapedness" of the integral without imposing any particular condition on the atomic part of the integral, but it also points out that convexity is not the property that in general we can hope for.

Example 2. Consider the non-cooperative game in which the index of players is set $I=N_1\cup N_2\cup A=[0,1.8)\cup[1.8,1.9]\cup\{2\}$. On I the measure μ is defined which is

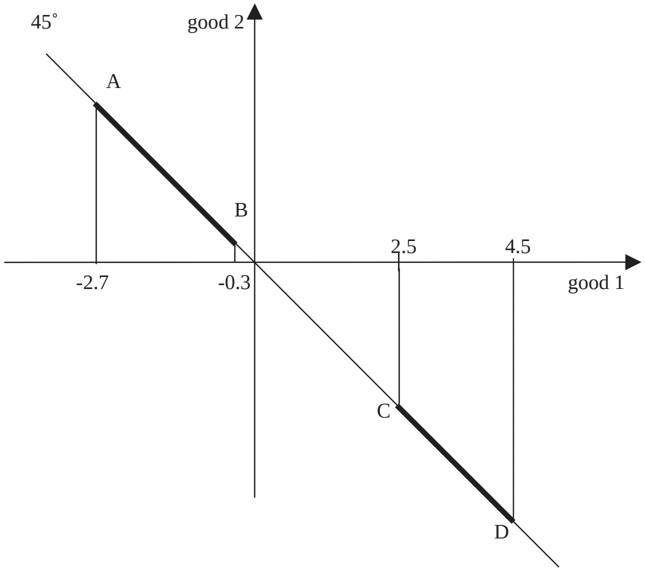

Fig. 3. For $\alpha = 5$, the excess demand set is represented by the set $AB \cup CD$. There is no equilibrium since $(0, 0) \in AB \cup CD$.

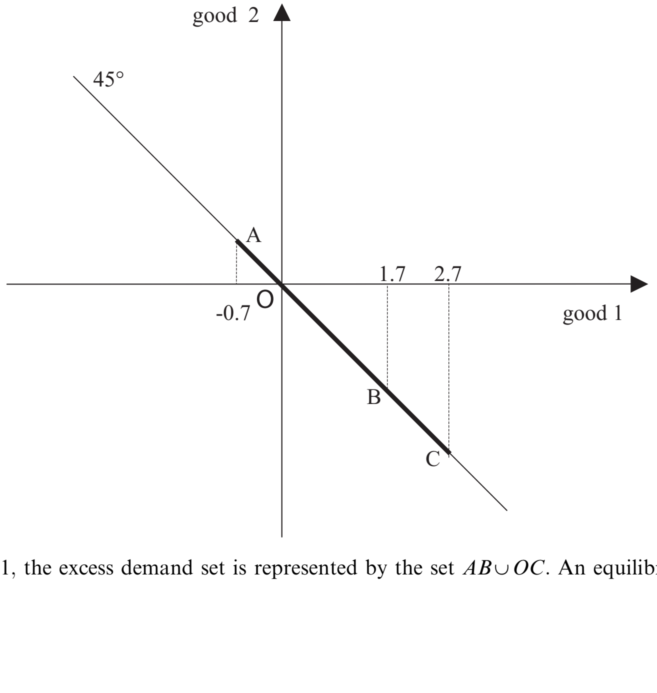

Fig. 4. For $\alpha = 1$, the excess demand set is represented by the set $AB \cup OC$. An equilibrium exists since $(0, 0) \in AB \cup OC$.

Lebesgue on set $N_1 \cup N_2 = [0, 1.9]$ and $\mu\{(2)\} = 1$. The strategy set of player i is

$$X(i) = \{(x_1, x_2) \mid x_1 \geqslant 0, x_2 \geqslant 0, x_1 + x_2 = 1\} \quad \text{for } i \in N_1,$$

$$X(i) = \{(x_1, x_2) \mid 0 \leqslant x_1 \leqslant 0.5, 0 \leqslant x_2 \leqslant 0.5, x_1 = x_2\} \quad \text{for } i \in N_2,$$

$$X(i) = \{(0, 1), (1, 1)\} \quad \text{for } i \in A.$$

Denote by $x(i) = (x_1(i), x_2(i))$ the generic element of the set $X(i)$, and by X the set of all measurable strategy profiles.

Given a feasible strategy profile $x^\circ = (x^\circ(i))_{i \in I} \in X$ interpreted as the status quo, the pay off of player i when he/she chooses strategy $x(i)$ is

$$P_i(x(i), A(x^\circ)) = x_2(i) + (A(x^\circ) - 2)x_1(i) \quad \text{for } i \in N_1,$$

$$P_i(x(i), A(x^\circ)) = x_2(i) - (A(x^\circ) - 2)x_1(i) \quad \text{for } i \in N_2,$$

$$P_i(x(i), A(x^\circ)) = x_2(i) - A(x^\circ)x_1(i) + 3x_1(i)^2 \quad \text{for } i \in A,$$

where $A(x^\circ) = \int_{N_1} x_1^\circ(i)\,d\mu + \int_{N_1} x_2^\circ(i)\,d\mu + \int_{N_2} x_1^\circ(i)\,d\mu + \int_{N_2} x_2^\circ(i)\,d\mu + \int_A x_1^\circ(i)\,d\mu + \int_A x_2^\circ(i)\,d\mu$. Given a non-negative real number α, define the game Γ_α where $\mu_\alpha = \mu\chi_{I\backslash A} + \alpha\mu(1 - \chi_{I\backslash A})$, and where $\chi_{I\backslash A}$ denotes again the characteristic function with respect to set $I\backslash A$. Notice that Γ_0 is the usual non-atomic game considered by the literature. Also note that the integral of the strategy spaces, $\int_{N_1} X(i)\,d\mu + \int_{N_2} X(i)\,d\mu + \int_A X(i)\,d\mu$, is the set A in Figs. 5 and 6 for $\alpha = 0$, the set $B \cup C$ in Fig. 5 for $\alpha = 1$ and again the set $B \cup C$ in Fig. 6 for $\alpha = 0.05$.

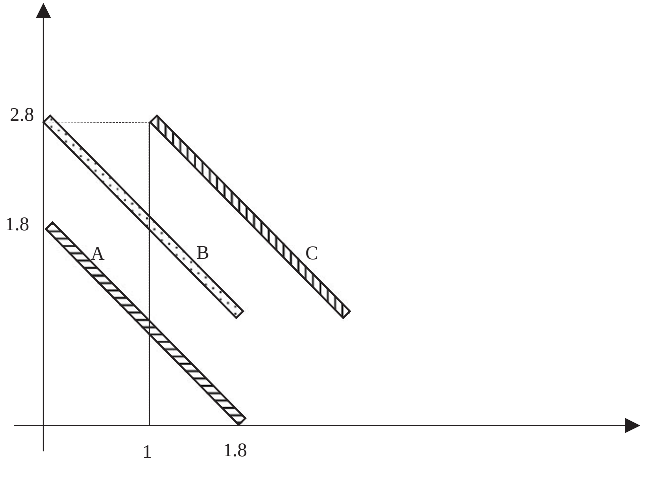

Fig. 5. For $\alpha = 1$, if $A(x^\circ) = 3$, the atom has two best-reply strategies $x = (0, 1)$ and $y = (1, 1)$. Set A is the integral of the best-reply correspondence of the non-atomic part; $B = \alpha x + A$ and $C = \alpha y + A$ for $\alpha = 1$. The integrals of the best-reply correspondence and of the strategy sets, therefore, are the union of sets B and C.

