Economic Theory **25**, 933–956 (2005)
DOI: 10.1007/s00199-004-0508-2

Economic Theory

Intertemporal substitution, risk aversion and ambiguity aversion*

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Received: August 5, 2003; revised version: March 12, 2004

Summary. This paper axiomatizes a form of recursive utility on consumption processes that permits a role for ambiguity as well as risk. The model has two prominent special cases: (i) the recursive model of risk preference due to Kreps and Porteus [18]; and (ii) an intertemporal version of multiple-priors utility due to Epstein and Schneider [8]. The generalization presented here permits a three-way separation of intertemporal substitution, risk aversion and ambiguity aversion.

Keywords and Phrases: Generalized recursive multiple-priors utility, Risk aversion, Ambiguity aversion.

JEL Classification Numbers: D80, D81, D90.

1 Introduction

1.1 Outline

This paper axiomatizes a form of recursive utility on consumption processes that permits a role for ambiguity as well as risk. Moreover, the model allows for the separation of three properties of preference, namely, intertemporal substitution, risk aversion and ambiguity aversion.

The model has two prominent special cases: (i) the recursive model of risk preference by Kreps and Porteus [18], Epstein and Zin [10]; (ii) the intertemporal model of multiple-priors utility by Epstein and Schneider [8], which is a dynamic extension of the atemporal model by Gilboa and Schmeidler [14]. Recursive utility permits the disentangling of risk aversion from intertemporal substitution, whereas

* I am grateful to Larry Epstein for his guidance and invaluable advice, and to a referee for helpful comments and suggestions.

they are entangled in the standard intertemporally additive model of expected utility. However, the risk models contain only unambiguous objects of choice. As is shown by the Ellsberg Paradox [5], the distinction of ambiguity from risk is behaviorally meaningful and intuitive. The Epstein-Schneider form of intertemporal multiple-priors utility permits a role for ambiguity, and it permits the distinction of ambiguity aversion from risk aversion or intertemporal substitution. But it does not allow the distinction between risk aversion and intertemporal substitution.

This paper generalizes the above two models to permit a role for ambiguity *and* to permit the three-way separation. We call the resulting model *generalized recursive multiple-priors utility*. For a consumption process c, the conditional utility at time t and history ω^t is expressed in the recursive form

$$U_{t,\omega^t}(c) = W\left(c_t,\ \phi^{-1} \circ \min_{\mu \in M_{t,\omega^t}} E_\mu \left[\phi \circ U_{t+1,\omega^{t+1}}(c)\right]\right), \tag{1}$$

where M_{t,ω^t} is the set of priors at (t, ω^t) over 'one-step-ahead' events, W aggregates current consumption and a certainty equivalent of the future, and where ϕ is used to compute the certainty equivalent. This is the discrete-time counterpart of the model by Chen and Epstein [3] developed in the continuous-time setting.

To interpret further, note that in the certainty case (c is deterministic), the form reduces to

$$U_t(c) = W(c_t, U_{t+1}(c)),$$

which is the stationary utility by Koopmans [17]. The case of all M_{t,ω^t}'s being singletons (denoted by μ_{t,ω^t}'s) corresponds to a subjective version of Kreps-Porteus model in the form,

$$U_{t,\omega^t}(c) = W\left(c_t,\ \phi^{-1} \circ E_{\mu_{t,\omega^t}} \left[\phi \circ U_{t+1,\omega^{t+1}}(c)\right]\right).$$

Finally, when W is linear in the second argument and ϕ is linear, the model coincides with the Epstein-Schneider model

$$U_{t,\omega^t}(c) = u(c_t) + \alpha \min_{\mu \in M_{t,\omega^t}} E_\mu \left[U_{t+1,\omega^{t+1}}(c)\right].$$

The paper consists of three parts. First, we construct a hierarchical domain of choice that is suitable for our analysis. The objects of choice are called *compound lottery-acts*. They are dynamic counterparts of horse-race roulette-wheel acts introduced by Anscombe and Aumann [1]. A compound lottery-act is a random variable that maps today's state of the world into a joint lottery over current consumption and a compound lottery-act for tomorrow.

Next, we consider a process of conditional preferences, and axiomatize the representation (1). The main axioms are roughly that the preference process is dynamically consistent and satisfies at each history (i) the Gilboa-Schmeidler [14] axioms for the choice of 'one-step-ahead' acts, and (ii) the independence axiom for the resolution of 'timeless gambles.' Also we describe how recursive utility and intertemporal multiple-priors utility can be characterized axiomatically as special cases.

Finally, we show how our model allows the noted three-way separation. Proofs and some technical arguments are collected in the appendix.

1.2 Related literature

A subjective version of recursive utility is axiomatized by Skiadas [21] using a different approach. His model uses a domain of consumption processes which allows variable information structures. Klibanoff [16] axiomatizes an intertemporal model of utility that allows a role for ambiguity. He considers preferences over a hierarchical domain consisting of pairs of current consumption and a menu of acts for the future.

Representation results similar to ours are presented by Wang [22]. He axiomatizes several classes of utility that allow a role for ambiguity as well as risk. He considers preferences over the dynamic counterpart of Savage acts, but in which the information structure is variable and consumption plans may not be adapted.

In this paper, we fix the information structure, and also we adopt an Anscombe-Aumann approach that allows randomization through the presence of objective lotteries. Incorporating lotteries into the model is the cost we pay.[1]

2 The domain

Consider an infinite horizon discrete-time model. Let Ω be a finite set, which is taken to be the state space for each period. Thus, the full state space is Ω^∞. The consumption space for each period is denoted by C. It is assumed to be a compact metric space. For any compact metric space X, let $\mathcal{B}(X)$ denote the family of its Borel subsets, and $\Delta(X)$ denote the set of Borel probability measures over X. It is also a compact metric space with respect to the Prokhorov metric. Let $L(X) = X^\Omega$ denote the set of functions $h : \Omega \to X$. It is also compact metric with respect to the product topology. Thus, $L(\Delta(X))$ denotes the domain of lottery-acts, or Anscombe-Aumann acts over X.

The domain of *compound lottery-acts*, $\mathcal{H}$, is constructed in the following way. Inductively define a family of lottery-act domains $\{\mathcal{H}_0, \mathcal{H}_1, ...\}$ by

$$\begin{aligned}
\mathcal{H}_0 &= L(\Delta(C)) \\
\mathcal{H}_1 &= L(\Delta(C \times \mathcal{H}_0)) \\
&\cdots \\
\mathcal{H}_t &= L(\Delta(C \times \mathcal{H}_{t-1}))
\end{aligned}$$

and so on. An element of $\mathcal{H}_t$ is called a t-stage act. By induction, $\Delta(C \times \mathcal{H}_{t-1})$ is a compact metric space and $\mathcal{H}_t$ is also, for every $t \geq 0$. Let $\mathcal{H}^* = \prod_{t=0}^\infty \mathcal{H}_t$.

[1] Ghirardato et al. [13] establish a system of axioms in which one can formulate 'subjective mixture' of acts without randomization. Adopting their approach might make it possible to get our result without incorporating lotteries.

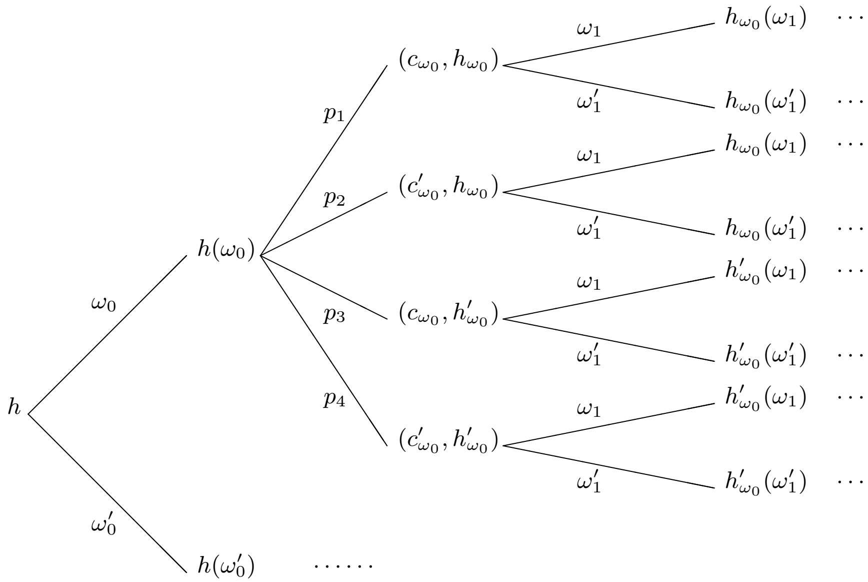

Figure 1. Compound lottery-act h

This is the set of sequences of finite-stage acts. It is a compact metric space with respect to the product metric. We require each sequence of acts to be *coherent*, that is, the first t-stage consumption process induced by a $t+1$-stage act must coincide with the t-stage act. The domain of coherent acts, a subset of $\mathcal{H}^*$, is denoted by $\mathcal{H}$. We put the details of the definition of coherent acts and formal construction of the domain in the appendix.

The domain $\mathcal{H}$ satisfies a homeomorphism analogous to those shown in [4, 10, 22].

Theorem 1 $\mathcal{H} \simeq L(\Delta(C \times \mathcal{H}))$.

Thus a compound lottery-act may be identified as a random variable that maps today's state of the world into a joint lottery over current consumption and a compound lottery-act for tomorrow. Figure 1 shows how a compound lottery-act is represented graphically.

