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A NON-SUBSTITUTION THEOREM WITH NON-CONSTANT RETURNS TO SCALE AND EXTERNALITIES

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ABSTRACT

An input–output model with non-constant returns to scale and externalities is presented, and it is shown that in this model the non-substitution theorem is still valid. More precisely, the quantity side of the theorem, i.e. the proposition on efficiency, remains valid, while there can be no equilibrium prices independent of final demand.

1. INTRODUCTION

In a classroom of linear economic models, it is usually explained that the non-substitution theorem holds when (1) there is only one primary production factor called labor, (2) constant returns to scale prevail, (3) there exists no joint production, (4) there exist no externalities and (5) the production period is uniform among processes. Manning (1981) showed the theorem is valid even when there are many primary factors, if those primary factors are used in fixed proportions. Schefold (1978a) argued that if a given net output matrix is non-negative invertible, then the non-substitution theorem survives in spite of the existence of joint production. Herrero and Villar (1988) proved in a rigorous way that the non-negative invertibility of the net output matrix

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is necessary and sufficient for the non-substitution theorem to hold under the above conditions except (3) concerning the absence of joint production. This proposition was obtained more than two decades ago by Dasgupta and Sinha (1979), but its publication was delayed until 1992 (see Dasgupta (1992), Dasgupta and Sinha (1992)).[1]

Now the problem remains whether the theorem holds good when the two conditions, (2) constant returns to scale and (4) absence of externalities, are relaxed at the same time. The answer is in the affirmative concerning the efficiency, and it is almost a tautology, but should be reported in an explicit way so that economists in general realize that the non-substitution theorem can survive in an economy with non-constant returns to scale and externalities, albeit of special types. In the next section, a generalized model is explained and our theorems are presented. In the final section, some additional comments are given.

2. MODELS WITH NON-CONSTANT RETURNS TO SCALE AND EXTERNALITIES

Our model is very much like an ordinary linear input–output model without joint production. First we explain our notation. The symbol R^n means the real Euclidean space of dimension n ($n \geq 2$), and R^n_+ the non-negative orthant of R^n. Let $S^{n-1} \equiv \{x | x \in R^n, \|x\| = 1\}$, where x is a column vector and $\|\cdot\|$ is the Euclidean norm, and $S^{n-1}_+ \equiv S^{n-1} \cap R^n_+$. A given real $m \times n$ non-negative matrix A maps from R^n_+ into R^m_+. The symbol $\mathbf{1}^n$ is the row vector in R^n whose entries are all unity. The inequality signs for vector comparison are as follows:

$$x \geq y \quad \text{iff} \quad x - y \in R^n_+$$
$$x > y \quad \text{iff} \quad x - y \in R^n_+ - \{0\}$$
$$x >> y \quad \text{iff} \quad x - y \in \text{int}(R^n_+)$$

Now the matrix A is understood as the input coefficient matrix. m is the number of commodities, while n represents the number of processes avail-

[1] Bidard (1991) presented a necessary and sufficient condition for a real square matrix to have a strictly positive inverse. Following this, Erreygers (1996) rediscovered propositions due to Dasgupta (1992), and Bidard (1996) and Bidard and Erreygers (1998) gave a necessary and sufficient condition for a square matrix to have a non-negative inverse. (An alternative way of treating joint production was investigated by Hinrichsen and Krause (1978, 1981), Nermuth (1984) and Takeda (1989). For more recent contributions, see also Kuga (2001), Hasfura-Buenaga *et al.* (2002), Fujimoto *et al.* (2002a, 2002b). For inverse positive matrices, the reader is referred to Berman and Plemmons (1979).)