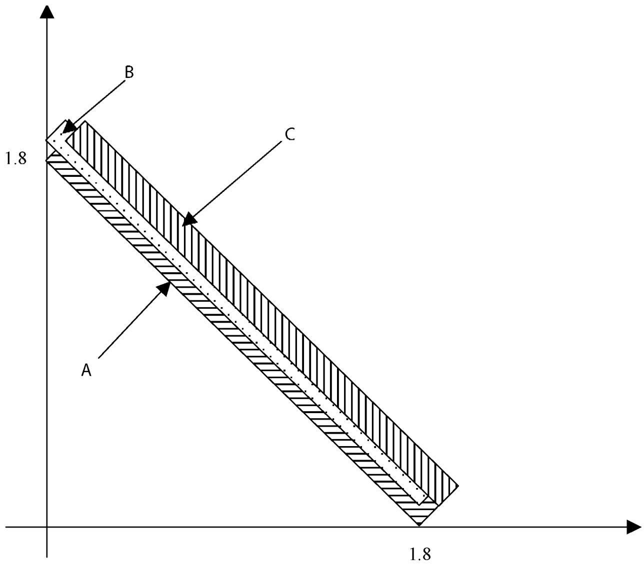

Fig. 6. For $\alpha = 0.05$ and $A(x^\circ) = 3$, the horizontally shaded area is set A, the dotted area is $B = \alpha x + A$, while the vertically shaded area is $C = \alpha y + A$. Sets A and B have a common side. In this case, the integrals of the best-reply correspondence and of the strategy sets, $B \cup C$, are not convex but star shaped.

Denote by $B(i) : X \to X(i)$ the best-reply correspondence of player i. It is possible to show that

$$B(i)(x) = \begin{cases} \{(0,1)\} & \text{if } A(x) < 3, \\ \{(x_1, x_2) | 0 \leqslant x_1, 0 \leqslant x_2, x_1 + x_2 = 1\} & \text{if } A(x) = 3, \\ \{(1,0)\} & \text{if } A(x) > 3 \end{cases}$$

for $i \in N_1$,

$$B(i)(x) = \begin{cases} \{(0.5, 0.5)\} & \text{if } A(x) < 3, \\ \{(x_1, x_2) | 0 \leqslant x_1 \leqslant 0.5, 0 \leqslant x_2 \leqslant 0.5, x_1 = x_2\} & \text{if } A(x) = 3, \\ \{(0,0)\} & \text{if } A(x) > 3 \end{cases}$$

for $i \in N_2$, and

$$B(i)(x) = \begin{cases} \{(1,1)\} & \text{if } A(x) < 3, \\ \{(0,1), (1,1)\} & \text{if } A(x) = 3, \\ \{(0,1)\} & \text{if } A(x) > 3 \end{cases}$$

for $i \in A$.

If $\alpha > 0$, the game Γ_α has atoms and the usual existence results cannot be straightforwardly applied, so a Nash equilibrium *may not* exist. Actually, it is easily seen that the game has no equilibrium for $\alpha = 1$ while it has an equilibrium if atoms are "small enough", i.e. for $0 \leqslant \alpha \leqslant 0.6$. The latter assertion is proved in Appendix A.

Obviously, it is natural to ask why this is so and if it is possible to generalize this result. Like the preceding example, the following graphical analysis suggests that the integral of the best-reply correspondence and of the strategy spaces have nice mathematical properties if the size of the atom is smaller than a threshold size $\alpha^* = 0.1$ (for $0.1 \leqslant \alpha \leqslant 0.6$, although an equilibrium exists, we have not been able to find any property of the relevant integral implying that an equilibrium exists). The set A in Figs. 5 and 6 is also the integral of the best-reply correspondence for $\alpha = 0$; clearly, it is convex. The set $B \cup C$ in Fig. 5 illustrates the integral of the best-reply correspondence for $\alpha = 1$ when $A(x^{\circ}) = 3$ (for any other value of $A(x^{\circ})$, the image of this correspondence is a point). This set is the union of two non-coincident convex sets; thus, it is not convex. Moreover, since B and C are disjoint sets, $B \cup C$ has apparently no property which can ensure the application of fixed point theorems stronger than Kakutani's. The set $B \cup C$ in Fig. 6 also illustrates the integral of the best-reply correspondence for $\alpha = 0.05$ when $A(x^{\circ}) = 3$ (for any other value of $A(x^{\circ})$, the image of this correspondence is again a point). In this case, the image is again the union of two convex, non-coincident sets. However, because of the small value of the parameter α (i.e. the small "size" of the atom), these sets are close enough to ensure that they have a point in common; their union is a star-shaped set and, therefore, fixed-point theorems are applicable. It is also intuitively clear that the integrals of the best-reply correspondence and of the strategy sets are always star shaped or convex if the size of the atom is below a threshold level (actually, for $0 \leqslant \alpha \leqslant 0.1$).

In conclusion, this example suggests that the integrals of the best-reply correspondence and of the strategy sets may not be convex when defined on a measure space with atoms; however, if atoms are "small enough", then they are star shaped, and it might be possible to recover the existence of an equilibrium for the game by using more general fixed-point theorems. The next section will provide a technical result which is preliminary for a rigorous foundation to this intuition.

3. Some preliminary mathematical results

Let $(I, \mathscr{B}, \mu)$ be a measure space, where I is a set, $\mathscr{B}$ is a σ-field of subsets of I and μ is a finite, complete, positive σ-additive measure on $\mathscr{B}$. Let $A_1, A_2, \ldots, A_n \ldots$ be an enumeration of all atoms of I (which is possible from finiteness of μ) and set $A = A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots$, and $N = I \backslash A$. With abuse of notation, we shall also use symbol A to denote the index set of atoms. By using the Lebesgue decomposition, measure μ can be written as $\mu = \mu_A + \mu_N$, where $\mu_A = \chi_A \mu$ and $\mu_N = (1 - \chi_A)\mu$, and where χ_A is the characteristic function with respect to set A. Given a $\alpha \in [0, \infty)$, define the measure $\mu_\alpha = \alpha\mu_A + \mu_N$. Clearly, μ_0 is non-atomic on I and $\mu_1 = \mu$. From now on we shall refer to the measure space $(I, \mathscr{B}, \mu_\alpha)$.

Let $F: I \to \to R^m$ be a correspondence with bounded values, and denote by F_A and F_N the restrictions of F to A and N, respectively. We also assume that F is integrable.[3]

The Richter–Aumann integral (see e.g., [1,3,16]) of F is: $\int_I F\, d\mu_\alpha = \int_A F_A\, d(\alpha\mu_A) + \int_N F_N\, d\mu_N = \bigcup_{s\in S}\{\sum_{i\in A} \alpha\mu(A_i)s(i) + \int_N F_N\, d\mu_N\} = \bigcup_{s\in S}\{\alpha b(s) + \int_N F_N\, d\mu_N\}$, where S is the set of selections of F_A, s is its generic element and $b(s) \in \{x \in R^m \mid x = \sum_{i\in A} \mu(A_i)s(i),\ s \in S\}$ (the sum $\sum_{i\in A} \mu(A_i)s(i)$ is well defined because μ is finite and F is bounded). By Richter [16] and Aumann [3], the integral $\int_N F_N\, d\mu_N$ is a convex set.