By the above identification, the domain $\mathcal{H}$ is well-defined as a mixture space. For $h, h' \in \mathcal{H}$ and $\lambda \in [0,1]$, the mixture $\lambda h \oplus (1-\lambda)h' \in \mathcal{H}$ is defined by state-wise mixture,

$$[\lambda h \oplus (1-\lambda)h'](\omega) \equiv \lambda h(\omega) + (1-\lambda)h'(\omega) \;\text{ for each }\; \omega \in \Omega.$$

Note that $\mathcal{H}$ contains the subdomains D, $\mathcal{G}$, $\mathcal{F}$ where:

(i) multi-stage lotteries

$$D \simeq \Delta(C \times D),$$

(ii) adapted processes of consumption lotteries

$$\mathcal{G} \simeq L(\Delta(C) \times \mathcal{G}),$$

(iii) adapted consumption processes

$$\mathcal{F} \simeq L(C \times \mathcal{F}).$$

Relations among the subdomains are expressed as,

$$\begin{array}{ccccccc} \mathcal{H} & & \supset & & \mathcal{G} & \supset & \mathcal{F} \\ \cup & & & & \cup & & \cup \\ D & \supset & \Delta(C^\infty) & \supset & [\Delta(C)]^\infty & \supset & C^\infty. \end{array}$$

The subdomain D is attained by taking constant acts. The class of recursive utility for the risk case ([10,4]) is established over this domain. A multi-stage lottery $d \in D$ is viewed as a lottery over current consumption and a multi-stage lottery for tomorrow.

The subdomain $\mathcal{G}$ is attained by randomizing current consumption only, and not the act. The domain adopted by Epstein-Schneider [8] corresponds to this in our setting. One obtains adapted processes in which the values are consumption lotteries. The critical difference between the domain $\mathcal{H}$ and the subdomain $\mathcal{G}$ is that in $\mathcal{H}$ we allow randomization of both act and current consumption. This permits the distinction between ‘mixing’ acts by randomization before the realization of one-step-ahead uncertainty, and mixing by randomizing outcomes after the realization, whereas this distinction is precluded in the subdomain $\mathcal{G}$ (see Fig. 2). The richer domain $\mathcal{H}$ is essential for our axiomatization of a utility function that provides the three-way separation described above. This parallels closely the fact that in the risk context, recursive utility, which provides the two-way separation of both attitudes towards risk and attitudes towards variability across time, has been axiomatized only in the domain D of multi-stage lotteries rather than in the subdomain $\Delta(C^\infty)$.

As discussed later, if the decision maker is indifferent between the two ways of mixing noted above, it turns out that her preference can be represented only by subjective expected utility. That is, there is no role for ambiguity, and no distinction between risk aversion and intertemporal substitution is permitted.

Finally, $\mathcal{F}$ is attained by taking degenerate lotteries. It consists of adapted consumption processes.

We define another important subdomain. Let

$$\mathcal{H}_{+1} = \{h_{+1} \in L(\Delta(C \times \mathcal{H})) : \forall\, \omega \in \Omega,\ h_{+1}(\omega) \in D\}$$

be the subdomain consisting of acts in which uncertainty resolves in one period. By the above recursive homeomorphism, we can embed $\mathcal{H}_{+1}$ into $\mathcal{H}$. In choosing an act from this subdomain, the decision maker faces subjective uncertainty just for one-step-ahead movement of the world. Thus, we call it the domain of *one-step-ahead* acts.

Finally, the domain of objects of choice is given by $\Delta\,(C \times \mathcal{H})$. That is, it consists of lotteries over current consumption and compound lottery-act.

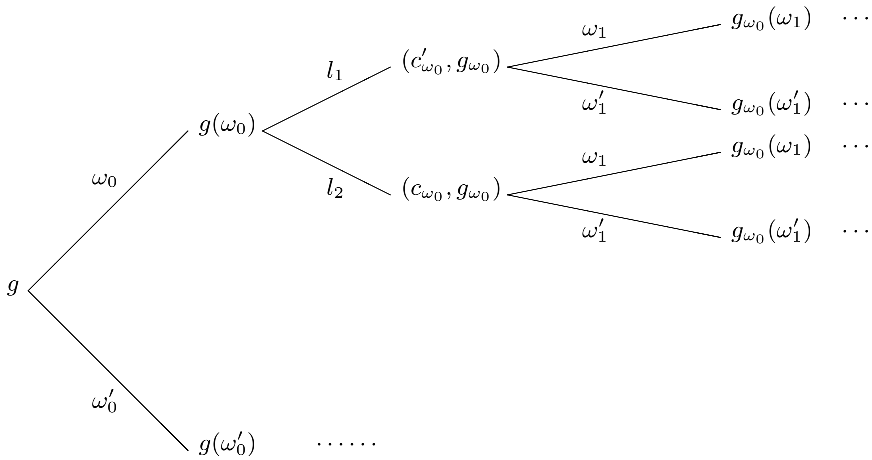

Figure 2. Adapted process of consumption lottery g

We introduce more notation. When no confusion arises, the degenerate lottery that gives a consumption sequence $y \in C^\infty$, is simply denoted y. The lottery that gives a current consumption $c \in C$ with certainty and distribution $p_1 \in \Delta(\mathcal{H})$ over compound lottery-act, is denoted $(\delta[c], p_1)$. The degenerate lottery that gives current consumption c and a compound lottery-act $h \in \mathcal{H}$, is denoted $\delta[c, h]$. The degenerate lottery that gives current consumption c and a compound lottery (beginning at next period) $d \in D$ is denoted $\delta[c, d]$. When no confusion arises, for example $\delta[c, h]$ is written more simply as (c, h).

3 Axioms and representation theorems

3.1 General model

At each time t and history $\omega^t = (\omega_1, ..., \omega_t) \in \Omega^t$, the decision maker has a preference ordering $\succsim_{\omega^t}$ on $\Delta(C \times \mathcal{H})$. Thus, consider a process of preferences $\{\succsim_{\omega^t} : \omega^t \in \Omega^t, t \geq 0\}$.

First, we assume weak order (complete and transitive), continuity and sensitivity.

Axiom 1 (Order): For any $\omega^t \in \Omega^t$, $\succsim_{\omega^t}$ is a continuous weak order over $\Delta(C \times \mathcal{H})$, and there exist $y, y' \in C^\infty$ such that $y \succ_{\omega^t} y'$.

The second axiom says that preference over temporal acts is independent of current consumption.

Axiom 2 (Consumption separability): For any $\omega^t \in \Omega^t$ and any $c, c' \in C$, $h, h' \in \mathcal{H}$,

$$\delta[c, h] \succsim_{\omega^t} \delta[c, h'] \quad \text{if and only if} \quad \delta[c', h] \succsim_{\omega^t} \delta[c', h']$$

Third, we assume that preference over risky consumption is independent of history and unchanged by a time delay.

Axiom 3 (Risk preference): For any $\omega^t, \omega'^t \in \Omega^t$ and any $c \in C$, $d, d' \in D$;

(i) (History-Independence)

$$d \succsim_{\omega^t} d' \qquad \text{if and only if} \qquad d \succsim_{\omega'^t} d',$$

(ii) (Stationarity)

$$\delta[c,d] \succsim_{\omega^t} \delta[c,d'] \qquad \text{if and only if} \qquad d \succsim_{\omega^t} d'.$$

The first states that history affects choice only through subjective uncertainty, and hence does not affect the ranking of risky prospects. The latter says that the ranking of risky consumption streams is unchanged if they are postponed (see Koopmans [17]).

Axiom 4 (Risk equivalence preservation): For any $\omega^t \in \Omega^t$, and any $p, p' \in \Delta(C \times \mathcal{H})$ and $d, d' \in D$, $\lambda \in (0,1)$,

$$\begin{array}{ll} \text{if} & p \sim_{\omega^t} d,\ p' \sim_{\omega^t} d', \\ \text{then} & \lambda p + (1-\lambda)p' \sim_{\omega^t} \lambda d + (1-\lambda)d'. \end{array}$$

We assume that risk equivalence is preserved by taking mixtures. This means that 'timeless gambles' over elements of $\Delta(C \times \mathcal{H})$ are equivalent to those over corresponding risk equivalents.

The choice of *one-step-ahead acts* (acts in $\mathcal{H}_{+1}$) is made in the manner of multiple-priors utility. That is, assume the following version of Gilboa-Schmeidler [14] axioms at each history[2].

Axiom 5 (One-step-ahead multiple-priors): For any $\omega^t \in \Omega^t, c \in C, h_{+1}, h'_{+1} \in \mathcal{H}_{+1}$, and $d \in D$, $\lambda \in (0,1)$:

(i) (Certainty Independence)

$$\begin{array}{l} \delta[c, h_{+1}] \succsim_{\omega^t} \delta[c, h'_{+1}] \\ \text{if and only if} \qquad \delta[c, \lambda h_{+1} \oplus (1-\lambda)d] \succsim_{\omega^t} \delta[c, \lambda h'_{+1} \oplus (1-\lambda)d]; \end{array}$$

(ii) (Ambiguity Aversion)

$$\begin{array}{ll} \text{if} & \delta[c, h_{+1}] \sim_{\omega^t} \delta[c, h'_{+1}], \\ \text{then} & \delta[c, \lambda h_{+1} \oplus (1-\lambda)h'_{+1}] \succsim_{\omega^t} \delta[c, h_{+1}]. \end{array}$$

[2] A similar analysis is possible also when we replace the One-step-ahead Multiple-priors axiom by comonotonic independence (Schmeidler [20]) applied to one-step-ahead acts

This axiom characterizes attitude towards ambiguity about one-step-ahead events. By condition (i), we assume that mixing with a risky prospect is neutral to ambiguity hedging. Since risky prospects do not involve subjective uncertainties, attitude toward ambiguity seems irrelevant to such mixing. Condition (ii) says that the decision maker prefers hedging ambiguity by smoothing variation of acts.

Finally, we impose dynamic consistency.

Axiom 6 (Dynamic consistency): For any $\omega^t \in \Omega^t$ and any $c \in C$, $h, h' \in \mathcal{H}$,

$$\begin{array}{lll} \text{if} & h(\omega) \succsim_{\omega^t,\omega} h'(\omega) & \text{for every} \quad \omega \in \Omega, \\ \text{then} & \delta[c,h] \succsim_{\omega^t} \delta[c,h'], & \end{array}$$

and the latter relation is strict if $h(\omega) \succ_{\omega^t,\omega} h'(\omega)$ for some ω.

The axiom states first that an act whose (interim) outcome is preferred at every new situation should be preferred ex-ante. This excludes preference reversal by arrival of new information. Additionally, if such act has an outcome strictly preferred at some new situation, it must be strictly preferred ex-ante. This implies the full-support property discussed later, which means that the decision maker conceives every state tomorrow to be possible.

We state the main theorem.

