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Applying exterior differential calculus to economics: a presentation and some new results[☆]

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Abstract

The paper provides an introduction to exterior differential calculus, and an application to the standard problem of the characterization of aggregate demand in an economy in which the number of agents is smaller than the number of goods.

Keywords: Exterior differential calculus; Aggregation; Aggregate demand; Cartan–Kähler theorem

1. Introduction: aggregation and gradient structures

In many situations, economists are interested in the behavior of aggregate variables that stem from the addition of several elementary demand or supply functions. In turn, each of these elementary components results from some maximizing decision process at the 'individual' level.[1] From a mathematical standpoint, the two ideas of maximization and aggregation have a natural translation in terms of combination of gradients. Specifically, they require that some given function $\boldsymbol{X}(\boldsymbol{p})$, mapping $\mathbb{R}^n_+$ to $\mathbb{R}^n$ and representing aggregate

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[1] A standard illustration is the characterization of aggregate market or excess demand in an exchange economy, a problem initially raised by Sonnenschein (1973a,b) and to which a number of author contributed, including Debreu (1974), McFadden et al. (1974), Mantel (1974, 1976, 1977), Diewert (1977), Geanakoplos and Polemarchakis (1980) and Chiappori and Ekeland (2000). A different but related example is provided by Browning and Chiappori (1994) and Chiappori and Ekeland (2002) who consider the demand function of a two-person household, where each member is characterized by a specific utility function and decisions are only assumed to be Pareto-efficient.

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behavior, can be decomposed as a linear combination of gradients $D_{\boldsymbol{p}}V^k(\boldsymbol{p})$, where the $V^k, k = 1, \ldots, K$ are functions defined on $\mathbb{R}^n_+$. Formally:

$$\boldsymbol{X}(\boldsymbol{p}) = \lambda_1(\boldsymbol{p})D_{\boldsymbol{p}}V^1(\boldsymbol{p}) + \cdots + \lambda_K(\boldsymbol{p})D_{\boldsymbol{p}}V^K(\boldsymbol{p}) \tag{1.1}$$

To give an example, consider an economy with K consumers, each of them characterized by some nominal income that can, without loss of generality, be normalized to 1. For a given price vector $\boldsymbol{p}$, consumer k solves the program

$$\max U^k(\boldsymbol{x}^k), \quad \boldsymbol{p} \cdot \boldsymbol{x}^k = 1 \tag{1.2}$$

The value of this program, denoted $V^k(\boldsymbol{p})$, is k's indirect utility; under adequate assumptions, it is decreasing, quasi-convex, and differentiable. The envelope theorem implies that $D_{\boldsymbol{p}}V^k(\boldsymbol{p}) = -\alpha_k(\boldsymbol{p}) \cdot \boldsymbol{x}^k(\boldsymbol{p})$, where $\alpha_k(\boldsymbol{p})$ is the Lagrange multiplier associated with the budget constraint. Substituting into the sum $X = \sum x^k$, we find the economy's aggregate demand at prices $\boldsymbol{p}$:

$$\boldsymbol{X}(\boldsymbol{p}) = \frac{1}{\alpha_1(\boldsymbol{p})}D_{\boldsymbol{p}}V^1(\boldsymbol{p}) + \cdots + \frac{1}{\alpha_K(\boldsymbol{p})}D_{\boldsymbol{p}}V^K(\boldsymbol{p}) \tag{1.3}$$

which is a linear combination of K gradients and satisfies $\boldsymbol{p}'X(\boldsymbol{p}) = K$. In addition:

- the V^k are (quasi) convex and decreasing;
- the α_k are negative;
- furthermore, the budget constraint implies:

$$\boldsymbol{p} \cdot D_{\boldsymbol{p}}V^k(\boldsymbol{p}) = \alpha_k(\boldsymbol{p}) \quad \forall k \tag{1.4}$$

A natural question, initially raised by Sonnenschein (1974), is the following: what does relation (1.1) imply upon the form of the function $\boldsymbol{X}$? In particular, are there testable necessary restrictions on the aggregate function $\boldsymbol{X}(\boldsymbol{p})$ that reflect its decomposability into individual maximizing behavior? And is it possible to find sufficient conditions on $\boldsymbol{X}(\boldsymbol{p})$ that guarantee the existence of a decomposition of the type (1.1)?

In practice, it is useful to decompose this problem into two subproblems. One, called the *mathematical integration* problem, can be stated as follows: given some function $\boldsymbol{X}$, when is it possible to find functions $V^k, k = 1, \ldots, K$, such that the decomposition (1.1) holds? The second problem, the *economic integration* problem, requires in addition that the V^k arise from maximizing some concave utilities, i.e. satisfy the conditions (quasi-convexity, (1.4), etc.) listed above.

In a previous contribution, Chiappori and Ekeland (1999) have argued that a particular subfield of differential topology, developed in the first half of this century by Elie Cartan and usually referred to as exterior differential calculus (from now on EDC), proved especially convenient to deal with problems of this type. Surprisingly enough, however, these tools have rarely been used in the field of economic theory, although other tools from differential topology have (see Sato and Ramachandran, 1998).[2] One possible explanation

[2] The first application of exterior differential calculus to economics can be found in a book edited by Chipman et al. (1971); there EDC was applied to the integrability of individual demand functions. Further applications include Sato (1981), Russell (1994) and Russell and Farris (1993).

for this absence is the lack of familiarity of most economists with these concepts, and the technical difficulty of the main results.

The main goal of the present paper is to propose an introduction to EDC. In particular, we describe in some details two very powerful theorems, respectively due to Darboux (with an improvement due to the authors and Nirenberg, see Ekeland and Nirenberg, 2000) and to Cartan and Kähler. To our knowledge, these results have never been used so far in economics, with the only exception of contributions by Chiappori and Ekeland (1999, 2002). We believe, however, that they may reveal extremely helpful in many contexts, and they should profitably be included in mathematical economists' toolbox. We show that Pfaff's theorem provides a direct answer to the mathematical integration problem, while Cartan–Kähler provides the tools that allow to solve (locally) the economic integration problem.

In the final section of the paper, we present an applications of this approach to Sonnenschein's problem. The standard version, where the number of agents is (at least) as large as the number of goods, has been solved, using the techniques described below, by Chiappori and Ekeland (1999). A more difficult problem deals with the characterization of the aggregate demand of a market economy when the number of agents is *smaller* than the number of commodities—a problem that has been discussed by various authors, including Diewert (1977) and Geanakoplos and Polemarchakis (1980), but has not been solved so far. We show how our approach allows to solve (locally) this problem.

2. Exterior differential calculus: an economist's toolkit

In this section, we introduce the basic notions of exterior differential calculus. Our purpose is exclusively pedagogical. At many places, in particular, our presentation is somewhat intuitive, and skips most technicalities, while many precautions are deliberately left aside. For a much more exhaustive and rigorous presentation, the interested reader is referred to Cartan's book (1945), or to the recent treatise by Bryant et al. (1991) (see also Sato, 1981).

2.1. Linear and differential forms

The basic notion is that of forms. A linear form (or a 1-form) on $E = \mathbb{R}^n$ is a linear mapping from E to $\mathbb{R}$:

$$\omega : \xi \in \mathbb{R}^n \mapsto \langle \omega, \xi \rangle = \sum_{i=1}^{n} \omega^i \xi_i$$

The set of linear forms on E is the dual E^* of E. A basic example of linear form is the projection π_i, which, to any vector, associates its ith coordinate. These form a basis of E^*; any form ω can be decomposed as:

$$\omega = \sum \omega^i \pi_i$$

In what follows, we are especially interested in *differential forms*. Consider a smooth manifold U, and let T_pU denote its tangent space at some point $\boldsymbol{p}$. A differential 1-form is, for every $\boldsymbol{p} \in U$, a 1-form $\omega(\boldsymbol{p})$ on the tangent space $T_{\boldsymbol{p}}U$ to U at $\boldsymbol{p}$ (with, say, $\langle \omega(\boldsymbol{p}), \xi \rangle = \sum \omega^i(\boldsymbol{p})\xi_i$), such that the coefficients $\omega^i(\boldsymbol{p})$ depend smoothly on $\boldsymbol{p}$. A local coordinate system at $\boldsymbol{p}$, say $\boldsymbol{p} = (p_1, \dots, p_n)$, provides $T_{\boldsymbol{p}}U$ with a coordinate system as well. If U is n-dimensional, then $T_{\boldsymbol{p}}U$ is a copy of $\mathbb{R}^n$, and the projection π_i: $T_{\boldsymbol{p}}U \to \mathbb{R}$, which associates with a tangent vector ξ its ith coordinate ξ^i, will be denoted by $\mathrm{d}p_i$.

As a simple example of a differential 1-form, we may, for any smooth mapping U from E to $\mathbb{R}$, consider the differential form $\mathrm{d}V$ (called a *total differential*) defined at any point $\boldsymbol{p}$ by:

$$\mathrm{d}V(\boldsymbol{p}) = \sum \frac{\partial V}{\partial p^i}(\boldsymbol{p})\,\mathrm{d}p_i$$

so that

$$\mathrm{d}V(\boldsymbol{p}) : \xi \mapsto \langle \mathrm{d}V(\boldsymbol{p}), \xi \rangle = \sum \frac{\partial V}{\partial p_i} \cdot \xi_i$$

Of course, this form is extremely specific, for the following reason. Consider the hypersurface (that is, the $(n-1)$-dimensional submanifold) $M \subset U$ defined by

$$M = \{\boldsymbol{p} \in U | V(\boldsymbol{p}) = a\}$$

where a is a constant. Then, for any $\boldsymbol{p}$, the form $\mathrm{d}V(\boldsymbol{p})$—and, as a matter of fact, any form $\omega(\boldsymbol{p}) = \lambda(\boldsymbol{p})\,\mathrm{d}V(\boldsymbol{p})$ proportional to $\mathrm{d}V(\boldsymbol{p})$—vanishes on the tangent space $T_{\boldsymbol{p}}M$:

$$\forall \boldsymbol{p} \in M, \quad \forall \xi \in T_{\boldsymbol{p}}M, \quad \langle \omega(\boldsymbol{p}), \xi \rangle = 0 \tag{2.1}$$

The *integration* problem is exactly this. Starting from some given differential form $\omega(\boldsymbol{p})$, when is it possible to find a hypersurface (or failing that, a lower-dimensional submanifold) M such that, for any $\boldsymbol{p}$, the restriction of $\omega(\boldsymbol{p})$ to $T_{\boldsymbol{p}}M$ is zero? Such a submanifold will be called an *integrating submanifold* for ω.