Given two points x and y in a subset X of R^n, set: $[x, y] = \{x \in X \mid \alpha x + (1-\alpha)y,\ 0 \leqslant \alpha \leqslant 1\}$. A set X is said to be *convex* if for every $x, y \in X$, $[x, y] \subset X$. Set X is said to be *star shaped* (or *star convex*) if there exists a point $x \in X$ such that for every $y \in X$, $[x, y] \subset X$. Clearly, any convex set is star shaped. Given a point $x \in R^m$, $B_r(x) = \{y \in X \mid d(x, y) \leqslant r\}$ denotes the closed ball of radius r around x, where $d(x, y)$ is the Euclidean distance between points x and y. Moreover, given a subset $X \subset R^m$, let $Proj_X\{x\} = \{y \in X \mid d(x, y) \leqslant d(x, z),\ z \in X\}$ and denote by $span\{X\}$ the smallest subspace of R^m containing $X - \{x_0\}$, where x_0 is any element in X (therefore, $0 \in X - \{x_0\}$).[4] For future reference, we emphasize that $Proj_X\{x\}$ is singleton if X is convex.

The following remark makes it clear that the "well-shapedness" of the integral. $\int_I F\, d\mu_\alpha$ can be easily obtained by ensuring that $\int_A F_A\, d\mu_A$ (and therefore that $\int_A F_A\, d(\alpha\mu_A)$) is "well shaped". The numerical examples of the preceding section, however, point out that this approach is not always fruitful, for this reason Remark 1 is presented here only for the sake of completeness. In this remark we shall use the following terminology: two continuous functions f and g from a topological space X into a topological space Y are said *homotopic* if there is a continuous function h from $X \times [0, 1]$ into Y such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$; set X is said to be *contractible* if the identity map $i(x) = x$ is homotopic to the constant map $c(x) = x'$, for some $x' \in X$. Finally, given a subspace $S \subset R^m$ of dimension $k < m$, symbol $S^\perp$ indicates the complementary space of S in R^m, i.e. a $(m\text{–}k)$-dimensional subspace in R^m such that $span\{S \cup S^\perp\} = R^m$ (see, for example, [12, p. 8]).

Remark 1.

(i) If $\int_A F_A\, d\mu_A$ is convex, then $\int_I F\, d\mu_\alpha$ is convex for every $\alpha \in [0, \infty)$.
(ii) If $\int_A F_A\, d\mu_A$ is star shaped, then $\int_I F\, d\mu_\alpha$ is star shaped for every $\alpha \in [0, \infty)$.
(iii) If $\int_N F_N\, d\mu_N$ and $\int_A F_A\, d\mu_A$ are compact, $\int_A F_A\, d\mu_A$ is contractible and there exists a subspace $S \subset R^m$ such that if $\int_N F_N\, d\mu_N \subset S$ and $\int_A F_A\, d\mu_A \subset S^\perp$, then $\int_I F\, d\mu_\alpha$ is contractible for every $\alpha \in [0, \infty)$.

[3] Under the assumptions adopted here, the correspondence is integrable, for example, when F_N is integrable and the number of atoms is finite.

[4] It is easy to see that the $span\{X\}$ is invariant with respect to the chosen vector x_0.

Proof. See Appendix B.

Given the difficulties already pointed out in ensuring that $\int_A F_A \, d\mu_A$ satisfies in applications the conditions stated in the preceding remark and given that we are interested in the star-shapedness of the integral, from now on we shall consider only cases in which this integral is neither convex nor star shaped. In order to analyze these cases the following characterization of star-shaped sets will be useful:

Lemma 1. *A set is star shaped if and only if it is the union of a family of convex sets with non-empty intersection.*

Proof. Consider a family $\{X_\psi \mid \psi \in \Psi\}$ of convex sets with non-empty intersection, i.e. $\bigcap_{\psi \in \Psi} X_\psi \neq \emptyset$. Suppose $x \in \bigcap_{\psi \in \Psi} X_\psi$. Take any point $y \in \bigcup_{\psi \in \Psi} X_\psi$; thus, there must exist $\psi \in \Psi$ such that $y \in X_\psi$. Clearly, $x \in X_\psi$, hence, by convexity, $[x, y] \subset X_\psi \subset \bigcup_{\psi \in \Psi} X_\psi$. Suppose now that set X is star shaped. Then, there exists $x^* \in X$ such that $[x^*, y] \subseteq X$ for every $y \in X$. Thus, $X = \bigcup_{y \in X} [x^*, y]$ and $x^* \in \bigcap_{y \in X} [x^*, y]$. □

The following assumption introduces the "spanning" condition we referred to in the introduction.

Assumption 1. $Z_A \equiv span\{\int_A F_A \, d\mu_A\} \subset Z_N \equiv span\{\int_N F_N \, d\mu_N\}$.

Notice that $span\{\int_A F_A \, d(\alpha\mu_A)\} \subset span\{\int_A F_A \, d\mu_A\}$ for every $\alpha \in (0, \infty)$; hence, Assumption 1 also implies that $span\{\int_A F_A \, d(\alpha\mu_A)\} \subset Z_N$ for every $\alpha \in (0, \infty)$. By construction there must exist two vectors $v_N, v_A \in R^m$, such that $\int_N F_N \, d\mu_N \subset v_N + Z_N$ and $\int_A F_A \, d(\alpha\mu_A) \subset \alpha v_A + Z_A$ for every $\alpha \in (0, \infty)$. Fig. 7 illustrates a case in which Assumption 1 is satisfied, while Fig. 8 illustrates a case in which it is not satisfied. The next result simply means that $\int_N F_N \, d\mu_N$ has a relative interior in $v_N + Z_N$.

Lemma 2. *There exists a positive real number r and a vector* $x^* \in \int_N F_N \, d\mu_N$ *such that* $B_r(x^*) \cap (v_N + Z_N) \subseteq \int_N F_N \, d\mu_N$.

Proof. If $\int_N F_N \, d\mu_N$ is singleton, then Z_N is 0-dimensional. By definition, $Z_N = \{0\}$ and $v_N = \int_N F_N \, d\mu_N$. Take $x^* = \int_N F_N \, d\mu_N$. Obviously, the statement of Lemma 2 is verified for every positive real number r. Suppose now that Z_N is k-dimensional ($1 \leqslant k \leqslant m$), therefore, there exist k vectors $z_1, z_2, \ldots, z_k$ in Z_N which are linearly independent and satisfy the condition that $v_N + z_i \in \int_N F_N \, d\mu_N$ for every $i = 1, 2, \ldots, k$. Take $co\{z_1, z_2, \ldots, z_k\}$ and set $b = \sum_{i=1}^{k} (z_i/k)$. It is well known that b belongs to the relative interior of set $co\{z_1, z_2, \ldots, z_k\}$ (see [12, p. 8]); therefore, there exists a positive real number r such that $B_r(v_N + b) \cap \{v_N + Z_N\} \subset v_N + co\{z_1, z_2, \ldots, z_k\}$. By Richter–Aumann's convexity result we have that $v_N + co\{z_1, z_2, \ldots, z_k\} \subset \int_N F_N \, d\mu_N$. Therefore, the assertion is proved by setting $x^* = v_N + b$. □

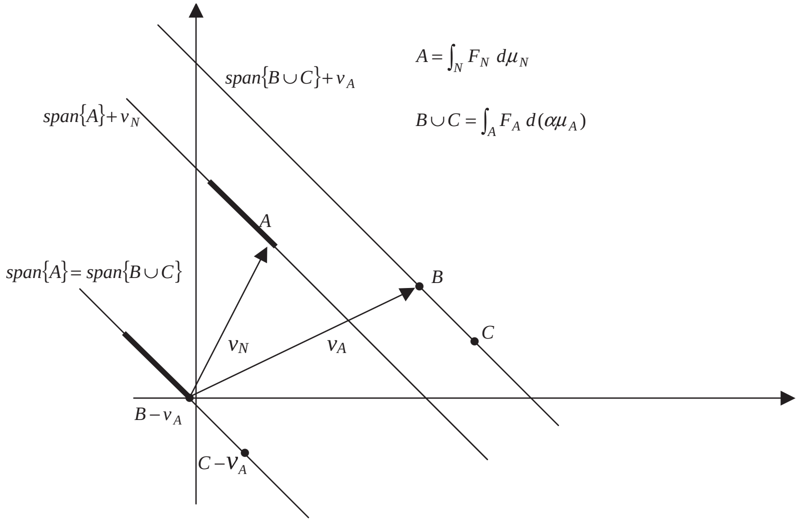

Fig. 7. We assume that the integral of the correspondence of the atomic part has only two values. Assumption 1 is satisfied since $span\{\int_N F_N\, d\mu_N\} = span\{\int_A F_A\, d(\alpha\mu_A)\}$.