able. The processes are normalized so that the labor input coefficient is unity in each process, implying that every process needs the sole primary input labor. Thus, the n-row vector $\mathbf{1}^n$ stands for the labor input coefficient vector. A column vector $\boldsymbol{x} \in R_+^n$ shows a corresponding activity vector with its jth entry x_j meaning the activity level of process j. Then $A\boldsymbol{x}$ stands for the material input vector as in normal input–output models, and $\mathbf{1}^n \cdot \boldsymbol{x}$ for the labor input. The vector $M\boldsymbol{x} \equiv \boldsymbol{x} - A\boldsymbol{x}$, however, does not represent the net output vector $v(\boldsymbol{x})$, which, in our model, is described by

$$v(\boldsymbol{x}, M) \equiv s\left(\frac{\mathbf{1}^n \cdot \boldsymbol{x}}{\mathbf{1}^n \cdot \boldsymbol{x}}, \frac{\boldsymbol{x} - A\boldsymbol{x}}{\|\boldsymbol{x} - A\boldsymbol{x}\|}\right) \cdot (\boldsymbol{x} - A\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in R_+^n \quad \text{such that } \boldsymbol{x} - A\boldsymbol{x} > 0$$

and

$$v(\boldsymbol{x}, M) \equiv 0 \quad \text{for } \boldsymbol{x} \in R_+^n \quad \text{such that } \boldsymbol{x} - A\boldsymbol{x} = 0$$

where $s(e, \boldsymbol{z})$ is a real function from $R_+ \times S_+^{n-1}$ into R_+, and $\boldsymbol{x}^{\mathrm{o}}$ is a given standard activity vector with $\mathbf{1}^n \cdot \boldsymbol{x}^{\mathrm{o}} > 0$. (Throughout this note, the rate of steady balanced growth is assumed to be zero.) Let us define two symbols here:

$$e(\boldsymbol{x}) \equiv \frac{\mathbf{1}^n - \boldsymbol{x}}{\mathbf{1}^n \cdot \boldsymbol{x}^{\mathrm{o}}}$$

$$\boldsymbol{z}(\boldsymbol{x}) \equiv \frac{M\boldsymbol{x}}{\|M\boldsymbol{x}\|} = \frac{\boldsymbol{x} - A\boldsymbol{x}}{\|\boldsymbol{x} - A\boldsymbol{x}\|}$$

The macroscopic synergetic effect, s, thus depends upon the total employment level relative to the standard size of employment as well as the relative composition of net output vector $\boldsymbol{z}$. The net output vector $v(\boldsymbol{x}, M)$ is from the domain

$$D \equiv \{\boldsymbol{x} | \boldsymbol{x} \in R_+^n, \boldsymbol{x} - A\boldsymbol{x} \geq 0\}$$

to R_+^m.

In ordinary linear models,

$$s(e(\boldsymbol{x}), z(\boldsymbol{x})) = 1$$

on its domain. Let us first consider the following linear programming problem:

$$\min \mathbf{1}^n \cdot \boldsymbol{x} \text{ subject to } M\boldsymbol{x} \equiv \boldsymbol{x} - A\boldsymbol{x} \geq d^{\circ} \text{ and } \boldsymbol{x} \in R_+^n$$

where $\boldsymbol{d}^{\circ}$ is an arbitrarily given column vector in R_+^m such that $\boldsymbol{d}^{\circ} >> 0$. We assume this minimization program has a solution $\boldsymbol{x}^*$ with its associated processes *utilized*, i.e. those processes with $x_j^* > 0$, forming $M^* \equiv I - A^*$. In addition, we assume the number of columns of M^* is not less than the number of rows. Then, without losing generality, we can suppose M^* is square, having m rows as well as m columns. (See Schefold (1978b).) Then we have $M^*\boldsymbol{x}^* = \boldsymbol{d}^{\circ}$, or $\boldsymbol{x}^* = (M^*)^{-1}\boldsymbol{d}^{\circ}$ with $(M^*)^{-1} \geq 0$ by a well-known theoremon M matrices.

We first prove the ordinary non-substitution theorem for the sake of self-containedness, and for easy understanding of the discussions below. The proof depends on the use of linear programming, and it is due to Chander (1974). (The method in Mirrlees (1969) is essentially the same as that of Chander (1974) with a growth factor or an interest factor preceding the input matrix.)

