Mathematical Social Sciences 27 (1994) 133–144
North-Holland

Equilibrium in a market of intellectual goods

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Abstract

We consider a simple model of economies of intellectual goods whose concept and way of modelling have been introduced by V. Makarov (1991). The main peculiarity of the model is the boolean character of variables. We prove that a competitive equilibrium exists and belongs to a core.

Key words: Equilibrium; Intellectual goods; Submodular function; Boolean lattice; Cores

1. Introduction

Modern markets are characterized by a greater and greater share of so called intellectual good(s) (IGs). Accordingly, one can watch an increasing flow of publications devoted to this phenomenon. The seminal work of Makarov (1991) has been our starting point. There, he gave a general description of economies of IGs (EIGs), namely, its main participants and markets; he provided a mathematical model of EIGs, and he carried out a preliminary analysis. Here we consider a version of EIGs and prove that an equilibrium in this model exists and belongs to a core.

Makarov (1991) defines four different types of agents in EIGs: creators (people who produce novelties such as scientific discoveries, scenarios, etc.), producers of IGs (people who transform novelties into IGs, i.e. new technologies, computer programs, etc.), producers of copies of IGs, and their consumers. Thus, three types of markets arise.

In the first market, creators sell novelties to producers of IGs. An essential point here is the indivisible nature of novelties.

In the second market, the producers of IGs sell them to producers of copies (as a rule, in the form of licenses which give the producers the right to produce and distribute copies of the IGs). An essential point here is that the production of copies is much cheaper than the production of an IG itself.

In the third market, the producers sell copies of IGs to each consumer personally. The peculiarity of this market is the phenomenon of individual prices. Equilibrium in EIGs means equilibria in these three markets.

SSDI 0165-4896(93)00721-6

To simplify our analysis and, at the same time, to retain the essence of the situation, we merge creators and producers of IGs and eliminate producers of copies, assuming that copying of IGs is costless. So, there is just one market involving producers of IGs and consumers.

The principal peculiarity of the model that distinguishes it from the Arrow–Debreu and Lindahl ones is the boolean character of IGs (in particular, the summation operation is maximization).

The convexity of a technology is replaced by submodularity of its cost function on the boolean lattice of subsets of IGs. Because of the 'irregularity' of the aggregation operation, the 'convexity' of individual technologies does not imply the same for the aggregate technology. So, in order to prove the existence of a competitive equilibrium in the market of IGs, we require the 'convexity' of the aggregate technology.

With regard to consumers, it is worthwhile mentioning the following. Since our model is not that of the general equilibrium, the consumers do not have budget restrictions. Their expenses are taken into account in their utilities which are assumed to be transferable and are supermodular functions on IGs.

2. A model

The main primitives of the model are IGs, producers of those and consumers. The list of all IGs is denoted by $\mathbb{L}$. We assume that $\mathbb{L}$ is a finite set. Its elements l are particular IGs, e.g. books or computer programs. A bundle of goods is a map $x:\mathbb{L}\to\{0,1\}$; its l-th coordinate x_l shows whether or not the corresponding good belongs to the bundle. So, one can identify each bundle of goods with a subset $X\subset\mathbb{L}$. The sum of two such bundles is obviously their union. We write $X\le Y$ if $X\subset Y$. Thus, the space of intellectual goods in our economy is a boolean lattice $\mathbb{B}=2^{\mathbb{L}}$ of all subsets of the set $\mathbb{L}$. Implicitly, one more (traditional) good–namely, money–is present in the model.

The set of producers is denoted by J. Each producer $j\in J$ can create a bundle of goods $Y\in\mathbb{B}$ having spent $a_j(Y)$ units of money. So, a producer is described by its cost function $a_j:\mathbb{B}\to\mathbb{R}\cup\{\infty\}$. We assume that, for all $j\in J$,

$$a_j(Y)\ge 0\,,\quad a_j(\emptyset)=0\,,$$

and every cost function a_j is monotone, i.e. $a_j(Y)\le a_j(Y')$ if $Y\subset Y'$.