As an example, take any demand function $\boldsymbol{x}(\boldsymbol{p})$ (where income is normalized to 1 by homogeneity). Assume it is invertible, let $\boldsymbol{p}(\boldsymbol{x})$ denote the inverse demand function, and consider the form $\omega(x) = \sum p_i(\boldsymbol{x})\,\mathrm{d}x^i$. Integrating ω means to look for a hypersurface M such that, for any $\boldsymbol{x}$ on M, the vector $\boldsymbol{p}(\boldsymbol{x})$ is orthogonal to the tangent subspace T_xM. More precisely, we are looking for a one-dimensional family of such hypersurfaces, so that every point $x \in M$ belongs to one hypersurface of the family and one only: this is called a *foliation*. In terms of consumer theory, each manifold of the foliation will be an indifference surface, the tangent subspace a budget constraint, and we are imposing that the price vector be orthogonal to the indifference curve at each point, which is the usual first order condition for utility maximization. This is nothing else than the standard integration problem for individual demand functions.[3] In the next section, we investigate this aspect in more details.[4]

[3] Note that additional restrictions must be imposed upon the manifolds, reflecting monotonicity and quasi-concavity of preferences.

[4] Alternatively, one may consider the form $\omega(\boldsymbol{x}) = \sum p_i(\boldsymbol{x})\,\mathrm{d}x_i$. Integration will then lead to recovering the indifference surfaces of the indirect utility function.

One point must, however, be emphasized. When $\omega(\boldsymbol{p})$ is proportional to some total differential dV, the submanifold M can be found of (maximum) dimension $(n-1)$. But, of course, life is not always that easy. Starting from an arbitrary form, it is in general impossible to find such an integrating submanifold of dimension $(n-1)$. On the other hand, as we shall see presently, it is easy to find integrating submanifolds of dimension 1: there is just an ordinary differential equation to solve. The integration problem then consists in finding a foliation of U by integrating submanifolds $M_t, t \in \mathbb{R}$, of *maximum dimension* (or minimum codimension). When this minimal codimension is one, the form is *completely integrable*; in general, the minimum codimension will be greater than 1.

In fact, this has an interesting translation in terms of our initial problem. Assume, indeed, that instead of being proportional to some total differential dV, the form $\omega(\boldsymbol{p})$ is a linear combination of k total differentials:

$$\omega(\boldsymbol{p}) = \lambda_1(\boldsymbol{p})\,\mathrm{d}V^1(\boldsymbol{p}) + \cdots + \lambda_k(\boldsymbol{p})\,\mathrm{d}V^k(\boldsymbol{p})$$

Then we can find an integrating submanifold of codimension (at most) k. Indeed, for every choice of $a = (a_1, \dots, a_k) \in \mathbb{R}^k$, define the submanifold M_a by:

$$M_a = \{\boldsymbol{p} \in E | V^1(\boldsymbol{p}) = a_1, \dots, V^k(\boldsymbol{p}) = a_k\}$$

Clearly, M_a is of dimension (at least) $n-k$. Also, the tangent space at $\boldsymbol{p}$ is the intersection of the tangent spaces to the k manifolds $M^i, \dots, M^k$ defined by

$$M^i = \{\boldsymbol{p} \in E | V^i(\boldsymbol{p}) = a_i\}$$

It follows that (2.1) is always fulfilled. So the $M_a, a = (a_1, \dots, a_k) \in \mathbb{R}^k$, constitute a foliation of U by k-dimensional integrating submanifolds. A reciprocal property will be given later.

2.2. Exterior k-forms and exterior product

Before addressing the integration problem in details, we must generalize our basic concept.

Definition 1. An exterior k-form is a mapping ω: $(E)^k \to \mathbb{R}$ which is:

- multilinear, i.e., linear w.r.t. each vector: for all $(\xi^1, \dots, \xi^{s-1}, \eta, \zeta, \xi^{s+1}, \dots, \xi^k) \in E^{k+1}$ and all $(a, b) \in \mathbb{R}^2$, we have:

$$\omega(\xi^1, \dots, \xi^{s-1}, a\eta + b\zeta, \xi^{s+1}, \dots, \xi^k) = a \cdot \omega(\xi^1, \dots, \xi^{s-1}, \dot{\eta}, \xi^{s+1}, \dots, \xi^k) + b \cdot \omega(\xi^1, \dots, \xi^{s-1}, \zeta, \xi^{s+1}, \dots, \xi^k)$$

- antisymmetric, i.e., the sign is changed when two vectors are permuted:

$$\forall(\xi^1, \dots, \xi^k) \in E^k, \omega(\xi^1, \dots, \xi^i, \dots, \xi^j, \dots, \xi^k) = -\omega(\xi^1, \dots, \xi^j, \dots, \xi^i, \dots, \xi^k)$$

It follows that for any permutation σ of $\{1, \dots, k\}$:

$$\forall(\xi^1, \dots, \xi^k) \in E^k, \omega(\xi^{\sigma(1)}, \dots, \xi^{\sigma(s)}, \dots, \xi^{\sigma(k)}) = (-1)^{\mathrm{sign}(\sigma)} \cdot \omega(\xi^1, \dots, \xi^s, \dots, \xi^k)$$

Note that, if $k = 1$, we are back to the definition of linear forms. Consider, for instance, the case $k = 2$. A 2-form is defined by a matrix Ω:

$$\omega(\xi, \eta) = \sum_{i,j} \omega^{i,j} \xi_i \eta_j = \xi' \Omega \eta$$

Additional restrictions are usually imposed upon the matrix Ω. A standard one is symmetry; i.e., $\Omega = \Omega'$. In EDC, on the contrary, since one considers *exterior* forms, *antisymmetry* is imposed. This gives $\Omega = -\Omega'$, i.e. $\omega_{i,j} = -\omega_{j,i}$ for all i, j; hence

$$\omega(\xi, \eta) = \sum_{i<j} \omega^{i,j} (\xi_i \eta_j - \xi_j \eta_i)$$

Another case of interest is $k = n$, where n is the dimension of the space E. Then the space of exterior n-form is of dimension one, and include the *determinant*. That is, any n-form ω is collinear to the determinant:

$$\omega(\xi^1, \dots, \xi^n) = \lambda \det(\xi^1, \dots, \xi^n)$$

Some well known properties of determinant are in fact due exclusively to multilinearity together with antisymmetry, and can thus be generalized to forms of any order. For instance, take any k-form ω, and take k vectors $(\xi^1, \dots, \xi^k)$ that are not linearly independent. Then $\omega(\xi^1, \dots, \xi^k) = 0$.[5] An important consequence is that, *for any* $k > n$, *any exterior k-form must be zero*.

2.3. Exterior product

The set of exterior forms on M is an algebra, on which the multiplication, called the *exterior product*, is formally defined by:

Definition 2. Let ω be a k-form, and γ be a ℓ-form, then $\omega \wedge \gamma$ is a $(k + \ell)$-form such that

$$\omega \wedge \gamma(\xi^1, \dots, \xi^{k+\ell}) = \sum_{\sigma} \frac{1}{k!\ell!} (-1)^{\mathrm{sign}(\sigma)} \omega(\xi^{\sigma(1)}, \dots, \xi^{\sigma(k)}) \gamma(\xi^{\sigma(k+1)}, \dots, \xi^{\sigma(k+\ell)})$$

where the sum is over all permutations σ of $\{1, \dots, k + \ell\}$.

The formula may seem complex. Note, however, that it satisfies two basic requirements: $\omega \wedge \gamma$ is multilinear and antisymmetric. To grasp the intuition, consider the case of two linear forms ($k = \ell = 1$). Then

$$\omega \wedge \gamma(\xi, \eta) = \omega(\xi)\gamma(\eta) - \omega(\eta)\gamma(\xi)$$

Obviously, this is the simplest exterior 2-form related to ω and γ and satisfying the two requirements above.

[5] Indeed, one vector (say, p_k) can be decomposed as a linear combination of the others. Multilinearity implies that $\omega(\boldsymbol{p}_1, \dots, \boldsymbol{p}_k)$ writes down as a linear combination of terms like $\omega(\boldsymbol{p}_1, \dots, \boldsymbol{p}_{k-1}, \boldsymbol{p}_s)$ with $s < k$.But antisymmetry imposes all these terms be zero.

