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Generic determinacy of equilibria with local substitution ☆

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Abstract

Most continuous-time models of trading presume a smooth cumulative consumption function, thereby neglecting stock and possibly singular consumption patterns. Hindy, Huang and Kreps pioneered an approach that captures the full possible variety of flow and stock (and more general) consumption patterns in continuous time. The stochastic version of their model poses significant challenges to General Equilibrium Theory as the price space is not a lattice. The present paper establishes generic determinacy for stochastic Hindy–Huang–Kreps economies. For smooth economies, almost all initial endowments admit only a finite number of competitive equilibria, and these equilibria vary (locally) smoothly with endowments; thus equilibrium comparative statics are locally determinate.

Keywords: Prevalence; Local substitution; Determinacy; Infinite-dimensional commodity space; Hindy–Huang–Kreps preferences; Local substitution

1. Introduction

In continuous-time stochastic models, three types of consumption patterns are possible. To illustrate these three types, consider the cumulative consumption function. As negative

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consumption is usually precluded, cumulative consumption has to be a nondecreasing function of time. First, there might be jumps in this function, meaning that the agent has taken a gulp or a lump. Second, the consumption function can be smooth implying consumption in rates and, in principle, the third and more fancy way of continuous, yet singular consumption is also possible.[1]

Most continuous-time models neglect the stock or singular component of dynamic consumption, and assume right away that consumption occurs in rates only. This tradition starts with the early work of Samuelson (1937) and continues to the most sophisticated models to date, see, e.g., stochastic differential utility (Duffie and Epstein, 1992) or habit formation (Ryder and Heal, 1973; Constantinides, 1990). Hindy et al. (1992) present the first continuous time model that takes the stock and the flow (and, in principle, also the singular) component of continuous-time consumption into account. Consequently, these authors free the consumption functions from the artificial smoothness restriction and allow *all* possible distribution functions on the time axis to be chosen by agents. Moreover, they convincingly argue that slight shifts of the consumption plan over time should not have dramatic effects on that plan's utility as consumption of a good at one point is a substitute for consumption of the same good an instant later or earlier. The appropriate topology that captures this economic requirement is the weak topology for distribution functions.[2]

The Hindy–Huang–Kreps model poses significant challenges for General Equilibrium Theory. The main difficulty is that the commodity space, the space of signed measures on the time axis, is not a lattice when endowed with the weak topology. In fact, this problem spurred the famous Mas-Colell–Richard Existence Theorem (Mas-Colell and Richard, 1991). As is well known, Mas-Colell and Richard use a disaggregated Negishi approach to establish existence. In this approach, equilibrium price candidates are weighted maxima of individual utility gradients. As prices should be continuous linear functionals, Mas-Colell and Richard assume that the price space, the topological dual of the commodity space, is a lattice. In the stochastic version of the Hindy–Huang–Kreps model, the price space loses the lattice property, and the Mas-Colell–Richard Theorem does not apply. Recently, the existence problem was solved by Bank and Riedel (2001a). There, one shows first existence of a price functional in the order dual of the commodity space. In a second step, arguments from stochastic analysis are used to show that equilibrium price functionals are continuous on the *consumption set*, the positive cone of the commodity space.

Besides existence of equilibrium, generic determinacy is a cornerstone of General Equilibrium Theory. In finite dimensions, Debreu (1970) shows that the number of equilibria is finite and the equilibrium set varies smoothly with endowments for almost all initial endowment vectors. For infinite-dimensional commodity spaces, the parallel result has re-

[1] To see that this last type of consumption is not as far fetched as it might appear, consider the standard model of continuous-time finance. There, the stochastics are typically modelled by some Brownian motion. In such a model, it can be perfectly sound when cumulative consumption is (a function of) the running maximum of Brownian motion (cf. Hindy and Huang, 1992; Bank and Riedel, 2001b). In this case, the consumption function is continuous and nondecreasing, but it does not have a density with respect to Lebesgue measure. Consumption rates do not exist.

[2] In fact, Hindy, Huang and Kreps propose to use norms that induce the same topology as the weak topology on the consumption set; however, the topological dual of the commodity space is smaller than the one induced by the weak topology.

cently been established by Shannon and Zame (2002). As in the finite-dimensional case, differential strict concavity is necessary for determinacy. In addition, one needs a constraint on the asymptotic behavior of the marginal rates of substitution. This is captured by the notion of *quadratic concavity* that Shannon and Zame introduce. A concave utility function is quadratically concave if it deviates from its linear approximation at a point proportionally to the square distance from that point. This property is implied by differential strict concavity in finite dimensions, but not in infinite-dimensional vector spaces.

In the tradition of the Mas-Colell–Richard Theorem, Shannon and Zame impose a lattice structure on the price space. Therefore, their theorem does not cover the stochastic version of the Hindy–Huang–Kreps model. The present paper proves generic determinacy for the stochastic Hindy–Huang–Kreps model. The strategy of proof follows the spirit of the Shannon–Zame Theorem, but has to replace those parts of the proof that rely on the lattice structure of the price space. To this end, I work on the *order interval* spanned by zero consumption and aggregate endowment. Maxima of continuous linear functionals are continuous when restricted to this order interval (Lemma 8). Fortunately, this kind of continuity suffices to establish generic determinacy along the lines of Shannon and Zame.