The relation $a_j(Y)=\infty$ means that the jth producer cannot produce the bundle Y. Other conditions on cost functions will be imposed later.

The set of consumers is denoted by I. A consumer $i\in I$ is described by his/her utility function $u_i:\mathbb{B}\to\mathbb{R}$ which specifies the utility of bundles of goods. Here, we suppose that values of u_i are measured in money units. We also assume that each consumer purchases no more than one copy of each IG.

Hereafter, the utility functions are assumed to be monotone and normalized by $u(\emptyset)=0$.

A state of an economy is described by a pair (Y, X), where $Y = (Y_j)_{j\in J}$ are production outputs and $X = (X_i)_{i\in I}$ are consumption bundles; Y_j and X_i belong to $\mathbb{B}$. A state (X, Y) is *feasible* if $\bigcup_{j\in J} Y_j \geq \bigcup_{i\in I} X_i$, i.e. the supply covers the demand. To be realizable, a feasible state has to be economically accessible; we introduce the corresponding concept in Section 3, where a notion of equilibrium is defined.

A market of IGs is specified by prices which are supposed to be individual for each consumer. A price, for a consumer i, is a bundle $p_i \in \mathbb{R}^L$; its lth coordinate p_{il} is a price of good l for the ith consumer. So, to purchase a bundle X, the consumer i ought to pay the sum of money $p_i(X) = \sum_{l\in X} p_{il}$. Denote by p ($p \in \mathbb{R}^L$) the sum of individual prices p_i, i.e. $p = \sum_{i\in I} p_i$. This is, in fact, a producer price. In other words, if producer j creates a good l, then he/she expects to obtain the sum of money $p_l = \sum_{i\in I} p_{il}$. Similarly, for a bundle Y, the expected sum to be obtained is $p(Y) = \sum_{l\in Y} p_l$.

In the light of the monotonicity of utility and cost functions, we may consider only non-negative prices.

Of course, one may ask why prices are not uniform, rather than personalized. The reason is that, in view of the free copying of IGs, a producer wishes to obtain something from each consumer, even if some of them have a very low evaluation of the producer's production. This enables the producer to collect a sum of money that covers its cost of production. For example, suppose the cost is \$5 and two consumers are ready to pay only \$4 and \$2, respectively. If a common price is \$4, only the first consumer will buy the product. For both consumers to participate in the bargain, the common price must be \$2. Obviously, both common prices fail to cover the cost of production. Moreover, individual prices ensure that equilibria are Pareto-optimal states. In general, one can observe here a similarity to Lindahl's individual prices. A more detailed discussion was provided by Makarov (1991).

3. The concept of equilibrium

Given prices p, a producer j aims to maximize its profit, i.e. it maximizes, on $\mathbb{B}$, the function

$$\pi_j(\cdot) := p(\cdot) - a_j(\cdot)\,. \tag{1}$$

At the same time, a consumer i wishes to maximize, over $\mathbb{B}$, his gain, i.e. the function

$$u_i(\cdot) - p_i(\cdot)\,. \tag{2}$$

A triple $(\mathbf{P}, \mathbf{X}, \mathbf{Y})$ of prices $\mathbf{P} = (p_i)_{i\in I}$ and a state $(\mathbf{X} = (X_i)_{i\in I}, \mathbf{Y} = (Y_j)_{j\in J})$ is called *an equilibrium* of the market of intellectual goods if each Y_j is a solution of (1), $j \in J$; each X_i is a solution of (2), $i \in I$; and the following two conditions hold:

$$\bigcup_{j\in J} Y_j \geq \bigcup_{i\in I} X_i$$

(the balance with respect to goods) and

$$\sum_{j\in J} p(Y_j) \leq \sum_{i\in I} p_i(X_i)$$

(the balance with respect to money). The latter means that the total sum obtained from all consumers is enough to cover the costs of all producers.

