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Economic Theory

Revealed preference and differentiable demand*

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Summary. We prove that, for finitely many demand observations, the Strong Axiom of Revealed Preference tests not only the existence of a strictly concave, strictly monotone and continuous utility generator, but also one that generates an infinitely differentiable demand function. Our results extend those of previous related results (Matzkin and Richter, 1991; Chiappori and Rochet, 1987), yielding differentiable demand functions but without requiring differentiable utility functions.

Keywords and Phrases: Nonparametric analysis, Revealed preference, Smooth demand.

JEL Classification Numbers: D11, D12.

1 Introduction

Economists – both applied and theoretical – often assume that a consumer has a differentiable (or even infinitely differentiable) demand function. What restrictions does this assumption impose on a finite set of observed consumption-purchase data? In this paper, we provide a complete answer: We need the Strong Axiom of Revealed Preference (SARP) and nothing more.[1]

One might think that the question of characterizing observations from a differentiable demand has been settled by previous work (e.g. Chiappori and Rochet, 1987;

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[1] This paper focuses only on the case in which there is a finite number of observations from a demand function. This finite approach has been developed in a series of papers by Afriat (1967), Diewert (1973), Varian (1982), Chiappori and Rochet (1987), Matzkin and Richter (1991), and others. For an infinite set of data, the SARP alone is not sufficient to yield a differentiable demand function.

Matzkin and Richter, 1991); but this is not true. In fact, Matzkin and Richter (1991) showed that the SARP ensures the existence of a strictly concave, strictly monotone and continuous utility function that generates a not-necessarily-differentiable demand function. However, in general, their demand function is not differentiable, but generically differentiable (see Remark 2). Chiappori and Rochet (1987) obtained an infinitely differentiable demand function for the special case, where they strengthened the SARP by adding the invertibility requirement for the demand data. (See Remark 3.) Our theorem sharpens these results, showing that the SARP alone is sufficient to ensure the existence of a utility function whose demand function is differentiable everywhere (not just generically differentiable), and we do not need any additional assumption, such as invertibility, on the data.[2]

In our proof of the theorem, we combine and extend the methods of Matzkin and Richter (1991) and Chiappori and Rochet (1987). In particular, for a given set of demand-data we start with a preference generator constructed by Matzkin and Richter (1991). This preference need not generate a differentiable demand and the hicksian functions of its indifference sets need not be differentiable. We use the method of Chiappori and Rochet (1987) to smooth these hicksian functions, and the images of these new functions form new indifference sets. These new indifference sets form a preference that generates an infinitely differentiable (Marshallian) demand function and rationalizes the data set.

2 Main result

The commodity space is $\mathbb{R}^l_+$, and the price space is $\mathbb{R}^l_{++}$, and the price-income space is $\mathbb{R}^l_{++} \times \mathbb{R}_{++}$. [3] A *budget set* is a set $B(p, w) = \{x \in \mathbb{R}^l_+ : p \cdot x \leq w\}$, where $(p, w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$.

A *utility function* is a function $U : \mathbb{R}^l_+ \to \mathbb{R}$. For any function $h : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to \mathbb{R}^l_+$, we say h is *generated (rationalized)* by a utility function $U(x)$ if for every $(p, w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$, the bundle $h(p, w)$ uniquely maximizes $U(x)$ over the budget set $B(p, w)$.

An observed *(demand) data set* is a (finite) set of price-consumption vectors, i.e. a set $D = \{(p^i, x^i)\}_{i=1}^T$, where $(p^i, x^i) \in \mathbb{R}^l_{++} \times (\mathbb{R}^l_+ \backslash \{0\})$, and $1 \leq T < \infty$. We say the data set D is *rational* if there is a pair (U, h) of utility function $U : \mathbb{R}^l_+ \to \mathbb{R}$ and demand function $h : \mathbb{R}^l_{++} \times \mathbb{R}_{++} \to \mathbb{R}^l_+$ such that U generates h, and h is

[2] With invertibility, Chiappori and Rochet (1987) also obtained an infinitely differentiable utility function. Without invertibility, our utility function need not be infinitely differentiable. See Figure 1 below.

[3] For all $x, y \in \mathbb{R}^l$, "$x \gg y$" means "$x_i > y_i$ for all i"; and "$x \geq y$" means "$x_i \geq y_i$ for all i", and "$x \gneqq y$" means "$x \geq y$ and $x \neq y$." We denote by $\mathbb{R}^l_+ = \{x \in \mathbb{R}^l : x \geq 0\}$, and $\mathbb{R}^l_{++} = \{x \in \mathbb{R}^l : x \gg 0\}$. The topology of $\mathbb{R}^l$ is induced by the Euclidean norm $\|\cdot\|$. We denote by e^i the i-th unit vector, i.e. its i-th coordinate $e^i_i = 1$, and all other j-th coordinates $e^i_j = 0$. We denote by $\triangle = \{x \in \mathbb{R}^l_+ : x_1 + \cdots + x_l = 1\}$, and $\text{int}(\triangle) = \{x \in \triangle : x \gg 0\}$. For any number $\epsilon > 0$, and any $x \in \mathbb{R}^l$, we denote by $\text{ball}(x, \epsilon) = \{y \in \mathbb{R}^l | \|y - x\| < \epsilon\}$. For any non-zero $x \in \mathbb{R}^l_+$, we denote by $\pi(x) = x / \sum_{i=1}^l x_i$. For any $x^1, \cdots, x^m \in \mathbb{R}^l$, we denote by $\text{co}\{x^1, \cdots, x^m\}$ the convex hull spanned by $x^1, \cdots, x^m$.

consistent with D (i.e. $x^i = h(p^i, w^i)$ for all $i = 1, \cdots, T$, where $w^i = p^i \cdot x^i$). Then, we also say (U, h) is a *rationalization* of D, and (U, h) *rationalizes* D.

We will obtain such a pair (U, h) that satisfies:

ID) (Infinite Differentiability) h is C^∞ (i.e. infinitely differentiable);
SP) (Special Property) U is continuous, strictly monotone,[4] and strictly concave.

(Note that even if h is C^∞, the function U need not be differentiable, e.g. consider the utility function of the Leontief preference.)

To characterize rationality with (ID) and (SP), we will use the standard Strong Axiom of Revealed Preference (see Houthakker, 1950; Richter, 1966). As usual, for a data set $D = \{(p^i, x^i)\}_{i=1}^T$, we say a bundle x is *(directly) revealed preferred* to a bundle y, if $x \neq y$ and there is an i such that $x^i = x$ and $y \in B(p^i, w^i)$, where $w^i = p^i \cdot x^i$; then we write xSy. We say D satisfies the *Strong Axiom of Revealed Preference (SARP)* if the S relation is acyclic (i.e. it is impossible to have any finite sequence $y_1 \, S \, y_2 \, S \, \cdots \, S \, y_1$).

We now give our main result.

Theorem 1 (Rational-smooth Demand) *For any finite data set* $D = \{(p^i, x^i)\}_{i=1}^T$, *the following (a), (b), and (c) are equivalent:*

a) *D satisfies SARP;*
b) *D has a rationalization (U, h);*
c) *D has a rationalization (U, h) satisfying (ID) and (SP).*

Remark 1. That (c) implies (b) is clear and that (b) implies (a) is well-known (see Richter, 1966). Hence, our main contribution is to show that (a) implies (c).

Remark 2. Matzkin and Richter (1991, Theorem 2) proved that for a data set D,

$$D \text{ satisfies SARP} \;\Rightarrow\; D \text{ has a rationalization } (U, h) \text{ satisfying (SP).}$$

We extend this by adding the (ID) conclusion for the demand function. In fact, their method generally fails to give a differentiable demand function; their utility construction generally yields a kinked utility function. Although a kinked utility function does not necessarily give a non-differentiable demand function, their kinked utility functions often give non-differentiable demand functions, shown by simple situations as given in Figure 1.

Remark 3. Chiappori and Rochet (1987) were interested in obtaining both a C^∞ utility function and a C^∞ demand function, so they needed to strengthen the SARP into a Strong Strong Axiom of Revealed Preference (SSARP), which requires that the data D satisfies the SARP and the assumption that D is invertible (i.e. $(\forall \lambda \in \mathbb{R}_{++})[p^i \neq \lambda p^j] \;\Rightarrow\; x^i \neq x^j$). The invertibility condition rules out choices in

[4] As usual, a function $U(x)$ is *strictly monotone* if: $x \geq y$ implies $U(x) \geq U(y)$, and $x \gneqq y$ implies $U(x) > U(y)$; $U(x)$ is *monotone* if $x \geq y$ implies $U(x) \geq U(y)$, and $x \gg y$ implies $U(x) > U(y)$.

