Insurance: Mathematics and Economics 12 (1993) 259–264
North-Holland

Asymptotic ordering of risks and ruin probabilities

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Received December 1992
Revised February 1993

Abstract: We introduce asymptotic orderings of distributions which imply orderings of ruin probabilities in the range of the Cramér case through some intermediate cases to large claims distributions. This allows to give bounds for ruin probabilities where it is not possible to give exact estimates.

Keywords: Ordering of risks; Ruin probability; Large claim distributions.

1. Introduction

Ordering of risks and its possible influence on ruin probabilities have been investigated in various papers and books [see e.g. Goovaerts et al. (1990) and van Heerwaarden (1991)]. This is an important subject since it allows to calculate bounds where it is not possible to calculate exact ruin probabilities.

In this paper we consider asymptotic estimates of the ruin probability as the initial risk reserve becomes large. In the Cramér case, van Heerwaarden (1991, p. 63f) states that the exponential order of claimsizes X and Y (i.e. $X \leq^{e} Y$ if $E\ \mathrm{e}^{sX} \leq E\ \mathrm{e}^{sY}$ for all $s \geq 0$ with $E\ \mathrm{e}^{sY} < \infty$ holds) implies an ordering of the Lundberg coefficients and thus an ordering of the ruin probabilities. We shall proceed further in this direction, extending this result in two ways. First we introduce asymptotic orderings of the claimsize distributions instead of the ordering of their moment generating functions and, second, we shall not only consider the Cramér case but cover the full range from the Cramér case via some intermediate cases to large claims distributions where the moment generating function does not exist for any positive value.

We consider the classical risk model where the claimsizes $(X_k)_{k \in \mathbb{N}}$ are iid non-negative random variables with common distribution function F and finite mean $\mu(F)$. The claim number process $(N(t))_{t \geq 0}$ is supposed to be a Poisson process with parameter λ, independent of $(X_k)_{k \in \mathbb{N}}$. Then $S(t) = \sum_{k=1}^{N(t)} X_k$ represents the accumulated claims in $[0, t]$. We further assume that the company has initial risk reserve $x > 0$ and that the intensity of the gross risk premium is $c > 0$. The safety loading should be positive, i.e. $\rho := \mu\lambda/c < 1$. Then the *ruin probability starting with initial risk reserve* x is defined as

$$\psi(x) = P(S(t) - ct > x \text{ for some } t > 0).$$

$\phi(x) = 1 - \psi(x)$ is called *survival probability* and is a proper distribution function on $[0, \infty)$. It is by now a standard result [see e.g. Gerber (1979) or Grandell (1991)] that ϕ has the representation

$$\phi(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n F_I^{n*}(x), \tag{1}$$

where $F_I(x) = \mu(F)^{-1} \int_0^x \overline{F}(t)\,\mathrm{d}t$ is the integrated tail distribution ($\overline{F} = 1 - F$), and F_I^{n*} denotes the n-fold convolution of F_I.

The paper is organized as follows: In Section 2 we state the asymptotic ruin probability for different classes of claimsize distributions. In Section 3 we introduce the asymptotic orders and give our main results.

2. Asymptotic ruin probabilities

For a distribution function F on $[0, \infty)$ we denote by

$$\hat{f}(s) = \int_0^\infty \mathrm{e}^{sx}\,\mathrm{d}F(x)$$

its moment generating function. Note that $\hat{f}(s)$ is finite for all $s < \gamma \leq \infty$ and, for $\gamma < \infty$, $\hat{f}(\gamma)$ can be finite or infinite.

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In the following we summarize known results on asymptotic ruin probabilities. We start with the famous Cramér–Lundberg result:

Definition 2.1. A distribution function F belongs to C if there exists some constant R such that

$$\int_0^\infty e^{Rz}\overline{F}(z)\,dz = \frac{c}{\lambda}.$$

R is called the *Lundberg coefficient*.