We assert that, modulo goods with zero prices, all X_i are the same and equal to $\bigcup_{j\in J} Y_j$, where the sets Y_j are disjoint. More exactly, the following proposition is valid.

Proposition 1. *Let* $((p_i)_{i\in I}, (Y_j)_{j\in J}, (X_i)_{i\in I})$ *be an equilibrium and* $X = \bigcup_i X_i$. *Then*:
(1) *if* $l \in X - X_i$, *then* $p_{il} = 0$;
(2) *if* $l \in Y_j \cap Y_j$, *and* $j \neq j$, *then* $p_l = 0$;
(3) *if* $l \in \bigcup_j Y_j \backslash X$, *then* $p_l = 0$;
(4) $\Sigma_{j\in J}\, p(Y_j) = \Sigma_{i\in I}\, p_i(X_i)$.

Proof. Let us write down the financial balance in detail:

$$\sum_j \sum_{l\in Y_j} \sum_i p_{il} \leq \sum_i \sum_{l\in X_i} p_{il} \,.$$

Consider one of the p_{il} for fixed i and l. From the left-hand-side (l.h.s.), it follows that there are as many of them as those j for which $l \in Y_j$. It follows from the right-hand-side (r.h.s.), that there is only one such term whenever, of course, $l \in X_i$. If $l \in X_i$, then according to the material balance there necessarily exists a j for which $l \in Y_j$. So, for every pair (i, l) the sum of terms p_{il} from the l.h.s. is not less than the sum of analogous terms from the r.h.s. Therefore, we have at once the equality (4) as well as the equality between the l.h.s. and the r.h.s. for each p_{il}. The last observation also implies assertions (1)–(3). □

Corollary 1. *If* $((p_i), (Y_j), (X_i))$ *is an equilibrium then the triple* $((p_i), (Y'_j), (X'_i))$ *is also an equilibrium, where* $X'_i = \bigcup_i X_i$, $Y'_j \subset Y_j$, $Y' \cap Y'_j = \emptyset$ *if* $j \neq j'$ *and* $\bigcup_j Y'_j = \bigcup_i X_i$.

Here we use the monotonicity of utility and cost functions.

The corollary shows that an equilibrium at the market of intellectual goods (if it exists) can always by realized in the refined form: the producers divide the production of all demanded goods X among themselves and the whole bundle of goods X is available for each consumer i.

In addition, the definition of equilibrium (in the refined form) implies two things:

(1) optimal allocation of production, which means that an equilibrium bundle Y is divided among producers by the rule:

$$A(Y) := \sum_j a_j(Y_j) = \min \sum_j a_j(Y'_j) ,$$

where the minimum is taken over all subdivisions $Y = \bigcup_j Y'_j$; and

(2) optimality of the total pure utility, which means that

$$\sum_i u_i(Y) - A(Y) = \min_{Z \in \mathbb{B}} \left(\sum_i u_i(Z) - A(Z) \right) .$$

The last property of pure utility suggests a way to find an equilibrium. One should find the maximum of the function $\Sigma_i u_i(Y) - A(Y)$ on $\mathbb{B}$ and then find corresponding supporting 'subgradients' of the functions $u_i(\cdot)$, $i = 1, \ldots, n$, and $-A(\cdot)$ at the point X. These subgradients are prices impelling the agents to adhere to their optimal plans. The question of existence of such 'subgradients' will be discussed in Section 5. Now we consider one more important property of equilibrium allocations.

4. Equilibrium allocations belong to a core

Consider an equilibrium. Can any coalition of participants organize its interaction somehow differently and be better off? We show this to be impossible.

First, we describe what states of coalitions are feasible. Let K be a coalition of consumers and producers who have decided to produce and consume separately just among themselves. An allocation within the coalition is the following: consumer $i \in K$ obtains a bundle X_i and spends amount of money τ_i; producer $j \in K$ creates a bundle Y_j and obtains amount of money σ_j. An allocation is feasible if

(a) $\bigcup_{j \in K} Y_j \geq \bigcup_{i \in K} X_i$ (the balance of intellectual goods) and

(b) $\Sigma_{j \in K} \sigma_j \leq \Sigma_{i \in K} \tau_i$ (the balance of money).