A few consequences of this definition must be kept in mind:

- Whenever ω is linear (or of odd order), $\omega \wedge \omega = 0$. More generally, let $\omega_1, \dots, \omega_s$ be any 1-forms, and consider the product:

  $$\omega_1 \wedge \cdots \wedge \omega_s$$

  If the forms are linearly dependent, this product is always zero.
- But whenever ω is a 2-form (or a form of even order), $\omega \wedge \omega$ need not be 0.
- For any k-form, $(\omega)^s = \omega \wedge \omega \wedge \cdots \wedge \omega$ is a (ks)-form. In particular, $(\omega)^s = 0$ as soon as $ks > n$.
- Any k-form can be decomposed into exterior products of 1-forms. If ω is a k-form, then:

  $$\omega = \sum_{\sigma} \omega_{\sigma(1),\dots,\sigma(k)}\, \mathrm{d}p_{\sigma(1)} \cdots \mathrm{d}p_{\sigma(k)} \tag{2.2}$$

  where the sum is over all ordered maps σ: $\{1, \dots, k\} \to \{1, \dots, n\}$

2.4. Differential forms and exterior differentiation

A *differential k-form* is, for every $\boldsymbol{p} \in U$, an exterior k-form $\omega(\boldsymbol{p})$ on the tangent space $T_{\boldsymbol{p}}U$ to U at $\boldsymbol{p}$, depending smoothly on $\boldsymbol{p}$. *Exterior differentiation* sends differential k-forms into differential $(k+1)$-forms. We first define it on 1-forms. Set:

$$\mathrm{d}(\boldsymbol{p}) = \sum \omega^j(\boldsymbol{p})\, \mathrm{d}p_j$$

To define the exterior differential of $\omega(\boldsymbol{p})$, we may first remark that the $\omega^j(\boldsymbol{p})$ are standard functions from E to $\mathbb{R}$. As such, they admit total differentials:

$$\mathrm{d}\omega^j(\boldsymbol{p}) = \sum_i \frac{\partial \omega^j}{\partial p_i}\, \mathrm{d}p_i$$

Then the exterior differential $\mathrm{d}\omega(\boldsymbol{p})$ of $\omega(p)$ is the differential 2-form defined by:

$$\begin{aligned} \mathrm{d}\omega(\boldsymbol{p}) &= \sum_j \mathrm{d}\omega^j(\boldsymbol{p}) \wedge \mathrm{d}p_j = \sum_{i,j} \frac{\partial \omega^j}{\partial p_i} \mathrm{d}p_i \wedge \mathrm{d}p_j \\ &= \sum_{i<j} \left(\frac{\partial \omega^j}{\partial p_i} - \frac{\partial \omega^i}{\partial p_j} \right) \mathrm{d}p_i \wedge \mathrm{d}p_j \end{aligned} \tag{2.3}$$

Note, again, that this formula guarantees that $\mathrm{d}\omega(\boldsymbol{p})$ is bilinear and antisymmetric.

Let us give a geometric interpretation, based on the antisymmetric side of the operation. In $E = \mathbb{R}^2$, consider a 1-form $\omega(\boldsymbol{p}) = \omega^1(\boldsymbol{p})\, \mathrm{d}p_1 + \omega^2(\boldsymbol{p})\, \mathrm{d}p_2$, where $\boldsymbol{p} = (p_1, p_2)$. Consider the four points depicted in Fig. 1, namely: $A = (p_1, p_2)$, $B = (p_1 + \delta p_1, p_2)$, $C = (p_1, p_2 + \delta p_2)$ and $D = (p_1 + \delta p_1, p_2 + \delta p_2)$, where the δp_i are ‘infinitesimally small’. Assume that we want to compute the following expressions:

$$I = \int_{\Gamma} \omega(\boldsymbol{p})\, \mathrm{d}\boldsymbol{p} \quad \text{and} \quad I' = \int_{\Gamma'} \omega(\boldsymbol{p})\, \mathrm{d}\boldsymbol{p}$$

where Γ (resp. Γ') is the infinitesimal curve ABD (resp. ACD).

Using the infinitesimal nature of the δp_i, we can compute

$$I = \int_A^B \omega(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} + \int_B^D \omega(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} = \omega^1(p_1, p_2)\delta p_1 + \omega^2(p_1 + \delta p_1, p_2)\delta p_2$$

and

$$I' = \int_A^C \omega(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} + \int_C^D \omega(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} = \omega^2(p_1, p_2)\delta p_2 + \omega^1(p_i, p_2 + \delta p_2)\delta p_1$$

What we are interested in is the difference $I - I'$. If ω was equal to the total differential $\mathrm{d}V$ for some smooth function V, this difference would be zero; in fact, the Jacobian matrix $D_{\boldsymbol{p}}\omega$ would then be symmetric. In the general case, using first order approximation, we find that:

$$I = I' = \left(\frac{\partial \omega^2}{\partial p_1} - \frac{\partial \omega^1}{\partial p_2}\right)\delta p_1\,\delta p_2$$

This is exactly the coefficient of the 2-form $\mathrm{d}\omega(\boldsymbol{b})$, as defined in (2.3).

Exterior differentiation is a linear operation, and there is a product formula:[6] if α is a differential p-form and ω a differential q-form, we have:

$$\mathrm{d}[\alpha + \omega] = \mathrm{d}\alpha + \mathrm{d}\omega$$
$$\mathrm{d}[\alpha \wedge \omega] = \mathrm{d}\alpha \wedge \omega + (-1)^p a \wedge \mathrm{d}\omega$$

This last property enables us to define exterior differentiation for differential forms of degree $k > 1$. Indeed, any such form can be decomposed into a product of 1-forms, and one just applies the two preceding rules to formula (2.2).

A last property that will turn out to be crucial in the sequel is naturalness with respect to pullbacks. To understand this property, take open subsets $V \subset R^q$ and $U \subset R^p$, and φ: $V \to U$ a smooth (non-linear) mapping. To any smooth f: $U \to R$, we can associate $f\mathrm{o}\varphi$, which is a smooth function on V. Similarly, to $\mathrm{d}f$, which is a 1-form on U, we associate $\mathrm{d}(f\mathrm{o}\varphi)$, which is a differential 1-form on V, called the *pullback* of f. As a particular case, take for f the ith coordinate map $x \to x^i$ on U; then the pullback of $\mathrm{d}x^i$ will be a 1-form on V that we denote by $\varphi^*(\mathrm{d}x_i)$.

The pullback φ^*, being defined for the $\mathrm{d}x^i$, is defined for all 1-forms on U by linearity, and for p-forms in the same way. Also, it is *natural* with respect to exterior products and exterior differentiation, in the following sense:

Proposition 1. *With the preceding definitions*:

$$\varphi^*(\alpha \wedge \omega) = (\varphi^*\alpha) \wedge (\varphi^*\omega)$$
$$\varphi^*(\mathrm{d}\omega) = \mathrm{d}\varphi^*\,\alpha(\omega)$$

[6] Exterior differentiation also has integration properties (Stokes' formula). Since we do not need this part of the theory, we shall nor enter into it in the paper.

These results, and the notion of pullback itself, will be important in studying integral manifolds of exterior differential systems in Section 4.

2.5. Poincaré's theorem

The construction detailed above has strong implications for the resolution of the type of equations we are interested in. Let us start with a simple problem: what are the conditions for a given exterior form ω to be the tangent form of some given, twice continuously differentiable function V? An immediate, necessary condition is given by the following result:

Proposition 2. *Assume* $\omega(\boldsymbol{p}) = \mathrm{d}V(\boldsymbol{p})$ *for some V, then*:

$$\mathrm{d}\omega = 0$$

Proof: Just note that,

$$\mathrm{d}\omega = \sum_{i<j} \left(\frac{\partial^2 V}{\partial p_j \partial p_i} - \frac{\partial^2 V}{\partial p_i \partial p_j} \right) \mathrm{d}p_i \wedge \mathrm{d}p_j = 0$$

This proposition admits a converse, due to Poincaré, that requires some topological condition upon U (there should be no 'hole' in U). For the sake of simplicity, let us just assume convexity (a sufficient property), and state the following result, valid for a differential form of arbitrary degree:

Theorem 2.1. *Let* ω *be a differential k-form on U such that* $\mathrm{d}\omega = 0$. *Assume U is convex. Then there exists a differential* $(k-1)$*-form on U, say* Ω, *such that*:

$$\omega = \mathrm{d}\Omega$$

Proof: See Bryant et al. (1991).

The simplest case is the following. Let $\omega^1(\boldsymbol{p}), \ldots, \omega^n(\boldsymbol{p})$ be given functions. Can we find V such that $\omega^i = \partial V / \partial p_i$? The answer is simple. Define the exterior form $\omega(\boldsymbol{p}) = \sum \omega^i(\boldsymbol{p})\, \mathrm{d}p_i$. Then, from the previous results, a necessary and sufficient condition is that:

$$\mathrm{d}\omega = \sum_{i,j} \frac{\partial \omega^i}{\partial p_j} \mathrm{d}p_i \wedge \mathrm{d}p_j = \sum_{i<j} \left(\frac{\partial \omega^i}{\partial p_j} - \frac{\partial \omega^j}{\partial p_i} \right) \mathrm{d}p_i \wedge \mathrm{d}p_j = 0$$

or

$$\frac{\partial \omega^i}{\partial p_j} = \frac{\partial \omega^j}{\partial p_i} \quad \forall i, j$$

2.6. The Darboux theorem

Poincaré's theorem provides necessary and sufficient conditions for a 1-form to be a total differential (or, equivalently, for vector field to be a gradient field). In this case, the integration problem is straightforward, as illustrated above. But, at the same time, these conditions are very strong. We now generalize this result, by giving necessary and sufficient conditions for a form to be a linear combination of k tangent forms. As discussed above, this means that the integration problem can be solved, but only with an integral manifold of dimension (at least) $(n-k)$. Assume that $\omega(\boldsymbol{p})$ can be written under the form:

$$\omega(\boldsymbol{p}) = \sum_{s=1}^{k} \lambda_s(\boldsymbol{p})\, \mathrm{d}V^s(\boldsymbol{p}), \quad \forall \boldsymbol{p} \in U \tag{2.4}$$

A first remark is that this structure has an immediate consequence. Indeed (forgetting the $\boldsymbol{p}$ for simplicity):

$$\mathrm{d}\omega = \sum_{s=1}^{k} \mathrm{d}\lambda_s \wedge \mathrm{d}V^s$$

Let us compute the exterior product $\omega \wedge (\mathrm{d}\omega)^k = \omega \wedge \mathrm{d}\omega \wedge \cdots \wedge \mathrm{d}\omega \wedge \mathrm{d}\omega$. We get first:

$$(\mathrm{d}\omega)^k = \mathrm{d}\lambda_1 \wedge \mathrm{d}V^1 \wedge \cdots \wedge \mathrm{d}\lambda_k \wedge \mathrm{d}V^k$$

hence

$$\omega \wedge (\mathrm{d}\omega)^k \left(\sum_{s=1}^{k} \mathrm{d}\lambda_s \wedge \mathrm{d}V^s \right) \wedge \mathrm{d}\lambda_1 \wedge \mathrm{d}V^1 \wedge \cdots \wedge \mathrm{d}\lambda_k \wedge \mathrm{d}V^k = 0$$

since we get the sum of k terms, each of whom includes twice the same 1-form.

