Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Mathematical Economics 41 (2005) 617–636

JOURNAL OF Mathematical ECONOMICS

www.elsevier.com/locate/jmateco

Existence and regularity of equilibria in a general equilibrium model with private provision of a public good

Antonio Villanacci[a,*], E. Ünal Zenginobuz[b]

[a] *Department of Mathematics for Decision Theory (Di.Ma.D), University of Florence, via Lombroso 6/17, 50134 Firenze, Italy*

[b] *Department of Economics, Boğaziçi University, 34342 Bebek, Istanbul, Turkey*

Received 14 May 2001; received in revised form 25 December 2004; accepted 31 December 2004
Available online 8 March 2005

Abstract

We prove existence and generic regularity of equilibria in a general equilibrium model of a completely decentralized pure public good economy. Competitive firms using private goods as inputs produce the public good, which is privately provided by households. Previous studies on private provision of public goods typically use one private good, one public good models in which the public good is produced through a constant returns to scale technology. As two distinguishing features of our model, we allow for the presence of *several* private goods and *nonlinear* production technology. In that framework, we use an homotopy argument to prove existence of equilibria and we show that economies are generically regular.

Keywords: General equilibrium model; Private provision of public good; Existence; Generic regularity

1. Introduction

In this paper, we prove the existence and regularity of equilibria in a general equilibrium model of a completely decentralized pure public good economy. The model studied extends

* Corresponding author. Tel.: +39 055 4796 817; fax: +39 055 4796 800.
E-mail addresses: villanac@unifi.it (Antonio Villanacci); zenginob@boun.edu.tr (E. Ünal Zenginobuz).

doi:10.1016/j.jmateco.2004.12.005

the standard pure exchange model with private goods by allowing households to make voluntary purchases of (or "privately provide") a public good that is produced by competitive firms using private goods as inputs.

The interest in a general equilibrium model with private provision of public goods lies in the fact that it serves as a benchmark extension of an analysis of completely decentralized private good economies to public good economies. Moreover, there are some relevant situations in which public goods are in fact privately provided, e.g., private donations to charity at a national and international level, campaign funds for political parties or special interests groups, and certain economic activities inside a family.

Previous studies on private provision of public goods typically use one private good, one public good models in which the public good is produced through a constant returns to scale technology. With only one private good, assuming constant returns to scale amounts to assuming linear production function, and as a result there is *no* loss of generality in normalizing prices of both the private and the public good to one. These assumptions also allow taking profits of firms equal to zero, with the implication that the presence of firms basically plays no role in the model. Therefore, as clearly explained by Bergstrom et al. (1986), the model reduces to a simple game where households are the players and the unique strategy variable is the choice of voluntary contribution. Then, proving existence is straight forward (see Bergstrom et al., 1986, Theorem 2, p. 33).[1]

As two distinguishing features of our model we have the presence of *several* private goods and we allow for *nonlinear* production technology for the public good. Therefore, the above-described price normalization will typically not be possible. Moreover, competitive firms will typically have non-zero profits, which are to be distributed among households. We are, therefore, analyzing a full-fledged general equilibrium model.

For given prices firms maximize profits. For given prices, initial endowments of private goods, ownership distribution of firms, as well as other households' choices of private provision of public goods, each household chooses a vector of private good consumption levels and a voluntary contribution level for the public good so as to maximize utility. Finally, markets for private and public goods clear. We show both existence and generic regularity of equilibria in this model.

We provide a proof of existence using a very simple homotopy argument, whose main characteristics consist in linking the true model with a fictitious one where the public good is treated as a private good. This strategy of proof turns out to be useful to show existence of equilibria in some related models as well.[2]

[1] The literature on voluntary contributions stems from Warr (1983), who studied the question of neutrality of taxes in a partial equilibrium public good model. Bergstrom et al. (1986) studied the same issue in a simple general equilibrium model with one private and one public good. The large literature that follows Bergstrom et al. (1986) mainly uses their framework and focuses on questions other than existence of equilibria in a full-fledged general equilibrium framework. A recent contribution by Cornes et al. (1999) on existence and uniqueness of equilibria in public good models also uses the framework in Bergstrom et al. (1986), i.e. a one private good, one public good economy with no relative price effects. The survey of papers on the existence of equilibria in private public good provision models by Cornes et al. (1999) confirms our observation that explicit work on existence of equilibrium in these models have indeed been limited to one public good, one private good case with no relative price effects.

[2] In a separate paper we prove, using the same approach, the existence of equilibria in a similar model where the public good is produced by a public firm subject to a budget constraint (Villanacci and Zenginobuz, 2004).

Regularity is an indispensable tool for carrying out comparative statics analyses. We show that for any vector of utility and transformation functions and ownership shares, in an open and full measure subset of the endowments, there is a finite number of equilibria and a local smooth dependence of the equilibrium variables on the exogenous variables. To show regularity we need to strengthen the assumptions on utility functions and production technology with which we prove existence.

Previous work on existence of competitive equilibrium with externalities in consumption and production relate to the model we study here. Public goods can be seen as a special case of consumption externalities. McKenzie (1955) studies a model of externalities with production, but he considers a linear production technology. Arrow and Hahn (1971) allow for decreasing returns to scale in production, but some of their assumptions do not apply to our case. Shafer and Sonnenschein (1975) show the existence of competitive equilibria in a pure exchange framework, hence their results do not apply to our model which incorporates production.[3]

Section 2 presents the setup of the model. In Section 3 the existence of equilibrium is proved. Section 4 contains the proof of generic regularity of the equilibria.[4]

2. Setup of the model

We consider a general equilibrium exchange model with private provision of a public good. There are $C, C \geq 1$, private commodities, labelled by $c = 1, 2, \ldots, C$. There are H households, $H > 1$, labelled by $h = 1, 2, \ldots, H$. Let $\mathcal{H} = \{1, \ldots, H\}$ denote the set of households. Let x_h^c denote consumption of private commodity c by household h; e_h^c embodies similar notation for the endowment in private goods.

The following standard notation is also used:

- $x_h \equiv (x_h^c)_{c=1}^C$, $x \equiv (x_h)_{h=1}^H \in \mathbb{R}_{++}^{CH}$.
- $e_h \equiv (e_h^c)_{c=1}^C$, $e \equiv (e_h)_{h=1}^H \in \mathbb{R}_{++}^{CH}$.
- p^c is the price of private good c, with $p \equiv (p^c)_{c=1}^C$; p^g is the price of the public good. Let $\hat{p} \equiv (p, p^g)$.
- $g_h \in \mathbb{R}_+$ is the amount of public good that consumer h provides. Let $g \equiv (g_h)_{h=1}^H$, $G \equiv \sum_{h=1}^H g_h$, and $G_{\backslash h} \equiv G - g_h$.

The preferences over the private goods and the public good of household h are represented by a utility function $u_h : \mathbb{R}_{++}^C \times \mathbb{R}_{++} \to \mathbb{R}$. Note that households' preferences are defined over the total amount of the public good, i.e. we have $u_h : (x_h, G) \mapsto u_h(x_h, G)$.

[3] The definition of equilibrium by Shafer and Sonnenschein (1975) is problematic due to the way they treat the (artificial) distinction between an household's actual consumption and her intended consumption (see pp. 84 and 85). Arrow and Hahn (1971) take account of this issue. On the other hand, they postulate the existence of a consumption vector whose utility is independent of other agents' choices, an assumption which fails to hold in our model (see p. 132 for details on both problems).

[4] While the paper was under revision for publication, we realized that Florenzano (2003) proves existence in a model with externalities which is general enough to include the case we analyze. However, the paper offers a different technique of proof which does not allow to address the generic regularity analysis we present.

Assumption 1. $u_h(x_h, G)$ is a C^2, differentiably strictly increasing (i.e., for every $(x_h, G) \in \mathbb{R}^{C+1}_{++}$, $Du_h(x_h, G) \gg 0$),[5] differentiably strictly quasiconcave function (i.e., $\forall (x_h, G) \in \mathbb{R}^{C+1}_{++}$, $\forall v \in \mathbb{R}^{C+1} \backslash \{0\}$, if $Du_h(x_h, G)v = 0$, then $vD^2u_h(x_h, G)v < 0$,[6] and for each $\underline{u} \in \mathbb{R}$ the closure (in the standard topology of R^{C+1}) of the set $\{(x_h, G) \in \mathbb{R}^{C+1}_{++} : u_h(x_h, G) \geq \underline{u}\}$ is contained in $\mathbb{R}^{C+1}_{++}$.