Let $(X_i, \tau_i, Y_j, \sigma_j)$, $i \in I$, $j \in J$, be a feasible allocation of all consumers and producers, $K' = I \cup J$. We say that it is (Pareto) dominated by a coalition K if there exists a feasible allocation $(X'_i, \tau'_i, Y'_j, \sigma'_j)$, i, $j \in K$, such that

$$u_i(X'_i) - \tau'_i \geq u_i(X_i) - \tau_i \quad \text{for each consumer } i \in K ,$$

$$\sigma'_j - a_j(Y'_j) \geq \sigma_j - a_j(Y_j) \quad \text{for each producer } j \in K ,$$

and at least one inequality is strict.

We say that a feasible allocation *belongs to the core* if it is not dominated by any coalition. This is the usual definition, except that we take the producers into account.

A producer's utility is its profit. Since our model is not that of a general equilibrium, we do not assume any redistribution of profit among consumers. However, such a redistribution would not affect allocation of IGs in an equilibrium because of the property of utility transferability. So, a producer is an independent agent.

Proposition 2. *Any equilibrium allocation* $(X_i, \tau_i, Y_j, \sigma_j)$, $i \in I$, $j \in J$ *(where* $\tau_i = p_i(X_i)$ *and* $\sigma_j = p(Y_j)$*) belongs to the core.*

Proof. Clearly, each equilibrium allocation is feasible. Assume that a coalition K dominates it by an allocation $(X'_i, \tau'_i, Y'_j, \sigma'_j)$, $i, j \in K$. Then, for each consumer $i \in K$, we have

$$u_i(X'_i) - \tau'_i \geq u_i(X_i) - p_i(X_i) .$$

In view of the maximality of the r.h.s. we obtain

$$\tau'_i \leq p_i(X'_i) \quad \text{for each } i \in K .$$

Summing up these inequalities, we have

$$\sum_{i \in K} \tau'_i \leq \sum_{i \in K} p_i(X'_i) . \tag{3}$$

Similarly,

$$\sigma'_j \geq p(Y'_j) \quad \text{for each } j \in K ,$$

and hence

$$\sum_{j \in K} \sigma'_j \geq \sum_{j \in K} p_j(Y'_j) . \tag{4}$$

Denoting by X' a union of X'_i, $i \in K$, we have

$$\sum_{i \in K} p_i(X'_i) \leq p(X') \leq \sum_{j \in K} p_j(Y'_j) . \tag{5}$$

In the last inequality we used the inclusion (a). Combining inequalities (3), (4) and (5) and taking into account that at least one of the inequalities (3) and (4) is strict, we have

$$\sum_{i \in K} \tau'_i < \sum_{j \in K} \sigma'_j ,$$

which contradicts (b). So, the equilibrium allocation is not dominated. □

5. Existence of an equilibrium

We begin with a simple example which shows that, without special assumptions on utility and cost functions, an equilibrium does not necessarily exist.

Example 1. Consider one consumer, one producer and two goods l and m. Utility function u is defined by $u(\emptyset)=0$, $u(l)=u(m)=u(\{l,m\})=1$. Let the cost function a be equal to u. We assert that, in this case, equilibria do not exist.

Indeed, let (p,X,Y) be an equilibrium. By Corollary 1, we may assume that $X=Y$. By the definition of equilibrium, functions $u-p$ and $p-a=p-u$ attain maximum at the point X. But this means that $u-p$ is a constant which obviously vanishes. So u is a linear function, but this is not the case. □

