THE PRESENT MAKROECONOMY AND THE THEORY OF OPTIMUM CONTROL

Michal SLAVÍK, Czech National Bank, Prague

1. Ú ú o d

The main stream of macroeconomic theory attempted to cope with the criticism of other economics schools, which was reckoned with weak microeconomic fundamentals and insufficient capture of the dynamics of the economic system. Traditional Keynesian macroeconomics worked with national-type national aggregates and to some extent completely ignored the microeconomic theory. Dynamic analysis was carried out using comparative statics, a process that compared two world states that change only one dimension (from the basic textbooks of the notorious principle, ceteris paribus, that is, everything else unchanged). Neoclassical macroeconomics attempted to incorporate microeconomic elements under the influence of critics. Instead of aggregated variables in their modes, they examine the behavior of disaggregated entities - typically individuals or households and firms. For simplicity, it is assumed that the economy is composed of many identical households and firms, and aggregate data are obtained by aggregating the same identical micro-entities. The behavior of aggregate indicators thus reflects behaviors that we would expect from the knowledge of microeconomics.

Deviation from aggregate values and involvement of individual entities allows us to use elements known from microeconomic analysis, static optimization with constraints, searching for an extreme (maximum or minimum) that is limited by a certain equality or inequality condition or conditions. This procedure leads to finding a static optimum at a certain time point and again does not say anything about the dynamics of the eco nomic system. As a tool for dealing with this problem, optimal control theory can be used in some cases. Instead of optimal optimum, optimum trajectory of the system (or control variable) is searched for in time. Since it is a mathematical optimization typical of neo-classical economics, the dynamics of the system is understood mechanically and does not cope with historical time. From the point of view of economic theory, this is a method for applying microeconomic approaches, for example. to dynamic macroeconomic problems, as some models of contemporary theory of growth are trying to do (see eg Agh, Ho, Witt, 1998 or Barro-Sala-i-Martin, 1999) or monetary economics Walsh, 1999).

The aim of the contribution is to bring the reader to the basics of optimal management theory and to draw attention to some of its applications in contemporary macroeconomics. The structure of the contribution is the following: in the first part some definitions are defined, in the second part are briefly introduced the basics of the theory of optimal control, then two generalizations are interpreted: the first concerns the transfer to the present value and the second infinite time horizon. In part 5, the method is approximated by two common economic applications and finally a brief conclusion is given. While I try to make the interpretation as clear as possible and I do not want to burden readers with too technical

details, knowledge of the basic mathematical apparatus used in master's courses in economics is necessary for understanding the method.1)

2. C o m a n d a n d a c tio n s

The classic role of static optimization can be seen as a search for the maximum profitability of a company's function. The company knows the demand for its products, knows its costs and faces the problem of producing a certain amount of product to maximize its profit. This quantity is determined by the fact that after the first deviation of the function is equal to zero and the equation is solved. Then, he finds the sign of the second derivative of the gain function and assures that the found extreme of the function is actually the maximum (the second derivative is negative). In fact, the company will not be able to look for an extreme function in the entire profiling function, but it can be limited by certain technological limitations that define the area in which the search term will be found (eg, the amount produced will be within a certain range of values).

Dynamic optimization will practically be an identical problem that the neoclassical firm will solve at any time of the time interval [0, T]. Our goal is to maximize profits for the whole time (not a profit in one single point - the firm would invest nothing and try to make the most of the existing stock of capital). A number of commodities can be input into the company's decision-making process, which can only be taken into account for the profitability of a company - for the sake of consistency, it is possible to consider only two of them. One can not directly influence the company and use it as an instrument - control variable. The second, the state variable that characterizes the system is not directly influenced by the company's immediate decisions. The company can influence this state variable by a control variable. In the context of this example, for example, pair of commodities: total company capital and investment.

The capital that determines how much the company is able to produce is the status quo (the firm can not directly influence the size of its stock of capital at some point in time, or • it is due to past developments). Investments that affect the total amount of capital, together with the depreciation of the capital, are the controlling variable through which a firm can exercise the size of its total capital. The state variable may be affected by one or more conditions. One of the most common conditions (for optimization of consumption or investment) can be, for example, budget constraints. The progress of the static quantity in time is determined by the differential equation - the law of motion. In order for the model to be solvable, the status parameter must be anchored in time, its starting and ending value must be known - we talk about the so- border conditions. The amount of total capital between 0 and T will be evolved in accordance with the law of motion.