Theorem 2 *(Existence): $\{\succsim_{\omega^t}\}$ satisfies Axioms 1–6 if and only if there exists a family of utility functions $\{U_{\omega^t}\}$, where $U_{\omega^t} : \Delta(C \times \mathcal{H}) \to R$ is continuous and non-constant, such that U_{ω^t} represents $\succsim_{\omega^t}$ for each ω^t and has the form*

$$U_{\omega^t}(p) = \int_{C\times\mathcal{H}} W\left(c, \min_{\mu \in M_{\omega^t}} \int_{\Omega} U_{\omega^t,\omega}(h(\omega))\, d\mu(\omega)\right) dp(c,h) \tag{1}$$

for $p \in \Delta(C \times \mathcal{H})$. Here, $W : C \times R_U \to R_U$ is continuous and strictly increasing in the second argument where $R_U = \bigcup_{\omega^t} range_{p\in\Delta(C\times\mathcal{H})} U_{\omega^t}(p)$. M_{ω^t} is a closed convex subset of $\Delta(\Omega)$, and every element of M_{ω^t} has full-support.

(Uniqueness): Moreover, $\{\succsim_{\omega^t}\}$ is represented by both $(\{U^1_{\omega^t}, M^1_{\omega^t}\}, W^1)$ and $(\{U^2_{\omega^t}, M^2_{\omega^t}\}, W^2)$ iff there exist constants A, B such that $A > 0$ and

$$\begin{aligned} U^2_{\omega^t} &= AU^1_{\omega^t} + B \\ W^2(c, AV^1 + B) &= AW^1(c, V^1) + B \\ M^2_{\omega^t} &= M^1_{\omega^t} \qquad \textit{for each } \omega^t \end{aligned}$$

Remark 1 If we strengthen axiom 5 by requiring subjective expected utility (in the sense of Anscombe-Aumann [1]) for one-step-ahead acts, then the set M_{ω^t} reduces to a singleton, which delivers a subjective version of recursive utility (Kreps-Porteus [18] and Epstein-Zin [10]).

Since risk preference is independent of histories, we denote its representation by $U : D \to R$, where $U_{\omega^t}|_D = U$ for every ω^t.

Over the domain of adapted consumption processes $C \times \mathcal{F}$, the representation reduces to

$$U_{\omega^t}(c, f) = W\left(c, \min_{\mu \in M_{\omega^t}} \int_{\Omega} U_{\omega^t,\omega}(f(\omega))\, d\mu(\omega)\right)$$

for $(c, f) \in C \times \mathcal{F}$.

3.2 Recursive multiple-priors utility with additive aggregator

In this subsection, we characterize the case of an additive aggregator. The resulting subclass of utility functions coincides with the recursive multiple-priors model in Epstein-Schneider [8] over the subdomain $\Delta(C) \times \mathcal{G}$.

First, we add an axiom that imposes indifference to the timing of resolution of risk.

Axiom 7 (Risk timing-indifference): For any $\omega^t, c \in C$, and $d, d' \in D, \lambda \in (0, 1)$,

$$\delta[c, \lambda d \oplus (1-\lambda)d'] \sim_{\omega^t} (\delta[c], \lambda\delta[d] + (1-\lambda)\delta[d']).$$

Second, assume that risk trade-offs of current and immediate future consumptions are independent of future consumption levels. For a two-stage consumption lottery $d_{0,1} \in \Delta(C \times \Delta(C))$ and a consumption sequence $y \in C^\infty$, let $(d_{0,1}, y)$ denote a consumption process which gives $d_{0,1}$ for the first two periods and y after that.

Axiom 8 (Future separability): For any ω^t and $d_{0,1}, d'_{0,1} \in \Delta(C \times \Delta(C))$, $y, y' \in C^\infty$,

$$(d_{0,1}, y) \succsim_{\omega^t} (d'_{0,1}, y) \quad \text{if and only if} \quad (d_{0,1}, y') \succsim_{\omega^t} (d'_{0,1}, y').$$

Corollary 1 $\{\succsim_{\omega^t}\}$ *satisfies Axioms 1-8 if and only if the function* $U_{\omega^t} : \Delta(C \times \mathcal{H}) \to R$ *in Eq. (1) has the form*

$$U_{\omega^t}(p) = \int_{C \times \mathcal{H}} u(c) + \alpha \min_{\mu \in M_{\omega^t}} \int_{\Omega} U_{\omega^t,\omega}(h(\omega))\, d\mu(\omega)\, dp(c, h) \qquad (2)$$

where u is continuous on C and non-constant, and $\alpha \in (0, 1)$. Moreover, u is unique up to positive affine transformations and α and $\{M_{\omega^t}\}$ are unique.

Remark 2 The representation (2) coincides with the recursive multiple-priors utility by Epstein-Schneider over the subdomain $\Delta(C) \times \mathcal{G}$. If we strengthen Axiom 5 as in Remark 1, then the set M_{ω^t} reduces to a singleton, which delivers the standard intertemporally additive subjective expected utility. Finally, if Axiom 8 is delited, then (2) remains valid except that the discount factor α is function of current consumption (see [6] for the risk case).

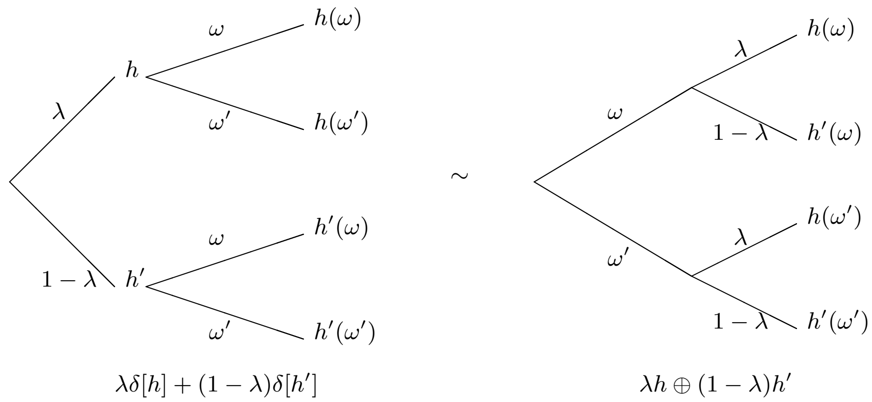

Figure 3. Act-timing indifference

Interchangeability of randomization of acts and realization of one-step-ahead uncertainty immediately excludes the role for ambiguity. In fact, a stronger form of timing-indifference implies subjective expected utility.

Axiom 9 (Act timing-indifference): For any $\omega^t, h_{+1}, h'_{+1} \in \mathcal{H}_{+1}$, and $\lambda \in (0, 1)$,

$$\delta[c, \lambda h_{+1} \oplus (1-\lambda)h'_{+1}] \sim_{\omega^t} (\delta[c], \lambda\delta[h_{+1}] + (1-\lambda)\delta[h'_{+1}]).$$

This timing-indifference condition states that the decision maker is indifferent in when to mix acts (see Fig. 3). Under Dynamic Consistency, Certainty-Independence and Risk-Equivalent Preservation, the Act Timing-Indifference condition implies One-step-ahead Subjective Expected Utility and Risk Timing-Indifference. That is, there is no role for ambiguity and no distinction between risk aversion and intertemporal substitution is permitted, if the decision maker is indifferent in the timing of mixing acts. Therefore, the addition of Act Timing-Indifference and Future Separability leads to intertemporally additive subjective expected utility. This observation is consistent with results by Epstein and LeBreton [7], and Ghirardato [11] that dynamic consistency together with other 'mild' axioms imply probabilistic sophistication (or more strongly, the Sure Thing Principle) when acts take values only over final outcomes.

Figure 4 shows the relation among the representation results.

4 Disentangling risk aversion and ambiguity aversion

In this section, we show how our model allows the three way distinction of intertemporal substitution, risk aversion and ambiguity aversion.

First, define comparative ambiguity aversion following Ghirardato-Marinacci [12], for one-step-ahead acts. Say that $\succsim^*_{\omega^t}$ is more ambiguity averse than $\succsim_{\omega^t}$ if for any $d \in D$ and $h_{+1} \in \mathcal{H}_{+1}$,

$$(c, d) \succsim_{\omega^t} (c, h_{+1}) \text{ implies } (c, d) \succsim^*_{\omega^t} (c, h_{+1}).$$

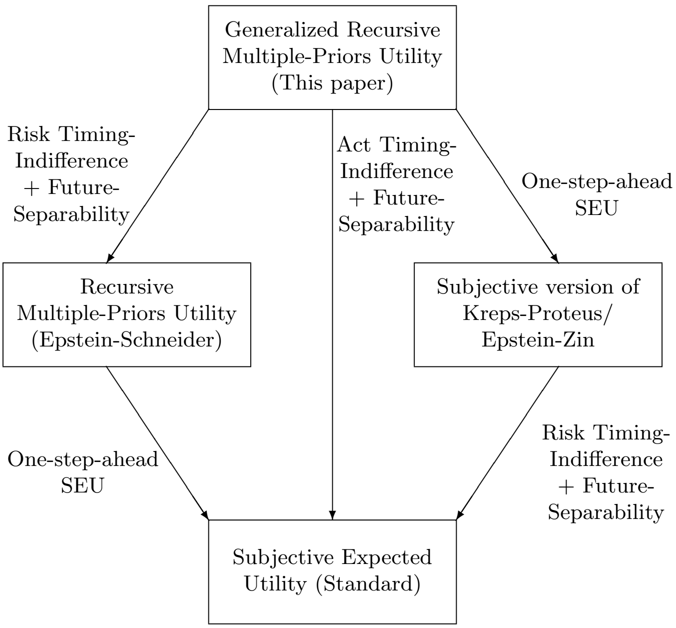

Figure 4. Relation among the results

Say that $\{\succsim^*_{\omega^t}\}$ is *more ambiguity averse* than $\{\succsim_{\omega^t}\}$ if $\succsim^*_{\omega^t}$ is more ambiguity averse than $\succsim_{\omega^t}$ for every ω^t. The interpretation is that the two preferences coincide over risky prospects, whereas at each history any one-step-ahead act disliked by one is disliked by the more ambiguity averse one.