Let $\mathcal{U}$ be the set of utility functions u_h satisfying Assumption 1.

There are F firms, indexed by subscript f, that use a production technology represented by a transformation function $t_f : \mathbb{R}^{C+1} \to \mathbb{R}$, where $t_f : (y_f, y^g_f) \mapsto t_f(y_f, y^g_f)$.

Assumption 2. $t_f(y_f, y^g_f)$ is a C^2, differentiably strictly decreasing (i.e., $Dt_f(y_f, y^g_f) \ll 0$), and differentiably strictly quasiconcave function,[7] with $t_f(0) = 0$.

For each f, define $\hat{y}_f \equiv (y_f, y^g_f)$, $\hat{y} \equiv (\hat{y}_f)^F_{f=1}$ and $Y_f \equiv \{\hat{y}_f \in \mathbb{R}^{C+1} : t_f(\hat{y}_f) \geq 0\}$ and $Y \equiv \sum^F_{f=1} Y_f$. Let also $t \equiv (t_f)^F_{f=1}$ and $\hat{p} \equiv (p, p^g)$.

The following assumption is sufficient to get boundedness of the set of feasible allocations.

Assumption 2' (Bounded reversibility). $Y \cap (-Y)$ is bounded.

Let $\mathcal{T}$ be the set of vectors t of transformation functions satisfying Assumptions 2 and 2'.

Remark 1. We avoided to assume irreversibility (i.e., $Y \cap (-Y) = \{0\}$) or no free lunch because either of the two together with free disposal and possibility of inaction would imply that the gradients of the transformation functions of all firms computed at zero are collinear, a quite restrictive assumption.

Using the convention that input components of the vector $\hat{y}_f$ are negative and output components are positive, the profit maximization problem for firm f is: For given $\hat{p} \in \mathbb{R}^{C+1}_{++}$,

$$\begin{array}{ll} \mathrm{Max}_{\hat{y} \in \mathbb{R}^{C+1}} & \hat{p}\hat{y} \\ \text{s.t.} & t_f(\hat{y}) \geq 0 (\alpha_f) \end{array} \tag{1}$$

where we follow the convention of writing the Kuhn-Tucker multipliers next to the associated constraints.

Remark 2. From Assumption 2, it follows that if problem (1) has a solution, it is unique and it is characterized by Kuhn-Tucker (in fact, Lagrange) conditions.

---

[5] For vectors y, z, $y \geq z$ (resp. $y \gg z$) means every element of y is not smaller (resp. strictly larger) than the corresponding element of z; $y > z$ means that $y \geq z$ but $y \neq z$.

[6] Observe that since u_h is differentiably strictly quasiconcave then the following condition holds: *if* $\forall (x'_h, G'), (x''_h, G'') \in \mathbb{R}^{C+1}_{++}$, $\forall \lambda \in (0, 1)$, $u_h((1-\lambda)(x'_h, G')' + \lambda(x''_h, G'')) > \min\{u_h((x'_h, G')), u_h((x''_h, G''))\}$. We use this condition in the proof of Theorem 9.

[7] Observe that since t_f is strictly quasiconcave then it is also the case that $\forall (y_f, y^g_f) \in \mathbb{R}^{C+1}$, $\det \begin{bmatrix} D^2t_f(y_f, y^g_f) & Dt_f(y_f, y^g_f) \\ Dt_f(y_f, y^g_f) & 0 \end{bmatrix} \neq 0$. We use this condition in the proof of Theorem 9 and Lemma 12.

Definition 3. s_{fh} is the share of firm f owned by household h. $s_f \equiv (s_{fh})_{h=1}^{H} \in \mathbb{R}^H$ and $s \equiv (s_f)_{f=1}^{F} \in \mathbb{R}^{FH}$. The set of all shares of each firm f is $S \equiv \{s_f \in [0,1]^H : \sum_{h=1}^{H} s_{fh} = 1\}$. $s \equiv (s_f)_{f=1}^{F} \in S^F$. The set of shares $s_h \equiv (s_{fh})_{f=1}^{F}$ of household h is $[0,1]^F$.

$s_{fh} \in [0,1]$ denotes the proportion of profits of firm f owned by household h. The definition of S simply requires each firm to be completely owned by some households.

Household's maximization problem is then the following: For given $\hat{p} \in \mathbb{R}^{C+1}_{++}$, $s_h \in [0,1]^F$, $e_h \in \mathbb{R}^C_{++}$, $G_{\backslash h} \in \mathbb{R}_+$, $\hat{y} \in \mathbb{R}^{(C+1)F}$,

$$
\begin{aligned}
&\mathrm{Max}_{(x_h, g_h) \in \mathbb{R}^C_{++} \times \mathbb{R}} && u_h(x_h, g_h + G_{\backslash h}) \\
&\text{s.t.} && -p(x_h - e_h) - p^g g_h + \hat{p} \sum_{f=1}^{F} s_{fh} \hat{y}_f \geq 0 \qquad (\lambda_h) \\
& && g_h \geq 0 \qquad (\mu_h) \\
& && g_h + G_{\backslash h} > 0
\end{aligned}
\tag{2}
$$

Remark 4. From Assumption 1, it follows that problem (2) has a unique solution characterized by Kuhn-Tucker conditions.

Market clearing conditions are:

$$
\begin{aligned}
&-\sum_{h=1}^{H} x_h + \sum_{h=1}^{H} e_h + \sum_{f=1}^{F} y_f = 0 \\
&-\sum_{h=1}^{H} g_h + \sum_{f=1}^{F} y_f^g = 0
\end{aligned}
$$

Remark 5. With Assumption 1 on u_h, at an equilibrium allocation we must have $\sum_h g_h > 0$, and hence $\sum_{f=1}^{F} y_f^g > 0$. Moreover, since $g_h \geq 0$ for all h, there exists h' such that $g_{h'} > 0$.

Since in the existence proof we consider an arbitrary $(s, u, t) \in S^F \times \mathcal{U}^H \times \mathcal{T}$ which remains fixed throughout the analysis, we define an economy as an endowment vector $e \in \mathbb{R}^{CH}_{++}$.

Summing up consumers' budget constraints, which from Assumption 1 hold with equality, we get

$$
-p\left(\sum_{h=1}^{H} x_h - \sum_{h=1}^{H}\left(e_h + \sum_{f=1}^{F} s_{fh} y_f\right)\right) - p^g\left(\sum_{h=1}^{H}\left(g_h - \sum_{f=1}^{F} s_{fh} y_f^g\right)\right) = 0
$$

i.e., the Walras law. Therefore, the market clearing condition for good C, for example, is redundant. Moreover, we can normalize the price of private good C without affecting the budget constraints of any household. With little abuse of notation, we denote the normalized private and public good prices with $p \equiv (p^{\backslash}, 1)$ and p^g, respectively.

Using Remarks 2 and 4, we can give the following definition:

Definition 6. A vector $(x, g, p^{\backslash}, p^g, \hat{y})$ is an equilibrium for an economy $e \in \mathbb{R}^{CH}_{++}$ if:

1. firms maximize, i.e., for each f, $\hat{y}_f$ solves problem (1) at $\hat{p} \in \mathbb{R}^{C+1}_{++}$;
2. households maximize, i.e., for each h , (x_h, g_h) solves problem (2) at $p^{\backslash} \in \mathbb{R}^{C-1}_{++}$, $p^g \in \mathbb{R}_{++}$, $e_h \in \mathbb{R}^C_{++}$, $G_{\backslash h} \in \mathbb{R}_+$, $s_h \in [0, 1]^F$, $\hat{y} \in \mathbb{R}^{(C+1)F}$; and
3. markets clear , i.e., $(x, g, \hat{y})$ solves

$$\begin{aligned} &-\sum_{h=1}^{H} x_h^{\backslash} + \sum_{h=1}^{H} e_h^{\backslash} + \sum_{f=1}^{F} y_f^{\backslash} = 0 \\ &-\sum_{h=1}^{H} g_h + \sum_{f=1}^{F} y_f^g = 0 \end{aligned} \tag{3}$$

where $x_h^{\backslash}, e_h^{\backslash} \in \mathbb{R}^{C-1}_{++}$ and $y^{\backslash} \in \mathbb{R}^{C-1}$.

