

# Markovian equilibrium in infinite horizon economies with incomplete markets and public policy

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## Abstract

We develop an isotone recursive approach to the problem of existence, computation, and characterization of *nonsymmetric* locally Lipschitz continuous (and, therefore, Clarke-differentiable) Markovian equilibrium for a class of infinite horizon multiagent competitive equilibrium models with capital, aggregate risk, public policy, externalities, one sector production, and incomplete markets. The class of models we consider is large, and examples have been studied extensively in the applied literature in public economics, macroeconomics, and financial economics. We provide sufficient conditions that distinguish between economies with isotone Lipschitz Markov equilibrium decision processes (MEDPs) and those that have only locally Lipschitz (but not necessarily isotone) MEDPs. As our fixed point operators are based upon order continuous and compact nonlinear operators, we are able to provide sufficient conditions under which isotone iterative fixed point constructions converge to extremal MEDPs via successive approximation. We develop a first application of a new method for computing MEDPs in a system of Euler inequalities using isotone fixed point theory even when MEDPs are not necessarily isotone. The method is a special case of a more general mixed monotone recursive approach. We show MEDPs are unique only under very restrictive conditions.

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Finally, we prove monotone comparison theorems in Veinott's strong set order on the space of public policy parameters and distorted production functions.

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## 1. Introduction

Over the last two decades researchers in macroeconomics and financial economics have proposed a number of new theoretical frameworks in an effort to assess the role of behavioral heterogeneity and incomplete markets on both the positive and the normative aspects of equilibrium economic fluctuations. Such models have proven to be important for researchers studying the interplay between public policy, incomplete markets, income risk (both aggregate and idiosyncratic), asset pricing, production uncertainty, quantitative welfare assessments of the costs of business cycles, financial economies, and macroeconomics. With a few notable exceptions, a majority of this work has focused on numerical characterizations of Markovian equilibrium. In addition even when theoretical work on Markovian equilibrium has been done, there has been little relationship provided to connect theoretical constructions that have used primarily pure topological fixed point methods to the study of numerical solutions for these models. This paper makes a significant contribution by bridging this large gap between theory and numerical applications for an important class of equilibrium models of economic fluctuations with behavioral heterogeneity, incomplete markets, public policy, and bounded production nonconvexities. In doing so, we suggest the possibility of broader applications of monotone methods to more complicated dynamic economies.

The problems associated with the tractability of heterogeneous agent economies (both theoretical and applied) from a mathematical perspective are well-known. One key conceptual question concerns the structure (or existence) of Markovian equilibrium in a multi-asset model. In models with many assets, little is known about the Markovian equilibrium with incomplete markets where the second welfare theorem fails, production is nonconvex, and there is public policy.<sup>1</sup> In the single-asset incomplete markets model of Bewley (1986), there have been two versions proposed. On the one hand, an important version of the Bewley model is presented in the work of Aiyagari (1994), Krusell and Smith (1998), and Miao (2002). A key feature of this class of single-asset models is that agents face a *continuum* of uninsured (agent specific) income risks without facing an aggregate risk. In some versions

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<sup>1</sup> One exception is Kubler and Schmedders (2003) where the issue of the existence of stationary Markov equilibrium in simple multi-asset incomplete markets model with collateral constraints is addressed. Their paper suggests the possibility of developing a general isotone iterative method based on “set-to-set” maps for computing Markov equilibrium. We remark first their methods do not work in the present single-asset context.

