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Optimal investment with lumpy costs

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Abstract

In this paper we analyze a continuous-time model of investment with uncertainty, irreversibility and a broad class of lumpy adjustment costs. We show that the two components of the optimal investment strategy, the investment trigger and the investment increment, can be found sequentially, and that the optimal investment increment maximizes a closed-form function. Solving the model numerically, we find that adding a relatively small amount of variable adjustment costs often leads firms to invest in much smaller increments. We derive a measure of user cost that incorporates lumpy investment, and use it to show that as firms invest in bigger increments, the investment trigger increases as well.

JEL classification: E22

Keywords: Investment; Lumpiness; Adjustment costs; User costs

1. Motivation and background

Many studies suggest that investment is lumpy. Peck (1974) concludes that turbogenerator investment is best described by a model with lumpiness. Working with the longitudinal research database, Doms and Dunne (1994) find that during a

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17-year period, plants made over half of their investment within their three busiest years. Using an extended version of the same data set, Cooper et al. (1999) find that the "top five investment years at each plant account for more than 50 percent of cumulative aggregate investment." Goolsbee and Gross (1997) study data from the Civil Aviation Board. They find that "on average, about 60 percent of the total acquisitions of an aircraft type by a given airline take place in the largest two-year investment episode." As Pindyck (1988) remarks, "the assumption that firms can continuously and incrementally add capital, though common in economic models, is extreme. Most investments are lumpy, and sometimes quite so."[1]

Several authors study investment cost structures that generate lumpiness; these include Caballero and Leahy (1996), Eberly (1997), Abel and Eberly (1998) and Caballero and Engel (1994, 1999).[2] Their focus, however, is on the case where adjustment costs have a single fixed component.[3] This represents a sharp break with much of the investment literature, where variable adjustment costs, most notably quadratic ones, play a large role. Although Abel and Eberly (1994) combine both sorts of costs in their generalized model of non-lumpy investment, in the context of lumpiness their 'augmented adjustment cost function' has received little attention.[4] Our contribution is to extend the analysis of lumpy investment in just this direction.

In particular, we consider a firm choosing the optimal pattern of investment under irreversibility and demand uncertainty. When undertaken, investment causes discrete jumps in the firm's capital stock and its total costs. The firm's decision rule thus consists of two quantities: the investment trigger, the level to which relative demand – equivalently, the marginal product of capital – must rise before the firm invests; and the investment amount, expressed as a fraction of capital at the trigger. These quantities relate to the firm's augmented adjustment costs in a surprisingly simple way. Our results are fairly general; the main restriction is that adjustment costs be linearly homogeneous in existing capital and investment.

The remainder of the paper is organized as follows. In Section 2, we describe the model and the basic features of the firm's decision rules. In Section 3, we solve the model. We show that the two elements of the firm's decision rule – the investment trigger and the investment amount – can be found sequentially, and that the optimal

[1] Cooper and Haltiwanger (2000) provide additional references.

[2] Extensive reviews of the investment literature can be found in Dixit and Pindyck (1994) and Abel and Eberly (1994). A closely related body of work is the analysis of consumer durables pioneered by Grossman and Laroque (1990), who derive an (S, s) rule for durables purchases. A recent example is Bertola et al. (2001).

[3] Caballero and Leahy (1996) note that their qualitative results hold when a convex adjustment charge is added to the fixed cost. Caballero and Engel (1994) consider a production function with two types of capital, with one type adjusting smoothly, and the other type adjusting lumpily.

[4] Abel and Eberly (1994) consider *flow* fixed costs, costs that are independent of the size of the investment flow, but proportional to its duration. Because flow fixed costs vanish as the duration of the investment flow goes to zero, they generate periods of inaction, but do not generate lumpy investment. We work with discrete or *stock* fixed costs, which are independent of both the size and the duration of the investment flow. The distinction between stock and flow fixed costs disappears when one moves from a continuous- to a discrete-time framework. Notable discrete-time studies of lumpy investment include Cooper et al. (1999), Cooper and Haltiwanger (2000) and Khan and Thomas (2003).

investment amount maximizes a closed-form function. Because our 'semi-analytical' solution is quite easy to solve numerically, it should be especially useful to those interested in simulation and/or structural estimation.

In Section 4, we use numerical exercises to analyze a variety of cost structures. We find that the interactions between fixed and variable adjustment costs are often quite important. For example, we find that variable adjustment costs can significantly reduce the degree of lumpiness that fixed adjustment costs induce – adding variable adjustment costs often leads firms to invest in much smaller increments. In Section 5, we use our semi-analytical solution to derive an implicit version of the user cost of capital. Analyzing this implicit user cost, we show that as investment becomes more lumpy, the investment trigger increases. We conclude in Section 6.

2. The model and its basic properties

We begin by constructing the firm's profit function, using a specification common in the investment literature. Consider a firm that uses capital and labor to produce output, S_t, with the Cobb–Douglas production function

$$S_t = L_t^{1-\beta} K_t^{\beta},$$

where $0<\beta<1$. Output is non-storable. The firm faces an isoelastic demand curve

$$S_t = X_t P_t^{-\varepsilon}$$

with $\varepsilon>1$. P_t is the firm's output price. X_t is an exogenous demand shifter following a geometric Brownian motion with drift:

$$\frac{\mathrm{d}X_t}{X_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}z, \tag{1}$$

where $\mathrm{d}z$ is an increment to a standard Wiener process.

The firm pays the constant wage w. Using this fact, it is straightforward to show that profits at time t are

$$\Pi(K_t, X_t) = \frac{h}{1-\gamma} K_t \left(\frac{X_t}{K_t}\right)^{\gamma}, \tag{2}$$

where

$$\gamma = \frac{1}{1+\beta(\varepsilon-1)} \quad \text{and} \quad h = (1-\gamma)(\gamma\varepsilon-1)^{\gamma\varepsilon-1}(\gamma\varepsilon)^{-\gamma\varepsilon} w^{1-\gamma\varepsilon}.$$

Since $\varepsilon>1$, $0<\gamma<1$ and $h>0$.

The law of motion for capital is

$$\mathrm{d}K_t = I_t - \delta K_t\,\mathrm{d}t. \tag{3}$$

Because investment, I_t, is defined as a total increment, rather than a flow rate (per unit of time), this equation embeds two cases. When there is no investment, capital depreciates at the constant rate δ:

$$\dot{K}_t = -\delta K_t.$$

But at the moment investment occurs, capital follows

$$K_t^+ = K_t^- + I_t,$$

where K_t^- and K_t^+ denote the capital stock immediately before and immediately after time t.

We assume firms cannot sell their capital, so that investment is always non-negative:

$$I_t \geqslant 0.$$

Although there are many reasons why investment might be irreversible – extreme specificity and "lemons" problems in resale markets are two – we have adopted this restriction in large part to simplify the analysis. One consequence of this assumption is that there is an option value to delaying investment.[5]

Normalizing the price of capital to 1, the costs of investing are given by

$$c(K_t, I_t) = v_t[I_t + G(K_t, I_t)], \tag{4}$$

where v_t is a 0–1 indicator variable equal to 1 only when investment is positive. v_t ensures that if there are fixed costs to investing, i.e., $G(K_t, 0) > 0$, the costs will be incurred only when investment is positive.[6] We assume that the adjustment cost $G(K_t, I_t)$ is linearly homogeneous in K and I. This implies that any fixed costs to investing are proportional to the existing capital stock.[7] Such a fixed cost structure could occur if, for example, upgrading a machine requires retooling the entire factory – as the factory grows, so do the costs of retooling it. Beyond the assumption of homogeneity, which appears throughout the investment literature, we impose few restrictions on G. Including fixed costs immediately implies that G need not be convex, and we even consider cases where G is decreasing in I_t. In the numerical treatment, G is usually specialized as $[a(I/K - b)^2/2 + c]K$, which generalizes the quadratic specification used in many recent empirical studies.

The firm's objective is to maximize the expected present value of its cash flow – profits less investment costs – discounted at rate r. Let C_t denote the undiscounted cumulation of all investment expenses through time t. The value of the firm is then given by

$$V(K_t, X_t) = \max_{\{I_s \geqslant 0\}} \mathrm{E}_t\left\{\int_t^\infty \mathrm{e}^{-r(s-t)}[\Pi(K_s, X_s)\,\mathrm{d}s - \mathrm{d}C_s]\right\}, \tag{5}$$

subject to Eq. (3). The final term in the expectation should be interpreted as a Stieltjes integral. When there are fixed costs to investment, the firm will find it

[5] Dixit and Pindyck (1994) discuss this point in some detail.

[6] Note that $G(K_t, I_t)$ gives stock adjustment costs, rather than flow costs. This means, for example, that the average cost of a new gas pipeline can vary with its diameter and length, but not with the speed with which it is built.