3. Existence

Given our assumptions, we can now characterize equilibria in terms of the system of Kuhn-Tucker conditions to problems (1) and (2), and market clearing conditions (3).

Define

$$\Xi \equiv \mathbb{R}^{(C+1)F} \times \mathbb{R}^F_{++} \times \mathbb{R}^{CH}_{++} \times \left\{ g \in \mathbb{R}^H : \sum_{h=1}^{H} g_h > 0 \right\} \times \mathbb{R}^H \times \mathbb{R}^H_{++} \times \mathbb{R}^{C-1}_{++} \times \mathbb{R}_{++}$$

$$\xi \equiv (\hat{y}, \alpha, x, g, \mu, \lambda, p^{\backslash}, p^g) \in \Xi$$

and

$F : \Xi \times \mathbb{R}^{CH}_{++} \to \mathbb{R}^{\dim \Xi}$, $F : (\xi, \pi) \mapsto$ left-hand side of (4) below

$$\begin{array}{ll} & \ldots \\ (h.1) & D_{x_h} u_h(x_h, g_h + G_{\backslash h}) - \lambda_h p = 0 \\ (h.2) & D_{g_h} u_h(x_h, g_h + G_{\backslash h}) - \lambda_h p^g + \mu_h = 0 \\ (h.3) & \min\{g_h, \mu_h\} = 0 \\ (h.4) & -p(x_h - e_h) - p^g g_h + \hat{p} \sum_{f=1}^{F} s_{fh} \hat{y}_f = 0 \\ (f.1) & \hat{p}. + \alpha_f D t_f(\hat{y}_f) = 0 \\ (f.2) & t_f(\hat{y}_f) = 0 \\ & \ldots \\ (M.1) & -\sum_{h=1}^{H} x_h^{\backslash} + \sum_{h=1}^{H} e_h^{\backslash} + \sum_{f=1}^{F} y_f^{\backslash} = 0 \\ (M.2) & -\sum_{h=1}^{H} g_h + \sum_{f=1}^{F} y_f^g = 0 \end{array} \tag{4}$$

Observe that $(\hat{y}, x, g, p^{\backslash}, p^g)$ is an equilibrium associated with an economy e if and only if there exists (α, μ, λ) such that $F(\hat{y}, \alpha, x, g, \mu, \lambda, p^{\backslash}, p^g, e) = 0$. With innocuous abuse of terminology, we will call $\xi \equiv (\hat{y}, \alpha, x, g, \mu, \lambda, p^{\backslash}, p^g)$ an equilibrium.

To show existence we are going to use the following homotopy theorem.

Theorem 7. *Let M and N be C^2 boundaryless manifolds of the same dimension, and let $f, g : M \to N$ be such that f is C^0 and g is C^0 and C^1 in an open neighbourhood of $g^{-1}(y)$, y is a regular value for g, $\# g^{-1}(y)$ is odd, and there exists a continuous homotopy H from f to g such that $H^{-1}(y)$ is compact. Then $f^{-1}(y) \neq \emptyset$.*

Theorem 7 is a consequence of a degree theorem presented, for example, in Geanakoplos and Shafer (1990). A self-contained account of the proof of the theorem can, for example, be found in Villanacci et al. (2002).

To apply Theorem 7, we consider a model in which the public good is treated as a private good and we construct a well-behaved vector of endowments (and the associated unique equilibrium). We then introduce the needed homotopy H, which "links" the true model to the above-mentioned fictitious one.

From Assumption 2', it follows that $Y \cap \mathbb{R}^{C+1}_{+}$ is bounded (a "bounded free lunch" property) and that Y is closed.[8] Then there exists a point $\hat{y}^*$ on the boundary of Y in $\mathbb{R}^{C+1}_{+}$. Let a price $\hat{p}^* \in \mathbb{R}^{C+1}_{++}$ in the normal cone[9] to Y at $\hat{y}^*$. Then $\forall \hat{y} \in Y$,

$$\hat{p}^* \hat{y} \leq \hat{p}^* \hat{y}^*$$

Since $Y(\hat{y}^*) \equiv \{(\hat{y}_f)_{f=1}^F \in \times_{f=1}^F Y_f : \sum_{f=1}^F \hat{y}_f = \hat{y}^*\}$, we then have

$$\exists\, (\hat{y}^*_f)_{f=1}^F \in Y(\hat{y}^*) \quad \text{such that } \forall f, \forall y_f \in Y_f, \quad \hat{p}^* \hat{y}_f \leq \hat{p}^* \hat{y}^*_f \tag{5}$$

Suppose that (5) does not hold, i.e., $\forall (\hat{y}_f)_{f=1}^F \in Y(\hat{y}^*)$, $\exists \tilde{f}$ and $\tilde{y}_{\tilde{f}} \in Y_f$ such that $\hat{p}^* \tilde{y}_{\tilde{f}} > \hat{p}^* \hat{y}^*_{\tilde{f}}$. But then choose

$$\tilde{\tilde{y}}_f \equiv \begin{cases} \tilde{y}_{\tilde{f}} \text{ if } f = \tilde{f} \\ \hat{y}^*_f \text{ if } f \neq \tilde{f} \end{cases}$$

and define $\tilde{\tilde{y}} \equiv (\tilde{\tilde{y}}_f)_f$. Then

$$\hat{p}^* \tilde{\tilde{y}} > \hat{p}^* \hat{y}^*$$

a contradiction. Therefore, from (5), $\forall f$, $\hat{y}^*_f$ is a profit maximizer at $\hat{p}^*$ and from Remark 2 we have that $\forall f$, there exists $\alpha^*_f \in \mathbb{R}_{++}$ such that

$$\begin{aligned} &\hat{p}^* + \alpha_f D t_f(\hat{y}^*) = 0 \\ &t_f(\hat{y}^*) = 0 \end{aligned} \tag{6}$$

[8] Closedness of Y can be obtained using Lemma A.1 and Corollary 2, p. 69, in Bergstrom et al. (1986).

[9] The normal cone to a convex set C at a is the set $\{\forall x^* \in \mathbb{R}^n : \forall x \in C,\ (x - a)x^* \leq 0\}$.

Then choose (x_h^*, g_h^*) in order to solve

$$\begin{array}{ll} \max_{(x_h, g_h)} & u_h(x_h, g_h) \\ \text{s.t.} & -p^* x_h - p^{g*} g_h + 1 \geq 0 \end{array}$$

whose associated first-order conditions are

$$\begin{array}{l} Du_h(x_h^*, g_h^*) - \lambda_h^* \hat{p}^* = 0 \\ -p^* x_h - p^{g*} g_h + 1 = 0 \end{array} \tag{7}$$

with $\lambda_h^* > 0$. Define

$$r^* \equiv \sum_{h=1}^{H} x_h^* - \sum_{f=1}^{F} y_f^* \quad \text{and} \quad \gamma^* \equiv \sum_{h=1}^{H} g_h^* - \sum_{f=1}^{F} y_f^{g*} \tag{8}$$

For each $e \in \mathbb{R}_{++}^{CH}$, the needed homotopy is then defined as follows.

$$H : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi},$$
$$H : (\xi, \tau) \mapsto \text{(left-hand side of system (9) below)}$$

$$\begin{array}{ll}
 & \ldots \\
(h.1) & D_{x_h} u_h(x_h, (1-\tau)G + \tau g_h) - \lambda_h p = 0 \\
(h.2) & D_{g_h} u_h(x_h, (1-\tau)G + \tau g_h) - \lambda_h p^g + \mu_h = 0 \\
(h.3) & \min\{g_h, \mu_h\} = 0 \\
(h.4) & -p x_h - p^g g_h + p[(1-\tau)e_h + \tau x_h^* + \sum_{f=1}^{F} s_{fh}(y_f - \tau y_f^*)] + p^g[\tau g_h^* \\
 & \quad + \sum_{f=1}^{F} s_{fh}(y_f^g - \tau y_f^{*g})] = 0 \\
 & \ldots \\
(f.1) & \hat{p} + \alpha_f D t_f(\hat{y}) = 0 \\
(f.2) & t_f(\hat{y}) = 0 \\
 & \ldots \\
(M.1) & -\sum_{h=1}^{H} x_h^{\backslash} + \sum_{f=1}^{F} y_f^{\backslash} + (1-\tau) \sum_{h=1}^{H} e_h^{\backslash} + \tau r^{\backslash *} = 0 \\
(M.2) & -\sum_{h=1}^{H} g_h + \sum_{f=1}^{F} y_f^g + \tau \gamma^* = 0
\end{array} \tag{9}$$

Remark 8. ξ is a solution to $H(\xi, 0) = 0$ if and only if ξ is an equilibrium at e, i.e., $F(\xi, e) = 0$.