Here as well as in continuous models, for an equilibrium to exist, one should require a 'convexity' property of utility functions and a technology. The following notion of a 'convex' function on a boolean lattice is being used in the theory of cooperative games: a function $f:\mathbb{B}\to\mathbb{R}$ is called convex if

$$f(X)+f(Y)\le f(X\cup Y)+f(X\cap Y)\,.$$

A certain natural continuation of such a function from a lattice $\mathbb{B}$ to $\mathbb{R}_+^{\mathbb{L}}$ generates a concave function on $\mathbb{R}_+^{\mathbb{L}}$ (see Lemma 1 on p. 142). So, it would be more reasonable to call such functions on $\mathbb{B}$ *concave*. We use also functions of the opposite type, viz. *convex* functions satisfying inequality

$$f(X)+f(Y)\ge f(X\cup Y)+f(X\cap Y)\,.$$

In discrete mathematics (see, for example, Edmonds (1970)) concave and convex functions are called *supermodular* and *submodular* respectively. If $f(X)+f(Y)=f(X\cup Y)+f(X\cap Y)$, then f is called a *modular* function. In what follows, we set $f(\emptyset)=0$.

Let us define a *technology* as a closed subset $Q\subset(\mathbb{R}\cup\{\infty\})\times\mathbb{B}$, its elements are $(t,X)\in Q$ where t is the amount of money needed to produce the bundle of goods X. We assume that all technologies satisfy the free disposal condition:

$$(t,X)\in Q\,,\quad X\subset X'\quad\text{and}\quad t'>t\Rightarrow(t',X')\in Q\,.$$

A technology Q is called *convex* if, for arbitrary (t,X) and (q,Y), there exist s and r such that $(s,X\cup Y)\in Q$, $(r,X\cap Y)\in Q$ and $t+q-r\ge s$. A technology Q is convex if and only if the corresponding cost function $f(X)=\min_{(t,X)\in Q}t$ is submodular and monotone.

If there are two technologies Q_1 and Q_2, then their sum is defined by

$$\begin{aligned}Q_1+Q_2=\{&(t,X),\,t=t_1+t_2,\,X=X_1\cup X_2\,,\\ &\text{where }(t_1,X_1)\in Q_1,(t_2,X_2)\in Q_2\}\,.\end{aligned}$$

Obviously, the cost function f of technology Q_1+Q_2 has the form

$$f(X)=\min_{X_1\cup X_2=X}(f_1(X_1)+f_2(X_2))$$

where f_1 and f_2 are the cost functions for Q_1 and Q_2, respectively. The function f is called the *convolution* of f_1 and f_2. We denote it by f_1*f_2. Similarly, one can define the convolution of a number of functions by

$$(*_{j\in J} f_j)(X) = \min_{\cup_j X_j = X} \sum f_j(X_j)\,.$$

So, we require the sum of technologies to be convex and utility functions to be concave (in other words, an aggregate cost function is submodular and utility functions are supermodular). These assumptions have the following economic interpretation.

The definition of supermodularity can be rewritten in the following form: $u(X \cup \{l\}) - u(X) \le u(X' \cup \{l\}) - u(X')$ where $X \subset X'$ and l does not belong to X'. This means that more IGs yield more pleasure for each additional IG.

Submodularity of the cost function can be clarified in a similar way: the cost of production of an additional IG decreases with the growth of the quantity of produced novelties.

Now we state our main result.

Theorem. *let u_i, $i \in I$, be utility functions, Q_j, $j \in J$, be technologies with cost functions a_j. Suppose that the following two conditions hold:*
(a) *all utility functions u_i are supermodular;*
(b) *the aggregate technology $Q = \Sigma_{j\in J} Q_j$ is convex (or, in other words, the convolution of the individual cost functions $A = *_{j\in J} a_j$ is submodular).*

Then there exists an equilibrium in the market of IGs at which all producers have zero profit.

One can ask why we require convexity of the aggregate technology Q but not the individual Q_j. The answer is that an equilibrium can fail to exist even if all individual technologies are convex. This can be illustrated by the following example.