In summary, we have a simple characterization: if ω can be written as in (2.4), then the product $\omega \wedge (\mathrm{d}\omega)^k$ must be zero.

This simple necessary condition admits an important converse.

Theorem 2.2 (Darboux). *Let ω be a linear form defined on some neighborhood U_0 of $\boldsymbol{p}$. Let k be such that*:

$$\omega \wedge (\mathrm{d}\omega)^{k-1} = \omega \wedge \mathrm{d}\omega \wedge \cdots \wedge \mathrm{d}\omega \neq 0, \quad \forall p \in U \tag{2.5}$$

$$\omega \wedge (\mathrm{d}\omega)^{k} = \omega \wedge \mathrm{d}\omega \wedge \cdots \wedge \mathrm{d}\omega = 0, \quad \forall p \in U \tag{2.6}$$

Then there exist a (possibly smaller) neighborhood U_1 of $\boldsymbol{p}$ and $2k$ smooth functions V^s and λ_s such that:

- *the V^s are linearly independent*;
- *none of the λ_s vanishes on U*;
- *and*

$$\omega(\boldsymbol{p}) = \sum_{s=1}^{k} \lambda_s(\boldsymbol{p})\, \mathrm{d}V^s(\boldsymbol{p}), \quad \forall \boldsymbol{p} \in U_1$$

Proof: See Bryant et al. (1991) (Chapter II, Section 3).

In other words, the Darboux theorem provides a necessary and sufficient condition for the mathematical integration problem.

3. Mathematical integration

In this section, we show, on two specific examples, how the tools previously described have very natural applications in consumer theory (see also Sato, 1981).

3.1. Maximization under linear constraint

The basic remark is the following. Consider the program that characterizes the behavior of an individual consumer facing a linear budget constraint:

$$V(\boldsymbol{p}) = \max_{\boldsymbol{x}} U(\boldsymbol{x}), \quad \boldsymbol{p}' \cdot \boldsymbol{x} = 1 \tag{3.1}$$

where the utility function U is continuously differentiable and strongly quasi-concave; note that, from now on, income is normalized to 1. Let $\boldsymbol{x}(\boldsymbol{p})$ denote the solution to (3.1). If α denotes the Lagrange multiplier, we have, from the envelope theorem, that:

$$DV(\boldsymbol{p}) = \alpha(\boldsymbol{p})\boldsymbol{x}(\boldsymbol{p}) \tag{3.2}$$

and $\boldsymbol{x}(\boldsymbol{p})$ is proportional to the gradient of the indirect utility V. Incidentally, (3.1) is equivalent to:

$$-U(\boldsymbol{x}) = \max_{\boldsymbol{p}}(-V(\boldsymbol{p})), \quad \boldsymbol{p}' \cdot \boldsymbol{x} = 1 \tag{3.3}$$

which implies, as above, that

$$DU(\boldsymbol{x}) = \beta(x)\boldsymbol{p}(\boldsymbol{x}) \tag{3.4}$$

where $\boldsymbol{p}(\boldsymbol{x})$ is the inverse demand function and $\beta(x)$ is the associated Lagrange multiplier; so $\boldsymbol{p}(\boldsymbol{x})$ is proportional to the gradient of U.[7]

So both $\boldsymbol{p}(\boldsymbol{x})$ and $\boldsymbol{x}(\boldsymbol{p})$ are proportional to a (single) gradient; we have a problem of the type (1.1) for $m = 1$. Actually, the programs (3.4) and (3.1) are exactly similar, so these two functions share exactly the same properties—a fact that has been known at least since Antonelli (1886).

Now, how does EDC enter the picture?[8] The idea is to define the linear form ω by:

$$\omega(\boldsymbol{p}) = \sum x^i(\boldsymbol{p})\,\mathrm{d}p^i \tag{3.5}$$

From the Darboux theorem, we know that $\boldsymbol{x}(\boldsymbol{p})$ is proportional to a gradient if and only if $\omega(\boldsymbol{p})$ satisfies:

$$\omega \wedge \mathrm{d}\omega = 0$$

which writes down:

$$\forall i, j, k \quad x_i\left(\frac{\partial x_j}{\partial p_k} - \frac{\partial x_k}{\partial p_j}\right) + x_k\left(\frac{\partial x_i}{\partial p_j} - \frac{\partial x_j}{\partial p_i}\right) + x_j\left(\frac{\partial x_k}{\partial p_i} - \frac{\partial x_i}{\partial p_k}\right) = 0 \tag{3.6}$$

[7] In fact, (3.4) can also be seen as the first order conditions of (3.1); this implies that $\beta\,[\boldsymbol{x}(\boldsymbol{p})] = \alpha(\boldsymbol{p})$.

[8] For a development on the links between EDC and Slutsky relations and the consequences upon Gorman forms, see Russell and Farris (1993).

We now show that (3.6) is nothing else than traditional Slutsky symmetry. To see why, note, first, that given our normalization (income is equal to 1), the Slutsky matrix is:

$$S = D_{\boldsymbol{p}}\boldsymbol{x}(I - \boldsymbol{p} \cdot \boldsymbol{x}') \tag{3.7}$$

where $D_{\boldsymbol{p}}\boldsymbol{x}$ is the Jacobian matrix of $\boldsymbol{x}(\boldsymbol{p})$. Take some fixed $\bar{\boldsymbol{p}}$. Slutsky symmetry is equivalent to the following condition: the restriction of $D_{\boldsymbol{p}}\boldsymbol{x}(\bar{\boldsymbol{p}})$ to the hyperplane orthogonal to $\boldsymbol{x}(\bar{\boldsymbol{p}}) = \bar{\boldsymbol{x}}$ must be symmetric. Formally:

$$\forall \boldsymbol{y}, \boldsymbol{z} \perp \bar{\boldsymbol{x}}, \quad \boldsymbol{y}'(D_{\boldsymbol{p}}\boldsymbol{x})\boldsymbol{z} = \boldsymbol{z}'(D_{\boldsymbol{p}}\boldsymbol{x})\boldsymbol{y} \Leftrightarrow \boldsymbol{y}'(D_{\boldsymbol{p}}\boldsymbol{x} - (D_{\boldsymbol{p}}\boldsymbol{x})')\boldsymbol{z} = 0 \tag{3.8}$$

Now, it can readily be seen that the vectors

$$\boldsymbol{y}_k = \begin{pmatrix} 0 \\ x_k(\bar{\boldsymbol{p}}) \\ -x_i(\bar{\boldsymbol{p}}) \\ 0 \end{pmatrix},$$

where $x_k(\bar{\boldsymbol{p}})$ (resp. $-x_i(\bar{\boldsymbol{p}})$) occupies the ith (resp. kth) row, form a basis of $\{\boldsymbol{x}(\bar{\boldsymbol{p}})\}^{\perp}$. It is thus sufficient to check (3.8) for any two $\boldsymbol{y}_j$ and $\boldsymbol{y}_k$. But this is exactly equivalent to (3.6).

Incidentally, given the similarity between (3.4) and (3.1), the same conclusion applies to the inverse demand function $\boldsymbol{p}(\boldsymbol{x})$; i.e., the matrix A defined by

$$A = D_{\boldsymbol{x}}\boldsymbol{p}(I - x.\boldsymbol{p}')$$

is symmetric. The reader can check that this is equivalent to the symmetry of the Antonelli matrix.

3.2. Aggregate demand

As a second example, consider the problem discussed in introduction, namely decomposition of aggregate market demand. Let $\boldsymbol{X}$ be some given function on $\mathbb{R}^n$; when is it possible to find scalar functions $\lambda_{\boldsymbol{k}}(\boldsymbol{p})$ and $V^k(\boldsymbol{p})$, $k = 1, \ldots, K$, such that

$$\boldsymbol{X}(\boldsymbol{p}) = \lambda_1(\boldsymbol{p})D_{\boldsymbol{p}}V^1(\boldsymbol{p}) + \cdots + \lambda_K(\boldsymbol{p})D_{\boldsymbol{p}}V^K(\boldsymbol{p}) \tag{3.9}$$

From the Darboux theorem, a necessary and sufficient condition for (3.9) is that:

$$\omega \wedge (\mathrm{d}\omega)^K = 0 \tag{3.10}$$

We now provide an equivalent but simpler statement of this property. Consider the Slutsky matrix associated to $\boldsymbol{X}$, namely $S(\boldsymbol{p}) = D_{\boldsymbol{p}}\boldsymbol{X}(I - \boldsymbol{p}\boldsymbol{X}')$. Then

$$S(\boldsymbol{p}) = \left(\sum_k D_{\boldsymbol{p}}\boldsymbol{x}^k\right)\left(\left(I - \boldsymbol{p}\sum_k \boldsymbol{x}^j\right)'\right) = \sum_k S_{\boldsymbol{p}} + \sum_j \left(\left(\sum_{k \neq j} D_{\boldsymbol{p}}\boldsymbol{x}^k\right)\boldsymbol{p}(\boldsymbol{x}^j)'\right) \tag{3.11}$$

where S_k is Slutsky matrix associated to $\boldsymbol{x}^k$. Hence $S(\boldsymbol{p})$ is of the form

$$S(\boldsymbol{p}) = \Sigma + \sum_j \boldsymbol{u}_j(\boldsymbol{v}_j)' \tag{3.12}$$

where

$$\boldsymbol{u}_j = \left(\sum_{k \neq j} D_{\boldsymbol{p}} \boldsymbol{x}^k\right) \boldsymbol{p} \quad \text{and} \quad \boldsymbol{v}_j = \boldsymbol{x}^j$$

and $S(\boldsymbol{p})$ is the sum of a symmetric matrix and a matrix of rank at most K. This shows that (3.12) is a necessary consequence of (3.9).