Next, define comparative risk aversion. Say that $\{\succsim^*_{\omega^t}\}$ is *more risk averse* than $\{\succsim_{\omega^t}\}$ if for every ω^t and any $y \in C^\infty$, $l \in \Delta(C^\infty)$,

$$y \succsim_{\omega^t} l \text{ implies } y \succsim^*_{\omega^t} l.$$

This says that the two preferences coincide over deterministic consumption streams (i.e., exhibit the same attitude toward intertemporal substitution), but any timeless risky prospect disliked by one is disliked by the more risk averse one.

Notice that these comparisons are *partial* in the sense that the degree of ambiguity (risk) aversion is comparable only between the preferences with the same ranking over risky prospects (deterministic consumptions).

The theorem below shows that ambiguity aversion is captured by size of the prior sets, and risk aversion is captured by curvature of the aggregators.

Theorem 3 *Maintain Axioms 1–6. Let* $(\{U^*_{\omega^t}, M^*_{\omega^t}\}, W^*)$ *and* $(\{U_{\omega^t}, M_{\omega^t}\}, W)$ *for* $\{\succsim^*_{\omega^t}\}$ *and* $\{\succsim_{\omega^t}\}$ *represent* $\{\succsim^*_{\omega^t}\}$ *and* $\{\succsim_{\omega^t}\}$*, respectively. Then, (i)* $\{\succsim^*_{\omega^t}\}$ *is more ambiguity averse than* $\{\succsim_{\omega^t}\}$ *if and only if* $M^*_{\omega^t} \supset M_{\omega^t}$ *at any* ω^t *and there exist constants* A, B *such that* $A > 0$ *and* $U^* = AU + B$, $W^*(c, AV + B) = AW(c, V) + B$*; and (ii)* $\{\succsim^*_{\omega^t}\}$ *is more risk averse than* $\{\succsim_{\omega^t}\}$ *if and* *only if there exists an increasing concave function* ϕ *such that* $U^* = \phi[U]$ *and* $W^*(c, \phi[V]) = \phi[W(c, V)]$.

Appendix

A Construction of the domain

Inductively define the family of lottery-act domains $\{\mathcal{H}_0, \mathcal{H}_1, \cdots\}$ by

$$\begin{aligned} \mathcal{H}_0 &= L(\Delta(C)) \\ \mathcal{H}_1 &= L(\Delta(C \times \mathcal{H}_0)) \\ &\cdots \\ \mathcal{H}_t &= L(\Delta(C \times \mathcal{H}_{t-1})) \end{aligned}$$

and so on. By induction, $\Delta(C \times \mathcal{H}_{t-1})$ is a compact metric space and $\mathcal{H}_t$ is also, for every $t \geq 0$. Let d_t be the metric over $\mathcal{H}_t$. Let $\mathcal{H}^* = \prod_{t=0}^{\infty} \mathcal{H}_t$. This is a compact metric space with respect to the product metric $d(h, h') = \sum_{t=0}^{\infty} d_t(h_t, h'_t)\alpha^t$ for $\alpha \in (0, 1)$.

We construct the domain of coherent acts. Define a mapping $G_0 : C \times \mathcal{H}_0 \to C$ by

$$G_0(c, h_0) = c$$

and a mapping $H_0 : \mathcal{H}_1 \to \mathcal{H}_0$ by

$$H_0(h_1)(\omega)[B_0] = h_1(\omega)[G_0^{-1}(B_0)]$$

for each $h_1 \in \mathcal{H}_1$ and $\omega \in \Omega$ and $B_0 \in \mathcal{B}(C)$. An element of $\mathcal{H}_0$ is a one-stage lottery-act, whereas that of $\mathcal{H}_1$ is a two-stage act. The relation $h_0 = H_0(h_1)$ imposes that the first-stage consumption given by the two-stage act h_1 has to coincide with the one-stage act h_0.

Similarly, define $G_1 : C \times \mathcal{H}_1 \to C \times \mathcal{H}_0$ by

$$G_1(c, h_1) = (c, H_0(h_1))$$

and $H_1 : \mathcal{H}_2 \to \mathcal{H}_1$ by

$$H_1(h_2)(\omega)[B_1] = h_2(\omega)[G_1^{-1}(B_1)]$$

for each $h_2 \in \mathcal{H}_2$, $\omega \in \Omega$, and $B_1 \in \mathcal{B}(C \times \mathcal{H}_0)$.

Inductively, define $G_t : C \times \mathcal{H}_t \to C \times \mathcal{H}_{t-1}$ by

$$G_t(c, h_t) = (c, H_{t-1}(h_t))$$

and $H_t : \mathcal{H}_{t+1} \to \mathcal{H}_t$ is defined by

$$H_t(h_{t+1})(\omega)[B_t] = h_{t+1}(\omega)[G_t^{-1}(B_t)]$$

for each $h_{t+1} \in \mathcal{H}_{t+1}$, $\omega \in \Omega$, and $B_t \in \mathcal{B}(C \times \mathcal{H}_{t-1})$.

Let $\mathcal{H} = \{h = (h_0, h_1, h_2, \cdots) \in \mathcal{H}^* : h_t = H_t(h_{t+1}),\ t \geq 0\}$. This is the set of coherent acts. We call this the domain of *compound lottery-acts*.

B Proofs

B.1 Proof of Theorem 1

Proof of Theorem 1 is quite analogous to those of the recursive homeomorphism results in Epstein-Zin [10] and Wang [22]. We put it below just for completeness.

For each $\omega \in \Omega$, the sequence $(h_0(\omega), h_1(\omega), h_2(\omega), \cdots) \in \prod_{t=0}^{\infty} \Delta(C \times \mathcal{H}_{t-1})$ is viewed as a sequence of constant acts since

$$\begin{array}{rcl} h_0(\omega) \in & \Delta(C) & \subset \mathcal{H}_0 \\ h_1(\omega) \in & \Delta(C \times \mathcal{H}_0) & \subset \mathcal{H}_1 \\ & \cdots & \\ h_t(\omega) \in & \Delta(C \times \mathcal{H}_{t-1}) & \subset \mathcal{H}_t \end{array}$$

The lemma below says that they are also coherent.

Lemma 1 *For any $h \in \mathcal{H}$ and $\omega \in \Omega$, the sequence $(h_0(\omega), h_1(\omega), h_2(\omega), \cdots) \in \prod_{t=0}^{\infty} \Delta(C \times \mathcal{H}_{t-1})$ satisfies $h_t(\omega) = H_t(h_{t+1}(\omega))$ for each $t \geq 0$.*

Proof. For any $\omega' \in \Omega$,

$$\begin{aligned} & h_t(\omega)(\omega')[B_t] \\ \text{(by definition of } h_t(\omega) \text{ as a constant act)} &= h_t(\omega)[B_t] \\ \text{(since } h \in \mathcal{H}) &= H_t(h_{t+1})(\omega)[B_t] \\ \text{(by definition of } H_t) &= h_{t+1}(\omega)[G_t^{-1}(B_t)] \\ \text{(by definition of } h_{t+1}(\omega) \text{ as a constant act)} &= h_{t+1}(\omega)(\omega')[G_t^{-1}(B_t)] \\ \text{(by definition of } H_t) &= H_t(h_{t+1}(\omega))(\omega')[B_t] \end{aligned}$$

for any $B_t \in \mathcal{B}(C \times \mathcal{H}_{t-1})$. □

The next lemma says that if a sequence of acts is state by state coherent, it is coherent as a whole.

Lemma 2 *For any $t \geq 0$, $h_t \in \mathcal{H}_t$, $h_{t+1} \in \mathcal{H}_{t+1}$,*

$$[\forall \omega \in \Omega : h_t(\omega) = H_t(h_{t+1}(\omega))] \Longrightarrow h_t = H_t(h_{t+1})$$

Proof. Pick any $\omega \in \Omega$. Then, for any $B_t \in \mathcal{B}(C \times \mathcal{H}_{t-1})$ and $\omega' \in \Omega$,

$$\begin{aligned} & H_t(h_{t+1})(\omega)[B_t] \\ \text{(by definition of } H_t) &= h_{t+1}(\omega)[G_t^{-1}(B_t)] \\ \text{(by definition of } h_{t+1}(\omega) \text{ as a constant act)} &= h_{t+1}(\omega)(\omega')[G_t^{-1}(B_t)] \\ \text{(by definition of } H_t) &= H_t(h_{t+1}(\omega))(\omega')[B_t] \\ \text{(by Lemma 1)} &= h_t(\omega)(\omega')[B_t] \\ \text{(by definition of } h_t(\omega) \text{ as a constant act)} &= h_t(\omega)[B_t] \end{aligned}$$

Since ω is arbitrary, we have $H_t(h_{t+1}) = h_t$. □

The above two lemmata motivate the following definition.

$$Z = \left\{ \{z_t\} \in \prod_{t=0}^{\infty} \Delta(C \times \mathcal{H}_{t-1}) : z_t = H_t(z_{t+1}), t \geq 0 \right\}$$

where Z is endowed with the product topology.

Lemma 3 $\mathcal{H} \simeq L(Z)$.

Proof. Define $\varphi : \mathcal{H} \to L(Z)$ by

$$\varphi(h)(\omega) = (h_0(\omega), h_1(\omega), h_2(\omega), \cdots)$$

$\varphi(h) \in L(Z)$ follows from Lemma 1.
One to one: Suppose $\varphi(h) = \varphi(h')$. This means $(h_0(\omega), h_1(\omega), h_2(\omega), \cdots) = (h'_0(\omega), h'_1(\omega), h'_2(\omega), \cdots)$ for every $\omega \in \Omega$. Then, it must be that $h = h'$
Onto: Take any $\tilde{h} \in L(Z)$. Its value is given by

$$\tilde{h}(\omega) = (\tilde{h}_0(\omega), \tilde{h}_1(\omega), \tilde{h}_2(\omega), \cdots) \in \prod_{t=0}^{\infty} \Delta(C \times \mathcal{H}_{t-1})$$

for each $\omega \in \Omega$. Then, $\varphi^{-1}(\tilde{h}) = (h_0, h_1, h_2, \cdots) \in \mathcal{H}^*$ is given by $h_t(\omega) = \tilde{h}_t(\omega)$ for each t and ω. By Lemma 2, the sequence $(h_0, h_1, h_2, \cdots)$ is coherent and thus $\varphi^{-1}(\tilde{h}) \in \mathcal{H}$.