Theorem 1: If $s(u, z) = 1$ everywhere on its domain, then the non-substitution theorem holds in the sense that the processes forming M^* can realize the most efficient production with respect to the use of the primary input, labor, for any final demand vector $\boldsymbol{d} \in R_+^m$.

Proof: Consider the programming problem dual to the problem above:

$$\max \boldsymbol{p} \cdot \boldsymbol{d}^{\circ} \text{ subject to } \boldsymbol{p}M \leq \mathbf{1}^n \text{ and } \boldsymbol{p} \in R_+^m$$

where $\boldsymbol{p}$ is understood as a row vector. A solution to this dual program can be obtained as $\boldsymbol{p}^* = \mathbf{1}^m \cdot (M^*)^{-1}$ because those constraints in the dual which correspond to the columns of M^* are strictly binding, and M^* has a non-negative inverse.

Then, let us take up an arbitrary production scheme

$$M\boldsymbol{x} \geq \boldsymbol{d} \text{ for some } \boldsymbol{x} \in R_+^n$$

where $\boldsymbol{d} \in R_+^m$ is a designated vector in this production scheme. Pre-multiplying this inequality by $\boldsymbol{p}^*$, we have

$$\boldsymbol{p}^* M\boldsymbol{x} \geq \boldsymbol{p}^* \cdot \boldsymbol{d} \tag{1}$$

Since the solution $\boldsymbol{p}^*$ satisfies the constraint, it follows that $\boldsymbol{p}^*M \leq \mathbf{1}^n$, and post-multiplying this by $\boldsymbol{x}$ we obtain

$$\boldsymbol{p}^*M\boldsymbol{x} \le \mathbf{1}^n \cdot \boldsymbol{x} \tag{2}$$

From these inequalities (1) and (2), we have

$$\boldsymbol{p}^* \cdot \boldsymbol{d} \le \mathbf{1}^n \cdot \boldsymbol{x} \tag{3}$$

On the other hand, we can find $\boldsymbol{y}^*$ such that $M^*\boldsymbol{y}^* = \boldsymbol{d}$ because M^* has a non-negative inverse. Thus,

$$\mathbf{1}^m \cdot \boldsymbol{y}^* = 1^m (M^*)^{-1} \boldsymbol{d} = \boldsymbol{p}^* \cdot \boldsymbol{d} \tag{4}$$

From the inequalities (3) and (4), we have $\mathbf{1}^m \cdot \boldsymbol{y}^* \le \mathbf{1}^n \cdot \boldsymbol{x}$. This is what should be proved. ∎

Next we deal with the case where $s(e, z) \neq 1$, and in addition make the following assumption.

Assumption A1: A given function s is continuous with respect to the first argument e and satisfies $\beta \ge s(e, z) \ge \alpha$ on its domain for two positive constants α and β, and $e \cdot s(e, z)$ is strictly increasing with respect to e.

It should be noticed that s itself need not be monotone non-decreasing or non-increasing with respect to the first argument. We can show the following proposition.

Proposition: The net output vector $v(\boldsymbol{x}, M^*)$ is one to one and onto the non-negative orthant R_+^m.

Proof: First we show that $v(\boldsymbol{x}, M^*)$ is injective. Suppose there exist two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in R_+^n such that $\boldsymbol{x} \neq \boldsymbol{y}$ and $v(\boldsymbol{x}, M^*) = v(\boldsymbol{y}, M^*) \in R_+^m$. By premultiplying this equation by $(M^*)^{-1}$, we have $s(e(\boldsymbol{x}), z(\boldsymbol{x})) \cdot \boldsymbol{x} = s(e(\boldsymbol{y}), z(\boldsymbol{y})) \cdot \boldsymbol{y}$. Thus, $\boldsymbol{x}$ and $\boldsymbol{y}$ are expressed as $\boldsymbol{x} = k\boldsymbol{y}$ with k being assumed greater than unity, $k > 1$, without loss of generality. This leads to $k \cdot s(ke(\boldsymbol{y}), z(\boldsymbol{y})) \cdot \boldsymbol{y} = s(e(\boldsymbol{y}), z(\boldsymbol{y})) \cdot \boldsymbol{y}$, which implies $ke \cdot s(ke(\boldsymbol{y}), z(\boldsymbol{y})) = e \cdot s(e(\boldsymbol{y}), z(\boldsymbol{y}))$, contradicting assumption A1 that $e \cdot s(e, z)$ is strictly increasing with respect to e.