Reffett (2005a) shows that the Kubler–Schmedder's approach is a special case of a more general monotone-map method based on interval condensing methods. Numerical methods for computing such set-to-set maps in principle can be based upon the methods of Rockafellar and Wets (1997).

of the model, aggregate sources of risk in production are allowed, and agents have at their disposal only a single asset (e.g., capital or fiat money) for which to self insure all sources of uninsured risk. In these models many computational problems are yet to be resolved (e.g., the approximation of a probability measure and/or the existence of Markovian equilibrium decision processes and Stationary Markovian equilibrium on the so-called “natural state space”). Recently, some existence issues have been resolved in a paper by Miao (2002) where he proves the existence of stationary Markovian equilibrium in the model of Krusell and Smith (1998) on an “expanded” state space. For this argument, the topological methods are very simple and involve a straight-forward application of a local convexity argument to the space of probability measures.<sup>2</sup> Unfortunately from our perspective, as these topological fixed point methods are nonconstructive, it is not clear how to tie Miao’s work directly to numerical approximation. In Mirman et al. (2004b), some of these issues are addressed; however, those methods are not directly applicable to the economy studied in this paper.<sup>3</sup>

A second type of heterogeneous agent model having many of the features of the model of Bewley (1986) has been proposed. This model is based on a stochastic version of the Ramsey model originally presented in the seminal work of Becker (1980). This class of models has the particular advantage that their state variables can be represented by a collection of functions. This is very appealing from a numerical perspectives as the solution methods can, in theory, be related to existing solution methods for systems of differential equations often encountered in the study of physics. In these models (as in the Bewley model), households face an uninsured income fluctuation problem. This problem does not have the complication of a continuum of idiosyncratic risk ex post. Therefore, in this setting, there is a finite number of households ex post and ex ante. Recursive versions of these models have been extended in a deterministic setting by Becker (1985a,b) (see Becker and Boyd, 1997 for a review of this literature) and Townsend (1980), and in a stochastic setting in Scheinkman and Weiss (1986), Becker and Zilcha (1997), Judd and Gasper (1997), Judd et al. (2000, 2003), and Kubler and Schmedders (2003). In related work, a class of multisector, stochastic, incomplete markets models in the “Ramsey style” has been developed in a series of interesting papers by Krebs (2003, 2004a). In this work, the use of a two sector incomplete markets economy with a finite number of agents that have access to both physical and human capital, but

<sup>2</sup> The approach of Miao (2002) is remarkably simple conceptually, and reminiscent of the approach taken for studying the existence of Markov equilibrium in large anonymous games as Bergin and Bernhardt (1992). Basically, given a compact support for the candidate sequence of probability measures, the space of probability measures on this compact support can be shown to be a weakly compact metric space. Then assuming all noise satisfies standard Feller properties, and after using a version of the LLN to aggregate of decision rules, one obtains a weakly continuous operator from the sequence of probability measures into itself. Existence then follows from a standard application of the Schauder–Tychonoff theorem (with, if necessary, one more step to construct payoff-equivalent Markovian equilibrium on an expanded state space including the “natural” state space and current period value functions). To date, this approach has had little to say about the structure of equilibrium policy functions which seems needed to obtain characterizations of numerical solutions for stationary Markov equilibrium.

<sup>3</sup> In Mirman et al. (2004b), it is shown for the economies studied in Miao (2002) that a new mixed-monotone map approach can compute the Markovian equilibrium decision process for such models as Aiyagari (1994) and Krusell and Smith (1998) using two step construction. In this paper, a much stronger existence result than in Miao (2002) is proved for Markovian equilibrium decision processes on the “natural” state space as in Krusell and Smith (1998).

face uninsured income risk is introduced. However, Krebs studies a very specialized CES class of economies with no public policy. In this setting, he provides a topological proof of existence of a Markovian equilibrium with no-trade.