[7] This property implies that fixed costs do not become irrelevant as the firm grows. Abel and Eberly (1998) and Cooper et al. (1999) achieve the same result by making the fixed cost proportional to the demand shifter X_t.

optimal to invest at distinct points of time s_i, $i = 1, 2, 3, \ldots$, and the final term can be written as

$$\mathrm{E}_t\left\{\int_t^{\infty} \mathrm{e}^{-r(s-t)}\,\mathrm{d}C_s\right\} = \mathrm{E}_t\left\{\sum_{i=1}^{\infty} \mathrm{e}^{-r(s_i-t)}[I_{s_i} + G(K_{s_i}, I_{s_i})]\right\}.$$

To bound the value of the firm, we assume that $r > \mu$, so that the firm's discount rate exceeds the growth rate of demand.

Let $\theta_t \equiv I_t/K_t$ denote the firm's relative investment. Because adjustment costs are homogeneous, we can rewrite Eq. (4) as

$$c(K_t, I_t) = v_t[\theta_t + g(\theta_t)]K_t.$$

Similarly, the capital accumulation equation can be rewritten as

$$K_t^+ = (1 + \theta_t)K_t^-.$$

Moreover, since X_t evolves continuously, it follows from Eq. (5) that when an investment occurs,

$$V(K_t^-, X_t) = V((1 + \theta_t)K_t^-, X_t) - [\theta_t + g(\theta_t)]K_t^-.$$

This identity is often referred to as the 'value matching condition.'

Now suppose, without loss of generality, that X is fixed, and let $V(K_t, X; K_u, \theta)$ denote the value of a firm that adopts the strategy of investing θK_u whenever capital depreciates to the level K_u. Even if K_u and θ are suboptimal, a variant of Eq. (5) will hold, so that the value matching condition still applies. Then when $K_t = K_u$, the value matching condition can be written as

$$V(K_u, X; K_u, \theta) = V((1 + \theta)K_u, X; K_u, \theta) - [\theta + g(\theta)]K_u. \tag{6}$$

As Dumas (1991) and Dixit (1991) note, and we confirm below, for the optimal policies K^* and θ^*,

$$\frac{\partial V(K_t, X; K^*, \theta^*)}{\partial K_u} = \frac{\partial V(K_t, X; K^*, \theta^*)}{\partial \theta} = 0,$$

for *all* values of K_t and X. Intuitively, adopting a better investment rule increases the firm's discounted cash flow at every value of capital and demand. Following Dumas (1991), one can then totally differentiate both sides of Eq. (6) with respect to θ and K_u to get

$$\frac{\partial V((1 + \theta^*)K^*, X; K^*, \theta^*)}{\partial K_t} = 1 + g'(\theta^*),$$

$$\frac{\partial V(K^*, X; K^*, \theta^*)}{\partial K_t} = \frac{\partial V((1 + \theta^*)K^*, X; K^*, \theta^*)}{\partial K_t}(1 + \theta^*) - [\theta^* + g(\theta^*)].$$

These are variants of the familiar 'smooth pasting conditions.'

To better interpret the first-order conditions, let $y_t \equiv X_t/K_t$ denote the relative demand for the firm's output. Since the firm's profits, investment costs and constraints are all linearly homogeneous in K_t and X_t, so is the value function $V(K_t, X_t)$. It follows that the marginal value of capital, q, is homogeneous of degree zero in K_t and X_t:

$$q(K_t, X_t) \equiv \frac{\partial V(K_t, X_t)}{\partial K_t} = q(y_t).$$

Inserting this result into the smooth pasting conditions shows that the optimal investment strategy is a trigger strategy in y_t; whenever relative demand y_t rises to the threshold u, the firm will invest θK^-, driving relative demand down to $v = u/(1+\theta)$.[8] In particular, the smooth pasting conditions can be written as

$$q\left(\frac{u^*}{1+\theta^*}\right) = 1 + g'(\theta^*), \tag{7}$$

$$q(u^*) = q\left(\frac{u^*}{1+\theta^*}\right) + \theta^*[g'(\theta^*) - g(\theta^*)/\theta^*]. \tag{8}$$

The intuition behind Eq. (7) is the standard one: once the firm has committed to invest, it invests until the marginal value of capital equals its marginal cost. The first term on the right-hand side of Eq. (8), $q(u^*/(1+\theta^*))$, arises because q is forward looking; the marginal value of capital immediately before an investment depends directly on its marginal value immediately after investment. The latter term, $\theta^*[g'(\theta^*) - g(\theta^*)/\theta^*]$, captures the way in which changing the capital stock before investing alters *non*-marginal adjustment costs, by comparing marginal and average adjustment costs. If, for example, investment involves only a fixed cost ($g(\theta) = c$) the latter term will be negative, so that the marginal value of capital increases when an investment is made. Immediately prior to an investment, each additional unit of capital entails c units of upcoming fixed adjustment costs; evidently, the marginal value of capital increases once the investment has been made.

This discontinuity in the marginal value of capital is, not surprisingly, a function of our particular specification. If investment costs depended only on investment, i.e., if $G(K_t, I_t) = G(I_t) = G(\theta_t K_t)$, the smooth pasting conditions would reduce to $q(K^*, X) = 1 + G'(I^*) = q(K^* + I^*, X)$, the standard condition. In either case, marginal q is of little help in predicting investment. As Caballero and Leahy (1996) point out, when investment is lumpy, $q(y)$ is not monotone in y, and therefore cannot be a sufficient statistic for investment.[9]

3. The model's solution

The model's solution involves three steps. The first step consists of finding the law of motion for the marginal value of capital, $q(y)$, when no investment is taken. In the second step, the investment trigger u is found as a function of the investment fraction θ. In the final step the optimal value of θ is found.

[8]We restrict our analysis to the interval $[0, u]$, which forms an absorbing set for y. The optimal investment strategy for $y > u$ is an interesting but involved topic that we do not pursue here.

[9]In the case where $q(u^*) = q(v^*)$ but $u^* \neq v^*$, Caballero and Leahy's result is immediate. Le (2002) shows that their result also holds for our specification, where $q(u^*)$ need not equal $q(v^*)$.

3.1. The value of capital when there is no investment

In those time intervals where no investment occurs, the value of capital evolves in the typical fashion. Our treatment most closely follows Abel and Eberly (1996). In the absence of investment, the following Bellman equation will hold:

$$rV(K_t, X_t) = \Pi(K_t, X_t) + \mathrm{E}_t\left\{\frac{\mathrm{d}V(K_t, X_t)}{\mathrm{d}t}\right\}.$$

Since X_t follows a geometric Brownian motion, one can apply Ito's Lemma to show

$$\begin{aligned} rV(K, X) = {} & \frac{h}{1-\gamma} X^{\gamma} K^{1-\gamma} - \delta K V_K(K, X) + \mu X V_X(K, X) \\ & + \frac{1}{2}\sigma^2 X^2 V_{XX}(K, X). \end{aligned} \tag{9}$$

Differentiating both sides with respect to K yields

$$\begin{aligned} rV_K(K, X) = {} & h\left(\frac{X}{K}\right)^{\gamma} - \delta V_K(K, X) - \delta K V_{KK}(K, X) \\ & + \mu X V_{XK}(K, X) + \frac{1}{2}\sigma^2 X^2 V_{XXK}(K, X). \end{aligned} \tag{10}$$

Recalling Eqs. (1) and (3), one can show that in the absence of investment, relative demand, y_t, follows a geometric Brownian motion:

$$\frac{\mathrm{d}y_t}{y_t} = \lambda\,\mathrm{d}t + \sigma\,\mathrm{d}z,$$
$$\lambda = \mu + \delta.$$

Eq. (10) is thus identical to

$$(r+\delta)q(y) = hy^{\gamma} + \lambda y q'(y) + \tfrac{1}{2}\sigma^2 y^2 q''(y). \tag{11}$$

As Dixit and Pindyck (1994, Chapters 5 and 6) show, the solution to this differential equation is

$$q(y) = Ay^{\gamma} + By^{\alpha_P} + Cy^{\alpha_N}, \tag{12}$$

where A, B and C are constants to be determined, and α_P and α_N are the roots of the following quadratic equation:

$$\rho(z) = -\tfrac{1}{2}\sigma^2 z^2 - (\lambda - \tfrac{1}{2}\sigma^2)z + r + \delta = 0.$$

It can be confirmed that $A = h/\rho(\gamma) > 0$, and that

$$\alpha_N < 0 < \gamma < 1 < \alpha_P. \tag{13}$$

This leaves the task of finding the constants B and C. Note that unless

$$C = 0,$$

the marginal value of capital will explode as relative demand, y, approaches zero. (With irreversible investment, such an outcome is possible.) We turn to finding B.

3.2. Finding the investment trigger u

Suppose that the firm follows the possibly suboptimal strategy given by the trigger u and investment percentage θ. Given this strategy, the value matching condition given by Eq. (6) provides a boundary condition that expresses the coefficient B in Eq. (12) as a function of u and θ. This relationship in turn allows one to find the optimal investment trigger as a function of the investment fraction θ, and from there find the optimal value of θ.

