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Characterization of the existence of maximal elements of acyclic relations*

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Summary. We obtain a sufficient condition for the existence of maximal elements of irreflexive binary relations that generalizes the theorem of Bergstrom and Walker by relaxing the compactness condition to a weaker one that is naturally related to the relation. We then prove that the sufficient conditions used both in the Bergstrom-Walker Theorem and in our generalization provide a characterization of the existence of maximal elements of acyclic binary relations. Other sufficient conditions for the existence of maximal elements obtained by Mehta, by Peris and Subiza and by Campbell and Walker are shown to be necessary too.

Keywords and Phrases: Maximal elements, Acyclicity, $\succ$-compactness.

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1 Introduction

In this paper we aim at two related purposes concerning maximality in acyclic binary relations. Firstly, we shall weaken the hypothesis of compactness in the classical Bergstrom-Walker Theorem. Secondly, this work will lead us to provide necessary and sufficient conditions for the existence of maximal elements for acyclic binary relations.

The existence of maximal elements of binary relations on compact sets has been the subject of several interesting approaches. In Sloss [11], Brown [5], Bergstrom [3] and Walker [15] it is proved that every nonempty compact subset of a space on which an upper semicontinuous acyclic binary relation is defined

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contains a maximal element. This is usually known as the Bergstrom-Walker Theorem. Peris and Subiza [9] offered a generalization of this result to irreflexive relations. On the other hand, continuity conditions weaker than upper semicontinuity are used by Mehta [7] and by Subiza and Peris [13] in the same framework as that considered in the Bergstrom-Walker Theorem. Further results in the line of this latter theorem are Theorems 1 and 2 in Campbell and Walker [6], where upper semicontinuity is also replaced by a weaker property but a condition stronger than acyclicity is assumed. An excellent survey of the sufficient conditions proposed to obtain maximal elements of binary relations in the setting of compact sets is Border [4], Chapter 7.

We have mentioned that the results listed above rely on the concept of compactness. However, this seems to be a strong requirement in the search for maximality. The reason is that the use of compactness involves dealing with far too many open sets; indeed, the relevant ones seem to be the so-called lower comprehensive or decreasing sets (cf. Nachbin [8]) alone. In Alcantud [1] we introduced the concept of $\succ$-upper (resp. lower) compactness of a topological space on which a binary relation $\succ$ is defined. Here we show that $\succ$-upper compactness is a necessary condition for the existence of maximum for any binary relation on a topological space, as it is not the case for compactness. It is also sufficient in the particular case of upper semicontinuous linear orders. Some further properties of $\succ$-compactness are set forth.

We then proceed to prove that the hypothesis of compactness may be weakened to $\succ$-upper compactness in Peris-Subiza's extension of the Bergstrom-Walker Theorem to the only irreflexive case. In addition to this, the sufficient conditions used both in the Bergstrom-Walker Theorem and in our generalization are shown to provide a characterization of the existence of maximal elements for acyclic binary relations. This constitutes a further argument to emphasize the importance of $\succ$-compactness (as well as compactness) as a natural condition in the study of maximality for acyclic binary relations. The above mentioned characterization will be refined by considering other sufficient conditions used by Mehta and by Subiza and Peris in the acyclic case, which are weaker than those used by Bergstrom and Walker.

We shall obtain refinements of the characterization obtained in the setting of a particular type of acyclic relations: namely, interval orders. In order to do this, and following the approach initiated above, we prove that Theorems 1 and 2 in Campbell and Walker [6] may also be extended to characterizations of the existence of maximal elements in the class of interval orders.

Moreover, the structure of the set of maximal elements will be analyzed in an Appendix. Analogously to the fact that the set of maximal elements is compact in the case considered by Sloss, Brown, Bergstrom and Walker (cf. 7.12 in Border [4]), in the event that we only assume $\succ$-compactness we shall show that the set of maximal elements is nonempty and $\succ$-compact even if $\succ$ satisfies a certain condition weaker than semicontinuity.