If we move to math, the following variables can be defined:

- continuous time t; t ∈ [0, T],

- state variable s (t); s (t) sS, s: [0, T]

- control (variable) variable c (t); c (t) ÎC, c: [0, T] mm, where n ³ m,

- law of motion2) characterizing the change of state variable with (t)

¶S (t) ·

= t (t), c (t), t), where g: S × C × [0, T]

¶t

- the target function V, a function that describes the value of the system for any

1) For the time being, we know that it is not easy to read Barrot and Sala-i-Martin (1999), which summarizes both the basics of static optimization and the basics of differential equations solving and dynamic optimization, or Chiang's monograph (1992), which provides a more detailed introduction to dynamic optimization. Also interesting is the work of Violante (2002), on which I am going.

2) The dash above the variable is the derivation of the given variable according to the time.

control and status variables, it is an integral yield (or utility, gain) function known from the static optimization u (·): 3)

(T), c (t), t) dt,

0

- the set of constraints f (s (t), c (t), t) ³ 0, which defines the values that the control variable c (t) can take.

Two basic prerequisites are required for the solvability of the task:

- all the functions listed are continuous and differentiable,

- the set w (s (t), t) = {c (t) Î C m. f (s (t), c (t), t) ³ 0} is compact and convex.

Following this definition of variables, we can approach the definition of an overall problem (CP):

max V = òT u (s (t), c (t), t) dt

{C (t)} 0

.

with respect to: s (t) = g (s (t), c (t), t)

s (t) Î S c (t) Î C

f (s (t), c (t), t) ³ 0

s (0) = s0, s (T) = sT

By resolving the CP, the optimal path is the control variable (and therefore the law of motion and knowledge of the start and end conditions of the trajectory state variable) for each time interval of the time interval. The basic difference compared to static optimization where the optimal value is solved at a single time point is in this type of task to find the optimal time trajectory of the control variable {c * (t)}.

In the long run, the text will be written on one of the top-level CPs - the maxim principle - designed by the Russian mathematician Pontryagin in the 1950s. The English translation of his work was published in 1962. The competitive approach to the solution is the dynamic programming method proposed by R. Belenman (1957). His method is more appropriate for solving discrete time problems, Pontryagion's in continuous. Prior to describing the Pontryagion method, it may be appropriate to approach the possibly abstractly-looking problem with one of the possible applications, which is the optimal allocation of consumption within the economic entity's lifecycle (see, for example, Arlt, Èutková, Radkovský, 2002).

Let us assume that a man or a household can only benefit from the use of c (t). We usually express utility by u (·) and in this case it will depend only on a single value - on consumption, ie u (c (t)). Let's assume that every eco-nomic sub-factor knows the length of his life - he knows he's been living for years. During his lifetime, he is able to accumulate the wealth w (·) and, for the sake of clarity, he assumes that at any given moment he receives a constant income that can either be forced to consume or part of it (s (t)) and consume one of the following periods. The waiver of immediate consumption is remunerated by interest, at an interest rate that is worth its wealth. Wealth increases with both the size of the savings at a given time and the rate of the previous level. For simplicity, let's assume that an economic entity is born and dying with zero wealth. The classical task of the consumer is to maximize the benefit, in the case of dynamic analysis, of the total benefit during life.

If we summarize the input, then consumption c (t) is a control variable, and wealth (savings, deferred consumption) w (t) is the status quo. The set of limitations consists of the equations

3) For simplicity, the target function can be conceived as a sum of all benefits (using consumer theory) at individual time points t from 0 to

the suave distribution of retirement and savings, and the law of motion is determined by the dynamics of wealth. The whole problem can be written as follows:

max T ¬maximized function

u (c (t)) dt

{c (t)} ò0

· The power of movement

with respect to: w (t) = s (t) + r × w (t)

y = c (t) + s (t) ¬the number of constraints

w (0) = 0, w (T) = 0 ¬ start / end condition

3. A c h a r d a n d a n i n a n d

The derivation of the maxima principle can be used similarly to the static Lagrangean CP optimization. For simplicity, it is advisable to negate the set of constraints f, resp. we can suppose that we are able to substitute all the constraints for the motion by the equation g. It is also appropriate to limit the dimensionality of the problem and to predict m = n = 1. For the simplicity of writing, the dynamic path can be varied, y, that is, {y (t)} t∈ [0, T], write it as {y (t)}.

Lagrangean corresponding to the overall problem can be written as:

T ÷ é · ü

L = ò ù

(t), c (t), t) + l (t) g (s (t), c (t)

0 ê ú

û û

þ

Similarly to static optimization, we also look for saddle points here. Now, however, these are points in function, ie the minimum with respect to {l (t)} and the minimum in relation to {c (t)}. The first minimum is obtained by examining the deviation {l (t)} to {l (t) + Dl (t)} and using the saddle point condition: DL = 0. The second minimum is obtained through the perte partition of the last member of Lagrangean.