One problem with the latter class of models is that little is known about the nature of Markovian equilibrium decision processes (or MEDPs, in short). A key innovation of this paper is to provide (i) a sharp characterization of MEDPs for each type of agent, and (ii) a description of a collection of isotone iterative methods for computing a particular Markov equilibrium from an initial guess for an MEDP (namely, an extremal MEDP). To accomplish this, we develop an application of order theoretic fixed point theory which is broadly related to the “Monotone-Map” methods for homogeneous agent economies. The complications introduced by heterogeneity are formidable. For the homogeneous agent case, the monotone map methods are found in a series of papers beginning with Bizer and Judd (1989) and Coleman (1991), and continuing with Greenwood and Huffman (1995), Datta et al. (2002), Morand and Reffett (2003), and Mirman et al. (2004a). The work in these papers can be viewed as a collection of methodological precursors to our work for heterogeneous agent economies.<sup>4</sup> This strand of literature, for the homogenous agent case, establishes existence and provides sufficient conditions for the global uniqueness of equilibrium within a large class of continuous Markovian equilibrium. In addition, it provides strong suggestions for computational methods for applications. Unfortunately, *none* of these existing methods for studying homogeneous agent economies work directly in analyzing nonsymmetric equilibrium in a multiagent setting. Our paper sets up an application of an order continuous version of the fixed point theory on complete lattices due to Tarski (1949, 1955) to study the existence and computation of continuous Markov equilibrium.

In this paper, we provide results for heterogeneous agent economies using an integrated topological and order-theoretic approach. We provide a set of sufficient conditions under which there exists a Markovian equilibrium within a set of equicontinuous functions. We distinguish two important cases: sufficient conditions of locally Lipschitz continuous and monotone equilibrium decisions, and sufficient conditions locally Lipschitz equilibrium decision processes. Both sets of sufficient conditions seem important. In the case of monotone MEDPs, it is well-known that long-run properties are very tractable (see, for example, Dubins and Freedman, 1966; Bhattacharya and Lee, 1988). As for the latter set of sufficient conditions, the Lipschitz continuity of MEDPs plays an important role in the approximation and limiting properties of MEDPs and stationary Markov equilibrium (see Santos and Vigo-Aguiar, 1998; Santos, 2000; Santos and Peralta-Alva, 2004). Therefore, both sets of sufficient conditions appear important for future work on the study and characterization stationary Markov equilibrium. Further, since our methods are constructive, we provide theoretical algorithms that could prove useful for computing extremal points of the set of Markovian equilibrium. As the trajectories of the operator is shown to be monotone in relevant policy parameters, we conduct monotone comparative analysis in the sense of Veinott (1992) (see also Milgrom and Shannon, 1994) on the entire equilibrium set for

<sup>4</sup> For an alternative approach, using monotone operators for deterministic Ramsey equilibrium problems, see Becker and Foias (1998). Their approach looks promising, although it is yet to be developed for stochastic Ramsey problems where more than one agent in the economy can make interior savings decisions in all states. The relationship between our method and Becker and Foias’ method needs to be explored in future work.

equicontinuous Markovian equilibrium. Finally, we show that the computational methods employed here converges to the unique equilibrium within the set of equicontinuous equilibrium points. In this sense, this paper provides an additional level of geometric equivalence for a particular class of heterogeneous agent, incomplete market economies with capital accumulation and inelastic labor supply. Introduction of elastic labor supply and analysis of the structure of Stationary Markov equilibrium is left for the future.

The remainder of the paper is laid out as follows. In the second section of the paper, we describe a benchmark model with incomplete markets and nonclassical production. In the third section, we describe agents' decision problems in a decentralized Markov equilibrium. Section 4 focuses on the existence and computation of monotone MEDPs with a collection of isotone recursive methods. In particular, we construct the operator and prove the existence of Markovian equilibrium decision processes and provide some basic monotone comparative analysis results on the set of MEDPs using Veinott's strong set order. In the fifth section, we provide sufficient conditions under which there exists a unique Markovian equilibrium decision process within an equicontinuous class. In Section 6, we show how the isotone recursive methods can be generalized to income processes associate with general concave production functions. We show that locally Lipschitz continuous MEDPs exist and can be computed with successive approximation algorithms. It is particularly important to point out that we develop a new "mixed" monotone approach to address the existence and computation questions using an expanded state space approach. The last section of the paper concludes.