The abstract functional analytic approach of Shannon and Zame requires that the order ideal generated by aggregate endowment is weakly dense in the commodity space in order to extend certain price functionals from the order ideal to the whole commodity space. In the present context, one can use another approach; since the structure of the model provides explicitly a candidate for a price functional, one just checks the desired properties for the candidate (cf. Bank and Riedel, 2001a; Duffie and Zame, 1989), and the abstract extension problem does not arise. This is important for applications as the theorem proved herein covers cases with positive supply of the consumption good at some random or fixed points in time only.[3]

As the commodity space is infinite-dimensional, let me say a few words on the notion of genericity used in this paper. As is well known from the finite-dimensional case (Debreu, 1970), the number of equilibria can be infinite, even with smooth preferences. However, the set of initial endowment vectors for which this occurs has Lebesgue measure zero, and it is said that *generically*, the number of equilibria is finite. In the absence of a Lebesgue measure (i.e., a translation invariant measure which assigns positive measure to all nonempty open sets) for infinite dimensional spaces, another concept of genericity is needed. Here, *generically* will mean *finitely prevalent*, a concept introduced into the economics literature by Anderson and Zame (2001). The notion of *prevalence* for complete metric vector spaces goes back to Hunt et al. (1992). Prevalence captures the *probabilistic* intuition one associates with 'largeness'. For example, a prevalent set is dense, a countable intersection of prevalent sets is again prevalent, and translates of a prevalent set are prevalent. A sufficient condition for prevalence of a set L is that there exists a finite-dimensional subspace P such that arbitrary translates of the complement L^c have Lebesgue measure zero in P. For finite-dimensional spaces, a property is prevalent if and only if it holds true Lebesgue almost everywhere.

In the current paper, an economy is characterized by a distribution of initial endowments, a 'small' convex subset of the commodity space. Therefore, one needs a sensible definition of being a large subset of some set which is itself small. This notion of prevalence *relative to*

[3] The order ideal generated by measures with finite support is not weakly dense in the commodity space.

some convex set has been provided by Anderson and Zame (2001). To establish prevalence, one first chooses a suitable finite-dimensional subspace of the commodity space that 'sees' the set of endowment distributions in the sense that the intersection of the set of endowment distributions with that subspace has positive Lebesgue measure. Then, it is shown that the intersection of the subset of nondeterminate economies with the subspace has Lebesgue measure zero. In this sense, the set of nondeterminate economies is small.

The paper is organized as follows. The next section presents the model and examples of utility functions covered in this paper. Section 3 states the main result and contains its proof. Appendices A and B collect some supplementary proofs.

2. Assumptions, preliminaries, examples

This section sets up a model for a stochastic pure exchange economy with a finite number m of agents in continuous time. A referee pointed out that potential readers might not be familiar with some of the advanced probability concepts used below. A good quick reference can be found in Duffie (1992). Karatzas and Shreve (1991) and Protter (1990) provide more details on the probabilistic background. Most of the arguments below do not need the theory of stochastic integration or martingale theory, though.

Let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t, 0 \le t \le T), \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right continuity and completeness; $\mathcal{F}_0$ is $\mathbb{P}$-a.s. trivial.

A nonnegative, nondecreasing, rightcontinuous and adapted process $C = (C_t)_{0 \le t \le T}$ is called an optional random measure. If $X = C - C'$ for two optional random measures, then X is called a signed optional random measure. Agents' consumption set $\mathcal{X}_+$ consists of all optional random measures C with $\mathbb{E}C_T < \infty$, and the commodity space $\mathcal{X}$ is the vector space spanned by $\mathcal{X}_+$. An ordering $\le$ is given on $\mathcal{X}$ via $X \le Y$ if $X - Y \in \mathcal{X}_+$. For $C \in \mathcal{X}_+$, the order interval is

$$[0, C] = \{D \in \mathcal{X}_+ : 0 \le D \le C\}.$$

The Hindy–Huang–Kreps norm on $\mathcal{X}$ is given by $\|X\|_{\mathrm{HHK}} = \mathbb{E}\int_0^T |X_t \,\mathrm{d}t + \mathbb{E}|X_T|$. On the consumption set, this norm induces the topology of weak convergence in probability plus L^1-convergence of total cumulative consumption. This topology captures the notion of local substitutability of consumption as shown by Hindy and Huang (1992) and Hindy et al. (1992).

The m agents are described by their utility functions $U^i : \mathcal{X}_+ \to \mathbb{R}, i = 1, \ldots, m$. Throughout, we take utility functions and aggregate endowment $\bar{E} \in \mathcal{X}_+$, $\bar{E} \ne 0$ as fixed, and we vary the vector of initial endowments $(E^i)_{i=1,\ldots,m}$ in the set

$$\mathcal{D} = \left\{ (E^i)_{i=1,\ldots,m} \in \mathcal{X}_+^m : E^i \ne 0, \sum_{i=1}^m E^i = \bar{E} \right\}.$$

Therefore, one can identify an economy with an endowment vector $(E^i) \in \mathcal{D}$ in the sequel.