Continuity: Let $h^\nu \to h$ as $\nu \to \infty$. By nature of product topology in $\mathcal{H}$, $h_t^\nu(\omega) \to h_t(\omega)$ for every t and ω. By nature of product topology in Z, $(h_0^\nu(\omega), h_1^\nu(\omega), h_2^\nu(\omega), \cdots) \to (h_0(\omega), h_1(\omega), h_2(\omega), \cdots)$ for every ω, which means that $\varphi(h^\nu) \to \varphi(h)$. □

Now define the following object:

$$M^c = \left\{ \{m_t\} \in \prod_{t=0}^{\infty} \Delta\left(C \times \prod_{\tau=0}^{t-1} \mathcal{H}_\tau\right) : mrg_{C \times \prod_{\tau=0}^{t-1} \mathcal{H}_\tau} m_{t+1} = m_t, t \geq 0 \right\}$$

Lemma 4 *For any $\{m_t\} \in M^c$, there exist a unique $m \in \Delta(C \times \mathcal{H}^*)$ such that for any $t \geq 0$,*

$$mrg_{C \times \prod_{\tau=0}^{t-1} \mathcal{H}_\tau} m = m_t$$

Moreover, there exists a homeomorphism ψ, $M^c \overset{\psi}{\simeq} \Delta(C \times \mathcal{H}^*)$, and $\psi(\{m_t\})$ is the Kolmogorov extension of $\{m_t\}$.

Proof. It follows from Lemma 1 in Brandenberger and Dekel [2]. □

Define the following objects.

$$\mathcal{H}^t = \left\{ (h_0, ..., h_t) \in \prod_{\tau=0}^{t} \mathcal{H}_\tau : h_\tau = H_\tau(h_{\tau+1}),\ \tau = 0, ..., t-1 \right\}$$
$$M = \left\{ \{m_t\} \in M^c : m_{t+1}(C \times \mathcal{H}^t) = 1, t \geq 0 \right\}$$

Lemma 5 *$\psi(M) = \Delta(C \times \mathcal{H})$. As a result, $M \overset{\psi}{\simeq} \Delta(C \times \mathcal{H})$.*

Proof. '$\subset$' part: Let $m = \psi(\{m_t\})$ for some $\{m_t\} \in M$. Define $\Gamma_t = C \times \mathcal{H}^t \times \prod_{\tau=t+1}^{\infty} \mathcal{H}_\tau$ for each $t \geq 0$. Then, $C \times \mathcal{H} \subset \Gamma_t \subset C \times \mathcal{H}^*$ for every t, $\{\Gamma_t\}$ is decreasing, and $\bigcap_{t \geq 0} \Gamma_t = C \times \mathcal{H}$. Since m is the Kolmogorov extension of $\{m_t\}$,

$$m(\Gamma_t) = m_t(C \times \mathcal{H}^t) = 1$$

for any $t \geq 0$. Thus, $m(C \times \mathcal{H}) = m(\bigcap_{t \geq 0} \Gamma_t) = \lim m(\Gamma_t) = 1$.

'$\supset$' part: Pick any $m \in \Delta(C \times \mathcal{H})$ so that $m(C \times \mathcal{H}) = 1$. Let $\{m_t\}$ be the sequence of its marginals defined by $mrg_{C \times \prod_{\tau=0}^{t-1} \mathcal{H}_\tau} m = m_t$ for each $t \geq 0$. Then, $m_t(C \times \mathcal{H}^t) = m(\Gamma_t) = (\geq)1$. The second equality follows from $\Gamma_t \supset C \times \mathcal{H}$ for every $t \geq 0$. Since m is the unique Kolmogorov extension of $\{m_t\}$, $m = \psi(\{m_t\})$. □

Lemma 6 *For any $\{z_t\} \in Z$, there exist a unique $\{m_t\} \in M$ such that for any $t \geq 0$,*

$$mrg_{C \times \mathcal{H}_{t-1}} m_t = z_t$$

Moreover, there exists a homeomorphism ϕ, $Z \overset{\phi}{\simeq} M$, and $\phi(\{z_t\})$ gives the value obtained above.

Proof. Define a sequence of mappings $\{\zeta_t\}$, $\zeta_t : C \times \mathcal{H}_{t-1} \to C \times \prod_{\tau=0}^{t-1} \mathcal{H}_\tau$ for each t, by

$$\zeta_t(c, h_{t-1}) = (c, \widehat{h}_0, \cdots, \widehat{h}_{t-1})$$

where $\widehat{h}_{t-1} = h_{t-1}$, $\widehat{h}_\tau = H_\tau(\widehat{h}_{\tau+1})$ for $\tau = 0, \cdots, t-2$. Note that ζ_0, ζ_1 are identity mappings.

By construction, ζ_t is one to one and $\zeta_t(C \times \mathcal{H}_{t-1}) = C \times \mathcal{H}^{t-1}$.

Next, define a sequence of mappings $\{\xi_t\}$, $\xi_t : C \times \mathcal{H}^{t-1} \to C \times \mathcal{H}_{t-1}$ for each t, by

$$\xi_t(c, h_0, \cdots, h_{t-1}) = (c, h_{t-1})$$

Since ξ_t is a projection map, we can continuously extend it to $C \times \prod_{\tau=0}^{t-1} \mathcal{H}_\tau$. By construction, we have $\xi_t = \zeta_t^{-1}$ as a mapping from $C \times \prod_{\tau=0}^{t-1} \mathcal{H}_\tau$ to $C \times \mathcal{H}_{t-1}$, and $\zeta_t^{-1}(C \times \mathcal{H}^{t-1}) = C \times \mathcal{H}_{t-1}$.

For $\{z_t\} \in Z$, define the corresponding sequence of probability measures $\{m_t\} \in M$ by

$$m_t[A_t] = z_t[\zeta_t^{-1}(A_t)]$$

for $A_t \in \mathcal{B}(C \times \prod_{\tau=0}^{t-1} \mathcal{H}_\tau)$, for each $t \geq 0$. We see that $m \in M$ since $m_t(C \times \mathcal{H}^{t-1}) = z_t(\zeta_t^{-1}(C \times \mathcal{H}^{t-1})) = z_t(C \times \mathcal{H}_{t-1}) = 1$. By construction, $mrg_{C \times \mathcal{H}_{t-1}} m_t = z_t$ for any $t \geq 0$.

Define $\phi : Z \to M$ by the value obtained above.

One to one: Let $\phi(z) = \phi(z')$. Then, for every t and $A_t \in \mathcal{B}(C \times \prod_{\tau=0}^{t-1} \mathcal{H}_\tau)$, $z_t[\zeta_t^{-1}(A_t)] = z'_t[\zeta_t^{-1}(A_t)]$. Since ζ_t^{-1} is a projection, this implies that $z_t[B_t] = z'_t[B_t]$ for every $B_t \in \mathcal{B}(C \times \mathcal{H}_{t-1})$. Thus, $z_t = z'_t$ for every t, implying $z = z'$.

Onto: Pick any $\{m_t\} \in M$, then the corresponding $\{z_t\} \in Z$ is obtained by

$$z_t[B_t] = m_t[\xi_t^{-1}(B_t)]$$

for $B_t \in \mathcal{B}(C \times \mathcal{H}_{t-1})$.

Continuity: It follows from continuity of ζ_t^{-1} and the definition of weak convergence. □

Thus, $Z \overset{\phi}{\simeq} M \overset{\psi}{\simeq} \Delta(C \times \mathcal{H})$. Hence $Z \overset{\psi\circ\phi}{\simeq} \Delta(C \times \mathcal{H})$. Finally, it is easy to see that homeomorphism is preserved under finite products.

Lemma 7 *If $X \simeq Y$ for two compact metric spaces X and Y, then $L(X) \simeq L(Y)$ in the product topology.*

We complete the proof of Theorem 1.

Proof of Theorem 1. Since $Z \simeq \Delta(C \times \mathcal{H})$, we get $\mathcal{H} \simeq L(Z) \simeq L(\Delta(C \times \mathcal{H}))$ by the previous lemmata. □

Let $\mathcal{H}_{+1} = \{h_{+1} \in L(\Delta(C \times \mathcal{H})) : \forall \omega \in \Omega,\ h_{+1}(\omega) \in D\}$. An element of $\mathcal{H}_{+1}$ is an act such that uncertainty resolves in one period. We call it a one-step-ahead act. Since $\mathcal{H} \simeq L(\Delta(C \times \mathcal{H}))$, we can embed $\mathcal{H}_{+1}$ into $\mathcal{H}$. Inductively, define $\mathcal{H}_{+\tau} = \{h \in L(\Delta(C \times \mathcal{H})) : \forall \omega \in \Omega,\ h_{+\tau}(\omega) \in \Delta(C \times \mathcal{H}_{+(\tau-1)})\}$. An element of $\mathcal{H}_{+\tau}$ is an act such that uncertainty resolves in τ periods. Inductively, we can embed $\mathcal{H}_{+\tau}$ into $\mathcal{H}$. We call $\bigcup_{\tau\geq 1} \mathcal{H}_{+\tau}$ the domain of *finite-step acts*. The following proposition says that any compound lottery-act is a limit point of a sequence of finite-step acts.

Proposition 1 *The domain of finite-step acts $\bigcup_{\tau\geq 1} \mathcal{H}_{+\tau}$ is a dense subset of $\mathcal{H}$. Also, $\bigcup_{\tau\geq 1} \Delta(C \times \mathcal{H}_{+\tau})$ is dense in $\Delta(C \times \mathcal{H})$.*

Proof. Second claim follows from the first one. Fix some $c \in C$. For each τ, a mapping $\mathrm{Pr}_{+\tau} : \mathcal{H} \to \mathcal{H}_{+\tau}$ is defined as follows. For $h = (h_0, h_1, \cdots) \in \mathcal{H}$, $h_{+\tau} = \mathrm{Pr}_{+\tau}(h)$ is a sequence of acts $(\widehat{h}_0, \widehat{h}_1, \cdots, \widehat{h}_{t-1}, \widehat{h}_t, \widehat{h}_{t+1}, \cdots)$ such that $\widehat{h}_t = h_t$ for $t = 0, \cdots, \tau - 1$, and $\widehat{h}_t = (h_{\tau-1}, \delta[c^{t-\tau+1}])$ for $t = \tau, \tau+1, \cdots$, where $(h_{\tau-1}, \delta[c^{t-\tau+1}])$ denotes a process which follows $h_{\tau-1}$ for first $\tau - 1$ stages, and follows a deterministic and constant sequence $(c, \overset{t-\tau+1}{\cdots}, c)$ with length $t-\tau+1$. First $\tau-1$ coordinates of $h_{+\tau}$ coincide with those of h. Thus, $d(h, h_{+\tau}) = \alpha^\tau \sum_{t=\tau}^{\infty} d_t(h_t, \widehat{h}_t)\alpha^{t-\tau}$. This becomes arbitrarily small by taking τ sufficiently large. □

B.2 Proof of Theorem 2

Necessity of the axioms is routine. We show sufficiency.