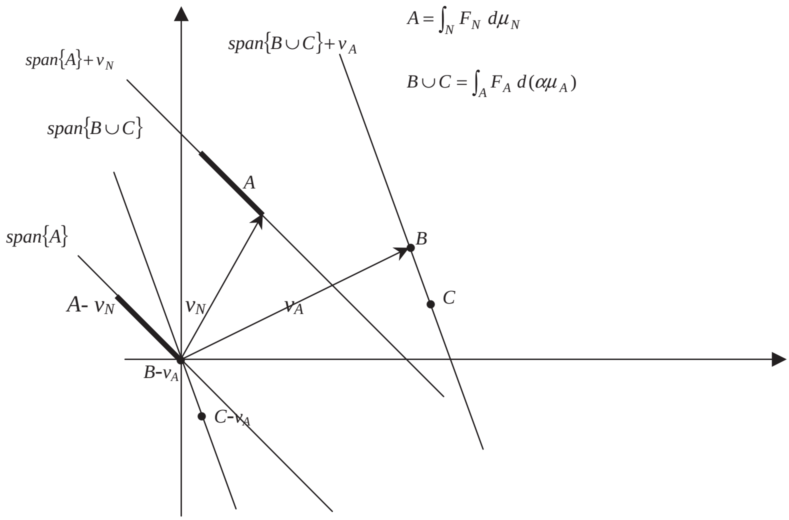

Fig. 8. In the case illustrated in the figure Assumption 1 is not satisfied since $span\{\int_N F_N\, d\mu_N\} \neq span\{\int_A F_A\, d(\alpha\mu_A)\}$.

The ensuing Theorem 1, which is the main result of the paper, characterizes the star-shapedness of the integral of a correspondence with atoms. In the applications we provide, however, only the sufficient condition turns out to be important.

Theorem 1. *Suppose that $\int_A F_A \, d\mu_A$ is neither convex nor star shaped. Then set $\int_I F \, d\mu_\alpha$ is star shaped for every $\alpha \in [0, r/\sup_{s\in S} ||b(s)||]$ with r defined as in Lemma 2, if and only if Assumption 1 holds true.*

Proof. Suppose Assumption 1 is satisfied. For every $\alpha \in [0, r/\sup_{s\in S} ||b(s)||]$ it follows that $\alpha||b(s)|| \leqslant r$; thus, $d(x^*, \alpha b(s) + x^*) \leqslant r$ for every $s \in S$, where r has been defined in Lemma 2. Now set $x^{**} = Proj_{\alpha v_A + v_N + Z_N}\{x^*\}$ (since Z_N is convex, $Proj_{\alpha v_A + v_N + Z_N}\{x^*\}$ is singleton); we show that for every $\alpha \in [0, r/\sup_{s\in S} ||b(s)||]$, $d(\alpha b(s) + x^*, x^{**}) \leqslant r$ for every $s \in S$. In fact, by the properties of the inner product we have: $\cos\theta = ||(\alpha b(s) + x^*) - x^{**}||/||(\alpha b(s) + x^*) - x^*||$, where θ is the angle between $(\alpha b(s) + x^*) - x^{**}$ and $(\alpha b(s) + x^*) - x^*$. Since $\cos\theta \leqslant 1$, $d(\alpha b(s) + x^*, x^{**}) \leqslant d(\alpha b(s) + x^*, x^*) \leqslant r$, where the last inequality follows from the assumption on α. The fact that $d(\alpha b(s) + x^*, x^{**}) \leqslant r$ for every $s \in S$ implies that $x^{**} \in B_r(\alpha b(s) + x^*) = \alpha b(s) + B_r(x^*)$ for every $s \in S$. Note now that for every $s \in S$ and $\alpha \in [0, r/\sup_{s\in S} ||b(s)||]$, the vector $\alpha b(s)$ can be written as follows: $\alpha b(s) = \alpha v_A + z(s)$, where $z(s) \in Z_A$; hence, $\alpha v_A = \alpha b(s) - z(s)$. Since by construction, $x^{**} \in \alpha v_A + v_N + Z_N$, it follows that $x^{**} \in \alpha b(s) - z(s) + v_N + Z_N = \alpha b(s) + v_N + Z_N$, where the equality follows from the fact that $z(s) \in Z_A \subseteq Z_N$ and from the fact that Z_N is a subspace. Therefore, $x^{**} \in \alpha b(s) + \{B_r(x^*) \cap (v_N + Z_N)\} \subset \alpha b(s) + \int_N F_N \, d\mu_N$ for every $s \in S$, where the last inclusion follows from Lemma 2. Since $\int_N F_N \, d\mu_N$ is convex and $\bigcup_{s\in S}\{\alpha b(s) + \int_N F_N \, d\mu_N\} = \int_I F \, d\mu_\alpha$, Lemma 1 ensures that $\int_I F \, d\mu_\alpha$ is star shaped for every $\alpha \in [0, r/\sup_{s\in S} ||b(s)||]$.

Suppose now that for every $\alpha \in [0, r/\sup_{s\in S} ||b(s)||]$ the integral $\int_I F \, d\mu_\alpha$ is star shaped, but that Assumption 1 is not satisfied. Therefore, $Z_A = span\{\int_A F_A \, d(\alpha\mu_A)\} \not\subset Z_N = span\{\int_N F_N \, d\mu_N\}$. This means there must exist at least two selections s and s' in S such that if $b(s) \in Z_N$ then $b(s') \notin Z_N$. Assuming that Z_N is k-dimensional, this means that if $b(s) - v_A = \sum_{i=1}^k \lambda_i y_i$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are not all zero and where $y_1, y_2, \ldots, y_k$ are linear independent vectors in Z_N, then it is not possible to write $b(s') - v_A = \sum_{i=1}^k \lambda'_i y_i$, where λ'_i are not all zero; the following equality does not hold possible $b(s) - b(s') = \sum_{i=1}^k (\lambda_i - \lambda'_i) y_i$. We shall show that $\{\alpha b(s) + \int_N F_N \, d\mu_N\} \cap \{\alpha b(s') + \int_N F_N \, d\mu_N\} = \emptyset$ for every $\alpha > 0$. Suppose not; i.e., suppose that there exists $x' \in R^m$ such that $x' \in \{\alpha b(s) + \int_N F_N \, d\mu_N\} \cap \{\alpha b(s') + \int_N F_N \, d\mu_N\}$. Therefore, $x' = \alpha b(s) + \sum_{i=1}^k \gamma_i y_i$ and $x' = \alpha b(s) + \sum_{i=1}^k \delta_i y_i$ for real numbers $\gamma_1, \gamma_2, \ldots, \gamma_k$ and $\delta_1, \delta_2, \ldots, \delta_k$ not all zero. That is, $\alpha(b(s') - b(s)) = \sum_{i=1}^k (\gamma_i - \delta_i) y_i$. Since $\alpha > 0$, the last equality contradicts what was said above. $\square$

4. Existence of a Walrasian equilibrium in mixed economies

In this section we shall use Theorem 1 to provide sufficient conditions for the existence of a competitive equilibrium in mixed economies. This result fills a gap in the literature of general equilibrium with imperfect competition. In fact, on the one hand the concept of competitive equilibrium in mixed economies has been systematically used by an extensive literature in comparing, for example, competitive and core allocations (see, e.g. [6,8,9,20]. On the other hand, however, to the best of our knowledge, no existence result concerning competitive equilibria for these economies has yet been provided.