Let $Q(y) \equiv V(K, X)/K$ denote Tobin's (1969) q, the average value of capital. Recall that the value matching condition restricts the *level* of the value function. What we are seeking, however, is the coefficient B in the expression for $q(y)$, the value function's *slope*. To convert the value matching condition from a restriction on levels to a restriction on slopes, we need the following result:

Lemma 1. *Tobin's q can be expressed in terms of* $q(y)$ *and* $q'(y)$:

$$Q(y) = \frac{1}{r-\mu}\left[\frac{h}{1-\gamma}\, y^{\gamma} - \lambda q(y) - \frac{1}{2}\,\sigma^2 y q'(y)\right]. \tag{14}$$

The proof, which appears in Appendix A, is a straightforward application of linear homogeneity.

We can now solve for B. Let $Q(y; u, \theta)$ denote the average value of a firm that adopts the strategy (u, θ). We will assume throughout that the investment fraction is strictly positive, with $\theta = 0$ as a limiting case. Let $\phi(\theta) \equiv \theta + g(\theta)$ denote investment costs per unit of existing capital. Using this notation, the value matching condition given by Eq. (6) can be written as

$$Q(u; u, \theta) = (1+\theta)Q\left(\frac{u}{1+\theta}; u, \theta\right) - \phi(\theta). \tag{15}$$

Applying Lemma 1, with $q(y; u, \theta) = Ay^{\gamma} + By^{\alpha_P}$, yields[10]

$$Q(y; u, \theta) = \frac{1}{r-\mu}[Hy^{\gamma} - BEy^{\alpha_P}],$$

$$(1+\theta)Q\left(\frac{y}{1+\theta}; u, \theta\right) = \frac{1}{r-\mu}\left[\frac{H}{(1+\theta)^{\gamma-1}}\, y^{\gamma} - \frac{BEy^{\alpha_P}}{(1+\theta)^{\alpha_P-1}}\right], \tag{16}$$

where $D \equiv \lambda + \frac{1}{2}\gamma\sigma^2$, $E \equiv \lambda + \frac{1}{2}\alpha_P\sigma^2$ and $H \equiv h/(1-\gamma) - AD$. Inserting these results into Eq. (15) (so that $y = u$), the value matching condition becomes[11]

$$u^{\gamma}[(1+\theta)^{1-\gamma} - 1]H - BEu^{\alpha_P}[(1+\theta)^{1-\alpha_P} - 1] = (r-\mu)\phi(\theta). \tag{17}$$

[10] Although Lemma 1 technically applies to the value function for the optimal strategy, $Q(y) = Q(y; u^*, \theta^*)$, the homogeneity properties that make it hold also apply to $Q(y; u, \theta)$. Similarly, the law of motion for $q(y)$ given by Eq. (12) also holds for $q(y; u, \theta)$.

[11] Even if $V(\cdot)$ is not differentiable at the investment trigger u, one can apply Lemma 1 by taking limits as y approaches u from below.

We show in Appendix B that

$$H = \frac{h}{1-\gamma} - AD = \frac{2h(\mu - r)}{(\gamma - \alpha_P)(\gamma - \alpha_N)(1-\gamma)\sigma^2}, \tag{18}$$

so that Eq. (17) implies

$$B = \frac{\mu - r}{E[(1+\theta)^{1-\alpha_P} - 1]}\left[\frac{2h[(1+\theta)^{1-\gamma} - 1]u^{\gamma-\alpha_P}}{(\gamma - \alpha_P)(\gamma - \alpha_N)\sigma^2(1-\gamma)} + \phi(\theta)u^{-\alpha_P}\right]. \tag{19}$$

It follows from Eq. (16) that u and θ affect the value of the firm only to the extent they affect B. This implies that the first-order condition for the optimal investment trigger, $u(\theta)$, is

$$\frac{\partial Q(y; u(\theta), \theta)}{\partial u} = -\frac{1}{r-\mu} Ey^{\alpha_P} \frac{\partial B(u(\theta), \theta)}{\partial u} = 0$$

or

$$\frac{\partial B(u(\theta), \theta)}{\partial u} = 0.$$

In Appendix C, we show that this first-order condition implies

$$u(\theta)^\gamma = \frac{\alpha_P \phi(\theta)(\gamma - \alpha_N)(1-\gamma)\sigma^2}{2h[(1+\theta)^{1-\gamma} - 1]}, \tag{20}$$

which can also be expressed as

$$u(\theta)^\gamma = \left(1 - \frac{\gamma}{\alpha_N}\right)\frac{(1-\gamma)(r+\delta)\phi(\theta)}{h[(1+\theta)^{1-\gamma} - 1]}. \tag{21}$$

In Appendix D, we confirm that Eq. (21) identifies a maximum.

It follows from Eq. (13) that $u(\theta)^\gamma$ and thus $u(\theta)$ are strictly positive. Moreover, combining Eqs. (19) and (20) yields

$$Bu(\theta)^{\alpha_P} = \frac{\mu - r}{E[(1+\theta)^{1-\alpha_P} - 1]}\,\phi(\theta)\left[\frac{\gamma}{\gamma - \alpha_P}\right]. \tag{22}$$

This expression can be shown to be negative, so that B is negative.

Combining Eqs. (12) and (22) (and recalling the definition of $\phi(\theta)$) reveals that for any investment fraction θ, the marginal value of capital is

$$q(y;\theta) = Ay^\gamma + \frac{\gamma(r-\mu)[\theta + g(\theta)]}{(\alpha_P - \gamma)E[(1+\theta)^{1-\alpha_P} - 1]} \times \left(\frac{y}{u(\theta)}\right)^{\alpha_P}, \tag{23}$$

while the average value of the firm, $Q(y;\theta)$, as calculated in Appendix E, is

$$Q(y;\theta) = \frac{1}{\alpha_P - \gamma}\left[\frac{2hy^\gamma}{\sigma^2(\gamma - \alpha_N)(1-\gamma)} + \frac{\gamma[\theta + g(\theta)]}{1 - (1+\theta)^{1-\alpha_P}} \times \left(\frac{y}{u(\theta)}\right)^{\alpha_P}\right]. \tag{24}$$

3.3. Finding the investment fraction θ

Having found the marginal and average value of capital for arbitrary values of θ, we turn to finding the optimal investment fraction. Recalling Eq. (13), it follows immediately from Eq. (24) that maximizing $Q(y;\theta)$ boils down to solving

$$\max_{\theta \geqslant 0} \frac{\theta + g(\theta)}{1-(1+\theta)^{1-\alpha_P}} u(\theta)^{-\alpha_P}.$$

But the optimal investment trigger $u(\theta)$ is given by Eq. (21), so that

$$\theta^* = \arg\max_{\theta \geqslant 0} \frac{[(1+\theta)^{1-\gamma}-1]^{\alpha_P/\gamma}}{1-(1+\theta)^{1-\alpha_P}} [\theta + g(\theta)]^{1-\alpha_P/\gamma}. \tag{25}$$

In Appendix F, we verify that the first-order conditions to this maximization problem satisfy the smooth pasting conditions given by Eqs. (7) and (8).

In short, the optimal investment policy can be found by solving Eq. (25) for θ^*, and inserting the optimizing value into Eq. (21) to find the investment trigger $u^* = u(\theta^*)$. In general, the optimal investment fraction θ^* must be found numerically. Fortunately, Eq. (25) is quite easy to solve. In the next section, we use this numerical solution to analyze a variety of cost structures.

4. Numerical analyses

Using the methodology described immediately above, we can find the optimal investment fraction, θ^*, for any well-behaved adjustment cost structure. To facilitate comparison with the existing literature, we examine, separately and then jointly, the two most familiar adjustment cost structures: the pure fixed cost, $g(\theta) = c$; and the quadratic variable cost $g(\theta) = a(\theta - b)^2/2$.

4.1. Calibration

Throughout the numerical analyses, we utilize the following parameter values:

1. $\gamma = 0.431$: This follows from a capital share, β, of 0.33, from Abel and Eberly (1999), and a demand elasticity, ε, of 5, from Cooper and Haltiwanger (2000).
2. $\sigma = 0.05$ and $\mu = 0.03$: We derive demand volatility, σ, and growth, μ, from Abel and Eberly (1999) and from data on U.S. post-tax corporate profits (DRI variable name: NANICP01NCALSA@C111) over the period 1960–2000.
3. $\delta = 0.06$ and $r = 0.04$: These values are fairly standard. (See, for example, Veracierto, 1998.)
4. $\alpha_P = 1.1094$: This is the positive solution to $\rho(z) = 0$, using the values for μ, σ, r and δ listed above.

