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A Test for Symmetry with Leptokurtic Financial Data

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ABSTRACT

Most of the tests for symmetry are developed under the (implicit or explicit) null hypothesis of normal distribution. As is well known, many financial data exhibit fat tails, and therefore commonly used tests for symmetry (such as the standard $\sqrt{b_1}$ test based on sample skewness) are not valid for testing the symmetry of leptokurtic financial data. In particular, the $\sqrt{b_1}$ test uses third moment, which may not be robust in presence of gross outliers. In this article we propose a simple test for symmetry based on the Pearson type IV family of distributions, which take account of leptokurtosis explicitly. Our test is based on a function that is bounded over the real line, and we expect it to be more well behaved than the test based on sample skewness (third moment). Results from our Monte Carlo study reveal that the suggested test performs very well in finite samples both in terms of size and power. Simulation results also support our conjecture of the tests to be well behaved and robust to excess kurtosis. We apply the test to some selected individual stock return data to illustrate its usefulness.

KEYWORDS: $\sqrt{b_1}$ test, kurtosis, Monte Carlo study, Pearson family of distributions, Rao's score test, skewness, $\tan^{-1}(\bullet)$ function.

The objective of this article is to propose a test for symmetry in the presence of excess kurtosis. It is a widely accepted fact that most financial data exhibit leptokurtosis and sometimes asymmetry [see, e.g., Kon (1984), Mills (1995), Peiró (1999), Premaratne and Bera (2000), and Patton (2004)]. The commonly used test of symmetry uses the standardized third central moment. Several studies, including

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Lin and Mudholkar (1980), Horsewell and Looney (1993), Rayner, Best, and Mathews (1995), and Kim, Mittnik, and Rachev (1996), have found that the test based on the sample skewness measure (say, $\sqrt{b_1}$) is adversely affected by leptokurtosis regardless of whether or not the distribution is symmetric. Several Monte Carlo studies found a considerably inflated size of the test when used against Student's t distribution [see, e.g., Horsewell and Looney (1993), Rayner, Best and Mathews (1995), and Peiró (1999)]. Therefore $\sqrt{b_1}$ does not provide a reliable test to discriminate between the symmetric and asymmetric distributions. Hence there is need for a simple test for symmetry that is robust to extreme values or the fat tail of the distribution for analyzing economic and financial data.

The literature has highlighted the importance of higher moments on financial and other socioeconomic variables for some time. For example, the occurrence of the asymmetric distribution was noted more than a century ago by Karl Pearson (1895:344). He stated:

> An asymmetrical frequency curve may arise from two quite distinct classes of causes. In the first place the material measured may be heterogeneous and may consist of a mixture of two or more homogeneous materials. Such frequency curves, for example, arise when we have a mixed population of two different races, a homogeneous population with a sprinkling of diseased or deformed members, a curve for the frequency of matrimony covering more than one class of the population, or in economics, a frequency of interest curve for securities of different types of stability-runaways and government stocks mixed with mining and financial companies.
> The second class of frequency curve arises in the case of homogeneous material when the tendency to deviation on the side of the mean is unequal to the tendency to deviation on the other side. Such curves arise in many physical, economic and biological investigations, for example, in those for prices and rates of interest of securities of the same class, in income tax and house duty returns.

However, there was no noticeable contribution characterizing financial data until Mandelbrot (1963). Mandelbrot observed the presence of leptokurtosis in the empirical distribution of price changes and also suggested the use of symmetric stable distribution to capture this excess kurtosis. Consequently there was significant development in the modeling of financial data using nonnormal distributions. For instance, distributions such as Student's t [Blattberg and Gonedes (1974)], generalized beta of the second kind [Bookstaber and McDonald (1987)], mixtures of normal distributions [Ali and Giacotta (1982)], discrete mixtures of normal distributions [Kon (1984)], and Tukey's g and h distributions [Badrinath and Chatterjee (1988, 1991)] have all been used to characterize the data.

Asymmetry of financial data has also been noted in many studies based on hypothesis tests. Kon (1984) applied the $\sqrt{b_1}$ test to 30 individual stocks and 3 major indices to investigate the asymmetry. He found that 26 of the 30 stocks were positively skewed. Affleck-Graves and McDonald (1989) and Richardson and Smith (1993) also found significant asymmetry in monthly returns data from 30 Dow Jones companies. In a recent article, Premaratne and Bera (2000) highlighted the simultaneous presence of excess kurtosis and asymmetry in financial data. Christie-David and Chaudhry (2001) observed similar results for futures returns.

With these studies, the adequacy of commonly used financial models, such as the capital asset pricing model [Sharpe (1964) and Lintner (1965)] and the options pricing model [Black and Scholes (1973)], has become an issue, since these models are developed based on the symmetry assumption. In the financial literature, skewness of returns is interpreted as an indicator of risk, since it measures the concentration of probability in downside and upside returns. Economics theory describes skewness in relation to the preference of an investor. All else constant, there is a view that typical investors would prefer positive skewness, that is, portfolios with a larger probability of very large returns [see Harvey and Siddique (2000:1264)].

Given the importance of skewness in the economics and financial literature, it is useful to have a test that can correctly identify the asymmetry of data while allowing for excess kurtosis. As we noted earlier, the standard $\sqrt{b_1}$ test is not reliable when excess kurtosis is present. The purpose of this article is to suggest a simple test for symmetry that can be used in financial data. The main ingredient of our approach is the Pearson type IV distribution [Pearson (1895)], which incorporates asymmetry and excess kurtosis in a simple way. The next section provides a brief introduction to the type IV distribution and its relationship with some common densities. The test statistic is derived in Section 2. Section 3 describes the results from a Monte Carlo study on the finite sample size and power of the suggested and $\sqrt{b_1}$ tests. We illustrate the usefulness of our procedure by applying it to some individual stock return data in Section 4. The last section concludes the article.

