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# Green Accounting for an Externality, Pollution at a Mine

ROBERT D. CAIRNS
*Department of Economics, McGill University, 855 Sherbrooke St. W., Montreal, Canada H3A 2T7 (E-mail: robert.cairns@mcgill.ca)*

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**Abstract.** This paper takes a value-added approach to "green" accounting at an individual microeconomic unit, a mine. Capacities for extraction and for abatement of pollution are chosen subject to an environmental regulation. The implications for accounting for resource and environmental degradation are discussed. Depreciation is not quantitatively unique, but can be compared qualitatively with a condition involving shadow prices. The costs of defensive expenditures contribute to increasing green NNP, but depreciation of the resource is a charge against GNP in computing green NNP. Income from capital is the return on the undepreciated values of extractive capacity, abatement capacity and the resource, and is a part of net domestic income.

**Key words:** compliance cost, depreciation, green accounting, pollution

**JEL classifications:** Q3, Q2, E2, C6

## 1. Introduction

In the mining industry, explicit, daily trade-offs are made among (a) making a significant contribution to society's consumption possibilities, (b) the use of a nonrenewable resource, (c) the use of large investments, and (d) the possibility of harm to renewable resources such as water or the atmosphere. The present paper discusses green accounting for the exploitation of a mine which is subject to a regulation prescribing a maximum level of pollutant in a stream. The flow of pollutant is proportional to current production.

Typically, theoretical studies of green accounting begin with a macro-growth model of the economy, with welfare being equal to the discounted, instantaneous utility of a representative consumer. Insights for national accounting are based on the results from an optimal control, and accounting (shadow) values are obtained by linearizing the Hamiltonian. In a simple growth model, Weitzman (1976) provides an interpretation of green net national product (NNP) in an ideal economy as a measure of welfare, and his findings have been extended by many.

Cairns (2000, 2001a) demonstrates that a macroeconomic model may neglect important considerations and thereby lead to inadequate accounting prices. The influential book, *Nature's Numbers* (Nordhaus and Kokkelenberg, eds. 1999,

p. 59ff.), points to the validity of a microeconomic approach to the accounts, at least for nonrenewable resources. In practice, the data for the national accounts are gathered by observing microeconomic units, and the accounting values used are market prices. Once gathered, the results are aggregated to form a macroeconomic statistic. The correspondence of market prices to shadow prices in an ideal economy provides a loose theoretical rationale for this practice.

For the mining industry, a partial-equilibrium model can be used as a guide to valuation in the national accounts in much the same way that the macro models are used in much of the literature. It preserves Weitzman's welfare interpretation of green NNP in an ideal economy. It also corresponds to (a) practice in using market prices and (b) theory in using estimates of shadow prices for nonmarketed goods.

As in the conventional accounts, if an accounting price is not a true shadow price, the evaluation is imperfect. Some current methods of nonmarket valuation are, to varying degrees, controversial, even within the economics profession. The present paper proposes a practical method to value pollution, using market observations to generate estimates of nonmarketed values.

Some environmentalists have held that defensive expenditures on the environment are incorrectly accounted as contributing to NNP, whereas they ought to be accounted as deductions. For the case studied in the present paper, this challenge to prevailing practice is not justified. Even so, the deterioration of stocks of nonrenewable resources should be treated similarly to that of manufactured capital. It is not possible to use shadow prices to determine the depreciation of sunk capital or the resource.

## 2. Mining with Abatement

We assume for simplicity that background conditions, including price, remain stationary. Roan and Martin (1996) analyze the dynamic path for this problem, given predetermined stocks of capital for extraction and abatement of pollution. In the spirit of their study, we assume that capital is irreversibly invested at time $t = 0$. This assumption, for mining capital at least, is made in mining-engineering textbooks, and Cairns (2001b) shows it to be optimal when there are decreasing unit costs of installing capacity. For abatement capacity we simply assume it to hold. Mathematically, the capital stocks are *control parameters* (Takayama 1974, p. 654ff.).[1]

We view mining and abatement as separate processes, so that variable cost is a separable and additive function of the levels of output and abatement. The variable cost of production, $q$, at any time is $C(q)$, and of abatement, $x$, is $A(x)$. The following conditions are assumed to hold:

$$\begin{aligned} &C'(q) > 0 \text{ and } C''(q) > 0, \text{ for } q \geq 0; \\ &A'(x) > 0 \text{ and } A''(x) > 0, \text{ for } x \geq 0; \\ &\qquad C(0) = 0; A(0) = 0. \end{aligned}$$

The cost of installation of the capital stocks, $K \geq 0$ for extraction and concentration and $X \geq 0$ for abatement, is $\psi(K, X)$ (which may be separable), with $\psi_K > 0$ and $\psi_X > 0$.[2] The stocks provide constraints on production of ore and on abatement, so that $x \leq X$ and $q \leq K$.

The quantity of waste in the stream, $W$, increases with output, and decreases with abatement effort and natural cleansing. It is governed by the differential equation,

$$\dot{W} = q - x - kW. \tag{1}$$

Let the maximal level of pollution permitted by the regulation be $\bar{W}$, and cumulative ore production by the firm up to time $t$ be $Q_t$.[3] Also let the time at which the project is abandoned be a free variable represented by $T$, and the initial stock of reserves be represented by $S$. The problem of the firm is to

$$\max_{\{T,[q]_0^T,[x]_0^\infty,K,X\}} \int_0^T \left[pq - C(q) - A(x)\right] e^{-rt} dt - \psi(K, X) = \Phi(Q_0, W_0), \tag{2}$$

such that $\dot{Q} = q \geq 0$; $\dot{W} = q - x - kW$; $0 \leq x \leq X$; $0 \leq q \leq K$; $W \leq \bar{W}$; $W_0 = 0$; $Q_0 = 0$; and $Q_T \leq S$. We assume that the sufficient conditions for existence of a solution (Cairns 1998) are satisfied.

