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Numerical representability of semiorders

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Abstract

In the framework of the analysis of orderings whose associated indifference relation is not necessarily transitive, we study the structure of a semiorder, and its representability through a real-valued function and a threshold. Inspired in a recent characterization of the representability of interval orders, we obtain a full characterization of the existence of numerical representations for semiorders. This is an extension to the general case of the classical Scott–Suppes theorem concerning the representability of semiorders defined on finite sets.

Keywords: Orderings on a set; Semiorders; Numerical representations of orderings

JEL classification: 213

1. Introduction

The concept of a semiorder was introduced by R. Duncan Luce in 1956 (Fishburn, 1968; Fishburn, 1970; Krantz, 1967; Luce, 1956; Scott, 1964; Scott and Suppes, 1958; Suppes and Zinnes, 1963; Tversky, 1969). The original idea was that of presenting a mathematical model of preferences enable to capture situations of 'intransitive indifference with a threshold of discrimination':

> *Suppose, for instance, that a man is not able to declare different two quantities of a same thing when such two quantities do not differ more than a threshold of*

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discrimination or perception, α. This threshold is a non-negative real number, and it is supposed to be the same for every individual. That is, if $a \prec b$ means here 'a man is able to realize that the quantity a is smaller than the quantity b', then we have $a \prec b \Leftrightarrow a + \alpha < b$.

A classical example, attributed to Armstrong (1950), *considers a man that prefers a cup of coffee with a whole portion of sugar, to a cup of coffee with no sugar at all. If such man is forced to declare his preference between a cup with no sugar at all and a cup with only one molecule of sugar, he will declare them indifferent. The same will occur if he compares a cup with n molecules and a cup with $n + 1$ molecules of sugar. However, after a very large number of intermediate comparisons we would finally confront him with a cup that has a whole portion of sugar that he is able to discriminate from the cup with no sugar at all. Here, we observe a clear intransitivity of the indifference.*

Classical studies on semiorders appeared in Bosi and Isler (1995), Fishburn (1970a–c, 1973), Gensemer (1987a,b, 1988), Pirlot (1990, 1991), Pirlot and Vincke (1997). In some of those works were obtained either necessary or else sufficient conditions for the existence of a numerical representation of a semiorder $\prec$ defined on a nonempty set X, by means of a real-valued function u: $X \to \mathbb{R}$ and a threshold $\alpha \geq 0$, $(\alpha \in \mathbb{R})$, such that $x \prec y \Leftrightarrow u(x) + \alpha < u(y)$ $(x,y \in X)$. This type of representations of semiordered structures has been also applied in choice theory under risk (Fishburn, 1968), modellization of choice with errors (Agaev and Aleskerov, 1993), social welfare theory (Ng, 1975), general equilibrium theory (Jamison and Lau, 1977) and expected utility theory on mixture spaces (Vincke, 1980).

However, despite those interesting approaches and the available results about the existence of suitable representations of semiordered structures, the crucial (as well as abstract) question of finding, for the general case, a full characterization of the representability, as defined above, of a semiordered structure $(X, \prec)$, remained still open (see, e.g. the introductory sections in Gensemer (1987, 1988)).

Since a semiorder is a particular case of interval order, a previous crucial question was to characterize the representability of interval orders. Such problem was solved by Fishburn (1973). Also, a different characterization was obtained by Doignon et al. (1984). Furthermore, a new alternative solution using only one ordinal condition that characterizes the representability of a structure of interval order has been recently obtained in Oloriz et al. (1998).

The main goal of the present paper is that of achieving a full characterization of the representability of a semiorder. It extends to the general case the well-known Scott–Suppes theorem formerly stated for semiorders defined on finite sets.

2. Previous concepts and results

Let X be a nonempty set. In what follows '$\prec$' will denote an asymmetric binary relation defined on X. Associated to $\prec$ we will also consider the binary relations '$\precsim$' and '$\sim$', respectively, defined as $x \precsim y \Leftrightarrow \neg(y \prec x)$ and $x \sim y \Leftrightarrow x \precsim y$, $y \precsim x$. The relation

$\prec$ is usually called strict preference. The relation $\precsim$ is said to be the weak preference, and $\sim$ is called the indifference, associated to $\prec$.

Given $x \in X$, the sets $L(x) = \{y \in X : y \prec x\}$ and $U(x) = \{y \in X : x \prec y\}$ are called, respectively, the lower contour set and the upper contour set relative to $\prec$.

Definition. The binary relation $\prec$ is said to be an interval order if $(x \prec y,\, a \prec b) \Rightarrow$ either $x \prec b$, or $a \prec y$. An interval order $\prec$ is said to be a semiorder if in addition $a \prec b \prec c \Rightarrow a \prec d$ or $d \prec c$ $(a,b,c,d \in X)$.

Observe that because $\prec$ is asymmetric, if it is an interval order then it must be transitive. (To prove it, just take $a = y$, $b = z$ in the definition of interval order, and observe that the possibility $y \prec y$ must be rejected because every asymmetric relation is in particular irreflexive.) Notice also that $\prec$ being an interval order, the associated relations $\precsim$ and $\sim$ may fail to be transitive. An example is the relation $\prec$ defined on the real line $\mathbb{R}$ as $x \prec y \Leftrightarrow x + 1 < y$. However an interval order $\prec$ is always pseudo-transitive: That is $x \prec y \precsim z \prec t \Rightarrow x \prec t$ $(x,y,z,t \in X)$.

It is straightforward to see that if $\prec$ is an asymmetric binary relation defined on X, then pseudotransitivity is equivalent to the fact of $\prec$ being an interval order.

Moreover:

An asymmetric binary relation $\prec$ defined on X is a semiorder if and only if it satisfies the condition of generalized pseudotransitivity, namely, for every $x,y,z,t \in X$ the following three conditions hold true:

1. $x \prec y \precsim z \prec t \Rightarrow x \prec t$,
   $x \precsim y \prec z \prec t \Rightarrow x \prec t$,
   $x \prec y \prec z \precsim t \Rightarrow x \prec t$.

See Gensemer (1987) for details.

An interval order $\prec$ defined on X is said to be representable if there exist two real-valued functions $u,v:X \to \mathbb{R}$ such that $u(x) \le v(x)$ and $x \prec y \Leftrightarrow v(x) < u(y)$ $(x,y \in X)$. Since $\prec$ is asymmetric, this is equivalent to associate to each element $x \in X$ a real-interval (that eventually may collapse to a single point), $I_x = [u(x),v(x)]$. Thus $x \prec y$ if and only if I_x is located on the left of I_y, and I_x does not meet I_y. This kind of ‘interval-representation’ gave rise to the nomenclature of interval order. However, not every interval order is representable (see Oloriz et al. (1998) for details).

Inspired by Scott and Suppes (1958), we will say that a semiorder $\prec$ defined on X is representable in the sense of Scott and Suppes (or representable as a semiorder) if there exist a real-valued function $u:X \to \mathbb{R}$ and a non-negative real number α (called threshold) such that $x \prec y \Leftrightarrow u(x) + \alpha < u(y)$. Observe that this is a particular case of representation of an interval order, in which $v(x)$ can be defined as $u(x) + \alpha$ $(x \in X)$. Observe also that this is equivalent to associate to each element $x \in X$ a real-interval $I_x = [u(x),u(x) + \alpha]$. In this case, all the intervals have the same length α. Obviously, all of them will collapse to a single point if and only if $\alpha = 0$.

