Linear and non-linear price decentralization

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Abstract

Compendious and thorough solutions to the existence of a linear price equilibrium problem, the second welfare theorem, and the limit theorem on the core are provided for exchange economies whose consumption sets are the positive cone of arbitrary ordered Fréchet spaces—dispensing entirely with the assumption that the vector ordering of the commodity space is a lattice. The motivation comes from economic applications showing the need to bring within the scope of equilibrium theory vector orderings that are not lattices, which arise in the typical model of portfolio trading with missing options. The assumptions are on the primitives of the model. They are bounds on the marginals of non-linear prices and for ω-proper economies they are both sufficient and necessary.
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1. Introduction

The second half of the 20th century saw an explosion of interest in the Walrasian model of general equilibrium. Foundational results include the existence of at least one competitive equilibrium, the characterization of valuation equilibria as efficient allocations, and the characterization of competitive equilibria as Edgeworth equilibria.

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For several decades a primary research program in general equilibrium theory has been to establish the classical theorems on price decentralization and the price equilibrium existence problem in a context sufficiently general to encompass as particular instances the important general equilibrium models which arise in applications. This program was well articulated by Mas-Colell [33]. The idea is to separate the very difficult mathematical problems associated with price decentralization from their applications.

The principal contributors to this program are, of course, Mas-Colell [33] and Mas-Colell and Richard [34]. In those papers the authors replace the pervasive finite dimensional assumption on the interiority of endowments in consumption sets with two requirements:

(1) The commodity space is ordered by a vector lattice ordering that is compatible with the topology of the space.[1] The traders' consumption sets coincide with the positive cone of this ordering and preferences are monotone with respect to this ordering and continuous with respect to the topology of the commodity space.
(2) Preferences satisfy an assumption termed ω-uniform properness (where ω is the total endowment of resources), which is a cone condition that could in some settings be interpreted as a bound on the marginal rates of substitution (see for instance [18,12,38,29]).

The work of Mas-Colell [33] was extended in [3,4,16,17,43,44] and the work of Mas-Colell and Richard [34] was extended in [14,21,24,32,36,37,41,42]. In all these papers some variant of ω-properness is assumed and the lattice theoretic arguments are used in a non-trivial way. However, the use of lattice-theoretic techniques comes at a price: the very structure of the economy is expressed in terms of the vector lattice ordering of the commodity space. For instance, the constraints on consumption sets, the notion of monotone preferences, the properness assumptions, free disposal, the notion of a free disposal equilibrium, the topology of the commodity space, and even the compactness assumptions on the set of feasible allocations are all defined with respect to the vector lattice ordering of the commodity space. Therefore, the economic meaning of these results hinges on the interpretive efficacy of the ordering of the space.

Commodity spaces that are not lattice ordered arise naturally in many economic models and the large literature on price decentralization in vector lattices has little, that is obvious, to say in such a setting. An example of such an economic model is portfolio trading when markets are incomplete. It is known that in such models all the decentralization results can fail even if preferences are uniformly proper and the commodity space is finite dimensional. In these models consumers are motivated by the payoff of a portfolio. Therefore, the meaningful natural ordering of the portfolio space is the one that compares portfolio payoffs and which is closely related to the notion of first-order stochastic domination. In fact, the notion of *arbitrage free prices* is an order theoretic notion that induces this natural ordering of the portfolio space. Unfortunately, this ordering is rarely a vector lattice ordering when markets are not complete. The basic intuition for this is the following. Generally, when markets are not complete some call and put options cannot be replicated as the payoff of a portfolio of available securities. However, call and put options are closely related to the order structure of the portfolio space. Indeed, every marketed option is a lattice operation in the portfolio

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[1] Mas-Colell assumes that the lattice operations are uniformly continuous and Mas-Colell–Richard assume that the topological dual of the commodity space is a Riesz subspace of the order dual.

space and every lattice operation in the portfolio space is related to what is termed in the finance literature a *minimum-cost super replicating* portfolio of a call or put option (which need not exist).

Can we extend the existing results on price decentralization and the price equilibrium existence problem to ordered commodity spaces that are not vector lattices? We have the following four pointers:

(a) It is known that when the commodity space is an Archimedean ordered vector lattice the price decentralization results hold true but with possibly discontinuous prices. Jones [27] and Aliprantis and Burkinshaw [7] show that if the commodity space is a vector lattice, then to ensure that the decentralizing prices are continuous we need some lattice structure on the space of continuous prices.

(b) In the very general context where the commodity space is an ordered vector space that need not be a vector lattice, Aliprantis–Tourky–Yannelis [13] introduced an alternate value theory that arises from a personalized pricing system which induces a possibly non-linear value function. The classical price decentralization problems are solved by means of these non-linear value functions.

(c) Aliprantis–Monteiro–Tourky [9] give an example of a uniformly proper model of port-folio trading, with three securities and two traders (i.e., with a three-dimensional commodity space), whose portfolio space is not lattice ordered and in which all the decentralization theorems fail for linear prices. In that example, optimal allocations can be supported by the Aliprantis–Tourky–Yannelis non-linear prices.

(d) Aliprantis and Tourky [10] show that the dual cone of every ordered topological vector space can be embedded in a larger lattice ordered cone that contains non-linear func-tionals, which they term the *super order dual* of the ordered vector space. Crucially, this embedding preserves lattice operations. By means of this embedding, they show that many of the useful structures of vector lattices have analogues in general ordered vector space settings. That is, ordered vector spaces are basically not so different from vector lattices.

Our purpose in this paper is to provide a thorough solution to the (linear) price decentral-ization and equilibrium existence problems in commodity spaces that are ordered Fréchet spaces, which includes the important class of ordered Banach spaces. The paper provides both necessary and sufficient conditions for linear price decentralization.

We deal with well behaved exchange economies that we call ω-proper economies, which include the class of ω-uniformly proper economies. We show that the linear decentralization of Pareto optimal allocations and Edgeworth equilibria as well as the existence of a linear price equilibrium are all basically equivalent to the properness of functionals in the super-order dual of the commodity space. More precisely, we consider the embedding of the dual cone of the commodity space in the super order dual, which is a lattice. Then we denote by $\mathcal{P}$ the semi-lattice generated by the cone of positive linear prices and call $\mathcal{P}$ our *price space*. Each possibly non-linear price p in $\mathcal{P}$ can be written as the supremum $p = \bigvee_{i=1}^{m} f_i$, where $f_1, \ldots, f_m$ are positive linear prices. Now each function p in $\mathcal{P}$ is a concave function defined on the positive cone of the commodity space. Therefore, the standard properness notions can be defined for these functions like any utility function. Surprisingly, it turns out that if each price in $\mathcal{P}$ satisfies a properness assumption, then the following

hold true:

(1) The second welfare theorem is valid, i.e., Pareto optimal allocations can be decentralized as (linear) Walrasian valuation equilibria.
(2) The limit theorem on the core holds, i.e., Edgeworth equilibria are (linear) Walrasian equilibria.
(3) With the standard compactness assumptions, there exists a (linear) Walrasian equilibrium.

The second major set of results of the paper shows that these same properness assumptions on the price space $\mathcal{P}$ are also necessary conditions for linear decentralization and the existence of equilibria.

The linear price decentralization theorems provide positive results in a host of infinite dimensional settings that are not vector lattice ordered. We note that the ω-properness of prices in $\mathcal{P}$ depends on the order structure of the commodity space and the "location" of the total endowment of resources. Furthermore, it is relatively easy to check if a price $p = \bigvee_{i=1}^{m} f_i$ satisfies our properness assumption. Indeed, in a large class of settings, properness is equivalent to the existence of a solution to the following convex programming problem:

$$\min g(\omega) \quad \text{subject to}: \quad g \text{ is linear and } g \geqslant f_1, \ldots, f_m.$$

In the finance framework this minimization problem translates into the following linear minimization problem:

$$\min g(T(\omega)) \quad \text{subject to}: \quad T'(g) \geqslant f_i \quad \text{for all } i = 1, \ldots, m,$$

where each f_i is an arbitrage free securities price, T is a portfolio returns operator that associates with each portfolio a state contingent claim, and T' is the adjoint of T that associates with each Arrow–Debreu price a securities price. This second minimization problem is a variant of the *canonical linear minimization problem* and is the subject of much investigation in the mathematical literature on linear programming and operations research.

The main surprise of the analysis in the present paper is that the theory of value with linear prices can be articulated in terms of the cone conditions ubiquitously called properness in the economics literature. That is, the main contribution in the literature that follows the work of Mas-Colell [33] is the application of cone conditions to the primitive data of the economy rather than the application of lattice theoretic ideas to general equilibrium analysis.