The current-value Hamiltonian for this problem is

$$H = pq - C(q) - A(x) + \lambda q + \mu (q - x - kW), \tag{3}$$

and the Lagrangean is

$$L = H + u(X - x) + v(K - q) + w(\bar{W} - W) + yx + zq. \tag{4}$$

All of $u$, $v$, $w$, $y$, and $z$ are nonnegative, and by complementary slackness, $u(X - x) = v(K - q) = w(\bar{W} - W) = yx = zq = 0$. The following first-order, adjoint and transversality conditions are necessary:

$$\begin{aligned}
p - C'(q) + \lambda + \mu - v + z &= \frac{\partial L}{\partial q} = 0; && (5)\\
-A'(x) - \mu - u + y &= \frac{\partial L}{\partial x} = 0; && (6)\\
\dot{\lambda} - r\lambda &= -\frac{\partial L}{\partial Q} = 0; && (7)\\
\dot{\mu} - r\mu &= -\frac{\partial L}{\partial W} = k\mu + w; && (8)\\
H|_T &= 0; && (9)\\
\lambda_T(S - Q_T) &= 0; && (10)\\
\mu_T W_T &= 0; && (11)\\
\int_0^T v e^{-rt} dt &= \frac{\partial \psi}{\partial K} > 0; && (12)
\end{aligned}$$

$$\int_0^T ue^{-rt}dt = \frac{\partial \psi}{\partial X} > 0. \tag{13}$$

Roan and Martin's solution applies only when capacity constraints are not effective. Equations (12) and (13) imply that each capacity constraint must bind on some interval.[4]

The remaining (or undepreciated) value of the program at time $t \in (0, T)$ is

$$V\left(Q, W, K, X\right)|_t = \int_t^T \left[pq - C(q) - A(x)\right] e^{-r(s-t)}ds, \tag{14}$$

where the variables $q$ and $x$ follow the optimal path. We assume that $V(Q, W, K, X)$ is differentiable with respect to all of its arguments. Then,

$$\lambda_t = \frac{\partial V}{\partial Q}|_t < 0;$$
$$\mu_t = \frac{\partial V}{\partial W}|_t < 0.$$

Solving problem (2) is intricate because there are two state variables, $Q$ and $W$; two control variables, $q$ and $x$; and two control parameters, $K$ and $X$. Results are derived in the appendix and summarized as follows.

**Proposition 1**

1. *Initially, output is at capacity, and abatement is below capacity. The expression $p - C'(K) - A'(x) - v$ rises at the rate of interest.*
2. *Abatement reaches capacity at the time that the pollution constraint is reached.*
3. *Beginning at the point when the pollution constraint is effective, there is an interval on which the capacity constraints on output and abatement and the pollution constraint are effective. The expression $p - C'(K) - A'(X) - u - v$ rises at the rate of interest.*
4. *Eventually, a time is reached when output and abatement simultaneously begin to decline. They decline at equal rates, so that the capacity constraints are not effective but the pollution constraint continues to bind, until abatement is zero. At that point, output is equal to the natural level of cleansing when pollution is at its maximal level: $q = k\bar{W}$.*
5. *While output is declining from $q = K$ to $q = k\bar{W}$, the value of the expression $p - C'(q) - A'(x)$ rises at the rate of interest.*
6. *Output remains at $k\bar{W}$ on an interval. The expression $p - C'(q)$ is constant and the expression $p - C'(q) + \mu$ rises at the rate of interest.*
7. *After that interval, output declines to zero at the time of closure. The expression $p - C'(q)$ rises at the rate of interest.*
8. *When output begins to decline from $k\bar{W}$, the pollution constraint ceases to be effective.*
9. *Reserves are exhausted at the time of abandonment, $T$.*

*10. At the time of abandonment, the level of pollution is positive. Thereafter it declines asymptotically toward zero through natural cleansing.*

For a period beginning with the opening of the mine, the r-percent rule for resource rent $(-\lambda)$ is masked by other variables; only toward the end of the mine's life does $(-\lambda) = p - C'(q)$.

Mining engineers typically assume constant marginal costs: $C(q) = cq$ and $A(x) = ax$. When $C(q) = cq$ and there is no pollution constraint, production is at capacity through to abandonment (Crabbé 1982). This production profile is usually assumed by engineers. When there is abatement, the optimal pattern is modified as follows.

**Corollary 1** *When marginal costs are constant, production is at capacity on an interval, then falls discretely to $k\bar{W}$ and remains at that level on an interval with no abatement effort, and finally falls discretely to zero.*

## 3. Accounting for Flows

Proposition 1 and Corollary 1 provide a context for analyzing the roles of shadow values in accounting for various forms of depreciation. There are five shadow values that have instructive economic interpretations:

- $\partial V/\partial Q = \lambda$, the shadow value of a unit of resource use;
- $\partial V/\partial W = \mu$, the shadow value of a unit of pollution in the stream;
- $\partial L/\partial \bar{W} = w$, the shadow value of the pollution constraint;
- $\partial L/\partial K = v$, the shadow value of a unit of productive capital;
- $\partial L/\partial X = u$, the shadow value of a unit of abatement capital.

In Lemma 1 of the Appendix, we show that

$$\mu_t = -\int_t^T w_s e^{-(r+k)(s-t)} ds. \tag{15}$$

The shadow value of pollution is an aggregate of the instantaneous shadow values of the pollution constraint, $w$, discounted for both the time value of money and the natural regenerative capacity of the stream.

By equation (6),

$$-\mu \leq A'(x) + u.$$

When the level of abatement, $x$, is positive, equality holds: the shadow value of pollution is equal to the sum of the marginal abatement cost, $A'(x)$, and the shadow value of abatement capacity $u$. By equation (13), the integral of the discounted shadow values of abatement capacity must equal the marginal cost of expansion of abatement capacity. Thus, the marginal user cost of abatement capacity is $u$. Therefore, *the marginal cost to the firm of compliance with the regulation* at any instant is $A'(x) + u$. When abatement is occurring (when $x > 0$), the marginal cost of pollution (its shadow value, $\mu$) is fully captured in the marginal compliance cost.

Adding equations (5) and (6) when $q > 0$ and $x > 0$ and rearranging yield that

$$p = \left(C' + v\right) + \left(A' + u\right) + (-\lambda) : \tag{16}$$

Price is the sum of marginal extraction cost (including the shadow value of extractive capacity), marginal compliance cost, and resource rent.

By equation (14), depreciation of the value of the program is

$$\begin{aligned} D &= -\dot{V} \\ &= -\left[\frac{\partial V}{\partial K}\dot{K} + \frac{\partial V}{\partial X}\dot{X} + \frac{\partial V}{\partial Q}\dot{Q} + \frac{\partial V}{\partial W}\dot{W}\right] \\ &= (-\lambda)\, q + (-\mu)\,(q - x - kW)\,. \end{aligned} \tag{17}$$

(Recall that $\dot{K} = 0$ and $\dot{X} = 0$.) When the stock of pollution is increasing (when $\dot{W} = q - x - kW > 0$), its shadow value, $\mu$, enters the computation, accounting for the increase in the discounted value of the environmental constraint when that constraint is effective. However, $D = -\lambda q$ when pollution is constant (since $q - x - kW = 0$) or decreasing (since $\mu = 0$). In many studies, the resource rent, $-\lambda q$, has been considered the depreciation of the resource. However, the formula for $D$ gives the depreciation of the program, not just of the resource.

Some environmentalists consider command-and-control regulation to be a superior policy tool. Physical limits on pollution have an ironic accounting implication. The shadow (accounting) cost of pollution, $-\mu\dot{W}$, is positive only when pollution is rising toward its limit. At the limit, the accounting cost is zero. (It is 'very high' if pollution exceeds the limit.) When pollution is declining, its shadow value is nil ($w = 0$ for $t \geq t_\mu$ and hence $\mu = 0$). Such declines have no imputed social value, because the regulation is satisfied.