Following Fishburn (1970a,b), we shall associate to an interval order $\prec$ two new binary relations, respectively denoted by $\prec^*$ and $\prec^{**}$ and defined by $x \prec^* y \Leftrightarrow x \prec$

$z \precsim y$ for some $z \in X$ $(x,y \in X)$, and, similarly, $x \prec^{**} y \Leftrightarrow x \precsim z \prec y$ for some $z \in X$ $(x,y \in X)$. If we denote $x \precsim^{*} y \Leftrightarrow \neg(y \prec^{*} x)$ and $x \precsim^{**} y \Leftrightarrow \neg(y \prec^{**} x)$ $(x,y \in X)$ then it is straightforward to see that $x \precsim^{*} y \Leftrightarrow (y \prec z \Rightarrow x \prec z \quad (z \in X))$ and similarly $x \precsim^{**} y \Leftrightarrow (z \prec x \Rightarrow z \prec y \quad (z \in X))$. Observe also that in terms of the contour sets, it follows that $x \precsim^{*} y \Leftrightarrow U(y) \subseteq U(x)$ and also $x \precsim^{**} y \Leftrightarrow L(x) \subseteq L(y)$ $(x,y \in X)$.

A preorder $\precsim$ defined on a nonempty set X is a reflexive and transitive binary relation defined on X. If it is also complete (i.e. either $x \precsim y$ or else $y \precsim x$ for every $x,y \in X$), $\precsim$ is said to be a total preorder. A total preorder $\precsim$ is said to be representable if there exists a real-valued function $u:X \to \mathbb{R}$ such that $x \precsim y \Leftrightarrow u(x) \leq u(y)$. If we denote $x \prec y \Leftrightarrow \neg(y \precsim x)$ and $x \sim y \Leftrightarrow x \precsim y \precsim x$ $(x,y \in X)$, then it is clear that a representable total preorder provides a representation of the weak preference $\precsim$ associated to a semiorder $\prec$, for the special case in which the threshold α equals zero. (In classical applications, this corresponds to situations of 'perfect discrimination or perception'.) An antisymmetric total preorder $\precsim$ on a set X (i.e. $x \precsim y \precsim x \Rightarrow x = y$ $(x,y \in X)$) is said to be a total order.

If $\precsim$ is a total preorder on a set X, and $x \prec y$ $(x,y \in X)$, then we say that the pair (x,y) defines a jump if there is no $z \in X$ such that $x \prec z \prec y$. A subset $Y \subseteq X$ is said to be cofinal if for every $x \in X$ there exists $y \in Y$ such that $x \prec y$. Similarly Y is said to be coinitial if for every $x \in X$ there exists $y \in Y$ such that $y \prec x$. An element $z \in X$ is said to be minimal (respectively: maximal) with respect to $\precsim$ if $z \precsim x$ (respectively: $x \precsim z$) for every $x \in X$. Consider now the quotient space $X/\sim$ of X through the equivalence relation $\sim$. The ordering $\precsim$ is compatible with this quotient, so that $X/\sim$ becomes a totally ordered set. The total preorder $\precsim$ is said to be Dedekind complete if in $X/\sim$ every subset $C \subseteq X/\sim$ that is bounded above with respect to $\prec$has a supremum (i.e.: a smallest upper bound) denoted by *sup* C in $X/\sim$.

When dealing with a total order $\precsim$ defined on X we can endow X with the order topology whose subbasis is defined by the lower and upper contour sets relative to $\prec$.

It can be proved that:

Lemma 1.

(*i*) *See Gillman and Jerison* (1960), *Section* 0.6 *on p*. 3, *Birkhoff* (1967), *p*. 200, *or else Candeal and Induráin* (1999): *Every total preorder* $\precsim$ *has a Dedekind complete extension without jumps that has neither minimal nor maximal elements*. *This extension is unique up to order isomorphism*.

(*ii*) *See*, *for example*, *Birkhoff* (1967), *p*. 243: *The order topology relative to a total order* $\precsim$ *on a set X is connected if and only if* $\precsim$ *is Dedekind complete and has no jumps*.

Remark 1. The extension corresponding to Lemma 1 (i) may produce a set that is much bigger than the given set. Notice that if we start with a single $X = \{x\}$, with the trivial total ordering, then we arrive to an extension that is isotonic to the real line $\mathbb{R}$.

Coming back to the study of interval orders and semiorders, it is well-known (see e.g. Proposition 2.1 in Bridges (1985)) that:

Lemma 2. *Let $\prec$ be an asymmetric binary relation defined on a nonempty set X. Then the following statements are equivalent:*

(i) $\prec$ is an interval order,
(ii) $\precsim^*$ is a total preorder,
(iii) $\precsim^{**}$ is a total preorder.

In addition, $\precsim$ is transitive if and only if $\precsim$, $\precsim^*$, and $\precsim^{**}$ coincide.

In what concerns semiorders among interval orders, suppose that an interval order $\prec$ has been defined on a set X, and define the following new binary relation, introduced in Fishburn (1970): $x \prec^0 y \Leftrightarrow x \prec^* y$ or else $x \prec^{**} y$ $(x,y \in X)$. In Bosi and Isler (1995) it is proved the following fact:

Lemma 3. *The interval order $\prec$ is actually a semiorder if and only if $\prec^0$ is asymmetric. Moreover in this case the associated relation $\precsim^0$ defined as $x \precsim^0 y \Leftrightarrow \neg(y \prec^0 x)$; $(x,y \in X)$ becomes a total preorder.*

A key concept, used in Oloriz et al. (1998) to get a characterization of the representability of interval orders, is that of interval order separability (henceforward i.o.-separability). An interval order $\prec$ on a set X is said to be i.o.-separable if there exists a countable subset $D \subseteq X$ such that for every $x,y \in X$ with $x \prec y$ there exists an element d in D such that $x \prec d \precsim^{**} y$.

Remark 2. This is clearly a generalization of the condition of perfect separability that characterizes the representability of a totally preordered structure $(X, \precsim)$. (See the first chapters in Bridges and Mehta (1995) for details). A total preorder $\precsim$ on a set X is said to be perfectly separable if there exists a countable subset $D \subseteq X$ such that for every $x,y \in X$ with $x \prec y$ there exists an element d in D such that $x \prec d \precsim y$.

The nub result on the representability of interval orders is in order now:

Proposition 1. *See Oloriz et al.* (1998): *Let X be a nonempty set endowed with an interval order $\prec$. Then, the following statements are equivalent*:

(i) $\prec$ is i.o.-separable,
(ii) there exists a bivariate map $F: X \times X \to \mathbb{R}$ such that $x \precsim y \Leftrightarrow F(x,y) \geq 0$ and $F(x,y) + F(y,z) = F(x,z) + F(y,y)$ for every $x,y,z \in X$,
(iii) $\prec$ is representable.