We shall show that the economy in Example 1 in Section 2 satisfies sufficient conditions for the existence of a competitive equilibrium which can be drawn from the technical result in the preceding section.

Consider an m-good mixed economy: $\mathscr{E}((I, \mathfrak{J}, \mu), (\precsim(i))_{i\in I}, (\omega(i))_{i\in I})$, where $I = [0,1] \cup \{2\} \cup \{3\} \cup \cdots \cup \{n\} \cup \cdots = N \cup A$, measure μ is non-atomic on N, and $\mu = \mu_A + \mu_N$, with the usual meaning of symbols. We adopt the usual assumptions of desirability, continuity and measurability on the preference relations $\precsim(i)$ and that $\int_I \omega \, d\mu \gg 0$ (see [4]). The consumption set is the non-negative orthant of R^m, while the price space is the $(m-1)$-dimensional *open* unit simplex Δ. As usual, given a measurable function f on I, we denote by f_N (by f_A) the restriction of f on N (on A).

Given any positive real number α, we define the economy $\mathscr{E}_\alpha = \mathscr{E}((I, \mathfrak{J}, \mu_\alpha), (\precsim(i))_{i\in I}, (\omega(i))_{i\in I})$, where $\mu_\alpha = \mu_N + \alpha\mu_A$. Clearly, $\mathscr{E}_0$ is the economy considered by Aumann (see [2,4]). Define the following correspondences from $I \times \Delta$ into $R^m_+ : L(i)(p) = \{x \in R^m_+ \mid p \cdot x \leqslant p \cdot \omega(i)\}$, $L^=(i)(p) = \{x \in R^m_+ \mid p \cdot x = p \cdot \omega(i)\}$ and $B(i)(p) = \{x \in L(i)(p) \mid x \succsim(i) y, \ \ y \in L(i)(p)\}$. Define the correspondences $B : \Delta \to R^m_+$ and $L : \ \Delta \to R^m_+$ as follows:

$$B_\alpha(p) = \int_I B(i)(p) \, d\mu_\alpha = \int_N B_N(i)(p) \, d\mu_N + \int_A B_A(i)(p) \, d(\alpha\mu_A)$$

and

$$L(p) = \int_I L(i)(p) \, d\mu_\alpha = \int_N L_N(i)(p) \, d\mu_N + \int_A L_A(i)(p) \, d(\alpha\mu_A).$$

Finally, denote by $S_B(p)$ the set of all selections of correspondence $B_A(\cdot)(p)$. An *allocation* is a measurable function $f : \ I \to R^m_+$ such that $\int_I f \, d\mu_\alpha = \int_I \omega \, d\mu_\alpha$. A *competitive equilibrium* is a non-negative price vector $p^* \in \Delta$ and an allocation f^* such that for every agent $i \in I : f^*(i) \in B(i)(p^*)$. After the main result of this section we will provide an interesting class of economies satisfying the following assumption.

Assumption 2. For every $p \in \Delta$, $Z_A(p) \equiv span(\int_A B_A(i)(p) \, d\mu_A) \subset Z_N(p) \equiv span(\int_N B_N(i)(p) \, d\mu_N)$.

Assumption 2 introduces the spanning condition we refer to in the introduction. Intuitively, it means that for any price at which the atomic part of the demand is not single valued, the atomless part of the demand is also not single valued and the latter set is "bigger" than the former (in terms of the linear spaces generated by these sets).[5] It is clear that for every $p \in \Delta$ there exist in R^m two vectors $v_A(p)$ and $v_N(p)$ (take $v_A(p) = \int_A \omega_A(i)(p) \, d(\alpha\mu_A)$ and $v_N(p) = \int_N \omega_N(i)(p) \, d\mu_N$) such that

[5] I owe this interpretation to an anonymous referee.

$\int_N B_N(i)(p)\,d\mu_N \subset v_N(p) + Z_N(p)$ and $\int_A B_A(i)(p)\,d(\alpha\mu_A) \subset \alpha v_A(p) + Z_A(p)$ for every $\alpha \in [0, \infty)$. Lemma 2 holds also in this case, now it becomes:

Lemma 3. *For every $p \in \Delta$ there exist a vector $x(p)$ and a positive real number r_p such that $B_{r_p}(x(p)) \cap \{v_N(p) + Z_N(p)\} \subset \int_N B_N(i)(p)\,d\mu_N$.*

Proof. The proof is the same as the proof of Lemma 2. Therefore it is omitted. □

Assumption 3. There exists a positive real number ε such that $r_p \geqslant \varepsilon$ for every $p \in \Delta$.

Lemma 3 ensures that $\int_N B_N(i)(p)\,d\mu_N$ is "big enough" to contain a "ball" with a positive "radius" (r_p) and whose dimension is equal to the dimension of subspace $Z_N(p)$. This means, for example, that if $v_N(p) + Z_N(p)$ is one (resp., two, three) dimensional, then $\int_N B_N(i)(p)\,d\mu_N$ contains a segment (resp., a disk, a sphere) with a positive length (resp., radius) equal to r_p. As already pointed out at the beginning of the proof of Lemma 2, if $\int_N B_N(i)(p)\,d\mu_N$ is single valued, then Lemma 3 is satisfied for every $r_p > 0$.

Assumption 3 requires only that there exists a positive lower bound to these magnitudes (lengths, areas, volumes) as p runs over the simplex.

Theorem 2. *Under Assumptions* 2 *and* 3 *there exists a positive real number α^* such that for $\alpha \in [0, \alpha^*]$, the economy $\mathscr{E}_\alpha$ has a competitive equilibrium.*

Proof. The proof is an adaptation from Hildenbrand [10, Section 2.2]. It is easy to check that for every $\alpha \in [0, \infty)$, $\int_I B(i)(p)\,d\mu_\alpha$ satisfies the usual properties of Walras' Law, upper hemi-continuity, compact valuedness and the further condition that if $\{p_n\}$ is a sequence of strictly positive price vectors converging to a vector that is not strictly positive, then $\inf\{\sum_{h=1}^m z_h \mid z \in \int_I (B(i)(p) - \omega(i))\,d\mu_\alpha\} > 0$ for n large enough. Now define the set $\Delta_n = \{p \in \Delta \mid \sum_{i=1}^m p_i$ and $p_i \geqslant 1/n\}$, $n \geqslant m$. From Assumption 2 and Theorem 1 it follows that given $p \in \Delta_n$ the integral $\int_I B(i)(p)\,d\mu_\alpha$ is star shaped for every $\alpha \in [0, \alpha_n^*(p)]$ where $\alpha_n^*(p) = r_p / \sup\|b(p)\|_{b(p) \in S_B(p)}$ and where r_p is defined as in Lemma 3. From Assumption 3 and from the boundedness of the optimal consumption sets $B(i)(p)$, it follows that given n, $\alpha_n^* \equiv \inf\{\alpha_n^*(p) \mid p \in \Delta_n\} > 0$. Therefore, for every $\alpha \in [0, \alpha_n^*]$, the integral $\int_I B(i)(p)\,d\mu_\alpha$ is star shaped for every $p \in \Delta_n$. Hence, by following the same argument used by Hildenbrand [10, Lemma 1, Section 2.2] but by using Eilenberg and Montgomery's fixed-point theorem (see [7]) rather than Kakutani one, it is possible to show that for every n there exists a price vector $p_n \in \Delta_n$ and an aggregate excess demand vector z_n such that $z_n \in \int_I (B(i)(p) - \omega(i))\,d\mu_\alpha$ and $p \cdot z_n \leqslant 0$ for every $p \in \Delta_n$. Moreover, it is possible to show that there exists a number $n^* > 0$ and a price vector $p_n^* \in \Delta_{n^*}$ such that $z_{n^*} = 0$. This means that there exists a measurable function f^* on I such that $z_{n^*} = \int_I (f^*(i)(p) - \omega(i))\,d\mu_\alpha$ where $f^*(i) \in B(i)(p_{n^*})$ for every i. Clearly, f^* is an allocation and p_n^* and f^* is a competitive equilibrium. □