Policy is unlikely to be optimal in practice. Furthermore, the shadow value of pollution, $\mu$, is a private value. In social accounting, the use of a private shadow value emerging from what may be a sub-optimal policy may be controversial. Ideally in a welfare measure, the marginal *social* cost of pollution would be the shadow value used. It is not, however, for an accountant or an economist to prescribe what is optimal. If a command-and-control policy of the sort analyzed above is in place, then it is a type of revealed preference of society. An implication of implementation of the policy is that it defines implicit shadow values. An analogous practice in the national accounts is the convention of recording governmental expenditures at market value.

As is implicitly maintained in publishing the conventional accounts, optimal control serves as a *guide* to accounting method and interpretation, even though the world is not optimal and observed prices may not be ideal.

## 4. Depreciation

Green accounting requires that depreciation of the resource, which is negative the rate of change of the value of the resource, be deducted from the gross product. The investment of manufactured capital, however, transforms three assets (productive capital, abatement capital and mineral deposit) into a single, new, sunk asset, a mine. For $t \in (0, T)$, the three former assets are not independent. By equations (2) and (14),

$$\begin{aligned}\Phi\left(Q_0, W_0\right) + \psi\left(K, X\right) &= \int_0^T \left[pq - C\left(q\right) - A(x)\right]_s e^{-rs} ds \\ &= \lim_{t \downarrow 0} V\left(Q, W, K, X\right)|_t .\end{aligned}$$

The value of the resource, $\Phi\left(Q_0, W_0\right)$, and the value of manufactured capital, $\psi\left(K, X\right)$, are defined only at time $t = 0$. For $t > 0$, only the value of the composite asset or mine, $V_t$, is defined, and hence only depreciation of the mine, $D_t = -\dot{V}_t$, is uniquely defined. Green accounting, then, requires a method of separating depreciation of the resource from depreciation of the manufactured assets.

To derive depreciation of manufactured capital, what is needed is a schedule of *payments* to each of the two types of capital, $[\kappa_K\left(t\right)]_0^T$ and $[\kappa_X(t)]_0^T$ (Baumol, Panzar and Willig 1982: ch. 13). These payments must have total discounted value equal to the total original investment:

$$\int_0^T \left[\kappa_K(t) + \kappa_X(t)\right] e^{-rt} dt = \psi(K, X). \tag{18}$$

The payment to capital would be the rental value if there were a competitive market for all vintages of capital and if the capital were not sunk. Defining the payments to sunk capital requires an accounting convention. *There is no unique, "economic" allocation*; only condition (18) and the firm's net cash flow constrain the accounting convention. Because of nonlinearities and nonseparabilities, the payments may not bear a clear relation to the shadow values in equations (12) and (13). The payments can be equated to $vK$ and $uX$ only if $\psi(K, X)$ is linear in $K$ and $X$. The values of $\kappa_K$ and $\kappa_X$ depend on conventions for (a) allocating portions of the investment $\psi(K, X)$ to $K$ and $X$ and (b) allocating those portions through time. It is possible, for example, to let $\kappa_K > 0$ when $q < K$ (and hence $v = 0$) or $\kappa_X > 0$ when $x < X$ (and hence $u = 0$).

Given an appropriate accounting convention, which is not unique, the undepreciated (remaining) value of an asset at time $t > 0$ is

$$J_i(t) = \int_t^T \kappa_i(s) e^{-r(s-t)} ds, i = K, X,$$

and depreciation is negative the rate of change of the asset's value,

$$D_i = -\dot{J}_i = \kappa_i - rJ_i,$$

so that $\kappa_i(t) = rJ_i(t) + D_i(t)$: the payment to capital is interest on the undepreciated value plus depreciation, as prescribed by capital theory. Given the path of $\kappa_i(t)$ or $D_i\,(t)$, $0 \le t \le T$, the path of the other is uniquely defined.

Depreciation of the resource also has to be defined using capital theory. In problem (2), the total value of the resource is the maximized value,

$$\begin{aligned}
&\int_0^T \left[pq - C(q) - A(x)\right] e^{-rt} dt - \psi(K, X) \\
&\quad = \int_0^T \left[pq - C(q) - A(x) - \kappa_K - \kappa_X\right] e^{-rt} dt \\
&\quad = \int_0^T (-\lambda)\, q e^{-rt} dt + Z \\
&\quad = (-\lambda_0)\, S + Z,
\end{aligned}$$

where (by the necessary conditions)

$$\begin{aligned}
Z \ = \ &\int_0^T \left\{\left[qC' - C\right] + \left[xA' - A\right] + \left[ux + vq - \kappa_X - \kappa_K\right]\right. \\
&\left. + \,(q - x)\,(A' + u)\right\} e^{-rt} dt.
\end{aligned}$$

Only for very special cases does $Z = 0$. For convex functions, $Z > 0$: the value of the program at time $t = 0$, and hence of the resource, is greater than the stock evaluated at the resource rent when all functions are convex. It may be that $Z < 0$, however – for example, for proportional operating costs and sufficiently concave investment costs.[5] In general, if there are non-constant returns to scale, the reserves are not valued at a price equal to the marginal rent.

Analogously to the definitions for the undepreciated values of productive and abatement capital, let $J_R(t)$ be the discounted value of payments to the resource,

$$J_R(t) = \int_t^T \kappa_R(s) e^{-r(s-t)} ds = V(Q, W, S, K)\,|_t \, - J_K(t) - J_X(t). \qquad (19)$$

By the definitions of $\kappa_K$ and $\kappa_X$,

$$J_R(0) = \Phi\left(Q_0, W_0\right).$$

Also, $\kappa_R = rJ_R - \dot{J}_R = rJ_R + D_R$, where $D_R$ is the depreciation of the reserves. Since the depreciation of the manufactured and natural capital are interrelated and the former is not unique, the depreciation of the resource is not unique. At any time,

$$pq = \left[C(q) + D_K + rJ_K\right] + \left[A\,(x) + D_X + rJ_X\right] + \left[D_R + rJ_R\right]. \qquad (20)$$

In equation (20), (accounted) extraction costs are $\left[C(q) + D_K + rJ_K\right]$, compliance costs are $\left[A\,(x) + D_X + rJ_X\right]$, and the cost of resource use is $\left[D_R + rJ_R\right]$.

These costs include the returns to the various types of capital. Compare equations (16) and (20).

For the selected set of payments to capital and the implied depreciation payments (or vice versa), the current accounting profit at any time before shutdown is

$$\begin{aligned}\pi &= pq - C - A - D_K - D_X \\ &= r\,(J_K + J_X + J_R) + D_R.\end{aligned}$$

The return on the three types of capital, $r\,(J_K + J_X + J_R)$, is the contribution of the project to green net domestic income at any time. Current accounting profit overstates the contribution of the project by the depreciation of the resource, $D_R$, which is equal to the total depreciation of the program minus depreciation of the two types of manufactured capital.