By examining the deviation D1 (t) we derive an implied effect on Lagrangean:

T é é

(T), c (t), t) - s (t) dt

0 ê ú

ë û

By using the necessary condition for the extreme DL = 0 we deduce from the previous expression,

.

(t) = g (s (t), c (t), t).

Now define Hamiltonian as H (s, c, l, t) = u (s, c, t) + l (t) g (s, c, t). Lagrangean

Hamiltonian can be written as:

L = Té

(t), t (t), l (t), t) + l (t) s (t) dt -

at

ë û

If we take into account any deviation of the path of the control variable {c (t)}, which by law moves to deviate the state variable's trajectory {s (t)}, the implicit change of Lagrangian will be as follows:

DL = òT ¶H (t)

0 ¶c (t)

é¶H (t)

Dc (t) +

ës (t)

Ù

+ l (t) Ds (t)

at

at

For the saddle point, it must pay DL = 0, ie:

H (t) H (t) ·

= 0 and = - 1 (t)

(T)

¶C (t)

Maximum principle is the optimal solution to the overall problem, ie the triplet

{t * (t)}, {l * (t)}}, the optimal trajectories for the status parameter {s * (t)}, multipliers {l * (t)} which for Hamiltonian defined as H (s, c, l, t) = u (s, c, t) + l (t) perform the following set of 4 conditions at any given time t:

H (s *, c *, 1 *, t)

¶C

H (s *, c *, 1 *, t)

¶with

H (s *, c *, 1 *, t)

¶L

= 0

.

= - l * (t)

.

= s * (t)

boundary conditions s (0) = s0, s (T) = sT

The advantage of the maxima principle lies in dividing the dynamic problem into a section of simpler static optimization problems, which, moreover, look very similar. By maximizing one Hamiltonian for general time t, you will get to solve a dynamic problem. The four conditions above are necessary, but they can be sufficient.

A rigorous description of the method would require mathematical proof that the step is too tangent and that it leads to a maximum. However, this is not the intention of this - the contribution and the interested can find the relevant evidence in the literature (see, for example, Leonard-Long, 1992).

4. C o m e n c a n i n i n a n d

A number of economic problems are based on slightly different assumptions than those on which previous conclusions were derived. The most commonly mentioned conception of the time value of a particular farm (such as utility or money) that takes into account the time in which we benefit from the state (for example, a benefit of 100 Kè today is higher than the benefit of the same amount of living ra). Money today can be borrowed at the interest rate for the day (before risk-free) and tomorrow we will have both interest and interest. For comparability in individual time periods, therefore, benefits (future income, etc.) are discounted. Another deviation from the above-mentioned assumptions are applications that assume an infinite horizon, ie T ¥. Both modifications - discounting to the current value and the infinite horizon - and their consequences for the optimization task by the extended principle of maxima will now be described in more detail.

4. 1 Discounting

In many economic applications, practically in all time workers, the continuous profitable (target, payroll, utility) function is discounted by the discontinuous factor e-t where the discrete rate r> 0. If we modify a task in the form of a discounted control problem (DCP) we get:

T

max (e (t), c (t)) dt

{c (t)} 0

.

with respect to: s (t) = g (s (t), c (t), t)

f (s (t), c (t), t) ³ 0

s (0) = s0, s (T) = sT,

where the only deviation compared to CP is in the different target function V (·). Instead of Hamiltonian H, it is possible to define Hamiltonian current values of Hc (s, c, l) as:

(T), c (t)), where m (t) = e (t) (t), respectively. l (t) = e-t m (t).

For the principle of maximum discounting, then, the optimal solution of the discounted control problem is the triplet {{s * (t)}, {c * (t)}, {m * (t)}} , c, m, t) = u (s (t), c (t)) + m (t) g (s (t), c (t)

H c (s *, c *, m *, t)

¶C

H c (s *, c *, m *, t)

¶with

H c (s *, c *, m *, t)

¶M

= 0

.

= rm (t) - m * (t)

.