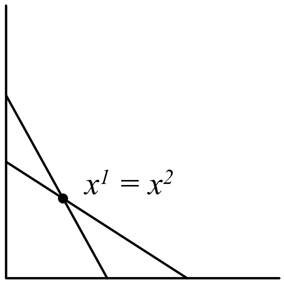

Figure 1. A data set violating invertibility

which two budget lines cross each other and have the same chosen bundles. (See Fig. 1.) They essentially showed that (1987, Theorem and Corollary):[5]

$$D \text{ satisfies SSARP } \Rightarrow D \text{ has a rationalization } (U, h) \text{ satisfying (SP) and (ID), and such that } U \text{ is } C^\infty.$$

Our Theorem 1 shows that if we drop the C^∞ conclusion just on U, then we can drop the invertibility assumption on the data.

Remark 4. We can prove that the SARP ensures the rationalization with (SP) and (ID) because the data set is finite and we have a lot of freedom in choosing a utility function and an infinitely differentiable demand function. With infinitely many data points, this situation fails. It is still true that h satisfies the SARP if and only if it is generated by a preference (see Richter, 1966). However, the SARP on h does not ensure that a preference generator must satisfy (SP); indeed h need not have a utility function (see Richter, 1971, p. 46, Example 2). In addition, the SARP on h does not ensure that h must be differentiable, or even continuous (see Richter, 1971, p. 47, Example 3).

To illustrate the main ideas of our proof, we will now provide a proof sketch. A detailed proof is given in the next section.

We will make use of the following notion. For any non-empty set $I \subseteq \mathbb{R}^l_+$, we say a function $\zeta : \mathbb{R}^l_{++} \to \mathbb{R}^l_+$ is the *hicksian function* for I if ζ is continuous, and such that for every $p \in \mathbb{R}^l_{++}$, the vector $\zeta(p) \in I$ and $\zeta(p)$ uniquely minimizes $p \cdot x$ over all $x \in I$.[6] (Clearly, a hicksian function is homogeneous of degree 0.)

Proof Sketch for Theorem 1. By Remark 1, it clearly suffices to prove that (a) implies (c).

We consider a data set $D = \{(p^i, x^i)\}_{i=1}^T$ that satisfies the SARP. We choose a utility function that rationalizes the data set, from which we choose (compact) indifference sets $I^1, \cdots, I^k$ containing $x^1, \cdots .x^T$. Then, we modify these $I^1, \cdots, I^k$ into new (compact) indifferent sets $\tilde{I}^1, \cdots, \tilde{I}^k$, which still contain $x^1, \cdots, x^T$.

[5] The utility function U obtained by Chiappori and Rochet (1987, Theorem 1) is defined only on a compact set $A \subseteq \mathbb{R}^l_{++}$, not all of the consumption space $\mathbb{R}^l_+$.

[6] If I is an indifference set with utility level u of some strictly monotone utility function U, then the hicksian function of I is precisely the hicksian demand function at that utility level u.

Moreover, their hicksian functions $\tilde{\zeta}^1, \cdots, \tilde{\zeta}^k$ are infinitely differentiable[7] (and satisfy the (CP) property (defined below)). Each $\tilde{I}^j$ is the image of its hicksian function $\tilde{\zeta}^j$. We extend $\tilde{\zeta}^1, \cdots, \tilde{\zeta}^k$ into a family $\{\tilde{\zeta}^u\}_{u\in\mathbb{R}_+}$ of hicksian functions $\tilde{\zeta}^u$. The family $\{\tilde{I}^u\}_{u\in\mathbb{R}_+}$ of the images $\tilde{I}^u$ of $\tilde{\zeta}^u$ forms the indifference sets of the new preference. This preference gives a rationalization satisfying (SP) and (ID). □

3 Proof of Theorem 1

We require two main tools in our proof.

First, we need to smooth hicksian functions. Our approach is to obtain smooth "expenditure" functions, the derivatives of which are the desired smooth hicksian functions. To obtain these smooth "expenditure" functions, we will extensively use the smoothing method through convolution that was introduced by Chiappori and Rochet (1987). This method transforms a piecewise linear function into an infinitely differentiable function. It requires the standard "pole" function $\rho : \mathbb{R}^l \to \mathbb{R}$ defined by

$$\rho(x) = \begin{cases} \exp\left(\frac{1}{\|x\|^2-1}\right) \Big/ \int_{\mathbb{R}^l} \exp\left(\frac{1}{\|x\|^2-1}\right) dx & \text{if } \|x\| \leq 1 \\ 0 \quad \text{otherwise.} \end{cases} \tag{1}$$

In particular, we will choose a small number $\eta > 0$, and use the function $\rho_\eta : \mathbb{R}^l \to \mathbb{R}$ defined by

$$\rho_\eta(x) = \frac{1}{\eta}\rho\left(\frac{x}{\eta}\right). \tag{2}$$

Fact 1 (Chiappori and Rochet, 1987)[8] *Let functions* $W^1, \cdots, W^k : \mathbb{R}^l \to \mathbb{R}$ *be linear, and function* $W : \mathbb{R}^l \to \mathbb{R}$ *be defined by* $W(x) = \min_{i=1,\cdots,k} W^i(x)$*. Let number* $\eta > 0$*. Then the function* $\tilde{W} : \mathbb{R}^l \to \mathbb{R}$ *defined by* $\tilde{W}(y) = \int_{\mathbb{R}^l} W(y - x)\rho_\eta(x)dx$ *is infinitely differentiable, and such that for all* $y \in \mathbb{R}^l$*, if* W *is linear on* $\text{ball}(y, \eta)$*, then* $\tilde{W}(y) = W(y)$*. Moreover, if* W *is concave, then* $\tilde{W}$ *is concave; if* W *is monotone, then* $\tilde{W}$ *is monotone.*

We then need to construct compact indifference sets. Our approach is to obtain a corner property for a hicksian function, and a related property for an "expenditure" function. We say a function $\zeta : \mathbb{R}^l_{++} \to \mathbb{R}^l_+\backslash\{0\}$ satisfies the *corner property* (CP) if:

$$\text{for every } \bar{p} \in \mathbb{R}^l_{++} \text{ and every set } J \subseteq \{1, \cdots, l\}, \text{ there is a number } N > 0 \text{ such that if } p \in G(\bar{p}, N, J), \text{ then } (\zeta(p))_j = 0 \text{ for all } j \in J, \tag{CP}$$

7 Note that the $\tilde{I}^1, \cdots, \tilde{I}^k$ sets need not be smooth, as the data set may have "kinked" points as illustrated in Figure 1.

8 For a proof for this fact, see Chiappori and Rochet (1987, p. 690).

where $G(\bar{p}, N, J) = \{p \in \mathbb{R}^l_{++} : (\forall j \notin J)[p_j = \bar{p}_j] \ \& \ (\forall j \in J)[p_j > N]\}$. We say a function $E : \mathbb{R}^l_{++} \to \mathbb{R}_{++}$ is *eventually constant* if

$$\begin{array}{c}\text{for every } \bar{p} \in \mathbb{R}^l_{++} \text{ and every set } J \subseteq \{1, \cdots, l\}, \text{ there is a number } N > 0 \\ \text{and a constant } C \text{ such that if } p \in G(\bar{p}, N, J), \text{ then } E(p) = C.\end{array} \tag{3}$$

Clearly, if E is homogeneous of degree 1 and $\partial_p E(p) = \zeta(p)$ for all $p \in \mathbb{R}^l_{++}$, then: E is eventually constant if and only if ζ satisfies (CP).

Intuitively, if one views ζ as a hicksian demand function, then (CP) means that if p_j is sufficiently large (where $j \in J$), then the demand $(\zeta(p))_j$ for the j-th commodity is zero; a similar case is true for the intuition of eventual constancy. Technically, they ensure that the image set $\zeta(\mathbb{R}^l_{++})$ is compact even when the domain $\mathbb{R}^l_{++}$ is an open set; and they also allow us to obtain a continuous, strictly monotone, and strictly convex preference.[9]

Lemma 1 *Let function $\zeta : \mathbb{R}^l_{++} \to \mathbb{R}^l_{+} \backslash \{0\}$ be homogeneous of degree 0, and differentiable. Let the function $E(p) = p \cdot \zeta(p)$ be concave. Let set $I = \cup_{p \in \mathbb{R}^l_{++}} \zeta(p)$, and set $W = \cap_{p \in \mathbb{R}^l_{++}} W_p$, where $W_p = \{x \in \mathbb{R}^l_{+} : \ p \cdot x \geq E(p)\}$. If ζ satisfies (CP),[10] then the set I is compact, and ζ is the hicksian function of W and I, and the sets I and W satisfy:*

$$\begin{array}{l} a)\ I = \mathrm{bdy}(W)\,, \\ b)\ \text{if } x \in I, \text{ and } y \gneqq x, \text{ then } y \in \mathrm{int}(W)\,, \\ c)\ \text{if } x \in I,\ y \in W,\ x \neq y,\ 0 < \lambda < 1, \text{ then } \lambda x + (1-\lambda) y \in \mathrm{int}(W), \end{array} \tag{4}$$

where $\mathrm{bdy}(W)$ and $\mathrm{int}(W)$ denote the boundary and interior of W with respect to the subspace topology of $\mathbb{R}^l_{+}$.[11]

Proof of Theorem 1. By Remark 1, it suffices to prove that (a) implies (c). We consider a data set $D = \{(p^i, x^i)\}_{i=1}^T$ that satisfies the SARP.