Step 1: We begin with proving some lemmata.

History-Independence of risk preference and Dynamic Consistency together imply monotonicity for one-step-ahead acts.

Lemma 8 ***(Monotonicity)***: *For any ω^t, $c \in C$, $h_{+1}, h'_{+1} \in \mathcal{H}_{+1}$:*

$$\begin{array}{lll} \textit{if} & \delta[c, h_{+1}(\omega)] \succsim_{\omega^t} \delta[c, h'_{+1}(\omega)] & \textit{for every} \quad \omega \in \Omega, \\ \textit{then} & \delta[c, h_{+1}] \succsim_{\omega^t} \delta[c, h'_{+1}]; & \end{array}$$

Proof. Let $\delta[c, h_{+1}(\omega)] \succsim_{\omega^t} \delta[c, h'_{+1}(\omega)]$ for every $\omega \in \Omega$. Since $h_{+1}(\omega), h'_{+1}(\omega) \in D$ for every $\omega \in \Omega$, History-Independence of risk preference imply $\delta[c, h_{+1}(\omega)] \succsim_{\omega^t,\omega} \delta[c, h'_{+1}(\omega)]$ for every $\omega \in \Omega$. Thus, Dynamic Consistency implies that $\delta[c, h_{+1}] \succsim_{\omega^t} \delta[c, h'_{+1}]$. □

Next, we show that risk-equivalent always exists.

Lemma 9

(i) For any ω^t and τ, $h_{+\tau} \in \mathcal{H}_{+\tau}$, $p_{+\tau} \in \Delta(C \times \mathcal{H}_{+\tau})$, there exist risk equivalents $d, d' \in D$ such that $\delta[c, h_{+\tau}] \sim_{\omega^t} \delta[c, d]$, $p_{+\tau} \sim_{\omega^t} d'$.
(ii) For any ω^t, $h \in \mathcal{H}$, $p \in \Delta(C \times \mathcal{H})$, there exist risk equivalents $d, d' \in D$ such that $\delta[c, h] \sim_{\omega^t} \delta[c, d]$, $p \sim_{\omega^t} d'$.

Proof. Take any $h_{+1} \in \mathcal{H}_{+1}$. For each $\omega \in \Omega$, $h_{+1}(\omega) \succsim_{\omega^t,\omega} \underline{d}$ since $h_{+1}(\omega) \in D$. By Dynamic Consistency, $\delta[c, h_{+1}(\omega)] \succsim_{\omega^t} \delta[c, \underline{d}]$ for each ω. By Monotonicity, $\delta[c, h_{+1}] \succsim_{\omega^t} \delta[c, \underline{d}] \succsim_{\omega^t} \underline{d}$. Thus, $p_{+1} \succsim_{\omega^t} \underline{d}$ for any $p_{+1} \in \Delta(C \times \mathcal{H}_{+1})$. Similarly, $\delta[c, \overline{d}] \succsim_{\omega^t} \delta[c, h_{+1}]$, $\overline{d} \succsim_{\omega^t} p_{+1}$ for any h_{+1}, p_{+1}. Continuity delivers that there exist risk equivalents $d, d' \in D$ such that $\delta[c, h_{+1}] \sim_{\omega^t} \delta[c, d]$, $p_{+1} \sim_{\omega^t} d'$.

Next, pick any $h_{+2} \in \mathcal{H}_{+2}$. By the previous result, For each $\omega \in \Omega$, there exists $d_\omega \in D$ such that $h_{+2}(\omega) \sim_{\omega^t,\omega} d_\omega$, since $h_{+2}(\omega)$ is viewed as an element of $\Delta(C \times \mathcal{H}^{+1})$. Dynamic Consistency delivers that $\delta[c, h_{+2}] \sim_{\omega^t} \delta[c, h_{+1}]$, where h_{+1} is defined by $h_{+1}(\omega) = d_\omega$ for each ω. Since $\delta[c, h_{+1}] \succsim_{\omega^t} \delta[c, \underline{d}] \succsim_{\omega^t} \underline{d}$, we get $\delta[c, h_{+2}] \succsim_{\omega^t} \delta[c, \underline{d}] \succsim_{\omega^t} \underline{d}$. Thus, the same argument and continuity again delivers the existence of risk equivalents.

Induction argument delivers $\delta[c, \overline{d}] \succsim_{\omega^t} \delta[c, h_{+\tau}] \succsim_{\omega^t} \delta[c, \underline{d}]$ and $\overline{d} \succsim_{\omega^t} p_{+\tau} \succsim_{\omega^t} \underline{d}$ and the existence of risk equivalents for any $h_{+\tau} \in \mathcal{H}_{+\tau}$, $p_{+\tau} \in \Delta(C \times \mathcal{H}_{+\tau})$. Since $\bigcup_{\tau \geq 1} \mathcal{H}_{+\tau}$ is dense in $\mathcal{H}$, for any $h \in \mathcal{H}$, there exist a convergent sequence $\{h_{+k}\}$ with $h_{+k} \in \mathcal{H}_{+\tau}$ for some τ for each k. Continuity delivers the desired result. Existence of risk equivalent for $p \in \Delta(C \times \mathcal{H})$ is shown in the similar way. □

The following lemma says every act has an *equivalent one-step-ahead act*.

Lemma 10 *For any $h \in \mathcal{H}$, there exists a one-step-ahead act $h_{+1} \in \mathcal{H}_{+1}$ such that:*

(i) $h(\omega) \sim_{\omega^t,\omega} h_{+1}(\omega)$ *for each* $\omega \in \Omega$,
(ii) $\delta[c,h] \sim_{\omega^t} \delta[c,h_{+1}]$.

Proof. Fix any ω^t and pick any $h \in \mathcal{H}$. By the previous result, for each ω, there exists a risky prospect $d_\omega \in D$ such that $h(\omega) \sim_{\omega^t,\omega} d_\omega$. Define $h_{+1} \in \mathcal{H}_{+1}$ by $h_{+1}(\omega) = d_\omega$ for each ω. Then, the condition (i) is met by construction. By Dynamic Consistency, (ii) is satisfied. □

Next lemma says that Certainty Independence implies independence (in the sense of von-Neumann-Morgenstern) for timeless risky prospects.

Lemma 11 *Assume Certainty Independence. Then, at every* ω^t*, for any* $d, d', d'' \in D$ *and* $\lambda \in (0,1)$,

$$d \succsim_{\omega^t} d' \quad \textit{if and only if} \quad \lambda d + (1-\lambda)d'' \succsim_{\omega^t} \lambda d' + (1-\lambda)d''$$

Proof. Fix some $c \in C$. By Certainty Independence, $\delta[c,d] \succsim_{\omega^t} \delta[c,d']$ iff $\delta[c,\lambda d + (1-\lambda)d''] \succsim_{\omega^t} \delta[c,\lambda d' + (1-\lambda)d'']$. By Stationarity, we get the claim. □

By combining independence for timeless risky prospects and Risk Equivalence Preservation, we get the independence condition for general timeless gambles.

Lemma 12 ***(Independence for timeless lotteries)****: Assume Certainty Independence and Risk Equivalence Preservation. Then, at any* ω^t*, for any* $p, p', p'' \in \Delta(C \times \mathcal{H})$ *and* $\lambda \in (0,1)$,

$$p \succsim_{\omega^t} p' \quad \textit{if and only if} \quad \lambda p + (1-\lambda)p'' \succsim_{\omega^t} \lambda p' + (1-\lambda)p''$$

By History-Independence and Stationarity of risk preference and Dynamic Consistency, we have $d \succsim_{t+1} d'$ iff $\delta[c,d] \succsim_t \delta[c,d']$ iff $d \succsim_t d'$ for any $d, d' \in D$ and $c \in C$. Thus, risk preference is uniquely determined by a history-independent relation $\succsim$ over D. Since $\succsim$ satisfies weak order, continuity, and independence for timeless risky prospects, it is represented by an vNM expected utility $U : D \to R$, which is unique up to positive linear transformations and represented in the form $U(d) = E_d[u(c,d')]$. Fix one representation U. Without loss, we set U so that $U(d^*) = 0$ for some $d^* \in D$, such that there exist $\overline{d}, \underline{d} \in D$ with $\overline{d} \succ d^* \succ \underline{d}$. Since D is compact, we can set $\overline{d}$ ($\underline{d}$) be a best (worst) element in D with respect to $\succsim$. Let $R_U = [U(\underline{d}), U(\overline{d})]$.

Step 2: Continuity and Independence for Timeless Lotteries deliver a family of vNM expected utility $\{U_{\omega^t}\}$ for $\{\succsim_{\omega^t}\}$ with the form

$$U_{\omega^t}(p) = E_p[u_{\omega^t}(c,h)]$$

where u_{ω^t} is continuous on $C \times \mathcal{H}$.

By Consumption Separability, u_{ω^t} has the form

$$\widetilde{u}_{\omega^t}(c,h) = W_{\omega^t}\left[c, \widetilde{u}_{\omega^t}(\widehat{c},h)\right]$$

for an arbitrarily fixed $\widehat{c} \in C$.

We show that W_{ω^t} is taken to be independent of any history ω^t and time t.