= s * (t)

boundary conditions s (0) = s0, s (T) = sT

4. 2 Endless horizon

In this case, the change occurs only in the boundary condition, which will be with (T) ³ 0. There is, of course, another variant that consists of changing the upper limit of the maximized integral. Integral would not have the upper boundary at T, but in non-neene. It is possible to prove that the condition with (T) ³ 0 can be transformed into eva (t) × s (T) = 0, which is nothing more than a known condition of transversality. If T ¥, the transversality condition can be written as a limit:

lim l (t) s (t) = 0

t ® ¥

Thus, when applying the original boundary condition with (T) = sT, write l (t) s (t) = 0.

t ¥

The condition of transversality also has an economic interpretation. If we are approaching the end of the planning horizon, the optimal solution requires that either (T) be equal to zero (ie nothing is lost if it expresses the benefit (T)) or if s (T) is positive, so that the quoted price of the state to m (T) is zero (and again nothing has been shed).

5. C ompa tio n c o c o c tio n a c o c tio n a n c o c tio n s

optimalization

a) Investment theory

An exemplary example of using the principle of maxima principle is the application of optimal management to investment theory by R. Dorfman (1969). Simplified, the problem can be characterized as follows: a neoclassical firm is not required to decide how much it invests in the time interval [0, T] to maximize its profit, which is given by the classical gain function P (k (t) x (t)), where for currency

k is the stock of capital and x is the investment expenditure.4) Standard economic assumptions are made on the profitable function: the first derivative by capital is positive and the first derivative by investment is negative. Changing capital stock (motion law) is determined by the function g (k (t), x (t)), eg. g (·) = x (t) -dk (t), where d is the degree of devaluation of the capillary. The boundary conditions have the form: k (0) = k0 and k (T) ³. The company's problem can then be written as follows:

max ∫T P (k (t), x (t)) dt

{x (t)} 0

.

with respect to k (t) = g (k (t), x (t))

k (0) = k0, k (T) ³ 0

Using the procedure outlined in the previous part, we obtain:

Hamiltonian: H (k (t), x (t), t) = P (k (t), x (t)

and consequently the conditions of the principle of maximum:

(X) ¶p (×) ¶g (×) ¶P (×) ¶g (×)

= 0 Þ + l (t) = 0 Þ = - l (t)

¶X (×)

Ũx (×) Ũx (×) ¶x (×) ¶x (×)

¶H (×) · ¶P (×) ¶g (×)

= - l (t) Þ + l (t)

¶K (×) ¶k (×)

¶K (×)

.

= - l (t)

.

Law of motion: k (t) = g (k (t), x (t)). The initial and final condition: k (0) = k0, l (T) .K (T) = 0.

If we know the functional shape of the gain function P (k (t), x (t)) and the law of motion g (k (t), x (t)), we can solve the equation of these equations and obtain the optimal trajectory of investment costs in time to maximize profit in a given period. In the first step, we try to eliminate the multiplier l (t) from the system, which can be achieved by generating the first constraint of time t. The municipality complains that if we can simplify the system of equations in two differential equations, that the problem will be analytically solvable. However, this is not the rule, and more complicated systems will be solvable only numerically using a computer.

(b) Consumption during the life cycle

Frequent cases of optimal control theory are growth models that describe the economic agent's consumption trajectory at a time. Its goal is to maximize the utility function with a given budget constraint and respecting the underlying conditions. This model was a community written above, now we approach its variant for a specified utility function and start and end conditions.

Suppose that the economic agent maximizes the utility function u (c (t)) which is equal to the natural logarithm of consumption at a given time t, that is: u (c (t)) = ln (c (t)) and equipped with the initial supply the wealth w (0) = 0, ie it does not have to start any boats. However, its wealth can be influenced by work, not at any time receiving a constant retirement income, either by switching to the c (·) switch or by converting it to the next period using savings (). The state of wealth w (t) acts as a saving with (t) at individual time moments, so exogenously given

4) Capital stock k (t) is state variable, investment costs x (t) are control variable.

the interest rate r, which we assume is constant throughout the life cycle. For simplicity, let's assume that the entity does not leave a legacy

and all the accumulated wealth he himself consumes during his lifetime (w (T) = 0). The problem that the economic agent solves in the time interval between time 0 to T can be formally written as:

T

max ò e rt ln c (t) dt

{c (t)} 0

.

with respect to: w (t) = s (t) + rw (t)

y = c (t) + s (t)

w (0) = 0, w (T) = 0

The economic agent discovers his utility function in the (c) discounting factor e-ţa maximizes it throughout his life cycle with respect to the law of motion which determines the dynamics of growth / decline of wealth, to the budget constraint y = c (t) + s (t) and with respect to the initial and ultimate limitation of wealth.

In most cases, it seems appropriate to reduce the number of equations of constraints by mutual substitution. In this task, it is possible to express the budget constraint as s (t) = y - c (t) and substitute it for the law of motion. We will get intertemporal

.

the budget constraint in the form: w (t) + c (t) = y + rw (t).