1 THE PEARSON TYPE IV DISTRIBUTION

Karl Pearson (1895) introduced a generalized system of curves to capture a wide range of probability distributions. This family of curves is well known as the Pearson family of distributions. This is not a very restrictive family; many distributions, such as normal, Student's t, gamma, beta, and F are nested within this family. The Pearson type IV distribution is one of the main types within this system. The probability density function $f(\bullet)$ of the Pearson family of distributions can be represented by the following differential equation [see, e.g., Stuart and Ord (1994:215–226)]:

$$\frac{d\log f(\varepsilon_i)}{d\varepsilon_i} = \frac{\alpha - \varepsilon_i}{b_0 - b_1\varepsilon_i + b_2\varepsilon_i^2} \tag{1}$$

The shapes of distributions belonging to this family depend on the values of parameters α, b_0, b_1, and b_2. These different shapes are classified as types depending on the roots of he equation $b_0 - b_1\varepsilon_i + b_2\,\varepsilon_i^2 = 0$. Let us assume $b_1 \neq 0$, $b_2 \neq 0$, and the roots of the quadratic equation $b_0 - b_1 z + b_2 z^2 = 0$ are imaginary, say, $b + ia$ and $b - ia$, with $i = \sqrt{-1}$. Then from Equation (1), we can derive the type IV density as [see Stuart and Ord (1994:221–222)],

$$f(\varepsilon_i) = constant.\left(1+\frac{\varepsilon_i^2}{a^2}\right)^{-m} exp\left[\delta\ \tan^{-1}\left(\frac{\varepsilon_i}{a}\right)\right], \tag{2}$$

with

$$constant = \frac{1}{a\int_{-\pi/2}^{\pi/2}\cos^{2m-2}(t)e^{\delta t}dt}, \tag{3}$$

where $m = -1/2b_2$ and $\delta = (b - \alpha)/ab_2$.

As mentioned earlier, the Pearson type IV distribution is an *asymmetric* and *leptokurtic* density function. It is also referred to as the *skew-t* distribution [see Skates (1993)]. The type IV distribution is solely described by three parameters: δ, m, and a. Nonzero values of the parameter δ represent the asymmetry, whereas low values of parameter m indicate high leptokurtosis. The spread of the distribution is represented by the scale parameter a. When $\delta = 0$, the type IV distribution reduces to a *symmetric* Pearson type VII distribution with the density function

$$f(\varepsilon_i) = \frac{1}{a\sqrt{\pi}}\frac{\Gamma(m)}{\Gamma\left(m-\frac{1}{2}\right)}\left(1+\frac{\varepsilon_i^2}{a^2}\right)^{-m}. \tag{4}$$

In Equation (4), replacing a and m, respectively, by $\sqrt{\nu}$ and $(\nu + 1)/2$, we get a Student's t distribution with ν degrees of freedom:

$$f(\varepsilon_i) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}\left[1+\frac{\varepsilon_i^2}{\nu}\right]^{-\frac{(\nu+1)}{2}}. \tag{5}$$

As is well known, when $\nu \to \infty$, Equation (5) becomes a normal distribution. Therefore Student's t and normal, the two most commonly used distributions in modeling financial data, are special cases of the Pearson type IV distribution. In Figures 1 and 2 we plot the type IV densities for different values of m and δ with $a = 1$. Figure 1 shows the impact of m on the tails, after keeping δ fixed at two. As

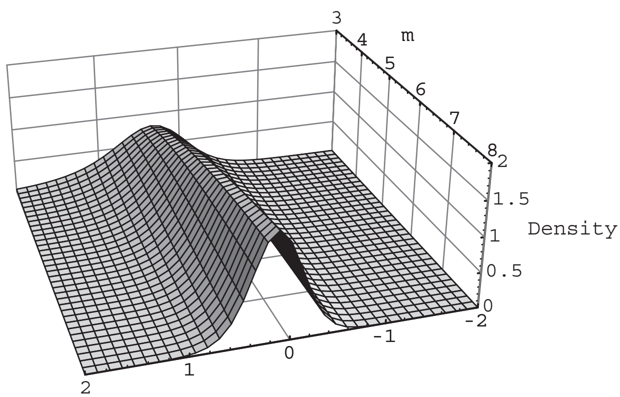

Figure 1 Density of Pearson type IV with $\delta = 2$ and different values of m.

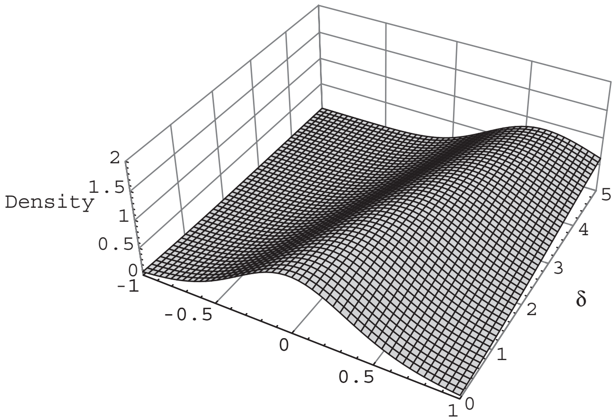

Figure 2 Pearson type IV with m = 4 and different values of δ.

m decreases the distribution has thicker tails. The density is clearly positively skewed, but the skewness pattern changes with m even with this fixed δ. Therefore there is a clear link between skewness and kurtosis when the data are assumed to follow a Pearson type IV distribution. The distribution in Figure 2, where we kept m fixed at four, becomes more positively skewed as we increase the value of δ. We also notice that with higher values of δ, the mass of the distribution moves away from zero toward positive values.

As given in Pearson (1895:378), the parameters m, δ, and a can be expressed in terms of population skewness $\left(\sqrt{\beta_1}\right)$ and kurtosis (β_2) measures as follows:

$$m = \frac{1}{2}\left[\frac{6(\beta_2 - \beta_1 - 1)}{2\beta_2 - 3\beta_1 - 6} + 2\right], \tag{6}$$

$$\delta = \frac{2(m-1)(m-2)\sqrt{\beta_1}}{\sqrt{\left[4(2m-3) - \beta_1(m-2)^2\right]}}, \tag{7}$$

$$a = \sqrt{\frac{\mu_2}{4}\left[4(2m-3) - \beta_1(m-2)^2\right]}, \tag{8}$$

where μ_2 is the population variance.