## 2. A benchmark economy with incomplete market and nonclassical production

We first consider a benchmark economy where we can provide sufficient conditions under which there exist isotone locally Lipschitz continuous MEDPs that can be computed by a successive approximation algorithm. It is well known that the studying of the long-run properties of Markovian models is greatly simplified when the Markov processes are isotone (e.g., see Dubins and Freedman, 1966; Bhattacharya and Lee, 1988). Then later in the paper, we allow for more general production processes, and prove similar existence results using isotone operators; but in these more general production cases, MEDPs are not necessarily isotone. To do accomplish our first objective of isotone MEDPs, we initially consider a multiagent version of the growth under uncertainty model originally proposed by Brock and Mirman (1972) as modified by Becker and Zilcha (1997). Relative to Becker and Zilcha (1997), we extend their results in many ways. First, we allow for more general production processes (e.g., nonclassical technologies) and public policy (e.g., fiscal policy and monetary policy). In particular, we amend their classical stochastic production setup (which is a generalization of the deterministic Ramsey model of Becker, 1980) in a number of ways: we allow for (i) Markov shocks to both aggregate technologies and individual labor income streams, (ii) equilibrium distortions and public policy, (iii) production nonconvexity in social returns, and, (iv) labor income risk that is uninsured. Becker and Zilcha consider the existence of a stationary Markov equilibrium, but we focus on the existence and computation of MEDPs. Our notion of uninsured idiosyncratic income risk is related to the notion in Aiyagari (1994), Krusell and Smith (1998), Datta and Reffett (2005b), and Mirman et al. (2004b), yet different since we generate a continuum of income heterogeneity in each

period. Our model represents a version of the classical income fluctuation problem proposed in [Scheinkman and Escudero \(1977\)](#) and developed, in continuous time, by [Scheinkman and Weiss \(1986\)](#).<sup>5</sup> Second, our results focus on the structure of MEDPs in these environments; a question that [Becker and Zilcha \(1997\)](#) do not consider. Our methods are also constructive.

Time is discrete. There are a continuum of infinitely-lived household/firm agents of each type and there are finite number of types. To abstract from the complications introduced by elastic labor supply, we assume that households do not have preferences over leisure; therefore, each agent supplies its endowment of labor in fixed supply. The households face varying forms of equilibrium distortions in addition to incomplete markets for risk-sharing and this is summarized using a reduced-form production function. The equilibrium frictions can be explicitly formulated as cases of economies with: (i) public sector tax and transfer policies, (ii) market structures consistent with monopolistic competition or (iii) production nonconvexity, for example. The interested reader might want to refer to the comprehensive discussion on alternative decentralizations of our environment found in [Greenwood and Huffman \(1995\)](#), [Datta et al. \(2002\)](#), and [Morand and Reffett \(2003\)](#). For production, we assume there are a large number of identical firms are endowed with an identical stochastic production technology that is (among other things) constant returns to scale in each firm's private inputs. Unlike all aforementioned work in the literature on heterogeneous agent models, we allow this reduced-form aggregate production technology to also possess nonconvexities in social returns. Each individual firm produces a single perishable good for sale in a competitive market using constant returns to scale production technologies, and it rents the inputs of capital and labor in competitive factor markets. As in [Becker and Zilcha \(1997\)](#), [Scheinkman and Weiss \(1986\)](#), [Aiyagari \(1994\)](#), and [Krusell and Smith \(1998\)](#), we allow households to smooth consumption by accumulating a single asset. As this asset is assumed to be productive, we refer to it as capital.

There are  $J$  types of infinitely-lived household/firm agents, each of whom owns a constant fraction of capital in a firm with constant returns to scale technology. They are endowed with an initial capital holding and a unit of labor in each period. Agents of the same type are treated identically, and are identified by the subscript  $j \in \mathbf{J} = \{1, 2, \dots, J\}$ . Here,  $\mathbf{J}$  is a finite set of agents. For convenience (and without loss of generality), we normalize the mass of agents to be the unit interval, and further assume that  $\eta_j = (1/J) > 0$ , of each type  $j$ .<sup>6</sup>

Uncertainty comes each period in the form of a vector of exogenous shocks  $\theta \in \Theta \subset \mathbb{R}_{++}^{J+1}$ . The stochastic process of shocks is assumed to follow a first-order Markov process with stationary transition matrix  $\chi(\theta, d\theta')$ . To avoid technical issues associated with measurability, we assume that  $\Theta$  is denumerable (e.g., either having finite or countable cardinality).