Theorem 2.2. *Suppose*

$$F \in C \quad \textit{and} \quad \mu^* = (\lambda/c)\textstyle\int_0^\infty z\, e^{Rz}\overline{F}(z)\,dz < \infty,$$

then

$$\psi(x) \sim \frac{1-\rho}{R\mu^*} e^{-Rx}, \quad x \to \infty,$$

and if $\mu^* = \infty$, *then*

$$\psi(x) = o(e^{-Rx}), \quad x \to \infty.$$

We proceed with some intermediate cases where the claimsize distribution has an exponentially decreasing tail but the Lundberg coefficient does not exist.

Definition 2.3. A distribution function F with $F(x) < 1$ for all $x > 0$ belongs to $S(\gamma)$ for $\gamma > 0$ if
(i) $\lim_{x\to\infty} \overline{F^{2*}}(x)/\overline{F}(x) = 2\hat{f}(\gamma) < \infty$,
(ii) $\lim_{x\to\infty} \overline{F}(x-y)/\overline{F}(x) = e^{\gamma y}\ \forall y \in \mathbb{R}$.

These classes have been introduced by Chover et al. (1973a, b); for some characterizations see Klüppelberg (1989a). They have been applied to risk theory in Embrechts and Veraverbeke (1982) and in Klüppelberg (1989b).

Condition (ii) of Definition 2.3 implies that

$$\gamma = \sup\{s \in \mathbb{R};\ \hat{f}(s) < \infty\} = \sup\left\{s \in \mathbb{R};\ \int_0^\infty e^{sz}\overline{F}(z)\,dz < \infty\right\}.$$

Furthermore, as $x \to \infty$, $e^{sx}\overline{F}(x) \to 0$ for all $x < \gamma$ and $e^{sx}\overline{F}(x) \to \infty$ for all $s > \gamma$.

Theorem 2.4 [Klüppelberg (1989a, Theorem 4.1, Corollary 4.2)]. *Suppose* $F \in S(\gamma)$ *for some* $\gamma > 0$ *and* $\int_0^\infty e^{sz}\overline{F}(z)\,dz < c/\lambda$ *for all* $s \in (0, \gamma]$. *Then*

$$\psi(x) \sim \frac{\lambda(1-\rho)}{\gamma c}\left(1 - \frac{\lambda}{c}\int_0^\infty e^{\gamma z}\overline{F}(z)\,dz\right)^{-2}\overline{F}(x),$$

$x \to \infty$.

The last class we consider is a class of heavy-tailed distributions which contains e.g. the Pareto, lognormal and loggamma distributions.

Definition 2.5. A distribution function F with $F(x) < 1$ for all $x > 0$ belongs to the class S^* if

$$\lim_{x\to\infty}\int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)}\overline{F}(y)\,dy = 2\mu(F) < \infty.$$

The class has been introduced in Klüppelberg (1988) where also some conditions for $F \in S^*$ can be found. In particular, $F \in S^*$ implies that $\lim_{x\to\infty}\overline{F}(x-y)/\overline{F}(x) = 1$ and $\hat{f}(s) = \infty$ for all $s > 0$. Hence $e^{sx}\overline{F}(x) \to \infty$ as $x \to \infty$ for all $s > 0$. The class S^* can be used to model large claims since for $(X_k)_{k\in\mathbb{N}}$ iid with common distribution function $F \in S^*$ the following property holds:

$$P\left(\sum_{k=1}^n X_k > x\right) \sim P\left(\max_{1\le k\le n} X_k > x\right), \quad x \to \infty,$$

hence the sum of n claims gets large if and only if their maximum gets large. Asymptotic ruin probabilities can be calculated for S^* [Klüppelberg (1989b, Corollary of Theorem 4); see also Embrechts and Veraverbeke (1982)]

Theorem 2.6. *Suppose* $F \in S^*$, *then*

$$\psi(x) \sim \frac{\rho}{1-\rho}\overline{F}_I(x), \quad x \to \infty.$$

3. Asymptotic ordering of ruin probabilities

Asymptotic orderings have been used e.g. in Embrechts and Goldie (1980) and Klüppelberg (1990) to investigate convolution closure of certain classes of distribution functions. We shall show that they are also appropriate orderings to compare asymptotic ruin probabilities and derive bounds in cases not easily established by the results in Section 2. In the following we repeat some definitions given in Klüppelberg (1990), where we adjust the notation to the established one in risk ordering.