(Stage 1: Obtaining hicksian functions $\zeta^1, \cdots, \zeta^k$ and indifferent sets $I^1, \cdots, I^k$.) By Matzkin and Richter (1991, proof of Lemma 2), there is a strictly concave, strictly monotone, and continuous utility function $U : \mathbb{R}^l \to \mathbb{R}$ such that for all $i = 1, \cdots, T$, and $y \in \mathbb{R}^l$ (not just $y \in \mathbb{R}^l_{+}$): if $y \neq x^i$ and $p^i \cdot y \leq p^i \cdot x^i$, then $U(x^i) > U(y)$.

We choose utility values $u^1 < \cdots < u^k$, and indifference sets $I^1, \cdots, I^k$ that contain all of the chosen bundles $x^1, \cdots, x^T$, where each $I^j = \{x \in \mathbb{R}^l_{+} : U(x) = u^j\}$. Clearly, if $j > j'$, then indifferent set I^j lies strictly above $I^{j'}$, i.e. for all $x \in \triangle$, we have $s^j(x) > s^{j'}(x)$, where $s^j(x)$ and $s^{j'}(x)$ are the unique

[9] We can avoid special assumptions used in several existing methods. For example, we no longer require the expenditure function to be defined only on a compact set of $p \in \mathbb{R}^l_{++}$ (required by Blackorby and Diewert, 1979, Theorem 1). We also no longer require the expenditure function to have a continuous extension to all $p \in \mathbb{R}^l_{+} \backslash \{0\}$ (required by Blackorby et al., 1978, Theorem A.4).

[10] Equivalently, if E is eventually constant.

[11] That is, $\mathrm{bdy}(W)$ is the set of all $x \in W$ such that for all $\epsilon > 0$, there is an $y \in \mathbb{R}^l_{+} \backslash W$ such that $\|x - y\| < \epsilon$. Also, $\mathrm{int}(W) = W \backslash \mathrm{bdy}(W)$.

positive numbers such that $s^j(x)x \in I^j$ and $s^{j'}(x)x \in I^{j'}$ respectively.[12] Hence, for all $i = 1, \cdots, T$, we have:

$$x^i = \zeta^j(p^i) \qquad \text{where } j \text{ is the index with } x^i \in I^j. \tag{5}$$

Because U is a continuous and strictly monotone function defined on all of $\mathbb{R}^l$, each I^j is compact. Moreover, it is clear that each I^j has a well-defined hicksian function ζ^j and $I^j = \cup_{p \in \mathbb{R}^l_{++}} \zeta^j(p)$ (i.e ζ^j maps $\mathbb{R}^l_{++}$ onto I^j).

(Stage 2: Obtaining smooth "expenditure" functions $\tilde{E}^1, \cdots, \tilde{E}^k$, smooth hicksian functions $\tilde{\zeta}^1, \cdots, \tilde{\zeta}^k$, and indifference sets $\tilde{I}^1, \cdots, \tilde{I}^k$.) The hicksian functions $\zeta^1, \cdots, \zeta^k$ need not be differentiable. We will modify them into infinitely differentiable functions $\tilde{\zeta}^1, \cdots, \tilde{\zeta}^k : \mathbb{R}^l_{++} \to \mathbb{R}^l_+ \backslash \{0\}$ that satisfies (CP), and such that $\zeta^j(p^i) = \tilde{\zeta}^j(p^i)$ for all $i = 1, \cdots, T$ and all $j = 1, \cdots, k$.

We consider any ζ^j and take the following steps:

(Step 2.1) We choose a finite set $F^j \subseteq \mathbb{R}^l_{++}$ such that

$$\text{for every } i = 1, \cdots, T, \text{ there is a number } t > 0 \text{ with } tp^i \in F^j. \tag{6}$$

Let $F^j = \{q^1, \cdots, q^m\}$, and we choose the image set $K^j = \{y^1, \cdots, y^m\}$, where vectors $y^i = \zeta^j(q^i)$. We also choose the "corner" vectors $z^1, \cdots, z^l \in I^j$, whose coordinates $(z^i)_i > 0$ and $(z^i)_{i'} = 0$ for all $i' \neq i$. Because ζ^j is an onto function, by expanding F^j and K^j if necessary, we can assume that $z^1, \cdots, z^l \in K^j$. Finally, by replacing each q^i by some scalar multiplication of q^i if necessary, we can assume that $q^i \cdot y^i = 1$ for all $i = 1, \cdots, m$.

(Step 2.2) We choose the continuous, monotone, and concave function $E^j : \mathbb{R}^l \to \mathbb{R}$ by

$$E^j(p) = \min_{y \in K^j} p \cdot y. \tag{7}$$

(See Fig. 2.) The function E^j is piecewise linear, and is not infinitely differentiable. We choose a small number $\eta > 0$, and define the function $\hat{E}^j : \mathbb{R}^l \to \mathbb{R}$ by

$$\hat{E}^j(p) = \int_{\mathbb{R}^l} E^j(p - x)\rho_\eta(x)dx. \tag{8}$$

The function $\hat{E}^j$ is infinitely differentiable, monotone, and concave, but is not necessarily homogenous of degree 1. We need to modify it (see Fig. 2).

Lemma 2 *If $\eta > 0$ is sufficiently small, then:*

a) *For each $p \in \mathbb{R}^l_{++}$, there is a unique number $\alpha^j(p) > 0$ such that $\hat{E}^j(\alpha^j(p)p) = 1$.*
b) *The function $\tilde{E}^j : \mathbb{R}^l_{++} \to \mathbb{R}_{++}$ defined by $\tilde{E}^j(p) = 1/\alpha^j(p)$ is infinitely differentiable, concave, homogeneous of degree 1, eventually constant, has derivative $\partial_p \tilde{E}^j(p) \gneq 0$ for all $p \in \mathbb{R}^l_{++}$, and such that for all $i = 1, \cdots, m$:*

$$\tilde{E}^j(q^i) = q^i \cdot y^i \qquad \textit{and} \qquad \partial_p \tilde{E}^j(q^i) = y^i\ . \tag{9}$$

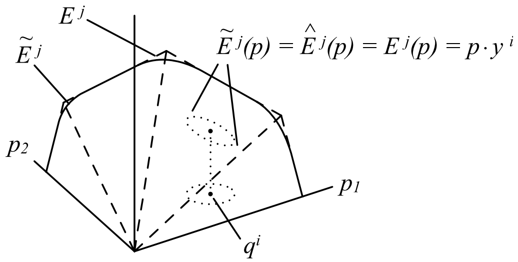

Figure 2. Functions E^j and $\tilde{E}^j$

(Step 2.3) We choose a $\eta > 0$ and the function $\tilde{E}^j$ as given in Lemma 2. We define the function $\tilde{\zeta}^j(p) = \partial_p \tilde{E}^j(p)$. Then $\tilde{\zeta}(p)$ is infinitely differentiable, satisfies (CP), and such that $\tilde{\zeta}^j(p^i) = \zeta^j(p^i)$ for all $i = 1, \cdots, T$ (by (6), (9), and 0-homogeneity of ζ^j and $\tilde{\zeta}^j$). Moreover, by Lemma 1, the function $\tilde{\zeta}^j$ is the hicksian function of its image set $\tilde{I}^j = \cup_{p \in \mathbb{R}^l_{++}} \tilde{\zeta}^j(p)$, and $\tilde{I}^j$ is compact. Furthermore, we can ensure that each $\tilde{I}^j$ is very close to I^j, as follows:

Claim 1 *a) For every $j = 1, \cdots, k$, and every $x \in \triangle$, there is a unique positive number $\tilde{s}^j(x)$ such that $\tilde{s}^j(x)x \in \tilde{I}^j$.*[13] *b) For any small number $\delta > 0$, the functions $\tilde{E}^1, \cdots, \tilde{E}^k$ can be chosen so that $\sup_{x \in \triangle} |s^j(x) - \tilde{s}^j(x)| < \delta$ for all $j = 1, \cdots, k$.*

Then, we can ensure that the indifference sets $\tilde{I}^j$ lies strictly above $\tilde{I}^{j'}$ when $j > j'$, i.e. $\tilde{s}^1(x) < \cdots < \tilde{s}^k(x)$ for all $x \in \triangle$ (as $s^1 < \cdots < s^k$, and s^j and $\tilde{s}^j$ are continuous on the compact domain $\triangle$). As $\tilde{I}^j$ are compact, it follows that:

$$\tilde{E}^1(p) < \tilde{E}^2(p) < \cdots < \tilde{E}^k(p) \qquad \text{for all } p \in \mathbb{R}^l_{++}. \tag{10}$$

(Stage 3: Obtaining a single expenditure function $\tilde{E}(p, u)$, a family $\{\tilde{\zeta}^u\}_{u \in \mathbb{R}_+}$ of hicksian functions, and a family $\{\tilde{I}^u\}_{u \in \mathbb{R}_+}$ of indifference sets.)