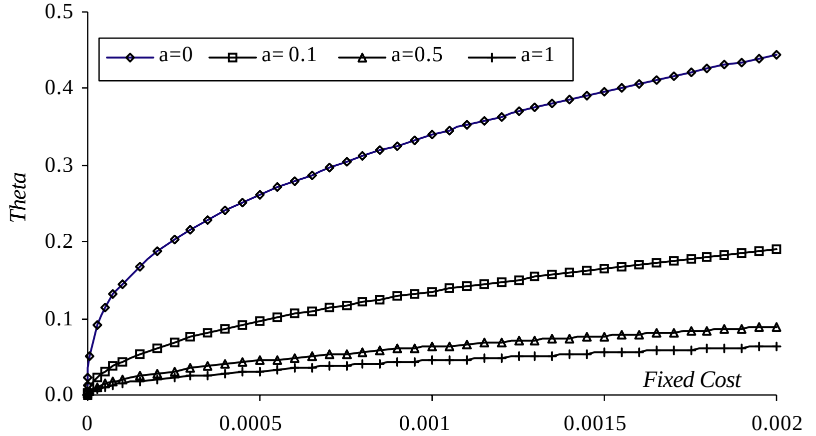

Fig. 1. θ^* as a function of the fixed cost c.

4.2. Pure fixed cost

We begin with the adjustment cost structure that has received the most attention in the study of lumpiness, the pure fixed cost: $g(\theta) = c$. This case appears as the top ($a = 0$) line of Fig. 1, which shows θ^* as a function of the fixed cost c. (We discuss the rest of Fig. 1 further below.) Fig. 1 shows that fixed adjustment costs can have large effects: for example, a fixed cost equal to 0.2 percent of the capital stock generates an investment fraction of over 40 percent.

In Appendix G, we show that one can approximate the first-order condition for θ around 0 to get

$$\theta^* \cong \left(\frac{12}{\gamma\alpha_P} c\right)^{1/3}.$$

Abel and Eberly (1998) derive similar cubic approximations while analyzing their analogs to u and $u/(1+\theta)$.[12] These approximations confirm that θ^* can be quite sensitive to fixed costs. In particular, the derivative of θ^* with respect to c is infinite at $c = 0$.

[12]The approximations differ because Abel and Eberly (1998) assume that the fixed cost c is proportional to the demand shifter X_t, rather than the capital stock. This seemingly minor difference in specification also changes the way in which the model must be solved. In particular, when adjustment costs are given by cX_t, the value matching condition given by Eq. (15) becomes

$$Q(u; u, \theta) = (1+\theta)Q\left(\frac{u}{1+\theta}; u, \theta\right) - [\theta + cu].$$

With this change, θ^* and u^* must be found simultaneously, rather than in sequence.

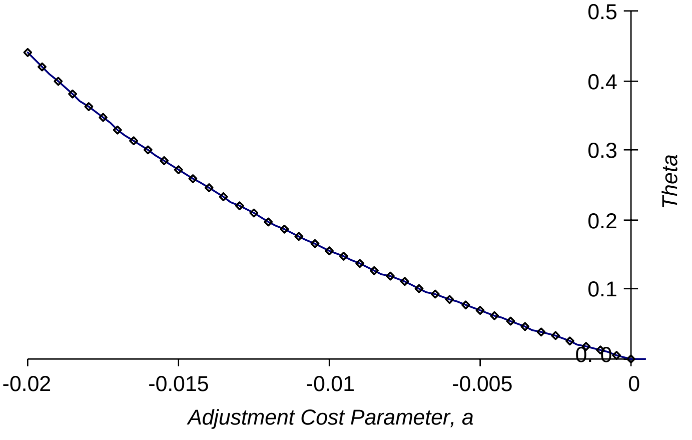

Fig. 2. θ^* as a function of the adjustment cost parameter a.

4.3. Pure variable cost

The familiar quadratic adjustment cost, $g(\theta) = a(\theta - b)^2/2$, appears extensively in the empirical literature.[13] As shown below, the parameter b in many ways acts as a fixed cost. We thus begin our analysis with the purest case, $g(\theta) = a\theta^2/2$, where adjustment costs are completely variable. For any $a \geqslant 0$, the solution is $\theta^* = 0$; if a firm wants to invest the total amount Θ in M increments, the total adjustment cost, $M \times a(\Theta/M)^2/2$, is minimized with $\theta \equiv \Theta/M = 0$. Investment follows what is known as barrier control.[14] Moreover, applying L'Hôpital's rule to Eq. (21) shows that

$$\lim_{\theta \to 0} hu(\theta)^{\gamma} = \left(1 - \frac{\gamma}{\alpha_N}\right)(r + \delta),$$

which is the trigger condition for smooth irreversible investment shown in Abel and Eberly (1996).

Lumpiness re-emerges when a is negative, so that there are increasing returns to investment. This is illustrated in Fig. 2, which shows θ^* for a range of negative values

[13] See, for example, Chirinko (1993), Cummins et al. (1994, 1996), Gilchrist and Himmelberg (1995), Hassett and Hubbard (1996), and Oliner et al. (1996).

[14] An intuitive discussion of barrier control can be found in Dixit and Pindyck (1994, Chapter 11). Recall that $\theta = I/K$, where I measures the total investment increment, rather than the flow rate of investment per unit of time. Under barrier control any such investment increments are infinitesimal, but when they occur their flow rate is infinite. Barrier control is therefore optimal for the convex stock adjustment costs we consider here (as it minimizes our version of θ), even though it would not be optimal for convex flow adjustment costs. A formal treatment of barrier control appears in Harrison and Taksar (1983). Although the control variable in Harrison and Taksar's framework is the cumulation of all investment, while our control variable is the investment increment I, in the limit the two frameworks yield the same investment process; as Harrison et al. (1983) point out, as fixed costs approach zero 'impulse control' reduces to barrier control.

of a. As the analysis of the pure fixed cost already showed, increasing returns do not rule out a bounded solution; the key regularity condition is not that the adjustment cost $g(\theta)$ be convex, but rather that the function maximized in Eq. (25) be well-behaved.[15] Intuitively, if the profit function $\Pi(K, X)$ is sufficiently concave, the firm's problem will have an interior solution even when $g(\theta)$ is non-convex.

4.4. Full quadratic form

The most interesting specification is the full quadratic form, $g(\theta) = a(\theta - b)^2/2 + c$. To analyze this case, we calibrate two additional parameters:

1. $a = 1$: Relating this parameter to the coefficient on Tobin's q in an investment equation, Hassett and Hubbard (1996) suggest that a is between 1 and 2. Estimating a structural model, Cooper and Haltiwanger (2000) find a to be less than 0.1.
2. $-0.25 \leqslant b \leqslant 0.25$: This covers the range of reasonable reported estimates. For example, Cummins et al. (1994) find that $|b|$ is less than 0.25, and is usually less than 0.03. Another common approach is to assume that there are no adjustment costs along a balanced growth path, so that $b = \delta + \mu$: in our case, this would imply that $b = 0.09$.

Fig. 3 shows the values of θ^* generated by this calibration, for different values of the fixed cost c. The bottom ($c = 0$) line of this figure deserves mention. When c is set to 0, $g(\theta)$ reduces to $a(\theta - b)^2/2$. In this case the optimal value of θ has a surprisingly precise and simple approximation: $\theta^* = |b|$. Since $\theta = b$ minimizes the adjustment cost $g(\theta)$, it is not surprising that the approximation holds for $b > 0$. What is surprising is that it holds for $b < 0$. To understand this symmetry further, define the function $h(\theta)$ such that Eq. (25) can be written as

$$\theta^* = \arg\max_{\theta \geqslant 0} (\theta + g(\theta))^{1-\alpha_P/\gamma} h(\theta).$$

When $g(\theta) = a(\theta - b)^2/2$, the first-order condition for this maximization problem is

$$-\frac{h'(\theta^*)}{(1 - \alpha_P/\gamma)h(\theta^*)} = J(\theta^*, b),$$

$$J(\theta, b) \equiv \frac{1 + a(\theta - b)}{\theta + a(\theta - b)^2/2} = \frac{1 + g'(\theta)}{\theta + g(\theta)}.$$

Note that

$$J(b_0, b_0) = J(b_0, -b_0) = \frac{1}{b_0}.$$

It immediately follows that if $\theta^* = b_0$ satisfies the first-order condition when $b = b_0$, it also satisfies the first-order condition when $b = -b_0$.

[15] An important early study of smooth, as opposed to fixed-cost-driven, concave adjustment costs is Manne (1961). Dixit and Pindyck (1994, Chapter 11) provide many other references.