2 Notation and definitions

Let $\succ$ denote an irreflexive binary relation on X. We define the *transitive closure* $\succ\succ$ of $\succ$ by the expression

$$x \succ\succ y \Leftrightarrow \text{ for some } n \geq 2 \text{ there exist } x_1, ..., x_n \in X \\ \text{such that } x = x_1 \succ ... \succ x_n = y$$

Furthermore, we define an equivalence binary relation $\approx$ on X by

$$x \approx y \Leftrightarrow x = y \text{ or } x \succ\succ y \succ\succ x$$

The quotient set by this equivalence relation will be denoted by $\frac{X}{\approx}$, and its elements (equivalence classes) by $[x]$. We can define on this set an asymmetric, transitive binary relation P by: for each $[x], [y] \in X$,

$$[x]P[y] \Leftrightarrow [x] \neq [y] \text{ and there are } x' \in [x],\ y' \in [y] \text{ such that } x' \succ\succ y'$$

This is called the *quotient relation* of $\succ$. We remark that

$$[x]P[y] \Leftrightarrow x \succ\succ y \text{ and not } y \succ\succ x \tag{1}$$

and that therefore a member z of X is $\succ$-maximal if and only if $[z]$ is a singleton which is P-maximal.

We say that $\succ$ is *acyclic* if for each $x_1, ..., x_n \in X : x_1 \succ x_2 \succ ... \succ x_n$ implies $x_1 \neq x_n$. If $\succ$ is acyclic then $\approx$ coincides with =, and P coincides with $\succ\succ$. The binary relation $\succ$ is a *partial order* if it is irreflexive (or asymmetric) and transitive. Partial orders are acyclic. Both the transitive closure of an acyclic binary relation and the quotient relation of an irreflexive binary relation are partial orders.

For an asymmetric binary relation $\succ$ on a set X, $\succeq$ will denote the completion of $\succ$; i.e., $x \succeq y$ means that $y \succ x$ is false.

An acyclic binary relation $\succ$ on X is *pseudotransitive* if $x_1 \succ x \succeq y_1 \succ y$ implies $x_1 \succ y$ whenever $x \neq y_1$. It is an *interval order* if $x_1 \succ x \succeq y_1 \succ y$ implies $x_1 \succ y$ (even when $x = y_1$). And it is *negatively transitive* if whenever $x, y, z \in X$ and $x \succ y$ then either $x \succ z$ or $z \succ y$. Interval orders are pseudotransitive (cf. Campbell and Walker [6]). Also the converse is true, because pseudotransitive relations are transitive as pointed out in Remark 3 in Zhou and Tian [16].

Let $\succ$ be an asymmetric binary relation on a set X. The *indifference* associated with $\succ$, which is commonly denoted by $\sim$, is defined by $x \sim y$ iff not $x \succ y$, not $y \succ x$.

Now, let $\succ$ denote an irreflexive binary relation on X. A subset S of X is a *lower comprehensive set* if $x \in S$ and $x \succ y$ together imply that $y \in S$; dually we would define *upper comprehensive set*. For the sake of conciseness, we shall also refer to them as lower/upper sets. Let τ denote a topology on X. We say that $\succ$ is *upper (lower) semicontinuous* if for all $x \in X$ the lower (upper) contour set $\{y \in X : x \succ y\}$ ($\{y \in X : y \succ x\}$) is open; and it is *continuous* if it is both upper and lower semicontinuous.

The binary relation $\succ$ on X is *weakly upper semicontinuous* if whenever $x \succ y$ there is a neighborhood V of y such that $v \succ x$ is false for each $v \in V$. It is obvious that upper semicontinuity implies weak upper semicontinuity if $\succ$ is asymmetric.

Furthermore, whenever τ is a topology on X we say that (X,τ) is $\succ$-*upper compact* if for each collection of lower open sets which covers X there exists a finite subcollection that also covers X. We would define dually $\succ$-*lower compact*. It is clear that compact spaces are in particular $\succ$-upper (lower) compact. The converse is not generally true: even a space which is both $\succ$-upper compact and $\succ$-lower compact need not be compact, as Example 1 demonstrates below. However, Proposition 1 gives conditions under which such spaces are compact.

Let (X,τ) be a topological space. A function $f : X \to \mathbb{R}$ is *upper* (respectively *lower*) *semicontinuous* if for each $\alpha \in \mathbb{R}$ the set $f^{-1}((-\infty,\alpha))$ (respectively $f^{-1}((\alpha,+\infty))$) is open.