3. Scott–Suppes representability of semiorders

We are interested in finding representations for a semiorder $\prec$, defined on a set X, whose associated indifference $\sim$ is not transitive. (If $\sim$ were transitive, by Lemma 2, $\precsim$

would be a total preorder, and characterizations of the representability of total preorders are well-known, see e.g. the first chapters in Bridges and Mehta (1995)).

Let us assume that $\prec$ is representable through a real-valued function $u:X \to \mathbb{R}$ and a threshold $\alpha > 0$ such that $x \prec y \Leftrightarrow u(x) + \alpha < u(y)$ $(x,y \in X)$ (observe also that $\alpha = 0$ would imply that $\sim$ is transitive, against the hypothesis). We can enlarge the set X in the following way: For each $a \in \mathbb{R} \backslash u(X)$, add an extra element y_a to the set X. Let $Y = \{y_a : a \in \mathbb{R} \backslash u(X)\}$ the set of new elements added to X. Let $\bar{X} = X \cup Y$. The function $u:X \to \mathbb{R}$ can easily be extended to a map $\bar{u}:\bar{X} \to \mathbb{R}$ in the natural way, that is $\bar{u}(x) = u(x)$ if $x \in X$ and $\bar{u}(y_a) = a$ if $y_a \in Y$. Thus, the semiorder $\prec$ can also be extended to $\bar{X}$. We simply declare $\bar{x} \mathrel{\bar{\prec}} \bar{y} \Leftrightarrow \bar{u}(\bar{x}) + \alpha < \bar{u}(\bar{y})$ $(\bar{x}, \bar{y} \in \bar{X})$. Then the extension $\bar{\prec}$ is plainly a representable semiorder.

Let us analyze the properties of this extension $\bar{\prec}$:

Taking into account that $\bar{u}(\bar{X}) = \mathbb{R}$, if we consider the orders $\bar{\prec}^{*}$, $\bar{\prec}^{**}$ and $\bar{\prec}^{0}$ on $\bar{X}$, we have $\bar{x} \mathrel{\bar{\prec}^{*}} \bar{y} \Leftrightarrow$ there exists $\bar{z} \in \bar{X}$ such that $\bar{x} \mathrel{\bar{\prec}} \bar{z} \mathrel{\bar{\precsim}} \bar{y}$. This implies $\bar{u}(\bar{x}) + \alpha < \bar{u}(\bar{z}) \leq \bar{u}(\bar{y}) + \alpha$. Also $\bar{x} \mathrel{\bar{\prec}^{**}} \bar{y} \Leftrightarrow$ there exists $\bar{z} \in \bar{X}$ such that $\bar{x} \mathrel{\bar{\precsim}} \bar{z} \mathrel{\bar{\prec}} \bar{y}$, and this implies $\bar{u}(\bar{x}) \leq \bar{u}(\bar{z}) + \alpha < \bar{u}(\bar{y})$.

Hence it is straightforward to see that $\bar{x} \mathrel{\bar{\prec}^{*}} \bar{y} \Leftrightarrow \bar{u}(\bar{x}) + \alpha < \bar{u}(\bar{y}) + \alpha \Leftrightarrow \bar{u}(\bar{x}) < \bar{u}(\bar{y}) \Leftrightarrow \bar{x} \mathrel{\bar{\prec}^{**}} \bar{y}$ $(\bar{x}, \bar{y} \in \bar{X})$.

Hence $\bar{\prec}^{*}$, $\bar{\prec}^{**}$ and $\bar{\prec}^{0}$ coincide.

As a consequence, notice that $\bar{x} \mathrel{\bar{\sim}^{*}} \bar{y} \Leftrightarrow \bar{x} \mathrel{\bar{\sim}^{**}} \bar{y} \Leftrightarrow \bar{x} \mathrel{\bar{\sim}^{0}} \bar{y} \Leftrightarrow \bar{u}(\bar{x}) = \bar{u}(\bar{y})$ $(\bar{x}, \bar{y} \in \bar{X})$.

Let Z be now the quotient $\bar{X}/\bar{\sim}^{0}$, where every equivalence class on $\bar{X}$ relative to $\bar{\sim}^{0}$ has been identified to a point. The ordering $\bar{\prec}$ and the map $\bar{u}$ are plainly compatible with this quotient. Working directly on Z, on which $\bar{\precsim}^{0}$ is now a total order, observe that any element $z \in Z$ has a successor. This successor, denoted $S(z)$, is defined as the element in Z such that $\bar{u}(S(z)) = \bar{u}(z) + \alpha$. In a completely analogous way, any element $z \in Z$ has an antecedent, that we denote $A(z)$, defined through the condition $\bar{u}(A(z)) = \bar{u}(z) - \alpha$. Obviously $A(S(z)) = z = S(A(z))$ $(z \in Z)$. Call $S_0(z) = z$, $S_1(z) = S(z)$, $S_2(z) = S(S(z))$, $S_3(z) = S(S_2(z))$, $\ldots$, $S_{n+1}(z) = S(S_n(z))$ $(n \in \mathbb{N})$ and similarly $A_0(z) = z$, $A_1(z) = A(z)$, $A_2(z) = A(A(z)), \ldots$, $A_{n+1}(z) = A(A_n(z))$ $(n \in \mathbb{N})$.

Now observe that for every $z \in Z$, the family $C_z = \{z\} \cup \{S_{2k}(z) : k \in \mathbb{N}, k > 0\} \cup \{A_{2k}(z) : k \in \mathbb{N}, k > 0\}$ is, by definition, a maximal totally ordered subset in Z with respect to $\bar{\prec}$. In particular it is coinitial and cofinal in Z.

For any $z \in Z$ the element $S(z)$ is the infimum, relatively to the total preorder $\bar{\precsim}^{0}$, of the set $\{y \in Z : z \mathrel{\bar{\prec}} y\} = U_{\bar{\prec}}(z)$. Moreover, $S(z) \mathrel{\bar{\precsim}} z \mathrel{\bar{\prec}^{0}} S(z)$. Similarly the element $A(z)$ is the supremum, relatively to the total preorder $\bar{\precsim}^{0}$, of the set $\{y \in Z : y \mathrel{\bar{\prec}} z\} = L_{\bar{\prec}}(z)$. In addition, $A(z) \mathrel{\bar{\prec}^{0}} z \mathrel{\bar{\precsim}} A(z)$.

Finally, notice that by definition of successor and antecedent elements, it holds that $z_1 \mathrel{\bar{\prec}} z_2 \Leftrightarrow S(z_1) \mathrel{\bar{\prec}^{0}} z_2 \Leftrightarrow z_1 \mathrel{\bar{\prec}^{0}} A(z_2)$ $(z_1, z_2 \in Z)$.

A new glance at $\bar{X}$ shows that the extended total order $\bar{\prec}^{0}$, considered in the previous construction, is the Dedekind complete, without jumps and neither minimal nor maximal elements, extension of the total preorder $\precsim^{0}$ defined on X. By Lemma 1, this extension is unique up to order isomorphism.

Remark 3. In the classical Scott–Suppes result (Scott and Suppes, 1958) it is proved that every finite semiordered structure $(X, \prec)$ admits a representation through a utility function $u:X \to \mathbb{R}$ and a threshold $\alpha \geq 0$ such that $a \prec b \Leftrightarrow u(a) + \alpha < u(b)$ $(a,b \in X)$. Moreover $\precsim$ is a total preorder if and only if there exists a representation of such kind with $\alpha = 0$.