Optional, nonnegative processes ψ with

$$0 < \langle \psi, \bar{E} \rangle \overset{\Delta}{=} \mathbb{E} \int_0^T \psi_t \, \mathrm{d}\bar{E}_t < \infty$$

are called price processes. The set of all price processes is denoted Ψ. An allocation is a vector $(C^i)_{i=1,\ldots,m} \in \mathcal{X}_+^m$. It is feasible if $\sum_{i=1}^m C^i \leq \bar{E}$. The set of feasible allocations will be denoted by $\mathcal{Z}$. An (Arrow–Debreu) equilibrium for the economy $(E^i) \in \mathcal{D}$ consists of a feasible allocation $(C^i)_{i=1,\ldots,m}$ and a price process ψ such that, for any $i = 1, \ldots, m$, the consumption plan C^i maximizes agent i's utility over all D^i satisfying the budget-constraint $\langle \psi, D^i \rangle \leq \langle \psi, E^i \rangle$.

An economy $(E^i) \in \mathcal{D}$ is called determinate if the number of equilibria is finite and the equilibrium allocation correspondence

$$\mathcal{E} : (F^i) \in \mathcal{D} \mapsto \{(C^i) \in \mathcal{D} : (C^i) \text{ is equilibrium allocation for } (F^i)\}$$

is continuous at (E^i). Here, I show that almost every economy is determinate in the sense that the set of determinate economies

$$\mathcal{D}_d \overset{\Delta}{=} \{(E^i) \in \mathcal{D} : (E^i) \text{ is determinate}\}$$

is *finitely prevalent in* $\mathcal{D}$. A Borel subset $\mathcal{D}_0 \subset \mathcal{D}$ is called finitely prevalent in $\mathcal{D}$ if there exists a finite-dimensional subspace $\mathcal{V}$ of $\mathcal{X}^m$ and an element $F \in \mathcal{X}^m$ such that the intersection $\mathcal{V} \cap (\mathcal{D} + F)$ is not a Lebesgue null set in $\mathcal{V}$, and for all $G \in \mathcal{X}$ the sets $\mathcal{V} \cap ((\mathcal{D} \backslash \mathcal{D}_0) + G)$ *are* Lebesgue null sets in $\mathcal{V}$. The subspace $\mathcal{V}$ is called a probe (cf. Hunt et al., 1992; Anderson and Zame, 2001). As for terminology, I will say that a property holds for *almost every* endowment, if it holds true on a finitely prevalent subset of $\mathcal{D}$.

Prevalence is defined for completely metrizable sets only, so we note as a preliminary fact proved in the appendix:

Lemma 1. *The metric space* $(\mathcal{X}_+, \|\cdot\|_{\mathrm{HHK}})$ *is complete.*

Another requirement for determinacy is that utility functionals be smooth and strictly concave. In the infinite dimensional case, one has to *require* an additional property that is automatically satisfied in finite dimensions. It is important that the marginal rates of substitution do not vanish asymptotically. This is captured by *quadratic concavity*, as Shannon and Zame (2002) explain carefully. Formally, the following properties are imposed.

Assumption 2. The utility functions U^i have the following properties:

(1) U^i is strictly increasing and continuous.
(2) There exist a mapping $\nabla U^i : \mathcal{X}_+ \to \Psi$, a norm $\|\cdot\|_i$ on $\mathcal{X}$ and constants $B, K > 0$, such that:
   (a) on $[0, \bar{E}]$, the topology induced by $\|\cdot\|_i$ coincides with the topology induced by $\|\cdot\|_{\mathrm{HHK}}$,
   (b) for all $C \in [0, \bar{E}]$ and all $X \in \mathcal{X}$
   $$|\langle \nabla U^i(C), X \rangle| \leq B \|X\|_i,$$

(c) for all $C, C' \in \mathcal{X}_+$, $C'' \in [0, \bar{E}]$
$$|\langle \nabla U^i(C) - \nabla U^i(C'), C'' \rangle \leq B\|C - C'\|_i$$
(d) for all $C, C' \in [0, \bar{E}]$
$$U^i(C') - U^i(C) \leq \langle \nabla U^i(C), C' - C \rangle - K\|C' - C\|_i^2.$$

Whenever assumptions are made, it is important to check that one does not talk about the empty set. Hindy, Huang and Kreps studied additive utility functions that are functions of a certain level of satisfaction instead of the consumption rate. These preferences satisfy the above assumption if the associated period utility function has the appropriate properties.

Example 3. Hindy–Huang–Kreps utility functions satisfy Assumption 2. For $\beta, \eta > 0$ and $X \in \mathcal{X}$ set

$$z_t^X = \mathrm{e}^{-\beta t} \int_0^t \mathrm{e}^{\beta s} \, \mathrm{d}X_s$$
$$y_t^X = \eta \, \mathrm{e}^{-\beta t} + z_t^X.$$

Let $u, v : \mathbb{R}_+ \to \mathbb{R}$ be twice continuously differentiable with strictly positive first and strictly negative second derivative. Set for $\delta > 0$ and $C \in \mathcal{X}_+$

$$U(C) = \mathbb{E} \int_0^T \mathrm{e}^{-\delta t} u(y_t^C) \, \mathrm{d}t + \mathbb{E} v(y_T^C).$$

Define a norm on $\mathcal{X}$ via

$$\|X\|_R \triangleq \mathbb{E} \int_0^T |z_t^X| \, \mathrm{d}t + \mathbb{E}|z_T^X|.$$

The subgradient of U at $C \in \mathcal{X}_+$ is

$$\nabla U(C)_t = \mathbb{E}\left[\int_t^T \mathrm{e}^{-\delta s} \, u'(y_s^C) \, \mathrm{e}^{-\beta(s-t)} \, \mathrm{d}s | \mathcal{F}_t\right] + \mathbb{E}[\mathrm{e}^{-\beta(T-t)} \, v'(y_T^C) | \mathcal{F}_t].$$

We claim that U, ∇U *and* $\|\cdot\|_R$ satisfy Assumption 2 if aggregate endowment $\bar{E}$ is bounded a.s. This is shown in Appendix B.