The reason for this result is that the value of the mineral reserves at time zero is identified with the value of the project. If the firm purchases the undeveloped orebody in a competitive market, the purchase price is $\Phi(Q_0, W_0)$. If that initial outlay is depreciated in the corporate accounts, then the firm's accounting profit estimates the contribution to green NNP. Otherwise, to obtain a consistent measure of the contribution of the project to green NNP, one must deduct depreciation of the resource, $D_R$, from accounting profits $\pi$.

The negative effects of pollution are approximated in the compliance cost, $A(x_t) + \kappa_X(t)$, and hence in the outlays of the firm in meeting the regulation. To see this, we differentiate both sides of equation (14) to find (recall equation (17)) that

$$-\lambda q - \mu \dot{W} = -\dot{V} = pq - C - A - rV.$$

Simplifying using equations (19) and (20) yields

$$\begin{aligned}&rV - \lambda q - \mu \dot{W} = r(J_K + J_X + J_R) + (D_K + D_X + D_R), \text{ or} \\ &D = -\lambda q - \mu \dot{W} = D_K + D_X + D_R + r\,(J_K + J_X + J_R - V) \\ &\quad = D_K + D_X + D_R.\end{aligned}$$

Recall that when pollution is constant or declining, $\mu \dot{W} = 0$ and $D = -\lambda q$. Therefore, $D_R < -\lambda q$. Even so, economic theory tells us that "green" depreciation, $D$, is equal to accounting depreciation, $D_K + D_X + D_R$. Imposition of the regulation reduces the value of the resource and the level of resource rent because the compliance cost, $A(x_t) + \kappa_X(t)$, increases the cost of gaining the value $p$ per unit product.[6] If $A(x)$ is a payment to a factor, e.g., labor, it should be accounted as part of the factor's income in the national-income account in the same way as $C(q)$.

When output and abatement are at capacity, the net product of this mine is the sum of the net value of output of the mine, $pq - D$, and of the net value (zero) of tendencies at the margin to overpollute and to reduce any overpollution. Net income is $pq - D$ as well, as it is equal to shareholders' income of $\left[pq - C(q) - A(x) - D\right]$ plus extraction workers' income $C(q)$, plus abatement workers' income $A(x)$. Defensive expenditures, $A(x)$, are included. But the income of shareholders is reduced by the requirement to limit pollution. If there is no change in pollution, there is no change in consumption possibilities. But, while pollution is rising toward its mandated limit, there is a change in consumption possibilities which is measured by $\mu$ and is recognized as depreciation in NNP.

Our main findings are summarized in the following proposition.

**Proposition 2**

1. *The sum of accounting depreciation of manufactured capital ($D_K + D_X + D_R$) and of the resource is equal to economic depreciation ($D = -\dot{V}$).*
2. *The cost of pollution is captured in the compliance cost, exactly at the margin and appropriately but not uniquely in the accounts.*
3. *The value of the project (present value of the resource) is diminished by the regulation. Therefore, the requirement to abate pollution diminishes the resource's contribution ($rJ_R$) to green NNP.*
4. *(a) Returns on, (b) net investments in, and (c) the compensation of primary factors used in defensive activity contribute to green NNP.*

One might object that, in the full accounts, the defensive activity would not enter into NNP because it is comparable to maintenance. But, by reducing depreciation, maintenance expenditures do reduce the difference between NNP and GNP, without affecting GNP. Therefore, in a model of the entire economy they would contribute somewhere to NNP.

If the metal is durable rather than perishable, the resource component of whatever is produced from the resource is an intermediate product to an investment that adds full value to NNP in the year made. During the useful lifetime, interest on the depreciated balance is earned and depreciation is assessed. The current price is the present value of periodic payments attributed to it, in the same way as for the other investments in this paper. No insight into accounting is lost by thinking of the resource as perishable.

## 5. Other Policies

Other policies have been proposed for pollution, including tradeable pollution permits and pollution taxes. Because of their greater simplicity, analysis of these cases is less informative; they do not require non-market valuation because they provide a market value of pollution. They too require the assumption made above concerning the appropriateness of policy. Their implications for green accounting can be noted easily.

### 5.1. TRADEABLE PERMITS

This policy requires that there be a large number of firms that can trade permits among themselves in meeting an aggregate constraint, rather than separate constraints affecting individual enterprises. An aggregate number of permits corresponding to $\bar{W}$ are issued. They have a foreseeable market price, $m(t)$. The firm must buy $\nu$ permits at time $t$ to satisfy the constraint,

$$\nu - (q - x) \geq 0. \tag{21}$$

Then the firm maximizes the present value of profit net of the cost of permits.

The optimization does not involve condition (1) for the path of $W$, nor the variables $\mu$ and $w$. In its place is a condition that equates the shadow value of constraint (21) to the cost of compliance, $A'(x) + u$. Any firm with abatement capacity will utilize it ($x > 0$) unless $A'(0) > m$.

In the accounting, the payments, $m\nu$, are transfers. The marginal social value of pollution at time $t$ is $m$, determined in the market, and is equal to $A' + u$ for each firm (unless $A'(0) > m$). This is also the marginal contribution of the (quality of the) stream to social welfare. As usual, there remains an implicit planning problem of determining the appropriate number of permits, or the appropriate amount by which to cut back pollution.

The greater efficiency of this policy over command and control arises not only because of differences among firms in the curvature of abatement cost, $A(x)$, as stressed in the literature, but also in the curvature of extraction cost, $C(q)$, and in the cost of capacity, $\psi(K, X)$.

### 5.2. A POLLUTION TAX

Suppose a tax, $\tau(W)$, is imposed on emissions, $q - x$, and depends on (aggregate) pollution $W$. This tax is clearly a social valuation of pollution at the margin. If it is applied on many firms, the model again does not account for the dynamics of $W$. There are similar conclusions to those for permits. If, as in the main model, it is applied to a firm which is the sole source of this type of pollution, the firm must concern itself with the dynamics of $W$. The condition corresponding to condition (8) is

$$\dot{\mu} - (r + k)\mu = -\tau'(W)(q - x),$$

so that the cost of compliance is $\mu = \int_t^T \tau'(W)(q - x)\, e^{-(r+k)(s-t)} ds$. The effects of the tax on resource value are particularly easy to see:

$$J_R(0) = \int_0^T \left[pq - C(q) - A(x) - \tau(W)(q - x)\right] e^{-rt} dt - \psi(K, X).$$

Provisions for pollution reduce the value of the resource by $\int_0^T \tau(W)$ $(q - x)\, e^{-rt} dt$, and hence reduce the resource rent and the resource's total contribution to national product. As above, the reason is that the value of the resource is a residual, the excess of project value over capital value.

In the full accounts, the tax is a transfer. This is consistent with the fact that the resources freed by the tax can be used to produce other goods.