<sup>5</sup> We can incorporate period-by-period uninsured idiosyncratic shocks, as for example, in [Krebs \(2004a\)](#), into our framework without changing the structure of our argument. We choose the form of idiosyncratic risk based upon making the environment as close as possible to [Becker and Zilcha \(1997\)](#).

<sup>6</sup> Given the nonnegative vector of equilibrium capital stocks, and given our assumptions on bounded returns to the mean capital stock, we can find an upper bound for any collection of weights  $\eta_j$  for  $j \in \mathbf{J}$ . We use this normalization only to facilitate formulation of the upper bound for the capital stock, in equilibrium, for any agent  $j \in \mathbf{J}$ .

Specifically,  $\theta = [\theta_{\text{ind}}, \theta_a]$  where  $\theta_{\text{ind}} = [\theta_1, \dots, \theta_J] \subset R_{++}^J$  represents the vector of type-specific idiosyncratic shocks ( $\theta_j$  is received by the households of type  $j$ ), and  $\theta_a \subset R_{++}$  a shock to aggregate production technology. To keep things simple, we assume that  $\chi(\theta, d\theta')$  is such that the efficiency unit shocks do not generate aggregate risk in any state, i.e.,  $\sum_{j=1}^J \eta_j \theta_j = \sum_{j=1}^J (\theta_j/J) = \hat{\theta} = 1$ , in each state of the world.

For each period and state, preferences for households of type  $j$  are represented by a period utility index  $u_j(c)$ , where the commodity space is assumed to be a positive interval in  $R_+$ . Letting  $\theta^t = (\theta_1, \dots, \theta_t)$  denote the history of the shocks until period  $t$ , the household's lifetime preferences are assumed to be additively separable and defined over infinite sequences, indexed by dates and histories  $c = (c_{\theta^t})$ , and are given as,

$$U_j(\theta) = E_0 \left\{ \sum_{t=0}^{\infty} \beta_j^t u_j(c_{\theta^t}) \right\}, \quad (1)$$

where the summation in the mathematical expectation in (1) is with respect to the probability structure of future histories of the shocks  $\theta^t$ , given the transition matrix  $\chi$ , and  $\beta_j \in (0, 1)$ . The assumptions on the period utility function  $u_j : R_+ \mapsto R$  for  $j = 1, \dots, J$  are standard for models where interior solution is an issue, and are given as,

**Assumption 1.** The vector of period utility functions  $u(c) = [u_1(c_1), \dots, u_J(c_J)]$  satisfy the following:

- (i) Each utility function  $u_j(c_j)$  is at least twice continuously differentiable, strictly increasing, strictly concave;
- (ii) The derivatives  $u'_j(c_j)$  satisfy the Inada conditions,

$$\lim_{c_j \rightarrow 0} u'_j(c_j) = \infty \text{ and } \lim_{c_j \rightarrow \infty} u'_j(c_j) = 0.$$

**Assumption 1** is standard in the literature. Notice that there is heterogeneity in both period utility and the discount rate.<sup>7</sup>

Each household of type  $j$  is endowed with an initial stock of capital,  $k_{j0} > 0$ , and a unit of time. The firms are assumed to use capital and labor in the production of a single perishable output good according to a constant returns to scale technology in private inputs. We summarize the distorted production technology by the continuous function  $f(k, n, K_m, N, \theta_a)$  where  $f : \mathbf{K} \times [0, 1] \times \mathbf{K} \times [0, 1] \times \Theta_a \rightarrow R_+$  and the firm's decision for capital and labor given by  $(k, n) \in \mathbf{K} \times [0, 1]$  where  $\mathbf{K} \subset R_+$  is a compact interval (to be described immediately after **Assumption 2** below),  $\Theta_a \subset R_{++}$  is compact,  $K_m = \sum_{j=1}^J \eta_j K_j = \sum_{j=1}^J (K_j/J)$  is the mean capital stock, and  $N$  is the average stock of labor. Let the cross-sectional distribution of initial individual capital stocks (or, initial aggregate capital stocks) by  $k_0 = (k_{10}, \dots, k_{J0}) \in \mathbf{K}^J$  (or,  $K_0 = (K_{10}, \dots, K_{J0}) \in \mathbf{K}^J$ ). In the sequel, we are more concise about where the initial capital endowments are assumed to reside. We make the following assumptions on the function  $f$ :