The paper is organized as follows. Section 2 motivates the study in the paper by introducing an abstract framework of portfolio trading with incomplete markets. Our analysis begins in Section 3. The main results of the paper are contained in Sections 4 and 5. We return to financial models in the last section of the paper and apply our results.

2. A motivating framework

The purpose of this section is to introduce a very general securities model that includes as special cases many of the securities models studied in the literature. Our approach will

be an operator theoretic study that is closely related to the setup of Jarrow et al. [26] and Galvani [25]. For a study of positive operators see the monographs [1,6]. We refer the reader to the next section of this paper for definitions concerning ordered vector spaces.

Let us summarize the basic framework. In a typical finance model one begins with a pair of function spaces L and X, where L is the portfolio space and X is the space of all contingent claims, together with a *portfolio returns operator* $R: L \to X$ that satisfies the following two properties.

R1: *There are no redundant securities* (*that is, R is one-to-one*) *and R is positive.*
R2: *The operator R pulls back order intervals of $R(L)$ to closed and bounded subsets of L.*

The operator $R: L \to X$ reorders the portfolio space L according to the ordering of X. This new ordering of the portfolio space is called the portfolio dominance ordering (a notion introduced in [5]), which in general is not a vector lattice ordering even if X is a vector lattice. Portfolio traders are motivated and constrained by this new ordering rather than the canonical ordering of L, which is basically redundant. Crucially, the arbitrage freeness of securities prices is relative to this new generally non-lattice portfolio dominance ordering. Some important examples of portfolio returns operators are provided by embeddings $R: L \hookrightarrow X$. That is, when $R(L)$ is closed or, equivalently, when R is bounded below, i.e., there exists some constant $c > 0$ such that $\|R\theta\| \geqslant c\|\theta\|$ holds for all $\theta \in L$; see [1, Theorem 2.5].

In a two period model the Banach lattice X could be for instance one of the following:

(a) The Banach lattice $X = L_2(\mathcal{S}, \Sigma, \pi)$ of all square integrable functions on $\mathcal{S}$, where $(\mathcal{S}, \Sigma, \pi)$ is a probability space representing period one uncertainty.
(b) Given a compact topological space $\mathcal{S}$ representing the states of the world tomorrow, the Banach lattice $X = C(\mathcal{S})$ of all continuous functions on $\mathcal{S}$.
(c) Any of the Banach lattice commodity spaces from the existing literature.

In a continuous time model, we are given a probability space $(\mathcal{S}, \Sigma, \pi)$ and an interval $T \subseteq \mathbb{R}_+$ such that the space X is the space of all stochastic processes of the form $f(t, s)$, which are square integrable functions from T to $L_2(\mathcal{S}, \Sigma, \pi)$.

Two examples in the finance literature of portfolio spaces are as follows:

(a) In a two-period model the available securities are indexed by a positive σ-finite measure space $(\mathcal{J}, \mathcal{F}, \mu)$. The portfolio space is the Banach space $L = L_2(\mathcal{J}, \mathcal{F}, \mu)$. A portfolio is any vector in the portfolio space L.
(b) In a continuous time model where T is a subinterval of $\mathbb{R}$, the space of portfolios is the space of portfolio processes from T to $L_2(\mathcal{J}, \mathcal{F}, \mu)$. For more on portfolio processes we refer the reader to [28].

Two important examples of assets returns operators are presented next.

(1) When there are finitely many states of the world and finitely many available securities the assets returns operator R is simply a non-negative matrix whose columns delineate the available non-redundant assets. For more on this example see for instance [31,39].
(2) In the case the portfolio space is the Banach function space $L = L_2(\mathcal{J}, \mathcal{F}, \mu) = L_2(\mathcal{J})$ and the space of contingent claims is $X = L_2(\mathcal{S}, \Sigma, \pi) = L_2(\mathcal{S})$, R is an integral operator. Here, the payoff of a security j at state s is denoted $P(j, s)$. The

function $P\colon \mathcal{J} \times \mathcal{S} \to \mathbb{R}_+$ is called the *assets returns kernel*, which is assumed to be $\mu \times \pi$-measurable. We assume that for each portfolio $\theta \in L$ the fnction $j \mapsto P(j,s)\theta(j)$ is μ-integrable for π-almost all $s \in \mathcal{S}$. Moreover, the payoff $[R\theta](s)$ of a portfolio θ for π-almost all states of the world s is given by

$$[R\theta](s) = \int_{\mathcal{J}} P(j,s)\theta(j)\, d\mu(j)\,.$$

We assume that the range of the operator R lies in $L_2(\mathcal{S})$. Therefore, R defines an integral linear operator from the portfolio space $L_2(\mathcal{J})$ to the contingent claims space $L_2(\mathcal{S})$. In this case, notice that this operator is positive if and only if the assets returns kernel is non-negative.

In (2) above it is not always true that the portfolio returns operator R pulls back order intervals of $R(L)$ to closed and bounded subsets of L. However, an example of a portfolio returns operator that pulls back order intervals of $R(L)$ to closed and bounded subsets of L will be presented in Section 6.

Let us formalize the general framework. We are given a Banach lattice X that represents the space of all real (possibly multiperiod) contingent claims. A vector in X is called a *contingent claim*. The positive cone of this space is X_+ and, as usual, a consumer's consumption decision is restricted to this cone. Agents trade portfolios of available securities that pay contingent claims in X. They choose a *portfolio* which is a vector in a given Banach lattice L called the *portfolio space*. The payoffs of portfolios are given by means of the portfolio returns operator $R\colon L \to X$. So, the *payoff* of a portfolio $\theta \in L$ is $R(\theta) \in X$ or simply $R\theta$.

A contingent claim $x \in X$ is said to be *marketed* if there exists a portfolio $\theta \in L$ such that $R\theta = x$. The vector subspace of all marketed contingent claims is denoted $\mathcal{M}$, and clearly, $\mathcal{M} = R(L)$, the range of R. We say that the market is *complete* if $\overline{\mathcal{M}} = X$, where $\overline{\mathcal{M}}$ denotes the closure of $\mathcal{M}$. If $\overline{\mathcal{M}} \neq X$, then the market is *incomplete*.

The portfolio returns operator induces a new ordering $\geqslant$ on the portfolio space L as follows:

$$\theta \geqslant \eta \iff R\theta \geqslant R\eta\,.$$

This ordering is termed *portfolio dominance*. Denote the positive cone of this ordering of the portfolio space by $L_+ = \{\theta \in L\colon R\theta \geqslant 0\}$, which is in general different from the canonical positive cone of the portfolio space L.

From now on when we talk about the ordering of the portfolio space we are referring to the portfolio dominance ordering.

The portfolio space under this ordering fits perfectly into the analysis in the sequel.

Lemma 1. *Under portfolio dominance the portfolio space is an ordered Banach space with a generating positive cone and bounded order intervals. If L is reflexive or if $\mathcal{M}$ is closed and X has an order continuous norm, then order intervals in the portfolio space are weakly compact.*

Proof. Notice first that the portfolio cone is generating since it contains the original positive cone of L, which is generating. Moreover, $L_+ = R^{-1}(X_+)$ is closed.

If L is reflexive, then the dominance order intervals of L are closed and bounded by assumption, and hence they are weakly compact. When $\mathcal{M}$ is closed, then by the open mapping theorem the operator $R: L \to \mathcal{M}$ is a topological and order isomorphism. Moreover, since X has order continuous norm the order intervals of X are weakly compact and so are the order intervals of $\mathcal{M}$. By the continuity of $R^{-1}: \mathcal{M} \to L$ it follows that the dominance order intervals of L are likewise weakly compact. $\square$

Notice that commodity spaces that are Banach lattices with order continuous norms are ubiquitous in modern general equilibrium analysis.

Clearly, if $\mathcal{M} = X$, then portfolio dominance is a vector lattice ordering. This is so since the portfolio returns operator is surjective and therefore it reorders the portfolio space in a vector lattice manner. However, if the market is incomplete, then portfolio dominance is unlikely to be a vector lattice ordering. In fact, one can characterize the very rare situations in which this ordering is a vector lattice ordering.

For any set Y in the space of contingent claims let $\mathcal{R}(Y)$ be the Riesz subspace of X generated by Y, i.e., $\mathcal{R}(Y)$ is the smallest Riesz subspace containing Y.