## 6. Conclusion

An optimization model of a single firm is the counterpart of the aggregate optimization models that provide the theoretical rationale for green accounting in an ideal economy, as well as a guide to what variables should appear in green NNP, at what prices. The microeconomic view can provide much that the macroeconomic ones do not. In this paper, we have accounted for sunkenness of assets, production constraints and increasing returns.

Depreciation of the program includes depreciation of the resource and of capital. Except in the very special case of linear investment cost, the marginal magnitudes do not provide measures of depreciation of the resource, nor of the social cost of pollution. Only by use of an accounting convention can the appropriate repayments to capital be determined at any particular instant. The pollution cost (the cost of environmental degradation) is not explicitly involved. Rather, the pollution constraint causes the firm to incur costs – out-of-pocket costs for abatement, and opportunity costs of altering its optimal program – which feed back into the present value of the resource and into its rent. The compliance cost measures the implicit, marginal valuation placed by society on damage due to pollution, and accounting cost approximates it.

Contrary to a widespread view, all expenditures on defensive measures are incorporated into NNP in the same way that other expenditures are incorporated. The way in which the cost of environmental damage affects NNP is to reduce resource rent.

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## Notes

1. The stationarity of conditions such as price simplifies the analysis but does not affect the method of accounting. Cf. Cairns (2001b). Under uncertainty, the assumption that all investment is at time $t = 0$ amounts to an stylization of reality. Even where there is expansion of a mine after opening (under uncertainty), the expansion is lumpy (Lasserre and Harchaoui 2001). For lumpy investments, the optimality conditions are comparable to the ones considered herein, and the accounting considerations are similar.

2. Nonseparability of investment costs could be due to complementarities (or their opposite) involved in the simultaneous investments. The possibility may complicate the accounting below. Assuming separability of operating costs is comparatively innocuous for our purposes, simplifies the analysis, and preserves a link to Roan and Martin's paper.
3. To simplify the notation, we suppress subscripts representing time – for example, in $Q_t$, $q_t$, $x_t$, etc. – whenever the point in time is general or obvious from the context.
4. Many instances in which the solution is not internal give either $K = 0$ or $X = 0$ when the appropriate constraints are explicitly modelled, and both of these will be considered uninteresting in the current context. If $X = 0$, the problem is the one of a mine operating without environmental constraint. If $K = 0$, the orebody is not exploited.
5. In spite of concavity of investment cost, Cairns (1998) shows that an extension to control theory can be used.
6. In problem (2), $\Phi\left(K_0, W_0 \,|\bar{W}\right) < \Phi\left(K_0, W_0\,|\infty\right)$, the initial value of the resource in the problem with no regulation and hence no abatement.

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## Appendix

The solution is characterized in part by the relationships among certain critical times. The following are critical points:

- $t_u$ is the earliest point at which $u$ is positive on an interval;
- $t_w$ is the earliest point at which $w$ is positive on an interval;
- $t_K$ is the earliest point at which $q < K$ on an interval;
- $t_x$ is the earliest point at which $x = 0$ on an interval, after having been positive;

- $t_\mu$ is the earliest point at which $\mu = 0$; and
- $T$ is the point at which production ceases.

**Lemma 1** *There is an interval on which the shadow value, $w$, of the pollution constraint is positive, and hence that constraint is effective: $W = \bar{W}$. The interval begins at some point $t_w > 0$. The shadow value of a unit of pollution of the stream, $\mu$, is nonpositive, and reaches zero by the time of abandonment, $T$. Furthermore, its rate of change, $\dot{\mu}_t$, is nonnegative only if (a) $w_t > 0$, and hence the pollution constraint is effective ($W_t = \bar{W}$), or else (b) $w_t$ is equal to zero on the interval $(t, T)$, and hence $\mu = 0$ and $\dot{\mu} = 0$. Also, $\mu_t = 0$ if and only if $w = 0$ on the interval $(t, T)$.*

**Proof.** If $w_t = 0$ for all $t$, then the pollution constraint is never effective, and hence does not constrain the activities of the firm. For the problem to be interesting, it must be that $w > 0$ (and hence $W = \bar{W}$) on some interval. For $t > T$, since $q_t = 0$ by definition of $T$ and $x_t \geq 0$ by assumption, whenever $W > 0$ it must be by equation (1) that $\dot{W} \leq -kW < 0$, and hence $W < \bar{W}$ and $w = 0$. Therefore, $\mu_t = 0$ when $t \geq T$, and, by equation (8),

$$\mu_t = -\int_t^T w_s e^{-(r+k)(s-t)} ds \leq 0.$$

This is equation (15). (Since $t_\mu \leq T$, in this formula one could substitute $t_\mu$ for $T$.) Also, $W(0) = 0$ and $W$ is continuous. Therefore, $t_w > 0$. Since $w > 0$ on an interval $(t_w, t_w + \theta)$ where $\theta > 0$, therefore $\mu_t < 0$ for $t \in [0, t_w)$. Also, if $t < t_w$, then by equations (8) and (15), $\dot{\mu} = (r + k)\,\mu + w = (r + k)\,\mu < 0$.

By equation (8), $\dot{\mu} \leq w$. Hence, $\dot{\mu} > 0$ only if $w > 0$ and $\dot{\mu} = 0$ only if $w = -(r + k)\,\mu \geq 0$. If $w = 0$ when $\dot{\mu} \geq 0$ then $\mu = 0$ and hence $w = 0$ and $\dot{\mu} = 0$ thereafter. ■

**Lemma 2** *The shadow value of resource use, $\lambda_t$, is negative and decreasing. Reserves are exhausted at the time of abandonment, $T$. Abatement and output at $T$ are nil.*

**Proof.** By Lemma 1, $\mu_T = 0$. By equation (6), $y_T - u_T = A'(x_T) > 0$. Hence, $y_T > 0$ and $x_T = 0$. By equation (9)

$$(p + \lambda_T)q_T = C(q_T) - z_T q_T.$$

By equation (5) and the fact that $\mu_T = 0$,

$$(p + \lambda_T)q_T = (C'(q_T) + v_T)q_T.$$

Therefore,

$$C(q_T) = \left[C'(q_T) + v_T + z_T\right] q_T \geq q_T C'(q_T).$$

But $C(q) \leq qC'(q)$, and $C(q) = qC(q)$ only if $q = 0$. Hence, $q_T = 0 < K$, and so $v_T = 0$.

By equation (5), then, $\lambda_T \leq -\left[p - C'(0)\right]$, and hence $\lambda_T < 0$ in an interesting problem. By equation (10), $Q_T = S$. By equation (7), $\lambda_t < 0$, all $t$, and $\dot{\lambda} = r\lambda < 0$. ■

**Lemma 3** *Except for times at which they become or cease to be equal to their capacity levels, output, $q$, and abatement, $x$, are differentiable with respect to time; they are continuous everywhere. Also, $u$ and $v$ are continuous everywhere and differentiable except for those times, and $w$ is continuous.*

**Proof.** Suppose that $q > 0$ and hence that $z = 0$. By equation (5), $p+\lambda+\mu = C'(q)+v$. Since $p$ is constant and $\lambda$ and $\mu$ are differentiable, $C'(q) + v$ is differentiable for all $t$.