Next, let us prove $v(\boldsymbol{x}, M^*)$ is onto. At the outset we note $v(0, M^*) = 0$. Then, for an arbitrary $\boldsymbol{d} > 0$ in R_+^m, we can find two vectors $\boldsymbol{y} = (M^*)^{-1} \cdot (\boldsymbol{d}/\alpha)$ and $\boldsymbol{w} = (M^*)^{-1} \cdot (\boldsymbol{d}/\beta)$. It follows that $\boldsymbol{y} \ge \boldsymbol{w}$ and by assumption A1

$$\begin{aligned} v(\boldsymbol{y}, M^*) &= s(e(\boldsymbol{y}), z(\boldsymbol{y})) \cdot M^*\boldsymbol{y} \ge \alpha \cdot M^*\boldsymbol{y} = \boldsymbol{d} \\ &= \beta \cdot M^*\boldsymbol{w} \ge s(e(\boldsymbol{w}), z(\boldsymbol{w})) \cdot M^*\boldsymbol{w} = v(\boldsymbol{w}, M^*) \end{aligned}$$

Noting that $v(k\boldsymbol{y}, M^*)$ and $v(k\boldsymbol{w}, M^*)$ are always proportional to $\boldsymbol{d}$ for any scalar $k > 0$, and $s(e, \boldsymbol{z})$ is continuous, there should be an $\boldsymbol{x}^*$ such that $v(\boldsymbol{x}^*, M^*) \equiv s(e(\boldsymbol{x}^*), \boldsymbol{z}(\boldsymbol{x}^*)) \cdot M\boldsymbol{x}^* = \boldsymbol{d}$, thanks to the intermediate value theorem. ■

Now it is not difficult to prove theorem 2.

Theorem 2: Given assumption A1, the non-substitution theorem holds in the sense that, for an arbitrarily given production scheme, the optimal processes in M can produce at least the final demand vector in that scheme with the same amount of labor.*

Proof: Let a given activity vector be $\boldsymbol{x} \in R^n_+$, and define the resulting final demand vector as

$$\boldsymbol{d} \equiv s(e(\boldsymbol{x}), \boldsymbol{z}(\boldsymbol{x})) \cdot (\boldsymbol{x} - A\boldsymbol{x})$$

Then calculate $\boldsymbol{x}^* \in R^m_+$ as

$$\boldsymbol{x}^* = (M^*)^{-1}\boldsymbol{d} \big/ (s(e(\boldsymbol{x}), \boldsymbol{z}(\boldsymbol{x})))$$

Thus we have

$$s(e(\boldsymbol{x}), \boldsymbol{z}(\boldsymbol{x})) \cdot (\boldsymbol{x}^* - A^*\boldsymbol{x}^*) = \boldsymbol{d}$$

From theorem 1, we should have $\mathbf{1}^m \cdot \boldsymbol{x}^* \leq \mathbf{1}^n \cdot \boldsymbol{x}$ because both $\boldsymbol{x}^*$ and $\boldsymbol{x}$ can be thought of as producing $\boldsymbol{d}/s(e(\boldsymbol{x}), \boldsymbol{z}(\boldsymbol{x}))$. Since

$$\boldsymbol{z}(\boldsymbol{x}) = \boldsymbol{z}(\boldsymbol{x}^*) = \frac{\boldsymbol{d}}{\|\boldsymbol{d}\|}$$