Lemma 2. *Portfolio dominance is a vector lattice ordering if and only if there exists a positive projection from $\mathcal{R}(\mathcal{M})$ to $\mathcal{R}(\mathcal{M})$ whose range is the marketed space $\mathcal{M}$.*

Proof. Clearly, portfolio dominance is a vector lattice ordering if and only if $\mathcal{M}$ is a Riesz space under its canonical ordering as a subspace of X. But by Miyajima's result [1, Theorem 5.64] this is the case if and only if there exists a positive projection [2] from $\mathcal{R}(\mathcal{M})$ to $\mathcal{R}(\mathcal{M})$. $\square$

The lattice operations in the portfolio space are closely related to the presence of marketed call options. Recall, that a contingent claim $x \in X$ is called *marketed* if there exists a portfolio $\theta \in L$ satisfying $R\theta = x$. Now take a portfolio θ and a strike price $k \in X$. Assume that k is marketed and let $\kappa \in L$ satisfy $R\kappa = k$. The *call option* on the portfolio θ with strike price k is the contingent claim $[R\theta - k]^+ = (R\theta - k) \vee 0 \in X$. The call option is *non-trivial* if neither $R\theta \geqslant k$ nor $R\theta \leqslant k$ hold. If such an option is marketed, then the supremum $[\theta - \kappa]^+ = (\theta - \kappa) \vee 0$ exists in L for the portfolio dominance ordering and $R[\theta - \kappa]^+ = [R\theta - k]^+$.

In other words, a call option is a vector lattice operation in X and if it is marketed, then it is replicated by a vector lattice operation in L. So, the marketability of call options transfers the lattice structure of X to L. However, we know the following results from the finite dimensional setting.

Lemma 3. *Suppose that there are J finitely many nominal securities and S finitely many states of the world. Let Φ be the space of all assets returns matrices for which the bond $\mathbf{1}$*

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[2] An operator $T: V \to V$ on a vector space is called a *projection* if $T^2 = T$. If V is an ordered vector space a projection that is also a positive operator is called a *positive projection*.

is marketed. The following results hold true:

(1) *Generically in Φ the market is complete if and only if all call options are marketed.*
(2) *If $J \leqslant \frac{1}{2}(S+1)$, then generically in Φ no non-trivial call option is marketed.*

Proof. For (1) see [39] and for (2) see [11]. □

Finally for this section, let us define the notion of arbitrage free securities prices. A *securities price* is simply a continuous linear functional on the portfolio space. Therefore, the dual space L' is called the space of *securities prices*. A securities price p is *weakly arbitrage free* if for any portfolio $\theta \in L_+$ (i.e., $R\theta \in X_+$) we have $p(\theta) \geqslant 0$. A securities price p is *arbitrage free* if for any non-zero portfolio $\theta \in L_+ \backslash \{0\}$ we have $p(\theta) > 0$. Therefore, arbitrage freeness is defined in terms of the portfolio dominance ordering of L.

In fact, portfolio dominance induces an ordering on the space of securities prices as follows: $p \geqslant q$ if $p(\theta) \geqslant q(\theta)$ for all $\theta \in L_+$. This ordering is a vector ordering that makes L' an ordered Banach space. As usual denote by L'_+ the positive cone of L'. That is, $L'_+ = \{f \in L': f \geqslant 0\}$.

Lemma 4. *A securities price p is weakly arbitrage free if and only if $p \in L'_+$. It is arbitrage free if and only if it is a strictly positive linear functional (i.e., $p \gg 0$).*

3. The general equilibrium model

In this section, we introduce the commodity space, the price space, proper economies, and the various notions of equilibria. For technical details not covered in this paper, we refer the reader to the following monographs. For general infinite dimensional analysis see [2,23]. For the theory of ordered topological vector spaces see [35,40]. For vector lattices see [8,30]. We shall extensively use results obtained in [10] regarding the super order dual of an ordered vector space.

THE COMMODITY SPACE is an ordered Fréchet space L with a closed and generating positive cone whose order intervals are topologically bounded. That is:

(a) L is a topological vector space endowed with a complete metrizable locally convex topology τ.
(b) L is equipped with an order relation $\geqslant$ such that $x \geqslant y$ imply $\alpha x + z \geqslant \alpha y + z$ for all $z \in L$ and all scalars $\alpha \geqslant 0$.
(c) The *positive cone* $L_+ = \{x \in L: x \geqslant 0\}$ is closed and generating, i.e., $L_+ - L_+ = L$.
(d) Each *order interval* $[0, x] = \{y \in L_+: 0 \leqslant y \leqslant x\}$ is τ-bounded.

The *topological dual* of (L, τ) is denoted L'. It is also an ordered vector space under its canonical ordering, i.e., $f \geqslant g$ if $f(x) \geqslant g(x)$ for all $x \in L_+$. We shall require that at least one of Mas-Colell's [33] quasi-concave uniformly proper functions that is also strictly monotone exists, or equivalently:

- *There exists a strictly positive linear functional $f \in L'$, i.e., $f(x) > 0$ for all $x > 0$.*

The dual of our commodity space is well behaved.

Lemma 5. *Every positive linear functional is continuous and the cone L'_+ is generating.* [3]

If the *least upper bound* (*supremum*) (resp. the *greatest lower bound* (*infimum*)) of a subset S of an ordered space exists, then it may be denoted $\sup S$ or $\bigvee_{i=1}^{m} x_i$ when $S = \{x_1, \dots, x_m\}$ or $x \vee y$ when $S = \{x, y\}$ (resp. $\inf S$ or $\bigwedge_{i=1}^{m} x_i$ or $x \wedge y$). As usual, we shall say that L is a *Riesz space* (or a *vector lattice*) if $x \vee y$ and $x \wedge y$ exist in L for all $x, y \in L$.

THE PRICE SPACE will be a semi-lattice of the super order dual of L_+. We begin with a few definitions. We say that a function $f: L_+ \to \mathbb{R}$ is *positive* if $f(x) \geqslant 0$ holds for each $x \in L_+$; is *super-additive*, if $f(x) + f(y) \leqslant f(x + y)$ holds for all $x, y \in L_+$; *additive* if $f(x + y) = f(x) + f(y)$ holds for all $x, y \in L_+$; and is *positively homogeneous*, if $f(\alpha x) = \alpha f(x)$ holds for every $\alpha \geqslant 0$ and all $x \in L_+$.

The price space will be a subset of the following collection of real-valued non-negative functions on L_+ given by

$$\Pi = \left\{ f \in \mathbb{R}_+^{L_+}: \ f \text{ is super-additive and positively homogeneous} \right\}.$$

It should be noticed that every function $f \in \Pi$ is *monotone* in the sense that $0 \leqslant x \leqslant y$ implies $0 = f(0) \leqslant f(x) \leqslant f(y)$. Clearly, Π is a cone of $\mathbb{R}^{L_+}$. The cone Π is called the *super order dual cone* of L_+ and was introduced and studied in detail in [10]. Notice that Π inherits the natural pointwise order of $\mathbb{R}^{L_+}$. In other words, for $f, g: L_+ \to \mathbb{R}$ we write $f \geqslant g$ (and say that f *dominates* g) whenever $f(x) \geqslant g(x)$ holds for each $x \in L_+$.

The important result for us here is that the cone Π is a lattice even though L may not be a vector lattice; see [10, Corollary 4.4].

Theorem 6. *The super order dual cone Π with its natural ordering is a lattice. Moreover, the supremum operation for any pair f_1, $f_2 \in \Pi$ and any $x \in L_+$ is given by*

$$[f_1 \vee f_2](x) = \sup \{ f_1(y_1) + f_2(y_2) : \ y_1, y_2 \in L_+ \ and \ y_1 + y_2 = x \}.$$

The dual cone L'_+ can be viewed as a subcone of the cone Π. In particular, throughout our discussion we shall use the following simple but very useful result, whose proof follows from the Kantorovich extension theorem; see for instance [1, Theorem 1.15].

Lemma 7. *The positive linear functionals on L are precisely the extensions of the additive functions in Π. So, we can write $L'_+ = \{ f \in \Pi: \ f \text{ is additive} \}$.*

As a matter of fact, there is a natural embedding $J: L'_+ \to \Pi$ defined by

$$J(f)(x) = f(x)$$

for all $x \in L_+$. This embedding is clearly one-to-one and monotone. In addition, we have the following remarkable result proved in [10, Theorem 5.5].

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[3] From [10, Theorem 7.2] we know that $L' = L^r = L^\sim$ where L^r is the *order regular dual* of L comprising of linear functionals that can be written as differences of positive linear functionals and $L^\sim$ is the *order dual* of L consisting of all order bounded linear functionals on L.

Theorem 8 (*Aliprantis–Tourky*). *Assume that the order intervals of L are weakly compact. If the supremum $\bigvee_{i=1}^{n} f_i$ of the positive continuous linear functionals $f_1, \ldots, f_n$ on L exists in L', then*

$$J\left(\bigvee_{i=1}^{n} f_i\right) = \bigvee_{i=1}^{n} J(f_i)$$

holds in Π. That is, the embedding $J: L'_+ \to \Pi$ preserves finite suprema.

