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A Note on the Performance of Simple Specification Tests for the Tobit Model

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I. INTRODUCTION

To explain microeconomic behaviour, complex models have been developed. It is oftten difficult to transform these models to fit a standard linear regression framework, and maximum likelihodd (ML) estimation has become a standard tool. The difficulty with ML estimation is that the distributional assumptions may be crucial, and several specification tests have been proposed to check them. However, to date it has not become a standard practice to apply these tests, since most of them are burdensome to implement, and little is known about their performance in small samples.

The aim in this note is to investigate the small sample properties of a selection of specification tests for the Tobit model. Emphasis have been put solely on tests which are computationally tractable. The size of these tests are presented, as well as their ability to detect certain kinds of heteroscedasticity and non-normality in the error term.

In the next section, the Tobit model is introduced together with the specification test statistics associated with it. The sampling experiments are reported in section III, and this note ends with some concluding comments in section IV.

II. TEST STATISTICS

The censored regression model in its simplest form for an individual i ($i = 1, \ldots, n$) can be written as

$$\begin{aligned} y_i &= \beta' x_i + u_i \ \ if \ \ u_i > -\beta' x_i \\ &= 0 \ \ otherwise \end{aligned} \tag{1}$$

where β is a (k $\times$ 1) vector of parameters. Assuming that $u_i \sim NID(0, \sigma^2)$,

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we obtain the familiar Tobit model. Defining an indicator function, such that $I_i = 1$ if $u_i > \beta' x_i$, and $I_i = 0$ otherwise, we can write the log-likelihood function ($L_n(\theta)$) for the Tobit model as

$$L_n(\theta) = \sum_{i=1}^{n} I_i \left[-\frac{1}{2} \ln \sigma^2 - \frac{1}{2\sigma^2}(y_i - \beta' x_i)^2 \right] + (1 - I_i) \ln \left[1 - \Phi\left(\frac{\beta' x_i}{\sigma} \right) \right] \tag{2}$$

where $\theta = [\beta', \sigma^2]'$ is a $(k + 1 \times 1)$ vector of parameters and Φ is the standard normal distribution function. The score vector, $S_n(\theta)$, and the sample information matrix, B_θ, are defined as

$$\frac{\partial}{\partial\theta} \frac{1}{n} L_n(\theta) = S_n(\theta)$$

$$E_\theta \left\{ \frac{-\frac{1}{n} \partial^2 L_n(\theta)}{\partial\theta \, \partial\theta'} \right\} = B_\theta \tag{3}$$

The Hausman (H-) test

Newey (1987) proposed a Hausman specification test for the Tobit model using the ML estimator of β along with the symmetrically censored least squares (SCLS) estimator proposed by Powell (1986). The main criterion for the SCLS estimator to be consistent is that the error distribution must be symmetric. The Hausman test statistic then takes the form

$$H = (\hat{\beta}_{SCLS} - \hat{\beta}_{ML})' [\hat{V}(\hat{\beta}_{SCLS} - \hat{\beta}_{ML})]^{-1} (\hat{\beta}_{SCLS} - \hat{\beta}_{ML}) \tag{4}$$

Since $\hat{\beta}_{ML}$ is at least as efficient as $\hat{\beta}_{SCLS}$ under the null hypothesis, an estimate of the covariance matrix is given by $\hat{V}(\hat{\beta}_{SCLS}) - \hat{V}(\hat{\beta}_{ML})$. A practical problem is that the difference in estimates can be singular. In the sampling experiment, we adopt the approach suggested by Newey, who derived a simple expression for a consistent estimator of the covariance matrix which is positive semi-definite.