A symmetric binary relation $\mathscr{R}$ on the topological space (X,τ) *saturates* τ if $x\mathscr{R}y$ and $y \in A \in \tau$ together imply that $x \in A$ (cf. Alcantud [2]).

Let $\succ$ be an acyclic binary relation on a set X. Let $\mathscr{S}$ denote the collection of all upper- and lower- contour sets for $\succ$- i.e., all sets of the form $\{a \in X : x \succ a\}$ and $\{a \in X : a \succ x\}$. The *order topology* associated with $\succ$ is the topology generated by the subbase $\mathscr{S}$. We have proved in Alcantud [2] that the indifference associated with any asymmetric and negatively transitive binary relation $\succ$ saturates its order topology.

3 Results

We begin by pointing out that if a binary relation $\succ$ on a topological space (X,τ) has a maximum x (i.e. an element satisfying $x \succ y$ for any $y \in X$ with $y \neq x$) then the space is $\succ$-upper compact; however it may not be compact, as shown by the interval (0,1] with the usual orden relation and topology. Indeed, if $\{U_a\}_{a\in A}$ is a collection of lower open sets that covers X then $x \in U_a$ for some $a \in A$, and therefore $X = U_a$. Besides, for an upper semicontinuous linear order $\succ$ on (X,τ) the converse implication also holds: $\succ$-upper compactness implies the existence of a maximum (otherwise, the collection $\{\{x \in X : y \succ x\}\}_{y\in Y}$ of lower open sets covers X and admits no finite subcollection that covers X).

Remark 1. A type of orders which generalize linear orders are asymmetric and negatively transitive binary relations; some authors use the term *preference* to refer to them. They are of particular interest in mathematical economics. In some situations, among which we may mention the analysis of intrinsic topologies (cf. Alcantud [2]), they exhibit properties similar to those of linear orders. In this line, one can easily adapt the reasoning given above for linear orders to show that preferences satisfy an analogous property to that enunciated for them. Namely, that any upper semicontinuous preference $\succ$ on (X,τ) which is $\succ$-upper compact possesses a maximal element. Furthermore, let us recall that the indifference associated to a preference is an equivalence relation. This permits to complete the result by noticing that a maximal element of a preference is a maximum exactly when its indifference class consists of a single element.

The following example shows that a $\succ$-upper compact space may not possess a maximum, even though $\succ$ is a partial order. It is also another example that $\succ$-upper compact spaces may fail to be compact.

Example 1. *Let $X = \mathbb{Z} \setminus \{0\}$, and define $\succ$ on X according to the following rule (see Figure 1):*

$$x \succ y \text{ if and only if } x > 0 \text{ and either } y = -x \text{ or } y = -x - 1$$

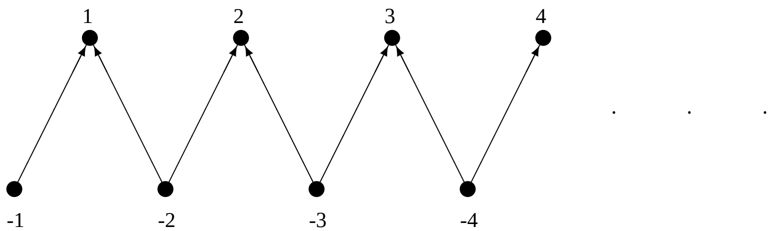

Figure 1. The partial order in Example 1

The binary relation is a partial order and therefore acyclic, but it clearly has no maximum. Let τ be the topology on X generated by the base $\{\{x, -x\} : x \in X, x > 0\}$. Notice that the open lower comprehensive sets are precisely the sets of the form $X \setminus \{-x, -x+1, \ldots, x\}$. Then X is obviously not compact (the base itself is an open cover which can not be reduced), but every open cover consisting of lower comprehensive sets includes the singleton $\{X\}$, and X is therefore $\succ$-upper compact.

It has been mentioned that compactness implies $\succ$-upper and lower compactness in any space X ordered by an irreflexive relation $\succ$. The following result provides conditions under which the converse holds.