Let us say that a non-empty subset of a lattice is a *semi-lattice* if it is closed under finite suprema. Every non-empty subset A of a lattice is included in a smallest semi-lattice $\mathcal{U}(A)$, called *the semi-lattice generated* by A. Clearly, $\mathcal{U}(A)$ consists of all finite suprema of the set A. With this definition in mind, we are now ready to define our price space.

Definition 9. The *price space* $\mathcal{P}$ is the semi-lattice $\mathcal{U}(L'_+)$, i.e., $\mathcal{P}$ consists of all finite suprema of L'_+ in the super dual cone Π.

The pair $\langle L, \mathcal{P} \rangle$ will be our *commodity-price duality*. In the special case of a Riesz commodity space, i.e., when L is a Fréchet lattice, the price space $\mathcal{P}$ coincides with the positive cone of L' and we regain the standard commodity-price duality in the literature. We have the following result concerning the price space $\mathcal{P}$.

Lemma 10. *If $\mathcal{P} = L'_+$, then L' is a Riesz space. Conversely, $\mathcal{P} = L'_+$ whenever one of the following conditions holds true:*

(1) *L is a Riesz space.*
(2) *L has the Riesz decomposition property, i.e., $[0, x] + [0, y] = [0, x + y]$ holds for all $x, y \in L_+$.*
(3) *L is a Banach space and L' is a Riesz space in its own right.*
(4) *The order intervals of L are weakly compact and L' is a Riesz space in its own right.* [4]

Proof. First suppose that $\mathcal{P} = L'_+$. Fix $f \in L'$. Since $L' = L^r$ holds, it follows that there exists some $g \in L'_+$ satisfying $-g \leqslant f$. By our assumption $(f + g) \vee g = g + f \vee 0$ exists in $\mathcal{P} = L'_+$. This implies that $f \vee 0$ exists in L'. Therefore, L' is a Riesz space.

The proofs of (1)–(4) follow from the fact that in those settings for any $f_1, f_2 \in L'_+$ and for any $x \in L_+$ the supremum is given by the Riesz–Kantorovich formula:

$$[f_1 \vee f_2](x) = \sup \{f_1(y_1) + f_2(y_2): y_1, y_2 \in L_+ \text{ and } y_1 + y_2 = x\},$$

in both L' and in Π. See [15] for (3) and [10] for (4). $\quad\square$

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[4] We note that every Riesz space has the Riesz decomposition property, but not every space that has the Riesz decomposition property is a Riesz space. The following is a open question in the mathematical literature: *Does there exist an ordered vector space that does not have the Riesz decomposition property and whose regular dual L^r is a vector lattice?*

Each price in $\mathcal{P}$ can be written as the supremum $\bigvee_{i=1}^{n} f_i$ of linear functions $f_1, \ldots, f_n$ in L'_+. So, from now on, $\bigvee_{i=1}^{n} f_i$ shall be understood to be the price in $\mathcal{P}$ which is the supremum of the linear functions $f_1, \ldots, f_n \in L'_+$.

Perhaps we can appreciate why in general the suprema in $\mathcal{P}$ are not additive by looking at the special case of a three dimensional commodity space. The half-ray of the cone L_+ determined by a vector $x \in L_+$ is the set $\mathcal{L}_x = \{\alpha x \colon \alpha \geqslant 0\}$. A half-ray $\mathcal{L}_x$ of L_+ is called an *extremal ray* of the cone L_+ if for each $y \in \mathcal{L}_x$ we have $[0, y] = \{\delta y \colon 0 \leqslant \delta \leqslant 1\}$. The following result is easy to prove; see [10].

Lemma 11. *Let L be a three dimensional vector space. There exist at least one pair of non-comparable linear functionals $f, g \in L'_+$ such that the function $f \vee g \in \mathcal{P}$ is additive if and only if L_+ has at most four extremal rays.*

Domination in $\mathcal{P}$ is characterized as follows; for a proof see [10, Lemma 3.7].

Lemma 12 (Aliprantis–Tourky). *Assume that $f, g \in \mathcal{P}$. If $f(x_0) > 0$ holds for some $x_0 \in L_+$, then the following statements are equivalent.*

(1) *The function g dominates f, i.e., $g(y) \geqslant f(y)$ holds for each $y \in L_+$.*
(2) *For each $y \in L_+$ satisfying $f(y) = f(x_0)$ we have $g(y) \geqslant f(x_0)$.*

THE ECONOMY has m consumers designated by the index $i = 1, \ldots, m$. As usual, the consumption set of consumer i is L_+. The bundle $\omega_i > 0$ is the ith consumer's *initial endowment* and $\omega = \sum_{i=1}^{m} \omega_i$ is the *total endowment*.

The correspondence $P_i \colon L_+ \twoheadrightarrow L_+$ denotes the ith consumer's (strict) preferences. We assume that each mapping P_i satisfies the standard assumptions. That is, P_i is *irreflexive*, i.e., $x \notin P_i(x)$ for each $x \in L_+$; is *strictly monotone*, i.e., $x + y \in P_i(x)$ for each $y > 0$; is *convex-valued*; has *open values* in L_+ for some linear topology on L; and is ω-*proper* in the sense of Tourky [41]:

- There exists another convex-valued correspondence $\widehat{P}_i \colon L_+ \twoheadrightarrow L$ such that for each $x \in L_+$:

  (a) The vector $x + \omega$ is an interior point of $\widehat{P}_i(x)$.
  (b) $\widehat{P}_i(x) \cap L_+ = P_i(x)$.

An exchange economy with commodity-price duality $\langle L, \mathcal{P} \rangle$ and with the above characteristics will be referred to as a *proper economy*. We will say that the economy is *compact* if order intervals of L are weakly compact and preferences have *weakly open lower* sections, that is, for each bundle $y \in L_+$ the lower section $P_i^{-1}(y) = \{x \in L_+ \colon y \in P_i(x)\}$ is weakly open in L_+.

For any $x \in L_+$ let $\mathcal{A}_x = \{y \in L_+^m \colon \sum_{i=1}^{m} y_i = x\}$. As usually, $(x_1, \ldots, x_m) \in \mathcal{A}_\omega$ is said to be an *allocation*. In our general setting, we say that *endowments are decomposable* if

$$\mathcal{A}_{\omega_1} + \mathcal{A}_{\omega_2} + \cdots + \mathcal{A}_{\omega_m} = \mathcal{A}_\omega.$$

This is a very natural assumption in economics. It means that each allocation can be viewed as the outcome of bilateral trading. Notice that in the presence of the Riesz decomposition property this condition is always true. However, this equality can be true for some choices of initial endowments in any ordered vector space.

Definition 13. For a given commodity space L and any fixed $\omega > 0$, we shall denote the collection of all proper economies with commodity-price duality $\langle L, \mathcal{P} \rangle$ and total endowment ω by $\mathcal{E}(L, \omega)$. The collection of economies in $\mathcal{E}(L, \omega)$ for which the endowments are decomposable will be denoted $\mathcal{E}_\mathrm{D}(L, \omega)$.

The next result informs us that we are not working with an empty set of economies.

Lemma 14. *For any commodity space L and any fixed $\omega > 0$ the collection of all decomposable economies $\mathcal{E}_\mathrm{D}(L, \omega) \subseteq \mathcal{E}(L, \omega)$ is non-empty and contains economies with arbitrarily many consumers. Further, if L has the Riesz decomposition property, then $\mathcal{E}(L, \omega) = \mathcal{E}_\mathrm{D}(L, \omega)$.*

Proof. To see that $\mathcal{E}_\mathrm{D}(L, \omega)$ is non-empty, let $\omega_i = \frac{1}{m}\omega$ and assume that every consumer i has the utility function $u_i(x) = f(x)$, where $f \in L'$ is strictly positive. $\square$

Let us introduce the standard notions of efficiency. An allocation $(x_1, \ldots, x_m)$ is said to be *weakly Pareto optimal*, if there is no allocation $(y_1, \ldots, y_m)$ satisfying $y_i \in P(x_i)$ for each $i \in I$; a *core allocation*, if it cannot be blocked by any allocation in the sense that there is no allocation $(y_1, \ldots, y_m)$ and a coalition S such that

(a) $\sum_{i \in S} y_i = \sum_{i \in S} \omega_i$, and
(b) $y_i \in P_i(x_i)$ for all $i \in S$;

an *Edgeworth equilibrium*, if it belongs to the core of every r-fold replica economy.[5]

A PRICE EQUILIBRIUM in this paper is the generalization in [13] of the standard notion of Walrasian equilibrium to the case of the extended commodity price duality $\langle L, \mathcal{P} \rangle$.