For claimsize distributions F and G with infinite support we define

$$F \le^{t} G \Leftrightarrow \lim_{x\to\infty}\overline{F}(x)/\overline{G}(x) = a_t < \infty.$$

$\leq^{t}$ defines a partial ordering and is called *tail ordering*. Consistent with the above definition we also say $F \leq^{t} G$ if F has finite support, also if $\operatorname{supp} F \subset \operatorname{supp} G$.

Similarly, the *weak tail ordering* $\leq^{w}$ is defined by

$$F \leq^{w} G \Leftrightarrow \overline{F}(x)/\overline{G}(x) \leq a_{w} < \infty \quad \forall x \geq 0.$$

Obviously, $\leq^{w}$ defines a partial ordering which is weaker than $\leq^{t}$. Furthermore, if $F \leq^{t} G$ with a_{t}, then $F \leq^{w} G$ with $a_{w} \geq a_{t}$. Also $\leq^{w}$ is essentially an ordering of the tails since the quotient $\overline{F}(x)/\overline{G}(x)$ is obviously bounded on any compact subset of the support of G. This also explains why any change of the distributions on a compact subset of their support does not affect the ordering relation. Notice that $F \leq^{w} G$ with $a_{w} = 1$ is equivalent to *stochastic order*.

From (1) it becomes obvious that certain closure properties are required to conclude from ordering of the claimsizes to ordering of the ruin probabilities. A first step is the transition to integrated tail distributions.

Lemma 3.1. *(a)* $F \leq^{t} G \Rightarrow F_{I} \leq^{t} G_{I}$.
(b) $F \leq^{w} G \Rightarrow F_{I} \leq^{w} G_{I}$.

Proof. (a) is a consequence of l'Hospital's rule and (b) is obvious. □

Notice that $F \leq^{w} G$ with $a_{w} \leq 1$ implies *stop-loss order*, i.e.

$$\int_{d}^{\infty} \overline{F}(x)\, \mathrm{d}x \leq \int_{d}^{\infty} \overline{G}(x)\, \mathrm{d}x$$

holds for all $d \geq 0$.

For ease of notation we assume that the safety load ρ is chosen to be equal for all models considered. This means that the premium state c is compensating differences in mean μ and arrival rate λ.

The following result enables us to compare asymptotic ruin probabilities.

Proposition 3.2. *Suppose F and G are arbitrary claimsize distributions with ruin probabilities ψ_{F} and ψ_{G} respectively. Then*

$$\limsup_{x\to\infty} \frac{\psi_{G}(x)}{\psi_{F}(x)} \leq (\rho^{-1} - 1) \limsup_{x\to\infty} \frac{\psi_{G}(x)}{\overline{F}_{I}(x)}.$$

Proof. Suppose $(Z_{k})_{k\in\mathbb{N}}$ are iid non-negative with common distribution function H. Then

$$\overline{H}^{n*}(x) = P\left(\sum_{k=1}^{n} Z_{k} > x\right) \geq P\left(\max_{1\leq k\leq n} Z_{k} > x\right)$$
$$= 1 - H^{n}(x),$$

and hence

$$\frac{\overline{H}^{n*}(x)}{\overline{H}(x)} \geq \frac{1 - H^{n}(x)}{1 - H(x)} = 1 + \sum_{k=1}^{n-1} H^{k}(x)$$