Proposition 1. *Let $\succ$ be an asymmetric and negatively transitive binary relation on a set X, and endow X with the associated order topology. If this space is connected, then it is compact if and only if it is both $\succ$-upper and lower compact.*

Proof. Since any compact space is always both $\succ$-upper and $\succ$-lower compact, we only need to prove that the converse holds under the hypotheses of the Proposition. For this we apply Alexander's theorem, which ensures that X will be compact provided that every cover of some subbase has a finite subcover. We work with the subbase $\mathscr{S}$ for the order topology, as defined in Section 2.

Take an open covering of $\mathscr{S}$. Because these sets are either upper or lower comprehensive sets by definition, two cases may arise.

Case 1. If all the sets in the covering are lower (respectively upper) sets then $\succ$-upper (resp. lower) compactness ensures the existence of a finite subcovering.

Case 2. If the open covering contains both upper and lower comprehensive sets, let U denote the union of all the upper comprehensive sets in the covering, and let V denote the union of all the lower comprehensive sets in the covering. From the connectedness of the topology, there exists $x \in U \cap V$. It follows that $X = U \cup V$: either $y \succ x$ (in which case $y \in U$), or $x \succ y$ (in which case $y \in V$), or else $y \sim x$, in which case $y \in U \cap V$, because $\sim$ saturates the order topology. □

We now proceed to show that $\succ$-upper compactness provides a result of existence of maximal elements for irreflexive relations. Recall the well-known theorem:

Theorem 1. *(*Sloss, Brown, Bergstrom, Walker*) If an acyclic binary relation defined on a topological space is upper semicontinuous then every non-empty compact subset of the space contains a maximal element.*

We already mentioned that the theorem of Bergstrom and Walker was generalized by a result of Peris and Subiza, which says as follows:

Theorem 2. *(*Peris and Subiza*) Let (X, τ) be a compact topological space, and let $\succ$ be an irreflexive relation such that for each $x \in X$ the set*

$$\{y \in X : x \succ\succ y \text{ and not } y \succ\succ x\}$$

is open. Then, the quotient set $\frac{X}{\approx}$ has a maximal element for the relation P.

In order to generalize the latter result to consider $\succ$-compact spaces we need a Lemma:

Lemma 1. *Let $\succ$ be a partial order on a topological space (X, τ). If $\succ$ is upper semicontinuous then every $\succ$-upper compact subset of X has a maximal element.*

Proof. Suppose that $Y \subseteq X$ is $\succ$-upper compact but has no maximal element. Thus $Y = \cup_{y \in Y}(\{x \in X : y \succ x\} \cap Y)$. By transitivity the sets $\{x \in X : y \succ x\} \cap Y$ are lower comprehensive sets for every $y \in Y$, and they are clearly open in the relative topology of Y. Then there exist $\{y_1, \ldots, y_n\}$ such that $Y = \cup_{i=1,\ldots,n}\{y \in Y : y_i \succ y\}$, by $\succ$-upper compactness of Y. This yields a contradiction because acyclicity would imply that one of the y_i is maximal in Y. □

Theorem 3. *Theorem 2 remains true if we only assume $\succ$-upper compactness instead of compactness.*

Proof. We shall show that $(\frac{X}{\approx}, P)$ with the quotient topology satisfies the hypotheses of Lemma 1, which would complete the proof.

(a) P *is a partial order* by definition.

(b) P *is upper semicontinuous*; that is, the lower contour set $\{[y] : [x]P[y]\}$ is open in the quotient topology for each $[x] \in \frac{X}{\approx}$. Indeed, if p denotes the projection onto the quotient space, then $\{[y] : [x]P[y]\}$ is open if and only if $p^{-1}(\{[y] : [x]P[y]\}) \in \tau$. Now $p^{-1}(\{[y] : [x]P[y]\}) = \{y \in X : [x]P[y]\}$ and thus (1) yields $p^{-1}(\{[y] : [x]P[y]\}) = \{y \in X : x \succ\succ y \text{ and not } y \succ\succ x\} \in \tau$.

(c) $\frac{X}{\approx}$ *is P-upper compact*. Let $\frac{X}{\approx} = \cup_{\alpha \in A} U_\alpha$ with U_α lower open set (respect to P and the quotient topology). Then $X = \cup_{\alpha \in A} p^{-1}(U_\alpha)$, $p^{-1}(U_\alpha) \in \tau$ by definition and $p^{-1}(U_\alpha)$ is a lower set (respect to $\succ$): indeed, if $y \in p^{-1}(U_\alpha)$ and $y \succ x$ then either $[x] = [y] \in U_\alpha$ or $[y]P[x]$; both cases imply $[x] \in U_\alpha$ and thus $x \in p^{-1}(U_\alpha)$.