Example 1 in Section 2 satisfies all the conditions adopted here. In particular, Lemma 3 is satisfied with $span\{\int_A B_A(i)(p)\, d(\alpha\mu_\alpha)\} = span\{\int_N B_N(i)(p)\, d\mu_N\} = R$, and Assumption 2 is satisfied as well by taking $\varepsilon = 0.5$

Note that in the proof of Theorem 2, Assumption 3 has the role only to ensure that $\alpha_n^* > 0$, by setting a positive lower bound to r_p. However, the positivity of α_n^* can be ensured also by the following assumption, which is more restrictive but may be more convenient in applications:

Assumption 3*. For every n, the set $F_n \equiv \{p \in \Delta_n \mid B_A(i)(p)$ is not convex for at least one index $i \in A\}$ has a finite number of elements.

Under Assumption 3*, in fact, $\int_I B(i)(p)\, d\mu_\alpha$ is clearly convex for $p \in \Delta_n \backslash F_n$, while it turns out to be star shaped for every $p \in F_n$ whenever $\alpha \in [0, \alpha_n^*]$ and where α_n^* has been defined in the proof of Theorem 2. Now notice that if $\alpha_n^* \equiv \inf\{\alpha_n^*(p) \mid p \in \Delta_n\}$ is replaced by $\alpha_n^* \equiv \inf\{\alpha_n^*(p) \mid p \in F_n\}$, then α_n^* turns out to be positive if Assumption 3* holds true, and the argument in the remaining part of the proof of Theorem 2 still applies.

It is immediate to check that Example 1 in Section 2 satisfies Assumption 3* since $F_n = \{(1, 1)\}$.

The following example shows that Assumption 2 is satisfied in a class of economies widely considered by the literature on general equilibrium with imperfect competition (see the references at the beginning of this section).

Example 3. Consider a mixed economy in which there are k atoms and in which the non-atomic sector is partitioned into t elements with positive measure, $P_1, P_2, \ldots, P_t$ with $t \geqslant k$. Suppose that for every atom A_i there exists an element of the partition P_i (possibly after change of indices) such that $\precsim(A_i) = \precsim(j)$ and $\omega(A_i) = \omega(j)$ with $j \in P_i$. Suppose that $\mu(A_i) = \beta_i \mu(P_i)$ for $i = 1, 2, \ldots, k$ and set $\beta = \min\{\beta_i \mid i = 1, 2, \ldots, k\}$. Clearly, $\beta > 0$. By non-atomicity, it is possible to decompose every P_i as follows: $P_i = Q_i \cup R_i$, where $\mu(A_i) = \beta\mu(Q_i)$. Clearly, $B(A_i)(p)\mu(A_i) = \beta \int_{Q_i} B(i)(p)\, d\mu$ for every $i = 1, 2, \ldots, k$ and, therefore, $\sum_{i=1}^k B(A_i)(p)\mu(A_i) = \beta \sum_{i=1}^k \int_{Q_i} B(i)(p)\, d\mu_{|Q_i}$, where $d\mu_{|Q_i}$ denotes the restriction of measure μ to Q_i. This clearly means that for every $p \in \Delta$, $span\{\int_A B_A(i)(p)\, d\mu_A\} = span\{\int_{\cup_{i=1}^k \{Q_i\}} B_N(i)(p)\, d\mu_N\}$, which in turns implies that for every $p \in \Delta$, $span\{\int_A B_A(i)(p)\, d\mu_A\} \subset span\{\int_N B_N(i)(p)\, d\mu_N\}$.

5. Existence of pure strategy equilibrium in games with atoms

By using Theorem 1 in Section 3, we now provide sufficient conditions for the existence of a pure strategy Nash equilibrium in games with an atomic measure set of players. This result should fill a gap in the existing literature of non-cooperative games. In fact, while there is a quite extensive literature on games defined on non-atomic measure spaces of agents (see, e.g. [13–15,19] and references herein), to the best of our knowledge, there is no existence result for games defined upon a measure space of agents with atoms. We shall also show that the game considered in Example 2 in Section 2 satisfies these conditions. In analyzing this case, we closely follow Rath [15]. Let us consider the game $\Gamma_\alpha((I,\mathfrak{I},\mu_\alpha),\ (X(i))_{i\in I},\ (\Pi(i))_{i\in I})$, where $I = N \cup A$ is the index set of players, with $N = [0,1]$ indicating the set of non-atomic players and $A = \{2\} \cup \{3\} \cup \cdots \cup \{n\} \cup \cdots$ indicating the set of atoms. Set I is assumed to be endowed with a measure $\mu_\alpha = \mu_N + \alpha\mu_A = \mu\chi_N + \alpha\mu\chi_A$, where μ is Lebesgue measure on N. Symbol $X(i)$ denotes the strategy set of player i. We assume (see [15, p. 432]) that sets $X(i)$ define the strategy set correspondence $X: I \to\to R^m$ which has non-empty and compact (but not necessarily convex) values, has measurable graph and is integrably bounded. Denote by X_A (by X_N) the restriction of X to set A (to set N). A strategy profile is a measurable function $f: I \to X$ such that $f(i) \in X(i)$ for every $i \in I$. Denote by F_X the set of all strategy profiles, let $q_\alpha(f) = \int_I f\, d\mu_\alpha$, where $f \in F_X$ and let $Q_\alpha = \{q_\alpha(f) | f \in F_X\}$. Symbol $\Pi(i)$ denotes the payoff function of player i, which is a real-valued function defined on $X(i) \times Q$, where $Q = \bigcup_{\alpha\in[0,1]} Q_\alpha$.[6] We assume that this function is continuous and that for every $q \in Q$, $\Pi(\cdot)(\cdot, q)$ is measurable from the graph of X to R (see [15, p. 432]). Clearly, Γ_0 is a standard game with a non-atomic set of players (see e.g. [15,19]).

Define the correspondence $B: I \times Q \to\to \bigcup_{i\in I} X(i)$ by $B(i)(q) = \{a \in X(i) \mid \Pi(i)(a,q) \geqslant \Pi(i)(x,q) \text{ for every } x \in X(i)\}$. A *Nash equilibrium* of game Γ_α is a strategy profile $f^* \in F_X$ such that for almost all i, $f^*(i) \in B(i)(q(f^*))$. Given $\alpha \in [0,1]$ and $q \in Q$, define the correspondence $\Phi_\alpha(q) = \int_I B(i)(q)\, d\mu_\alpha = \int_N B_N(i)(q)\, d\mu_N + \int_A B_A(i)(q)\, d(\alpha\mu_A)$ with obvious meaning of symbols. Denote by $S_{B(q)}$ the set of all selections of correspondence $B_A(\cdot)(q)$ and by S_X the set of all selections of correspondence X_A.

Assumption 4. For every $q \in Q$, $Z_{B_A}(q) \equiv span\{\int_A B_A(i)(q)\, d\mu_A\} \subset Z_{B_N} \equiv span\{\int_N B_N(i)(q)\, d\mu_N\}$.