which implies that

$$\liminf_{x\to\infty} \frac{\overline{H}^{n*}(x)}{\overline{H}(x)} \geq n.$$

We apply this to F_{I} to obtain

$$\limsup_{x\to\infty} \frac{\psi_{G}(x)}{\psi_{F}(x)}$$
$$= \limsup_{x\to\infty} \frac{\psi_{G}(x)}{\overline{F}_{I}(x)} \left((1-\rho) \sum_{n=0}^{\infty} \rho^{n} \frac{\overline{F}_{I}^{n*}(x)}{\overline{F}_{I}(x)}\right)^{-1}$$
$$\leq \limsup_{x\to\infty} \frac{\psi_{G}(x)}{\overline{F}_{I}(x)}$$
$$\times \left(\liminf_{x\to\infty} (1-\rho) \sum_{n=0}^{\infty} \rho^{n} \frac{\overline{F}_{I}^{n*}(x)}{\overline{F}_{I}(x)}\right)^{-1}$$
$$\leq (\rho^{-1} - 1) \limsup_{x\to\infty} \frac{\psi_{G}(x)}{\overline{F}_{I}(x)}$$

where the last inequality follows from Fatou's Lemma. □

We first consider the Cramér case.

Theorem 3.3. *Suppose $G \in C$ with Lundberg coefficient R and denote $E^{R}(x) = 1 - \mathrm{e}^{-Rx}$, $x \geq 0$. If $E^{R} \leq^{w} F$, then $\phi_{G} \leq^{w} \phi_{F}$.*

Proof. We use Proposition 3.2:

$$\limsup_{x\to\infty} \frac{\psi_{G}(x)}{\overline{F}_{I}(x)} = \lim_{x\to\infty} \frac{\psi_{G}(x)}{\mathrm{e}^{-Rx}} \limsup_{x\to\infty} \frac{\mathrm{e}^{-Rx}}{\overline{F}_{I}(x)}.$$

Now the first limit exists by Theorem 2.2, and the second is bounded by assumption, the fact that $E_{I}^{R} = E^{R}$, and Lemma 3.1. □

From the proof we obtain immediately the following result.

Corollary 3.4. *Suppose $G \in C$ with Lundberg coefficient R and, either $\mu^* = (\lambda/c)\int_0^\infty z\, \mathrm{e}^{Rz}\overline{G}(z)\,\mathrm{d}z = \infty$ and $E^R \leq^{\mathrm{w}} F$, or $\lim_{x\to\infty}\overline{E}^R(x)/\overline{F}(x) = 0$, then $\lim_{x\to\infty}\psi_G(x)/\psi_F(x) = 0$.*

Example. It is well-known that $E^R \in C$ with ruin probability

$$\psi_{E^R}(x) = \rho \exp\{-R(1-\rho)x\}, \quad x \geq 0.$$

Then for any claimsize distribution F whose tail $\overline{F}$ decreases slower to 0 than e^{-Rx} the asymptotic ruin probability is larger than ψ_{E^R}.

Notice that it is well possible to have two claimsize distributions F and G such that $\psi_F(x) \sim c\psi_G(x)$ as $x \to \infty$ for some $c > 0$; i.e. ϕ_F and ϕ_G are tail-equivalent, but F and G not even weakly tail-equivalent. Take e.g. F exponential and G as the distribution function with density

$$g(x) = \frac{(x+1-a)}{\Gamma(a)} x^{a-2}\,\mathrm{e}^{-x}, \quad x \geq 0,$$

for $a \in (0, 1]$. Then

$$\overline{G}(x) = \frac{1}{\Gamma(a)} x^{a-1}\,\mathrm{e}^{-x}, \quad x \geq 0$$

and

$$\hat{g}_a(s) = \frac{1}{(1-s)^a}.$$

This implies that the Lundberg coefficient is

$$R_a = 1 - \rho^{1/a}$$

and hence

$$\overline{F}(x) = \exp\left\{-\frac{1-\rho^{1/a}}{1-\rho}x\right\}$$

has the same Lundberg coefficient as G. Since $\rho < 1$ for $a < 1$ we obtain $\overline{F}(x) = \mathrm{e}^{-(1+\epsilon)x}$ for some $\epsilon > 0$. Now

$$\frac{\overline{G}(x)}{\overline{F}(x)} = \frac{x^{a-1}}{\Gamma(a)}\mathrm{e}^{\epsilon x} \to \infty, \quad x \to \infty,$$

and hence even $F \leq^{\mathrm{t}} G$.