From Equation (7), the population skewness can be expressed as

$$\sqrt{\beta_1} = 4\delta\sqrt{\frac{(2m-3)}{2(m-2)\left[4(m-1)^2 + \delta^2\right]}}. \tag{9}$$

Therefore, when δ = 0, the population skewness measure $\sqrt{\beta_1} = 0$ for any $m > 2$. We can also see from Equation (2) that the density becomes symmetric with δ = 0. Therefore the parameter δ can be viewed as the asymmetry parameter. Using this idea, in the next section we develop Rao's (1948) score (RS) or the

Lagrange multiplier (LM) test for asymmetry by testing $H_0 : \delta = 0$ within the family of Pearson type IV. Since, under H_0, Equation (2) reduces to the type VII density in Equation (4), which allows for excess kurtosis and is more general than the standard Student's t distribution, the resulting test will take account of the presence of leptokurtosis explicitly, while the standard $\sqrt{b_1}$ test ignores it completely.

2 A RAO SCORE TEST FOR SYMMETRY

Let us consider having a set of n independent observations, $y_1, y_2, \ldots, y_n$ on a random variable Y distributed with mean μ. We define $\varepsilon_i = y_i - \mu$, $i = 1 \ldots n$, which is assumed to follow the Pearson type IV distribution. Using Equation (2), it can be shown that

$$E(\varepsilon_i) = \frac{\delta a}{2(m-1)}. \tag{10}$$

Under symmetry $\delta = 0$, we have $E(\varepsilon_i) = 0$; therefore we will apply out test to ε_i, which are the observations measured from their mean. In this way we allow for the nonzero mean in the original data.

The log-likelihood function $l(\theta)$, based on the density of Equation (2), can be written as

$$l(\theta) = -n \log \psi(m,\delta,0) - n \log a - m \sum_{i=1}^{n} \log\left(1 + \frac{\varepsilon_i^2}{a^2}\right) + \delta \sum_{i=1}^{n} \tan^{-1}\left(\frac{\varepsilon_i}{a}\right), \tag{11}$$

where $\theta = (\delta, \mu, a, m)'$ and

$$\psi(m,\delta,\lambda) = \int_{-\pi/2}^{\pi/2} t^{\lambda} \cos^{(2m-2)}(t) e^{\delta t} dt. \tag{12}$$

The score test will be based on (for derivation, see the appendix)

$$d_\delta = \frac{\partial l(\theta)}{\partial \delta} = n \frac{\psi(m,\delta,1)}{\psi(m,\delta,0)} + \sum_{i=1}^{n} \tan^{-1}\left(\frac{\varepsilon_i}{a}\right), \tag{13}$$

which, under the null hypothesis of $\delta = 0$, reduces to

$$d_\delta^0 = \sum_{i=1}^{n} \tan^{-1}\left(\frac{\varepsilon_i}{a}\right). \tag{14}$$

We will denote

$$T_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \tan^{-1}\left(\frac{\varepsilon_i}{a}\right) \tag{15}$$

to emphasize that this quantity depends on the elements of $\theta = (0, \mu, a, m)'$ under the null hypothesis. The $\tan^{-1}(\bullet)$ function is an odd function; therefore it is easy to see that $E(T_n(\theta)) = 0$ whenever ε_i has a symmetric distribution. Also, as we plot in

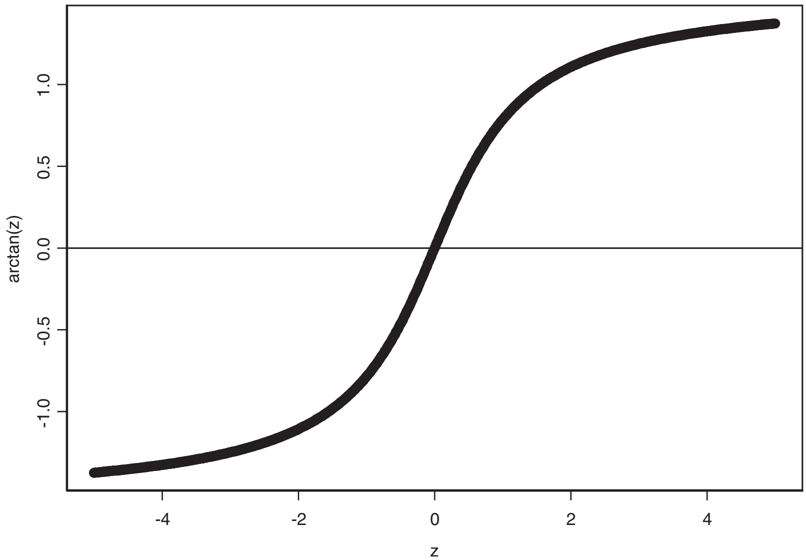

Figure 3 A plot of the $\tan^{-1}(\bullet)$ function.

Figure 3, this is a bounded function over the whole real line. That means the tests based on d_δ^0 will be robust to outliers. In contrast, as we know, the test $\sqrt{b_1}$ uses the average of ε_i^3 values, and these are very sensitive to outliers.

The RS test is given by

$$\mathrm{RS} = n\frac{T_n^2\left(\hat{\theta}\right)}{\hat{V}}, \tag{16}$$

where $\hat{V}$ is the consistent estimator of asymptotic variance of $\sqrt{n}T_n\left(\hat{\theta}\right)$. Under H_0, RS will be asymptotically distributed as χ_1^2. The $\mathrm{var}\left(\sqrt{n}T_n\left(\hat{\theta}\right)\right)$ can be obtained using the Pierce (1982) result,

$$\mathrm{var}\left[\sqrt{n}T_n\left(\hat{\theta}\right)\right] = n\ \mathrm{var}[T_n(\theta)] - \lim_{n\to\infty} E\left[\frac{\partial \mathrm{T_n}(\theta)}{\partial\theta}\right]' \mathrm{var}\left[\sqrt{n}\hat{\theta}\right] \lim_{n\to\infty} E\left[\frac{\partial \mathrm{T_n}(\theta)}{\partial\theta}\right]. \tag{17}$$