If $v > 0$ then $q = K$. Since $C'(q)+v = C'(K)+v$ is differentiable, $v$ is differentiable. In particular, $v \to 0$ smoothly as $t \to t_K$. If $q < K$ on an interval ending at $t_0$ and $q = K$ on an interval beginning at $t_0$, then $v$ increases smoothly from 0 just after $t_0$. (ii) If $q < K$ then $v = 0$ and $v$ is differentiable. Therefore, $v$ is differentiable except for possible kinks when it ceases to be or becomes equal to 0, at which points $v$ is continuous. Since $C'(q)$ is differentiable, $q$ is continuous everywhere, and also is differentiable except where there is a kink in $v$, i.e., when production begins to be or ceases to be at capacity.

By equation (6), $\mu = -A'(x) - u$ when $x > 0$. By a similar argument, $u$ and $x$ are continuous and $x$ has a kink only where there is a kink in $u$, i.e., when abatement ceases to be or begins to be at capacity.

By the maximum principle, $w$ is continuous at points of continuity of $q$ and $x$, and hence is continuous everywhere. ■

**Lemma 4** *Output never rises; thus, $\dot{q} \leq 0$.*

**Proof.** Suppose $\dot{q} > 0$. Then $0 < q < K$, and so $v = 0$ and $\dot{v} = 0$. By differentiation of equation (5), $\dot{\lambda} + \dot{\mu} = C''\dot{q} > 0$, so that $\dot{\mu} > -\dot{\lambda} > 0$. By Lemma 1, $\mu \leq 0$. By equation (8), $w > -(r+k)\mu \geq 0$, and so $W = \bar{W}$. Since $W$ is fixed at $\bar{W}$, $q - x - k\bar{W} = \dot{W} = 0$. By further differentiation, $\dot{x} = \dot{q} > 0$. Thus, $0 < x < X$; $u = 0 = y$; and $\dot{u} = 0 = \dot{y}$. By differentiation of equation (6), $\dot{\mu} = -A''\dot{x} < 0$. But this contradicts the condition $\dot{\mu} > 0$ found above. ■

It is intuitively clear that, for the regulation to be a real constraint to the firm, the optimal productive capacity must be great enough to imply that the pollution constraint is effective on some interval. (See also Lemma 1.) That is the subject of the following lemma.

**Lemma 5** *In an interesting problem, productive capacity is greater than the quantity of pollutant cleansed when the pollution constraint holds: $K > k\bar{W}$.*

**Proof.** Suppose that $K \leq k\bar{W}$. Then

$$\dot{W} = q - x - kW \leq q - kW \leq K - kW.$$

If $W < \bar{W}$, then $w = 0$. If $W = \bar{W}$ then $\dot{W} \leq K - k\bar{W} \leq 0$.

- Suppose $K < k\bar{W}$. Then $\dot{W} < 0$ whenever $W = \bar{W}$, and so $w = 0$ on all but sets of measure zero; $\mu_t = 0$, all $t$; and this is an uninteresting problem: regulation does not matter.
- Suppose $K = k\bar{W}$. By equation (13), $u > 0$ on some interval. This implies that $x = X > 0$ on that interval, and hence, by equation (1), that $\dot{W} = q - x - kW \leq K - X - kW$. If $W < \bar{W}$ then $w = 0$. If $W = \bar{W}$ then $\dot{W} \leq -X < 0$, and $w = 0$ (except possibly at a single point). Therefore, by equation (8), $\dot{\mu} = (r+k)\mu \leq 0$,

and so by differentiation of equation (6) with $x = X > 0$ and $\dot{x} = 0$, $\dot{u} = -\dot{\mu} \geq 0$. Thus, $\dot{u} \geq 0$ whenever $u > 0$. When $u = 0$, $\dot{u} = 0$. But, $u_T = 0$ because $x_T = 0$ from the proof of Lemma 2. Also, $u$ is continuous by Lemma 3. Therefore, $u = 0$ at all times. But this contradicts equation (13).

Therefore, $K > k\bar{W}$. ■

**Lemma 6** *Production is at capacity ($q = K$) on the interval* $[0, t_K]$*, where* $0 < t_K < T$.

**Proof.** By Lemma 4, if $q < K$ at $t = 0$, then $q < K$ for all $t$. Then $v = 0$ for all $t$. Equation (12) implies that $v > 0$ on some interval. This contradiction establishes that $q = K$ at time $t = 0$. But (a) $q = K$ at $t_K$, (b) $q_T = 0$, and (c) $q$ is continuous by Lemma 3. Therefore, $t_K < T$. ■

**Lemma 7** *When the pollution constraint is effective (when $W = \bar{W}$) on an interval, then abatement is nonincreasing and $q = x + k\bar{W}$ on that interval.*

**Proof.** By equation (1), on an interval on which $W = \bar{W}$, $\dot{W} = 0$ and

$$q - x - k\bar{W} = \dot{W} = 0. \tag{22}$$

By direct differentiation and by Lemma 4, $\dot{x} = \dot{q} \leq 0$. ■

**Lemma 8** *If abatement is decreasing then output is decreasing at the same rate.*

**Proof.** If $\dot{x} < 0$ then $0 < x < X$; $u = 0 = y$; and $\dot{u} = 0 = \dot{y}$. By differentiation of equation (6), $\dot{\mu} = -A''\dot{x} > 0$. By Lemma 1, $w > 0$, and hence $W = \bar{W}$. By differentiation of equation (22), $\dot{q} = \dot{x} < 0$. ■

Therefore, if $\dot{x} < 0$, then (a) $\dot{\mu} > 0$; (b) $w > 0$; (c) $W = \bar{W}$; (d) $\dot{q} = \dot{x} < 0$; (d) $q < K$; (e) $t > t_K$; and (f) $x < X$. Recall that $t_x$ is the time at which $x$ is first equal to 0 on an interval.