Claim 2 *If a number $\beta > 1$ is sufficiently close to 1, then $(1/\beta)\tilde{E}^{j+1}(p) > \beta \tilde{E}^j(p) > 0$ for all $p \in \mathbb{R}^l_{++}$ and all $j = 1, \cdots, k-1$.*

Then, we fix such a β as given in Claim 2. We choose a strictly positive number $\gamma < 1$, and choose the numbers $v^0, \cdots, v^{2k}$ where $v^0 = 0$, $v^1 = \gamma$, and $v^j = v^{j-1} + (\gamma)^j$, where $j = 2, \cdots, 2k$ (and as standard, $(\gamma)^j$ denotes the j-th power of γ). For each $p \in \mathbb{R}^l_{++}$, we define $E(p, u)$ over all $u \in \mathbb{R}$. First, we define (see Fig. 3):

$$\begin{aligned} E(p, v^0) &= 0 \\ E(p, v^{2j-1}) &= \frac{1}{\beta} \tilde{E}^j(p) \qquad j = 1, \cdots, k \\ E(p, v^{2j}) &= \beta \tilde{E}^j(p) \qquad j = 1, \cdots, k. \end{aligned} \tag{11}$$

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[12] Each $s^j : \triangle \to \mathbb{R}_{++}$ is continuous. This is because the the set I^j is compact (see below) and projection $\pi|_{I^j} : I^j \to \triangle$ (where $\pi(x) = x / \sum_{i=1}^l x_i$) is a continuous bijection, so its inverse $x \mapsto s^j(x)x$ is continuous, hence s^j is continuous.

[13] Similar to s^j (see Footnote 12), the function $\tilde{s}^j$ is continuous.

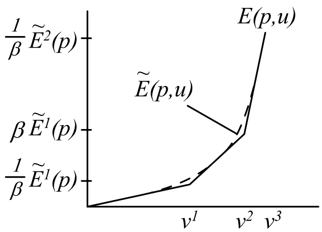

Figure 3. Functions $E(p,\cdot)$ and $\tilde{E}(p,\cdot)$

Then, we define $E(p,u)$ on all $u \in \mathbb{R}$ by choosing $E(p,u)$ to be the piecewise linear function, where the only kinked points are $v^1, \cdots, v^{2k-1}$. We choose a small $\mu > 0$, and for every $(p,u) \in \mathbb{R}^l_{++} \times \mathbb{R}$, we define

$$\tilde{E}(p,u) = \int_{\mathbb{R}} E(p,u-w)\rho_\mu(w)dw, \tag{12}$$

where $\rho_\mu(w)$ is defined by (1) and (2) for μ (with $\mathbb{R}^l = \mathbb{R}$).

Lemma 3 *. If a number $\beta > 1$ is sufficiently close to 1, then there are small numbers $\gamma, \mu > 0$ such that the function $\tilde{E} : \mathbb{R}^l_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ is infinitely differentiable, concave in p, homogeneous of degree 1 in p, convex in u, and $\partial_u \tilde{E}(p,u) > 0$ for all $(p,u) \in \mathbb{R}^l_{++} \times \mathbb{R}_+$, and $\tilde{E}(\cdot,u)$ is eventually constant for all u, and $\partial_p \tilde{E}(p,u) \gneq 0$ for all $(p,u) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}$, and there are $0 < \tilde{u}^1 < \cdots < \tilde{u}^k$ such that $\tilde{E}(\cdot,\tilde{u}^j) = \tilde{E}^j(\cdot)$ for all $j = 1,\cdots,k$.*

We choose the family $\{\tilde{\zeta}^u\}_{u\in\mathbb{R}_+}$ of functions $\tilde{\zeta}^u(p) = \partial_p \tilde{E}(p,u)$. Then we treat the sets $\tilde{I}^u = \cup_{p\in\mathbb{R}^l_{++}} \tilde{\zeta}^u(p)$ as indifference sets and treat the sets W^u as weakly preferred sets, where $W^u = \cap_{p\in\mathbb{R}^l_{++}} W^u_p$ and $W^u_p = \{x \in \mathbb{R}^l_+ : p\cdot x \geq \tilde{E}(p,u)\}$. (As shown in Claim 3 below, this $\{\tilde{I}^u\}_{u\in\mathbb{R}_+}$ is the family of indifference sets of a continuous, strictly convex, and strict monotone preference.)

(Stage 4: Obtaining a rationalization that satisfies (SP) and (ID).

For every $x \in \mathbb{R}^l_+$, we define

$$\tilde{U}(x) = \sup\{u : x \in W^u\}. \tag{13}$$

Clearly, we have $\tilde{U}(x) = 0$ for $x = 0$.

Claim 3 *The function $\tilde{U} : \mathbb{R}^l_+ \to \mathbb{R}_+$ is well-defined, and such that $x \in \tilde{I}^{\tilde{U}(x)}$ for all $x \in \mathbb{R}^l_+\backslash\{0\}$. Moreover, the function $\tilde{U}$ is continuous, concave, strictly quasiconcave, and strictly monotone.*

Although the function $\tilde{U}(x)$ need not be strictly concave, we can take the continuous monotone transformation $\sqrt{\tilde{U}(x)}$, which is strictly concave (as $\tilde{U}$ is concave and strictly quasiconcave). Thus, $\sqrt{\tilde{U}}$ satisfies (SP). We choose the function (Marshallian) demand function $h(p,w)$.

Claim 4 *The pair* $(\sqrt{\tilde{U}}, h)$ *is a rationalization satisfying (ID).*

Therefore, the pair $(\sqrt{\tilde{U}}, h)$ is a rationalization that satisfies (SP) and (ID). $\square$

4 Proofs of Lemmas 1–3 and Claims 1–4

Proof of Lemma 1. Let functions ζ, and E, and sets I and W be as given in Lemma 1.

First, we will show that for every $p \in \mathbb{R}^l_{++}$,

$$\text{the vector } \zeta(p) \text{ uniquely minimizes } p \cdot x \text{ over } W. \tag{14}$$

Note that $\zeta(p) = \partial_p E(p)$ for all p. The set W is non-empty, closed, convex, and monotone (i.e if $x \in W$ and $y \geq W$, then $y \in W$). Recall that E is concave, so for all $p, p' \in \mathbb{R}^l_{++}$ we have $E(p') \leq E(p) + \partial_p E(p)(p' - p)$, hence $p' \cdot \zeta(p') \leq p' \cdot \zeta(p)$. Therefore, for all p, we have $\zeta(p) \in W$, and $\zeta(p)$ minimizes $p \cdot x$ over W. Now, consider any $p \in \mathbb{R}^l_{++}$, and suppose that $\bar{x} \in \mathrm{bdy}(W)$ minimizes $p \cdot x$ over W. Then, we have $p' \cdot \bar{x} \geq E(p')$ for all p', and $p \cdot \bar{x} = E(p)$. So, p minimizes $p' \cdot \bar{x} - E(p')$ over all $p' \in \mathbb{R}^l_{++}$, and the first order necessary condition of minimization ensures that $\bar{x} = \partial_p E(p) = \zeta(p)$.

Because each $\zeta(p)$ is a minimizer of $p \cdot x$ over W, it follows that $I \subseteq \mathrm{bdy}(W)$. Therefore, to show that $I = \mathrm{bdy}(W)$ it remains to prove that

$$\text{for each } \bar{x} \in \mathrm{bdy}(W)\text{, there is a } p \gg 0 \text{ with } p \cdot \bar{x} \leq p \cdot y \text{ for all } y \in W. \tag{15}$$

Consider any $\bar{x} \in \mathrm{bdy}(W)$. Because W is closed, convex and monotone, by a supporting hyperplane theorem there exists a non-negative and non-zero p such that $p \cdot \bar{x} \leq p \cdot y$ for all $y \in W$. It remains to prove that $p \gg 0$. There are two cases.