Define also

$$G : \Xi \to \mathbb{R}^{\dim \Xi}, \qquad G : \xi \mapsto H(\xi, 1)$$

Theorem 9. *For every economy $e \in \mathbb{R}^{CH}_{++}$, an equilibrium exists.*

Proof. The proof consists in checking that the assumptions of Theorem 7 are verified, which is done below in four steps.

Step 1: There exists ξ^* such that $G(\xi^*) = 0$

Define

$$\xi^* = (\hat{y}^*, \alpha^*, x_h^*, g^*, \mu^* = 0, \lambda_h^*, \hat{p}^*)$$

$G(\xi^*) = 0$ is clearly true because of the definition of ξ^* and (6), (7) and (8).

Step 2: $\{\xi^*\} = G^{-1}(0)$.

Suppose there exists $\xi' \equiv (y', y^{g\prime}, \alpha', x', g', \mu', \lambda', p^{\backslash\prime}, p^{g\prime}) \in G^{-1}(0)$, with $\xi' \neq \xi^*$. Observe that $(y', y^{g\prime}, x', g')$ satisfies equations $(M.1)$ and $(M.2)$ in system (9) for $\tau = 1$. Moreover, summing up equations $(h.4)$ with respect to h, and using Definition (8) and equation $(M.2)$, we get

$$-\sum_{h=1}^{H} x'_h + r^* + \sum_{f=1}^{F} y'_f = 0 \tag{10}$$

Consider now $(x'', g'', y'', y^{g''}) \equiv \frac{1}{2}(x', g', y', y^{g\prime}) + \frac{1}{2}(x^*, g^*, y^*, y^{g*})$, then $(x'', g'', y'', y^{g''})$ satisfies (10) and $(M.2)$ and $t_f(y'', y^{g''}) \geq 0$, since t_f is a quasiconcave function. Now, observe from equations $(h.1) - (h.4)$ that (x'_h, g'_h) solves

$$\begin{aligned} &\mathrm{Max}_{(x_h, g_h) \in \mathbb{R}^C_{++} \times \mathbb{R}} \; u_h(x_h, g_h) \\ &\text{s.t.} \qquad p' x_h + p^{g\prime} g_h \leq p^{g\prime} g_h^* + p' x_h^* + \sum_{f=1}^{F} s_{fh}^* \hat{p}'(\hat{y}'_f - \hat{y}_f^*) \\ &\qquad\qquad g_h \geq 0 \end{aligned}$$

Since $f(y^*, y^{g*}) = 0$ and

$$\hat{y}' = \arg\max \hat{p}' \hat{y}_f \quad \text{s.t. } f(\hat{y}) \geq 0$$

we have that

$$\hat{p}' \hat{y}_f - \hat{p}' \hat{y}_f^* \geq 0$$

Therefore, $u_h(x'_h, g'_h) \geq u_h(x_h^*, g_h^*)$, with $g_h^* > 0$. Finally, since the utility function is strictly quasiconcave, we have

$$u_h(x''_h, g''_h) > \min\{u_h(x'_h, g'_h), u_h(x_h^*, g_h^*)\} \geq u_h(x_h^*, g_h^*)$$

Note that $u_h(x''_h, g''_h) > u_h(x^*_h, g^*_h)$ implies that $p^* x''_h + p^{g*} g''_h > p^* x^*_h + p^{g*} g^*_h$. Furthermore, since y^*_f is a solution of the firm problem at price $\hat{p}^*$ and $t_f(y''_f) \geq 0$, by definition of y''_f and by quasiconcavity of t_f, $\hat{p}^* \hat{y}''_f \leq \hat{p}^* \hat{y}^*_f$. Summarizing and using the definition of ξ'', we have

$$p^* x''_h + p^{g*} g''_h > p^* x^*_h + p^{g*} g^*_h \tag{11}$$

$$-\sum_h x''_h + \sum_f y''_f + r^* = 0 \tag{12}$$

$$-\sum_h g''_h + \sum_f y^{g''}_f + \gamma^* = 0 \tag{13}$$

$$-p^* y''_f - p^{g*} y^{g''}_f \geq -p^* y^*_f - p^{g*} y^{g*}_f \geq 0 \tag{14}$$

From (11) and (14),

$$p^* \left(\sum_h x''_h - r^* - \sum_f y''_f \right) + p^{g*} \left(\sum_h g''_h - \gamma^* - \sum_f y^{g''}_f \right) > 0 \tag{15}$$

But (15), contradicts (12) and (13). Thus $(x', y', y', y^{g\prime}) = (x^*, g^*, y^*, y^{g*})$, and the uniqueness of multipliers and prices follows easily.

Step 3: $D_\xi G(\xi^*)$ has full rank.

$D_\xi G(\xi^*)$ is

$$\begin{bmatrix}
\ddots & & & & & & & \\
& D^2 u_h(x^*_h, g^*_h) & & \hat{p}^* & & & & \lambda^*_h \tilde{I} \\
& & 1 & & & & & \\
& \hat{p}^* & & & & & & \\
& & & & \ddots & & & \\
& & & & & \alpha^*_f D^2 t_f(\hat{y}^*) & Dt_f(\hat{y}^*) & \tilde{I} \\
& & & & & Dt_f(\hat{y}^*) & & \\
& & & & & & \ddots & \\
& -\tilde{I}^T & & & & \tilde{I}^T & &
\end{bmatrix}$$

where $\tilde{I} \equiv \begin{bmatrix} I_{C-1} \\ 0 \\ 01 \end{bmatrix}$.

We want to show that $\ker D_\xi G(\xi^*) = \{0\}$. Defining

$$\Delta \equiv ((\Delta u_h, \Delta\mu_h, \Delta\lambda_h)_{h=1}^H, (\Delta t_f, \Delta\alpha_f)_{f=1}^F, \Delta\hat{p}^\backslash)$$

$$\in (\mathbb{R}^{C+1} \times \mathbb{R} \times \mathbb{R})^H \times (\mathbb{R}^{C+1} \times \mathbb{R})^F \times \mathbb{R}^C$$

we prove that if $D_\xi G(\xi^*) \cdot \Delta = 0$, then $\Delta = 0$. The proof proceeds through the following steps: (a) There exists h' such that $\Delta u_{h'} \neq 0$; (b) $\forall h,\ Du_h(x_h^*, g_h^*) \cdot \Delta u_h = 0$; (c) $\forall f,\ Dt_f(\hat{y}^*) \cdot \Delta t_f = 0$; (d) $\sum_h \Delta u_h \cdot D^2 u_h(x_h^*, g_h^*) \cdot \Delta u_h + \sum_f \Delta t_f \cdot \alpha_f^* D^2 t_f(\hat{y}^*) \cdot \Delta t_f = 0$.

But then (a)–(c), and the fact that u_h and t_f are differentiably strictly quasiconcave contradict (d).

Step 4: For every $e \in \mathbb{R}_{++}^{CH}$, $H^{-1}(0)$ is compact.