The information matrix (IM-) test

The IM test was introduced by White (1982) as a general specification test. The test is based on the fact that for a correctly specified model

$$E_\theta[S_n(\hat{\theta}) S_n(\hat{\theta})'] = E_\theta \left\{ \frac{-\frac{1}{n} \partial^2 L_n(\hat{\theta})}{\partial\theta \, \partial\theta'} \right\} \tag{5}$$

The basis of the test statistic, $\text{IM}(\hat{\theta})$, is the sample representation of (5),

$$IM(\hat{\theta}) = n^{-1} \sum_{i=1}^{n} \left[S_i(\hat{\theta}) S_i(\hat{\theta})' + \frac{\partial^2 L_n(\hat{\theta})}{\partial\theta\,\partial\theta'} \right] \tag{6}$$

where $\hat{\theta}$ is the ML estimate of θ. The difficulty with this test is to obtain the variance of $IM(\hat{\theta})$, which involves the higher-order derivatives of the log-likelihood function. However, Chesher (1983) and Lancaster (1984) showed that this test statistic can be obtained as n times the R^2 from the following artificial regression

$$i = \gamma_1' G + \gamma_2' Z + e \tag{7}$$

where i is a $(n \times 1)$ vector of ones, G is a $(n \times p)$ matrix whose elements are

$$\frac{\partial L_i(\theta)}{\partial \theta_j} \quad (j = 1, \ldots, p),$$

and Z is a $(n \times v)$ matrix whose elements are

$$\frac{\partial L_i(\theta)}{\partial \theta_j} \frac{\partial L_i(\theta)}{\partial \theta_k} + \frac{\partial^2 L_i(\theta)}{\partial \theta_j\,\partial \theta_k} \tag{8}$$

for $j = 1, \ldots, p$ and $k = j, \ldots, p$. Both G and Z are evaluated at the ML estimates, and v is the $p(p+1)/2$ redundant elements in

$$\frac{\partial^2 L_i(\theta)}{\partial \theta_j\,\partial \theta_k}$$

Chesher and Irish (1987) showed that this test, by modifying the elements in Z, can also be used as a separate test for heteroscedasticity (IMH) and non-normality (IMN). In a test for heteroscedasticity, the columns in Z are replaced by

$$\frac{\partial L_i(\theta)}{\partial \sigma} x_i x_i',$$

and in a test for non-normality, the elements in Z are based on the third and fourth moments of generalized residuals. Generalized residuals are used since the dependent variable is censored.

The conditional moment (CM-) test

The CM test was suggested by Newey (1985) and Tauchen (1985). Pagan and Vella (1989) considered this test in limited-dependent variable models. When the model is correctly specified, $E[m(y, \theta)] = 0$. The test statistic can then be formulated as

$$CM = n r_j \hat{\Sigma}^{-1} r_j$$

where

$$r_j = n^{-1} \sum_{i=1}^{n} m_i(y_i, \hat{\theta}) \quad (9)$$

Again, the problem is to derive an estimate of the covariance matrix, Σ. Pagan and Vella (1989) showed that for ML estimates, Σ can be estimated by

$$\hat{\Sigma} = n^{-1}(M'M - M'D(D'D)^{-1}D'M) \quad (10)$$

where M is a $(n \times j)$ matrix whose i:th element is $m_i(y_i, \hat{\theta})$, and D is a $(n \times p)$ matrix with the i:th row equal to $S_i(\hat{\theta})$. In a test for heteroscedasticity (CMH), the moment restriction is

$$E[x_i[(y_i - \beta' x_i)^2 - \sigma^2]] = 0 \, \forall i \quad (11)$$

and when testing for non-normality (CMN), the moment restrictions are

$$E[(y_i - \beta' x_i)^3] = 0$$

and

$$E[(y_i - \beta' x_i)^4 - 3\sigma^4] = 0 \, \forall i \quad (12)$$

III. RESULTS FROM SAMPLING EXPERIMENTS

A simple model for female labour supply is used to generate data. The dependent variable is hours of work, explained by the gross wage (w_i), non-labor income, and an intercept term. In order to measure the size of the test statistics, and the ability of detecting heteroscedasticity and non-normality, four experiments have been set up:

1. $u_i \sim NID(0, \sigma_u^2)$
2. $u_i \sim NID(0, \sigma_u^2(\alpha_0 + \alpha_1 w_i)^2)$
3. $u_i = Xv_1 + (1 - X)v_2$, where $v_1 \sim NID(-\mu, \sigma_v^2)$, $v_2 \sim NID(\mu, \sigma_v^2)$, and X is Bernoulli distributed with $p = 0.5$.
4. $u_i \sim t_3$

The parameters are chosen to yield approximately the same variance of the error term in all experiments, and about 15 percent zeros in the dependent variable. All results stem from 1000 replications with a sample size of 640.

To measure how well the test statistics corresponds to the nominal size, the error term in the first experiment is normally distributed and homoscedastic. In Table 1, the rejection rates give the percentage of these test statistics exceeding the critical values at the 1, 5 and 10 percent levels. Clearly, all statistics reject the null hypothesis more often than the nominal level.

By letting the variance in the error term be a function of the gross wage, we introduce heteroscedasticity into the model: this generally leads to inconsistent ML estimates. The results from this experiment are found in Table 2. As can be seen, both the IM and CM tests detect this kind of misspecification with a high power, while the Hausman test has a very low rejection rate.

In the third experiment the error term is drawn from a normal distribution with a random mean. This is one way of modelling random preferences in labour supply models. The parameters are chosen to preserve the symmetry of the error distribution. In experiment four, the t_3-distribution is used in the data-generating process. A motivation for this is that in empirical studies of labour supply, one often encounters a wider range of hours of work than what is observed if data were generated by a normal distribution.

Table 3 summarizes the ability of the different tests to detect the false distributional assumption. In the first trial the true distribution is far from normal, and this is detected by an almost 100 percent rejection rate at all levels for the IM and CM tests, while the Hausman test has a considerably lower power. Although the t_3-distribution is similar to the normal, the power is still high for the IM tests, while moderate for the CM test, and very low for the Hausman test.

TABLE 1
Size performance of Tobit specification tests

	Rejection rates		
Statistic	*1%*	*5%*	*10%*
H	2.7	8.4	13.3
IM	9.6	18.8	25.7
IMH	2.7	8.7	13.8
IMN	5.5	13.8	19.4
CMH	4.1	11.4	19.2
CMN	3.2	8.0	12.8

TABLE 2
Performance of Tobit specification tests; heteroscedastic errors

	Rejection rates		
Statistic	*1%*	*5%*	*10%*
H	5.9	14.2	21.1
IM	71.2	86.0	91.3
IMH	75.2	89.6	94.0
CMH	61.4	86.7	92.4

TABLE 3
Performance of Tobit specification tests; non-normal errors

	Rejection rates		
Statistic	*1%*	*5%*	*10%*
		Mixed normal	
H	33.1	47.7	56.6
IM	100.0	100.0	100.0
IMN	100.0	100.0	100.0
CMN	99.8	100.0	100.0
		t_3-distribution	
H	1.7	6.8	12.7
IM	81.0	91.9	95.0
IMN	73.5	87.8	93.2
CMN	23.4	42.5	54.0

IV. CONCLUDING REMARKS

In this note, we have investigated the small sample properties of a selection of specification tests for the Tobit model. Emphasis have been put solely on tests which are computationally tractable, and could easily be applied in empirical work. The size of these tests were presented, as well as their ability to detect certain kinds of model misspecifications.

Our conclusions are as follows: First, the Hausman test proposed by Newey (1987) had a very low power, both when data was generated with heteroscedastic error terms and when the error terms were assumed to be non-normally distributed. Second, all the IM tests (both general and specific) performed well in all experiments. Third, when the error terms were drawn from the t_3-distribution, the power for the CM test was moderate. In all other circumstances, the CM test performed well. Hence, our suggestion to empirical economists using the Tobit model is to use the general IM test for potential misspecifications of their model.

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