Conversely, let us show that (3.12) is sufficient for the necessary and sufficient condition (3.10) to hold. Indeed, (3.12) is equivalent to

$$\mathrm{d}\omega - \omega \wedge a = \sum_j b_j \wedge c_j$$

where a, b_j and c_j ($j = 1, \dots, K$) are 1-forms whose definition is clear. Then

$$(\mathrm{d}\omega)^K = \omega \wedge a \wedge b_1 \wedge c_1 \dots \wedge b_K \wedge c_K$$

and (3.10) is fulfilled.

Finally, still another equivalent statement is the following. Consider the linear subspace $\mathscr{S}$, of codimension K, orthogonal to the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_K$. For any vectors $\boldsymbol{x}, \boldsymbol{y}$ in $\mathscr{S}$, one has that

$$xS(\boldsymbol{p})\boldsymbol{y} = \boldsymbol{x}' \sum \boldsymbol{y} = \boldsymbol{y}' \sum \boldsymbol{x} = \boldsymbol{y}' S(\boldsymbol{p}) \boldsymbol{x}$$

implying that the restriction of $\mathscr{S}(\boldsymbol{p})$ to $\mathscr{S}$ is symmetric. Hence there exists a subspace of codimension K such that the restriction of $\mathscr{S}$ $(\boldsymbol{p})$ to this subspace is symmetric.

An interesting property is the following. Assume $K \geq n/2$. Then $\omega \wedge (\mathrm{d}\omega)^k$ is a s-form with $s \geq n + 1$ (remember that $\mathrm{d}\omega$ is a 2-form). It follows that it is identically zero, so that the condition is always fulfilled. We conclude that when the number of consumers is larger than $n/2$, the mathematical integration problem can always be solved.

Note, however, that these conditions are necessary and sufficient for mathematical integration only. Although they guarantee the existence of functions V^k, $k = 1, \dots, K$, such that the decomposition (3.9) holds, one does not expect in general that these functions will satisfy the additional restrictions required by the economic interpretation in terms of maximization under budget constraint. Economic integration is in fact a much harder problem, that requires the full strength of a very deep theorem in EDC, due to Cartan and Kähler. Note, however, that in some situations, it is enough for economic integration that, in the decomposition (3.9), the λ_k are positive and the V^k are quasi-concave. In other words, one needs a convex version of the Darboux theorem. Together with Nirenberg, the authors have proved the following:

Theorem 3.1. *A necessary and sufficient condition for* (3.9) *to hold in some convex neighborhood of* $\bar{\boldsymbol{p}}$, *with the* λ_k *positive and the* V^k *strictly quasi-concave*, *is that the Darboux conditions* (2.5) *and* (2.6) *are satisfied*, *and that there exists a subspace* $\mathscr{S}$, *of codimension* K, *such that the restriction to* $\mathscr{S}$ *of the Slutsky matrix of* $\boldsymbol{X}$ *at* $\bar{\boldsymbol{p}}$ *is symmetric and negative definite*.

We refer to Chiappori and Ekeland (1997) and to Ekeland and Nirenberg (2000) for the general statement and for the proof.

4. Exterior differential systems on manifolds: the Cartan–Kähler theorem

We now present the key result upon which our approach relies. This theorem, due to Cartan and Kähler, solves the following, general problem. Given a certain family of differential forms (not necessarily 1-forms, nor even of the same degree), a point $\bar{\boldsymbol{p}}$ and an integer $m \geq 1$, can one find some m-dimensional submanifold M containing $\bar{\boldsymbol{p}}$ and on which all the given forms vanish?

4.1. An introductory example

As an introduction, let us start from a simple version of our problem, namely the Cauchy–Lipschitz theorem for ordinary differential equations. It states that, given a point $\bar{\boldsymbol{p}} \in \mathbb{R}^{n-1}$ and a C^1 function f, defined from some neighborhood $\mathcal{U}$ of $\bar{\boldsymbol{p}}$ into $\mathbb{R}^{n-1}$, there exists some $\epsilon > 0$ and a C^1 function $\varphi :]-\epsilon, \epsilon[\rightarrow \mathcal{U}$ such that

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = f(\varphi(t)), \qquad \forall t \in]-\epsilon, \epsilon[, \quad \varphi(0) = \bar{\boldsymbol{p}} \tag{4.1}$$

It follows that $\mathrm{d}\varphi/\mathrm{d}t(0) = f(\bar{\boldsymbol{p}})$. If $f(\bar{\boldsymbol{p}}) = 0$, the solution is trivial, $\varphi(t) = \bar{\boldsymbol{p}}$ for all t so we assume that $f(\bar{\boldsymbol{p}})$ does not vanish.

This theorem can be rephrased in a geometric way. Consider the graph M of φ:

$$M = \{(t, \varphi(t)) | -\epsilon < t < \epsilon\}$$

which is a one-dimensional submanifold of $]-\epsilon, \epsilon[\times \mathcal{U}$. Let us introduce the 1-forms ω^i defined by:

$$\omega^i = f^i(\boldsymbol{p})\,\mathrm{d}t - \mathrm{d}p^i, \quad 1 \leq i \leq n-1$$

Clearly φ solves the differential Eq. (4.1) if and only if the ω^i all vanish on M. More precisely, substituting $p^i = (\varphi^i(t)$ into formula (4) yields the pullbacks:

$$\varphi^* \omega^i = \left[f^i(\varphi(t)) - \frac{\mathrm{d}\varphi^i}{\mathrm{d}t}(t) \right] \mathrm{d}t$$

which vanish if and only if φ solves Eq. (4.1).

So the Cauchy–Lipschitz theorem tells us how to find a one-dimensional submanifold of $\mathbb{R} \times \mathbb{R}^n$ on which certain 1-forms vanish.

4.2. The general problem

The Cauchy–Lipschitz theorem deals with 1-forms of a specific nature. By extension, the general problem can formally be stated as follows.

Definition 3. Let ω^k, $1 \leq k \leq K$, be differential forms on an open subset of R^n, and $M \subset R^n$ a submanifold. We call M an integral submanifold of the exterior differential system:

$$\omega^1 = 0, \ldots, \omega^K = 0 \tag{4.2}$$

if the pullbacks of the ω^k to M all vanish:

$$\omega^k(\boldsymbol{p})(\xi^1, \ldots, \xi^{d_k}) = 0, \quad 1 \le k \le K \tag{4.3}$$

whenever $\boldsymbol{p} \in M$, ω^k has degree d_k, and $\xi^i \in T_{\boldsymbol{p}}M$ for $1 \le i \le d_k$.

Given a point $\bar{\boldsymbol{p}} \in \mathbb{R}^n$, the Cartan–Kähler theorem will give necessary and sufficient conditions for the existence of an integral manifold containing $\bar{x}$.

Necessary conditions are easy to find. Assume an integral manifold $M \ni \bar{\boldsymbol{p}}$ exists, and let m be its dimension. Then its tangent space at $\bar{\boldsymbol{p}}$, denoted by $T_{\bar{\boldsymbol{p}}}M$, is m-dimensional, and all the $\omega^j(\bar{\boldsymbol{p}})$ must vanish on $T_{\bar{\boldsymbol{p}}}M$, because of formula (7). Any subspace $E \subset T_{\bar{\boldsymbol{p}}}M$ with this property will be called an *integral element* of system (6) at $\bar{\boldsymbol{p}}$. The set of all m-dimensional integral elements at $\bar{\boldsymbol{p}}$ will be

$$G^m(\bar{\boldsymbol{p}}) = \left\{ E \,\middle|\, \begin{array}{l} E \subset T_{\bar{\boldsymbol{p}}}M \text{ and} \dim E = m \\ \omega^1(\bar{\boldsymbol{p}}), \ldots, \omega^K(\bar{\boldsymbol{p}}) \text{ vanish on } E \end{array} \right\}$$

Our first necessary condition is clear:

$$G^m_{\bar{x}} \neq \emptyset \tag{4.4}$$

4.3. Differential ideals

To get the second one, let us ask a strange question: have we written all the equations? In other words, does the system:

$$\omega^1 = 0, \ldots, \omega^K = 0 \tag{4.5}$$

exhibit all the relevant information?

The answer may be no. To see why, recall that M is a submanifold of $\mathbb{R}^n$, and denote by φ_M: $M \to \mathbb{R}^n$ the standard embedding ($\varphi_M(x) = x$ for all $x \in M$. Then M is an integral manifold of system (4.5) if:

$$\varphi_M^*\omega^1 = 0, \ldots, \varphi_M^*\omega^K = 0 \tag{4.6}$$

But we know that exterior differentiation is natural with respect to pullbacks, that is, that d commutes with φ_M^*. So (4.6) implies that:

$$\varphi_M^*(\mathrm{d}\omega^1) = 0, \ldots, \varphi_M^*(\mathrm{d}\omega^K) = 0$$

In other words, M is also an integral manifold of the larger system:

$$\begin{cases} \omega^1 = 0, \ldots, \omega^K = 0 \\ \mathrm{d}\omega^1 = 0, \ldots, \mathrm{d}\omega^K = 0 \end{cases} \tag{4.7}$$

which is different from (4.5). If integral elements of (4.7) are different from integral elements of (4.5), it is not clear which ones we should be working with.