Take an arbitrary sequence $(\xi^v, \tau^v)_{v\in\mathbb{N}}$ in $H^{-1}(0)$. The main difficulties are in showing that, up to subsequences, $\{((x^n, g^n), \hat{y}^n) : n \in \mathbb{N}\}$ are contained in a bounded set, and that $(x^n, G^n) \to (\overline{x}, \underline{\overline{G}}) \in \mathbb{R}_{++}^{CH+1}$.

First of all observe that, since $\{\tau^v : v \in \mathbb{N}\} \subseteq [0, 1]$, up to a subsequence, $(\tau^n)_{n\in\mathbb{N}}$ converges. Then, so does $(\overline{r}^n)_{n\in\mathbb{N}}$, where

$$\overline{r}^n \equiv \left((1-\tau^n) \sum_{h=1}^{H} e_h + \tau^n r^*, \tau^n \gamma^* \right)$$

say to $\overline{\overline{r}}$. Defining

$$\text{for } c = 1, \ldots, C, g, \quad \tilde{r}^c \equiv \max_n (\{\overline{r}^{cn} : n \in \mathbb{N}\} \cup \{\overline{\overline{r}}^c\}) \quad \text{and}$$

$$\tilde{r} \equiv (\tilde{r}^c)_{c=1}^C, \hat{\tilde{r}} \equiv (\tilde{r}, \tilde{r}^g)$$

we have

$$\{((x^n, g^n), \hat{y}^n) : n \in \mathbb{N}\} \subseteq A_{\hat{\tilde{r}}}$$
$$\equiv \left\{ (x, g, \hat{y}) \in \mathbb{R}_{++}^{CH} \times \mathbb{R}_+^H \times \mathbb{R}^{(C+1)F} : -\sum_h x_h + \sum_f y_f + \tilde{r} \geq 0, \right.$$
$$\left. -\sum_h g_h + \sum_f y_f^g + \tilde{r}^g \geq 0 \text{ and } \forall f,\ t_f(\hat{y}_f) \geq 0 \right\}$$

which is proved to be bounded in Lemma A.1. Therefore, $\hat{y}$ and g_h converges to elements of $\mathbb{R}^{(C+1)F}$ and $\mathbb{R}$, respectively, and x converges to an element of $\mathbb{R}^{CH}$.

In order to show that $(\overline{x}, \underline{\overline{G}}) \in \mathbb{R}_{++}^{CH+1}$, it is enough to prove that, for sufficiently large n, for each h, there exists $\overline{\overline{e}}_h \in \mathbb{R}_{++}^{C+1}$ such that $u_h(x_h^n, \sum_h g_h^n) \geq u_h(\overline{\overline{e}}_h)$. The above result can be proved as follows. Observe that from equations $(h.1)$ and $(h.2)$ in system (9), x_h^n solves the problem

$$\begin{aligned} &\mathrm{Max}_{(x_h, g_h)} && u_h(x_h, g_h + (1-\tau) G_{\backslash h}) \\ &\text{s.t.} && p^n x_h + p^{gn} g_h \leq p^n[(1-\tau^n)e_h + \tau^n x_h^*] + \tau^n p^{gn} g_h^* \\ &&& + \left\{ \hat{p}^n \sum_{f=1}^{F} s_{fh}(\hat{y}_f^n - \tau^n y_f^*) \right\} \end{aligned}$$

Moreover, $\forall n$, $(1-\tau^n)e_h + \tau^n x_h^*$ belongs to a compact set contained in $\mathbb{R}^C_{++}$ and the term in the curly brackets is nonnegative, because $\hat{y}^n_f$ is a solution to firm *f* problem at prices $\hat{p}^n$, $0, y^{*g}_f \in Y_f$, Y_f is convex, and therefore $\hat{p}^n \hat{y}^n_f \geq \tau^n \hat{p}^n \hat{y}^*_f$.

4. Regularity

In this section, we prove that there is a large set of the endowments (the so-called regular economies) for which associated equilibria are finite in number, and that equilibria change smoothly with respect to endowments - see Theorem 11 below. To do this, we need first of all to add the following assumptions.

Assumption 3. $\forall h$, u_h is differentiably strictly concave, i.e., $\forall (x_h, G) \in \mathbb{R}^{C+1}_{++}$, $D^2 u_h(x_h)$ is negative definite.

Assumption 4. For all *h* and $(x_h, G) \in \mathbb{R}^{C+1}_{++}$,

$$\det \begin{bmatrix} D_{x_h x_h} u_h(x_h, G) & D_{x_h} u_h(x_h, G) \\ D_{G x_h} u_h(x_h, G) & D_G u_h(x_h, G) \end{bmatrix} \neq 0$$

The above assumption has an easy and appealing economic interpretation. Let $w_h \in \mathbb{R}_{++}$ denote the wealth of household *h*, and $g_h : \mathbb{R}^{C+2}_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$, $(\hat{p}, w_h, G_{\backslash h}) \mapsto g_h(\hat{p}, w_h, G_{\backslash h})$ denote the private contribution function of household *h*, i.e. part of the solution function to problem

$$\begin{array}{ll} \max_{(x_h, g_h)} & u_h(x_h, g_h + G_{\backslash h}) \\ \text{s.t.} & p x_h + p^g g_h \leq w_h \text{ and } g_h \geq 0 \end{array} \tag{16}$$

Lemma 10. Assumption 4 *is equivalent to*

for each $(\hat{p}, w_h, G_{\backslash h}) \in \mathbb{R}^{C+2}_{++} \times \mathbb{R}_+$ *such that* $g_h(\hat{p}, w_h, G_{\backslash h}) > 0$,

$$D_{w_h} g_h(\hat{p}, w_h, G_{\backslash h}) \neq 0$$

Proof. See Appendix A.

Call $\tilde{\mathcal{U}}$ the subset of $\mathcal{U}$ whose elements satisfy Assumptions 3 and 4. Define

$$pr : F^{-1}(0) \to \mathbb{R}^{CH}_{++}, \quad pr : (\xi, e) \mapsto e$$

We can state the main theorem of this section.

Theorem 11. *For each* $(s, u, t) \in S^F \times \tilde{\mathcal{U}}^H \times \mathcal{T}$, *there exists an open and full measure subset* $\mathcal{R}$ *of* $\mathbb{R}^{CH}_{++}$ *such that*

1. *there exists* $r \in \mathbb{N}$ *such that* $F_e^{-1}(0) = \{\xi^i\}_{i=1}^r$;
2. *there exist an open neighborhood* Y *of* e *in* $\mathbb{R}^{CH}_{++}$, *and for each* i *an open neighborhood* U_i *of* (ξ^i, e) *in* $F^{-1}(0)$, *such that* $U_j \cap U_k = \emptyset$ *if* $j \neq k$, $pr^{-1}(Y) = \cup_{i=1}^r U_i$ *and* $pr_{|U_i} : U_i \to Y$ *is a diffeomorphism.*

A key ingredient in the proof of the above theorem is to show that zero is a regular value for *f*. An obvious immediate problem is that the min function used in defining the equilibrium function *f* is not even differentiable when both the constraint and the multiplier are equal to zero, which can be called a "border line" case. We therefore show that border line cases occur outside an open and full measure subset D^* of the economy space.

Lemma 12. *For each* $(s, u, t) \in S^F \times \tilde{\mathcal{U}}^H \times \mathcal{T}$, *there exists an open and full measure subset* D^* *of* $\mathbb{R}^{CH}_{++}$ *such that* $\forall e \in D^*$ *and* $\forall \xi$ *such that* $F(\xi, e) = 0$, *it is the case that*

$$\forall h \in \mathcal{H},\ either\ g_h > 0\ or\ \mu_h > 0$$

Proof. Define the set

$$C \equiv \{(\xi, e) \in F^{-1}(0) : \exists\, h \text{ such that } g_h = \mu_h = 0\}$$

and observe that $D^* = \mathbb{R}^{CH}_{++} \setminus pr(C)$. Since C is a closed set, openness of D^* follows from properness of pr, which can be proven following the same arguments contained in Step 4 of Theorem 9. The proof of full measure proceeds in three steps.