Now we are ready to present a generalized Scott–Suppes theorem of representability of semiorders, in which we characterize the existence of a Scott–Suppes representation of semiorders defined on sets which may or may not be finite.

Generalized Scott–Suppes Representability Theorem (GSSRT). *Let X be a set endowed with a semiorder* $\prec$. *Let* $(\overline{X}, \overline{\precsim}^0)$ *be the Dedekind complete, without jumps and neither minimal nor maximal elements, extension of the totally preordered structure* $(X, \precsim^0)$.

Then the semiorder $\prec$ is representable through a function $u:X \to \mathbb{R}$ and a strictly positive threshold $\alpha > 0$ such that $x \prec y \Leftrightarrow u(x) + \alpha < u(y)$ $(x,y \in X)$ if and only if the extension $(\overline{X}, \overline{\prec}^0)$ satisfies the following properties:

1. Let $Z = \overline{X}/\overline{\sim}^0$. Then for a given $z \in Z$ there exist two elements $S(z)$ and $A(z)$ in Z such that $A(S(z)) = z = S(A(z))$, and for every $a,b \in Z$ it holds that $S(a)\overline{\prec}^0 b \Leftrightarrow a \overline{\prec}^0 A(b)$.
2. For every $z \in Z$ the family $F_z = \{z\} \cup \{S_k(z):k \in \mathbb{N},\ k > 0\} \cup \{A_k(z):k \in \mathbb{N},\ k > 0\}$ is coinitial and cofinal with respect to the totally ordered structure $(Z, \overline{\prec}^0)$, (where $A_0(z) = z = S_0(z)$, $S_{k+1}(z) = S(S_k(z))$, $A_{k+1}(z) = A(A_k(z))$ $(k \in \mathbb{N})$).
3. For every $p,q \in X$, let $\overline{p}$ and $\overline{q}$ be their respective equivalent classes in $Z = \overline{X}/\overline{\sim}^0$. Then it holds that $p \prec q \Leftrightarrow S(\overline{p})\overline{\prec}^0 \overline{q}$.
4. The totally preordered structure $(\overline{X}, \overline{\precsim}^0)$ is representable.

In what follows, if the conditions 1–4 of the statement of the GSSRT are fulfilled, we can extend the semiorder $\prec$ defined on X to the relation $\overline{\prec}$ defined on $\overline{X}$ as follows:

$$\overline{x}\ \overline{\prec}\ \overline{y} \Leftrightarrow S(\overline{x})\ \overline{\prec}^0\ \overline{y}\ (\overline{x}, \overline{y}\ \in \overline{X})$$

This extension $\overline{\prec}$ is also a semiorder whose restriction to X is $\prec$.

In order to prove the GSSRT, we need the following preliminary lemma:

Lemma 4. *Let X be a set endowed with a semiorder* $\prec$. *Define* $(\overline{X}, \precsim^0)$ *as in the statement of the GSSRT, and suppose that the condition* 1 *is satisfied. Then for every* $z,t \in Z$ *the following three conditions are equivalent*:

(*i*) $z \overline{\prec}^0 t$,
(*ii*) $S(z)\overline{\prec}^0 S(t)$,
(*iii*) $A(z)\overline{\prec}^0 A(t)$.

Proof of Lemma 4. It follows that: $z \overline{\prec}^0 A(S(t)) \Leftrightarrow z \overline{\prec}^0 t \Leftrightarrow S(A(z)) \overline{\prec}^0 t$. Also, $z \overline{\prec}^0 A(S(t)) \Leftrightarrow S(z) \overline{\prec}^0 S(t)$ and $S(A(z)) \overline{\prec}^0 t \Leftrightarrow A(z) \overline{\prec}^0 A(t)$. $\square$

Proof of the GSSRT. By the discussion just before Remark 3, it is clear that if $(X, \prec)$ is representable then the extension $(\overline{X}, \overline{\prec}^0)$ satisfies the conditions 1–4.

Let us prove the converse: Fix an element $z \in Z$. Given any element $y \in Z$, by conditions 1–3, one and only one of the following situations must hold true:

(i) There exists $k \in \mathbb{N}$, such that $A_{k+1}(z) \overline{\prec}^0 y \overline{\precsim}^0 A_k(z)$,
(ii) there exists $k \in \mathbb{N}$, such that $S_k(z) \overline{\prec}^0 y \overline{\precsim}^0 S_{k+1}(z)$.

By Lemma 4 we can find an element y_z in the family $F_y = \{y\} \cup \{S_k(y):k \in \mathbb{N},\ k > 0\} \cup \{A_k(y):k \in \mathbb{N},\ k > 0\}$ such that $z \overline{\prec}^0 y_z \overline{\precsim}^0 S(z)$.

Now by condition 4, and using Remark 2, take a countable subset $D(Z) \subseteq Z$ such that whenever $z_1, z_2 \in Z$ are such that $z_1 \overline{\prec}^0 z_2$, there exists $d \in D(Z)$ with $z_1 \overline{\prec}^0 d \overline{\precsim}^0 z_2$.

Define $D'(Z) = \{d \in D(Z):z \overline{\prec}^0 d \overline{\precsim}^0 S(z)\}$. Since $D'(Z)$ is also countable, we can label its elements as $D'(Z) = \{d_n:n \in \mathbb{N}\}$.

Define now the real number:

$$f(y_z) = \sum_{\{n \in \mathbb{N},\ z \overline{\prec}^0 d_n \overline{\precsim}^0 y_z\}} 2^{-(n+1)}$$

Finally set

$$f(y) = f(y_z) + k \text{ if } S_k(z) \overline{\prec}^0 y \overline{\precsim}^0 S_{k+1}(z) \text{ for some } k \in \mathbb{N}$$

$$f(y) = f(y_z) - k - 1 \text{ if } A_{k+1}(z) \overline{\prec}^0 y \overline{\precsim}^0 A_k(z) \text{ for some } k \in \mathbb{N}$$

This function f that has been defined on Z easily extends to a map F defined on $\overline{X}$: Let $\overline{x} \in \overline{X}$ and $z(\overline{x})$ be the correspondent class in $Z = \overline{X}/\overline{\sim}^0$ to which $\overline{x}$ belongs. Then just define $F(\overline{x}) = f(z(\overline{x}))$.

A final checking of condition 3 shows that the restriction of F to the semiordered structure $(X, \prec)$ furnishes an Scott–Suppes representation with threshold 1, as follows:

$$x \prec y \Leftrightarrow F(x) + 1 < F(y)(x, y \in X).$$

This concludes the proof. □

Remark 4. The classical Scott–Suppes Representability Theorem states that every finite semiordered structure $(X, \prec)$ admits a Scott–Suppes representation. Following the original proof appeared in Scott and Suppes (1958), we can interpret this setting as a particular case of GSSRT.

Thus if X is a finite set endowed with a semiorder $\prec$, the quotient set $Y = X/\sim^0$ is also finite and $\prec^0$ is a total order on Y. Let $Y = \{y_0, y_1, y_2, \ldots, y_k\}$ where $y_i \prec^0 y_{i+1}$ ($i = 1, \ldots k$).