To resolve this quandary, we shall assume that systems (4.5) and (4.7) have the same integral elements. In other words, the second equations in (4.7) must be *algebraic* consequences of the first ones. The precise statement for this is as follows:

Definition 4. The family $\{\omega^k | 1 \le k \le K\}$ is said to generate a differential ideal if there are forms $\{\alpha_j^k | 1 \le j, k \le K\}$ such that:

$$\forall k, \mathrm{d}\omega^k = \sum_j \alpha_j^k \wedge \omega^j \tag{4.8}$$

Our second necessary condition is that the ω^k, $1 \le k \le K$, must generate a differential ideal. If this is the case, we say that the exterior differential system is *closed*.

Note that if the given family $\{\omega^k | 1 \le k \le K\}$ does not satisfy this condition, the enlarged family $\{\omega^{\mathrm{k}}, \mathrm{d}\omega^k | 1 \le k \le K\}$ certainly will (because $\mathrm{d}\omega^k = 0$). So the condition that the system is closed can be understood as saying that the enlargement procedure has already taken place.

Unfortunately, conditions (4.4) and (4.8) are not sufficient. We give two counter-examples to show that an additional condition is needed.

4.4. A first counter example

Consider two functions f and g from $\mathbb{R}^{n-1}$ into itself, with $f(0) = g(0) \neq 0$, and $f(\boldsymbol{p}) \neq g(\boldsymbol{p})$ for $\boldsymbol{p} \neq 0$. Define α^i and β^i, $1 \le i \le n-1$, by

$$\alpha^i = f^i(\boldsymbol{p})\,\mathrm{d}t - \mathrm{d}p^i$$
$$\beta^i = g^i(\boldsymbol{p})\,\mathrm{d}t - \mathrm{d}p^i$$

and consider the exterior differential system in $\mathbb{R}^n$:

$$\alpha^i = 0, \qquad \beta^i = 0, \quad 1 \le i \le n-1$$

The α^i and the β^i generate a differential ideal, and there is an integral element at 0, namely the line carried by (1, *N*), so $G^1(0) \neq \varnothing$. However, finding an integral manifold of the initial system containing 0 amounts to finding a common solution of the two Cauchy problems:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = f(\boldsymbol{p}), \quad \boldsymbol{p}(0) = 0$$
$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = g(\boldsymbol{p}), \quad \boldsymbol{p}(0) = 0$$

which does not exist in general. The problem clearly is that the equality $f(\boldsymbol{p}) = g(\boldsymbol{p})$ holds at $\boldsymbol{p} = 0$ only. So we need a regularity condition which will exclude such pathological situations—technically, that guarantees that the required equality hold true at *ordinary points*, a concept we formally define below.

4.5. A second counterexample

Let us work in $\mathbb{R}^2$, and let us find all functions $\phi(x, y)$ which can be written as

$$\phi(x, y) = f(x) + g(y)$$

It is well known that a necessary and sufficient condition for such a decomposition to be possible, at least for smooth ϕ, is that the cross derivative vanishes:

$$\frac{\partial^2 \phi}{\partial x \, \partial y} \equiv 0 \tag{4.9}$$

Consider the exterior differential system in $\mathbb{R}^2 = (x, y, f, g)$:

$$\frac{\partial \phi}{\partial x} \mathrm{d}x + \frac{\partial \phi}{\partial y} \mathrm{d}y = \mathrm{d}f + \mathrm{d}g \tag{4.10}$$

$$0 = \mathrm{d}f \wedge \mathrm{d}x \tag{4.11}$$

$$0 = \mathrm{d}g \wedge \mathrm{d}y \tag{4.12}$$

Any two-dimensional integral submanifold M of this system will be the graph of a pair of functions (f, g) which solve the problem, provided only that it is not vertical, that is, that neither $\mathrm{d}x$ nor $\mathrm{d}y$ vanish on M. Let us try to find such an integral submanifold. The system is obviously closed. We then look for non-vertical integral elements, at 0 say. They are defined by a set of linear equations:

$$\mathrm{d}f = A_1 \, \mathrm{d}x + B_1 \, \mathrm{d}y$$
$$\mathrm{d}g = A_2 \, \mathrm{d}x + B_2 \, \mathrm{d}y$$

and plugging them into the system, we get

$$B_1 = 0, \quad A_2 = 0, \quad A_1 = \frac{\partial \phi}{\partial x}(0,0), \quad B_2 = \frac{\partial \phi}{\partial y}(0,0)$$

So there is an integral element. However, there is no two-dimensional integral submanifold, unless condition (4.9) is satisfied.

4.6. *The last condition*

If all the $\omega_k(\boldsymbol{p})$ are 1-forms, the regularity condition is clear enough: the dimension of the space spanned by the $\omega_k(\boldsymbol{p})$, $1 \leq k \leq K$, should be constant on a neighborhood of $\bar{\boldsymbol{p}}$ (which is obviously not the case in the counterexample above). Note that, locally, this dimension can only increase, that is, the codimension can only decrease.

If some of the ω_k have higher degree, the regularity condition is more complicated. It is expressed in the following.

Pick a point $\bar{\boldsymbol{p}} \in \mathbb{R}^n$; from now on, we work in the tangent space $V = T_{\bar{\boldsymbol{p}}}\mathbb{R}^n$. Let $E \subset V$ be an m-dimensional integral element at $\bar{\boldsymbol{p}}$. Pick a basis $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ of V^* such that:

$$E = \{\xi \in V | \langle \xi, \bar{\alpha}_i \rangle = 0, \quad \forall i \geq m+1\}$$

For $n' \leq n$, denote by $\mathscr{I}(n', d)$ the set of all ordered subsets of $\{1, \ldots, n'\}$ with d elements. Denote by d_k the degree of ω_k. For every k, writing $\omega_k(\bar{\boldsymbol{p}})$ in the $\bar{\alpha}_i$, basis, we get

$$\omega_k(\bar{\boldsymbol{p}}) = \sum_{I \in \mathscr{I}(n, d_k)} c_I^k \bar{\alpha}_{i_1} \wedge \cdots \wedge \bar{\alpha}_{i_{d_k}}.$$

In this summation, it is understood that $I = \{i_1, \dots, i_{d_k}\}$. Since $\omega^k\{\bar{x}\}$ vanishes on E, each monomial must contain some $\bar{\alpha}_i$, with $i \geq m+1$. Let us single out the monomials containing one such term only. Regrouping and rewriting, we get the expression:

$$\omega^k(\bar{\boldsymbol{p}}) = \sum_{J \in \mathscr{I}(m, d_k - 1)} \bar{\beta}_J^k \wedge \bar{\alpha}_{j_1} \wedge \cdots \wedge \bar{\alpha}_{j_{d_k - 1}} + \text{remainder}$$

where $\bar{\beta}_J^k$ is a linear combination of the α_i for $i \geq m+1$, and all the monomials in the remainder contain $\bar{\alpha}_i \wedge \bar{\alpha}_i{}'$ for some $i > i' \geq m+1$.

Define an increasing sequence of linear subspace $H_0^* \subset H_1^* \subset \cdots \subset H_M^* \subset V^*$ as follows:

$$H_m^* = \operatorname{Span}[\bar{\beta}_J^k | 1 \leq k \leq K, J \in \mathscr{I}(m, d_k - 1)]$$
$$H_{m-1}^* = \operatorname{Span}[\bar{\beta}_J^k | 1 \leq k \leq K, J \in \mathscr{I}(m-1, d_k - 1)]$$
$$H_0^* = \operatorname{Span}[\bar{\beta}_J^k | 1 \leq k \leq K, J \in \mathscr{I}(0, d_k - 1)]$$

The latter is just the linear subspace generated by those of the $\omega^k(\bar{x})$ which happen to be 1-forms. We define an increasing sequence of integers $0 \leq c_0(\bar{x}, E) \leq \cdots \leq c_m(\bar{x}, E) \leq n$ by:

$$c_i(\bar{x}, E) = \dim H_i^*$$

We are finally able to express Cartan's regularity condition. Denote by $\mathbb{P}^m(\mathbb{R}^n)$ the set of all m-dimensional subspaces of $\mathbb{R}^n$ with the standard (Grassmannian) topology: it is known to be a manifold of dimension $m(n-m)$. Denote by G^m the set of all $(\boldsymbol{p}, E)$ such that E is an m-dimensional integral element at $\boldsymbol{p}$. Note that G^m is a subset of $\mathbb{R}^n \times \mathbb{P}^m(\mathbb{R}^n)$.