3. Generic determinacy

The ensuing analysis uses the Negishi (1960) approach. This method of proof relies on the fact that equilibrium allocations are efficient. The set of efficient allocations can be parametrized by a finite-dimensional simplex of utility weights because efficient allocations maximize a weighted sum of utilities over the set of feasible allocations. For the problem at hand, it is very important that *all* equilibria correspond to some utility weights vector.

This is shown in Subsection 3.1. The proof of the parallel results in Shannon and Zame (2002) uses their assumption that aggregate endowment is strictly positive. Here, I present a different proof of these facts, which relies on my previous work with Peter Bank (2001a).

Given a vector of weights, the associated efficient allocation, and the equilibrium price candidate, one can define the *excess spending map*, the value of agents' excess consumption in that allocation. Every equilibrium corresponds to a zero of the finite-dimensional excess spending map, and determinacy will follow from regularity of this map. As Shannon and Zame show, it suffices to have Lipschitz continuity in the utility weights and continuity in endowments. Lipschitz continuity in the utility weights follows from quadratic concavity of utility functions. For continuity in endowments, Shannon and Zame rely on the lattice structure of the price space. As the price space is not a lattice in the present model, one has to use a different argument. It turns out that it suffices to have continuity of the price functional on order intervals. This is a much weaker requirement than continuity on the commodity space and I show that it holds true in the present model (Section 3.2).

We can now state the main theorem of the paper.

Theorem 4. *Under Assumption 2, almost every economy* $(E^i)_{i=1,\dots,m} \in \mathcal{D}$ *is determinate.*

The proof is given in Section 3.3.

3.1. Efficient allocations and equilibria

Introduce the set

$$\Lambda \triangleq \left\{ \lambda \in \mathbb{R}^m : \forall i\, \lambda^i > 0, \sum_{i=1}^{m} \lambda^i = 1 \right\}$$

and denote its closure by $\overline{\Lambda}$. A feasible allocation $(C^i) \in \mathcal{Z}$ is called λ-efficient for $\lambda \in \overline{\Lambda}$ if it maximizes the weighted sum of utilities $\sum \lambda^i U^i(D^i)$ over all feasible allocations $(D^i) \in \mathcal{Z}$.

The following two lemmas are as in Shannon and Zame (2002). In Bank and Riedel (2001a), one can find a different proof that does not need strict positivity of aggregate endowment.

Lemma 5. *For* $\lambda \in \overline{\Lambda}$ *there exists a unique* λ*-efficient allocation* (C^i_λ)*. It satisfies*

$$\langle \psi_\lambda - \lambda^i \nabla U^i(C^i_\lambda), C^i_\lambda \rangle = 0 \quad (i = 1, \dots, m) \tag{1}$$

for

$$\psi_\lambda \triangleq \max_{i=1,\dots,m} \lambda^i \nabla U^i(C^i_\lambda). \tag{2}$$

Proof. Bank and Riedel (2001a), Lemma 1. □

Lemma 6.

(1) *Let* $(C^i) \in \mathcal{X}^m_+$ *and* $\psi \in \Psi$ *form an equilibrium. Then there exist* $\lambda \in \Lambda$ *and* $L > 0$ *such that*:

$$C^i = C^i_\lambda \tag{3}$$

$$\psi = L\psi_\lambda, \qquad \mathbb{P} \otimes \mathrm{d}\bar{E} - \text{a.e.} \tag{4}$$

$$\langle \psi_\lambda, C^i_\lambda - E^i \rangle = 0. \tag{5}$$

(2) *If (5) holds true for some $\lambda \in \Lambda$, then (C^i_λ) and ψ_λ form an equilibrium.*

Proof. For (1), let $((C^i), \psi)$ be an equilibrium. In particular, ψ supports the allocation (C^i). Lemma 3 in Bank and Riedel (2001a) yields nonnegative $k^i \geq 0$ such that:

$$\psi = \max k^i \nabla U^i(C^i), \qquad \mathbb{P} \otimes \mathrm{d}\bar{E} - \text{a.e.} \tag{6}$$

and

$$\langle \psi - k^i \nabla U^i(C^i), C^i \rangle = 0. \tag{7}$$

Note that not all k^i can be zero, because else $\psi = 0 \notin \Psi$. Moreover, strict monotonicity of utility functions implies that:

$$\sum C^i = \bar{E}. \tag{8}$$

Now, by Lemma 1 in Bank and Riedel (2001a), (6)–(8) show that (C^i) is λ–efficient for $\lambda^i \triangleq \dfrac{k^i}{\sum k^j}$, and by uniqueness, $C^i = C^i_\lambda$. I show next that $\lambda^i > 0$. $\lambda^i = 0$ implies $C^i_\lambda = 0$, and therefore $U^i(C^i) < U^i(E^i)$, a contradiction to (C^j) being an equilibrium. Hence, $\lambda \in \Lambda$ and $C^i = C^i_\lambda$, which establishes (3). From this, and (6), we obtain (4) and (5) is the equilibrium budget constraint.