Example 2. There are two producers and one consumer. The cost function a_1 is specified as follows:

$$a_1(\emptyset) = 0\,,\quad a_1(\{1\}) = a_1(\{2\}) = a_1(\{3\}) = a_1(\{1,3\}) = 1\,,$$

$$a_1(\{1,2\}) = a_1(\{2,3\}) = a_1(\{1,2,3\}) = 2\,.$$

The cost function a_2 is defined by equalities

$$a_2(\emptyset) = 0\,,\quad a_2(\{1\}) = a_2(\{2\}) = a_2(\{3\}) = a_2(\{1,2\}) = 1\,,$$

$$a_2(\{1,3\}) = a_2(\{2,3\}) = a_2(\{1,2,3\}) = 2\,.$$

The values of utility function u are specified by

$$u(\emptyset) = u(\{2\}) = u(\{3\}) = 0\,,\quad u(\{2,3\}) = 1\,,$$

$$u(\{1\}) = u(\{1,2\}) = u(\{1,3\}) = 2\,,\quad u(\{1,2,3\}) = 3\,.$$

One can check that u is supermodular and a_1, a_2 are submodular functions. For the values of the convolution $A = a_1 * a_2$

$$A(\emptyset)=0\,,\quad A(\{1\})=A(\{2\})=A(\{3\})=A(\{1,2\})=A(\{1,3\})=1$$
$$\text{and } A(\{2,3\})=A(\{1,2,3\})=2\,.$$

The function A is not submodular. We verify that no equilibrium exists. Indeed, suppose that (p, Y, X) is an equilibrium. By Corollary 1 one can assume that $X = Y$. Then, as was mentioned above, the function $u - A$ attains its maximum at the point X. The function $u - A$ is equal to 1 at points $\{1\}$, $\{1,2\}$, $\{1,3\}$, $\{1,2,3\}$ and is non-positive at all other points. So the point X has the form $\{1, \ldots\}$. At this point the function $u - p$ attains its maximum and the function $A - p$ attains its minimum. But, on the face $x_1 = 1$, the function u is equal to $A + 1$. So, the function $u - p$ is constant on the face $x_1 = 1$ and hence u is affine on it. But this is not the case. □

In the last example, the convolution $A = a_1 * a_2$ of two submodular functions a_1 and a_2 is not submodular.

In connection with the question about conditions ensuring submodularity of convolution, recall that the convolution of a submodular and a modular function is submodular (Lovasz, 1983).

6. Proof of the theorem

Consider the natural embedding of the lattice $\mathbb{B}$ in the coordinate vector space $\mathbb{R}^{\mathbb{L}}$. A continuation of functions from $\mathbb{B}$ to $\mathbb{R}^{\mathbb{L}}_{+}$ will play an important role in what follows.

We continue a *function* $f:\mathbb{B}\to\mathbb{R}$ *to a function* $\hat{f}:\mathbb{R}^{\mathbb{L}}_{+}\to\mathbb{R}$ *in the standard way* used in the cooperative game theory for computing gains of fuzzy coalitions. Let x be a point in $\mathbb{R}^{\mathbb{L}}_{+}$. Represent x (a fuzzy-bundle of intellectual goods) in the form of a weighted sum of 'pure' bundles, i.e. points of $\mathbb{B}$.

Write down the coordinates of x in decreasing order, i.e. choose an appropriate permutation of coordinates σ such that

$$x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(L)}\,,\quad L = |\mathbb{L}|\,.$$

Denote by $X_{\sigma(k)}$ the point of $\mathbb{B}$ with the following coordinates: the coordinates: $\sigma(l)$ $(l \leq k)$ are equal to 1 and all other coordinates vanish. Then x can be represented in the form

$$x = \sum_{k\in\mathbb{L}} \alpha_{\sigma(k)} \cdot X_{\sigma(k)}\,,$$

where $\alpha_{\sigma(k)} = x_{\sigma(k)} - x_{\sigma(k+1)}$, $\alpha_{\sigma(L)} = x_{\sigma(L)}$.

Set $\hat{f}(x) = \sum_{k=1}^{L} \alpha_{\sigma(k)} f(X_{\sigma(k)})$.