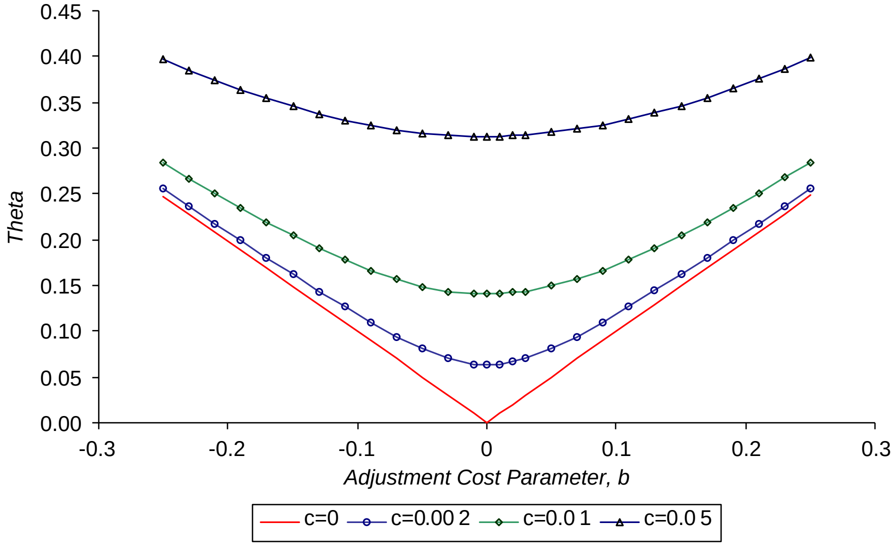

Fig. 3. θ^* as a function of the adjustment cost parameter b.

Note that $g(0) = ab^2/2 + c$, so that in many ways b is a fixed cost like c. It is therefore not surprising that the interactions between the two parameters depend on their relative sizes. It is also useful to recall Fig. 1, which shows that θ^* exhibits diminishing returns in c. This suggests that if c (or b) is large, the incremental effect of b (or c) will be small. For example, the incremental effect of b is the smallest on the top ($c = 0.05$) line of Fig. 3, where the fixed cost c is the largest.

The interactions between the fixed cost c and the parameter a, which determines the relative weight of the quadratic term, are also quite interesting. Fig. 1, which shows the values of θ^* that arise when $g(\theta) = c + a\theta^2/2$, reveals that even a small amount of variable adjustment costs ($a = 0.1$) can significantly dampen the effects of fixed adjustment costs. This is consistent with Cooper and Haltiwanger (2000), who fit a (discrete-time) structural model to their lumpy investment data, and find a to be less than 0.05. A similar set of implications appears in Fig. 4, which shows that when $b = 0$, higher values of a lead to considerably less investment. Fig. 4 also shows that when c is small (0.0 or 0.002) and b is large (0.25), the approximation $\theta^* = |b|$ holds for nearly all values of a; the slope of the adjustment cost function can be as important as its size.

In short, variable adjustment costs can significantly alter a firm's investment strategy. It immediately follows that for any given value of the fixed cost c, the standard practice of ignoring variable costs can lead to extremely inaccurate predictions. Even if c is calibrated or estimated to replicate some feature of the data, a specification that excludes variable adjustment costs is a reduced form, with limited value for policy analysis.

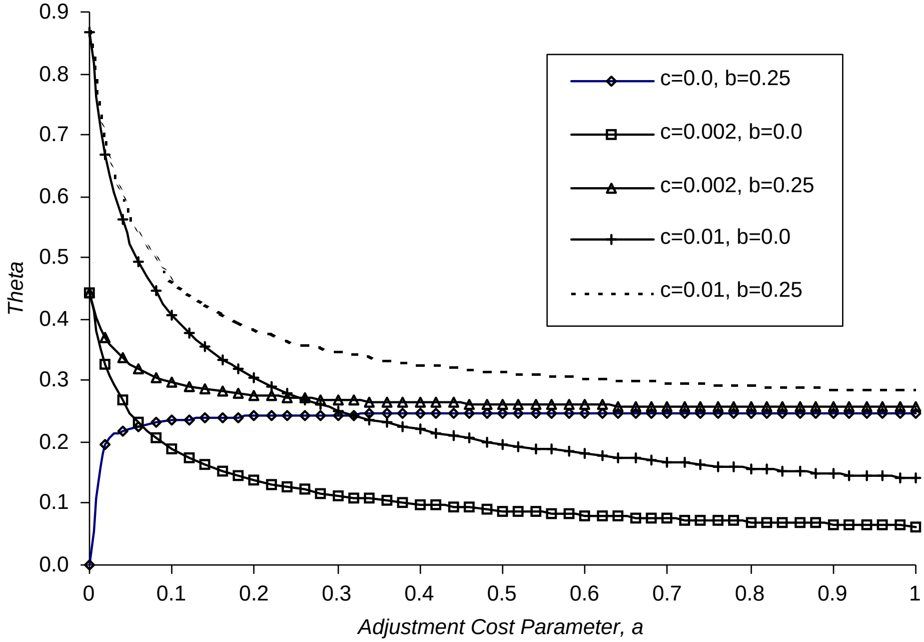

Fig. 4. θ^* as a function of the adjustment cost parameter a.

5. Augmented user cost

As formulated by Jorgenson (1963), the user cost of holding capital, $U(y)$, consists of interest and depreciation costs, less any gains in the price of capital. Abel and Eberly (1996) generalize this measure by replacing the price of capital (here normalized to 1) with $q(y)$, its marginal value to the firm:

$$U(y) = (r + \delta)q(y) - \frac{1}{\mathrm{d}t}\,\mathrm{E}_t\{\mathrm{d}q(y)\}.$$

Since y follows a geometric Brownian motion, it follows from Ito's Lemma that the expected change in the value of capital is

$$\frac{1}{\mathrm{d}t}\,\mathrm{E}_t\{\mathrm{d}q(y)\} = \lambda y q'(y) + \frac{1}{2}\,\sigma^2 y^2 q''(y),$$

so that

$$U(y) = (r + \delta)q(y) - [\lambda y q'(y) + \tfrac{1}{2}\sigma^2 y^2 q''(y)].$$

Imposing Eq. (11), the implicit user cost of capital reduces to

$$U(y) = hy^{\gamma}. \tag{26}$$

Recalling that profits are given by

$$\Pi(K, X) = \frac{h}{1-\gamma}\,K_t^{1-\gamma} X^{\gamma},$$

Eq. (26) shows that the implicit user cost of a unit of capital always equals its marginal revenue product, hy^{γ}.

Although this implicit user cost cannot be interpreted as the competitive rental rate of capital, as could the traditional user cost, it can still be used to find the marginal revenue product of capital at the time of an investment. This in turn can be used to interpret the investment trigger u^*. In particular, it follows from Eq. (21) that immediately before an investment, the implicit user cost of capital is

$$U(u^*) = \left(1 - \frac{\gamma}{\alpha_N}\right)(r+\delta)\left[1 + \frac{g(\theta^*)}{\theta^*}\right] \times \frac{(1-\gamma)\theta^*}{(1+\theta^*)^{1-\gamma} - 1}.$$

$U(u^*)$ can be broken into three components:

1. *The sum of interest and depreciation costs*:

   $$(r+\delta)[1 + g(\theta^*)/\theta^*].$$

   Note that the firm behaves as if interest and depreciation expenses are incurred on adjustment costs as well as the new capital actually installed.
2. *The effects of irreversibility*:

   $$1 - \frac{\gamma}{\alpha_N}.$$

   Abel and Eberly (1996) show that the implicit user cost for a firm facing irreversibility but no fixed costs is $(1-\gamma/\alpha_N)(r+\delta)$, as opposed to the standard measure of $r+\delta$. For the parameters used in the preceding subsection, the irreversibility effect is quite small: $-\gamma/\alpha_N \approx 0.006$.
3. *The effects of lumpiness*:

   $$\frac{(1-\gamma)\theta^*}{(1+\theta^*)^{1-\gamma} - 1}.$$

   With $0<\gamma<1$, $(1+\theta)^{1-\gamma}$ is a concave function, so that for positive θ, $(1+\theta)^{1-\gamma} < 1 + (1-\gamma)\theta$ and the ratio is bigger than 1. As θ^* increases, and investment grows lumpier, the implicit user cost at the time of investment, and thus the investment trigger u^*, both grow as well. This immediately implies that the trigger capital stock, $K^* = X_t/u^*$, is decreasing in θ^*: when investment is lumpy, firms let their capital depreciate further before they invest. By way of example, in the model with $c = 0.002$ and $a = 0$, the optimal value of θ is 0.444, and lumpiness increases user cost by 8.65 percent. Intuitively, a discrete increase in the capital stock causes a drop in relative demand, y, and thus a decrease in the marginal revenue product of capital. This upcoming drop in marginal product is a cost of investing that does not occur when capital evolves smoothly.

Note that the effects of irreversibility and lumpiness compound; as investments get larger, the flexibility losses they entail get larger as well, so that the investment trigger rises. For example, in the model with $c = 0.002$ and $a = 0$ (so that $\theta^* = 0.444$), the implicit user cost is roughly 0.11, about 10 percent higher than the conventional value of 0.10.