Because of history-independence of risk preference, we can without loss set $U_{\omega^t}(d) = U(d)$, $u_{\omega^t}(c,d) = u(c,d)$ for every ω^t and t. Since risk-equivalent always exists, we get $\bigcup_{\omega^t} range\ U_{\omega^t} = R_U$. Since $u(c,d) = W_{\omega^t}[c, u(\widehat{c}, d)]$ holds for every ω^t and t, W_{ω^t} is independent of ω^t and t. Without loss, we can normalize W_{ω^t} to a history-independent one, W. Thus,

$$U_{\omega^t}(p) = E_p[W[c, u_{\omega^t}(\widehat{c}, h)]]$$

where $W : C \times \hat{R}_U \to R_U$ and $\hat{R}_U = range_{d\in D}\hat{u}(\hat{c}, d)$.

Step 3: Consider the subdomain $\mathcal{H}_{+1}$ Let $\succsim_{\omega^t,\widehat{c}}$ be the preference ordering over $\mathcal{H}_{+1}$ induced by $\succsim_{\omega^t}$. Recall that here exist $y, y' \in C^\infty \subset \mathcal{H}$ such that $y \succ_{\omega^t} y'$. Since risk preference is history-independent and satisfies separability and stationarity, this implies $\delta[\widehat{c}, y] \succ_{\omega^t} \delta[\widehat{c}, y']$. Thus, $\succsim_{\omega^t,\widehat{c}}$ is a nondegenerate relation and it is continuous.

By One-step-ahead Multiple-Priors, $\succsim_{\omega^t,\widehat{c}}$ satisfies the conditions for multiple-priors utility over $\mathcal{H}_{+1}$. Thus, it follows from [14] that it has max-min expected utility representation

$$V_{\omega^t}(h_{+1}) = \min_{\mu\in M_{\omega^t}} \int v_{\omega^t} \circ h_{+1}\ d\mu$$

for each $h_{+1} \in \mathcal{H}_{+1}$.[3]

v_{ω^t} is a mixture-linear function representing $\succsim_{\omega^t,\widehat{c}}$ over D and is unique up to positive linear transformations, and $M(\omega^t)$ is a unique closed convex subset of $\Delta(\Omega)$. Note that V_{ω^t} satisfies for any $h_{+1}, h'_{+1} \in \mathcal{H}_{+1}$, $d \in D$ and $\lambda \in [0,1]$,

(C-additivity): $V_{\omega^t}(\lambda h_{+1} + (1-\lambda)d) = \lambda V_{\omega^t}(h_{+1}) + (1-\lambda)V_{\omega^t}(d)$,

(Super-additivity): $V_{\omega^t}(h_{+1}) = V_{\omega^t}(h'_{+1})$ implies $V_{\omega^t}\left(\lambda h_{+1} + (1-\lambda)h'_{+1}\right) \geq V_{\omega^t}(h_{+1})$.

Since v_{ω^t} represents also $\succsim_{\omega^t,\widehat{c}}$ over D, by history-independence of risk preference, without loss we set $v_{\omega^t} = U$ over D.

Thus, we have

$$V_{\omega^t}(h_{+1}) = \min_{\mu\in M_{\omega^t}} \int U \circ h_{+1}\ d\mu$$

for $h_{+1} \in \mathcal{H}_{+1}$, and $range_{h\in\mathcal{H}_{+1}} V_{\omega^t} = R_U$.

Since any act has an equivalent one-step-ahead act, we can extend V_{ω_t} to $\mathcal{H}$ by defining $V_{\omega_t}(h) = V_{\omega_t}(h_{+1})$ where h_{+1} is an equivalent one-step-ahead act of h.

Step 4 (Construction of the certainty equivalent function) Let $U_{\omega^t,h} \in L(R_U)$ be the random variable defined by $U_{\omega^t,h}(\omega) = U_{\omega^t,\omega}(h(\omega))$. Let $X_{\omega^t} = \{U_{\omega^t,h} \in L(R_U) : h \in \mathcal{H}\}$. Since equivalent one-step-ahead act always exists, we get the claim

[3] Although Gilboa-Schmeidler [14] show their representation theorem over acts whose outcome is a simple lottery, the claim is valid for general acts with non-simple lotteries as outcomes, since we assume full continuity instead of mixture continuity. Thus the result by Grandmont [15] is adapted to modify their Lemma 3.1.

Lemma 13 *For any $h \in \mathcal{H}$, there exists a one-step-ahead act $h_{+1} \in \mathcal{H}_{+1}$ such that $U_{\omega^t,h} = U_{\omega^t,h_{+1}}$.*

Dynamic Consistency delivers the following lemma.

Lemma 14 *$V_{\omega^t}(h)$ is represented in a form,*

$$V_{\omega^t}(h) = \Phi_{\omega^t}(U_{\omega^t,h})$$

where $\Phi_{\omega^t} : X_{\omega^t} \to R_U$ is a strongly monotone function.

Proof. Let $U_{\omega^t,h} \geq U_{\omega^t,h'}$. This is equivalent to that $h(\omega) \succsim_{\omega^t,\omega} h'(\omega)$ for every $\omega \in \Omega$. Dynamic Consistency implies $(\widehat{c}, h) \succsim_{\omega^t} (\widehat{c}, h')$ which leads to $V_{\omega^t}(h) \geq V_{\omega^t}(h')$. Strong monotonicity follows from the second assertion of Dynamic Consistency. □

Next, we show that the certainty equivalent function Φ_{ω^t} satisfies several properties. Say that Φ_{ω^t} is *homogeneous* if $\Phi_{\omega^t}(\lambda U) = \lambda \Phi_{\omega^t}(U)$ for any $\lambda \geq 0$ and $U \in X_{\omega^t}$. It is *C-additive* if $\Phi_{\omega^t}(\lambda U + (1-\lambda)c) = \lambda \Phi_{\omega^t}(U) + (1-\lambda)c$ for any $\lambda \in [0,1]$ and $U \in X_{\omega^t}$, $c \in X_{\omega^t}$ where $c(\omega) = c(\omega')$ for every $\omega, \omega' \in \Omega$. It is *super-additive* if $\Phi_{\omega^t}(U) = \Phi_{\omega^t}(U')$ implies $\Phi_{\omega^t}(\lambda U + (1-\lambda)U') \geq \Phi_{\omega^t}(U)$ for every $U, U' \in X_{\omega^t}$ and $\lambda \in [0,1]$.

Lemma 15 *Φ_{ω^t} is homogeneous, C-additive, super-additive. Moreover it has a unique extension to $L(R)$.*

Proof. By Lemma 13, any act has an equivalent one-step-ahead act. Hence it suffices to show the properties for random variables given by one-step-ahead acts.

(1) Homogeneity: Let $0 \leq \alpha \leq 1$. Since $U_{\omega^t,d^*} = \mathbf{0}$,

$$\begin{aligned}
& \Phi_{\omega^t}(\alpha U_{\omega^t,h_{+1}}) \\
&= \Phi_{\omega^t}(\alpha U_{\omega^t,h_{+1}} + (1-\alpha)U_{\omega^t,d^*}) \\
&= \Phi_{\omega^t}(U_{\omega^t,\alpha h_{+1} \oplus (1-\alpha)d^*}) \\
&= V_{\omega^t}(\alpha h_{+1} \oplus (1-\alpha)d^*) \\
\text{(by C-additivity of } V \text{ over } \mathcal{H}_{+1}) &= \alpha V_{\omega^t}(h_{+1}) + (1-\alpha)V_{\omega^t}(d^*) \\
&= \alpha V_{\omega^t}(h_{+1}) + (1-\alpha)U(d^*) \\
&= \alpha V_{\omega^t}(h_{+1}) \\
&= \alpha \Phi_{\omega^t}(U_{\omega^t,h_{+1}})
\end{aligned}$$

This implies homogeneity for $\alpha > 1$.

By homogeneity, we extend Φ_{ω^t} to $L(R)$. Because $d^* \in D$ is taken so that there exist $\overline{d}, \underline{d} \in D$ with $\overline{d} \succ d^* \succ \underline{d}$,

$$L(R) = \bigcup_{\lambda \geq 0} \lambda \{U(h_{+1}(\cdot)) : h_{+1} \in \mathcal{H}_{+1}\}$$

Since $\{U(h(\cdot)) : h \in \mathcal{H}_{+1}\} \subset X_{\omega^t}$,

$$L(R) = \bigcup_{\lambda \geq 0} \lambda X_{\omega^t}$$

Take a constant act d. Since $U_{\omega^t,d}(\omega) = U(d)$ for every ω and $\Phi_{\omega^t}(U_{\omega^t,d}) = V_{\omega^t,\omega}(d) = U(d)$, $\Phi_{\omega^t}(\mathbf{1}) = 1$.

(2) C-additivity:

$$\begin{aligned}
& \Phi_{\omega^t}(\lambda U_{\omega^t,h_{+1}} + (1-\lambda)U_{\omega^t,d}) \\
\text{(by independence for risky prospects)} &= \Phi_{\omega^t}(U_{\omega^t,\lambda h_{+1} \oplus (1-\lambda)d}) \\
&= V_{\omega^t}(\lambda h \oplus (1-\lambda)d) \\
\text{(by C-additivity of } V \text{ over } \mathcal{H}_{+1}) &= \lambda V_{\omega^t}(h_{+1}) + (1-\lambda)V_{\omega^t}(d) \\
&= \lambda \Phi_{\omega^t}(U_{\omega^t,h_{+1}}) + (1-\lambda)\Phi_{\omega^t}(U_{\omega^t,d})
\end{aligned}$$

(3) Super-additivity: Suppose $\Phi_{\omega^t}(U_{\omega^t,h_{+1}}) = \Phi_{\omega^t}(U_{\omega^t,h'_{+1}})$. Then,

$$\begin{aligned}
& \Phi_{\omega^t}\left(\lambda U_{\omega^t,h_{+1}} + (1-\lambda)U_{\omega^t,h'_{+1}}\right) \\
\text{(by independence for risky prospects)} &= \Phi_{\omega^t}\left(U_{\omega^t,\lambda h_{+1} \oplus (1-\lambda)h'_{+1}}\right) \\
&= V_{\omega^t}\left(\lambda h_{+1} \oplus (1-\lambda)h'_{+1}\right) \\
\text{(by super-additivity of } V \text{ over } \mathcal{H}_{+1}) &\geq V_{\omega^t}(h_{+1}) \\
&= \lambda V_{\omega^t}(h_{+1}) + (1-\lambda)V_{\omega^t}(h'_{+1}) \\
&= \lambda \Phi_{\omega^t}(U_{\omega^t,h_{+1}}) + (1-\lambda)\Phi_{\omega^t}(U_{\omega^t,h'_{+1}})
\end{aligned}$$

□

Since $L(R)$ is a finite dimensional Euclidian space, we can adapt the argument by [14], lemma 3.5. Thus, Φ_{ω^t} has the form

$$\Phi_{\omega^t}(U_{\omega^t,h}) = \min_{\mu \in Q_{\omega^t}} \int U_{\omega^t,h}\, d\mu$$

Over the subdomain $\mathcal{H}_{+1} \subset \mathcal{H}$, $U_{\omega^t,h_{+1}} = U \circ h_{+1}$. Hence,

$$\begin{aligned}
V_{\omega^t}(h_{+1}) &= \min_{\mu \in M_{\omega^t}} \int U \circ h_{+1}\, d\mu \\
&= \min_{\mu \in Q_{\omega^t}} \int U \circ h_{+1}\, d\mu
\end{aligned}$$

for $h_{+1} \in \mathcal{H}_{+1}$. Uniqueness of the prior-set leads to $M_{\omega^t} = Q_{\omega^t}$.