if $\mathbf{1}^m \cdot \boldsymbol{x}^* = \mathbf{1}^n \cdot \boldsymbol{x}$, we have $s(e(\boldsymbol{x}^*), \boldsymbol{z}(\boldsymbol{x}^*)) \cdot (\boldsymbol{x}^* - A^*\boldsymbol{x}^*) = \boldsymbol{d}$, and the theorem is confirmed. When $\mathbf{1}^m \cdot \boldsymbol{x}^* < \mathbf{1}^n \cdot \boldsymbol{x}$, we can find a scalar $k > 1$ such that $k \cdot \mathbf{1}^m \cdot \boldsymbol{x}^* = \mathbf{1}^n \cdot \boldsymbol{x}$, and consider an activity vector $k\boldsymbol{x}^*$, using the optimal technology M^*. By this production activity $k\boldsymbol{x}^*$, it is possible to yield

$$s(e(k\boldsymbol{x}^*), \boldsymbol{z}(k\boldsymbol{x}^*)) \cdot ((k\boldsymbol{x}^*) - A^*(k\boldsymbol{x}^*))$$

which is equivalent to

$$k \cdot s(e(\boldsymbol{x}), \boldsymbol{z}(\boldsymbol{x})) \cdot (\boldsymbol{x}^* - A^*\boldsymbol{x}^*) = k \cdot \boldsymbol{d} \geq \boldsymbol{d}$$

Thus, the theorem is proved. ■

Remark: There can be an irregular case in which for an $\boldsymbol{x} \in R_+^n$

$$s(e(\boldsymbol{x}), \boldsymbol{z}(\boldsymbol{x})) \cdot (\boldsymbol{x} - A\boldsymbol{x}) \geq s(e(\boldsymbol{x}^*), z(\boldsymbol{x}^*)) \cdot (\boldsymbol{x}^* - A^* \boldsymbol{x}^*) > 0 \tag{5}$$

with

$$\mathbf{1}^n \cdot \boldsymbol{x} < \mathbf{1}^m \cdot \boldsymbol{x}^* \tag{6}$$

(See figure 1 for how this anomaly occurs in a two-commodity case.) In plain words, there can be a set of processes which can produce a final demand vector with less labor than the optimal set of processes M^*. In this case, our theorem 2 simply asserts that there exists a $\boldsymbol{y} \in R_+^m$ such that

$$s(e(\boldsymbol{y}), \boldsymbol{z}(\boldsymbol{y})) \cdot (\boldsymbol{y} - A^* \boldsymbol{y}) \geq s(e(\boldsymbol{x}), \boldsymbol{z}(\boldsymbol{x})) \cdot (\boldsymbol{x} - A\boldsymbol{x}) \tag{7}$$

with

$$\mathbf{1}^m \cdot \boldsymbol{y} = \mathbf{1}^n \cdot \boldsymbol{x} \qquad \text{and} \qquad \boldsymbol{z}(\boldsymbol{y}) = \boldsymbol{z}(\boldsymbol{x}) \tag{8}$$

To prevent the above anomaly, let us consider one more assumption, which says that more output requires more labor if the same optimal set of processes is utilized.

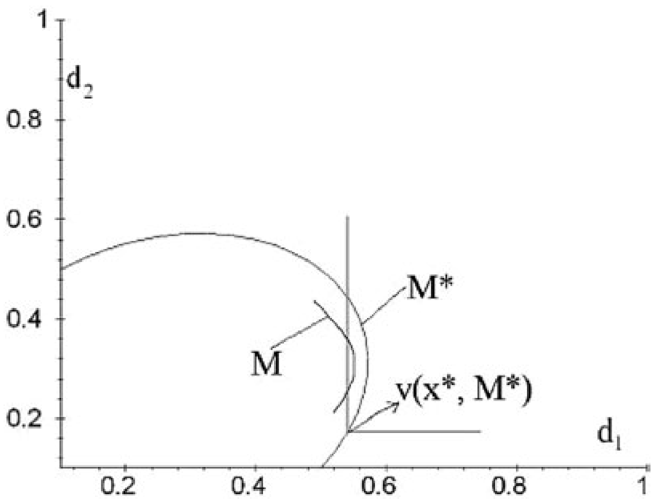

Figure 1

Assumption A2: If $s(e(\boldsymbol{y}), \boldsymbol{z}(\boldsymbol{y})) \cdot (\boldsymbol{y} - A^*\boldsymbol{y}) \geq s(e(\boldsymbol{x}), \boldsymbol{z}(\boldsymbol{x})) \cdot (\boldsymbol{x} - A^*\boldsymbol{x}) > \mathbf{0}$, then $\mathbf{1}^m \cdot \boldsymbol{y} \geq \mathbf{1}^m \cdot \boldsymbol{x}$.