This formula gives a simple way to find $\mathrm{var}\left(\sqrt{n}T_n\left(\hat{\theta}\right)\right)$ once we know $\mathrm{var}(\sqrt{n}T_n(\theta))$, which is always easy to find. The $\mathrm{var}(T_n(\theta))$ in Equation (17) can be replaced by the first element of the information matrix as

$$\begin{aligned} V_{\delta\delta} &= E\left(-\frac{\partial^2 l(\theta)}{\partial\delta^2}\right) \\ &= \frac{n\int_{-\pi/2}^{\pi/2} t^2 \cos^{(2m-2)}(t)dt}{\int_{-\pi/2}^{\pi/2}\cos^{(2m-2)}(t)dt} \end{aligned} \tag{18}$$

The integrals given in Equation (18) can be easily evaluated by numerical integration. However, using the closed form of these integrals for the closest integer value of m, denoted by m^*, $V_{\delta\delta}$ can be expressed as (see the appendix)

$$V_{\delta\delta} = 2n\frac{1}{\sqrt{\pi}}\frac{\Gamma(m^*)}{\Gamma(m^*-0.5)}$$

$$\left[\binom{2m^*-2}{m^*-1}\cdot\frac{\pi^2}{6}\frac{1}{2^{2m^*}}+\frac{\pi}{2^{(2m^*-1)}}\sum_{k=0}^{m^*-2}\binom{2m^*-2}{k}\frac{(-1)^{(m^*-1-k)}}{(m^*-1-k)^2}\right]. \tag{19}$$

Using Equation (15), we have

$$E\left[\frac{\partial \mathrm{T_n}}{\partial \mu}\right] = naE\left[\frac{1}{a^2+\varepsilon_i^2}\right]. \tag{20}$$

For simplicity, we denote $z_i = a^2 + \varepsilon_i^2$ and replace $E[1/z_i]$ by its sample counterpart $n^{-1}\sum_{i=1}^n (1/z_i)$ to lessen the computational difficulties. Since $T_n(\theta)$ in Equation (15) does not depend on the parameter m, we have $\frac{\partial \mathrm{T_n}}{\partial \mathrm{m}} = 0$. Also note that

$$\frac{\partial \mathrm{T_n}}{\partial a} = -\sum_{i=1}^{n}\left(\frac{\varepsilon_i}{\varepsilon_i^2+a^2}\right),$$

and hence, under symmetry of ε, $E\left[\frac{\partial \mathrm{T_n}}{\partial a}\right] = 0$. Therefore all we need is to consider $\frac{\partial \mathrm{T_n}}{\partial \mu}$ and only the $V_{\mu\mu}^{-1}$ element of $\mathrm{var}\left(\sqrt{n}\hat{\theta}\right)$. Then the second part of Equation (17) reduces to

$$\lim_{n\to\infty} E\left[\frac{\partial \mathrm{T_n}}{\partial \theta}\right]' \mathrm{var}\left[\sqrt{n}\hat{\theta}\right] \lim_{n\to\infty} E\left[\frac{\partial \mathrm{T_n}}{\partial \theta}\right] = a^2\left(\sum\nolimits_{i=1}^{n}\frac{1}{z_i}\right)^2 V_{\mu\mu}^{-1}, \tag{21}$$

where $V_{\mu\mu}$ is given by

$$\begin{aligned} V_{\mu\mu} &= E\left[-\frac{\partial^2 \log l(\theta)}{\partial \mu^2}\right] \\ &= E\left[2m\sum\nolimits_{i=1}^{n}\frac{2a^2-z_i}{z_i^2}\right]. \end{aligned} \tag{22}$$

Equation (21) is essentially a correction term due to the replacement of θ by $\hat{\theta}$, that is, due to the estimation of parameters μ, a, and m. A simplified version of our proposed RS test can now be written as

$$RS_\delta = \frac{\left[\sum_{i=1}^n \tan^{-1}\left(\frac{\varepsilon_i}{a}\right)\right]^2}{\left[\widehat{V_{\delta\delta}} - \hat{a}^2\left(\sum_{i=1}^n \frac{1}{z_i}\right)^2\left(2\hat{m}\sum_{i=1}^n\left(\frac{2\hat{a}^2-z_i}{z_i^2}\right)\right)^{-1}\right]}, \tag{23}$$

where "hats" denote that the parameters have been replaced by their corresponding maximum-likelihood estimates under the null hypothesis.

Our test could be legitimately criticized for the reason that a *specific* distribution, namely Pearson type IV, has been used for its derivation. From this point of view, the test appears to be very "parametric" in nature. However, the test can be interpreted in a semiparametric way. We do start with a specific distribution, but once we obtain the test function, $\tan^{-1}(\bullet)$, using the Rao score principle, its relation to the Pearson type IV distribution is no longer relevant. The essential property of this test indicator function is that $E[\tan^{-1}(\varepsilon_i/a)] = 0$ under symmetry, just like the test based on the skewness measure $\sqrt{\beta_1}$ uses the moment condition $E(\varepsilon^3) = 0$. We can also derive this third moment test starting from a number of parametric family of distributions [see Bera (1983)]. Basically any odd-order central moment (or certain combination of them) or, more generally, any odd function can serve as a test function for symmetry. Chen, Chou, and Kuan (2000) used the "sine" function.

Theoretically speaking, we can make our test completely independent of the type IV distribution by using a studentized version of the variance of $T_n(\hat{\theta})$, which is nothing but a sample average. We did not attempt that here mainly because our variance formula [Equation (17)] worked so well in a finite sample, as we will see in the results of our simulation studies in the next section. We should also mention that the asymmetry of any random variable is related to "all" odd-order central moments (such as $E(\varepsilon^3)$, $E(\varepsilon^5), \ldots$), and $\sqrt{\beta_1}$ uses just one of them. Asymmetry can also be characterize by many odd-function moments. As noted earlier, the benefit of an odd function like $\tan^{-1}(\varepsilon/a)$ is that it is *bounded* over the whole range and therefore such a transformed function will be somewhat immune to outliers and will possess moments of all orders. Consequently, tests based on this kind of bounded odd function will be more well behaved than a power-transform function (like ε^3). In our Monte Carlo study, we see ample evidence of this conjecture.