<sup>7</sup> We remark that the proofs in Becker and Zilcha (1997) show that their arguments work in a setting where both period utility and discount rates differ.



**Assumption 2.** The production function  $f(k, n, K_m, N, \theta_a)$  has the following properties:

- (i)  $f(k, n, K_m, N, \theta_a) > 0$  for all  $k > 0, \theta_a \in \Theta_a$  whenever  $k = K, n = N = 1$ ;
- (ii)  $f(k, n, K_m, N, \theta_a)$  is continuous, strictly increasing, twice continuously differentiable, and strictly concave in  $(k, n)$  for each  $(K_m, N, \theta_a)$ ;
- (iii) there exists  $\hat{k} > 0$  such that  $f(\hat{k}, n, \hat{k}, N, \theta_a) = \hat{k}$  and  $f(k, n, k, N, \theta_a) < k$  for all  $k > \hat{k}(\theta_a)$  for all  $\theta \in \Theta_a, k = K$  and  $n = N = 1$ .

Here, we assume that the production function  $f$  is defined net of capital depreciation. We next interpret [Assumption 2](#) in the context of our model. The types of equilibrium distortions allowed under [Assumption 2](#) are standard in the context of a homogeneous agent nonoptimal economy (e.g., [Coleman, 1991](#); [Greenwood and Huffman, 1995](#)). The assumption allows for more general production technologies than [Becker and Zilcha \(1997\)](#) (e.g., it is a reduced form settings with production nonconvexities of the sort discussed in [Romer, 1986](#); monopolistic competition, and public policy. See [Greenwood and Huffman, 1995](#), for a discussion).

Additionally, under [Assumption 2\(iii\)](#), there exists a maximum sustainable mean stock  $\sup_{\theta_a} \hat{k}(\theta_a)$  for this economy. We can therefore restrict the state space for capital stock to be in  $[0, \bar{k}]$  where  $\bar{k} = \max\{K_{10}, \dots, K_{J0}, \sup_{\theta_a} \hat{k}(\theta_a)\}$ . Recalling that the distribution of aggregate capital is given by  $K = [K_1, \dots, K_J] \in \mathbf{K}^J$ , as the aggregate capital stock is bounded by  $\sum_j \eta_j K_j$ , and we impose nonnegativity constraints on each component  $K_j \geq 0$ , we can use  $\sup_{\theta_a} \hat{k}(\theta_a)$  to construct an upper bound  $\bar{k}$  for any particular agent's capital holding in equilibrium under [Assumption 2](#).

Denote  $\mathbf{K}_+^J = \mathbf{K}^J \setminus \mathbf{0}$ , where  $\mathbf{0}$  the  $J$ -dimensional zero and  $\mathbf{K}^J = \Pi_n [0, \bar{k}]$ . Assume that all initial states for the economy have  $(k_0, K_0) \in [0, \sup_{\theta_a} \hat{k}(\theta_a)]^{2J} \subset \mathbf{K}_+^J \times \mathbf{K}_+^J$ . Denote the aggregate state space for the entire economy by  $S = [K, \theta] = [K_1, \dots, K_J, \theta] \in \mathbf{S} = \mathbf{K}^J \times \Theta$ . Notice that  $\mathbf{S}$  is compact. Finally, since the  $j$ th household enters a period with stock  $k_j$ , the state of an individual household is  $s_j = [k_j, S]$ .