**Lemma 9** *(i) At time $t_x$, $q = k\bar{W}$, $w > 0$ and $\mu(t_x) < 0$. (ii) Also, $t_x < T$ . (iii) If output is constant at a level less than capacity, then abatement is nil ($x = 0$) and output is equal to the maximal permitted by the level of cleansing ($q = k\bar{W}$). (iv) On the interval $(t_x, T)$, abatement, $x$, remains at zero. (v) There is a nondegenerate interval, $(t_x, t_\mu)$, where $t_\mu < T$, on which $q = k\bar{W}$. On that interval, $w > 0$ and $\mu < 0$, and $p - C'(q) + \mu = -\lambda$ rises at the rate of interest.*

**Proof.**

(i) Since $x$ is differentiable, there is an interval $(\tau, t_x)$ on which $\dot{x} < 0$. By Lemma 8 and its proof, $\dot{q} = \dot{x} < 0$ and $w > 0$. Therefore, $W = \bar{W}$ for $t \in (\tau, t_x]$. Since $x = 0$ at $t_x$ by definition, and $\dot{W} = 0$ at $t_x$, $q = x + k\bar{W} = k\bar{W}$ at $t_x$. Also, since $x \downarrow 0$ as $t \uparrow t_x$, therefore $\mu \to -A'(0) < 0$ (from the proof of Lemma 8). By the continuity of $w$ (from Lemma 3), there is an interval $(t_x, \tau')$ on which $w > 0$, and hence on which $W = \bar{W}$. Therefore, on $(t_x, \tau')$,

$$0 = \dot{W} = q - x - k\bar{W} = q - k\bar{W}.$$

(ii) Because (a) $q = k\bar{W}$; (b) $q$ is differentiable; and (c) $q_T = 0$, therefore $t_x < T$.

(iii) Suppose $\dot{q} = 0$ when $q < K$ (and hence, $t > t_K$). Then by differentiation of equation (5) with $v = 0 = \dot{v}$, we obtain that $\dot{\lambda} + \dot{\mu} = 0$, so that $\dot{\mu} = -\dot{\lambda} > 0$ and hence, by the proof of Lemma 1, $w > 0$. By Lemma 7, then, $\dot{x} = \dot{q} = 0$. We now prove that $x = 0$ by contradiction. Suppose that $x > 0$. Then by differentiation of equation (6), $A''\dot{x} + \dot{\mu} + \dot{u} = 0$, so that $\dot{u} = -\dot{\mu} < 0$. Therefore, $u > 0$, and hence $x = X$. Let $t_q$ be the time from which $\dot{q} = 0$. Then, $t_q > t_K$. Moreover, on an interval $(\tau, t_q)$ for some $\tau \geq t_K$, we have $\dot{q} < 0$. Also, at $t_q$, $w + (r + k)\mu = \dot{\mu} = -\dot{\lambda} = -r\lambda$. Therefore, $w > 0$, and in fact, by continuity of $w$, then, $t_w < t_q$. Let $t_0 = \max(t_w, t_K) < t_q$. On the interval $(t_0, t_q)$, $\dot{q} < 0$ and $w > 0$, and hence by Lemma 7, $\dot{x} = \dot{q} < 0$. But $x = X$ at $t_q$. This contradiction implies that $x = 0$ if $\dot{q} = 0$ and $q < K$. But then, since $w > 0$, $\dot{W} = q - k\bar{W} = 0$, and $q = k\bar{W}$, so that $t_q = t_x$.

(iv) We have shown that $q(t_x) = k\bar{W}$ and $x(t_x) = 0$. Also, by Lemma 4, $\dot{q} \leq 0$, all $t$. Therefore, for $t > t_x$, $q - x \leq q \leq k\bar{W}$. Integrating equation (1) for $t \geq t_x$ yields

$$W_t = \bar{W}e^{-k(t-t_x)} + \int_{t_x}^{t} (q - x)\, e^{-k(t-s)} ds.$$

It is easily checked by integration that $W_t \leq \bar{W}$, and that $W_t = \bar{W}$ only if $q = k\bar{W}$ and $x = 0$ throughout the interval $(t_x, t)$. Furthermore, whenever $q < k\bar{W}$ or $x > 0$ on an interval beginning at some time $t_q$, $w = 0$ for $t > t_q$ and hence $\mu = 0$ for $t > t_q$. If $\mu = 0$ then

$$H = pq - C(q) - A(x) + \lambda q$$

and the Hamiltonian is maximized by the choice $x = 0$. Therefore, $x = 0$ for $t > t_x$.

(v) By the proof of part (ii) and the continuity of $\mu$, $\mu(t_x) = -A'(0)$. Therefore, $w(t_x) > 0$; but $w$ must reach zero at some point $t_\mu \leq T$, because $\mu = 0$ for $t \geq T$. But, at any point $t$ just before $t_\mu$, $w > 0$, and hence $q = k\bar{W} > 0$. Because $q$ is continuous and $q_T = 0$, therefore $t_\mu < T$. Since $q = k\bar{W} \in (0, K)$, we have $v = 0$ and $z = 0$. By equation (5), $p - C'(q) + \mu = -\lambda$, which rises at the rate of interest. ■

At time $T$, the reserves are exhausted ($Q_T = S$). Thereafter, concentrations of effluent in the stream follow the equation, $\dot{W} = -kW$.

**Lemma 10** *(i) On $(t_\mu, T)$, output falls toward zero, such that the net price, $p - C'(q)$, rises at the rate of interest. (ii) The level of pollution, $W$, declines monotonically after time $t_\mu$.*

**Proof.**

(i) Since $\mu < 0$ at $t_x$, therefore $t_\mu > t_x$. But $t_x > t_K$ since $q = k\bar{W} < K$ at $t_x$. Thus, $v = 0$ at $t_\mu$.For $t \geq t_\mu$, $\mu = 0$, and hence by equation (5), $p - C' = -\lambda$. Therefore, $q$ declines steadily such that $[p - C'(q)]$ rises steadily at the rate of interest.

(ii) At $t_\mu$, $q = k\bar{W}$. For $t > t_\mu$, $q < k\bar{W}$ and, from the proof of Lemma 9 (iv),

$$W_t = \bar{W}e^{-k(t-t_\mu)} + \int_{t_\mu}^{t} qe^{-k(t-s)} ds < \bar{W}.$$

By continuity of $W$ and the fact that $W_t < \bar{W}$, there is an interval $(t_\mu, t')$ on which $\dot{W} < 0$, and hence $q_s < kW_{t_\mu}$ for $s > t' > t_\mu$. By similar reasoning, $W_s < W_{t'}$ for $s < T$, and hence, $-kW < \dot{W} < 0$ until $T$. At $T$, $W > \bar{W}e^{-k(T-t_\mu)} > 0$ and thereafter, $\dot{W} = -kW < 0$.

**Lemma 11** *(i) Abatement is at capacity by the time that the pollution constraint is effective ($t_u \leq t_w$). If the two occur simultaneously, then output is at capacity ($q = K$). (ii) The pollution constraint is reached while production is at capacity ($t_w < t_K$). (iii) Extractive capacity is the sum of abatement capacity and maximal natural cleansing: $K = X + k\bar{W}$.*

**Proof.**

(i) Suppose $t_w < t_u$. There is an interval $(t_w, \tau)$ with $\tau \leq t_u$ on which $W = \bar{W}$ and $u = 0 = \dot{u}$. Also, $\dot{W} = q - x - k\bar{W}$, and so $\dot{q} = \dot{x}$. If $x = 0$ then by the same proof as in Lemma 9, $x = 0$ up to time $T$. But then $u$ is never positive, contrary to equation (13). If $x > 0$, then by Lemma 4, $\dot{x} \leq 0$, and $w + (r+k)\mu = \dot{\mu} = -A''\dot{x} + \dot{u} \geq 0$, and hence $w \geq -(r+k)\mu$. By continuity of $w$, however, $W = 0$ at $t_w$. There is a time, then, sufficiently close to $t_w$ such that $w < -(r+k)\mu$. This contradiction implies that $t_u \leq t_w$.