3 THE MONTE CARLO STUDY

In this section we present the results of a Monte Carlo study to compare the performance of the proposed test with that of the standard $\sqrt{b_1}$ test. To study finite sample size behavior, we generated observations from two symmetric type IV distributions with mild and strong excess kurtosis, and Student's *t* distribution with seven degrees of freedom, for sample sizes $n = 100, 200, 300, 400$, and 500. We performed 10,000 replications; therefore the standard error of the estimated sizes reported in Table 1 will be much less than $\sqrt{0.5(1-0.5)/10000} = 0.005$.

We first note that for the integer m approximation version of the RS test, we obtain slightly larger sizes in some cases, whereas the other version has mostly smaller sizes. The estimated sizes, however, are not far from the nominal 1% and 5% in any case. On the other hand, the $\sqrt{b_1}$ test is highly sensitive to the excess kurtosis. Even for a moderate excess kurtosis of 1.2 (type IV distribution with $\delta = 0, a = 1, m = 5$), we reject more than 35% when the nominal size is only 5%. The excess rejection of the correct null hypothesis of symmetry by the $\sqrt{b_1}$ test reaches an exorbitant stage when used against simulated data with an excess kurtosis of 6

Table 1 Estimated sizes of the RS and $\sqrt{b_1}$ tests with 10,000 replications

	RS test (integer m)		RS test (real m)		$\sqrt{b_1}$	
Sample size	1%	5%	1%	5%	1%	5%
Type IV distribution ($\delta = 0$, $a = 1$, $m = 5$) with implied $\sqrt{\beta_1} = 0$, $\beta_2 = 4.2$						
100	0.0081	0.0494	0.0046	0.0396	0.1754	0.3514
200	0.0120	0.0519	0.0058	0.0399	0.1893	0.3520
300	0.0130	0.0553	0.0075	0.0460	0.2000	0.3579
400	0.0149	0.0591	0.0078	0.0504	0.2153	0.3755
500	0.0146	0.0551	0.0088	0.0440	0.2140	0.3762
Type IV distribution ($\delta = 0$, $a = 1$, $m = 3$) with implied $\sqrt{\beta_1} = 0$, $\beta_2 = 9.0$						
100	0.0134	0.0560	0.0062	0.0455	0.3022	0.4765
200	0.0150	0.0608	0.0076	0.0495	0.3856	0.5322
300	0.0168	0.0596	0.0092	0.0488	0.4179	0.5675
400	0.0195	0.0638	0.0094	0.0525	0.4541	0.5970
500	0.0164	0.0569	0.0072	0.0466	0.4675	0.6062
Student's t_7 distribution with implied $\sqrt{\beta_1} = 0$, $\beta_2 = 5.0$						
100	0.0078	0.0477	0.0035	0.0360	0.2392	0.4129
200	0.0115	0.0561	0.0073	0.0467	0.2736	0.4362
300	0.0140	0.0553	0.0076	0.0447	0.2867	0.4438
400	0.0131	0.0560	0.0068	0.0457	0.3049	0.4418
500	0.0134	0.0575	0.0074	0.0465	0.3132	0.4681

(type IV distribution with $\delta = 0$, $a = 1$, $m = 3$). The true size reaches as much as 60% with a sample size of 500. For the t_7 distribution, when the excess kurtosis is somewhat at medium level ($\beta_2 - 3 = 2$), we observe very high overrejection of symmetry. The main problem with the standard $\sqrt{b_1}$ test is that its variance calculation does not account for the excess kurtosis, therefore the variance is underestimated. Consequently the null hypothesis of symmetry is rejected too frequently. Our RS test explicitly takes into account the excess kurtosis through proper estimation of the variance, and hence does not suffer from the over rejection problem. Also our test is based on a bounded function, making it somewhat immune to outliers, which occurs more frequently with excess kurtic data.

To study the power performance, we generated data by introducing asymmetry (through nonzero δ) to our two type IV distributions used in Table 1. Since we wanted to keep the β_2 level at 4.2 and 9.0, different values of m were needed [see Equation (6)]. The results on power are reported in Table 2. Given that $\sqrt{b_1}$ is very large, it is only legitimate to compare the *size adjusted powers* of the two tests. Among the two versions of the RS tests, the "real m" version has the better power. And in general, for sample sizes larger than 100, the RS test has better power. The power gain by the RS test as we increase the sample size is quite dramatic. For example, as we move from sample size 100 to 500, the estimated power of the RS (real m version) test increases from 18.02% to 74.39%, while for $\sqrt{b_1}$, the movement is from 17.66% to 40.60% when implied $\sqrt{\beta_1} = 0.84$ and $\beta_2 = 9.0$. It is quite interesting that for $\sqrt{b_1}$, the

Table 2 Estimated unadjusted and size-adjusted power of the tests with 10,000 replications

		RS test (integer m)		RS test (real m)		$\sqrt{b_1}$	
Power	Sample size	1%	5%	1%	5%	1%	5%
Type IV distribution (δ = 3.12, a = 1, m = 6) with implied $\sqrt{\beta_1}$ = 0.44, β_2 = 4.2							
Unadjusted	100	0.0514	0.1782	0.0327	0.1727	0.3653	0.5836
	200	0.1447	0.3477	0.1252	0.3452	0.5309	0.7173
	300	0.2555	0.4990	0.2397	0.5038	0.6534	0.8121
	400	0.3705	0.6233	0.3637	0.6312	0.7530	0.8726
	500	0.4815	0.7279	0.4860	0.7406	0.8274	0.9247
Size adjusted	100	0.0586	0.1800	0.0668	0.2000	0.0423	0.2220
	200	0.1303	0.3424	0.1743	0.3800	0.0746	0.3535
	300	0.2255	0.4273	0.2769	0.5179	0.1093	0.4516
	400	0.3202	0.5917	0.3911	0.6299	0.1486	0.5423
	500	0.4343	0.7057	0.5075	0.7595	0.2208	0.6469
Type IV distribution (δ = 1.2526, a = 1, m = 3.2) with implied $\sqrt{\beta_1}$ = 0.84, β_2 = 9.0							
Unadjusted	100	0.0526	0.1715	0.0384	0.1697	0.4616	0.6347
	200	0.1446	0.3329	0.1303	0.3397	0.6522	0.7725
	300	0.2529	0.4796	0.2405	0.4912	0.7704	0.8604
	400	0.3672	0.6056	0.3686	0.6234	0.8396	0.9000
	500	0.4668	0.7081	0.4801	0.7321	0.8882	0.9336
Size adjusted	100	0.0388	0.1596	0.0565	0.1802	0.0327	0.1766
	200	0.1077	0.2920	0.1559	0.3418	0.0383	0.2423
	300	0.1904	0.4402	0.2556	0.4959	0.0604	0.2967
	400	0.2976	0.5612	0.3756	0.6146	0.0676	0.3283
	500	0.3229	0.6820	0.5135	0.7439	0.0906	0.4060