Thus,

$$V_{\omega^t}(h) = \min_{\mu \in M_{\omega^t}} \int U_{\omega^t,h}\, d\mu$$

for any $h \in \mathcal{H}$.

Since both $u_{\omega^t}(\widehat{c}, \cdot)$ and $V_{\omega^t}(\cdot)$ represent $\succsim_{\omega^t}$ over $\mathcal{H}$, they are ordinally equivalent, hence there exist a strictly increasing mapping $\varphi_{\omega^t} : R_U \to \hat{R}_U$ such that

$$u_{\omega^t}(\widehat{c}, h) = \varphi_{\omega^t}(V_{\omega^t}(h))$$

By restricting attention to the subdomain D, we see that φ_{ω^t} is history-independent.

Thus,

$$\begin{aligned} U_{\omega^t}(p) &= E_p[W[c, u_{\omega^t}(\widehat{c}, h)]] \\ &= E_p[W[c, \varphi(V_{\omega^t}(h))]] \end{aligned}$$

where $W : C \times \hat{R}_U \to R_U$ is a continuous and onto mapping. Without loss, we can rewrite this form as

$$U_{\omega^t}(p) = E_p[W[c, V_{\omega^t}(h)]]$$

where $W : C \times R_U \to R_U$.

Full-support property. Suppose for some ω^t there is a one-step-ahead conditional $\mu \in M_{\omega^t}$ such that $\mu[\{\omega^*\}] = 0$ for some $\omega^* \in \Omega$. Take two acts $h, h' \in \mathcal{H}$ defined by

$$\begin{aligned} h(\omega) &= \underline{d} \qquad \text{for every } \omega \\ h'(\omega) &= \begin{bmatrix} \overline{d} & \text{if } \omega = \omega^* \\ \underline{d} & \text{if } \omega \neq \omega^* \end{bmatrix} \end{aligned}$$

Then, the resulting representation gives $U_{\omega^t}(\delta[c, h]) = U_{\omega^t}(\delta[c, h'])$. However, this contradicts the second assertion of Dynamic Consistency.

Uniqueness. Suppose $\{\succsim_{\omega^t}\}$ is represented by $(\{U^1_{\omega^t}, M^1_{\omega^t}\}, W^1)$ and $(\{U^2_{\omega^t}, M^2_{\omega^t}\}, W^2)$, where $W^1 : C \times R^1_U \to R^1_U$, $W^1 : C \times R^2_U \to R^2_U$. Uniqueness of the prior-set follows from the previous argument. We show the relation between $\{U^1_{\omega^t}\}$, W^1 and $\{U^2_{\omega^t}\}$, W^2. Since $\{U^1_{\omega^t}\}$ and $\{U^2_{\omega^t}\}$ are vNM indices over $\Delta(C \times \mathcal{H})$, we get cardinal equivalence $U^2_{\omega^t} = A_{\omega^t} U^1_{\omega^t} + B_{\omega^t}$ for each ω^t. Since we have normalized $\{U^1_{\omega^t}\}$ and $\{U^2_{\omega^t}\}$ to history-independent ones U^1 and U^2 over D, A_{ω^t} and B_{ω^t} are history-independent. Thus, $U^2_{\omega^t} = AU^1_{\omega^t} + B$ for each ω^t.

Thus, $U^2(c, d) = W^2(c, U^2(d)) = W^2(c, AU^1(d) + B)$. On the other hand, $U^2(c, d) = AU^1(c, d) + B = AW^1(c, U^1(d)) + B$.

B.3 Proof of Corollary 1

First, we look at the implication of Risk Timing-Indifference. We show that $W(\cdot, \cdot)$ is linear in the second argument. Since every act has risk equivalent, it suffices to show linearity of risk equivalent function with respect to the utilities of risky prospects. Take values V, V'. There exist temporal lotteries $d, d' \in D$ such that $V_{\omega^t}(d) = V$, $V_{\omega^t}(d') = V'$. Then, for any $c \in C$,

$$\begin{aligned} W(c, \lambda V + (1-\lambda)V') &= W(c, \lambda V_{\omega^t}(d) + (1-\lambda)V_{\omega^t}(d')) \\ &= W(c, V_{\omega^t}(\lambda d \oplus (1-\lambda)d')) \\ &= U_{\omega^t}(\delta[c, \lambda d \oplus (1-\lambda)d']) \\ &= U_{\omega^t}(\delta[c], \lambda\delta[d] + (1-\lambda)\delta[d']) \\ &= \lambda U_{\omega^t}(\delta[c,d]) + (1-\lambda)U_{\omega^t}(\delta[c,d']) \\ &= \lambda W(c,V) + (1-\lambda)W(c,V') \end{aligned}$$

Thus, W is affine in the second argument, and is represented as $W(c,V) = u(c) + \alpha(c)V$. Continuity and uniqueness result of (u, α), non-constancy of $u/(1-\alpha)$, and $\alpha(\cdot) \in (0,1)$ follow from [6].

Addition of axiom 8 delivers the discount factor α as a constant (see [6]).

B.4 Proof of Theorem 3

Proof of (i). '$\Longrightarrow$' part : Over the subdomain of risky prospects D, both W^* and W represent the same ordering. Thus, the uniqueness result delivers $U^* = AU + B$ and $W^*(c, AV + B) = AW(c,V) + B$. Since $U_{\omega^t}(c,d) \geq U_{\omega^t}(c,h)$ is equivalent to $AU(d) + B \geq AV_{\omega^t}(h) + B$ and $U^*_{\omega^t}(c,d) \geq U^*_{\omega^t}(c,h)$ is equivalent to $AU(d) + B \geq V^*_{\omega^t}(h)$, the condition is met only if $AV_{\omega^t}(h) + B \geq V^*_{\omega^t}(h)$. Consider the subdomain $\mathcal{H}_{+1}$. Then, for $h_{+1} \in \mathcal{H}_{+1}$,

$$\begin{aligned} AV_{\omega^t}(h_{+1}) + B &= A \min_{\mu \in M_{\omega^t}} \int U \circ h_{+1}\, d\mu + B \\ V^*_{\omega^t}(h_{+1}) &= \min_{\mu \in M^*_{\omega^t}} \int U^* \circ h_{+1}\, d\mu \\ &= A \min_{\mu \in M^*_{\omega^t}} \int U \circ h_{+1}\, d\mu + B \end{aligned}$$

This is met only when $M^*_{\omega^t} \supset M_{\omega^t}$. '$\Longleftarrow$' part : Both represent the same ordering over risky consumptions. Take any ω^t and consider an act $h_{+1} \in \mathcal{H}_{+1}$ such that $U_{\omega^t}(c,d) \geq U_{\omega^t}(c, h_{+1})$. This holds iff $AU(d) + B \geq AV_{\omega^t}(h_{+1}) + B$. Since $AV_{\omega^t}(h_{+1}) + B \geq V^*_{\omega^t}(h_{+1})$ by the assumption, $AU(d) + B \geq V^*_{\omega^t}(h_{+1})$, which implies $U^*(c,d) \geq U^*_{\omega^t}(c, h_{+1})$.

Proof of (ii). '$\Longrightarrow$' part : Since both represent the same ranking over deterministic consumptions, there is an increasing transformation $\phi : R_U \to R_{U^*}$ such that $U^* = \phi \circ U$. Thus, $U^*(c,y) = \phi[U(c,y)] = \phi[W(c, U(y))]$. Since $U^*(c,y) = W^*(c, U^*(y)) = W^*(c, \phi[U(y)])$, we get $W^*(c, \phi[U(y)]) = \phi[W(c, U(y))]$.

Next, we show that ϕ is concave. Consider a one-shot lottery $l = \lambda\delta[y'] + (1-\lambda)\delta[y'']$. Thus,

$$\begin{aligned} U(l) &= \lambda U(y') + (1-\lambda)U(y'') \\ U^*(l) &= \lambda U^*(y') + (1-\lambda)U^*(y'') \\ &= \lambda\phi[U(y')] + (1-\lambda)\phi[U(y'')] \end{aligned}$$

Since $U(y) \geq U(l)$ is equivalent to $\phi(U(y)) \geq \phi\left[\lambda U(y') + (1-\lambda)U(y'')\right]$ and $U^*(y) \geq U^*(l)$ is equivalent to $\phi(U(y)) \geq \lambda\phi\left[U(y')\right] + (1-\lambda)\phi\left[U(y'')\right]$, the required condition is met only when

$$\begin{aligned}&\phi\left[\lambda U' + (1-\lambda)U''\right]\\ &\geq \lambda\phi\left[U'\right] + (1-\lambda)\phi\left[U''\right]\end{aligned}$$

for any $U', U'' \in R_U$. This delivers the concavity of ϕ.

'$\Longleftarrow$' part : Both represent the same ordering over deterministic consumptions. Take any lottery $l \in \Delta(C^\infty)$. Jensen's inequality and converse argument of the above delivers that $U(y) \geq U(l)$ implies $U^*(y) \geq U^*(l)$.

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