The following result gives bounds in the intermediate case.

Theorem 3.5. *Suppose $G \in S(\gamma)$ for some $\gamma > 0$ and $\int_0^\infty \mathrm{e}^{sz}\overline{G}(z)\,\mathrm{d}z < c/\lambda$ for all $s \in (0, \gamma]$. If $G \leq^{\mathrm{w}} F$, then $\phi G \leq^{\mathrm{w}} \phi_F$.*

Proof. We apply Proposition 3.2:

$$\limsup_{x\to\infty} \frac{\psi_G(x)}{\overline{F}_I(x)} = \lim_{x\to\infty} \frac{\psi_G(x)}{\overline{G}(x)} \frac{\overline{G}(x)}{\overline{G}_I(x)} \limsup_{x\to\infty} \frac{\overline{G}_I(x)}{\overline{F}_I(x)}.$$

The first limit is finite by Theorem 2.4 and the fact that $\lim_{x\to\infty}\overline{G}(x)/\overline{G}_I(x) = \gamma$ for $G \in S(\gamma)$, and the second by Lemma 3.1(b). □

For the tail order we obtain the following Corollary.

Corollary 3.6. *Suppose G satisfies the conditions of Theorem 3.5 and $\lim_{x\to\infty}\overline{G}(x)/\overline{F}(x) = a < \infty$, then $\lim_{x\to\infty}\psi_G(x)/\psi_F(x) = \tilde{a}$. If $a > 0$ and also F satisfies the conditions of Theorem 3.5 then $\tilde{a} = (c_1/c_2)a$ where c_1 is the constant of Theorem 2.4 applied to G and c_2 the corresponding constant for F; otherwise $\tilde{a} = 0$.*

Proof. If $a > 0$ then also $F \in S(\gamma)$ by Theorem 2.7 of Embrechts and Goldie (1982). If $\int_0^\infty \mathrm{e}^{sz}\overline{F}(z)\,\mathrm{d}z < c/\lambda$, then the result follows by Theorem 2.4 with constants c_1 and c_2 given there. Now suppose $\int_0^\infty \mathrm{e}^{sz}\overline{F}(z)\,\mathrm{d}z \geq c/\lambda$, then by Fatou's lemma we obtain

$$\liminf_{x\to\infty} \frac{\psi_F(x)}{\overline{F}_I(x)} \geq (1-\rho)\sum_{n=0}^{\infty} n\rho^n \left(\frac{1}{\mu}\int_0^\infty e^{\gamma x}\overline{F}(x)\,\mathrm{d}x\right)^{n-1} = \frac{1-\rho}{\mu}\sum_{n=0}^{\infty}\left(\frac{\lambda}{c}\right)^n \left(\int_0^\infty e^{\gamma x}\overline{F}(x)\,\mathrm{d}x\right)^{n-1} = \infty.$$

This implies that

$$\lim_{x\to\infty}\frac{\psi_G(x)}{\psi_F(x)} = \lim_{x\to\infty}\frac{\psi_G(x)}{\overline{G}(x)}\frac{\overline{G}(x)}{\overline{F}(x)}\frac{\overline{F}(x)}{\overline{F}_I(x)}\frac{\overline{F}_I(x)}{\psi_F(x)} = c_1 a\gamma \lim_{x\to\infty}\frac{\overline{F}_I(x)}{\psi_F(x)} = 0.$$

If $a = 0$ then the result is a consequence of Proposition 3.2 by an application of l'Hospital's rule. □

Example. The generalized inverse Gaussian distribution is asymptotically $\overline{G}(x) \sim Cx^{d-1} \times \exp\{-(b/2)x\}$ for $C>0$ and for $b>0$ and $d<0$ is $G \in S(b/2)$ [Klüppelberg (1989b), Embrechts (1983)]. Thus

$$\psi_G(x) \sim Dx^{d-1} \exp\{-(b/2)x\} \quad \text{for } D>0.$$

Notice that explicit forms for C and D can be given. Then for any claimsize distribution F whose tail $\overline{F}$ decreases slower to 0 than $Cx^{d-1} \exp\{-(b/2)x\}$ the asymptotic ruin probability is larger than ψ_G.