powers are in general higher when the degree of asymmetry is lower, but with more excess kurtosis. To see this, just compare the size-adjusted powers of $\sqrt{b_1}$ for all sample sizes, say at the 5% level. The powers of the test for $\sqrt{\beta_1}$ = 0.84 and β_2 = 9.0 are systematically much lower than those for $\sqrt{\beta_1}$ = 0.44 and β_2 = 4.2. Therefore excess kurtosis has a very detrimental effect on the power of $\sqrt{b_1}$ test, which is not very powerful to start with. Therefore, both in terms of finite sample size and power, the overall performance of the RS test is much better.

4 AN EMPIRICAL ILLUSTRATION

In this section we apply our test and $\sqrt{b_1}$ test to some selected daily individual stock returns data from the Center for Research in Security Prices (CRSP) database for a seven-year period from January 1990 to December 1996. There were 2276 observations for each stock. The kurtosis of all the returns is greater than three, and some are indeed very large, signifying highly leptokurtic data. The critical values used for the $\sqrt{b_1}$ test were 0.09 and 0.12, at the 1% and 5% levels of significance, respectively [see Pearson and Hartley (1976:183)].

Table 3 A comparison of the standard skewness test with the suggested test using daily stock returns from CRSP data, January 1990–December 1996

Company name	Mean	Variance	Kurtosis	$\sqrt{b_1}$	RS test
British Airways	0.000919	0.00029	6.992	$0.2036^{\dagger}$	2.055
Boeing	0.001030	0.00024	5.807	$0.0928^{\ddagger}$	$10.368^{\dagger}$
Best Buy	0.001267	0.00133	13.351	0.0122	$23.814^{\dagger}$
Bell Atlantic	0.000585	0.00017	6.026	$0.2751^{\dagger}$	$7.295^{\dagger}$
Bell South	0.000636	0.00017	5.274	$0.2432^{\dagger}$	$6.176^{\ddagger}$
Cannon Inc	0.000709	0.00031	8.088	$0.8456^{\dagger}$	$13.094^{\dagger}$
Disney Walt	0.000827	0.00025	6.317	$0.3129^{\dagger}$	$10.001^{\dagger}$
General Electric	0.000858	0.00017	5.191	−0.0306	3.660
General Mills	0.000715	0.00018	4.572	$0.4016^{\dagger}$	$21.187^{\dagger}$
GM	0.000586	0.00032	4.323	$0.1489^{\dagger}$	$11.150^{\dagger}$
GTE	0.000713	0.00017	5.357	$0.1037^{\dagger}$	1.544
Hitachi Ltd.	0.000197	0.00025	5.479	$0.6730^{\ddagger}$	$20.090^{\dagger}$
Honda	0.000633	0.00031	8.551	$0.7159^{\dagger}$	$11.490^{\dagger}$
IBM	0.003820	0.00026	10.409	$0.4141^{\dagger}$	$5.294^{\ddagger}$
Intel	0.001600	0.00057	5.405	$-0.3073^{\dagger}$	0.003
JP Morgan	0.000707	0.00022	5.203	$0.4128^{\dagger}$	$16.940^{\dagger}$
Coca Cola	0.001235	0.00020	5.440	$0.1412^{\dagger}$	$4.897^{\ddagger}$
Oneida Ltd.	0.000474	0.00029	7.070	$0.5243^{\dagger}$	$10.310^{\dagger}$
Proctor & Gamble	0.000894	0.00017	4.681	$0.1141^{\ddagger}$	$4.911^{\ddagger}$
Pioneer Electric	0.000279	0.00044	7.015	$0.6266^{\dagger}$	$14.580^{\dagger}$
Viacom	0.000766	0.00036	6.916	$0.6540^{\dagger}$	$30.450^{\dagger}$
Volvo	0.000729	0.00036	6.119	$0.5168^{\dagger}$	$14.920^{\dagger}$
Walgreen	0.000910	0.00024	4.147	$0.2341^{\dagger}$	$12.100^{\dagger}$
Exxon	0.000667	0.00014	5.270	0.0633	3.141
Xerox	0.000736	0.00024	10.984	$-0.1102^{\ddagger}$	$4.234^{\ddagger}$
Zenith	0.000504	0.00137	35.586	$3.1150^{\dagger}$	$35.940^{\dagger}$
Ford	0.000574	0.00029	4.511	$0.2424^{\dagger}$	$9.932^{\dagger}$
Sears	0.000870	0.00029	4.954	$0.2706^{\dagger}$	$6.521^{\ddagger}$
USAIR	0.000339	0.00100	10.542	$0.6351^{\dagger}$	$20.090^{\dagger}$

† and ‡ denote significance at the 1% and 5% levels, respectively.

In Table 3 we can see that neither test rejects the symmetry for General Electric and Exxon. There are three cases in which our test does not reject the symmetry but $\sqrt{b_1}$ does; these are British Airways, GTE, and Intel. However, we noticed that for the majority of stocks, both tests reject the symmetry. For Best Buy, the RS test rejects the symmetry. In Figure 4 we provide the empirical densities for British Airways, GTE, Best Buy, and Honda, for which we got the same results from both the tests. In this way we can compare the test results to see how responsive our tests are to excess kurtosis and asymmetry. We can see that British Airways and GTE return distributions look symmetric, whereas Best Buy and Honda returns do not. Therefore it appears that our test can identify the presence or absence of asymmetry, whereas the $\sqrt{b_1}$ possibly cannot.