(Case 1) Suppose that $\bar{x} \gg 0$, then we have $p \cdot x > 0$. Suppose that p has some j-th coordinate $p_j = 0$. Then, by (CP) we can choose some $z \in I \subseteq W$ whose j'-th coordinates $z_{j'} = 0$ for all $j' \neq j$. Hence, we have $p \cdot z = 0 < p \cdot \bar{x}$, a contradiction. Therefore, we must have $p \gg 0$.

(Case 2) Suppose that $\bar{x}$ has at least one coordinate equal to zero. Without loss of generality, we suppose that $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_m, 0, \cdots, 0)$, where $\bar{x}_1, \cdots, \bar{x}_m > 0$. Define $W' = W \cap \mathbb{R}^m_+ \times \{(0, \cdots, 0)\}$, where $(0, \cdots, 0)$ is the $l - m$ dimensional zero vector. Viewing W' as a subset of $\mathbb{R}^m_+$, it follows (as in Case 1) that there are numbers $\bar{p}_1, \cdots, \bar{p}_m > 0$ such that for the vector $\tilde{p} = (\bar{p}_1, \cdots, \bar{p}_m, 0, \cdots, 0)$, we have: $\tilde{p} \cdot x \geq \tilde{p} \cdot \bar{x}$ for all $x \in W'$. It remains to modify $\tilde{p}$ into a $p \gg 0$ so that $p \cdot x \geq p \cdot \bar{x}$ for all $x \in W$ (not just all $x \in W'$). We choose $p = (\bar{p}_1, \cdots, \bar{p}_m, N+1, \cdots, N+1)$, where N is as given in (CP). So $x' \in W'$, where $x' = \zeta(p)$. The vector x' uniquely minimizes $p \cdot x$ over W. Note that $p \cdot \bar{x} = \tilde{p} \cdot \bar{x} \leq \tilde{p} \cdot x'$ (as $x' \in W'$), and $\tilde{p} \cdot x' = p \cdot x'$. Therefore, we have $\bar{x} = x'$ and $p \cdot \bar{x} \leq p \cdot y$ for all $y \in W$. This establishes (Case 2).

This proves (15), and that $I = \mathrm{bdy}(W)$. Thus, (4a) holds.

In addiiton, (14) ensures that ζ is the hicksian function of W and I.

Because ζ satisfies (CP), we have $z^1, \cdots, z^l \in \zeta(\mathbb{R}^l_{++}) \subseteq W$ for some corner vectors $z^1, \cdots, z^l$, whose coordinates $(z^j)_j > 0$ and $(z^j)_i = 0$ for all $i \neq j$. Because W is convex, it follows that I is contained in the convex hull $\mathrm{co}\{z^1, \cdots, z^l, 0\}$ spanned by $z^1, \cdots, z^l$ and origin 0. This convex hull is compact, so $I = \mathrm{bdy}(W)$ is also compact.

Because $I = \mathrm{bdy}(W)$, to prove property (4b) it is sufficient to prove that if $y \gneqq x$ and $x \in I$, then $y \notin I$. Suppose not, then there is a $p \gg 0$ such that $y = \zeta(p)$, so y uniquely minimizes $p \cdot z$ for all $z \in W$; but we also have $x \in W$ and $p \cdot x \leq p \cdot y$, a contradiction. Thus, (4b) follows.

To see property (4c), consider any x, y, λ as given in (4c). Suppose that $z = \lambda x + (1 - \lambda) y \in I$. Then, there is a $p \gg 0$ with $z = \zeta(p)$, so z uniquely minimizes $p \cdot w$ over all $w \in W$. However, we have $p \cdot z = \lambda p \cdot x + (1 - \lambda) p \cdot y$, hence either $p \cdot x \leq p \cdot z$ or $p \cdot y \leq p \cdot z$, and a contradiction is derived. □

Proof of Lemma 2. First, it is clear that $E^j(p) = 0$ at $p = 0$. For all $p \gg 0$, we have $E^j(\alpha p) \to \infty$ as scalar $\alpha \to \infty$. In addition, for each i, the vector $y^i = \zeta(q^i)$ uniquely minimizes $p \cdot y$ over all $y \in I^j$. Then, there is a small number $\tilde{\eta} > 0$ such that and all $i = 1, \cdots, m$:

$$E^j(p) = p \cdot y^i \quad \text{and} \quad \partial_p E^j(p) = y^i \qquad \text{for all } p \in \mathrm{ball}(q^i, \tilde{\eta}). \tag{16}$$

Suppose that a small number $\eta > 0$ with $\tilde{\eta} > \eta$. By Fact 1, the function $\hat{E}^j : \mathbb{R}^l \to \mathbb{R}$ is infinitely differentiable, concave, and monotone. In addition, the derivative $\partial_p \hat{E}^j(p) = \int_{\mathbb{R}^l} \partial_p E^j(p - x) \rho_\eta(x) dx$, and so $\partial_p \hat{E}^j(p) \gneqq 0$ at every p. Moreover, by (16) there is a small $\eta^* > 0$ (e.g. $\eta^* < \tilde{\eta} - \eta$) such that for all $i = 1, \cdots, m$:

$$\hat{E}^j(p) = p \cdot y^i \quad \text{and} \quad \partial_p \hat{E}^j(p) = y^i \qquad \text{for all } p \in \mathrm{ball}(q^i, \eta^*) \,. \tag{17}$$

Clearly, for all $p \gg 0$, we have $\hat{E}^j(\alpha p) \to \infty$ as $\alpha \to \infty$. By taking η sufficiently small, we can ensure that $\hat{E}^j(p) < 1$ at $p = 0$. Therefore, for each $p \in \mathbb{R}^l_{++}$, there is a unique number $\alpha^j(p) > 0$ such that $\hat{E}^j(\alpha^j(p)p) = 1$. This establishes Lemma 2a.

Because $\hat{E}^j$ is infinitely differentiable and $\partial_\alpha \hat{E}^j(\alpha p) = \sum_{i=1}^l \partial_{p_i} \hat{E}^j(\alpha p) > 0$, by the equation $\hat{E}^j(\alpha^j(p)p) = 1$ and by an Implicit Function Theorem, the function $\alpha^j(p)$ is infinitely differentiable, with derivative $\partial_p \alpha^j(p) = -\partial_p \hat{E}^j(\alpha^j(p)p) / \partial_\alpha \hat{E}^j(\alpha^j(p)p)$. Then, $\tilde{E}^j(p)$ is infinitely differentiable.

Clearly, $\tilde{E}^j$ is homogeneous of degree 1, and monotone, with derivative $\partial_p \tilde{E}^j(p) \gneqq 0$ at every p. As $\hat{E}^j(p)$ is concave, it is easy to verify that the function $\alpha^j(p)$ is convex, so the function $\tilde{E}^j(p)$ is concave. Moreover, as $q^i \cdot y^i = 1$ for all i, by (17) we have

$$\tilde{E}^j(q^i) = q^i \cdot y^i = 1 \quad \text{and} \quad \partial_p \tilde{E}^j(q^i) = y^i \qquad \text{for all } i = 1, \cdots, m. \tag{18}$$

It remains to prove that $\tilde{E}^j$ is eventually constant. Consider any $J \subseteq \{1, \cdots, l\}$, and any $\bar{p} \in \mathbb{R}^l_{++}$. Without loss of generality, let $J = \{n + 1, \cdots, l\}$. We define K' as the set of all $x \in K$ whose i-th coordinates $x_i = 0$ for all $i = n + 1, \cdots, l$. (This set K' is non-empty because the corner bundles $z^1, \cdots, z^n \in K'$.) Define number $C = \min_{x \in K'} \bar{p} \cdot x$. As K is a finite set, it is easy to verify that if N is sufficiently large, then:

$$E^j(p') = c(p'_1, \cdots, p'_n) \qquad \text{for all } p \in G(\bar{p}/C, N, J), \\ \text{all } \alpha \geq 1 \text{ and all } p' \in \text{ball}(\alpha p, \eta), \tag{19}$$

where $c(p'_1, \cdots, p'_n) = \min_{x \in K'} \sum_{i=1}^n p'_i x_i$, and each p'_i is the i-th coordinate of p'.