Note that the monotonicity is required only for the optimal set of processes.

Theorem 3: Given assumptions A1 and A2, there can be no set of processes by using which a final demand vector can be produced with less labor than using the optimal technology M^*.

Proof: Suppose to the contrary, and that inequalities (5) and (6) hold. On the other hand, from theorem 2, we have inequalities (7) and (8) as explained above. Therefore it follows from these inequalities

$$s(e(\boldsymbol{y}), \boldsymbol{z}(\boldsymbol{y})) \cdot (\boldsymbol{y} - A^*\boldsymbol{y}) \geq s(e(\boldsymbol{x}^*), \boldsymbol{z}(\boldsymbol{x}^*)) \cdot (\boldsymbol{x}^* - A^*\boldsymbol{x}^*)$$

with $\mathbf{1}^m \cdot \boldsymbol{y} < \mathbf{1}^m \cdot \boldsymbol{x}^*$, a contradiction to assumption A2. ■

Our theorem 3 tells us that whatever the composition of the final demand is, and whatever the level of output, a set of processes can remain efficient as in the case of the non-substitution theorem with constant returns to scale and no externalities.

Concerning prices, we can obtain a unique vector of equilibrium prices as

$$\boldsymbol{p}^* = \mathbf{1}^m \cdot (M^*)^{-1} / s(e(\boldsymbol{x}^*), \boldsymbol{z}(\boldsymbol{x}^*))$$

when the rate of profit is zero and the activity vector is $\boldsymbol{x}^*$. As the rate of profit r becomes positive, in the above formula, M^* should be replaced by $M^*(r) \equiv I - (1 + r)A^*$, and this formula is valid while the rate of profit r is less than the Frobenius root of M^*, supposing wages are paid after production.

An important fact to note is that now the equilibrium prices (more precisely, wage-unit prices) depend on the final demand, its global scale and its composition, though relative prices other than the wage rate remain the same. Thus, we may say that the half facet, the quantity side of the non-substitution theorem, i.e. the proposition on efficiency, survives, while the other half, the price side, i.e. the unique wage-unit prices independent of final demand, is lost.

3. REMARKS AND A NUMERICAL EXAMPLE

Our theorems are trivial if we consider the matter in terms of geometry of contours. The equi-employment net production loci (or contours) in the linear case are expanded or contracted by the synergetic factor $s(e(\boldsymbol{x}), \boldsymbol{z}(\boldsymbol{x}))$

in the direction of $\boldsymbol{z}(\boldsymbol{x})$, i.e. the net output. Thus, relative efficiency of a particular set of processes does not change in that particular direction. So, the non-substitution theorem can survive. Salvadori (1987) is careful enough not to exclude such a proposition on the quantity side.

Then, we may allow for some, but not all, automatic processes with zero labor input. This is possible, for example, when we assume in addition the undecomposability of A^*.

Next, it is straightforward to allow for joint production of the type considered in Dasgupta and Sinha (1979) and Dasgupta (1992). (See also Bidard (1996).) All we have to do is replace $M \equiv I - A$ by $M \equiv B - A$ and $M^* \equiv I - A^*$ by $M^* \equiv B^* - A^*$, and to assume the non-negative invertibility of M^*, or one of its equivalent conditions.