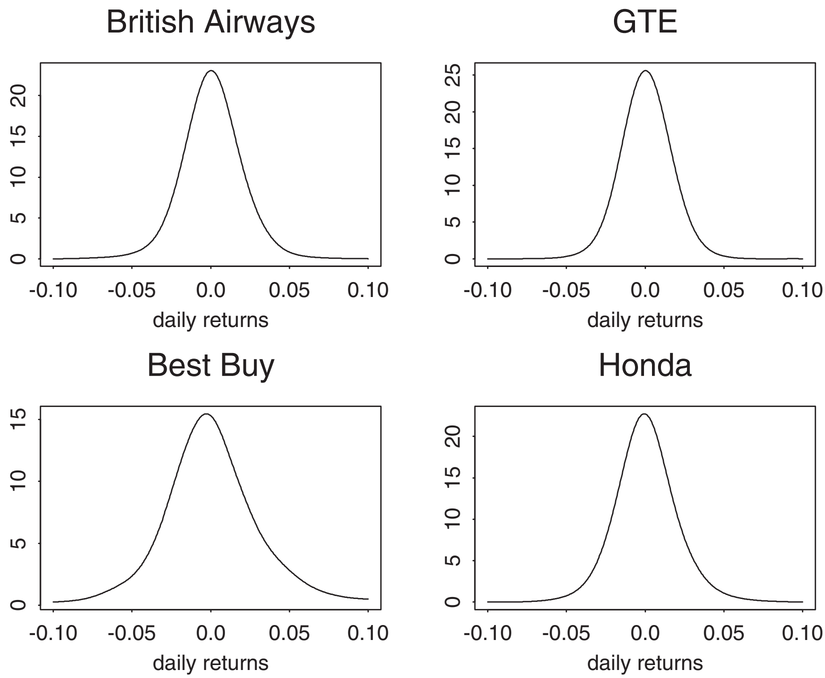

Figure 4 Empirical densities of selected cases.

5 CONCLUSION

In this article we proposed a simple test for symmetry that takes into account excess kurtosis. We carried out some simulations to study the performance of our test in comparison with the commonly used $\sqrt{b_1}$ test. The size and power comparisons of the Monte Corlo results indicate that our suggested test performs quite well in finite samples and is also robust to excess kurtosis. Further, its usefulness for financial data is highlighted with an application to individual stock return data. However, we need more simulation studies with a variety of other distributions to get further evidence on the finite sample properties. Applications to other financial data would also be useful.

APPENDIX

A. 1 The Derivatives of the Log-Likelihood Function and Other Derivations

The log-likelihood function given in Equation (11) and (12) is

$$l(\theta) = -n \log \psi(m,\delta,0) - n \log a - m \sum \log\left(1 + \frac{\varepsilon_i^2}{a^2}\right) + \delta \sum \tan^{-1}\left(\frac{\varepsilon_i}{a}\right),$$

where $\theta = (\delta, \mu, a, m)^{'}$ and

$$\psi(m, \delta, \lambda) = \int_{-\pi/2}^{\pi/2} t^{\lambda}\cos^{(2m-2)}(t)e^{\delta t}dt.$$

Let us also define

$$\phi(m, \delta, \lambda) = t^{\lambda}\cos^{(2m-2)}(t)e^{\delta t)}.$$

The first derivatives of $l(\theta)$ with respect to a different component of $\theta = (\delta, \mu, a, m)^{'}$ are given by

$$d_{\delta} = \frac{\partial l(\theta)}{\partial \delta} = n.\frac{\psi(m,\delta,1)}{\psi(m,\delta,0)} + \sum_{i=1}^{n}\tan^{-1}\left(\frac{\varepsilon_i}{a}\right),$$

$$d_m = \frac{\partial l(\theta)}{\partial m} = -\frac{2n}{\phi(m,\delta,0)}\int_{-\pi/2}^{\pi/2} \psi(m,\delta,0)\log(\cos(t))dt - \sum_{i=1}^{n}\log\left[1+\left(\frac{\varepsilon_i}{a}\right)^2\right],$$

$$d_{\mu} = \frac{\partial l(\theta)}{\partial \mu} = 2m\sum_{i=1}^{n}\frac{\varepsilon_i}{z_i} - \delta\sum_{i=1}^{n}\frac{a}{z_i},$$

$$d_a = \frac{\partial l(\theta)}{\partial a} = 2m\sum_{i=1}^{n}\frac{\varepsilon_i^2}{az_i} - \frac{n}{a} - \delta\sum_{i=1}^{n}\frac{\varepsilon_i}{z_i},$$

where $z_i = a^2 + (y_i - \mu)^2 = a^2 + \varepsilon_i^2$. Therefore our test will essentially be based on the score

$$\frac{\partial l(\theta)}{\partial \delta} = n\frac{\psi(m,\delta,1)}{\psi(m,\delta,0)} + \sum_{i=1}^{n}\tan^{-1}\left(\frac{\varepsilon_i}{a}\right).$$

Now consider

$$\begin{aligned}\psi(m,\delta,1) &= \int_{-\pi/2}^{\pi/2} t\cos^{(2m-2)}(t)e^{\delta t}dt \\ &= \int_{-\pi/2}^{\pi/2}\sum_{k=0}^{\infty}\frac{\delta^k t^k}{k!}t\cos^{(2m-2)}(t)dt \\ &= \sum_{k=0}^{\infty}\frac{\delta^k}{k!}\int_{-\pi/2}^{\pi/2} t^{(k+1)}\cos^{(2m-2)}(t)dt.\end{aligned}$$

Under the null, $H_0 : \delta = 0$, the above expression will be automatically zero for all k except for k = 0. However, the k = 0 term also vanishes, since

$$\int_{-\pi/2}^{\pi/2} t \cos^{(2m-2)}(t)dt = 0.$$

It can be shown that $\psi(m,\delta,0) \neq 0$. Therefore

$$d_\delta^0 = \frac{\partial l(\theta)}{\partial \delta}|_{\delta=0} = \sum_{i=1}^{n} \tan^{-1}\left(\frac{\varepsilon_i}{a}\right).$$

Now we consider the second derivative of the log-likelihood to find the element $V_{\delta\delta}$ of the information matrix $I(\theta)$.