If $t_u = t_w$ then, just after this point, $\dot{u} > 0$. Since $u > 0$, therefore $x = X$ and $\dot{x} = 0$, and hence $\dot{\mu} = -\dot{u} < 0$. Also, since $w > 0$, $W = \bar{W}$ and $\dot{W} = 0$. Also, $\dot{W} = q - X - k\bar{W}$. Hence, $q = X + k\bar{W}$, so that $\dot{q} = 0$. If $q < K$ then $\dot{\lambda} + \dot{\mu} = 0$, and so $\dot{\mu} > 0$, a contradiction. Therefore, $q = K$, and $\dot{v} = \dot{\lambda} + \dot{\mu} < 0$.

(ii) Suppose that $t_w \geq t_K$. Then $q(t_w) < K$. At time $t_w$, $\dot{\mu} = (r+k)\mu < 0$. By the continuity of $\dot{\mu} = (r+k)\mu + w$, there is an interval, $(t_w, \tau)$, on which $\dot{\mu} < 0$. Consider that interval in the following cases.

(a) Suppose that $u = 0$. If $x > 0$, then $-A''\dot{x} = \dot{\mu} < 0$. Therefore, by differentiation of equation (22), $\dot{q} = \dot{x} > 0$. This result contradicts Lemma 4. If $x = 0$, then $q = k\bar{W}$, and the equilibrium unfolds as in Lemma 9. Therefore, $u > 0$ on some interval ending before $t_w$. On *that* interval, $w = 0$ by hypothesis, and also the following hold: $x = X$; $\dot{x} = 0$; $0 = -A''\dot{x} = \dot{\mu} + \dot{u}$; $\dot{u} = -\dot{\mu}$. But, as $w = 0$, $\dot{\mu} < 0$, and so $\dot{u} > 0$, and $u > 0$. This holds true until $t_w$. But $u(t_w) = 0$. The implied jump is impossible by the continuity of $u$.

(b) Suppose that $u > 0$. Then $x = X$, and $q = X - k\bar{W}$. Hence, $\dot{q} = 0$. By Lemma 9 (iii), then, either $q = K$ or $q = k\bar{W}$. If $q = K$ then $t_w < t_K$, contrary to hypothesis. If $q = k\bar{W}$ then $x = 0$. This is contrary to the hypothesis $u > 0$.

(iii) Since $t_w < t_K$, on $(t_w, t_K)$, $W = \bar{W}$ and $q = K$. By differentiation of equation (1), $\dot{x} = 0$. But $x > 0$, since $t_K < t_x$. By differentiation of equation (6), $\dot{u} = -\dot{\mu} \neq 0$. Therefore, $u > 0$ and $x = X$. Hence, by equation (1), $K - X - k\bar{W} = 0$. ■

Therefore, $t_u \leq t_w < t_K$.

**Lemma 12** *(i) The level of pollution, W, is nondecreasing ($\dot{W} \geq 0$) until its shadow value drops to zero at time $t_\mu$. (ii) If the level of pollution is constant, then it is at its maximal level, $\bar{W}$.*

**Proof.**

(i) Suppose $\dot{W} < 0$. Then by equation (1), $q - x - kW < 0$, and hence, $q < x + kW = X + k\bar{W} = K$. But, when $q < K$, $W = \bar{W}$ until $t_\mu$, by Lemma 9.

(ii) Suppose $\dot{W} = 0$. Then, by differentiation of equation (1) and by Lemma 4, $\dot{x} = \dot{q} \leq 0$. If $\dot{q} = 0$, then $q = K$ or $q = k\bar{W}$, and $W = \bar{W}$. If $\dot{q} < 0$ then $W = \bar{W}$ also. ■

**Lemma 13** *Abatement reaches capacity when the pollution constraint is reached: $t_u = t_w$.*

**Proof.** It $t_u < t_w$, then there is an interval on which $x = X$ and $q = K$, but $W < \bar{W}$. On the interval $(t_u, t_w)$,

$$W = W_u e^{k(t-t_u)} + \int_{t_u}^{t} (K - X)e^{k(s-t)} ds < \bar{W}.$$

Therefore, $W$ never attains $\bar{W}$. (The approach is asymptotic.) This contradiction implies that $t_u = t_w$. ■

**Lemma 14** *The shadow value of capacity, $v$, is strictly decreasing to a point $t_v$, where $t_w < t_v \leq t_K$. If $t_v < t_K$, then on an interval contained in $(t_v, t_K)$, $v$ may increase. Still, on the interval $(t_w, t_K)$, $\dot{v} + \dot{u} = \dot{\lambda} < 0$.*

**Proof.** For $t \in (t_w, t_K)$, $q = K$ and $x = X$. By differentiation of equation (5), $\dot{v} = \dot{\lambda} + \dot{\mu}$. By equations (7) and (8), if $w > -r\lambda - (r + k)\mu$ then $\dot{v} > 0$. But $v$ is continuous and must reach zero at $t_K$. Also, since $x = X$, by differentiation of equation (6), $\dot{\mu} = -\dot{u}$, and hence $\dot{v} + \dot{u} = \dot{\lambda} < 0$. ■

**Proof of Corollary 1**

Suppose there is a jump from $q = K$ to $q = 0$ at time $t_K$, as found by Crabbé (1982), and hence from $x = X$ to $x = 0$. At a time just before $t_K$, say $t_K^-$, $x > 0$ and hence $\mu + a + u = 0$. But $u \to 0$ and hence $\mu \to -a$ as $t \to t_K$. Just after $t_K$, however, say at $t_K^+$, $\dot{W} = q - x - kW = -kW$. Hence, $W < \bar{W}$ and $w = 0$. But then $\mu = 0$. This contradicts the continuity of $\mu$.

Suppose $\dot{q} = \dot{x} < 0$ after $t_K$. Then $\dot{\mu} = -\dot{a} = 0$. But $\lambda + \mu = -(p - c - a)$, and hence $\dot{\lambda} = 0$. This contradicts equation (7).

Suppose there is a jump to $q = k\bar{W}$, and output remains at $k\bar{W}$ on a nondegenerate interval. Then $\lambda + \mu = -(p - c)$, a constant. But $\dot{\lambda} < 0$, and hence $\dot{\mu} > 0$. At $t_\mu$, $\mu = 0$ and hence $\lambda = -(p - c)$. Thereafter, $q = 0$, as $\dot{\lambda} < 0$ and we must have $q(p - c + \lambda + \mu - v) = q(p - c + \lambda) = 0$. ■