Therefore, by the definition of $\hat{E}^j$, for all $\alpha \geq 1$, and all $p \in G(\bar{p}/C, N, J)$, we have $\hat{E}^j(\alpha p) = \int_{\mathbb{R}^l} E((\alpha \bar{p}_1/C, \cdots, \alpha \bar{p}_n/C, \alpha p_{n+1}, \cdots, \alpha p_l) - x)\rho_\eta(x)dx = \int_{\mathbb{R}^l} c(\alpha \bar{p}_1/C - x_1, \cdots, \alpha \bar{p}_n/C - x_n)\rho_\eta(x)dx$. Set $p^* = (\bar{p}_1/C, \cdots, \bar{p}_n/C, N+1, \cdots, N+1)$. Then,

$$\hat{E}^j(\alpha p) = \hat{E}^j(\alpha p^*) \qquad \text{for all } p \in G(\bar{p}/C, N, J) \text{ and all } \alpha \geq 1. \tag{20}$$

Choose the $\alpha^* > 0$ such that $\hat{E}^j(\alpha^* p^*) = 1$. Note that $\hat{E}^j(p^*) = \int_{\mathbb{R}^l} c(\bar{p}_1/C - x_1, \cdots, \bar{p}_n/C - x_n)\rho_\eta(x)dx \leq c(\bar{p}_1/C, \cdots, \bar{p}_n/C) = 1$ (as $c(\cdot)$ is concave). Therefore, we have $\alpha^* \geq 1$. So,

$$\hat{E}^j(\alpha^* p/C) = \hat{E}^j(\alpha^* p^*) = 1 \qquad \text{for all } p \in G(\bar{p}, CN, J). \tag{21}$$

Hence, $\tilde{E}^j(p) = C/\alpha^*$ for all $p \in G(\bar{p}, CN, J)$. Thus, $\tilde{E}^j$ is eventually constant. This completes the proof for Lemma 2b. □

Proof of Claim 1. We fix any $j = 1, \cdots, k$. We choose the set $W = \cap_{p \in \mathbb{R}^l_{++}} \{x \in \mathbb{R}^l_+ : p \cdot x \geq \tilde{E}^j(p)\}$ (see Lemma 1). Consider any $x \in \triangle$. We have $\beta x \geq z^i$ for some $\beta > 0$ and some corner bundle $z^i \in K^j \subseteq \tilde{I}^j \subseteq W$. Hence, $\beta x \in W$. Therefore, we can choose the unique number $\tilde{s}^j(x)$ where $\tilde{s}^j(x) = \min\{\beta \in \mathbb{R}_+ : \beta x \in W\}$. Hence, $\tilde{s}^j(x)x \in \text{bdy}(W) = \tilde{I}^j$ (by Lemma 1). This proves Claim 1a.

It remains to prove Claim 1b. First, we choose any increasing sequence of finite sets $F_n \supseteq F^j$ such that $P = \cup_n \pi(F_n)$ is dense in $\text{int}(\triangle)$. Then we choose the corresponding $K_n = \{\zeta^j(p) : p \in F_n\}$. Let $\tilde{E}_n$, $\tilde{\zeta}_n$, $\tilde{s}_n$ and $\tilde{I}_n$ be the terms that correspond to $\tilde{E}^j$, $\tilde{\zeta}^j$, $\tilde{s}^j$, and $\tilde{I}^j$ for $K^j = K_n$. Then $\tilde{E}_n|_P$ point-wise converges to $E^*|_P$ as $n \to \infty$, where function $E^*(p) = p \cdot \zeta^j(p)$. In addition, for all n, we have $\tilde{I}_n, I \subseteq \text{co}\{z^1, \cdots, z^l, 0\}$, so $|\tilde{E}_n(p) - \tilde{E}_n(p')| \leq M\|p - p'\|$ for all $p, p' \in \mathbb{R}^l_{++}$, where $M = \sup_{x \in \text{co}\{z^1, \cdots, z^l, 0\}} \|x\|$. Then, it follows that $\lim_{n \to \infty} \tilde{E}_n(p) = E^*(p)$ for all $p \in \mathbb{R}^l_{++}$,

We will now prove that $\sup_{x \in \triangle} |s^j(x) - \tilde{s}_n(x)| < \delta$ for all large n. It suffices to prove that if $x_n \in \triangle$, $x_n \to \bar{x}$, and $\tilde{s}_n(x_n) \to t$, then $t = s^j(\bar{x})$. Suppose that $t < s^j(\bar{x})$, then $p \cdot t\bar{x} < E^*(p)$ at some p, so for all large n we have $p \cdot \tilde{s}_n(x_n)x_n < \tilde{E}_n(p)$, which contradicts the fact that $\tilde{s}_n(x_n)x_n \in \tilde{I}_n$ for all n. Suppose that $t > s^j(\bar{x})$, then we can choose a t^* such that $t^* > s^j(\bar{x})$ and $\tilde{s}_n(x_n) > t^*$ for all large n. Because $s^j(\cdot)$ is continuous, we can choose points $\tilde{x}^0, \cdots, \tilde{x}^l \in \triangle$ that are so close to $\bar{x}$ that $\max_{i=0,\cdots,l} s^j(\tilde{x}^i) < t^*$. Because K_n is increasing and $\cup_n K_n$ is dense in I^j, we can require that for all large n, we have $s^j(\tilde{x}^0)\tilde{x}^0, \cdots, s^j(\tilde{x}^l)\tilde{x}^l \in K_n$ and $\text{co}\{\tilde{x}^0, \cdots, \tilde{x}^l\} \supseteq \text{ball}(\bar{x}, \epsilon) \cap \triangle$ for some $\epsilon > 0$. So, $\tilde{s}_n(\tilde{x}^i) = s^j(\tilde{x}^i)$ and $x_n \in \text{co}\{\tilde{x}^0, \cdots, \tilde{x}^l\}$ for all large n. As each $\tilde{I}_n$ is convex to the origin, we have

$\tilde{s}_n(x_n) \leq \max_{i=0,\cdots,l} \tilde{s}_n(\tilde{x}^i) < t^*$ for all large n, contradicting the hypothesis that $\tilde{s}_n(x_n) > t^*$. Therefore, we must have $t = s^j(\bar{x})$. Hence, $\sup_{x\in\triangle} |s^j(x) - \tilde{s}_n(x)| < \delta$ for all large n.

Therefore, by setting $K^j = K_n$ for some large n, we can ensure that $\sup_{x\in\triangle} |s^j(x) - \tilde{s}_n(x)| < \delta$. This establishes Claim 1b. □

Proof of Claim 2. We choose the "normalized" price set $P = \{p \in \mathbb{R}^l_{++} | \tilde{E}^1(p) = 1\}$. For every $p \in \mathbb{R}^l_{++}$, we have $(1/\tilde{E}^1(p))p \in P$. We will prove that

$$\inf_{p\in P} \frac{\tilde{E}^{j+1}(p)}{\tilde{E}^j(p)} > 1 \qquad \text{for all } j = 1, \cdots, k-1. \tag{22}$$

Therefore, if β satisfies $1 < \beta^2 < \inf_{p\in P}(\tilde{E}^{j+1}(p)/\tilde{E}^j(p))$ for all $j = 1, \cdots, k-1$, then we have $(1/\beta)\tilde{E}^{j+1}(p) > \beta\tilde{E}^j(p)$ for all $p \in P$, so by homogeneity we have $(1/\beta)\tilde{E}^{j+1}(p) > \beta\tilde{E}^j(p)$ for all $p \in \mathbb{R}^l_{++}$.

To prove (22), note that for all $j = 1, \cdots, k$ we have:[14]

$$0 < \inf_{p\in P} \tilde{E}^j(p) \leq \sup_{p\in P} \tilde{E}^j(p) < \infty. \tag{23}$$

Now, suppose (22) is false. Then for some j, there exists a sequence of prices $p_n \in P$ such that $\lim_{n\to\infty} \tilde{E}^{j+1}(p_n) = \lim_{n\to\infty} E^j(p_n)$, which is a finite number (by (23)). Since $\tilde{I}^j$ and $\tilde{I}^{j+1}$ are compact, we can assume that $\tilde{\zeta}^j(p_n) \to \bar{x}^j$ and $\tilde{\zeta}^{j+1}(p_n) \to \bar{x}^{j+1}$, for some $\bar{x}^j \in \tilde{I}^j$, and some $\bar{x}^{j+1} \in \tilde{I}^{j+1}$. By taking subsequence if necessary, we can further assume that for all coordinates $i = 1, \cdots, l$, either the coordinates $(p_n)_i \to \bar{p}_i$ for some finite number $\bar{p}_i$, or $(p_n)_i \to \infty$. We define J as the set of all coordinates i where $(p_n)_i \to \infty$. As $\inf_{p\in P} \tilde{E}^j(p) \geq 1$, we have $\bar{p}_i > 0$ for all $i \notin J$. Also, as $\sup_{p\in P} \tilde{E}^j(p), \sup_{p\in P} \tilde{E}^{j+1} < \infty$, we have the coordinates $(\bar{x}^j)_i = (\bar{x}^{j+1})_i = 0$ for all $i \in J$. There are two cases:

(Case 1) Suppose $J = \emptyset$. Then $\bar{p} = (\bar{p}_1, \cdots, \bar{p}_l) \in \mathbb{R}^l_{++}$. By continuity we have: $\tilde{\zeta}^j(\bar{p}) = \bar{x}^j$, and $\tilde{\zeta}^{j+1}(\bar{p}) = \bar{x}^{j+1}$. Then we have: $\tilde{E}^j(\bar{p}) = \bar{p} \cdot \bar{x}^j = \lim_{n\to\infty} \tilde{E}^j(p_n) = \lim_{n\to\infty} \tilde{E}^{j+1}(p_n) = \bar{p} \cdot \bar{x}^{j+1} = \tilde{E}^{j+1}(\bar{p})$, which contradicts the fact that $\tilde{E}^{j+1}(p) > \tilde{E}^j(p)$ for all $p \in \mathbb{R}^l_{++}$.