Definition 15. An allocation $(x_1, \ldots, x_m)$ is said to be:

(1) A *valuation equilibrium*, if there exists some price $p \in \mathcal{P}$ (called a *supporting price*) such that:

   (a) $y \in P_i(x_i)$ implies $p(y) > p(x_i)$, and
   (b) the arbitrage-free condition $0 < p(\omega) \leqslant \sum_{i=1}^m p(x_i)$ holds.

(2) An *equilibrium*, if there exists some price $p \in \mathcal{P}$ (again called a *supporting price*) such that:

   (a) $p(\omega) > 0$,

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[5] The notion of Edgeworth equilibrium was introduced in [3] and was inspired by the notion of the *strict core* of Debreu and Scarf [20]. It was shown in [22] that Edgeworth equilibria exist in compact economies.

(b) $y \in P_i(x_i)$ implies $p(y) > p(x_i)$, and
(c) for each $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m_+$ we have $p\left(\sum_{i=1}^m \alpha_i \omega_i\right) \leqslant \sum_{i=1}^m \alpha_i p(x_i)$.

(3) A *personalized equilibrium* (resp. a *personalized valuation equilibrium*), whenever it is an equilibrium (resp. a valuation equilibrium) with respect to some supporting price of the form $p = \bigvee_{i=1}^m f_i$ with $f_i \in L'_+$ for each i that satisfies $p(\omega) = \sum_{i=1}^m f_i(x_i)$.
(4) A *Walrasian equilibrium* (resp. a *Walrasian valuation equilibrium*), if it is an equilibrium (resp. a valuation equilibrium) with an additive supporting price in $\mathcal{P}$.

Notice that condition (b) of part (1) of the above definition is equivalent to:

$$p(\omega) = \sum_{i=1}^m p(x_i) > 0.$$

In view of Lemma 7 it is easy to see that our definition of a Walrasian equilibrium coincides with the standard definition in the literature. The following theorem, established in [13, Theorems 5.1 and 7.5 and Lemma 6.3], is the basic result in the literature regarding the decentralization of efficient allocations by non-linear prices.

Theorem 16 (*Aliprantis–Tourky–Yannelis*). *For a proper economy the following hold true*:

(1) *An allocation is weakly Pareto optimal if and only if it is a valuation equilibrium.*
(2) *An allocation is an Edgeworth equilibrium if and only if it is an equilibrium.*
(3) *Every equilibrium is a personalized equilibrium and every valuation equilibrium is a personalized valuation equilibrium.*
(4) *If the economy is compact, then there exists an equilibrium.*

We also have the following companion to Theorem 16 that was obtained in [9].

Theorem 17. *There exists a compact proper economy in which an equilibrium with a non-linear price exists but no Walrasian equilibrium with* (*continuous or discontinuous*) *linear prices exists, and Pareto optimal allocations that are supported as equilibria by non-linear prices but not as Walrasian valuation equilibria with linear prices.*

4. Linear decentralization

In this section, we look at sufficient conditions on the order structure of the space that will guarantee the supportability of optimal allocations by linear prices. But first, let us recall some notions of properness for functions on L_+.

Definition 18. We say that a function $f : L_+ \to \mathbb{R}$ is:

(1) *ω-proper* at some $y \in L_+$, if there exists a convex set F such that:

(a) $y + \omega$ is an interior point of F, and

(b) $F \cap L_+ = \{z \in L_+ : f(z) > f(y)\}$.

(2) *Pointwise proper* at ω, if there exists an open wedge [6] Γ such that:

    (a) $(\omega - \Gamma) \cap L_+ \neq \emptyset$, and

    (b) $(\omega - \Gamma) \cap \{x \in L_+ : f(x) > f(\omega)\} = \emptyset$.

Some basic properties of the preceding notions of properness are included in the next result.

Lemma 19. *For a function $f : L_+ \to \mathbb{R}$ we have the following.*

(1) *If there exists a price $\pi \in L'_+$ that supports the set $\{z \in L_+ : f(z) > f(\omega)\}$ at ω (i.e., $z \in L_+$ with $f(z) > f(\omega)$ implies $\pi(z) \geqslant \pi(\omega)$) and satisfies $\pi(\omega) > 0$, then f is pointwise proper at ω.*

(2) *A price $p \in \mathcal{P}$ that satisfies $p(\omega) > 0$ is pointwise proper at ω if and only if there exists some price $\pi \in L'_+$ such that:*

    (a) $\pi(\omega) = p(\omega)$,

    (b) $\pi \geqslant p$, *and*

    (c) π *supports the convex set* $\{z \in L_+ : p(z) \geqslant p(\omega)\}$ *at* ω.

(3) *If f is ω-proper at ω, then f is pointwise proper at ω.*

Proof. (1) Assume that the set $\{z \in L_+ : f(z) > f(\omega)\}$ is supported at ω by a price $\pi \in L'_+$ that satisfies $\pi(\omega) > 0$. Now consider the open wedge $\Gamma = \{x \in L : \pi(x) > 0\}$ and note that $\omega - \frac{1}{2}\omega \in (\omega - \Gamma) \cap L_+$ and that if some $\gamma \in \Gamma$ satisfies $\omega - \gamma \in L_+$ and $f(\omega - \gamma) > f(\omega)$, then $\pi(\omega) \leqslant \pi(\omega - \gamma)$ or $\pi(\gamma) \leqslant 0$, which is impossible. Hence, $(\omega - \Gamma) \cap \{x \in L_+ : f(x) > f(\omega)\} = \emptyset$, and thus f is pointwise proper at ω.

(2) Assume that Γ is an open wedge such that $(\omega - \Gamma) \cap \{z \in L_+ : p(z) > p(\omega)\} = \emptyset$ and $(\omega - \Gamma) \cap L_+ \neq \emptyset$. By the separation theorem there exists some non-zero $\pi \in L'$ such that $\pi(\omega - \gamma) \leqslant \pi(z)$ holds for all $\gamma \in \Gamma$ and all $z \in L_+$ with $p(z) > p(\omega)$. From $p(\delta\omega) > p(\omega)$ for all $\delta > 1$ we get $\pi(\omega - \gamma) \leqslant \delta\pi(\omega)$ for all $\delta > 1$. This implies $\pi(\gamma) \geqslant 0$ for all $\gamma \in \Gamma$, and consequently $\pi(\gamma) > 0$ for all $\gamma \in \Gamma$. [7]

To see that $\pi \geqslant 0$ holds fix $z \in L_+$ and note that for each real number $\delta > 1$ we have $p(\delta\omega + z) > p(\omega)$. This implies $\pi(\omega - \gamma) \leqslant \delta\pi(\omega) + \pi(z)$ for all $\delta > 1$ and all $\gamma \in \Gamma$, and from this it follows that $\pi(z) \geqslant 0$, i.e., $\pi \geqslant 0$. Next, pick some $\gamma_1 \in \Gamma$ with $\omega - \gamma_1 \in L_+$ and note that $0 \leqslant \pi(\omega - \gamma_1) < \pi(\omega)$. Moreover, since p is concave, it should be clear that π supports the set $\{z \in L_+ : p(z) \geqslant p(\omega)\}$ at ω. If we replace π by $\frac{p(\omega)}{\pi(\omega)}\pi$ we have $\pi(\omega) = p(\omega)$. The fact that $\pi \geqslant p$ follows immediately from Lemma 12. The converse follows immediately from part (1).

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[6] An open *wedge* Γ is any nonempty open convex set such that $\alpha x \in \Gamma$ for any $\alpha > 0$ and $x \in \Gamma$.

[7] If $\pi(\gamma_0) = 0$ holds for some $\gamma_0 \in \Gamma$, then pick some circled neighborhood V of zero with $\gamma_0 + V \subseteq \Gamma$ and note that $\pi(\gamma_0 + v) \geqslant 0$ implies $\pi(v) \geqslant 0$ for all $v \in V$. The latter shows that $\pi = 0$, which is a contradiction.

(3) Assume that f is ω-proper at ω. Pick a convex set F such that 2ω in an interior point of F and $F \cap L_+ = \{z \in L_+ : f(z) > f(\omega)\}$. Since $\omega \notin \{z \in L_+ : f(z) > f(\omega)\}$, it follows that $\omega \notin F$. So, by the separation theorem, there exists some non-zero $\pi \in L'$ such that $\pi(\omega) \leqslant \pi(x)$ for all $x \in F$. Since 2ω is an interior point of F, it follows that $\pi(\omega) > 0$. Now notice that π supports the set $\{z \in L_+ : f(z) > f(\omega)\}$ at ω and use part (1). $\quad\square$

We also have the following useful characterization of pointwise proper prices.