Now assume that (5) holds true for some $\lambda \in \Lambda$. (C^i_λ) is a feasible allocation by definition. We must show that C^i_λ maximizes utility over agent i's budget set. So, assume that $\langle \psi_\lambda, D \rangle \leq \langle \psi_\lambda, E^i \rangle$. Concavity of U^i, (1), (5) and the budget constraint imply:

$$U^i(D) - U^i(C^i_\lambda) \leq \langle \nabla U^i(C^i_\lambda), D - C^i_\lambda \rangle \leq \frac{1}{\lambda^i} \langle \psi_\lambda, D - C^i_\lambda \rangle = \frac{1}{\lambda^i} \langle \psi_\lambda, D - E^i \rangle \leq 0.$$

Therefore, C^i_λ is agent i's demand given ψ_λ, and the proof is complete. □

3.2. Continuity of the excess spending map

The topological dual $\mathcal{X}^*_{\mathrm{HHK}}$ consists of semimartingales $\psi = M + A$, where M is a bounded martingale and A is an absolutely continuous process whose derivative A' is bounded a.e., see Hindy and Huang (1992). In general, this space is not a lattice, that is, it is not closed with respect to taking pointwise maxima as the following example shows.

Example 7. Let M be a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t, 0 \leq t \leq T), \mathbb{P})$ and suppose that $(\mathcal{F}_t, 0 \leq t \leq T)$ is the augmented filtration generated by M.[4] The topological dual space $\mathcal{X}^*_{\mathrm{HHK}}$ consists of all semimartingales $S = N + A$, where N

[4] For more details on this model, local time and the Tanaka formula, see, e.g., Section 3.6 in Karatzas and Shreve (1991).

is a bounded martingale and A an absolutely continuous process whose derivative A' it is bounded, see Hindy and Huang (1992). Set $\tau \triangleq \inf\{t \geq 0 : |M_t| = 1\}$ and $\psi_t \triangleq M_{\tau \wedge t}$, Brownian motion stopped at 1 or −1. As Brownian motion is a martingale, $\psi \in \mathcal{X}^*_{\mathrm{HHK}}$, and, of course, $0 \in \mathcal{X}^*_{\mathrm{HHK}}$. By the Tanaka formula, $\max\{\psi, 0\} = N + L$, where N is a bounded martingale, and L is Brownian local time stopped at τ. One of the remarkable features of diffusions is that their local time, while being increasing and continuous, is not absolutely continuous with respect to Lebesgue measure because local time is flat off the Lebesgue null set $\{t \leq T : M_t = 0\}$. Therefore, $\max\{\psi, 0\} \notin \mathcal{X}^*_{\mathrm{HHK}}$ —the topological dual is not a lattice.

Failure of the lattice property implies that the map $E \mapsto \langle \psi_\lambda, E\rangle$ will not be continuous in E on the whole consumption set $\mathcal{X}_+$. Fortunately, we need continuity only on the order interval $[0, \bar{E}]$, a much smaller set. As the next lemma shows, we do have continuity there.

Lemma 8. *For $\lambda \in \Lambda$, the mapping $E \mapsto \langle \psi_\lambda, E\rangle$ is continuous on $[0, \bar{E}]$.*

Proof. Let $\lambda \in \Lambda$ be given and assume that $\|E_n - E\|_{\mathrm{HHK}} \to 0$ in $[0, \bar{E}]$. Denote by $a_n \triangleq \langle \psi_\lambda, E_n\rangle$. By Lemma 5, (1), we have $0 \leq a_n \leq \langle \psi_\lambda, \bar{E}\rangle = \sum_i \lambda^i \langle \nabla U^i(C^i_\lambda), C^i_\lambda\rangle$. Since $\nabla U^i(C^i_\lambda)$ is a positive linear functional, and by Assumption 2, 2 (b), we have $\langle \nabla U^i(C^i_\lambda), C^i_\lambda\rangle \leq \langle \nabla U^i(C^i_\lambda), \bar{E}\rangle \leq B\|\bar{E}\|_i$. Thus, the sequence (a_n) is bounded, and has, therefore, limit points. Let a be such a limit point, that is, $a = \lim_k \langle \psi_\lambda, E_{n_k}\rangle$ for some subsequence (n_k). We must show that $a = \langle \psi_\lambda, E\rangle$. This is not clear, at first, since we do not know whether ψ_λ belongs to the HHK—dual or not. However, by the same reasoning as above, $\langle \psi_\lambda, \bar{E}\rangle \leq B \sum_i \lambda^i \|\bar{E}\|_i < \infty$. Therefore, ψ_λ belongs to $L^1(\mathbb{P} \otimes \mathrm{d}\bar{E})$, and is thus a continuous linear functional on the space $L^\infty(\mathbb{P} \otimes \mathrm{d}\bar{E})$.

The order interval $[0, \bar{E}]$ is compact in the weak*–topology on $L^\infty(\mathbb{P} \otimes \mathrm{d}\bar{E})$. Hence, we may assume without loss of generality that $\lim_k \langle \psi_\lambda, E_{n_k}\rangle = \langle \psi_\lambda, F\rangle$ for some $F \in [0, \bar{E}]$. It only remains to be shown that $F = E$. This follows from the fact that the HHK dual $\mathcal{X}^*_{\mathrm{HHK}} \subset L^1(\mathbb{P} \otimes \mathrm{d}\bar{E})$ because $\mathcal{X}^*_{\mathrm{HHK}}$ consists of bounded semimartingales. For every $\psi \in \mathcal{X}^*_{\mathrm{HHK}}$, we have therefore $\langle \psi, E\rangle = \lim \langle \psi, E_{n_k}\rangle = \langle \psi, F\rangle$, which implies $E = F$. □

For later use, we record the following continuity results from Shannon/Zame which do not use their critical assumption A2 (strict positivity of aggregate endowment) and A5 (price space is a lattice).