For each $i = 1, \ldots, k$ we take a representative $x_i \in X$ of the class y_i. Given $i, j \in \{1, \ldots, k\}$ either we have $x_i \prec x_j$ or else $x_j \precsim x_i$, and, by definition of $\sim^0$, this happens independently of the representatives chosen.

Associated to Y, we consider a set of rational numbers $A = \{a_0, a_1, a_2, \ldots, a_k\}$ whose elements are defined as follows by induction on i:

$$a_i = \frac{i}{i+1} \text{ if } x_i \precsim x_0$$

$$a_i = \frac{i}{i+1} \cdot a_j + \frac{1}{i+1} \cdot a_{j-1} + 1 \quad \text{if} \quad x_i \precsim x_j; x_{j-1} \prec x_i, (j \leq i)$$

In Scott and Suppes (1958) it is proved that:

1. $a_{i-1} < a_i \quad (i = 1, \ldots, k)$

2. $x_i \prec x_j \Leftrightarrow a_i + 1 < a_j \quad (i = 1, \ldots, k)$

This provides a Scott–Suppes representation for the preordered structure $(X, \prec)$. Moreover, the structure $(Y, \prec^0)$ can be obviously identified to the set $A \subset \mathbb{Q}$, endowed with the natural order. The extended structure $(Z, \bar{\prec}^0)$, with $Z = \bar{X}/\bar{\sim}^0$, can identified to $(\mathbb{R}, <)$, so that we can define successors and antecedents as $S(a) = a + 1$; $A(b) = b - 1$ $(a,b \in \mathbb{R})$. The extended semiordered structure $(Z, \bar{\prec})$ is then identified to $\mathbb{R}$ equipped with the semiorder $\mathcal{R}$ given by $a\mathcal{R}b \Leftrightarrow a + 1 < b$ $(a,b \in \mathbb{R})$. It is straightforward to see that this definition of successors and antecedents satisfies the conditions 1–4 that appear in the statement of GSSRT.

4. Discussion

Intending to analyze in detail the meaning of the GSSRT, we conclude the paper with a final discussion.

(i) The original Scott–Suppes theorem (see Scott and Suppes (1958)) deals with finite semiorders and, under this circumstance, it states that a semiordered structure $(X, \prec)$ always admits a representation through a real-valued function u and a threshold α such that $x \prec y \Leftrightarrow u(x) + \alpha < u(y)$ $(x,y \in X)$ where we can take $\alpha = 0$ if and only if $\precsim$ is a total preorder. In addition, it is straightforward to see that every finite semiordered structure $(X, \prec)$ also admits a representation:

$$x \prec y \Leftrightarrow v(x) + \beta \leq v(y) \, (x,y \in X),$$

with $v: X \to \mathbb{R}$ a real valued function, and $\beta \geq 0$ a real number.

To see this, observe that:

1. If $(X, \prec)$ admits a representation $x \prec y \Leftrightarrow u(x) + \alpha < u(y)$ $(x,y \in X)$, since X is finite the infimum of the set

   $$\{u(x_i) - u(x_j) : x_i, x_j \in X, \alpha < u(x_i) - u(x_j)\}$$

   is a real number β which is strictly greater than α, so that for every $x,y \in X$ it holds that $u(x) + \alpha < u(y) \Leftrightarrow u(x) + \beta \leq u(y)$. There are situations in which a representation $x \prec y \Leftrightarrow u(x) + \alpha < u(y)$ is equivalent to the representation $x \prec y \Leftrightarrow u(x) + \alpha \leq u(y)$.

An example is $X=\{1,2,3,4,5\}$ equipped with the semiorder $\prec$ defined by $i\prec j \Leftrightarrow i+\frac{3}{2}<j$. However, in general such an equivalence does not hold true: Let $X=\{1,2,3,4,5\}$ with the semiorder $\prec$ defined by $i\prec j \Leftrightarrow i+1<j$, and observe that $2 \precsim 1$.

2. If $(X,\prec)$ admits a representation $x\prec y \Leftrightarrow u(x)+\beta \leq u(y)$ $(x,y\in X)$, since X is finite the supremum of the set

$$\{u(x_i)-u(x_j){:}x_i, x_j \in X, u(x_i)-u(x_j)<\beta\}$$

is a real number α which is strictly smaller than β, so that for every $x,y\in X$ it holds that $u(x)+\alpha<u(y) \Leftrightarrow u(x)+\beta \leq u(y)$.

However, the situation is completely different in the infinite case. If, for instance, we consider the set $\mathbb{R}$ endowed with the semiorder $\prec$ given by $x\prec y \Leftrightarrow x+1\leq y$ $(x,y\in\mathbb{R})$ it is obvious that this structure $(\mathbb{R},\prec)$ admits a representation of the second kind above. However it does not admit a Scott–Suppes representation $x\prec y \Leftrightarrow u(x)+\alpha<u(y)$ $(x,y\in\mathbb{R})$. Actually, even if we consider $\prec$ only as an interval order, the structure $(\mathbb{R},\prec)$ is not representable (as an interval order) because it fails to be i.o.-separable (see Oloriz et al. (1998) for further details).

It is not difficult to produce examples of semiordered structures $(X,\prec)$ that neither admit a Scott–Suppes representation $x\prec y \Leftrightarrow u(x)+\alpha<u(y)$ $(x,y\in X)$ nor a representation $x\prec y \Leftrightarrow v(x)+\beta\leq v(y)$ $(x,y\in X)$. To do so, of course we could start thinking of a semiordered structure $(X,\prec)$ such that the cardinality of the set $X/\sim^0$ exceeds the cardinality $2^{\aleph_0}$ of $\mathbb{R}$, avoiding any possible representation. But there are simpler examples built on sets whose cardinality is not so big: Consider for instance the set $\mathbb{R}\times\{0,1\}$ equipped with the semiorder $\prec$ given by:

$$(a,0)\prec(b,0) \Leftrightarrow a+1\leq b$$

$$(c,0)\prec(d,1)$$

$$(g,1)\prec(h,1) \Leftrightarrow g+1<h$$

$(a,b,c,d,g,h\in\mathbb{R})$.

(ii) Second representability theorem. We could easily guess now which conditions would characterize the existence, for a semiordered structure $(X,\prec)$, of a representation $x\prec y \Leftrightarrow v(x)+\beta\leq v(y)$ $(x,y\in X)$, Let $v{:}X\to\mathbb{R}$ be a real valued function and $\beta\geq 0$ a real number. Proceeding as in the GSSRT, we should start again from the extension $\bar{\prec}^0$, but now the condition involving successors or antecedents must be changed to adapt the proof to this new kind of representation, in the following way:

Second Representability theorem for semiorders. *Let X be a set endowed with a semiorder $\prec$. Let $(\bar{X},\bar{\prec}^0)$ be the Dedekind complete, without jumps and neither minimal nor maximal elements, extension of the totally preordered structure $(X,\precsim^0)$. Then the semiorder $\prec$ is representable through a function $v{:}X\to\mathbb{R}$ and a strictly positive threshold $\beta\geq 0$ such that $x\prec y \Leftrightarrow v(x)+\beta\leq v(y)$ $(x,y\in X)$ if and only if the extension $(\bar{X},\bar{\prec}^0)$ satisfies the following properties*:

1. Let $Z=\bar{X}/\bar{\sim}^0$. Then for a given $z\in Z$ there exist two elements $S(z)$ and $A(z)$ in Z such that $A(S(z))=z=S(A(z))$, and for every $\bar{a},\bar{b}\in Z$ it holds that $S(a)\,\bar{\prec}^0 b \Leftrightarrow a\,\bar{\prec}^0 A(b)$.
2. For every $z\in Z$ the family $F_z=\{z\}\cup\{S_k(z):k\in\mathbb{N},\ k>0\}\cup\{A_k(z):k\in\mathbb{N},\ k>0\}$ is coinitial and cofinal with respect to the totally ordered structure $(Z,\bar{\prec}^0)$, (where $A_0(z)=z=S_0(z)$, $S_{k+1}(z)=S(S_k(z))$, $A_{k+1}(z)=A(A_k(z))$ $(k\in\mathbb{N})$).
3. For every $p,q\in X$, let $\bar{p}$ and $\bar{q}$ be their respective equivalent classes in $Z=\bar{X}/\bar{\sim}^0$. Then it holds that $p\prec q\Leftrightarrow S(\bar{p})\,\bar{\precsim}^0\,\bar{q}$.
4. The totally preordered structure $(\bar{X},\bar{\precsim}^0)$ is representable.

Observe that the condition 3 in the statement of GSSRT has been modified.

(iii) The conditions that appear in the statement of the GSSRT force the semiordered extended structure $(\bar{X},\bar{\prec})$ to have intransitive indifference $\bar{\sim}$. Notice that, given $z\in Z$ it holds that $z\,\bar{\prec}^0 S(z)\Leftrightarrow S(z)\,\bar{\prec}^0 S(S(z))\Leftrightarrow z\,\bar{\prec} S(S(z))$, but $S(S(z))\,\bar{\sim}\,S(z)\,\bar{\sim}\,z$. This justifies the fact of having found, through the GSSRT, a representation of $(X,\prec)$ whose threshold is different from zero (actually this threshold can be taken to be 1).

(iv) Another necessary condition for the Scott–Suppes representability of a semiorder. Looking again at the conditions that appear in the statement of the GSSRT, we observe that the successor $S(x)$ of an element $x\in X$ appears as:

$$S(x)=\inf_{\precsim^0}\{y\in Z:z_x\,\bar{\prec}\,y\}$$

where z_x denotes the class in Z corresponding to the element $x\in X$. It may happen that there exists an element $t\in X$ such that $z_t=S(x)$. If this is the case then if must hold that $t\precsim x$. Otherwise the semiordered structure $(X,\prec)$ will not be representable in the sense of Scott–Suppes since the condition 3 of the statement of GSSRT would be violated.

To illustrate these ideas, consider again $X=\mathbb{R}$ and the semiorder $\prec$ defined by $x\prec y\Leftrightarrow x+1\leq y$ $(x,y\in\mathbb{R})$. Here $\bar{X}$ and Z coincide with $X(=\mathbb{R})$, and $\precsim^0=\bar{\precsim}^0$ is the natural order on $\mathbb{R}$. Thus:

$$\inf_{\bar{\precsim}^0}\{y\in Z:z_x\,\bar{\prec}\,y\}=x+1$$

for every $x\in\mathbb{R}$.

Therefore $(X,\prec)$ is not representable in the sense of Scott–Suppes, because $x\prec x+1$ $(x\in\mathbb{R})$.

(v) Third representability theorem. The way in which we obtained GSSRT relies on the consideration of the extension $(\bar{X},\bar{\precsim}^0)$. If several conditions are satisfied, this allows us to get a representation of the given semiordered structure $(X,\prec)$. Following GSSRT we observe that there is also an extension $\bar{\prec}$, to the whole $\bar{X}$, of the semiorder $\prec$ that was formerly defined on X.

As the next result shows, we could have got, directly, an alternative version of the GSSRT working directly on an extension $(\bar{X},\bar{\prec})$ without a previous use of $(\bar{X},\bar{\precsim}^0)$.

Third representability theorem for semiorders. *Let X be a set endowed with a semiorder $\prec$. Then $\prec$ is representable through a function $u:X\rightarrow\mathbb{R}$ and a strictly* *positive threshold* $\alpha > 0$ *such that* $x \prec y \Leftrightarrow u(x) + \alpha < u(y)$ $(x,y \in X)$ *if and only if the structure* $(X, \prec)$ *can be extended to a structure* $(\bar{X}, \bar{\prec})$ *satisfying the following properties*:

1. $X \subseteq \bar{X}$ and $\bar{\prec}$ is a semiorder whose restriction to X is $\prec$.
2. Let $W = \bar{X}/\bar{\sim}^{**}$. Then for a given $w \in W$ there exist two elements $S(w)$ and $A(w)$ in W such that $A(S(w)) = w = S(A(w))$, $S(w) \bar{\precsim} w \bar{\prec}^{**} S(w)$, and $A(w) \bar{\prec}^{**} w \bar{\precsim} A(w)$.
3. For every $w \in W$ the family $F_w = \{w\} \cup \{S_k(w): k \in \mathbb{N},\ k > 0\} \cup \{A_k(w): k \in \mathbb{N},\ k > 0\}$ is coinitial and cofinal with respect to the totally ordered structure $(W, \bar{\prec}^{**})$, (where $A_0(w) = w = S_0(w)$, $S_{k+1}(w) = S(S_k(w))$, $A_{k+1}(w) = A(A_k(w))$ $(k \in \mathbb{N})$).
4. For every $a,b \in W$ it holds that $a \bar{\prec} b \Leftrightarrow S(a) \bar{\prec}^{**} b \Leftrightarrow a \bar{\prec}^{**} A(b)$.
5. The structure $(\bar{X}, \bar{\prec})$, considered as an interval order, is i.o.-separable.

The proof follows the steps of the one given for GSSRT: It is clear that if $(X, \prec)$ is representable then an extension $(\bar{X}, \bar{\prec})$ satisfying the conditions 1–5 must exist.

To prove the converse, fix an element $w \in W$. Given any element $y \in W$, by conditions 3 and 4 one and only one of the following situations must hold true:

(i) There exists $k \in \mathbb{N}$, such that $A_{k+1}(w) \bar{\prec}^{**} y \bar{\precsim}^{**} A_k(w)$,
(ii) there exists $k \in \mathbb{N}$, such that $S_k(w) \bar{\prec}^{**} y \bar{\precsim}^{**} S_{k+1}(w)$.

Similarly to Lemma 4 we can find an element y_w in the family $F_y = \{y\} \cup \{S_k(y): k \in \mathbb{N},\ k > 0\} \cup \{A_k(y): k \in \mathbb{N},\ k > 0\}$ such that $w \bar{\prec}^{**} y_w \bar{\precsim}^{**} S(w)$.