(Case 2) Suppose $J \neq \emptyset$. Since $\bar{x}^j \gneq 0$, there must exist some coordinates $i \notin J$. We choose a large N, and choose the price vector $\tilde{p}$ where $(\tilde{p})_i = (\bar{p})_i$ for all $i \notin J$, and $(\tilde{p})_i = N$ for all $i \in J$. By (CP), we the bundle $\tilde{x}^j = \tilde{\zeta}^j(\tilde{p})$ has its coordinates $(\tilde{x}^j)_i = 0$ for all $i \in J$. Note that $p_n \cdot \tilde{\zeta}^j(p_n) \leq p_n \cdot \tilde{x}^j$ for all n, so $\tilde{p} \cdot \bar{x}^j \leq \tilde{p} \cdot \tilde{x}^j$. Since $\tilde{x}^j$ uniquely minimizes $p \cdot x$ for all $x \in \tilde{I}^j$, we have $\bar{x}^j = \tilde{x}^j = \tilde{\zeta}^j(\tilde{p})$. Similarly, we have $\bar{x}^{j+1} = \tilde{\zeta}^{j+1}(\tilde{p})$. Because $\lim_{n\to\infty} p_n \cdot \tilde{\zeta}^j(p_n) \geq \lim_{n\to\infty} p_n \cdot \bar{x}^j$, and $p_n \cdot \tilde{\zeta}^j(p_n) \leq p_n \cdot \bar{x}^j$ for all large n, we have $\lim_{n\to\infty} p_n \cdot \tilde{\zeta}^j(p_n) = \lim_{n\to\infty} p_n \cdot \bar{x}^j = \tilde{p} \cdot \bar{x}^j = \tilde{E}^j(\tilde{p})$. Similarly, we also have $\lim_{n\to\infty} p_n \cdot \tilde{\zeta}^{j+1}(p_n) = \tilde{E}^{j+1}(\tilde{p})$. Then, as in (Case 1), we have $\tilde{E}^{j+1}(\tilde{p}) = \tilde{E}^j(\tilde{p})$, which contradicts to the fact that $\tilde{E}^{j+1}(p) > \tilde{E}^j(p)$ for all $p \in \mathbb{R}^l_{++}$.

This proves (22), and also Claim 2. □

[14] To see (23), note that for all $p \in P$, we have $\tilde{E}^j(p) \geq \tilde{E}^1(p) \geq 1$, so $\inf_{p\in P} \tilde{E}^j(p) \geq 1$. Also, for all $p \in \mathbb{R}^l_{++}$, we have $\tilde{E}^j(p) = p \cdot \tilde{\zeta}^j(p) \leq p \cdot \tilde{s}^j(x)x$ where $x = \pi(\tilde{\zeta}^1(p)) \in \triangle$. So, $\tilde{E}^j(p) \leq (\tilde{s}^j(x)/\tilde{s}^1(x))p \cdot \tilde{\zeta}^1(p)$. Therefore, we have: $\sup_{p\in P} \tilde{E}^j(p) \leq (\sup_{x\in\triangle} \tilde{s}^j(x))/(\inf_{x\in\triangle} \tilde{s}^1(x))$.

Proof of Lemma 3. We choose the set P as given in the proof of Claim 2 above. Let $\beta > 1$ be such that for all $j = 1, \cdots, k-1$, one has: $1 < \beta^2 < \inf_{p \in P}(\tilde{E}^{j+1}(p)/\tilde{E}^j(p))$. By (23), we have: $\inf_{p \in P}[(1/\beta)\tilde{E}^{j+1}(p) - \beta\tilde{E}^j(p)] > 0$. Then, by choosing γ sufficiently small,[15] we can ensure that for all $p \in P$ the function $E(p,u)$ is convex in u, so this is also true for all $p \in \mathbb{R}^l_{++}$. Hence, for all $p \in \mathbb{R}^l_{++}$, the function $\tilde{E}(p,u)$ is convex in u. In addition, for every p, the $E(p,u)$ function is piecewise linear and strictly monotone in u, so $\tilde{E}(p,u)$ is strictly increasing in u (with $\partial_u \tilde{E}(p,u) > 0$).

As each $\tilde{E}^j$ is concave and homogeneous of degree 1 in p, it follows that $\tilde{E}(p,u)$ is also concave and homogeneous of degree 1 in p.

In addition, note that for each j, there is a unique $\tilde{u}^j \in (v^{2j-1}, v^{2j})$ such that

$$E(p, \tilde{u}^j) = \tilde{E}^j(p) \qquad \text{and all } p \in \mathbb{R}^l_{++}. \tag{24}$$

By choosing μ sufficiently small, we can clearly ensure that at every p, the function $E(p,u)$ is linear on the interval $(\tilde{u}^j - \mu, \tilde{u}^j + \mu)$ for all j, and is linear on $(-\mu, \mu)$, hence

$$\tilde{E}(p, \tilde{u}^j) = \tilde{E}^j(p) \qquad \text{for all } p \in \mathbb{R}^l_{++} \text{ and all } j = 1, \cdots, k, \tag{25}$$

and $\tilde{E}(p,u) = E(p,u) = 0$ for $u = 0$ for all p.

It remains to prove that $\tilde{E}$ is infinitely differentiable, and is eventually constant and has $\partial p \tilde{E}(p,u) \gneqq 0$ for $u > 0$.

By definition of the $E(p,u)$ function, for all $(p,u) \in \mathbb{R}^l_{++} \times \mathbb{R}$ we actually have:

$$\begin{aligned} E(p,u) = \lambda^1(u)\frac{1}{\beta}\tilde{E}^1(p) + \lambda^2(u)\beta\tilde{E}^2(p) + \lambda^3(u)\frac{1}{\beta}\tilde{E}^2(p) + \cdots \\ \cdots + \lambda^{2k-1}(u)\frac{1}{\beta}\tilde{E}^k(p) + \lambda^{2k}(u)\beta\tilde{E}^k(p), \end{aligned} \tag{26}$$

where for every $j = 2, \cdots, 2k-1$, the function $\lambda^i : \mathbb{R} \to \mathbb{R}_+$ is defined by:

$$\lambda^j(u) = \frac{u - v^{j-1}}{v^j - v^{j-1}} \text{ for } u \in [v^{j-1}, v^j], \;\; \lambda^j(u) = 0 \text{ for } u \in (-\infty, v^{j-1}], \tag{27}$$

$$\lambda^j(u) = \frac{v^{j+1} - u}{v^{j+1} - v^j} \text{ for } u \in [v^j, v^{j+1}], \;\; \lambda^j(u) = 0 \text{ for } u \in [v^{j+1}, \infty), \tag{28}$$

and the function $\lambda^1 : \mathbb{R} \to \mathbb{R}$ is defined by (28) with $j = 1$ and by that $\lambda^1(u) = (u - v^0)/(v^1 - v^0)$ for $u \in (-\infty, v^1]$, and the function $\lambda^{2k} : \mathbb{R} \to \mathbb{R}_+$ is defined by (27) with $j = 2k$ and by that $\lambda^{2k}(u) = (u - v^{2k-1})/(v^{2k} - v^{2k-1})$. Then we have:

$$\begin{aligned} \tilde{E}(p,u) = \tilde{\lambda}^1(u)\frac{1}{\beta}\tilde{E}^1(p) + \tilde{\lambda}^2(u)\beta\tilde{E}^2(p) + \tilde{\lambda}^3(u)\frac{1}{\beta}\tilde{E}^2(p) + \cdots \\ \cdots + \tilde{\lambda}^{2k-1}(u)\frac{1}{\beta}\tilde{E}^k(p) + \tilde{\lambda}^{2k}(u)\beta\tilde{E}^k(p), \end{aligned} \tag{29}$$

[15] For example, by requiring that: $\gamma < (\beta - 1/\beta)$, and $\gamma(\beta - 1/\beta)[\sup_{p \in P} \tilde{E}^j(p)] < \inf_{p \in P}[(1/\beta)\tilde{E}^{j+1}(p) - \beta\tilde{E}^j(p)]$ for all $j = 1, \cdots, k-1$, and $\gamma \sup_{p \in P}[(1/\beta)\tilde{E}^{j+1}(p) - \beta\tilde{E}^j(p)] < (\beta - 1/\beta) \inf_{p \in P} \tilde{E}^{j+1}(p)$ for all $j = 1, \cdots, k-1$.

where each $\tilde{\lambda}^j(u) = \int_{\mathbb{R}} \lambda^j(u-w)\rho_\mu(w)dw$. Then, by Fact 1, all of the functions λ^j are infinitely differentiable, so $\tilde{E}$ is infinitely differentiable.