Lemma 9. (Shannon/Zame)

(1) *The mapping $\lambda \mapsto C^i_\lambda$ is locally Lipschitz continuous on Λ with respect to $\|\cdot\|_i$ and continuous with respect to $\|\cdot\|_{\mathrm{HHK}}$.*
(2) *the mappings $\lambda \mapsto (\langle \psi_\lambda, C^i_\lambda\rangle)$ is locally Lipschitz on Λ,*
(3) *For $E \in [0, \bar{E}]$, the mapping $\lambda \mapsto (\langle \psi_\lambda, E\rangle)$ is locally Lipschitz on Λ, uniformly in E.*

Proof. 1. is Lemma 5.1., 2. is Lemma 6.1., (i) and 3. is Lemma 6.1., (ii) in Shannon and Zame (2002). □

Corollary 10. *The mapping $(\lambda, E) \mapsto (\langle \psi_\lambda, E\rangle)$ is jointly continuous on $\Lambda \times [0, \bar{E}]$.*

Proof. Let $\lambda_n \to \lambda$ in Λ and $\|E_n - E\|_{\mathrm{HHK}} \to 0$ in $[0, \bar{E}]$. Set $\psi_n \overset{\Delta}{=} \psi_{\lambda_n}$. By Lemma 8, we know $|\langle \psi_\lambda, E_n - E\rangle| \to 0$. Lemma 9 yields a constant $L > 0$, independent of (E_n) and E, such that $|\langle \psi_n - \psi, E_n\rangle| \le L\|\lambda_n - \lambda\| \to 0$. Altogether, we obtain

$$|\langle \psi_n, E_n\rangle - \langle \psi, E\rangle| \le |\langle \psi_n - \psi, E_n\rangle| + |\langle \psi, E_n - E\rangle| \to 0. \quad \square$$

3.3. Proof of the Theorem

Proof. As a preliminary, we need to show that the sets $\mathcal{D}_f \overset{\Delta}{=} \{(E^i) \in \mathcal{D} : (E^i)\text{ has finitely many equilibria}\}$ and $\mathcal{D}_c \overset{\Delta}{=} \{(E^i) \in \mathcal{D} : \mathcal{E}\text{ is continuous at }(E^i)\}$ are Borel sets. The argument is as in Shannon and Zame (2002), p. 650. Note that joint continuity of the mapping $(\lambda, (E^i)) \mapsto \langle \Psi_\lambda, C^i_\lambda - E^i\rangle$ is needed for that. This follows from Lemma 9 and Corollary 10.

Introduce the $(m-1)$-dimensional subspace

$$\mathcal{V} \overset{\Delta}{=} \left\{(\alpha^i \bar{E})_{i=1,\dots,m} : \alpha^i \in \mathbb{R}, \sum \alpha^i = 0\right\} \subset \mathcal{X}^m$$

and the mapping

$$\begin{aligned} &\iota : \mathcal{V} \to \mathbb{R}^{m-1} \\ &(\alpha^i \bar{E})_{i=1,\dots,m} \mapsto (\alpha^i)_{i=1,\dots,m-1}. \end{aligned}$$

ι is an isomorphism between $\mathcal{V}$ and $\mathbb{R}^{m-1}$.

Set $F^*_i = \frac{1}{m}\bar{E}, i = 1, \dots, m$. The set $\mathcal{V} \cap (\mathcal{D} - F^*) = \{(\alpha^i \bar{E}) : \sum \alpha^i = 0, \alpha_i > -\frac{1}{m}\}$ has positive Lebesgue measure in $\mathcal{V}$ since

$$\iota(\mathcal{V} \cap (\mathcal{D} - F^*)) = \left\{\alpha \in \mathbb{R}^{m-1} : \text{ for all } i\ \alpha^i > -\frac{1}{m} \text{ and } \sum_{i=1}^{m-1} \alpha^i < \frac{1}{m}\right\}.$$

In order to establish $\mathcal{V}$ as a probe, it remains to be shown that for every $F \in \mathcal{X}^m$ the sets $\mathcal{V} \cap (\mathcal{D}\backslash\mathcal{D}_f + F)$ and $\mathcal{V} \cap (\mathcal{D}\backslash\mathcal{D}_c + F)$ have Lebesgue measure zero in $\mathcal{V}$.