Here is a numerical example of our model with joint production. Let

$$M \equiv \begin{pmatrix} 1 & -1 & 1 & 0.3 \\ 1 & 1 & -1 & 0.3 \\ -1 & 1 & 1 & 0.3 \end{pmatrix}$$

and let the net output vector be described as

$$s(e, \boldsymbol{z}) \cdot (M\boldsymbol{x})$$

where

$$s(e,\boldsymbol{z}) \equiv \begin{cases} \left[c\dfrac{(1+a-b)/e+b}{1+ae} + (1-c)\right]\left[\gamma\sum_{i=1}^{3}\left(\dfrac{z_i}{z_1+z_2+z_3} - \dfrac{1}{3}\right)^2 + 1\right] & \text{when } e > 1 \\ 1\left(\gamma\sum_{i=1}^{3}\left(\dfrac{z_i}{z_1+z_2+z_3} - \dfrac{1}{3}\right)^2 + 1\right) & \text{when } 0 \le e \le 1 \end{cases}$$

with $1 > \gamma > 0$, $b \ge a > 0$, $1 + a - b > 0$, $1 > c > 0$ and $e \equiv (\mathbf{1} \cdot \boldsymbol{x})/(\mathbf{1} \cdot \boldsymbol{x}^{\circ})$. Here, $\mathbf{1}$ is in R^4_+, $\boldsymbol{x}^{\circ} \equiv (1, 1, 1, 1)'$ and $\boldsymbol{z} \equiv (z_1, z_2, z_3)' \in S^2_+$, with a prime indicating transposition. This example shows overall scale diseconomies because s is decreasing with respect to the first variable e, when $e > 1$. It is not difficult to see that $e \cdot s(e, \boldsymbol{z})$ is strictly increasing with respect to e. Thus, assumption A1 is satisfied with $\alpha = 1 - c$ and $\beta = 1$. The function s is strictly convex with respect to the variable $\boldsymbol{z}$ on the simplex S^{n-1}_+, and so the iso-employment locus is also convex toward the origin, thus satisfying assumption A2. (The case depicted in figure 1 cannot take place.) In this numerical example, the first three columns of M form the optimal set of processes M^*, which has a non-negative inverse.

Now let us discuss the economic significance of our model. Our assumptions are not unrealistically arbitrary. First, ours is more general than normal linear input–output models: they are included as special cases when the function s is identically one. The factor s is not the combination of *individual* variable returns to scale, but can be interpreted as *global synergetic* economies/diseconomies, whose effect is determined by the level of *total* employment relative to a standard employment level and the relative composition of *economy-wide* net output. Realizing that these global synergetic effects do exist, our model can be understood as an econometric construct rather than a theoretical one. We may estimate the form of the function s, as data show changes in the employment levels and the composition of output. When an economy is specialized in a smaller number of industries, it may exhibit specialization economies/diseconomies: this does not depend upon the actual scales but the relative intensities of activity levels. Thus, the iso-employment locus need not be a hyper-plane, and may generally be a curved surface because of the second parameter z in the function s. On the other hand, there can be many firms within each industry with different levels of production; thus not only output but also *input* is influenced by global synergetic effects, making the factor s acting on the *net output* in the traditional linear models less unrealistic. Such a supposition should not be rejected as unrealistic in a model of a real national economy where variable returns and externalities are observed. Our assumption concerning variable returns to scale and externalities is certainly special in that they keep the relative composition of output. In a sense, the synergetic effects work on every process equally or, using a mathematical term, they are 'ray-preserving'. After all, our model is more general than most linear models. So far we have paid attention solely to individual production processes, and not much consideration has been given to the overall synergetic effects, except for some specific externalities among industries in microscopic models. In econometric research, however, the synergetic effect s may be difficult to detect simply because the variations in scales of processes within given data are not large enough for a period during which we can suppose technology is more or less constant.

Finally, since the equilibrium wage-unit prices depend upon final demands and some kinds of proper joint production are allowed for, we cannot use the methods of proof based on cost functions, which are employed in Morishima (1964), Stiglitz (1970), Johansen (1972) (as amended by Dasgupta (1974)), Fujimoto (1980, 1987) and Kuga (2001).

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