Since $\bar{\prec}$ is i.o.-separable, there exists a countable subset $\bar{D} \subseteq \bar{X}$ such that if $\bar{x} \bar{\prec} \bar{y}$ then there exists $d \in \bar{D}$ such that $\bar{x} \bar{\prec} d \bar{\precsim}^{**} \bar{y}$ $(\bar{x}, \bar{y} \in \bar{X})$. Let $\bar{a}, \bar{b} \in \bar{X}$ be such that $\bar{a} \bar{\prec}^{**} \bar{b}$. Then by definition of $\bar{\prec}^{**}$ there exists an element $\bar{c} \in \bar{X}$ such that $\bar{a} \bar{\precsim} \bar{c} \bar{\prec} \bar{b}$. Now, by i.o.-separability of $\bar{\prec}$ we can take $\bar{d} \in \bar{D}$ such that $\bar{a} \bar{\precsim} \bar{c} \bar{\prec} \bar{d} \bar{\precsim}^{**} \bar{b}$. Thus $\bar{a} \bar{\prec}^{**} \bar{d} \bar{\precsim}^{**} \bar{b}$.

Let $D(W)$ be a countable subset of W such that whenever $w_1, w_2 \in W$ are such that $w_1 \bar{\prec}^{**} w_2$, there exists $d \in D(W)$ with $w_1 \bar{\prec}^{**} d \bar{\precsim}^{**} w_2$.

Define $D'(W) = \{d \in D(W): w \bar{\prec}^{**} d \bar{\precsim}^{**} S(w)\}$. Since $D'(W)$ is also countable, we can label its elements as $D'(W) = \{d_n: n \in \mathbb{N}\}$.

Define now the real number:

$$f(y_w) = \sum_{\{n \in \mathbb{N},\ w \bar{\prec}^{**} d_n \bar{\precsim}^{**} y_w\}} 2^{-n-1}$$

Define also:

$$f(y) = f(y_w) + k \text{ if } S_k(w) \bar{\prec}^{**} y \bar{\precsim}^{**} S_{k+1}(w) \text{ for some } k \in \mathbb{N}$$

$$f(y) = f(y_w) - k - 1 \text{ if } A_{k+1}(w) \bar{\prec}^{**} y \bar{\precsim}^{**} A_k(w) \text{ for some } k \in \mathbb{N}$$

This function f that has been defined on W easily extends to a map F defined on $\bar{X}$: Let $\bar{x} \in \bar{X}$ and denote by $w(\bar{x})$ the correspondent class in $W = \bar{X}/\bar{\sim}^{**}$ to which $\bar{x}$ belongs. Then just define $F(\bar{x}) = f(w(\bar{x}))$.

A final checking shows that the semiorder $\bar{\prec}$ is representable by the function F with threshold 1, as follows:

$$\bar{x}\,\bar{\prec}\,\bar{y} \Leftrightarrow F(\bar{x}) + 1 < F(\bar{y}) \quad (\bar{x}, \bar{y} \in \bar{X})$$

Obviously the restriction of F to the semiordered structure $(X, \prec)$ is also a representation. This concludes the proof.

It is not difficult to get more alternative versions of GSSRT, in a completely analogous way to this Third Representability Theorem, but working with $V = \bar{X}/\bar{\sim}^{*}$ or else $Z = \bar{X}/\bar{\sim}^{0}$ instead of $W = \bar{X}/\bar{\sim}^{**}$.

If we compare this Third Representability Theorem and the GSSRT, we observe that if we start from $(\bar{X}, \bar{\sim}^0)$ as in GSSRT, and the conditions in the statement of GSSRT are fulfilled, we would finally obtain the semiordered extended structure $(\bar{X}, \bar{\prec})$ that can be proved to satisfy the conditions 1–5 of this Third Representability Theorem. Similarly, if we have an extension $(\bar{X}, \bar{\prec})$ accomplishing the conditions 1–5 of the Third Representability Theorem, it is straightforward to see that $(\bar{X}, \bar{\sim}^0)$ satisfies the conditions in the statement of GSSRT. A glance at both theorems will show us that the difficulty is, always, the definition of successors or antecedents in order to fulfill the suitable conditions of the statements of such theorems.

(vi) Structure Theorem. Observe also that by the construction given in the GSSRT, the semiordered structure $(Z, \bar{\prec})$ can be understood as the cartesian product $\mathbb{Z} \times [0,1)$ endowed with the following ordering $\prec$ (which is actually a semiorder):

$$(a,b) \prec (c,d) \Leftrightarrow c - a \geq 2 \text{ or else } c - a = 1,\, d - b > 0$$

Here an element $y \in Z$ is identified with a pair $(a,b) \in \mathbb{Z} \times [0,1)$, where, following the proof of the GSSRT, the first coordinate $a \in \mathbb{Z}$ equals:

$$-(k+1) \text{ if there exists } k \in \mathbb{N}, \text{ such that } A_{k+1}(z)\,\bar{\prec}^0 y\,\bar{\precsim}^0 A_k(z)$$

$$k \text{ if there exists } k > 0,\, k \in \mathbb{N}, \text{ such that } S_k(z)\,\bar{\prec}^0 y\,\bar{\precsim}^0 S_{k+1}(z)$$

Meanwhile, the second coordinate $b \in [0,1)$ equals $f(y_z)$ as given in the GSSRT. All the representable semiorders have a representable extension that can be described in this way.

Thus, we arrive to the following Structure Theorem:

Structure theorem. *Every representable semiorder is isotonic to a subset of the cartesian product* $\mathbb{Z} \times [0,1)$ *endowed with the following ordering* $\prec$:

$$(a,b) \prec (c,d) \Leftrightarrow c - a \geq 2 \textit{ or else } c - a = 1,\, d - b > 0$$

(vii) Observe that in Proposition 1 a characterization of the representability of interval orders is given in terms of a bivariate function that satisfies a suitable functional equation. Such functional equation is known to be the functional equation of separability (see Aczél (1987), p. 122). For the particular case of semiorders it was known that:

The following conditions are equivalent for a semiordered structure $(X, \prec)$:

1. $(X, \prec)$ *is representable*.
2. *There exists a bivariate map* $G: X \times X \to \mathbb{R}$ *such that* $x \precsim y \Leftrightarrow G(x,y) \geq 0$ *and* $G(x,y) + G(y,z) = G(x,z) + G(t,t)$ *for every* $x,y,z,t \in X$.
3. *There exists a bivariate map* $G: X \times X \to \mathbb{R}$ *such that* $x \precsim y \Leftrightarrow G(x,y) \geq 0$ *and* $G(x,y) + G(y,z) = G(x,t) + G(t,z)$ *for every* $x,y,z,t \in X$.

(See Candeal et al. (1996)).

However, nothing had been said about how to solve these functional equations.

Now observe that, following the proof of the GSSRT, if we define $G(a,b) = F(b) - F(a) + 1 (a,b \in \bar{X})$, it is clear that $a \precsim b \Leftrightarrow F(a) \leq F(b) + 1 \Leftrightarrow G(a,b) \geq 0$. In addition, $G(a,b) + G(b,c) = F(b) - F(a) + 1 + F(c) - F(b) + 1 = (F(c) - F(a) + 1) + (F(d) - F(d) + 1) = G(a,c) + G(d,d)$ $(a,b,c,d \in \bar{X})$. Similarly $G(a,b) + G(b,c) = F(b) - F(a) + 1 + F(c) - F(b) + 1 = (F(d) - F(a) + 1) + (F(c) - F(d) + 1) = G(a,d) + G(d,c)$ $(a,b,c,d \in \bar{X})$, so that the bivariate map G is a solution of the functional equations of semiorders.