For clarity only, we add that the state variable in the model so specified is the wealth w (t) and the control variable consumption c (t). Hamiltonian current

hour then you are then out:

Hc = ln c (t) + ? (t) [rw (t) + y - c (t)].

By applying the maximum principle, we obtain:

H c 1 = m (t)

= 0 Þ

Cc (t)

H c · ·

= rm (t) - m (t) Þ m (t) (r - r) = - m (t)

¶W

H c · ·

= w (t) = w (t) = rw (t) + y - c (t)

¶M

By combining the first two equations, we obtain a differential equation of the first order

Its solution is the optimal trajectory of consumption: c (t) = ce (r-r) t, where c is constant. We know, however, that in the initial period t = 0 is c (0) = c, and therefore it is no more than the initial value of consumption at time 0.

If the value c is of interest (the following interpretation is no longer related to the principle of maxima, or • the optimal trajectory is already detected, but it concerns the solution of differential equations) then it is necessary to involve the motion law also in the solution to more precisely substitute the optimal trajectory of consumption :

.

w (t) = y + rw (t) - c (0) e (r-) t

By multiplying the two sides of the previous e-rt equation we obtain:

· -Rt

¶W (t) e

w (t) e-rt - rw (t) e-rt = ye-rt-c (0) e-πτt where the left side is equal. Integrová-

No

we get: w (t) e-rt w ye-rt + c (0) e rt

= -.

r

r

The pair of integration constants w, c (0) can be used by using the start and end-

(0) = w (T) = 0. For t = 0, w y c (0)

= - and fit this one

r r

y c (0)

Reverse relationship, get: w (t) e-rt = (1-e-rt) - (1- e rt). For t = T leads to a series -

yr (1-e-rT) rr

c * (0) =

.

r

(I-erT)

The optimum trajectory of consumption with the knowledge of initial optimal consumption can be written as:

* yr (1-e-rT) (r-r) t

c (t) = e

r (1-tR)

We will arrive at a totally identical result by using Hamiltonian instead of Hamiltonian at the present value. For illustration, let us mention this solution path. Hamilto-niñan has the form: H = e (t) + t (t) (y - c (t) + rw (t)). By using the maximum principle, we get a triple of conditions:

HH

¶C

HH

¶W

HH

¶L

= 0 Þ e rt 1 = l (t) c (t)

. .

= - l (t) = rl (t) = - l (t)

. .

= w (t) = w (t) = rw (t) + y - c (t)

By solving the second of these equations we determine (t) as (t) = le-rt

By substituting (t) into the first equation we derive an optimal consumption at time t:

* e rt 1 (r-r) t (r-r) t 1

c (t) = -rt = l e = ce where c =

l

le

By comparing this equation with the previous result obtained by using Hamiltonian current values, we will verify that the optimal consumption trajectory is identical in both modes, namely c (t) = ce (r- r) t.

6. Z á v ě r

This article describes one of the possible ways to solve dynamic optimization tasks - the principle of maximum principle. Dynamic optimization tasks are one of the non-component parts of contemporary macroeconomics that seeks to respect micro-economic fundamentals (consumer theory and company theory) while taking into account the time factor. Understanding the principle of maximal principle opens the reader's door to a large part of the most modern economic literature, theory of growth and optimization models of monetary economics. The whole range of eco-nomic research works with dynamic optimization, the authors mostly specify the model and then present its results. The actual derivation of the results is very often left out and the reader is expected to have a sufficient control over the method used

The maximum principle method deduces from Hamiltonian a set of necessary conditions, which along with the initial and final condition of the state variable allow the optimal trajectory to derive the control variable. Following this optimal trajectory, the control variable will maximize the target function within a given timeframe. However, we do not have to follow the optimal trajectory of the variable variable - the system develops according to the internal dynamics given by the law of motion, which is one of the limiting conditions of the solution. Non-observance of the optimal trajectory of the control variable, of course, to the lower end of the function, which is the property of the very nature of optimization - other than the optimal solution must lead to a worse result.

The advantage of the principle of maximum principle is the possibility of its almost mechanical application to a wide range of dynamic (economic) problems in the continuous time. However, the principle of maximization can be successful in other ways (for example, physics). In addition to maximization tasks, the method can be used after trivial editing of the target function and minimization tasks. The limitation of the method lies in the fact that it is an approach based on working with continuous variables - not all economic tasks are easy or not appropriate at the same time. For tasks that have a discrete chakractor, most empirical studies, it is necessary to use the Bellman equation and dynamic programming.

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Abstract:

The paper presents the optimal control theory and the maximum principle technique