Introduce the compact simplex $\Delta \overset{\Delta}{=} \{\lambda \in \mathbb{R}^{m-1} : \lambda^i \ge 0, \sum_{i=1}^{m-1} \lambda^i \le 1\}$ in $\mathbb{R}^{m-1}$ and the function

$$\begin{aligned} &\sigma : \Delta \to \mathbb{R}^{m-1} \\ &\lambda \mapsto \left(\frac{\langle \psi_\lambda, C^i_\lambda + F_i\rangle}{\langle \psi_\lambda, \bar{E}\rangle}\right)_{i=1,\dots,m-1}. \end{aligned}$$

Note that we slightly abuse notation here because we identify the $m-1$-dimensional vector $\lambda \in \Delta$ with the m-dimensional vector $(\lambda_1, \dots, \lambda_{m-1}, 1 - \sum_{i=1}^{m-1} \lambda^i)$. σ is well-defined because $\langle \psi_\lambda, \bar{E}\rangle \ge \lambda^1 \langle \nabla U^1(C^1_\lambda), \bar{E}\rangle > 0$, since $\nabla U^1(C^1_\lambda) \in \Psi$ by Assumption 2. By Lemma

9, σ is locally Lipschitz continuous on the interior of Δ. Sard's theorem for Lipschitz functions yields that almost every $\alpha \in \mathbb{R}^{m-1}$ is a regular value for σ. From Lemma 6, we know that there are no λ in the boundary of Δ with $\sigma(\lambda) = \alpha$. We can thus apply Corollary 2 in Shannon (1994) to obtain that for every regular value α, the number of $\lambda \in \Delta$ with $\sigma(\lambda) = \alpha$ is finite.

Now fix some $F \in \mathcal{X}^m$ and set $\mathcal{W} \stackrel{\Delta}{=} \mathcal{V} \cap (\mathcal{D}\backslash\mathcal{D}_f + F)$. We have to show that $\mathcal{W}$ has Lebesgue measure zero in $\mathcal{V}$. For $\mathcal{W} = \emptyset$, there is nothing to show, so assume $\mathcal{W} \neq \emptyset$. Note that the set $\mathcal{W}$ can be identified with

$$\iota(\mathcal{W}) = \{\alpha \in \mathbb{R}^{m-1} : \text{ there exists } (E^i) \in \mathcal{D}\backslash\mathcal{D}_f \text{ such that } \alpha^i \overline{E} - F^i = E^i\}.$$

Hence, we identify in the following values $\alpha \in i(\mathcal{W})$ with the corresponding economy (A^i) given by $A_i \stackrel{\Delta}{=} \alpha^i \overline{E} - F^i$ for $i = 1, \ldots, m-1$ and $A_m =^{\Delta} \overline{E} - \sum_{i=1}^{m-1} A_i$. Lemma 6 states that the number of equilibria of the economy (A^i) is equal to the number of $\lambda \in \Delta$ with $\sigma(\lambda) = \alpha$. But this is a finite number for almost every $\alpha \in \mathbb{R}^{m-1}$, as we have seen above, and we conclude that $\iota(\mathcal{W})$ is a Lebesgue null set.

In order to show that $\mathcal{W}_c \stackrel{\Delta}{=} \mathcal{V} \cap (\mathcal{D}\backslash\mathcal{D}_c + F)$ has Lebesgue measure zero in $\mathcal{V}$, it suffices to prove that the equilibrium allocation correspondence is continuous at (A^i) (with $A_i \stackrel{\Delta}{=} \alpha^i \overline{E} - F^i$ for $i = 1, \ldots, m-1$ and $A_m \stackrel{\Delta}{=} \overline{E} - \sum_{i=1}^{m-1} A_i$ as above) for regular values α of σ. In view of Lemma 6, it is enough to do so for the correspondence

$$\mathbb{E} : \mathcal{D} \to \Delta \quad (E^i) \mapsto \{\lambda \in \Delta : (C^i_\lambda) \text{ is an equilibrium allocation for } (E^i)\}.$$

For upper hemicontinuity, take a sequence $(E^i_n) \subset \mathcal{D}$ with $E^i_n \to A^i$ for all i. Choose $\lambda_n \in \mathbb{E}(E^i_n)$ and suppose that $\lambda_n \to \lambda$. We have to show that $\lambda \in \mathbb{E}((A^i))$. To this end, it suffices to show that $\langle \psi_\lambda, C^i_\lambda + F_i - \alpha^i \overline{E}\rangle = 0$. This follows from Lemma 9 and Corollary 10.

Finally, turn to lower hemicontinuity of $\mathbb{E}$. Let α be a regular value of σ, and $\sigma(\lambda^*) = \alpha$. Fix $\epsilon > 0$. We have to find $\delta > 0$ such that the mapping $S(\lambda, (E^i)) =^{\Delta} \langle \psi_\lambda, C^i_\lambda - E^i\rangle$ has a zero in $B_\epsilon =^{\Delta} \{\lambda \in \Lambda : \|\lambda - \lambda^*\| < \epsilon\}$ for every $(E^i) \in \mathcal{D}$ with $\|E^i - A^i\|_{HHK} < \delta, i = 1, \ldots, m$. Since α is a regular value, we know from, e.g., Shannon (1994), Theorem 1, that λ^* is locally unique; choose $\eta \le \epsilon$ with $B_\eta \cap \{S(\cdot, (A^i)) = 0\} = \{\lambda^*\}$. As α is a regular value, the degree at zero $\deg(S(\cdot, (A^i)), B_\eta) \neq 0$ (Shannon (1994), Theorem 9). It is enough to show that this translates to $S(\cdot, (E^i))$ when (E^i) is close to (A^i). Let $\mu =^{\Delta} \min_{\|\lambda - \lambda^*\| = \eta} S(\lambda, (A^i))$. Set

$$T((E^i)) \stackrel{\Delta}{=} \max_{\lambda \in B_\eta} \|S(\lambda, (E^i)) - S(\lambda, (A^i))\| = \max_{\lambda \in B_\eta} \|\langle \psi_\lambda, A^i - E^i\rangle\|.$$