Therefore $X = \cup_{i=1,\ldots,n} p^{-1}(U_{\alpha_i})$ for some $\alpha_i \in A$, and thus $\frac{X}{\approx} = \cup_{i=1,..,n} U_{\alpha_i}$. This proves that $\frac{X}{\approx}$ is P-upper compact. □

Remark 2. Since the condition that the sets $\{y \in X : x \succ\succ y \text{ and not } y \succ\succ x\}$ are open is implied by upper semicontinuity, it follows that Theorem 1 also holds if $\succ$-upper compactness replaces compactness.

We are ready to use Theorem 3 in order to prove the following characterization of the existence of maximal elements in the whole class of acyclic binary relations:

Theorem 4. *Let $\succ$ be an acyclic binary relation on X. The following conditions are equivalent:*

(a) X has a maximal element

(b) there exists a compact topology on X such that $\succ$ is upper semicontinuous

(c) there exists a $\succ$-upper compact topology on X such that $\succ$ is upper semicontinuous.

Proof. It is obvious that (b) implies (c), and (c) implies (a) by Remark 2. We now prove that (a) implies (b).

Let z be a maximal element of $\succ$ in X. Let τ be the topology on X whose nontrivial open sets are all those subsets of X which do not contain z (this topology is called the excluded point topology). The set X is compact under τ: every open cover of X includes X itself, so that $\{X\}$ is always a finite subcover. Also, $\succ$ is upper semicontinuous with respect to τ because, wehenever $x \in X$, the subset $\{y \in X : x \succ y\}$ is open since it does not contain z. This completes the proof. □

The Theorem of Bergstrom and Walker was generalized in Mehta [7] and in Subiza and Peris [13] by relaxing the upper semicontinuity condition. This permits to refine the characterization above in the following terms.

We say that an acyclic binary relation $\succ$ on a set X is *transfer lower continuous* if:

$$\text{for all } x \in X \text{ such that } \exists y' \succ x \Rightarrow \exists y \text{ such that } x \in int\{z \in X : y \succ z\}$$

(as defined in Mehta [7]). This condition is obviously implied by upper semicontinuity. Some authors have dealt with the following alternative expression for transfer lower continuity (e.g. Sonnenschein [12] and Tian and Zhou [14]):

$$\text{for all } x \succ y \text{ there is } x' \text{ and a neighborhood } N(y) \text{ of } y \text{ such that } x' \succ N(y)$$

where $x' \succ N(y)$ means that $x' \succ z$ for all $z \in N(y)$. Then, an acyclic binary relation on a compact topological space that is transfer lower continuous has a maximal element (cf. Mehta [7]).

Following this approach, Subiza and Peris defined a condition which is still weaker than transfer lower continuity and provides a generalization to Mehta's result. They defined that the relation $\succ$ is *transfer lower quasi-continuous* if the following holds:

$$\text{for all } x \succ y \text{ there is } x' \text{ and a neighborhood } N(y) \text{ of } y \text{ such that } x' \succ\succ N(y)$$

An alternative way to introduce this concept, in the line of the original definition used by Mehta, would be the following:

$$\text{for all } x \in X \text{ such that } \exists y' \succ x \Rightarrow \exists y \text{ such that } x \in int\{z \in X : y \succ\succ z\}.$$

Then, an acyclic binary relation on a compact topological space that is transfer lower quasi-continuous has a maximal element (cf. Subiza and Peris [13]).

We now have:

Theorem 5. *Let $\succ$ be an acyclic binary relation on X. The following conditions are equivalent:*

(a) X has a maximal element

(b) there exists a compact topology on X such that $\succ$ is transfer lower continuous

(c) there exists a compact topology on X such that $\succ$ is transfer lower quasi-continuous.