Notice also that given $a,b \in \bar{X}$ the value of $G(a,b)$ can be directly obtained through the series used to define the values $f(y_z)$ in order to prove the GSSRT.

In this point we may also observe that this constitutes an abstract solution of the functional equations of semiorders, not an algorithmic one. Actually, computing the series used to define $f(y_z)$ seems uneasy in practice. The analysis of the algorithmic direction remains still open, and would complete the panorama.

(viii) Perhaps the main difficulty to get a characterization of the representability of semiorders was the fact that the known characterizations of the representability of ordered structures (see e.g. the first chapters in Bridges and Mehta (1995) for the case of total preorders, and Oloriz et al. (1998) for the case of interval orders) were given in terms of countable subsets that define in each case a suitable condition of ordinal separability. As a plain consequence, countable total preorders or countable interval orders were trivially representable. However, in the case of semiorders it was known that: Not all the countable semiorders are representable (see e.g. Scott and Suppes (1958), Induráin (2001) or Manders (1981)). An example is $X = \mathbb{Q} \times \mathbb{Q}$ (here $\mathbb{Q}$ denotes the set of rational numbers) endowed with the binary relation $\prec$ given by:

$$(a,b) \prec (c,d) \Leftrightarrow a < c \text{ or else } a = c,\ b + 1 < d$$

Observe that the associated indifference is not transitive:

$$(0,0) \sim \left(0, \frac{3}{4}\right), \left(0, \frac{3}{4}\right) \sim \left(0, \frac{3}{2}\right), \text{but } (0,0) \prec \left(0, \frac{3}{2}\right)$$

Moreover, the restriction of $\prec$ to the subset $\mathbb{Q} \times \{0\}$ is a total order. The existence of a representation (u,α) would imply $\alpha > 0$ because $\prec$ has intransitive indifference. This fact would carry the non-boundedness of u on, for instance, the vertical line $\{0\} \times \mathbb{Q}$ and this would force u to take a value $+\infty$ on any vertical line $\{q\} \times \mathbb{Q}$ $(q > 0,\ q \in \mathbb{Q})$.

(ix) If $(X, \prec)$ is a countable semiordered structure it was conjectured (see Induráin

(2001)) that such structure is representable if and only if the quotient $Z = X/\sim^0$ neither contains an isotonic copy of $\mathbb{N} \oplus 1$ nor of its dual (in fact, the idea of the existence of subsets whose mere occurrence provokes the non-representability of an ordered structure had already been considered in Mitas (1995)).

Now we know that the above conjecture is false:

Consider on $X = \mathbb{Q}$ the semiorder $\prec$ defined by $x \prec y \Leftrightarrow x + 1 \leq y$ $(x, y \in \mathbb{Q})$. We have that $\bar{X} = \mathbb{R}$, and $\bar{\prec}^0$ is the usual strict order $<$ on $\mathbb{R}$. Now it is easy to observe the impossibility of defining a successor $S(0)$ for the element $0 \in \mathbb{Q}$ in order to satisfy the condition 3 of the GSSRT.

Let us also point out that a necessary and sufficient condition for a countable semiorder to be representable had already been given in Manders (1981).

(x) Considered as an interval order, it is plain that a semiorder $\prec$ defined on a countable set X will always be representable as an interval order, through the corresponding pair of functions (u,v) such that $x \prec y \Leftrightarrow v(x) < u(y)$ (because it is trivially i.o.-separable). However, as the last example shows, it could be impossible to get a pair (u,v) with $v(x) = u(x) + \alpha$ for a certain real number $\alpha \geq 0$.

In other words:

The representability as an interval order of a countable semiordered structure $(X, \prec)$ *does not carry, in general, its representability as a semiorder.*

(xi) A representable semiordered structure $(X, \prec)$ verifies that on the extended structure $(\bar{X}, \bar{\prec})$ the total preorders $\bar{\precsim}^{*}$ and $\bar{\precsim}^{**}$ agree. Observe also that:

An interval order $(X, \prec)$ on which the total preorders $\precsim^{*}$ and $\precsim^{**}$ coincide is a particular case of a semiorder.

To see that simply observe that the coincidence of $\prec^{*}$ and $\prec^{**}$ forces the interval order $\prec$ to satisfy the property of generalized pseudotransitivity. However such condition of coincidence of $\prec^{*}$ and $\prec^{**}$ does not imply that a given semiorder be representable, even in the case in which it is representable as an interval order. An example is $(\mathbb{R}, \prec)$ where $x \prec y \Leftrightarrow 3|x| < 2|y|$ $(x,y \in \mathbb{R})$. Let $u,v:\mathbb{R} \to \mathbb{R}$ be the functions defined by $u(x) = 2|x|, v(x) = 3|x|$ $(x \in \mathbb{R})$. It is clear that the ordering $\prec^{**}$ (representable through the function u) and the ordering $\prec^{*}$ (representable through the function v) coincide. Nevertheless, $\prec$ is not representable as a semiorder: Observe that $\precsim$ is not transitive, so that if $(X, \prec)$ were representable through a function u and a threshold α, it would happen that $\alpha > 0$. Now observe that the sequence $(2^{-n})_{n \in \mathbb{N}} \subset X$ satisfies the condition $0 \prec 2^{-(n+1)} \prec 2^{-n}$ $(n \in \mathbb{N})$ and also $u(2^{-n}) + n\alpha < u(1)$. This would force $u(0)$ to be $-\infty$, which is impossible.

Actually if in this example we avoid the element 0 defining on $X = (0, +\infty)$ the semiorder $\prec$ as $x \prec y \Leftrightarrow 3x < 2y$ $(x,y > 0,\ x,y \in \mathbb{R})$ such semiorder becomes representable. Notice that $x \prec y \Leftrightarrow 3x < 2y \Leftrightarrow \log(3x) < \log(2y) \Leftrightarrow \log x + \log \frac{3}{2} < \log y$.

(xii) A semiordered structure $(X, \prec)$ can be representable even without the coincidence of the associated orderings $\prec^{*}$ and $\prec^{**}$ on X. An example is $X = \{1,2,3,4\}$ endowed with the semiorder $\prec$ defined as $x \prec y \Leftrightarrow x + 1 < y$ $(x,y \in X)$. Observe that $1 \prec^{*} 2$ but not $1 \prec^{**} 2$. Similarly $3 \prec^{**} 4$ but not $3 \prec^{*} 4$.

5. List of symbols

α, ∞, $\prec$, $\underline{\precsim}$, $\sim$, $\prec_{op}$, $\prec^{0}$, $\rightarrow$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\geq$, $\leq$, $\Leftrightarrow$, $\Rightarrow$, $\in$, $\subseteqq$, $\neg$, $\times$, $\cup$, $\ldots$, $\{,\}$, $\bar{\prec}$, $\bar{X}$, $\prec^{*}$, $\prec^{**}$, $\precsim^{*}$, $\precsim^{**}$, $\sim^{*}$, $\sim^{**}$, $\prec^{0}$, $\precsim^{0}$, $\sim^{0}$, /, $|$, Σ, $\oplus$.

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