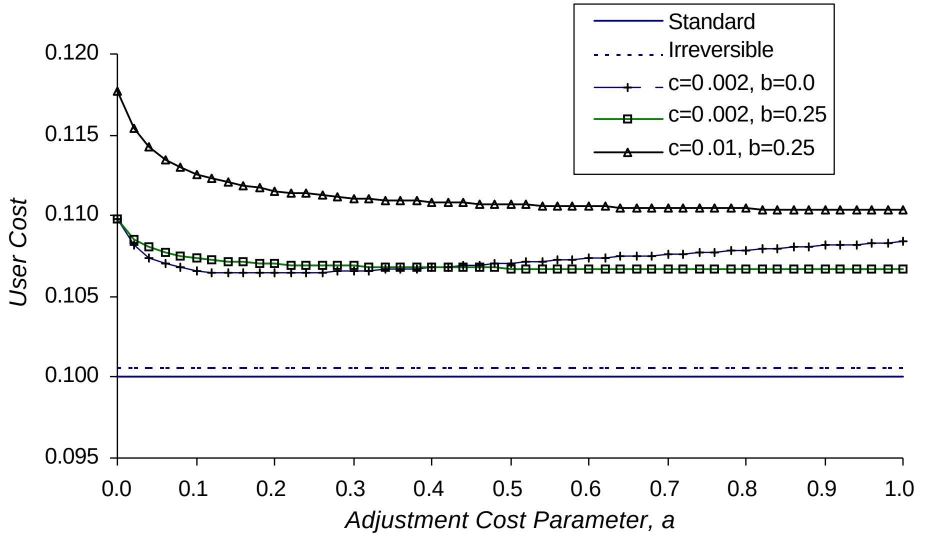

Fig. 5. User costs as functions of the adjustment cost parameter a.

The effects of irreversibility and lumpiness are illustrated further in Fig. 5, which shows different (mis-) measures of the user cost for the full quadratic form, $g(\theta) = a(\theta - b)^2/2 + c$, using three of the specifications shown in Fig. 4. In interpreting this figure, it is useful to note that the standard user cost, $(r + \delta)$, and implicit user cost under irreversibility, $(1 - \gamma/\alpha_N)(r + \delta)$, are the same for all the adjustment cost specifications.

6. Conclusion

For a broad class of adjustment costs, the optimal pattern of investment under lumpiness has a convenient analytical characterization. The firm's decision rule consists of an investment quantity, which can be found by maximizing a closed form function, and an investment trigger, which can be expressed as a function of the investment quantity. This makes the decision rule quite easy to study. Moreover, the user cost theory of investment can be extended to incorporate lumpiness in a very intuitive way.

While lumpy investment models with fixed adjustment costs have been studied extensively, our approach allows us to consider combinations of fixed and variable adjustment costs as well. Some straightforward numerical analyses show that the investment policy under a combination of fixed and variable adjustment costs is often quite different from the policy under a pure fixed cost. For example, adding a relatively small amount of convex adjustment costs can lead firms to invest in much smaller increments. These results suggest that the standard fixed-cost-only specification is an approximation that should be used with some caution.

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Appendix A. Proof of Lemma 1

Since $V(K,X)$ is linearly homogeneous, it follows from Euler's formula that

$$K\,\frac{\partial V(K,X)}{\partial K} + X\,\frac{\partial V(K,X)}{\partial X} = V(K,X), \tag{27}$$

so that

$$X\,\frac{\partial V(K,X)}{\partial X} = V(K,X) - Kq(y). \tag{28}$$

Differentiating both sides of (27) with respect to X yields

$$K\,\frac{\partial q(y)}{\partial X} + \frac{\partial V(K,X)}{\partial X} + X\,\frac{\partial V^2(K,X)}{\partial X^2} = \frac{\partial V(K,X)}{\partial X},$$

which implies that

$$\begin{aligned} X\,\frac{\partial V^2(K,X)}{\partial X^2} &= -K\,\frac{\partial q(y)}{\partial X} \\ &= -K\,\frac{\partial q(y)}{\partial y}\,\frac{\partial y}{\partial X} \\ &= -q'(y). \end{aligned} \tag{29}$$

Inserting Eqs. (28) and (29) into Eq. (9) yields

$$rV(K,X) = \frac{h}{1-\gamma}\,Ky^{\gamma} - \delta Kq(y) + \mu[V(K,X) - Kq(y)] - \frac{1}{2}\,\sigma^2 Kyq'(y),$$

which is equivalent to Eq. (14).

Appendix B. Derivation of Eq. (18)

Given that α_P and α_N are the roots of the quadratic equation

$$\rho(z) = -\tfrac{1}{2}\sigma^2 z^2 - (\lambda - \tfrac{1}{2}\sigma^2)z + r + \delta = 0, \tag{30}$$

it follows that

$$-\frac{2\rho(z)}{\sigma^2} = z^2 - \left(1 - \frac{2\lambda}{\sigma^2}\right)z - 2\frac{r+\delta}{\sigma^2} = (z-\alpha_P)(z-\alpha_N),$$

so that

$$\alpha_P\alpha_N = -2\,\frac{r+\delta}{\sigma^2}, \tag{31}$$

$$\alpha_P + \alpha_N = 1 - \frac{2\lambda}{\sigma^2}, \tag{32}$$

$$\rho(\gamma) = -\frac{\sigma^2}{2}(\gamma-\alpha_P)(\gamma-\alpha_N). \tag{33}$$

Recalling further that,

$$A = h/\rho(\gamma), \quad D = \lambda + \tfrac{1}{2}\gamma\sigma^2,$$

we have

$$\begin{aligned} h - AD(1-\gamma) &= h + h\,\frac{2D(1-\gamma)}{(\gamma-\alpha_P)(\gamma-\alpha_N)\sigma^2} \\ &= h\,\frac{(\gamma-\alpha_P)(\gamma-\alpha_N)\sigma^2 + 2D(1-\gamma)}{(\gamma-\alpha_P)(\gamma-\alpha_N)\sigma^2} \\ &= h\,\frac{\gamma^2\sigma^2 + 2\lambda\gamma - \gamma\sigma^2 - 2r - 2\delta + 2\lambda + \gamma\sigma^2 - 2\lambda\gamma - \gamma^2\sigma^2}{(\gamma-\alpha_P)(\gamma-\alpha_N)\sigma^2}, \end{aligned}$$

with the last line following from Eqs. (31) and (32). With $\lambda = \mu + \delta$, this simplifies to

$$h - AD(1-\gamma) = \frac{2h(\mu - r)}{(\gamma-\alpha_P)(\gamma-\alpha_N)\sigma^2},$$

which is equivalent to Eq. (18) in the main text.

Appendix C. Calculation of the upper trigger u

Differentiating Eq. (19), it follows that the first-order condition for optimality is equivalent to

$$\frac{\partial B}{\partial u} = F\left[\frac{(\gamma-\alpha_P)2h[(1+\theta)^{1-\gamma}-1]u^{\gamma-\alpha_P-1}}{(\gamma-\alpha_P)(\gamma-\alpha_N)\sigma^2(1-\gamma)} - \alpha_P\phi(\theta)u^{-\alpha_P-1}\right] = 0,$$

$$F \equiv \frac{\mu - r}{E[(1+\theta)^{1-\alpha_P}-1]} > 0, \tag{34}$$

which reduces to

$$u^\gamma = \frac{\alpha_P\phi(\theta)(\gamma-\alpha_N)(1-\gamma)\sigma^2}{2h[(1+\theta)^{1-\gamma}-1]}. \tag{35}$$

Imposing Eq. (31) yields

$$u^\gamma = \left(1 - \frac{\gamma}{\alpha_N}\right)(r+\delta)\phi(\theta)\frac{1-\gamma}{h[(1+\theta)^{1-\gamma}-1]}.$$

Appendix D. Second-order condition for optimal u

We confirm that Eq. (21) identifies a maximum by showing

$$\frac{\partial^2 Q(y;u,\theta)}{\partial u^2} = -\frac{1}{r-\mu} E y^{\alpha_P} \frac{\partial^2 B}{\partial u^2} < 0.$$

This boils down to showing that $\partial^2 B/\partial u^2$ is positive. It follows from Eq. (34) that

$$\frac{\partial^2 B}{\partial u^2} = F\left[\frac{(\gamma-\alpha_P-1)2h[(1+\theta)^{1-\gamma}-1]u^{\gamma-\alpha_P-1}}{(\gamma-\alpha_N)\sigma^2(1-\gamma)} + (\alpha_P+1)\alpha_P\phi(\theta)u^{-\alpha_P-1}\right],$$

the sign of which is given by the sign of

$$\frac{(\gamma-\alpha_P-1)2h[(1+\theta)^{1-\gamma}-1]u^{\gamma}}{(\gamma-\alpha_N)\sigma^2(1-\gamma)} + (\alpha_P+1)\alpha_P\phi(\theta).$$

Upon inserting Eq. (35), this expression simplifies to

$$(\gamma-\alpha_P-1)\alpha_P\phi(\theta) + \alpha_P(\alpha_P+1)\phi(\theta) > 0.$$

Appendix E. Derivation of $Q(y;\theta)$

The marginal value of capital is

$$q(y;\theta) = Ay^{\gamma} + \frac{\gamma(r-\mu)[\theta+g(\theta)]}{(\alpha_P-\gamma)E[(1+\theta)^{1-\alpha_P}-1]} \times \left(\frac{y}{u}\right)^{\alpha_P}.$$