Definition 5. Let $(\bar{\boldsymbol{p}}, \bar{E}) \in G^m$—that is, $\bar{E}$ is an m-dimensional integral element at $\bar{\boldsymbol{p}}$. We say that $(\bar{\boldsymbol{p}}, \bar{E})$ is *ordinary* if there is some neighborhood U of $(\bar{\boldsymbol{p}}, \bar{E})$ in $R^n \times P^m(R^n)$ such that $G^m \cap U$ is a submanifold of codimension

$$c_0(\bar{\boldsymbol{p}}, \bar{E}) + \cdots + c_{m-1}(\bar{\boldsymbol{p}}, \bar{E})$$

If all the ω^k are 1-forms, denote by $\mathrm{d}(\boldsymbol{p})$ the dimension of the space spanned by the $\omega^k(\boldsymbol{p})$. Then $c_i(\boldsymbol{p}, E) = \mathrm{d}(\boldsymbol{p})$ for every i, and $(\bar{\boldsymbol{p}}, \bar{E})$ is ordinary if $G^m \cap \mathscr{U}$ is a submanifold of codimension $m\mathrm{d}(\bar{\boldsymbol{p}})$ in $\mathbb{R}^n \times \mathbb{P}^m(\mathbb{R}^n)$. This implies that, for every $\boldsymbol{p}$ in a neighborhood of $\bar{\boldsymbol{p}}$, the set of $E \in G^m(\boldsymbol{p})$ (integral elements at $\boldsymbol{p}$) has codimension $m\mathrm{d}(\bar{\boldsymbol{p}})$ in $\mathbb{P}^m(\mathbb{R}^n)$. It can be seen directly to have codimension $m\mathrm{d}(\boldsymbol{p})$. So $\mathrm{d}(\boldsymbol{p}) = \mathrm{d}(\bar{\boldsymbol{p}})$ in a neighborhood of $\bar{\boldsymbol{p}}$: this is exactly the regularity condition we wanted for 1-forms.

In the general case, if $(\bar{\boldsymbol{p}}, \bar{E})$ is ordinary, the numbers c_i will also be locally constant:

$$c_i(\boldsymbol{p}, E) = c_i(\bar{\boldsymbol{p}}, \bar{E}) = c_i, \quad \forall (\boldsymbol{p}, E) \in \mathscr{U}.$$

The (non-negative) numbers

$$s_0 = c_0$$
$$s_i = c_i - c_{i-1}, \quad \text{for } 1 \leq i < m$$
$$s_m = n - m - c_{n-1}$$

are called the *Cartan characters*. We shall use them later on.

4.7. The main result

We are now in a position to state the Cartan–Kähler theorem. Recall that a real-valued function on $\mathbb{R}^n$ is called *analytic* if its Taylor series at every point is absolutely convergent.:

Theorem 4.1 (Cartan–Kähler). *Consider the exterior differential system*:

$$\omega^k = 0, \quad \lambda \le k \le K \tag{4.13}$$

Assume that the ω^k *are real analytic and that they generate a differential ideal. Let* $\bar{\boldsymbol{p}}$ *be a point and let* $\bar{E}$ *be an integral element at* $\bar{\boldsymbol{p}}$ *such that* $(\bar{\boldsymbol{p}}, \bar{E})$ *is ordinary. Then there is a real analytic integral manifold* M, *containing* $\bar{\boldsymbol{p}}$ *and such that*:

$$T_{\bar{\boldsymbol{p}}}M = \bar{E}. \tag{4.14}$$

Nothing should come as a surprise in this statement, except the real analyticity. It comes from the very generality of the Cartan–Kähler theorem. Indeed, every system of partial differential equations, linear or not, can be written as an exterior differential system, and there is a famous example, due to Hans Lewy, of a system of two first-order nonhomogeneous linear partial differential equations (with non-constant coefficients) for two unknown functions, which has no solution if the right-hand side is C^∞ but not analytic.

To conclude, let us mention the question of uniqueness. There is no uniqueness in the Cartan–Kähler theorem: there may be infinitely many analytic integral manifolds going through $\bar{\boldsymbol{p}}$ and having $\bar{E}$ as a tangent space at $\bar{\boldsymbol{p}}$. However, the theorem describes in a precise way (not given here) the set

$$\mathscr{M}_{\mathscr{U}} = \left\{ M \,\middle|\, \begin{array}{l} M \text{ is an integral manifold and there exists} \\ (\boldsymbol{p}, E) \in \mathscr{U} \text{ such that } p \in M \text{ and } T_{\boldsymbol{p}}M = E \end{array} \right\},$$

where $\mathscr{U}$ is a suitably chosen neighborhood of $(\bar{\boldsymbol{p}}, \bar{E})$. Loosely speaking, each M in $\mathscr{M}_{\mathscr{U}}$ is completely determined by the (arbitrary) choice of s_m analytic functions of m variables, the s_m being the Cartan character.

4.8. An example

Let us go back to the second counterexample. There is only one integral element at every point, so that G^2 is a point in $\mathbb{P}^2(\mathbb{R}^4)$ which has dimension 4. So its codimension is 4.

Let us now compute the Cartan characters. There is just one 1-form in the system, namely (4.10), so $c_0 = 1$. The integral element is defined by $\alpha_3 = \alpha_4 = 0$, where $\alpha_3 = \mathrm{d}f - (\partial\phi/\partial x)(0,0)\,\mathrm{d}x$ and $\alpha_4 = \mathrm{d}g - (\partial\phi/\partial y)(0,0)\,\mathrm{d}y$. Set $\alpha_1 = \mathrm{d}x$ and $\alpha_2 = \mathrm{d}y$ to get a basis. Substituting in the 2-forms (4.11) and (4.12), we get:

$$0 = \alpha_3 \wedge \alpha_1$$
$$0 = \alpha_4 \wedge \alpha_2$$

so that $H_1^* = \mathrm{Span}[\alpha_1, \alpha_2]$ and $c_1 = 2$. We get $c_0 + c_1 = 3 < 4$, so the Cartan criterion is not met, and the theorem does not obtain.

5. Application: aggregate market demand of a 'small' number of consumers

We now come back to the economic integration problem described in introduction (see Shafer and Sonnenschein, 1982 for a general presentation, and Momi (2002) for a recent extension). A first remark, due to Sonnenschein (1973a,b), is that, in contrast with the excess demand problem, the characterization of market demand will face complex non-negativity restrictions. In particular, he exhibits a counter-example of a function $\boldsymbol{X}$ that cannot be globally decomposed as (1.1) because of these constraints. However, the local version of the problem remains. From the mathematical integration problem, discussed above, we already know necessary conditions, stating that for any point $\bar{\boldsymbol{p}}$, there must exist a subspace $\mathscr{S}$, of codimension K, such that the restriction to $\mathscr{S}$ of the Slutsky matrix of $\boldsymbol{X}$ is symmetric. An additional, necessary condition is that this restriction should furthermore be negative, as can readily be checked; these necessary conditions have already be described by Diewert (1977), Mantel (1977) and Geanakoplos and Polemarchakis (1980). We now describe a strategy to prove that they are locally sufficient, a result that is new.

Consider the space $E = \{\boldsymbol{p}, \lambda_1, \dots, \lambda_K, \boldsymbol{\Delta}^1, \dots, \boldsymbol{\Delta}^K\} = \mathbb{R}^{n+K+K^2}$ (the vector $\boldsymbol{\Delta}^*$ will later be interpreted as the $D_{\boldsymbol{p}}V^i$). Clearly, if a solution exists, then the equations $\lambda_i = \lambda_i(\boldsymbol{p})$ and $\boldsymbol{\Delta}^* = \boldsymbol{\Delta}^*(\boldsymbol{p})$ define a (n-dimensional) submanifold $\mathscr{S}$ in E, and $\mathscr{S}$ is included in the K^2-dimensional manifold $\mathscr{M}$ defined by:

$$\begin{aligned} \boldsymbol{X}(\boldsymbol{p}) &= \sum_i \lambda_i \boldsymbol{\Delta}^i \\ \boldsymbol{p} \cdot \boldsymbol{\Delta}^* &= \frac{1}{\lambda_i}, \quad \forall i \end{aligned} \tag{5.1}$$

Conversely, assume that we have found functions $\lambda_i = \lambda_i(\boldsymbol{p})$ and $\boldsymbol{\Delta}^* = \boldsymbol{\Delta}^*(\boldsymbol{p})$ such that:

- for every $\boldsymbol{p}$, $(\boldsymbol{p}, \lambda_1(\boldsymbol{p}), \dots, \lambda_K(\boldsymbol{p}), \boldsymbol{\Delta}^1(\boldsymbol{p}), \dots, \boldsymbol{\Delta}^K(\boldsymbol{p})\}$ belong to $\mathscr{M}$;
- for every $i = 1, \dots, K$, $\boldsymbol{\Delta}^*(\boldsymbol{p})$ satisfies the equation:

$$\mathrm{d}\left(\sum_j \boldsymbol{\Delta}^{ij}\,\mathrm{d}p_j\right) = \sum_j \mathrm{d}\boldsymbol{\Delta}^{ij} \wedge \mathrm{d}p_j = \sum_{k<j} \left(\frac{\partial \boldsymbol{\Delta}^{ij}}{\partial p_k} - \frac{\partial \boldsymbol{\Delta}^{ik}}{\partial p_j}\right) \mathrm{d}p_k \wedge \mathrm{d}p_j = 0$$

Then $\boldsymbol{\Delta}^*(\boldsymbol{p})$ is the gradient of some function V^i and the $(V^1, \dots, V^K)$ solve the problem. In the language of the previous section, we are looking for an n-dimensional integral manifold of the exterior differential system:

$$\sum_j \mathrm{d}\boldsymbol{\Delta}^{ij} \wedge \mathrm{d}p_j = 0, \quad \forall i \tag{5.2}$$

Finally, the solution must be parameterized by $(p_1, \dots, p_n)$. The formal translation of this is:

$$\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n \neq 0 \tag{5.3}$$

An important remark, here, is that this system is closed, in the sense of the previous section (since all 1-forms involved are total differentials).

The idea, now, is to consider (5.2) and (5.3) as an exterior differential system to be solved on the manifold $\mathcal{M}$. Following the approach described in the previous section, the proof is in two steps.