Although the ordering of x is not unique, the continuation $\hat{f}$ specified in this way is well-defined. It follows from the definition, that the continuation operation is linear, i.e. $(f+g)\hat{} = \hat{f} + \hat{g}$.

Lemma 1. *Let* $f\colon \mathbb{B} \to \mathbb{R}_+$ *be supermodular,* $f \geq 0$ *and* $f(\emptyset) = 0$. *Then* $\hat{f}$ *is concave, monotonic, homogeneous and continuous on* $\mathbb{R}^L_+$.

Proof. The concavity of $\hat{f}$ is established if we show that for any point $x \in \mathbb{R}^{\mathbb{L}}_+$ there exists a linear function λ on $\mathbb{R}^{\mathbb{L}}$ such that $\lambda \geq \hat{f}$ and $\lambda(x) = \hat{f}(x)$. We define such a function in explicit form.

Let $x \in \mathbb{R}^{\mathbb{L}}_+$. Let us arrange the coordinates of x by $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(L)}$. Set $\lambda(X_{\sigma(k)}) = f(X_{\sigma(k)})$, $k = 1, \ldots, L$. Then the value of λ on each $\sigma(k)$th unit vector, i.e. $e_{\sigma(k)} = (0, \ldots, 1, \ldots, 0)$, is equal to $f(X_{\sigma(k)}) - f(X_{\sigma(k-1)})$. We complete the definition by linearity on $\mathbb{R}^{\mathbb{L}}$. In particular, we have $\lambda(x) = \sum_k \alpha_{\sigma(k)} \lambda(X_{\sigma(k)}) = \sum_k \alpha_{\sigma(k)} f(X_{\sigma(k)}) = \hat{f}(x)$. Now we check by induction that $\lambda \geq \hat{f}$ on $\mathbb{B}$.

Let $X \in \mathbb{B}$, and let $\sigma(k)$ be the smallest number for which $X_{\sigma(k)} \supset X$. Then, by supermodularity, we have the inequality

$$f(X) \leq f(X \cup X_{\sigma(k-1)}) + f(X \cap X_{\sigma(k-1)}) - f(X_{\sigma(k-1)}) .$$

Further, $X \cup X_{\sigma(k-1)} = X_{\sigma(k)}$ and by induction $f(X \cap X_{\sigma(k-1)}) \leq \lambda(X \cap X_{\sigma(k-1)})$. So, the r.h.s. of the previous inequality does not exceed

$$\begin{aligned} &f(X \cup X_{\sigma(k-1)}) + \lambda(X \cap X_{\sigma(k-1)}) - f(X_{\sigma(k-1)}) \\ &\quad = \lambda(X \cup X_{\sigma(k-1)}) + \lambda(X \cap X_{\sigma(k-1)}) - \lambda(X_{\sigma(k-1)}) = \lambda(X) . \end{aligned}$$

Thus, $f(X) \leq \lambda(X)$ on $\mathbb{B}$. But then, for any point $x' \in \mathbb{R}^{\mathbb{L}}_+$ and the corresponding representation $x' = \sum_k \alpha_{\sigma'(k)} X_{\sigma'(k)}$, one has:

$$\begin{aligned} \hat{f}(x') &= \sum_k \alpha_{\sigma'(k)} f(X_{\sigma'(k)}) \leq \sum_k \alpha_{\sigma'(k)} \lambda(X_{\sigma'(k)}) \\ &= \lambda\Big(\sum_k \alpha_{\sigma'(k)} X_{\sigma'(k)}\Big) \leq \lambda(x') , \end{aligned}$$

i.e. $\hat{f}(x') \leq \lambda(x')$.