1. Let $\mathcal{P}$ be the family of all partitions of $\mathcal{H}$ into three subsets $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$, with $\mathcal{H}_3 \neq \emptyset$. In the equilibrium system (4), in place of $\min\{g_h, \mu_h\} = 0$ substitute $\mu_h = 0$ for $h \in \mathcal{H}_1$, $g_h = 0$ for $h \in \mathcal{H}_2$, and $g_h = 0$ and $\mu_h = 0$ for $h \in \mathcal{H}_3$ (see the first column of Table 1 for demonstration of how this is done).
2. Define

$$F_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3} : \Xi \times \mathbb{R}^{CH}_{++} \to \mathbb{R}^{\dim \Xi + \#\mathcal{H}_3}$$

   which associates the left-hand side of the equilibrium system (4) modified as explained above to each (ξ, e) in the domain.
3. Define the set

$$B_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3} \equiv \{e \in \mathbb{R}^{CH}_{++} : \exists \xi \text{ such that } F_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}(\xi, e) = 0\}$$

   and observe that[10]

$$D^* \supseteq \mathbb{R}^{CH}_{++} \setminus \bigcup_{\{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3\} \in \mathcal{P}} B_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3} \tag{17}$$

That $B_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}$ is of measure zero follows from the parametric transversality theorem (see, e.g. Hirsch, 1976, Theorem 2.7, p. 79) and from the fact that zero is a regular value for $F_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}$ and $\#\mathcal{H}_3 > 0$, which is shown below.

In Table 1, the components of $F_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}$ are listed in the first column, the variables with respect to which derivatives are taken are listed in the first row, and in the remaining bottom

[10] We cannot use the equality sign in (17), because $B_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}$ contains economies e such that $F_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}(\xi, e) = 0$ and for which some components g_h and μ_h of ξ may be negative.

Table 1

	$\hat{y}_f$	α_f	x_{h_1}	g_{h_1}	λ_{h_1}	μ_{h_1}	x_{h_2}	g_{h_2}	λ_{h_2}	μ_{h_2}	x_{h_3}	g_{h_3}	λ_{h_3}	μ_{h_3}	$e_{h_1}^{\backslash}$	$e_{h_1}^C$	$e_{h_2}^C$	$e_{h_3}^C$
$\hat{p}+\alpha_f Dt_f$	$\alpha_f D^2 t_f$	Dt_f																
$t_f(\hat{y}_f)$	Dt_f																	
$D_{x_{h_1}} u_{h_1} + -\lambda_{h_1} p$			$D^{h_1}_{x_{h_1} x_{h_1}}$	$D^{h_1}_{x_{h_1} G}$	$-p$			$D^{h_1}_{x_{h_1} G}$				$D^{h_1}_{x_{h_1} G}$						
$D_G u_{h_1} + -\lambda_{h_1} p^g + \mu_{h_1}$			$D^{h_1}_{G x_{h_1}}$	$D^{h_1}_{GG}$	$-p^g$	1		$D^{h_1}_{GG}$				$D^{h_1}_{x_{h_1} G}$						
$-pz_{h_1} + -p^g g_{h_1} + \hat{p} \sum_f s_{fh_1} \hat{y}$	$s_{fh_1} \hat{p}$		$-p$	$-p^g$											$p^{\backslash}$	1		
μ_{h_1}						1												
$D_{x_{h_2}} u + -\lambda_{h_2} p$				$D^{h_2}_{x_{h_2} G}$			$D^{h_2}_{x_{h_2} x_{h_2}}$	$D^{h_2}_{x_{h_2} G}$	$-p$			$D^{h_2}_{x_{h_2} G}$						
$D_G u_{h_2} - \lambda_{h_2} p^g + \mu_{h_2}$				$D^{h_2}_{GG}$			$D^{h_2}_{G x_{h_2}}$	$D^{h_2}_{GG}$	$-p^g$	1		$D^{h_2}_{GG}$						
$-pz_{h_2} + -p^g g_{h_2} + \hat{p} \sum_f s_{fh_2} \hat{y}$	$s_{fh_2} \hat{p}$						$-p$	$-p^g$									1	
g_{h_2}								1										
$D_{x_{h_3}} u_{h_3} + -\lambda_{h_3} p$				$D^{h_3}_{x_{h_3} G}$				$D^{h_3}_{x_{h_3} G}$			$D^{h_3}_{x_{h_3}\ h_3}$	$D^{h_3}_{x_{h_3} G}$	$-p$					
$D_G u_{h_3} + -\lambda_{h_3} p^g + \mu_{h_3}$				$D^{h_3}_{GG}$				$D^{h_3}_{GG}$			$D^{h_3}_{G x_{h_3}}$	$D^{h_3}_{GG}$	$-p^g$	1				
$-pz_{h_3} + -p^g g_{h_3} + \hat{p} \sum_f s_{fh_3} \hat{y}$	$s_{fh_3} \hat{p}$										$-p$	$-p^g$						1
μ_{h_3}														1				
g_{h_3}												1						
$-z^{\backslash} + y^{\backslash}$	$I0$		$-I0$				$-I0$				$-I0$				$I0$			
$-G + y^g$	01			1				1				1						

right corner the corresponding partial Jacobian of $D_{(\xi,e)}F_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}(\xi,e)$ is displayed. Call M that partial Jacobian. Observe that in Table 1, $h_1 \in \mathcal{H}_1$, $h_2 \in \mathcal{H}_2$, and $h_3 \in \mathcal{H}_3$, and $z_h \equiv x_h - e_h$, $z \equiv \sum_h z_h$, $z^\backslash \equiv (z^c)_{c\neq C}$; for h_1, h_2, h_3, $D^h_{x_h x_h} \equiv D_{x_h x_h} u_h(x_h, G)$, $D^h_{x_h G} \equiv D_{x_h G} u_h(x_h, G)$, $D^h_{G x_h} \equiv D_{G x_h} u_h(x_h, G)$, $D^h_{GG} \equiv D_{GG} u_h(x_h, G)$. Note also that, to simplify the exposition of the proof, only one household from each set is presented in the Table 1.

We want to show that $\ker M^T = \{0\}$, i.e., $cM = 0 \Rightarrow c = 0$. Define

$$c \equiv ((c^{f1}, c^{f2})_{f=1}^F, \ldots, (c^1_{h1}, c^2_{h1}, c^3_{h1}, c^4_{h1}), \ldots, (c^1_{h2}, c^2_{h2}, c^3_{h2}, c^4_{h2}), \ldots,$$
$$(c^1_{h3}, c^2_{h3}, c^3_{h3}, c^4_{h3}, c^5_{h3}), c_{m1}, c_{m2})$$

with

$(c^{f1}, c^{f2})_{f=1}^F \in (\mathbb{R}^{C+1} \times \mathbb{R})^F$, $(c^1_{h1}, c^2_{h1}, c^3_{h1}, c^4_{h1})_{h=1}^H \in (\mathbb{R}^C \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})^H$, $c^5_{h3} \in \mathbb{R}$, $c_{m1} \in \mathbb{R}^{C-1}$, $c_{m2} \in \mathbb{R}$.