Lemma 20. *If order intervals of L are weakly compact, then for a price p of the form $p = \bigvee_{i=1}^m f_i$ such that $p(\omega) > 0$ the following statements are equivalent*:

(1) *The price p is pointwise proper at ω.*
(2) *There exists a solution to the following linear minimization problem*:

$$\min g(\omega) \quad \text{subject to}: \quad g \in L' \text{ and } g \geqslant f_i \text{ for all } i = 1, \dots, m. \qquad (\mathcal{S})$$

Proof. (1) $\implies$ (2) If $p = \bigvee_{i=1}^m f_i$ is pointwise proper at ω, then (according to part (2) of Lemma 19) there is some $\pi \in L'_+$ with $\pi \geqslant p$ and $\pi(\omega) = p(\omega)$. Clearly, $\pi \geqslant f_i$ for all i and π is a solution to the minimization problem $(\mathcal{S})$.

(2) $\implies$ (1) Let $p = \bigvee_{i=1}^m f_i$. Let π be a solution to the minimization problem $(\mathcal{S})$. From [10, Theorem 7.6] it follows that $p(\omega) = \pi(\omega) > 0$. From the fact that $\pi \geqslant f_i$ for each i we see that $\pi \geqslant p$ and (according to Lemma 12) π supports the convex set $\{y \in L_+ : p(y) > p(\omega)\}$ at ω. By Lemma 19(1), p is pointwise proper at ω. $\quad\square$

In the next result we show how pointwise properness of non-linear prices can guarantee decentralization by linear prices.

Lemma 21. *We have the following*:

(1) *A valuation equilibrium supported by a price that is pointwise proper at ω is a Walrasian valuation equilibrium.*
(2) *An equilibrium supported by a price that is ω-proper at ω is a Walrasian equilibrium.*
(3) *If the initial endowments are decomposable, then any equilibrium supported by a price that is pointwise proper at ω is a Walrasian equilibrium.*

Proof. (1) Let $(x_1, \dots, x_m)$ be a valuation equilibrium that is supported by a price $p \in \mathcal{P}$ which is pointwise proper at ω. We can assume $p(\omega) = 1$. By part (2) of Lemma 19 there exists some linear price $\pi \in L'_+$ satisfying $\pi(\omega) = p(\omega) = 1$ that supports the convex set $\{z \in L'_+ : p(z) \geqslant p(\omega)\}$ at ω and $\pi \geqslant p$. Now the relation

$$1 = \pi(\omega) = \sum_{i=1}^m \pi(x_i) \geqslant \sum_{i=1}^m p(x_i) = p(\omega) = 1$$

implies that $\pi(x_i) = p(x_i)$ for each i. It is now easy to see that π supports the allocation as a Walrasian valuation equilibrium.

(2) Let $(x_1, \ldots, x_m)$ be an equilibrium with a supporting price $p \in \mathcal{P}$ satisfying $p(\omega) = 1$. Since preferences are strictly monotone and $\alpha x_i \in P_i(x_i)$ holds for each $\alpha > 1$ it follows that $p(\alpha x_i) = \alpha p(x_i) > p(x_i)$. This implies that $p(x_i) > 0$ for each i.

Now consider the collection of bundles

$$X = \left\{ \frac{\omega_1}{p(x_1)}, \ldots, \frac{\omega_m}{p(x_m)} \right\} .$$

Since $\sum_{i=1}^{m} p(x_i) = p(\omega) = 1$, it is easy to see that $\omega \in \operatorname{co} X$. Also, from condition (c) of the definition of an equilibrium, it follows that

$$\operatorname{co} X \cap \{z \in L_+\colon p(z) > 1\} = \emptyset .$$

From the ω-properness of p we see that there exists some convex set F that contains 2ω in its interior such that $F \cap L_+ = \{z \in L_+\colon p(z) > 1\}$. Therefore, $\operatorname{co} X \cap F = \emptyset$. Let π be a non-zero continuous linear functional that separates these two sets, i.e., $\pi(x) \leqslant \pi(z)$ holds for all $x \in \operatorname{co} X$ and all $z \in F$ with $p(z) > 1$. Since $\omega \in \operatorname{co} X$ and 2ω is in the interior of F, we see that $\pi(2\omega) > \pi(\omega)$, which implies that $\pi(\omega) > 0$. Therefore, we can assume that $\pi(\omega) = 1$.

Now if $z \in L_+$ satisfies $p(z) = p(\omega) = 1$, then $\delta z \in F$ for all $\delta > 1$ and therefore we have $p(\omega) = \pi(\omega) \leqslant \delta \pi(z)$ for all $\delta > 1$. This implies $p(\omega) \leqslant \pi(z)$, and consequently from Lemma 12 it follows that $\pi \geqslant p$. Therefore, as in part (1), it must be the case that π supports the allocation as a Walrasian valuation equilibrium with $\pi(x_i) = p(x_i)$ for each i. Furthermore, it should be clear that $\pi \left(\frac{\omega_i}{p(x_i)} \right) \leqslant 1$ and therefore, $\pi(\omega_i) \leqslant p(x_i) = \pi(x_i)$. This implies $\pi(\omega_i) = \pi(x_i)$ and thus π supports the allocation as a Walrasian equilibrium.

(3) Assume that $\sum_{i=1}^{m} \mathcal{A}_{\omega_i} = \mathcal{A}_\omega$ and let $(x_1, \ldots, x_m)$ be an equilibrium. By part (3) of Theorem 16 we know that $(x_1, \ldots, x_m)$ is a personalized equilibrium. Let $p = \bigvee_{i=1}^{m} f_i$ be a personalized price supporting $(x_1, \ldots, x_m)$ with $p(\omega) = 1$. We know that

$$p(\omega) = \sup \left\{ \sum_{i=1}^{m} f_i(y)\colon y \in \mathcal{A}_\omega \right\} .$$

Now from the decomposability assumption, it follows that $\sum_{i=1}^{m} p(\omega_i) = p(\omega)$. In particular, $p(x_i) = p(\omega_i)$ for each i. Now as in (1) we can find some $\pi \in L'_+$ with $\pi(\omega) = 1$ and $\pi \geqslant p$. As above $\pi(x_i) = p(x_i)$ and $\pi(\omega_i) = p(\omega_i)$, which clearly implies that π is the desired supporting linear price. $\square$

We continue by introducing the notion of a proper price space.

Definition 22. We shall say that the price space $\mathcal{P}$ is:

(1) *ω-proper*, if each price $p \in \mathcal{P}$ satisfying $p(\omega) > 0$ is ω-proper at ω, and
(2) *pointwise proper at ω*, if each price $p \in \mathcal{P}$ satisfying $p(\omega) > 0$ is pointwise proper at ω.

From Lemma 19 we know that an ω-proper price space $\mathcal{P}$ is automatically pointwise proper at ω. We can now turn to the first of our major results. Remarkably, as expected

from the previous discussion, when the price space is proper we regain the classical Walrasian value theory with linear prices even outside of the realm of vector lattice commodity spaces.

Theorem 23. *For an ω-proper price space we have the following*:

(1) *The notions of equilibrium, Walrasian equilibrium, and Edgeworth equilibrium coincide.*
(2) *If the economy is compact, then there exists a Walrasian equilibrium.*

Proof. The validity of the statements follows immediately from Lemma 21 in conjunction with Theorem 16. □

There are many interesting examples of price spaces that are ω-proper for which the commodity space is not a Riesz space and lacks the Riesz decomposition property. Moreover, the properness of the price space depends on the position of ω in L_+. One can construct even finite dimensional examples in which the price space is proper for some positive vectors but not for others.

We conclude this section with an analogue of the preceding theorem for pointwise proper price spaces. Its proof is similar and is omitted.

Theorem 24. *For a price space that is pointwise proper at ω we have the following*:

(1) *The notions of valuation equilibrium, Walrasian valuation equilibrium, and weak Pareto optimality coincide.*
(2) *If the initial endowments are decomposable, then the notions of equilibrium, Walrasian equilibrium, and Edgeworth equilibrium coincide.*
(3) *If the economy is compact and the initial endowments are decomposable, then there exists a Walrasian equilibrium.*

5. The indispensability of proper price spaces

We have seen that the properness of prices provides sufficient conditions for the decentralization of optimal allocations by linear prices. In this section, we show that basically the properness of the non-linear prices also provides necessary conditions for linear decentralization. That is, one cannot hope to derive the classical Walrasian theory of value for proper economies unless the prices have some properness properties at the total endowment vector. We begin with a definition.

Definition 25. For a price of the form $p = \bigvee_{i=1}^{m} f_i$ with $f_i \in L'_+$ for each i, we say that an allocation $x \in A_\omega$ is an *optimizer* if $p(\omega) = \sum_{i=1}^{m} f_i(x_i)$. The allocation x is a *non-trivial optimizer* if $p(\omega) > 0$ and $x_i > 0$ for all i.