Assumption 5. $Z_{X_A} \equiv span\{\int_A X_A(i)\, d\mu_A\} \subset Z_{X_N} \equiv span\{\int_N X_N(i)\, d\mu_N\}$.

Assumptions 4 and 5 correspond to the spanning condition in Assumption 1, with Assumption 4 referring to the integral of the best-reply correspondence and Assumption 5 referring to the integral of the strategy set correspondence. Note that for every $q \in Q$ and every $\alpha \in [0,1]$, $span\{\int_A B_A(i)(q)\, d(\alpha\mu_A)\} \subset Z_{B_A}(q)$ and $span\{\int_A X_A(i)\, d(\alpha\mu_A)\} \subset Z_{X_A}$; moreover, there exist four vectors $v_N(q)$, $v_A(q)$, t_N and t_A in R^m such that $\int_N B_N(i)(q)\, d\mu_N \subset v_N(q) + Z_{B_N}$, $\int_A B_A(i)(q)\, d(\alpha\mu_A) \subset \alpha v_A(q) + Z_{B_A}$, $\int_N X_N(i)\, d\mu_N \subset t_N + Z_{X_N}$, $\int_A X_A(i)\, d(\alpha\mu_A) \subset \alpha t_N + Z_{X_A}$.

[6] Rath [15], assumes that function $\Pi(i)$ is defined on $X(i) \times Q_0$.

Lemma 4. *For $\alpha \in [0, 1]$ and for every $q \in Q$, there exist positive numbers r_q and r_X and vectors $b^*(q) \in \int_N B_N(i)\, d\mu_N$ and $x^* \in \int_N X_N(i)\, d\mu_N$ such that $B_{r_q}(b^*(q)) \cap (v_N + Z_{B_N}) \subset \int_N B_N(i)(q)\, d\mu_N$ and $B_{r_X}(x^*) \cap (v_N + Z_{B_N}) \subset \int_N X_N(i)\, d\mu_N$.*

Proof. The proof is similar to the proof of Lemma 2, therefore it is omitted. □

Assumption 6. There exists a positive real number ε such that $r_q \geqslant \varepsilon$ for $\alpha \in [0, 1]$ and for every $q \in Q$.

The meaning of Assumption 6 is similar to the meaning of Assumption 2.

Lemma 5. *Set $\int_I X_N(i)\, d\mu_\alpha = \int_N X_N(i)\, d\mu_N + \int_A X_A(i)\, d(\alpha\mu_A)$ is star shaped for every non-negative real number $\alpha \in [0, r_x/\sup ||b(s)||_{s \in S_X}]$ if and only if Assumption 5 holds true.*

Proof. The result follows immediately from Theorem 1 and Assumption 5, once noticing that $\int_N X_N(i)\, d\mu_N$ is convex (for this property, see [15, p. 432]). □

Theorem 3. *Under Assumptions 4, 5 and 6 there exists a positive real number α^* such that for $\alpha \in [0, \alpha^*]$ the game Γ_α has a Nash equilibrium.*

Proof. By continuity, for every $\alpha \in [0, 1]$ the set $B(i)(q)$ is non-empty for every $i \in I$ and $q \in Q$, and by the Maximum Theorem (see e.g. [5]) the correspondence $B(i)(\cdot)$ has also a closed graph for every $i \in I$. Define the correspondence $\Phi_\alpha : Q \to \to Q_\alpha$ by $\Phi_\alpha(q) = \int_I B(i)(q)\, d\mu_\alpha$. By using arguments similar to those used by Rath [15, pp. 430–431], it is possible to prove that for every $\alpha > 0$ the correspondence Φ_α has non-empty values and has a closed graph. Now we show that there exists a positive real number α' such that for $\alpha \leqslant \alpha'$ the set $\Phi_\alpha(q)$ is star shaped for every $q \in Q$. To this end, if $q \in Q$ is such that $B(i)(q)$ is convex for every $i \in A$, then $\Phi_\alpha(q) = \int_I B(i)(q)\, d\mu_\alpha$ is convex as well. By contrast, if $q \in Q$ is such that $B(i)(q)$ is not convex for some $i \in A$, then Assumption 4 and Theorem 1 ensure that $\Phi_\alpha(q)$ is star shaped for $\alpha \in [0, r_{B(q)}/\sup ||b(s(q))||_{s(q) \in S_{B(q)}}]$. So define $\alpha' = \min\{r_{B(q)}/\sup ||b(s(q))||_{s(q) \in S_{B(q)}} \mid q \in Q\}$; clearly, by the compactness of the strategy sets and from Assumption 6 it follows that $\alpha' > 0$. Take now $\alpha^* = \min\{\alpha', r_X/\sup||b(s)||_{s \in S_X}\}$. Again, the compactness of the strategy sets ensures that $\alpha^* > 0$. Therefore, for $0 \leqslant \alpha \leqslant \alpha^*$, the correspondence Φ_α maps $\bigcup_{\alpha \in [0,1]} \{\int_I X(i)\, d\mu_\alpha\}$ into $\int_I X(i)\, d\mu_\alpha$, where $\int_I X\, d\mu_\alpha$ is star shaped by Lemma 5. Moreover, Φ_α is upper hemi-continuous and has star-shaped values for every $q \in Q$. Take the restriction Φ'_α of Φ_α in $\int_I X(i)\, d\mu_\alpha$. Hence, by Eilenberg and Montgomery's theorem, correspondence Φ'_α has a fixed point $q^* \in \int_I X(i)\, d\mu_\alpha$. This implies that there exists a measurable function $f^* \in F_X$ such that $f^*(i) \in B(i)(q^*)$ for every $i \in I$. Clearly, f^* is a Nash equilibrium of game Γ_α. □

The game considered in Example 2 in Section 2 satisfies all the assumptions adopted here. More specifically, in that game, $span\{\int_A B_A(i)(q)\, d\mu_A\} = span\{\int_A X_A(i)(q)\, d(\alpha\mu_A)\} = R \subseteq span\{\int_N B_N(i)(q)\, d\mu_N\} = span\{\int_N X_N(i)(q)\, d\mu_N\} = R^2$, so Assumptions 4 and 5 are satisfied. Assumption 6 is satisfied as well by taking, for example, $\varepsilon = 0.01$.

6. Final remarks

If Assumption 1 is not satisfied, then Theorem 1 implies that if the integral of the atomic part is neither convex nor star shaped, then the integral of the correspondence cannot be star shaped, whatever the value of α (i.e. whatever the size of the atoms). (From an economic point of view this occurs, for example, when the objective function of the agents belonging to the non-atomic part is strictly concave, while the objective function of the atomic part is just continuous.) In this case, therefore, it is not possible to rely on our arguments in order to ensure the existence of an exact equilibrium. However, Assumption 1 is again satisfied if we replace the integral of the non-atomic part with the ball of radius r $(r>0)$ around it. In this case, in fact, $span\{B_r(\int_N F_N\, d\mu_N)\} = R^m \supseteq span\{\int_A F_A\, d(\alpha\mu_A)\}$; therefore, Theorem 1 is still valid. If we consider, for example, a game like that one analyzed in Section 5 and interpret the integral of the correspondence F as the aggregate best reply of players, then in replacing the integral of the non-atomic part with the ball around it we ensure that this correspondence has star-convex values for α "small enough". Therefore, the integral of correspondence F has a fixed point which can be interpreted as a Nash equilibrium of the game. However, this equilibrium is not an *exact* but an *approximate* Nash equilibrium since we have replaced the integral of the non-atomic part with the ball around it. Thus, for small enough α, there is an approximate equilibrium.