Then $cM = 0$ can be written as follows

$$\begin{cases}
(f.1) & c^{f1}\alpha_f D^2 t_f + c^{f2} D t_f + c^3_{h1} s_{fh_1}\hat{p} + c^3_{h2} s_{fh_2}\hat{p} + c^3_{h3} s_{fh_3}\hat{p} \\
 & \quad + c_{m1}(I0) + c_{m2}(01) = 0 \\
(f.2) & c^{f1} D t_f = 0 \\
(h_1.1) & c^1_{h1} D^{h_1}_{x_{h_1} x_{h_1}} + c^2_{h1} D^{h_1}_{G x_{h_1}} - c^3_{h1} p - c_{m1}(I0) = 0 \\
(h_1.2) & c^1_{h1} D^{h_1}_{x_{h_1} G} + c^2_{h1} D^{h_1}_{GG} - c^3_{h1} p^g + c^1_{h2} D^{h_2}_{x_{h_2} G} + c^2_{h2} D^{h_2}_{GG} + c^1_{h3} D^{h_3}_{x_{h_3} G} \\
 & \quad + c^2_{h3} D^{h_3}_{GG} + c_{m2} = 0 \\
(h_1.3) & -c^1_{h1} p - c^2_{h1} p^g = 0 \\
(h_1.4) & c^2_{h1} + c^4_{h1} = 0 \\
(h_2.1) & c^1_{h2} D^{h_2}_{x_{h_2} x_{h_2}} + c^2_{h2} D^{h_2}_{G x_{h_2}} - c^3_{h2} p - c_{m1}(I0) = 0 \\
(h_2.2) & c^1_{h1} D^{h_1}_{x_{h_1} G} + c^2_{h1} D^{h_1}_{GG} + c^1_{h2} D^{h_2}_{x_{h_2} G} + c^2_{h2} D^{h_2}_{GG} - c^3_{h2} p^g + c^4_{h2} \\
 & \quad + c^1_{h3} D^{h_3}_{x_{h_3} G} + c^2_{h3} D^{h_3}_{GG} + c_{m2} = 0 \\
(h_2.3) & -c^1_{h2} p - c^2_{h2} p^g = 0 \\
(h_2.4) & c^2_{h2} = 0 \\
(h_3.1) & c^1_{h3} D^{h_3}_{x_{h_3 h_3}} + c^2_{h3} D^{h_3}_{G x_{h_3}} - c^3_{h3} p - c_{m1}(I0) = 0 \\
(h_3.2) & c^1_{h1} D^{h_1}_{x_{h_1} G} + c^2_{h1} D^{h_1}_{x_{h_1} G} + c^1_{h2} D^{h_2}_{x_{h_2} G} + c^2_{h2} D^{h_2}_{GG} + c^1_{h3} D^{h_3}_{x_{h_3} G} \\
 & \quad + c^2_{h3} D^{h_3}_{GG} - c^3_{h3} p^g + c^5_{h3} + c_{m2} = 0 \\
(h_3.3) & -c^1_{h3} p - c^2_{h3} p^g = 0 \\
(h_3.4) & c^2_{h3} + c^4_{h3} = 0 \\
(M.1) & c^3_{h1} p^\backslash + c_{m1}(I0) = 0 \\
(M.2) & c^3_{h1} = 0 \\
(M.3) & c^3_{h2} = 0 \\
(M.4) & c^3_{h3} = 0
\end{cases} \tag{18}$$

From $(M.2), (M.3)$ and $(M.4)$, $c^3_{h1} = c^3_{h2} = c^3_{h3} = 0$. From $(M.1)$, $c_{m1} = 0$. From Assumption 4, $(h_1.1)$, $(h_1.3), (h_3.1)$, $(h_3.3)$, $c^1_{h1} = 0$, $c^2_{h1} = 0$ and $c^1_{h3} = 0$, $c^2_{h3} = 0$. From $(h_1.4)$ and $(h_3.4)$, $c^4_{h1} = 0$ and $c^4_{h3} = 0$. From $(h_2.4)$, $c^2_{h2} = 0$. Subtracting $(h_1.2)$ from $(h_2.2)$ and from $(h_3.2)$, we get $c^4_{h2} = 0$ and $c^5_{h3} = 0$. From Assumption 4, $D^2 u_h$ is negative definite and therefore each principal submatrix is negative definite and therefore of full rank. Then, from $(h_2.1)$, $c^1_{h2} = 0$. From $(h_1.2)$, $c_{m2} = 0$.

Finally, from Assumption 2, $(c^{f1}, c^{f2}) = 0$.

We can now prove the main theorem.

Proof of Theorem 11. Full measure follows from Sard's Theorem when it is applied to the projection defined on the differentiable equilibrium manifold $F^{-1}_{|\Xi\times D^*}(0)$. In fact, if $(\xi, e) \in \Xi \times D^*$ and $F(\xi, e) = 0$, f is differentiable at (ξ, e) since either $\mu_h > 0$ or $g_h > 0$. Consequently, 0 is a regular value of $F_{|\Xi\times D^*}$. This easily follows since the computation is the same as the one in Lemma A.1, except that there is no h in $\mathcal{H}_3$; hence, the associated system of equations is simpler than system (18), given that we have only to deal with the equations corresponding to households h_1 and h_2.

Acknowledgement

We would like to thank the editor and three anonymous referees for their helpful comments. In particular, we are very grateful to one of the referees for very stimulating and thoughtful comments and suggestions that helped improve the paper greatly. We would also like to express our gratitude to late Murat Sertel for a helpful discussion on the paper.

Appendix A

Lemma A.1. *$A_{\hat{\hat{r}}}$ is bounded.*

Proof. Since $\sum_{h=1}^{H} x_h \gg 0$ and $\sum_{h=1}^{H} g_h > 0$, if $(x, g, \hat{y}) \in A_{\hat{\hat{r}}}$, then

$$\hat{y} \in \hat{\mathcal{Y}}^{\hat{\hat{r}}} \equiv \left\{ \hat{y}' \equiv (\hat{y}'_f)_{f=1}^{F} \in \mathbb{R}^{(C+1)F} : \forall f,\ t_f(\hat{y}'_f) \geq 0 \text{ and } \sum_{f=1}^{F} \hat{y}'_f + \hat{\hat{r}} \gg 0 \right\}$$

Moreover, for each $h' \in \mathcal{H}$, if $(x, g, \hat{y}) \in A_{\hat{\tilde{r}}}$, then

$$0 \ll x_{h'} \ll \sum_{h=1}^{H} x_h \leq \sum_{f=1}^{F} y_f + \tilde{r} \text{ and } 0 \leq g_h \leq \sum_h g_h \leq \sum_f y_f^g + \tilde{r}^g$$

Therefore, to show the desired result it is enough to prove that $\hat{\mathcal{Y}}^{\hat{\tilde{r}}}$ is bounded, which is done below, where we denote $\hat{\mathcal{Y}}^{\hat{\tilde{r}}}$ simply by $\tilde{\mathcal{Y}}$ and we omit "$\widehat{\ }$" in any vector describing the production.

Suppose $\tilde{\mathcal{Y}}$ is not bounded. Then, we can find a sequence $(y^n)_{n\in\mathbb{N}} \equiv ((y_f^n)_{f=1}^F)_{n\in\mathbb{N}}$ in $\tilde{\mathcal{Y}}$ such that $\|y^n\| \to +\infty$. We can partition $\{1, \ldots, F\}$ into two subsets $\mathcal{F}^\infty \neq \emptyset$ and $\mathcal{F}^0$ such that, up to a subsequence, if $f \in \mathcal{F}^\infty$, then $\|y_f^n\| \to +\infty$, and if $f \in \mathcal{F}^0$, then there exists $k_f \in \mathbb{R}_+$ such that $\|y_f^n\| \to k_f$. Recall that from Assumption 2',

$$\exists a' > 0 \quad \text{such that } \forall y \in (Y \cap (-Y)),\ \|y\| < a' \tag{A.1}$$

Take $a \equiv 2Fa' > 0$. Then for sufficiently large n, for each $f \in \mathcal{F}^\infty$, $\|y_f^n\| > 0$, and we can define the sequence $\left(\frac{ay_f^n}{\|y_f^n\|}\right)_{n\in\mathbb{N}}$. Then

$$\forall f \in \mathcal{F}^\infty,\ \frac{ay_f^n}{\|y_f^n\|} \to \bar{y}_f \quad \text{and} \quad \|\bar{y}_f\| = a \tag{A.2}$$

Since $0, y_f^n \in Y_f$, and Y_f is convex, for sufficiently large n and each $f \in \mathcal{F}^\infty$, $\frac{a}{\|y_f^n\|} \in (0, 1)$ and

$$\frac{ay_f^n}{\left\|y_f^n\right\|} \in Y_f \tag{A.3}$$

Since, from Assumption 2, Y_f is closed, from (A.2) and (A.3)

$$\bar{y}_f \in Y_f \tag{A.4}$$

From the definition of $\tilde{\mathcal{Y}}$,

$$\sum_{f'\in\mathcal{F}^\infty} \frac{\|y_{f'}^n\|}{\sum_{f\in\mathcal{F}^\infty}\|y_f^n\|} \frac{ay_{f'}^n}{\|y_{f'}^n\|} + \sum_{f''\in\mathcal{F}^0} \frac{ay_{f''}^n}{\sum_{f\in\mathcal{F}^\infty}\|y_f^n\|} + \frac{ar}{\sum_{f\in\mathcal{F}^\infty}\|y_f^n\|} \gg 0 \tag{A.5}$$