We are ready to establish the equivalence between pointwise proper price spaces and the decentralization of Edgeworth equilibria by linear prices.

Theorem 26. *For the collection $\mathcal{E}_D(L, \omega)$ of decomposable economies the following statements are equivalent*:

(a) *Each price of the form $p = \bigvee_{i=1}^{m} f_i$ that has a non-trivial optimizer is pointwise proper at ω.*

(b) *Every Edgeworth equilibrium is a Walrasian equilibrium.*

Proof. $(a) \implies (b)$ Let $x = (x_1, \ldots, x_m)$ be an Edgeworth equilibrium for some economy in $\mathcal{E}_D(L, \omega)$. Clearly, x is a non-trivial allocation. According to Theorem 16, the allocation x is an equilibrium for some price of the form $p = \bigvee_{i=1}^{m} f_i$ with $p(\omega) > 0$ for which x is an optimizer.

Now, by our assumption, p is pointwise proper at ω and a glance at part (3) of Lemma 21 guarantees that x is a Walrasian equilibrium.

$(b) \implies (a)$ Let a price of the form $p = \bigvee_{i=1}^{m} f_i$ with $p(\omega) > 0$ have a non-trivial optimizer $x = (x_1, \ldots, x_m)$. That is, $x_i > 0$ for all i and $p(\omega) = \sum_{i=1}^{m} f_i(x_i)$.

Fix a strictly positive linear functional h on L and put $g_i = f_i + h$ for each i and consider the price $q = \bigvee_{i=1}^{m} g_i = p + h$. We claim that q is pointwise proper at ω.

Clearly, $x = (x_1, \ldots, x_m)$ is an optimizer for q. That is, $q(\omega) = \sum_{i=1}^{m} g_i(x_i) > 0$. From $q(x_i) \geqslant g_i(x_i)$ and the inequalities

$$q(\omega) = q\left(\sum_{i=1}^{m} x_i\right) \geqslant \sum_{i=1}^{m} q(x_i) \geqslant \sum_{i=1}^{m} g_i(x_i) = q(\omega),$$

it follows that $g_i(x_i) = q(x_i)$ holds for all i.

Next, we shall define an economy in $\mathcal{E}_D(L, \omega)$ for which x is an Edgeworth equilibrium. To this end, for each i let $\omega_i = \frac{g_i(x_i)}{q(\omega)} \omega > 0$ and note that $q(\omega_i) = g_i(x_i)$ holds for each i. Furthermore, for each trader $i = 1, \ldots, m$ let g_i be the trader's utility function. Clearly, this economy is in $\mathcal{E}_D(L, \omega)$.

Since $x \in L_+$ with $g_i(x) > g_i(x_i)$ implies $q(x) \geqslant g_i(x) > g_i(x_i) = q(x_i)$, it follows that q supports x as a valuation equilibrium. We need to show that q supports x as an equilibrium. To this end, let $\alpha_1, \ldots, \alpha_m$ be arbitrary non-negative scalars. Then we have

$$\sum_{i=1}^{m} \alpha_i q(x_i) = \sum_{i=1}^{m} \alpha_i g_i(x_i) = \sum_{i=1}^{m} \alpha_i q(\omega_i) = q\left(\sum_{i=1}^{m} \alpha_i \omega_i\right),$$

which tells us that x is an equilibrium and thus an Edgeworth equilibrium.

Therefore, by assumption x is a Walrasian equilibrium for some non-zero price π in L'_+, which is clearly strictly positive. We can assume without loss of generality that $\pi(\omega) = q(\omega)$. Clearly, $\pi(x_i) = \pi(\omega_i) = g_i(x_i)$. By Lemma 12 it must be the case that $\pi \geqslant g_i$ for each i and thus $\pi \geqslant q$.

Finally, notice that from the inequality $\pi \geqslant q = p + h$ we get $\pi - h \geqslant p > 0$. Moreover, from $(\pi - h)(\omega) = (q - h)(\omega) = p(\omega) > 0$ and the fact that if $y \in L_+$ satisfies $p(y) > p(\omega)$, then $(\pi - h)(y) \geqslant q(y) - h(y) = p(y) > p(\omega) = (\pi - h)(\omega)$, it follows

that $\pi - h$ supports the set $\{y \in L_+ : p(y) > p(\omega)\}$ at ω. Therefore, by part (1) of Lemma 19, we see that p is pointwise proper at ω. $\square$

Let us move on to the second welfare theorem. We begin by introducing one more notion of pointwise properness for prices.

Definition 27. We say that a price of the form $p = \bigvee_{i=1}^{m} f_i$ is ω-*scalable* at $x \in \mathcal{A}_\omega$ if there exist multipliers $\lambda_1, \dots, \lambda_m$ satisfying $0 < \lambda_i \leqslant 1$ for each i such that:

(1) the allocation x is an optimizer for the new price $q = \bigvee_{i=1}^{m} \lambda_i f_i$, and
(2) the price q is pointwise proper at ω.

Now we can state and prove the main result of this section. It presents necessary and sufficient conditions for the Walrasian decentralization of optimal allocations.

Theorem 28. *In the set of proper economies $\mathcal{E}(L, \omega)$ the following statements are equivalent*:

(a) *Every price of the form $p = \bigvee_{i=1}^{m} f_i$ with each f_i strictly positive is ω-scalable at each of its non-trivial optimizers.*
(b) *Every non-trivial Pareto optimal allocation is a Walrasian valuation equilibrium.*

Proof. (a) $\Longrightarrow$ (b) Let $x = (x_1, \dots, x_m)$ be a non-trivial Pareto optimal allocation for some economy in $\mathcal{E}(L, \omega)$. From parts (1) and (3) of Theorem 16 this allocation is supported as a valuation equilibrium by a price $p = \bigvee_{i=1}^{m} f_i$ for which x is an optimizer. So, we have $p(\omega) = \sum_{i=1}^{m} p(x_i) = \sum_{i=1}^{m} f_i(x_i)$. Since $p(x_i) \geqslant f_i(x_i)$ for each i, it follows that $p(x_i) = f_i(x_i)$ for each i. Furthermore, the proof of [13, Theorem 5.1(1)] shows that the functions f_i can be chosen in such a way that each f_i (strictly) supports $P_i(x_i)$ at x_i, i.e., $y \in P_i(x_i)$ implies $f_i(y) > f_i(x_i)$.

We need to show that each f_i is strictly positive. From $p(\omega) = \sum_{i=1}^{m} f_i(x_i) > 0$ it must be the case that $f_j(x_j) > 0$ for some j. Notice that if $z \in P_j(x_j)$ and $f_j(z) = f_j(x_j)$, then $f_j(\alpha z) < f_j(x_j)$ for all $\alpha < 1$; which is impossible since P_j has open values in some linear topology on L. Therefore, for $z > 0$, it must be the case that $f_j(x_j + z) > f_j(x_j)$, which implies that $f_j(z) > 0$. This in turn implies that f_j and p are strictly positive, which further implies that $f_i(x_i) = p(x_i) > 0$ for each i. By the same argument we see that each f_i is strictly positive.

Now by our assumption there exist scalars $\lambda_1, \dots, \lambda_m$ with $0 < \lambda_i \leqslant 1$ for each i such that x remains an optimizer of the price $q = \bigvee_{i=1}^{m} \lambda_i f_i$ that is pointwise proper at the point ω.

Next, we shall show that q also supports the allocation x as a valuation equilibrium. Clearly, $q(x_i) \geqslant \lambda_i f_i(x_i)$ for each i and since x is an optimizer for q we have

$$q(\omega) = \sum_{i=1}^{m} \lambda_i f_i(x_i) \leqslant \sum_{i=1}^{m} q(x_i) \leqslant q(\omega) .$$

Consequently, for each i we have $q(x_i) = \lambda_i f_i(x_i)$ and $\sum_{i=1}^{m} q(x_i) = q(\omega) > 0$. Now if $z \in P_i(x_i)$, then $q(z) \geqslant \lambda_i f_i(z) > \lambda_i f_i(x_i) = q(x_i)$, and this shows that q supports x as a valuation equilibrium. Now the validity of (b) follows from part (1) of Lemma 21.

(b) $\Longrightarrow$ (a) Assume that (b) is true and let an allocation $x = (x_1, \ldots, x_m)$ be a non-trivial optimizer for a price $p = \bigvee_{i=1}^{m} f_i$ with each f_i strictly positive.