Because each $\tilde{E}^j$ is eventually constant, it follows from (29) that $\tilde{E}(p,u)$ is eventually constant for all u.

Now, note that for all $u \geq 0$, we have $\tilde{\lambda}^1(u), \cdots, \tilde{\lambda}^{2k}(u) \geq 0$,[16] and if $u > 0$, then some $\tilde{\lambda}^j(u) > 0$. For all $p \in \mathbb{R}^l_{++}$, since $\partial_p \tilde{E}^j(p) \gneqq 0$ for every j, we have $\partial_p \tilde{E}(p,u) \gneqq 0$ if $u > 0$. □

Proof of Claim 3. Clearly, $\tilde{U}(x)$ is well-defined.

Consider any $x \in \mathbb{R}^l_+ \backslash \{0\}$. So $\bar{u} > 0$, where $\bar{u} = \tilde{U}(x)$. We will show that $x \in \tilde{I}^{\bar{u}}$. First, as $\tilde{E}(p,u)$ is continuous in u, we have $p \cdot x \geq \tilde{E}(p,u)$ for all $p \in \mathbb{R}^l_{++}$, so $x \in W^{\bar{u}}$. Suppose that $x \notin \tilde{I}^{\bar{u}}$. Then we have $p \cdot x > \tilde{E}(p, \bar{u})$ for all $p \in \mathbb{R}^l_{++}$ (as by Lemma 1, $\tilde{\zeta}^{\bar{u}}$ is the hicksian function of $W^{\bar{u}}$). Now, note that by the definition of $\tilde{U}$, we have $x \cap H = \emptyset$, where the convex and monotone set $H = \cup_{u > \bar{u}} W^u$. Then by a separating hyperplane theorem, there exists a $\bar{p} \in \mathbb{R}^l_+ \backslash \{0\}$ such that $\bar{p} \cdot x \leq p \cdot y$ for all $y \in H$. (Case 1) Suppose $x \gg 0$. Then as the corner vectors $Me^1, \cdots, Me^l \in H$ for all large number M, we have $\bar{p} \gg 0$. Therefore, we have $\tilde{E}(\bar{p}, \bar{u}) < \bar{p} \cdot x \leq \tilde{E}(\bar{p}, u)$ for all $u > \bar{u}$, contradicting the fact that $\tilde{E}(\bar{p}, u)$ is continuous in u. (Case 2) Suppose $x \not\gg 0$. Without loss of generality, let its coordinates $x_1, \cdots, x_n > 0$, and $x_{n+1} = \cdots = x_l = 0$. Since $\bar{p} \cdot x \leq \bar{p} \cdot y$ for all $y \in H \cap (\mathbb{R}^n \times \{(0, \cdots, 0)\})$, the vector $\bar{p}$ has its coordinates $\bar{p}_1, \cdots, \bar{p}_n > 0$. Then by replacing its coordinates $\bar{p}_{n+1}, \cdots, \bar{p}_l$ by some positive numbers if necessary, we can assume that $\bar{p} \gg 0$. As in Case 1, this derives a contradiction. Thus, we have shown that $x \in \tilde{I}^{\bar{u}}$.

For any $x \in \mathbb{R}^l_+ \backslash \{0\}$, we have the closed sets $\{y \in \mathbb{R}^l_+ : \tilde{U}(y) \geq \tilde{U}(x)\} = W^{\tilde{U}(x)}$, and $\{y \in \mathbb{R}^l_+ : \tilde{U}(y) \leq \tilde{U}(x)\} = \mathbb{R}^l_+ \backslash \mathrm{int}(W^{\tilde{U}(x)})$ (by (4a)), hence $\tilde{U}$ is continuous.

Clearly, by (4b) and (4c), the function $\tilde{U}$ is strictly monotone, and strictly quasiconcave.

To see that $\tilde{U}$ is concave, consider any $x, y \in \mathbb{R}^l_+$, any $\lambda \in (0,1)$, and let $z = \lambda x + (1-\lambda)y$. For all $p \in \mathbb{R}^l_{++}$, we have: $p \cdot x \geq \tilde{E}(p, \tilde{U}(x))$, and $p \cdot y \geq \tilde{E}(p, \tilde{U}(y))$, so $p \cdot z \geq \lambda \tilde{E}(p, \tilde{U}(x)) + (1-\lambda)\tilde{E}(p, \tilde{U}(y)) \geq \tilde{E}(p, \lambda \tilde{U}(x) + (1-\lambda)\tilde{U}(y)$ (as $\tilde{E}$ is convex in u). Therefore, $\tilde{U}(z) \geq \tilde{U}(x) + (1-\lambda)\tilde{U}(y)$. Thus, $\tilde{U}$ is concave. □

Proof of Claim 4. Because the functions $\tilde{U}$ and $\sqrt{\tilde{U}}$ have the same preference, they have the same (Marshallian) demand function $h(p,w)$. We can focus on the function $\tilde{U}$. Note that $\tilde{\zeta}^u$ is the hicksian demand function for this utility function $\tilde{U}$ at the utility level u. In particular, for all $i = 1, \cdots, T$, we have $x^i = \tilde{\zeta}^{\tilde{u}^j}(p^i)$, where index j is given in (5) and $\tilde{u}^j$ is given in Lemma 3.

Because $\tilde{\zeta}^u$ is the hicksian demand function for $\tilde{U}$, the function $\tilde{E}(p,u) = p \cdot \tilde{\zeta}^u(p)$ is the expenditure function for $\tilde{U}$, i.e. $\tilde{E}(p,u) = \min\{p \cdot x : \tilde{U}(x) \geq u\}$. Recall that $\tilde{E}(p,u) = p \cdot \tilde{\zeta}^u(p)$ is infinitely differentiable and $\partial_u \tilde{E}(p,u) > 0$ for all $(p,u) \in \mathbb{R}^l_{++} \times \mathbb{R}_+$. By a well-known duality fact (see Varian, 1984, p. 126), it

[16] As λ^1 is linear on $v^0 = 0$, we have $\tilde{\lambda}^1(0) = 0$, so $\tilde{\lambda}^1(u) \geq 0$ for all $u \geq 0$.

follows that the Marshallian) demand function $h(p, w)$ of $\tilde{U}(x)$ satisfies:

$$\begin{aligned}
&\text{a)}\ \ w = \tilde{E}(p, V(p, w)) \qquad \text{for every } (p, w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}, \\
&\text{b)}\ \ h(p, w) = \partial_p \tilde{E}(p, V(p, w)) \qquad \text{for every } (p, w) \in \mathbb{R}^l_{++} \times \mathbb{R}_{++}, \\
&\qquad \text{where (indirect utility) } V(p, w) = \max\{\tilde{U}(x) : x \in B(p, w)\}.
\end{aligned} \tag{30}$$

By (30a) the function $V(p, w)$ is defined implicitly by the equation $w - \tilde{E}(p, v) = 0$. Because $\tilde{E}(p, u)$ is C^∞ and $\partial_u \tilde{E}(p, u) > 0$, by an Implicit Function Theorem, it follows that the (indirect utility) function $V(p, w)$ is also C^∞. Therefore, the composite function $h(p, w) = \partial_p \tilde{E}(p, V(p, w))$ is C^∞, i.e. (ID) holds. For all $i = 1, \cdots, T$, we have $x^i = \partial_p \tilde{E}(p^i, \tilde{u}^j)$, where index j is given in (5) and $\tilde{u}^j$ is given in Lemma 3. Again, by (30a) we have $\tilde{u}^j = V(p^i, w^i)$, where $w^i = \tilde{E}(p^i, \tilde{u}^j) = p^i \cdot x^i$, hence $x^i = h(p^i, w^i)$. In other words, the demand function h is also consistent with the data set $D = \{(p^i, x^i)\}_{i=1}^T$. Thus, $(\tilde{U}, h)$ is a rationalization that satisfies (ID), and so is $(\sqrt{\tilde{U}}, h)$. $\square$

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