Imposing the expression for A given in Appendix B yields

$$q(y,\theta) = -\frac{2hy^{\gamma}}{\sigma^2(\gamma-\alpha_P)(\gamma-\alpha_N)} - \frac{\gamma(r-\mu)\phi(\theta)u^{-\alpha_P}}{(\gamma-\alpha_P)E[(1+\theta)^{1-\alpha_P}-1]} y^{\alpha_P},$$

so that

$$yq'(y,\theta) = -\frac{2\gamma hy^{\gamma}}{\sigma^2(\gamma-\alpha_P)(\gamma-\alpha_N)} - \frac{\alpha_P\gamma(r-\mu)\phi(\theta)u^{-\alpha_P}}{(\gamma-\alpha_P)E[(1+\theta)^{1-\alpha_P}-1]} y^{\alpha_P}.$$

Substituting these results into Eq. (14) (Lemma 1) yields

$$\begin{aligned}(r-\mu)Q(y;\theta) &= \frac{hy^{\gamma}}{1-\gamma} + \frac{2\lambda hy^{\gamma}}{\sigma^2(\gamma-\alpha_P)(\gamma-\alpha_N)} + \frac{\gamma\lambda(r-\mu)\phi(\theta)u^{-\alpha_P}}{(\gamma-\alpha_P)E[(1+\theta)^{1-\alpha_P}-1]} y^{\alpha_P} \\ &\quad + \frac{\gamma hy^{\gamma}}{(\gamma-\alpha_P)(\gamma-\alpha_N)} + \frac{\alpha_P\gamma\sigma^2(r-\mu)\phi(\theta)u^{-\alpha_P}}{2(\gamma-\alpha_P)E[(1+\theta)^{1-\alpha_P}-1]} y^{\alpha_P} \\ &= \frac{\sigma^2(\gamma-\alpha_P)(\gamma-\alpha_N) + 2\lambda(1-\gamma) + (1-\gamma)\gamma\sigma^2}{\sigma^2(\gamma-\alpha_P)(\gamma-\alpha_N)(1-\gamma)} hy^{\gamma} \\ &\quad + \frac{\gamma(r-\mu)\phi(\theta)u^{-\alpha_P}(2\lambda+\alpha_P\sigma^2)}{2(\gamma-\alpha_P)E[(1+\theta)^{1-\alpha_P}-1]} y^{\alpha_P}.\end{aligned}$$

Inserting Eqs. (31) and (32), and recalling that $E = \lambda + \frac{1}{2}\alpha_P\sigma^2$ and $\lambda = \mu + \delta$, we have

$$(r-\mu)Q(y;\theta) = \frac{-2r+2\mu}{\sigma^2(\gamma-\alpha_P)(\gamma-\alpha_N)(1-\gamma)}\,hy^{\gamma} + \frac{\gamma(r-\mu)\phi(\theta)u^{-\alpha_P}}{(\gamma-\alpha_P)[(1+\theta)^{1-\alpha_P}-1]}\,y^{\alpha_P}.$$

Thus,

$$Q(y;\theta) = \frac{1}{\alpha_P-\gamma}\left[\frac{2hy^{\gamma}}{\sigma^2(\gamma-\alpha_N)(1-\gamma)} - \frac{\gamma\phi(\theta)}{(1+\theta)^{1-\alpha_P}-1}\times\left(\frac{y}{u}\right)^{\alpha_P}\right].$$

Appendix F. Verification of the smooth pasting conditions

Recall from Eq. (25) that θ solves

$$\arg\max_{\theta_0\geqslant 0}(\gamma-\alpha_P)\log[\theta_0+g(\theta_0)] - \gamma\log[1-(1+\theta_0)^{1-\alpha_P}] + \alpha_P\log[(1+\theta_0)^{1-\gamma}-1].$$

The first-order condition for this problem is

$$(\gamma-\alpha_P)\frac{1+g'(\theta)}{\theta+g(\theta)} = -\left[\frac{\gamma(1-\alpha_P)(1+\theta)^{-\alpha_P}}{1-(1+\theta)^{1-\alpha_P}} + \frac{\alpha_P(1-\gamma)(1+\theta)^{-\gamma}}{(1+\theta)^{1-\gamma}-1}\right], \tag{36}$$

which can be written as

$$\frac{(1+\theta)^{-\alpha_P}}{(1+\theta)^{1-\alpha_P}-1} = \frac{1}{\gamma(1-\alpha_P)}\left[\frac{(\gamma-\alpha_P)[1+g'(\theta)]}{\theta+g(\theta)} + \alpha_P\,\frac{(1-\gamma)(1+\theta)^{-\gamma}}{(1+\theta)^{1-\gamma}-1}\right]. \tag{37}$$

Now it follows from Eq. (23) that

$$q(v) = q\left(\frac{u}{1+\theta}\right) = Au^{\gamma}(1+\theta)^{-\gamma} + \frac{\gamma(r-\mu)[\theta+g(\theta)]}{(\alpha_P-\gamma)E[(1+\theta)^{1-\alpha_P}-1]}(1+\theta)^{-\alpha_P}.$$

Combining this with Eq. (37) yields

$$q(v) = Au^{\gamma}(1+\theta)^{-\gamma} + \frac{(r-\mu)[1+g'(\theta)]}{(1-\alpha_P)E}(-1) + \frac{(r-\mu)[\theta+g(\theta)]}{(\alpha_P-\gamma)(1-\alpha_P)E}\,\alpha_P\,\frac{(1-\gamma)(1+\theta)^{-\gamma}}{(1+\theta)^{1-\gamma}-1}. \tag{38}$$

Applying Eq. (32), we have

$$E = \lambda + \tfrac{1}{2}\alpha_P\sigma^2 = \tfrac{1}{2}\sigma^2(1-\alpha_N).$$

Moreover, it follows from Appendix B and the definition of λ that

$$(1-\alpha_P)(1-\alpha_N) = 2\,\frac{\mu-r}{\sigma^2}.$$

Combining these results yields

$$(1-\alpha_P)E = \mu - r.$$

Inserting this result, Eq. (38) reduces to

$$q(v) = Au^{\gamma}(1+\theta)^{-\gamma} + 1 + g'(\theta) - \frac{\theta + g(\theta)}{\alpha_P - \gamma}\,\alpha_P\,\frac{(1-\gamma)(1+\theta)^{-\gamma}}{(1+\theta)^{1-\gamma} - 1}. \tag{39}$$

Imposing Eq. (35) and the definition of A given in Appendix B shows that

$$\begin{aligned} Au^{\gamma} &= \frac{h}{-\sigma^2(\gamma-\alpha_P)(\gamma-\alpha_N)/2} \times \frac{\alpha_P\phi(\theta)(\gamma-\alpha_N)(1-\gamma)\sigma^2}{2h[(1+\theta)^{1-\gamma}-1]} \\ &= \frac{\alpha_P\phi(\theta)(1-\gamma)}{(\alpha_P-\gamma)[(1+\theta)^{1-\gamma}-1]}. \end{aligned} \tag{40}$$

But $\phi(\theta) = \theta + g(\theta)$, so that

$$Au^{\gamma}(1+\theta)^{-\gamma} - \frac{\theta + g(\theta)}{\alpha_P - \gamma}\,\alpha_P\,\frac{(1-\gamma)(1+\theta)^{-\gamma}}{(1+\theta)^{1-\gamma} - 1} = 0$$

and

$$q(v) = 1 + g'(\theta). \tag{41}$$

This verifies the first of the smooth pasting conditions.

Repeating the steps that led to Eq. (39) yields

$$q(u) = Au^{\gamma} + \left[1 + g'(\theta) - \frac{\theta + g(\theta)}{\alpha_P - \gamma}\,\alpha_P\,\frac{(1-\gamma)(1+\theta)^{-\gamma}}{(1+\theta)^{1-\gamma} - 1}\right](1+\theta)^{\alpha_P}.$$

Upon inserting Eq. (40), this expression reduces to

$$q(u) = (1+\theta)^{\alpha_P}[1 + g'(\theta)] + \frac{\alpha_P(1-\gamma)[\theta + g(\theta)]}{(\alpha_P-\gamma)[(1+\theta)^{1-\gamma}-1]}[1 - (1+\theta)^{\alpha_P-\gamma}]. \tag{42}$$

Using Eqs. (41) and (42), we have

$$\begin{aligned} \frac{(\alpha_P-\gamma)[q(u) - (1+\theta)q(v)]}{\theta + g(\theta)} &= [(1+\theta)^{\alpha_P} - (1+\theta)]\frac{(\alpha_P-\gamma)[1+g'(\theta)]}{\theta+g(\theta)} \\ &\quad + \frac{\alpha_P(1-\gamma)}{(1+\theta)^{1-\gamma}-1}[1 - (1+\theta)^{\alpha_P-\gamma}]. \end{aligned} \tag{43}$$

Note that Eq. (36) can be written as

$$\begin{aligned} &[(1+\theta)^{\alpha_P} - (1+\theta)]\frac{(\alpha_P-\gamma)[1+g'(\theta)]}{\theta+g(\theta)} \\ &\quad = \gamma(1-\alpha_P) + \frac{\alpha_P(1-\gamma)[(1+\theta)^{\alpha_P-\gamma} - (1+\theta)^{1-\gamma}]}{(1+\theta)^{1-\gamma}-1}. \end{aligned}$$

Inserting this result into Eq. (43) yields

$$\frac{(\alpha_P-\gamma)[q(u)-(1+\theta)q(v)]}{\theta+g(\theta)} = \gamma(1-\alpha_P)+\frac{\alpha_P(1-\gamma)}{(1+\theta)^{1-\gamma}-1}[1-(1+\theta)^{1-\gamma}]$$
$$= \gamma(1-\alpha_P)-\alpha_P(1-\gamma)=\gamma-\alpha_P$$

or

$$q(u)=(1+\theta)q(v)-[\theta+g(\theta)].$$

This verifies the second smooth pasting condition.