Next, we consider an economy with m consumers, total endowment ω, and utility functions given by the strictly positive linear functionals f_i. Thus, the preference correspondence $P_i : L_+ \twoheadrightarrow L_+$ of consumer i is defined by $P_i(y) = \{x \in L_+ : f_i(x) > f_i(y)\}$. Clearly, this economy is ω-proper and since x is an optimizer it is Pareto optimal. So, according to our hypothesis, there exists a non-zero price $\pi \in L'_+$ that supports x as a valuation equilibrium. That is, if $z \in L_+$ satisfies $f_i(z) > f_i(x_i)$, then $\pi(z) > \pi(x_i)$. By the strict positivity of f_i we see that $f_i(2x_i) > f_i(x_i)$ and so $2\pi(x_i) > \pi(x_i)$. Consequently, $\pi(x_i) > 0$ holds for each i. Without loss of generality we can assume that $0 < \pi(x_i) < f_i(x_i)$ also holds for each i.

Now for each i let $\lambda_i = \frac{\pi(x_i)}{f_i(x_i)}$ and note that $0 < \lambda_i < 1$. For each i consider the strictly positive linear functional $g_i = \lambda_i f_i$. Observe that $g_i(x_i) = \pi(x_i)$ for each i. Moreover, if some vector $z \in L_+$ satisfies $g_i(z) = g_i(x_i) > 0$, or $f_i(z) = f_i(x_i)$, then we have $f_i(\delta z) > f_i(x_i)$ for all $\delta > 1$ and so $\delta \pi(z) > \pi(x_i)$ for all $\delta > 1$, from which it follows that $\pi(z) \geqslant \pi(x_i) = g_i(x_i)$. Now a glance at Lemma 12 guarantees that $\pi \geqslant g_i$ for each i. This implies $\pi \geqslant \bigvee_{i=1}^{m} g_i$, and so from

$$\pi(\omega) \geqslant \left[\bigvee_{i=1}^{m} g_i \right](\omega) \geqslant \sum_{i=1}^{m} g_i(x_i) = \sum_{i=1}^{m} \pi(x_i) = \pi(\omega),$$

we see that $(x_1, \ldots, x_m)$ is an optimizer for $\bigvee_{i=1}^{m} g_i$ and that $\pi(\omega) = \left[\bigvee_{i=1}^{m} g_i \right](\omega) > 0$.

Now if an arbitrary vector $z \in L_+$ satisfies $\left[\bigvee_{i=1}^{m} g_i \right](z) > \left[\bigvee_{i=1}^{m} g_i \right](\omega)$, then we have the inequalities $\pi(z) \geqslant \left[\bigvee_{i=1}^{m} g_i \right](z) > \left[\bigvee_{i=1}^{m} g_i \right](\omega) = \pi(\omega)$. This shows that π supports the set $\left\{ z \in L_+ : \left[\bigvee_{i=1}^{m} g_i \right](z) > \left[\bigvee_{i=1}^{m} g_i \right](\omega) \right\}$ at ω and from this and part (a) of Lemma 19, we see that the price $\bigvee_{i=1}^{m} g_i$ is pointwise proper at ω. $\square$

We conclude this section by showing that the existence of an equilibrium implies that prices are ω-scalable at each of their non-trivial optimizers.

Theorem 29. *If for each economy in $\mathcal{E}(L, \omega)$ there exists a Walrasian equilibrium, then every price $p = \bigvee_{i=1}^{m} f_i$ with each f_i strictly positive is ω-scalable at each of its non-trivial optimizers.*

Proof. Let an allocation $x = (x_1, \ldots, x_m)$ be a non-trivial optimizer for a price $p = \bigvee_{i=1}^{m} f_i$ with each f_i strictly positive. Consider an economy with m consumers, initial endowments $\omega_i = x_i$ for each i, and utility functions given by the strictly positive linear functionals f_i. Clearly, this economy is ω-proper.

According to our hypothesis, there exists a Walrasian equilibrium $y = (y_1, \ldots, y_m)$ for this economy supported by non-zero price $\pi \in L'_+$. Since y is individually rational and

since x is an optimizer for the price p and $f_i(y_i) \geqslant f_i(\omega_i) = f_i(x_i)$ holds for each i it must be the case that $f_i(y_i) = f_i(x_i)$ for each i. Since $\pi(y_i) = \pi(\omega_i) = \pi(x_i)$ for each i, we see that if $z \in L_+$ satisfies $f_i(z) > f_i(x_i)$, then $\pi(z) > \pi(x_i)$. It is now easy to follow the proof of Theorem 28 and show that p is ω-scalable at x. $\square$

6. Properness of prices in financial models

The previous theorems demonstrate that we need some pointwise properness of prices if we are to derive the Walrasian value theory. We start with an example that guarantees the ω-properness of prices in $\mathcal{P}$.

Recall that in our model of portfolio trading we have two spaces linked by the portfolio returns operator as follows:

(1) The portfolio space L is ordered by the portfolio dominance ordering.
(2) The space of contingent claims X is a Banach lattice.
(3) The portfolio returns operator $R \colon L \hookrightarrow X$ is one-to-one and positive and pulls back order intervals of the marketed space $\mathcal{M}$ to closed and bounded dominance intervals of L.

Here is an example of our finance model with an ω-proper price space.

Example 1. Let $L = \ell_2$ and $X = L_2[0, 1]$ and consider the operator $R \colon L \to X$ defined on the basic vectors $e_1, e_2, \ldots$ of ℓ_2 by

(1) $R(e_1) = \chi_{[\frac{1}{2}, 1]} + \chi_{[\frac{1}{5}, \frac{1}{3}]}$,
(2) $R(e_2) = \chi_{[\frac{1}{3}, 1]}$,
(3) $R(e_3) = \chi_{[\frac{1}{4}, \frac{1}{2}]}$, and
(4) $R(e_n) = \lambda_n \chi_{[\frac{1}{n+2}, \frac{1}{n+1}]}$ for each $n \geqslant 4$, where the scalars λ_n satisfy $\lambda_n > 0$ and they are appropriately chosen so that R is indeed an operator from ℓ_2 to $L_2[0, 1]$.

As usual, if $(\alpha_1, \alpha_2, \ldots) \in \ell_2$, then we have

$$R(\alpha_1, \alpha_2, \ldots) = \sum_{n=1}^{\infty} \alpha_n R(e_n).$$

Clearly, the operator R is one-to-one and positive (and hence continuous).

It should not be difficult to see that if $\lambda_n = 1$ for each $n \geqslant 4$, then the operator R is not bounded below. On the other hand if $\lambda_n = \sqrt{(n+1)(n+2)}$ for each $n \geqslant 4$, then the operator R is bounded below. In fact, it is easy to verify that when $\lambda_n = 1$ for each $n \geqslant 4$ the operator R satisfies the following properties:

(a) As in every model, the portfolio dominance cone on ℓ_2 is closed and generating. More-over, R pulls back order intervals of $\mathcal{M}$ to closed and bounded dominance intervals in L.

(b) The portfolio dominance ordering on L is not a lattice ordering.
(c) The price space $\mathcal{P}$ is ω-proper for every positive portfolio $\omega \in L_+$.

Therefore our results on linear decentralization apply to this example.   □

Let $\mathcal{M}'$ be the dual space of the marketed space $\mathcal{M}$ and assume that R is bounded below. Then considering $R \colon L \to \mathcal{M}$ as an order isomorphism, we see that its adjoint $R' \colon \mathcal{M}' \to L'$, defined via the duality identity

$$\langle R'(f), m \rangle = \langle f, R(m) \rangle = f(R(m))$$

for all $m \in \mathcal{M}$ and all $f \in \mathcal{M}'$, is also an order isomorphism. This operator associates each state dependent price system with a securities price. We call this operator the *securities prices operator*.

We conclude the paper with the following very useful characterization of pointwise proper prices in terms of a solution to a linear minimization problem. Its proof follows immediately from Lemma 20.

Theorem 30. *Assume that order intervals of L are weakly compact and that the payoff operator R is bounded from below. Let $f_1, \ldots, f_m \in L'_+$ and $\omega > 0$ be such that the price $p = \bigvee_{i=1}^{m} f_i$ satisfies $p(\omega) > 0$. Then the following three conditions are equivalent:*

(1) *The price p is pointwise proper at ω.*
(2) *There exists a solution to the following linear minimization problem:*

$$\min g(R(\omega)) \quad subject\ to: \quad g \in \mathcal{M}' \ and \ R'(g) \geqslant f_i \ for\ all \ i = 1, \ldots, m.$$

(3) *There exists a solution to the following linear minimization problem:*

$$\min h(R(\omega)) \quad subject\ to: \quad h \in X' \ and \ R'(h) \geqslant f_i \ for\ all \ i = 1, \ldots, m,$$

where now R' denotes the adjoint of the operator $R \colon L \to X$.

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