We have

$$\frac{\partial^2 l(\theta)}{\partial \delta^2} = -n\frac{\psi(m,\delta,0)\psi(m,\delta,2) - (\psi(m,\delta,1))^2}{[\psi(m,\delta,0)]^2}.$$

With the previous result, $\psi(m,\delta,1) = 0$ under the null, this simplifies to

$$\frac{\partial^2 l(\theta)}{\partial \delta^2} = -n\frac{\psi(m,\delta,0)\psi(m,\delta,2)}{(\psi(m,\delta,0))^2}$$

$$= -n\frac{\psi(m,\delta,2)}{\psi(m,\delta,0)},$$

where

$$\psi(m,\delta,2) = \int_{-\pi/2}^{\pi/2} t^2\cos^{(2m-2)}(t)e^{\delta t}dt$$

$$= \sum_{k=0}^{\infty}\left[\int_{-\pi/2}^{\pi/2} \frac{\delta^k}{k!} t^{(k+1)}\cos^{(2m-2)}(t)dt\right].$$

Under the null, $H_0 : \delta = 0$, we have

$$\psi(m,0,2) = \int_{-\pi/2}^{\pi/2} t^2\cos^{(2m-2)}(t)dt$$

$$= 2\int_{0}^{\pi/2} t^2\cos^{(2m-2)}(t)dt.$$

We can evaluate this integral for an integer m using Gradshteyn and Ryzhik (1994:225–226, Equations 2.631-6 and 2.633-2):

$$\psi(m,0,2) = \int_0^{\pi/2} t^2 \cos^{(2m-2)}(t)dt$$

$$= \binom{2m-2}{m-1} \frac{t^3}{3.2^{(2m-2)}} |_0^{\pi/2}$$

$$+ 2^{-(2m-3)} \sum_{k=0}^{m-2} \binom{2m-2}{k} \int_0^{\pi/2} t^2 \cos(2(m-1-k)t)dt$$

$$= \binom{2m-2}{m-1} \frac{\pi^3}{6 \cdot 2^{2m}} + \frac{\pi}{2^{2m-1}} \sum_{k=0}^{m-2} \binom{2m-2}{k} \frac{(-1)^{(m-1-k)}}{(m-1-k)^2}.$$

Similarly Gradshteyn and Ryzhik's (1994:412) Equation 3.621-3 gives

$$\psi(m,0,0) = 2 \int_0^{\pi/2} \cos^{(2m-2)}(t)dt$$

$$= \frac{1 \cdot 3 \cdot 5 ... (2(m-1)-1)}{2 \cdot 4 \cdot 6 ... 2(m-1)} \cdot \frac{\pi}{2}$$

$$= \frac{\Gamma(m-0.5)}{\Gamma(m)} \cdot \frac{\sqrt{\pi}}{2}.$$

Collecting the above terms, we have

$$d_{\delta\delta} = \frac{\partial^2 l(\theta)}{\partial \delta^2} |_{\delta=0} = -2n \frac{1}{\sqrt{\pi}} \frac{\Gamma(m)}{\Gamma(m-0.5)} \left[\binom{2m-2}{m-1} \cdot \frac{\pi^2}{6} \frac{1}{2^{2m}} \right.$$
$$\left. + \frac{\pi}{2^{(2m-1)}} \sum_{k=0}^{m-2} \binom{2m-2}{k} \frac{(-1)^{(m-1-k)}}{(m-1-k)^2} \right].$$

The other second derivatives, under $\delta = 0$, are

$$d_{\mu\mu} = \frac{\partial^2 l(\theta)}{\partial \mu^2} = 2m \sum_{i=1}^{n} \left[\frac{z_i - 2a^2}{z_i^2} \right],$$

$$d_{\delta\mu} = \frac{\partial^2 l(\theta)}{\partial \delta \partial \mu} = a \sum_{i=1}^{n} \left[\frac{1}{z_i} \right],$$

$$d_{aa} = \frac{\partial^2 l(\theta)}{\partial a^2} = -2m \sum_{i=1}^{n} \left[\frac{(2a^2 + z_i)}{z_i^2} \cdot \frac{\varepsilon_i}{a} \right] + \frac{n}{a^2},$$

$$d_{mm} = \frac{\partial^2 l(\theta)}{\partial m^2} = -4\left[\frac{\pi^2}{12} + \sum_{k=1}^{2(m-1)} \frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2(m-1)} \frac{(-1)^k}{k} + ln2\right]^2 - \left[\sum_{k=1}^{2(m-1)} \frac{(-1)^{(k-1)}}{k} - ln2\right]^2\right],$$

$$d_{ma} = \frac{\partial^2 l(\theta)}{\partial m \partial a} = 2\sum_{i=1}^{n} \frac{\varepsilon_i^2}{az_i}.$$

Then the information matrix evaluated under the null hypothesis $H_0 : \delta = 0$ can be written in the following form:

$$I(\theta) = \begin{bmatrix} V_{\delta\delta} & V_{\delta\mu} & 0 & 0 \\ V_{\mu\delta} & V_{\mu\mu} & 0 & 0 \\ 0 & 0 & V_{aa} & V_{am} \\ 0 & 0 & V_{ma} & V_{mm} \end{bmatrix},$$

where

$$V_{rs} = E\left[-\frac{\partial^2 \log l(\theta)}{\partial r \partial s}\right],$$

with r and s denoting certain parameters from $\theta = (\delta, \mu, a, m)'$. Note that the second derivative $d_{\delta\delta}$ is a "constant" and there is no need to take expectation for this element. Then, for an integer m, we have

$$V_{\delta\delta} = 2n\frac{1}{\sqrt{\pi}}\frac{\Gamma(m)}{\Gamma(m-0.5)} \left[\binom{2m-2}{m-1} \cdot \frac{\pi^3}{6}\frac{1}{2^{2m}} + \frac{\pi}{2^{(2m-1)}}\sum_{k=0}^{m-2}\binom{2m-2}{k}\frac{(-1)^{(m-1-k)}}{(m-1-k)^2}\right].$$

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