Appendix G. Local approximation of θ

We begin with some background results. It follows from the Taylor series expansion of $(1+\theta)^{\alpha_P}$ that

$$(1+\theta)^{\alpha_P}-(1+\theta)=(\alpha_P-1)\theta\left[1+\frac{\alpha_P}{2}\theta+\theta^2J(\theta)\right],$$
$$J(\theta)=\frac{\alpha_P(\alpha_P-2)}{6}+\frac{\alpha_P(\alpha_P-2)(\alpha_P-3)}{24}\theta+\cdots. \tag{44}$$

This implies that

$$\frac{\theta}{(1+\theta)^{\alpha_P}-(1+\theta)}-\frac{\theta}{(\alpha_P-1)\theta[1+\frac{\alpha_P}{2}\theta]}$$
$$=-\frac{\theta^2J(\theta)}{(\alpha_P-1)[1+\frac{\alpha_P}{2}\theta][1+\frac{\alpha_P}{2}\theta+\theta^2J(\theta)]},$$

so that after some polynomial division,

$$\frac{\theta}{(1+\theta)^{\alpha_P}-(1+\theta)}=\frac{1}{(\alpha_P-1)[1+\frac{\alpha_P}{2}\theta]}+j_0\theta^2+\theta^2[j_1\theta+j_2\theta^2\cdots],$$
$$j_0\equiv\frac{\alpha_P(\alpha_P-2)}{6(1-\alpha_P)},$$

and the remaining j's are coefficients on a bounded remainder. It follows that

$$\frac{\gamma(1-\alpha_P)\theta}{(1+\theta)^{\alpha_P}-(1+\theta)}=\frac{-\gamma}{1+\frac{\alpha_P}{2}\theta}+\frac{\gamma\alpha_P(\alpha_P-2)}{6}\theta^2$$
$$+\gamma(1-\alpha_P)\theta^2[j_1\theta+j_2\theta^2\cdots]. \tag{45}$$

Similarly, it can be shown that

$$\frac{\theta}{1+\theta-(1+\theta)^{\gamma}}=\frac{1}{(1-\gamma)(1+\frac{\gamma}{2}\theta)}+j_0'\theta^2+\theta^2[j_1'\theta+j_2'\theta^2\cdots],$$
$$j_0'\equiv\frac{\gamma(2-\gamma)}{6(1-\gamma)},$$

so that

$$\frac{\alpha_P(1-\gamma)\theta}{1+\theta-(1+\theta)^{\gamma}} = \frac{\alpha_P}{1+\frac{\gamma}{2}\theta} + \frac{\gamma\alpha_P(2-\gamma)}{6}\theta^2 + \alpha_P(1-\gamma)\theta^2[j_1'\theta + j_2'\theta^2 \cdots]. \tag{46}$$

Combining Eqs. (45) and (46) shows that

$$\frac{\gamma(1-\alpha_P)\theta}{(1+\theta)^{\alpha_P}-(1+\theta)} + \frac{\alpha_P(1-\gamma)\theta}{1+\theta-(1+\theta)^{\gamma}} = \frac{-\gamma}{1+\frac{1}{2}\alpha_P\theta}$$
$$+ \frac{\alpha_P}{1+\frac{1}{2}\gamma\theta} + (\alpha_P-\gamma)r_0\theta^2 + \theta^2 R(\theta), \tag{47}$$

where

$$r_0 = \frac{1}{\alpha_P-\gamma}\left[\frac{\gamma\alpha_P(\alpha_P-2)}{6} + \frac{\gamma\alpha_P(2-\gamma)}{6}\right] = \frac{\gamma\alpha_P}{6},$$
$$R(\theta) = \gamma(1-\alpha_P)[j_1\theta + j_2\theta^2 \cdots] + \alpha_P(1-\gamma)[j_1'\theta + j_2'\theta^2 \cdots]. \tag{48}$$

Returning to the main problem, recall from Eq. (36) that the first-order condition for θ is

$$(\gamma-\alpha_P)\,\frac{\theta[1+g'(\theta)]}{\theta+g(\theta)} = -\left[\frac{\gamma(1-\alpha_P)\theta(1+\theta)^{-\alpha_P}}{1-(1+\theta)^{1-\alpha_P}} + \frac{\alpha_P(1-\gamma)\theta(1+\theta)^{-\gamma}}{(1+\theta)^{1-\gamma}-1}\right]. \tag{49}$$

When $g(\theta)=c$, this simplifies to

$$\frac{\alpha_P-\gamma}{\theta+c} = \frac{\gamma(1-\alpha_P)}{(1+\theta)^{\alpha_P}-(1+\theta)} + \frac{\alpha_P(1-\gamma)}{1+\theta-(1+\theta)^{\gamma}}.$$

Imposing Eqs. (47) and (48), one can rewrite the first-order condition as

$$\frac{\alpha_P-\gamma}{\theta+c} = \frac{-\gamma}{\theta(1+\frac{1}{2}\alpha_P\theta)} + \frac{\alpha_P}{\theta(1+\frac{1}{2}\gamma\theta)} + (\alpha_P-\gamma)\,\frac{\gamma\alpha_P}{6}\,\theta + \theta R(\theta)$$
$$= \frac{(\alpha_P-\gamma)[1+\frac{\theta}{2}(\alpha_P+\gamma)]}{\theta(1+\frac{\alpha_P}{2}\theta)(1+\frac{\gamma}{2}\theta)} + (\alpha_P-\gamma)\,\frac{\gamma\alpha_P}{6}\,\theta + \theta R(\theta).$$

Thus,

$$\frac{1}{\theta+c} = \frac{1+\frac{\theta}{2}(\alpha_P+\gamma)}{\theta(1+\frac{\alpha_P}{2}\theta)(1+\frac{\gamma}{2}\theta)} + \frac{\gamma\alpha_P\theta}{6} + \frac{\theta}{\alpha_P-\gamma}R(\theta),$$

which can be written as

$$\frac{1}{\theta+c} = \frac{1+\frac{\theta}{2}(\alpha_P+\gamma)+\frac{1}{6}\gamma\alpha_P\theta^2+S(\theta)}{\theta(1+\frac{\alpha_P}{2}\theta)(1+\frac{\gamma}{2}\theta)},$$

where $S(\theta)$ is a polynomial of third- and higher-order terms. This is equivalent to

$$(\theta+c)\left[1+\frac{\theta}{2}(\alpha_P+\gamma)+\frac{1}{6}\,\gamma\alpha_P\theta^2+S(\theta)\right] = \theta\left(1+\frac{\alpha_P}{2}\,\theta\right)\left(1+\frac{\gamma}{2}\,\theta\right)$$

or

$$\theta + c + \frac{\theta^2}{2}(\alpha_P + \gamma) + \frac{\alpha_P + \gamma}{2}\, c\theta + \frac{1}{6}\, c\gamma\alpha_P\theta^2 + \frac{1}{6}\,\gamma\alpha_P\theta^3 + (\theta + c)S(\theta)$$
$$= \theta + \theta^2\,\frac{\alpha_P + \gamma}{2} + \frac{\alpha_P\gamma}{4}\,\theta^3.$$

The latter expression simplifies to

$$c + c\,\frac{\theta}{2}(\alpha_P + \gamma) + \frac{1}{6}\, c\gamma\alpha_P\theta^2 + \frac{1}{6}\,\gamma\alpha_P\theta^3 + (\theta + c)S(\theta) = \frac{\alpha_P\gamma}{4}\,\theta^3$$

or

$$c = \frac{\frac{1}{12}\gamma\alpha_P\theta^3 - \theta S(\theta)}{1 + \frac{\alpha_P+\gamma}{2}\theta + \frac{1}{6}\gamma\alpha_P\theta^2 + S(\theta)}.$$

Employing some polynomial division, and dropping fourth- and higher-order terms, one gets

$$c \cong \tfrac{1}{12}\gamma\alpha_P\theta^3,$$

so that

$$\theta \cong \left(\frac{12c}{\gamma\alpha_P}\right)^{1/3}.$$

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