### 3. Decision problems and definition of equilibrium

Firms hire capital and labor in competitive markets: let  $r(S)$  and  $w(S)$  represent the rental rate on capital and the wage rate, respectively. The technologies are allowed to be non-convex in social returns (e.g., see [Romer, 1986](#)). Given these prices, profit maximizing firms solve the standard optimization problem;

$$\Delta(S) = \max_{k \geq 0, n \in [0, 1]} f(k, n, K_m, N, \theta_a) - rk - wn.$$

Following [Becker and Zilcha \(1997\)](#), we assume that each household owns an identical share of the firm. Under constant returns to scale technologies, anticipating an equilibrium with  $k = K$  and  $n = N = 1$ , necessary and sufficient conditions are,

$$w(S) = f(K_m, \theta_a) - f_1(K_m, \theta_a)K_m, \quad (2)$$

$$r(S) = f_1(K_m, \theta_a), \quad (3)$$



where by homogeneity property in [Assumption 2](#),  $f(K_m, \theta_a) = f(K_m, 1, K_m, 1, \theta_a)$  is the output and  $f_1(K_m, \theta_a) = f_1(K_m, 1, K_m, 1, \theta_a)$  is marginal product of capital both along an equilibrium path.

We allow the government to tax both capital and wage income. Assume these policy interventions have the standard separable form:

$$\bar{r}(K, \theta_a) = [1 - \pi_k(S)]r(S) \text{ and } \bar{w}(K, \theta_a) = [1 - \pi_n(S)]w(S)$$

where the vector of distortion,  $\pi = [\pi_k, \pi_n]$  belongs to a function space,  $\Pi$ , which is a space of continuous mappings  $\mathbf{S} \rightarrow [0, 1] \times [0, 1]$ . We assume the dual partial order for a function lattice is placed on  $\Pi$ , i.e.,  $\pi' \geq \pi$  if  $\pi'(S) \leq \pi(S)$ , for all  $S$ . In addition, we restrict attention to  $\pi$  which satisfy the regularity conditions on the space of admissible distorted prices,

**Assumption 3.** The vector of distortions  $\pi = [\pi_k, \pi_n]$  and  $f$  are such that the distorted wage  $\bar{w}$  and the distorted rental rate  $\bar{r}$  satisfy the conditions:

- (i) (a)  $\bar{r}$  is decreasing in the vector  $K$ ; (b) given  $\hat{k}$  in [Assumption 2](#), there exists  $K_0$  such that  $\hat{k} > K_0 > 0$  each  $\theta$ ;  $\beta^u \int \bar{r}(K_0, \theta') \chi(\theta, d\theta') \leq 1$  where  $\beta^u = \max_{j \in \mathbf{J}} \{\beta_j\}$ ; and  $\beta_l \int \bar{r}(0, \theta') \chi(\theta, d\theta') > 1$  for  $\beta_l = \min_{j \in \mathbf{J}} \{\beta_j\}$ .
- (ii) the distorted income in equilibrium for the  $j$ th household,  $g_j(k_j, K, \theta) = \theta_j \bar{w}(K, \theta_a) + \bar{r}(K, \theta_a) k_j + J(K, \theta)$ , is increasing in  $(K, \theta)$ , when  $k_j = K_j$ , and  $g(0, 0, \theta) = 0$  for each  $\theta$ .

Here,  $J(K, \theta)$  is a continuous function that describes the average (and equal across all households) lump sum transfers from the government. [Assumption 3\(i.a\)](#) is similar to an assumption made in [Becker and Zilcha \(1997\)](#) except that we do not require the (reduced-form) production function to be “classical” and the distorted wage rate to be concave in the mean capital-labor ratio  $K_m$ .<sup>8</sup> Notice that under [Assumption 3\(i.a\)](#)  $f_1(k, 1, K_m, 1, \theta_a)$  to be decreasing in  $k$ , with  $k = K$ . This is the same assumption as in homogeneous agent economies of [Coleman \(1991\)](#) and [Greenwood and Huffman \(1995\)](#). The interpretation is that it bounds the size of production nonconvexity in equilibrium. For the methods we use in this paper, this assumption cannot be relaxed. Further, given the recent work on the nonexistence of continuous Markovian equilibrium ([Santos, 2002](#); [Krebs, 2004b](#); [Mirman et al., 2004a](#), Section 4), this assumption seems necessary for the existence of continuous Markovian equilibrium. We conjecture that [Assumption 3\(i.a\)](#) could be relaxed if one seeks a semicontinuous Markov equilibrium that is isotone in the vector  $K$  (see [Mirman et al., 2004a](#), for one possible attack on that problem.) We leave this discussion for future work.