By the Maximum Theorem (Berge (1997)) and Corollary 10, T is continuous, and $T((A^i)) = 0$. Hence, there exists $\delta > 0$ such that $|T((E^i))| < \mu$ whenever $\|E^i - A^i\|_{HHK} < \delta$. Therefore, we have $\|S(\cdot, (E^i)) - S(\cdot, (A^i))\| < \mu = \min_{\|\lambda - \lambda^*\| = \eta} S(\lambda, (A^i))$. Therefore, the usual invariance property of the degree under small perturbations yields $\deg(S(\cdot, (E^i)), B_\eta) = \deg(S(\cdot, (A^i)), B_\eta) \neq 0$, which concludes the proof. $\square$

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Appendix A. Proof of Lemma 1

Proof. Let (C_n) be a Cauchy sequence in $\mathcal{X}_+$. The metric space $(\mathcal{X}_+, \|\cdot\|_{HHK})$ is a subspace of the metric space $(L^1(\Omega \times [0, T], \mathcal{O}, \mathbb{P} \otimes (\mathrm{d}t + \delta_T)), \|\cdot\|_{HHK})$, where $\mathcal{O}$ denotes the optional σ-field. Since L^1-spaces are complete, there exists an optional process $Z \in L^1(\Omega \times [0, T], \mathcal{O}, \mathbb{P} \otimes (\mathrm{d}t + \delta_T))$ with $\lim \|C_n - Z\|_{HHK} = 0$. By passing to a subsequence if necessary, we may assume without loss of generality that $C_n \to Z\, \mathbb{P} \otimes (\mathrm{d}t + \delta_T)$–a.e. This shows that Z is nonnegative, nondecreasing, and rightcontinuous a.e. By the usual conditions on the filtered probability space, there exists an optional, nonnegative, and nondecreasing process $\bar{Z}$ which satisfies $\bar{Z} = Z\, \mathbb{P} \otimes (\mathrm{d}t + \delta_T)$– a.e. Then $\lim \|C_n - \bar{Z}\|_{HHK} = 0$ and $\bar{Z} \in \mathcal{X}_+$, which shows completeness. □

Appendix B. On Hindy–Huang–Kreps utility functions

I prove here the claim made in Example 3. The utility function U is strictly increasing because so are u and v. Continuity of U with respect to $\|\cdot\|_{HHK}$ is shown in Bank and Riedel (2001a) (for $v = 0$, but it is straightforward to adapt the argument). It is easy to see that $\|\cdot\|_R$ is indeed a norm on $\mathcal{X}$, and that the topology coincides with the $\|\cdot\|_{HHK}$–topology on $\mathcal{X}_+$.

Therefore, I focus on the remaining conditions 2, (b)–(d) in Assumption 2. y^C is uniformly bounded away from zero since $\eta > 0$. Therefore, $u'(y_t^C)$ and $v'(y_T^C)$ are uniformly bounded by some constant B. Note that by partial integration $\langle \nabla U(C), X\rangle = \mathbb{E}\int_0^T e^{-\delta t} u'(y_t^C) z_t^X\, \mathrm{d}t + \mathbb{E} v'(y_T^C) z_T^X$. Therefore,

$$|\langle \nabla U(C), X\rangle| \le B\mathbb{E}\int_0^T |z_t^X|\, \mathrm{d}t + \mathbb{E}|z_T^X|,$$

which is 2(b).

Next, assume that aggregate endowment is bounded a.e. This implies, of course, that $z^{\bar{E}}$ is bounded a.e., say by $B > 0$. Lipschitz continuity of u' and v' yields a constant L such that:

$$\begin{aligned}|\langle \nabla U(C) - \nabla U(C'), C''\rangle| &\le L\mathbb{E}\int_0^T |y_t^C - y_t^{C'}| z_t^{\bar{E}}\, \mathrm{d}t + L, \mathbb{E}|y_T^C - y_T^{C'}| z_T^{\bar{E}} \\ &\le LB\|C - C'\|_R.\end{aligned}$$

u and v are quadratically concave since they are twice continuously differentiable and strictly concave (compare Shannon and Zame (2002)). Hence, there exists a constant $K > 0$ such that:

$$u(y) - u(z) \le u'(z)(y - z) - K(y - z)^2$$
$$v(y) - v(z) \le v'(z)(y - z) - K(y - z)^2$$

for all $y, z \in [0, B]$. This yields quadratic concavity, as the following inequalities show for $C, D \in [0, \bar{E}]$:

$$\begin{aligned} U(D) - U(C) &= \mathbb{E} \int_0^T \mathrm{e}^{-\delta t} \, (u(y_t^D) - u(y_t^C)) \, \mathrm{d}t + \mathbb{E}(v(y_T^D) - v(y_T^C)) \\ &\le \mathbb{E} \int_0^T \mathrm{e}^{-\delta t} \, u'(y_t^C)(y_t^D - y_t^C) \, \mathrm{d}t + \mathbb{E} v'(y_T^C)(y_T^D - y_T^C) \\ &- K\mathbb{E} \int_0^T \mathrm{e}^{-\delta t} (y_t^D - y_t^C)^2 \, \mathrm{d}t - K\mathbb{E}(y_T^D - y_T^C)^2 \\ &\le \langle \nabla U(C), D - C \rangle - K e^{-\delta T} \| D - C \|_R^2. \end{aligned}$$

The last line uses the Cauchy–Schwarz inequality.

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