Appendix A

In order to show that game Γ_α in the Example 2 of Section 2 has a Nash equilibrium for $0 \leqslant \alpha \leqslant 0.6$, we denote by $A(B(A(x^\circ)))$ the sum of the elements of the vectors belonging to the integral of the best-reply correspondence associated with the initial strategy profile x°; i.e. $A(B(A(x^\circ))) = \int_{N_1} B_1(i)(x^\circ)\, d\mu + \int_{N_1} B_2(i)(x^\circ)\, d\mu + \int_{N_2} B_1(i)(x^\circ)\, d\mu + \int_{N_2} B_2(i)(x^\circ)\, d\mu + \int_A B_1(i)(x^\circ)\, d\mu + \int_A B_2(i)(x^\circ)\, d\mu$. Consider any strategy profile x° such that $A(x^\circ) < 3$, we obtain $A(B(A(x^\circ))) = 1.8 + 0.1 + 2\alpha$. Consider now any strategy profile x° such that $A(x^\circ) = 3$, we obtain: $A(B(A(x^\circ))) = [1.8 + \alpha, 1.9 + \alpha] \cup [1.8 + 2\alpha, 1.9 + 2\alpha]$. Consider finally any strategy profile x° such that $A(x^\circ) > 3$. Then $A(B(A(x^\circ))) = 1.8 + \alpha$. It follows that for $0 \leqslant \alpha \leqslant 0.55$, an equilibrium $x^* \in X$ exists with $A(x^*) < 3$ (take: $x^*(i) = (0,1)$ for $i \in N_1$, $x^*(i) = (0.5, 0.5)$ for $i \in N_2$, $x^*(i) = (1,1)$ for $i \in A$), if $0.55 \leqslant \alpha \leqslant 0.6$ or if $1.1 \leqslant \alpha \leqslant 1.2$, then an equilibrium exists with $A(x^*) = 3$ (take: $x^*(i) = (1,0)$ for $i \in N_1$, $x^*(i) = (0.5, 0.5)$ for

$i \in N_2$, $x^*(i) = (1,1)$ for $i \in A$); finally, if $\alpha > 1.2$ an equilibrium exists with $A(x^*) > 3$ (take: $x^*(i) = (1,0)$ for $i \in N_1$, $x^*(i) = (0,0)$ for $i \in N_2$, $x^*(i) = (0,1)$ for $i \in A$).

Appendix B

Proof of Remark 1. Notice first that if $\int_A F_A \, d\mu_A$ is convex, star shaped or contractible, then $\int_A F_A \, d(\alpha\mu_A)$ is convex, star shaped or contractible as well, since $\int_A F_A \, d(\alpha\mu_A) = \alpha \int_A F_A \, d\mu_A$.

(i) This assertion is well known (see, for example, [20, Theorem 1]).

(ii) It is sufficient to prove the following more general fact: let X and Y be two star-shaped subsets in R^m, then $X + Y$ is star shaped. In fact, since X and Y are star shaped, there exist $x^* \in X$ and $y^* \in Y$ such that $[x^*, x] \subset X$ for every $x \in X$, and $[y^*, y] \subset Y$ for every $y \in Y$. Take $z^* = x^* + y^*$. Clearly, $z^* \in X + Y$. We shall show that $[z^*, z] \subset X + Y$ for every $z \in X + Y$. In fact, take any $z \in X + Y$; then $z = x + y$ for $x \in X$ and $y \in Y$. However, $\alpha x^* + (1-\alpha)x \in X$ and $\alpha y^* + (1-\alpha)y \in Y$ for $\alpha \in [0,1]$, by star-shapedness, therefore, $\alpha z^* + (1-\alpha)z = \alpha x^* + (1-\alpha)x + \alpha y^* + (1-\alpha)y \in X + Y$ for $\alpha \in [0,1]$.

(iii) It is sufficient to prove the following more general result if (A) X and Y are two compact sets in R^m, X is star shaped and Y is contractible and (B) there exists a k-dimensional subspace S in R^m (with $k < m$) such that $X \subset S$ and $Y \subset S^\perp$, then the set $X + Y$ is contractible.

As for (A), star-shapedness of X implies that there exists x^* in X such that $[x, x^*] \subset X$ for every x in X, contractibility of Y implies that there exists a continuous function $G\colon Y \times [0,1] \to Y$ such that $G(y,0) = y$ and $G(y,1) = y^*$, with $y^* \in Y$. Assumption (B) implies that for every $z \in X + Y$ there exists a unique couple of vectors $x(z) \in X$ and $y(z) \in Y$ such that $x(z) + y(z) = z$; moreover, the functions $x\colon X + Y \to X$, $y\colon X + Y \to Y$ defined by $x(z)$ and $y(z)$ are continuous. In fact, denote by $x_1, x_2, \ldots, x_k$ a base of S and by $y_1, y_2, \ldots, y_{m-k}$ a base of $S^\perp$. It is standard that any vector x in X and any vector y in Y have a unique continuous representation in terms of the chosen base i.e. $x = \sum_{i=1}^k \lambda_i(x)x_i$ and $y = \sum_{j=1}^{m-k} \delta_j(y)y_j$ where the vectors $\{\lambda_i(x)\}$ and $\{\delta_j(y)\}$ are unique and each component is a continuous functions of its argument. Suppose now that $z \in X + Y$ and that z does not have a unique representation in terms of elements of X and Y; i.e. suppose that $z = x + y = x' + y'$ where $x, x' \in X$ and $y, y' \in Y$ with $x \neq x'$ and $y \neq y'$. Then, $\sum_{i=1}^k (\lambda_i(x) - \lambda_i(x'))x_i - \sum_{j=1}^{m-k} (\delta_j(y) - \delta_j(y'))y_j = 0$ which, by the linear independence of $x_1, x_2, \ldots, x_k,\ y_1, y_2, \ldots, y_{m-k}$, implies that $\lambda_i(x') = \lambda_i(x')$ and $\delta_j(y) = \delta_j(y')$ for every i and every j, which contradicts the assumption that $x \neq x'$ and $y \neq y'$. On the other hand, take a sequence $\{z_n\}$ in $X + Y$ converging to z°; by compactness, there are sequences $\{x(z_n)\}$ and $\{y(z_n)\}$ in X and Y converging, respectively, to $x^\circ \in X$ and $y^\circ \in Y$, and such that for every n: $z_n = x(z_n) + y(z_n)$. Now, on the one hand $z^\circ = x(z^\circ) + y(z^\circ)$; on the other hand, continuity yields that $z^\circ = x^\circ + y^\circ$.

Therefore, by the fact that z is uniquely represented by elements in X and Y one obtains that $x^\circ = x(z^\circ)$ and $y^\circ = y(z^\circ)$.

Consider now the function $H: X + Y \times [0,1] \to x^* + Y$ defined as follows: $H(z,t) = tz + (1-t)(x^* + y(z))$ and where x^* has been defined above. Function H is clearly continuous; moreover, $H(z,1) = z$ and $H(z,0) = x^* + y(z)$; finally, $H(z,t) = tx(z) + (1-t)x^* + y(z) \in X + Y$ for every $t \in [0,1]$, where the last relation holds true because of the star shapedness of X and by the way in which x^* has been defined. We now show that function H is homotopic to a constant function. In fact, consider the function $Z: x^* + Y \to Y$ defined as follows: $Z(x^* + y, t) = x^* + G(y,t)$, where G has been defined above. Clearly, Z is continuous; moreover, by the properties of function G, $Z(x^* + y, 0) = x^* + y$, $Z(x^* + y, 1) = x^* + y^*$ and $Z(x^* + y, t) = x^* + G(y,t) \in x^* + Y$ for every $t \in [0,1]$. Hence, $X + Y$ is contractible.

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