It follows from the above that $\hat{f}(x) = \min_{\lambda_\sigma \geq f} \lambda_\sigma(x)$, where minimum is taken over the set of linear functions λ_σ, $\sigma \in \mathcal{S}_L$, such that the values of λ_σ coincide with those of f on the chain $V_\sigma = \{\emptyset \subset X_{\sigma(1)} \subset X_{\sigma(2)} \subset \cdots \subset X_{\sigma(L)}\}$. Since the set of such functionals is finite and $f(\emptyset) = 0$, the function $\hat{f}$ is continuous on $\mathbb{R}^{\mathbb{L}}_+$. It is also obvious that $\hat{f}$ is concave, homogeneous and monotone. □

We now seek an equilibrium. Consider the function $\hat{f}(x) = \min_{\lambda \geq f} \lambda(x)$ on the entire space $\mathbb{R}^{\mathbb{L}}$. Set

$$W = \sum_i \hat{u}_i - \hat{A} .$$

(Recall that u_i are utility functions and A is an aggregated cost function.) Since $W = (\sum_i u_i - A)\hat{}$ on $C = [0, 1]^{\mathbb{L}}$ (see the definition of operation $\hat{}$ at the beginning of this section), there exists a point X in $\mathbb{B}$ at which W reaches maximum. So, there exist corresponding subgradients x^*_i, x^*_A and a functional x^*_C such that

$$\sum_i x_i^* + x_C^* = x_A^* . \tag{6}$$

We have

$$x_A^* \geq 0 , \quad x_A^*(X) = A(X) \quad \text{and}\, x_A^*(X') \leq A(X') \quad \text{for any}\, X' \in \mathbb{B} ; \tag{7}$$

$$x_i^* \geq 0 , \quad x_i^*(X) = u_i(X) \quad \text{and}\, x_i^*(X') \geq u_i(X') \quad \text{for any}\, X' \in \mathbb{B} ; \tag{8}$$

$$x_C^* = (\lambda_1, \ldots, \lambda_L) , \quad \text{where}\ \lambda_t \geq 0 \quad \text{if}\, (X)_t = 0 \quad \text{and}\ \lambda_t \leq 0 \quad \text{if}\, (X)_t = 1 . \tag{9}$$

Set $p_{it} = x_{it}^* + \delta_{it}$, where $\delta_{it} = x_{Ct}^* x_{it}^* (\Sigma_i x_{it}^*)^{-1}$, $t = 1, \ldots, l$; (if $\Sigma_i x_{it}^* = 0$ we set $p_{it} = x_{Ct}^*/L$);

$$p = x_A^* . \tag{10}$$

It follows from (6)–(9) that $p_i \geq 0$ and $\Sigma_{i \in I}\, p_i = p$. Since δ_t separates C from the point X (see (9)), we have:

$$\delta_i(X) \leq \delta_i(X') \quad \text{for any}\, X' \in \mathbb{B} , \quad i \in I .$$

Besides, property (8) implies that

$$u_i(X') - x_i^*(X') \leq u_i(X) - x_i^*(X) .$$

Summing up these inequalities, we obtain the following one:

$$u_i(X) - p_i(X) \geq u_i(X') - p_i(X') \quad \text{for each}\, X' \in \mathbb{B} .$$

It remains only to show that the price p is the supporting price for each producer.

Let $X = \cup_j Y_j$ be an optimal partition of X among the producers. So, $A(X) = \Sigma_j\, a_j(Y_j)$. For any j and all $Z \in \mathbb{B}$, we have $p(Z) - a_j(Z) \leq 0$ because otherwise $p(Z) - A(Z) > 0$, which contradicts (7) and (10). On the other hand, we have $p(Y_j) - a_j(Y_j) = 0$ for all j because the whole sum is equal to 0. So, for any $j \in J$,

$$0 = p(Y_j) - a_j(Y_j) \geq p(Z) - a_j(Z) \quad \text{for any}\, Z \in \mathbb{B} .$$

Finally, since the sets Y_j are disjoint, one has:

$$\sum_{j \in J} p(Y_j) = p(X) = \sum_{i \in I} p_i(X) ,$$

(the financial balance).

Thus, $((p_i)_{i \in I}, (Y_j)_{j \in J}, (X_i = X)_{i \in I})$ is an equilibrium. The proof is completed.

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