Up to a subsequence, we also get

$$\lambda_{f'}^n \equiv \frac{\|y_{f'}^n\|}{\sum_{f\in\mathcal{F}^\infty}\|y_f^n\|} \to \bar{\lambda}_{f'} \in [0, 1] \tag{A.6}$$

Moreover,

$$\sum_{f'\in\mathcal{F}^\infty} \lambda_{f'}^n \to \sum_{f'\in\mathcal{F}^\infty} \bar{\lambda}_{f'} = 1 \tag{A.7}$$

Therefore, $\exists f^* \in \mathcal{F}^\infty$, such that $\overline{\lambda}_{f^*} \geq \frac{1}{F}$. (Suppose otherwise, then $\forall f \in \mathcal{F}^\infty$, $\overline{\lambda}_f < \frac{1}{F}$, and $1 = \sum_{f' \in \mathcal{F}^\infty} \overline{\lambda}_{f'} < \sum_{f' \in \mathcal{F}^\infty} \frac{1}{F} \leq 1$.) Then, from (A.2),

$$\|\overline{\lambda}_{f^*}\overline{y}_{f^*}\| = |\overline{\lambda}_{f^*}| \cdot \|\overline{y}_{f^*}\| \geq \frac{1}{F} 2Fa' > a' \tag{A.8}$$

Taking limits of both sides of (A.5), we get

$$\sum_{f \in \mathcal{F}^\infty} \overline{\lambda}_f \overline{y}_f \geq 0$$

Since $\forall f$, $0, \overline{y}_f \in Y_f$, and Y_f is convex, we have

$$\overline{\lambda}_{f^*}\overline{y}_{f^*} \in Y \tag{A.9}$$

Moreover,

$$\overline{\lambda}_{f^*}\overline{y}_{f^*} + \sum_{f \in \mathcal{F}^\infty \setminus \{f^*\}} \overline{\lambda}_f \overline{y}_f \geq 0$$

and therefore

$$-\overline{\lambda}_{f^*}\overline{y}_{f^*} \leq \sum_{f \in \mathcal{F}^\infty \setminus \{f^*\}} \overline{\lambda}_f \overline{y}_f \in Y$$

Then, from free disposal

$$-\overline{\lambda}_{f^*}\overline{y}_{f^*} \in Y \tag{A.10}$$

But, from (A.9), (A.10) and (A.1)

$$\|\overline{\lambda}_{f^*}\overline{y}_{f^*}\| < a'$$

contradicting (A.8).

Proof of Lemma 10. The result follows from an application of the implicit function theorem to the first-order conditions of the household maximization problem. Computing the partial Jacobian of the left-hand sides of the first-order conditions of problem (16) with respect to $(x_h, g_h, \lambda_h, w_h)$, we get

$$\begin{bmatrix} D_{x_h x_h} u_h & D_{x_h G} u_h & -p^T & 0 \\ D_{x_h G} u_h & D_{GG} u_h & -p^g & 0 \\ -p & -p^g & 0 & 1 \end{bmatrix}$$

Since D^2u_h is negative definite, we get

$$[D_{w_h}(x_h, g_h, \lambda_h)(\hat{p}, w_h, G_{\backslash h})] = -\begin{bmatrix} D^2u_h & -\hat{p} \\ -\hat{p} & 0 \end{bmatrix}^{-1}_{(C+2)\times 2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{(C+2)\times 1}$$

Then,[11]

$$\begin{bmatrix} D^2u_h & -\hat{p} \\ -\hat{p} & 0 \end{bmatrix}^{-1}_{(C+2)\times 2} = \begin{bmatrix} \left[D^2u_h\right]^{-1}(I + \hat{p}\delta_h^{-1}\hat{p}\left[D^2u_h\right]^{-1}) & \left[D^2u_h\right]^{-1}\hat{p}\delta_h^{-1} \\ \delta_h^{-1}\hat{p}\left[D^2u_h\right]^{-1} & \delta_h^{-1} \end{bmatrix}$$

where $\delta_h \equiv -\hat{p}\left[D^2u_h\right]^{-1}\hat{p} \in \mathbb{R}_{++}$, and

$$[D_{w_h}(x_h, g_h)(\hat{p}, w_h, G_{\backslash h})] = -\delta_h^{-1}\left[D^2u_h\right]^{-1}\begin{bmatrix} p \\ p^g \end{bmatrix}$$

Define

$$\left[D^2u_h\right]^{-1} \equiv \begin{bmatrix} D_{xx} & D_{Gx} \\ D_{xG} & D_{GG} \end{bmatrix}^{-1} \equiv \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Observe that $(D_{GG} - D_{xG}D_{xx}^{-1}D_{Gx})^{-1}$ exists (see Horn and Johnson, 1985, 0.8.5, p. 21). Then, using first-order conditions and the fact that $g_h > 0$ and therefore $\mu_h = 0$, we get

$$\begin{aligned} -\delta_h \cdot D_{w_h}g_h &= B_{21}p + B_{22}p^g \\ &= (D_{GG} - D_{xG}D_{xx}^{-1}D_{Gx})^{-1}\frac{1}{\lambda_h}(-D_{Gx}D_{xx}^{-1}D_xu_h + D_Gu_h) \end{aligned}$$

Finally, again from 0.8.5, in Horn and Johnson (1985), we get

$$\det\begin{bmatrix} D_{xx} & D_xu_h \\ D_{xG} & D_Gu_h \end{bmatrix} = \det D_{xx} \cdot \det[D_Gu_h - D_{Gx}D_{xx}^{-1}D_xu_h]$$

the desired result.

[11] Given $A_{n\times n} \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with A_{11} a square n_1-dimensional matrix, A_{22} a square n_2-dimensional matrix, $n_1 + n_2 = n$, both A_{11} and $C \equiv A_{22} - A_{21}A_{11}^{-1}A_{12}$ invertible, then it can be easily checked that

$$A^{-1} \equiv \begin{bmatrix} A_{11}^{-1}(I + A_{12}C^{-1}A_{21}A_{11}^{-1}) & -A_{11}^{-1}A_{12}C^{-1} \\ -C^{-1}A_{21}A_{11}^{-1} & C^{-1} \end{bmatrix}$$

References

Arrow, K.J., Hahn, F., 1971. General Competitive Analysis. San Francisco, Holden-Day.

Bergstrom, T., Blume, L., Varian, H., 1986. On the private provision of public goods. Journal of Public Economics 29, 25–49.

Cornes, R., Hartley, R., Sandler, T., 1999. Equilibrium existence and uniqueness in public good models: an elementary proof via contraction. Journal of Public Economic Theory 1 (4), 499–509.

Florenzano, M., 2003. General Equilibrium Analysis, Kluwer Academic Press.

Geanakoplos, J., Shafer, W., 1990. Solving systems of simultaneous equations in economics. Journal of Mathematical Economics 19, 69–93.

Hirsch, M., 1976. Differential Topology. Springer-Verlag, New York, NY.

Horn, R.A., Johnson, C.R., 1985. Matrix Analysis. Cambridge University Press, Cambridge, UK.

McKenzie, L.W., 1955. Competitive equilibrium with dependent consumer preferences. National Bureau of Standards and Department of the Air Force, The Second Symposium on Linear Programming, Washington, DC.

Shafer, W., Sonnenschein, H., 1975. Some theorems on the existence of competitive equilibrium. Journal of Economic Theory 11, 83–93.

Villanacci, A., Carosi, L., Benevieri, P., Battinelli, A., 2002. Differential Topology and General Equilibrium with Complete and Incomplete Markets. Kluwer Academic Publishers.

Villanacci, A., Zenginobuz U., 2004. Subscription equilibria with public production, manuscript.

Warr, P., 1983. The private provision of public goods is independent of the distribution of income. Economics Letters 13, 207–211.