Finally we consider the large claims problem.

Theorem 3.7. *Suppose $G \in S^*$ and $G \leq^{\mathrm{w}} F$, then $\phi_G \leq^{\mathrm{w}} \phi_F$.*

Proof. Again we apply Proposition 3.2:

$$\limsup_{x\to\infty} \frac{\psi_G(x)}{\overline{F}_I(x)} \leq \lim_{x\to\infty} \frac{\psi_G(x)}{\overline{G}_I(x)} \liminf_{x\to\infty} \frac{\overline{G}_I(x)}{\overline{F}_I(x)}.$$

The first limit is finite by Theorem 2.6 and the second by Lemma 3.1(b). □

For the tail order we obtain a more precise result [see Gmür (1992)].

Theorem 3.8. *Suppose $G \in S^*$ and*

$$\lim_{x\to\infty} \overline{G}(x)/\overline{F}(x) = a < \infty,$$

then

$$\lim_{x\to\infty} \psi_G(x)/\psi_F(x) = \frac{\mu(F)}{\mu(G)} a.$$

Proof. If $a>0$ then also $F \in S^*$ by Theorem 2.1 of Klüppelberg (1988), and the result follows by Theorem 2.6 applied to ψ_G and ψ_F. If $a=0$ then by l'Hospital also $\lim_{x\to\infty}\overline{G}_I(x)/\overline{F}_I(x)=0$ and hence the result follows with Proposition 3.2. □

For $F \in S^*$ the constant $(\rho^{-1}-1)$ in Lemma 3.1 is sharp; this is an immediate consequence of (1) and Theorem 2.6.

While Theorem 2.6 allows to calculate asymptotic ruin probabilities for distributions in S^*, Theorems 3.7 and 3.8 go much further since they provide bounds also for distributions not in S^*. The following examples of distributions in S^* cover a wide range of tail behaviour of large claims distributions and may serve as reference distributions for those not known to belong to S^*. For more details on these distributions and their asymptotic ruin probabilities see Klüppelberg (1989b). With SV we denote the class of slowly varying functions.

Examples. (i)

$$\overline{G}_a(x) \sim l(x)x^{-a}, \quad x>0,\ l \in SV, \text{ for } a>1.$$

Special examples here are *Pareto distributions, Loggamma distributions, and Benktander-Type-I distributions.* If $a_1 > a_2$ then $G_{a_1} \leq^{\mathrm{t}} G_{a_2}$, indeed $\lim_{x\to\infty}\overline{G}_{a_1}(x)/\overline{G}_{a_2}(x)=0$.

(ii)

$$\overline{G}(x) \sim \frac{1}{\sqrt{2\pi}} \frac{\alpha}{\ln x - \ln \beta} \exp\left\{-\frac{(\ln x - \ln \beta)^2}{2\alpha^2}\right\},$$

$x>0$, for $\alpha>0$, $\beta \geq 1$.

An example here is the *Lognormal distribution*.

(iii)

$$\overline{G}_b(x) \sim l(x)x^{\alpha} \exp\{-cx^b\},$$

$$x>0,\ l \in SV, \text{ for } \alpha \in \mathbb{R},\ c>0,\ 0<b<1.$$

Special examples here are *Weibull distributions and Benktander-Type-II distributions.* If $b_1 > b_2$ then $G_{b_1} \leq^{\mathrm{t}} G_{b_2}$, indeed $\lim_{x\to\infty}\overline{G}_{b_1}(x)/\overline{G}_{b_2}(x) = 0$.

Furthermore, for G_a, G, and G_b as above $G_b \leq^{\mathrm{t}} G \leq^{\mathrm{t}} G_a$ holds for all $a>0$ and $0<b<1$. These orderings of the claimsize distributions yield the orderings of the ruin probabilities by the results given above. Again it reveals the well-known fact that the Pareto distributions are a very dangerous class of claimsize distributions.

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