Proof. It is obvious that (b) implies (c), and (a) implies (b) by Theorem 4. The fact that (c) implies (a) is Subiza and Peris's generalization of Mehta's theorem. □

Theorems 1 and 2 in Campbell and Walker [6] also provide sufficient conditions to ensure the existence of maximal elements, for certain types of acyclic binary relations: pseudotransitive relations and interval orders (that they call extratransitive relations instead) respectively. Since these classes of binary relations coincide (as shown in Section 2), those results are included in the following:

Theorem 6. *(*Campbell and Walker*) Let X be a compact topological space. Then, every weakly upper semicontinuous interval order on X has a maximal element.*

Following the approach given by Theorem 4, we obtain further conditions equivalent to the existence of maximal elements for the particular case of interval orders:

Theorem 7. *Let $\succ$ be an interval order on X. The following statements are equivalent:*

(a) X has a maximal element

(b) there exists a compact topology on X such that $\succ$ is weakly upper semicontinuous.

Proof. Implication (b) $\Rightarrow$ (a) is Theorem 6. The converse follows from Theorem 4, since upper semicontinuous interval orders are weakly upper semicontinuous. □

Appendix: The structure of the set of maximal elements

In the case considered by Sloss, Brown, Bergstrom and Walker, the underlying set is compact and the set of maximal elements obtained is compact too (cf. 7.12 in Border [4]). Similarly, in the event that we only assume $\succ$-compactness we shall show that the set of maximal elements is nonempty and $\succ$-compact whenever $\succ$ satisfies a condition weaker than semicontinuity. Therefore, Proposition 2 below will constitute a still sharpened form of the Bergstrom-Walker Theorem. Firstly we need to prove the following auxiliary result:

Lemma 2. *Let $\succ$ be an irreflexive binary relation on (X, τ). Suppose that X is $\succ$-upper compact, and let F be a closed, upper comprehensive set. Then F is $\succ$-upper compact.*

Proof. Let $\{U_\alpha : \alpha \in A\}$ be a collection of lower open sets in F whose union is F. Then $X = \cup\{U_\alpha \cup (X - F) : \alpha \in A\}$. For each $\alpha \in A$ there is an open set V_α in X such that $U_\alpha = V_\alpha \cap F$. Thus, for every $\alpha \in A$ the set $U_\alpha \cup (X - F)$ satisfies:

- It is open because $U_\alpha \cup (X - F) = V_\alpha \cup (X - F)$

- It is a lower comprehensive set in X. For this, let $y \in U_\alpha \cup (X - F)$ and take $x \in X$ such that $y \succ x$. We must show that $x \in U_\alpha \cup (X - F)$. This is obvious if $x \notin F$; assume, therefore, that $x \in F$. Two cases arise. If $y \in X - F$ then $x \in X - F$ because F is an upper comprehensive set, and we are done. Otherwise, $y \in U_\alpha$ lower comprehensive set in F; thus, $x \in U_\alpha$ because $x \in F$.

Therefore, there exists a finite subset B of A such that $X = \cup\{U_\alpha \cup (X - F) : \alpha \in B\}$. It follows that $F = \cup\{U_\alpha : \alpha \in B\}$ because $U_\alpha \subseteq F$. □

Proposition 2. *Let $\succ$ be an acyclic binary relation on (X, τ). Suppose that X is $\succ$-upper compact and that the transitive closure $\succ\succ$ of the relation is upper semicontinuous. Then, the set of maximal elements is non-empty and $\succ$-upper compact.*

Proof. The fact that there are maximal elements follows from a straightforward argument: if there is no maximal element in X then this set is the union of all the subsets $\{y \in X : x \succ\succ y\}$ with x belonging to X. However, those sets constitute a collection of open and lower comprehensive sets, for which there is no finite subcollection that covers X, by the acyclicity of $\succ$. This contradicts the fact that X is $\succ$-upper compact.

On the other hand, it is immediate that the set of maximal elements of $\succ$ is the intersection of all the subsets $X - \{y \in X : x \succ\succ y\}$ with x ranging over X; furthermore, all those subsets are closed, upper comprehensive sets. Because the intersection of closed upper comprehensive sets is again a closed upper set then the set of maximal elements is $\succ$-upper compact, by Lemma 2. □

Remark 3. Notice that the condition that the transitive closure $\succ\succ$ of the relation is upper semicontinuous is weaker than the upper continuity of $\succ$. Both conditions coincide for partial orders.

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