- As a first step, one must find an integral element at $\bar{\boldsymbol{p}}$, which amounts to solving the *linearized* problem (at some given point $\bar{\boldsymbol{p}}$). Specifically, choose (arbitrarily) the values (at $\bar{\boldsymbol{p}}$) of λ_i and $\boldsymbol{\Delta}^* = D_{\boldsymbol{p}} V^i$. In particular, one may choose $\lambda_i < 0$, $\boldsymbol{\Delta}^* \leqslant 0$ and $\boldsymbol{\Delta} = (\boldsymbol{\Delta}^1, \dots, A^K)$ invertible; if these properties hold at $\bar{\boldsymbol{p}}$, they will hold by continuity on a neighborhood as well. Also, these values must satisfy (again at $\bar{\boldsymbol{p}}$) the relations:

$$\boldsymbol{X}(\bar{\boldsymbol{p}}) = \sum_i \lambda_i \boldsymbol{\Delta}^i$$
$$\boldsymbol{p} \cdot \boldsymbol{\Delta}^i = \frac{1}{\lambda_i}, \quad \forall i$$

- Now, linearize λ_i and $\boldsymbol{\Delta}^*$ (as functions of $\bar{\boldsymbol{p}}$) around $\bar{\boldsymbol{p}}$:

$$\frac{\partial \lambda_i}{\partial p_j} = N_j^i$$
$$\frac{\partial \boldsymbol{\Delta}_k^i}{\partial p_j} = M_{kj}^i$$

  Solving the linearized problem is equivalent to finding vectors N^i and matrices M^i that satisfy the integration equations, i.e., (5.2) and (5.3), plus the equations expressing that λ_i and Δ^i remain on the manifold $\mathcal{M}$; in addition, we want the V^i to be convex. Formally, we write that:
  - $\boldsymbol{\Delta}^i$ is the gradient of a convex function; this implies that M^i symmetric positive, $i = 1, \dots, K$;
  - we differentiate the relations (5.1)), which express the fact that λ_i and $\boldsymbol{\Delta}^i$ remain on the manifold $\mathcal{M}$:

$$D_{\boldsymbol{p}} \boldsymbol{X}(\bar{\boldsymbol{p}}) = \sum_i (\boldsymbol{\Delta}^i D_{\boldsymbol{p}} \lambda_i' + \lambda_i D_{\boldsymbol{p}} \boldsymbol{\Delta}^i) = \sum_i (\boldsymbol{\Delta}^i N_i' + \lambda_i M^i)$$
$$M^i \boldsymbol{p} + \boldsymbol{\Delta}^i = \frac{1}{\lambda_i^2} N_i \Leftrightarrow N_i' = -\lambda_i^2 (\boldsymbol{p}' M^i \boldsymbol{\Delta}^{i'})$$

  One can show that this linear system has a solution if and only if the necessary conditions described above hold true, a property that is linked to the results derived in the linear context by Diewert (1977) and Geanakoplos and Polemarchakis (1980);
- the second, and more tricky step is to show that the Cartan criterion is met. This is crucial in order to go from a solution to the linearized version at each point to a solution to the general, non-linear problem; a move that may not be possible otherwise, as illustrated by the counter-examples in the previous section.

A full proof will be given elsewhere. Once these conditions have been checked, the Cartan–Kähler theorem applies, to yield the following statement:

Theorem 5.1. *Consider some open set* $\mathscr{U}$ in $\mathbb{R}^n - \{0\}$ *and some analytic mapping* $\boldsymbol{X}$: $\mathscr{U} \to \mathbb{R}^n$ *such that* $\boldsymbol{p} \cdot \boldsymbol{X}(\boldsymbol{p}) = 1$. *Choose some* $\bar{\boldsymbol{p}} \in U$ *such that for all* $\boldsymbol{p}$ *in some neighborhood of* $\bar{\boldsymbol{p}}$, *the Slutsky matrix of* $\boldsymbol{X}$ *at* $\boldsymbol{p}$, $S\,(\boldsymbol{p})$, *is such that there exists a subspace* $\mathscr{E} \subset \mathbb{R}^n$ *of codimension K to which the restriction of* $S\,(\boldsymbol{p})$ *is symmetric and negative. Then for all* $(\bar{\boldsymbol{x}}_i, \dots, \bar{\boldsymbol{x}}_K) \in \mathbb{R}^{nK}$ *and* $(\bar{\lambda}_1, \dots, \bar{\lambda}_K) \in \mathbb{R}^K$ *that satisfy:*

$$\bar{\boldsymbol{x}}_1 + \cdots + \bar{\boldsymbol{x}}_K = \boldsymbol{X}(\bar{\boldsymbol{p}})$$
$$\forall i, \quad \lambda_i > 0$$

there exist K functions $U^1, \dots, U^K$, *where each* U_i *is defined in some convex neighborhood* $\mathscr{U}_\mathrm{i}$ of $\bar{\boldsymbol{x}}_i$ *and is analytic and strictly quasi-concave in* $\mathscr{U}_\mathrm{i}$, *K mappings* $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_K)$ *and K functions* $(\lambda_1, \dots, \lambda_K)$, *all defined in some neighborhood V of* $\bar{\boldsymbol{p}}$ *and analytic in* $\mathscr{V}$, *such that, for all* $p \in \mathscr{V}$:

$$\boldsymbol{p} \cdot \boldsymbol{x}_i(\boldsymbol{p}) = 1/K, \quad i = 1, \dots, K$$
$$U_i(\boldsymbol{x}_i(\boldsymbol{p})) = \max\{U_i(\boldsymbol{x}) | \boldsymbol{x} \in \mathscr{U}_i, \quad \boldsymbol{p} \cdot \boldsymbol{x} \le 1/K\}, \quad i = 1, \dots, K$$
$$\frac{\partial U_i}{\partial x^j}(\boldsymbol{x}_i(\boldsymbol{p})) = \lambda_i(\boldsymbol{p}) p_j, \quad i = 1, \dots, K, \; j = 1, \dots, K$$
$$\sum_{i=1}^{K} \boldsymbol{x}_i(\boldsymbol{p}) = \boldsymbol{X}(\boldsymbol{p})$$
$$\boldsymbol{x}_i(\bar{\boldsymbol{p}}) = \bar{\boldsymbol{x}}_i, \quad i = 1, \dots, K$$
$$\lambda_i(\bar{\boldsymbol{p}}) = \bar{\lambda}_i, \quad i = 1, \dots, K$$

Note that, for $K \ge N$, there condition on the Slutsky matrix is automatically satisfied. We then get an earlier result of the authors (see Chiappori and Ekeland, 1999).

Note that both the individual demands and the Lagrange multipliers (i.e., each agent's marginal utility of income) can be freely chosen at $\bar{\boldsymbol{p}}$. In particular, non-negativity constraints can be forgotten, since one can choose individual demands to be strictly positive at $\bar{\boldsymbol{p}}$, and they will remain positive in a neighborhood.

References

Antonelli, G.B., 1886. Sulla teoria matematica della economia politica, Pisa.

Browning, M., Chiappori, P.A., 1994. Efficient Intra-Household Allocations: a General Characterization and Empirical Tests, Mimeo, DELTA.

Bryant, R.L., Chern, S.S., Gardner, R.B., Goldschmidt, H.L., Griffiths, P.A., 1991. Exterior Differential Systems, Springer-Verlag, New York.

Cartan, E., 1945. Les systèmes différentiels extérieurs et leurs applications géométriques, Hermann, Paris.

Chiappori, P.A., Ekeland, I., 1997. A Convex Darboux theorem. Annali della Scuola Normale Superiore di Pisa 4.25, 287–297.

Chiappori, P.A., Ekeland, I., 1999. Aggregation and market demand: an exterior differential calculus viewpoint. Econometrica 67 (6), 1435–1458.

Chipman, J., Hurwicz, L., Sonnenschein, H. (Eds.), 1971. Preferences, Utility, and Demand. Harcourt Brace Jovanovich, New York.

Debreu, G., 1974. Excess demand functions. Journal of Mathematical Economics 1, 15–23.

Diewert, W.E., 1977. Generalized Slutsky conditions for aggregate consumer demand functions. Journal of Economic Theory 15, 353–362.

Ekeland, I., Nirenberg, L., 2000. The Convex Darboux Theorem. Methods and Applications of Analysis.

Geanakoplos, J., Polemarchakis, H., 1980. On the disaggregation of excess demand functions. Econometrica, 315–331.

McFadden, D., Mas-Colell, A., Mantel, R., Richter, M., 1974. A characterization of community excess demand functions. Journal of Economic Theory 7, 361–374.

Mantel, R., 1974. On the characterization of aggregate excess demand. Journal of Economic Theory 7, 348–353.

Mantel, R., 1976. Homothetic preferences and community excess demand functions. Journal of Economic Theory 12, 197–201.

Mantel, R., 1977. Implications of economic theory for community excess demand functions, Cowles Foundation Discussion Paper no. 451, Yale University.

Momi, T., 2002. Excess Demand Functions with Incomplete Markets—A Global Result, Mimeo, University of Tokyo.

Russell, T., 1994. How Quasi-Rational Are You? Mimeo, Santa Clara University.

Russell, T., Farris, F., 1993. The geometric structure of some systems of demand functions. Journal of Mathematical Economics 22, 309–325.

Sato, R., 1981. Theory of Technical Change and Economiuc Invariance. Academic Press (new edition, Edward Elgar, 1999).

Sato, R., Ramachandran, R., 1998. Symmetry and Economic Invariance, An Introduction. Kluwer.

Shafer, W., Sonnenschein, H., 1982. Market demand and excess demand functions. In: Arrow, K., Intriligator, M. (Eds.), Handbook of Mathematical Economics, vol. 2, North Holland, Amsterdam, Chapter 14, pp. 670–693.

Sonnenschein, H., 1973a. Do Walras' identity and continuity characterize the class of community excess demand functions. Journal of Economic Theory, 345–354.

Sonnenschein, H., 1973b. The utility hypothesis and market demand theory. Western Economic Journal 11, 404–410.

Sonnenschein, H., 1974. Market excess demand functions. Econometrica 40, 549–563.