[Assumption 3\(i.b\)](#) (along with [Assumption 1](#)) guarantees that in any MEDP, at least one agent holds a strictly positive amount capital such that each agent consumption decisions are strictly interior. All of the conditions are satisfied for example when  $f_1(k, 1, k, 1, \theta)$  satisfies an Inada condition as  $k \rightarrow 0$ . We remark that neither this assumption nor for that

<sup>8</sup> By a “classical” production technology, we are referring to the standard production case considered in [Brock and Mirman \(1972\)](#) of a production function consistent with convex aggregate production sets.

matter [Assumption 1](#) is necessary to construct our nonlinear operator used in our fixed point construction in the next section; they are used, however, to guarantee the existence of non-trivial Markovian equilibrium; e.g., equilibrium where output is produced in all states  $(k, K) \in \mathbf{K}_+^J \times \mathbf{K}_+^J$  and, at least, one agent holds capital.

[Assumption 3](#)(i.b) and (ii) are *joint* restrictions on the reduced form production function  $f$ , the state space  $\Theta$ , the transition  $\chi$ , and the equilibrium distortions  $\Pi$  along the *equilibrium trajectory* of the economy, where  $k = K$ . [Assumption 3](#)(ii) is restrictive, but it is certainly not vacuous.<sup>9</sup> This assumption is a natural multiagent generalization of assumptions used in the work on representative agent models of [Coleman \(1991\)](#), [Greenwood and Huffman \(1995\)](#), and [Datta et al. \(2002\)](#). Also, [Assumption 3](#)(ii) requires that as the *vector* of capital stocks goes to the origin (null vector) along an income trajectory of  $k = K$ , the vector of household incomes  $g(k, K, \theta) \rightarrow \mathbf{0}$ .<sup>10</sup> We relax [Assumption 3](#)(ii) in Section 6.

Under [Assumptions 1–3](#), we study each household's dynamic program. Each household of type  $j \in J$  faces the following ex ante individual feasibility constraint given by a well-defined correspondence,

$$\Upsilon(s_j) = \{(c_j, k'_j) : c_j + k'_j = g_j(k_j, K, \theta), c_j, k'_j \geq 0\},$$

where  $g_j(k_j, K, \theta) = \theta_j \bar{w}(K, \theta_a) + k_j \bar{r}(K, \theta_a) + J(K, \theta)$ . Notice that under [Assumptions 2 and 3](#),  $\Upsilon(s_j)$  is well-defined and expanding in  $(k_j, K)$  in the set inclusion partial order on the powersets of  $R_+^2$  (i.e.,  $\Upsilon(k_j, K, \theta) \subset \Upsilon(k'_j, K', \theta)$  for  $(k_j, K) \leq (k'_j, K')$ ). Further, since  $g_j$  is continuous in each of its arguments,  $\Upsilon(s_j)$  is a non-empty, compact and convex-valued for each  $s_j$ , and a continuous correspondence in  $k_j$ .

Next, we develop each household's dynamic program. Denote by  $\mathbf{C}^+(\mathbf{S})$  the space of bounded, positive continuous functions  $h = (h_1, \dots, h_J)$ ,  $h_j : \mathbf{K}^J \times \Theta \rightarrow \mathbf{K}^J$  defined on the compact set  $\mathbf{S} = \mathbf{K}^J \times \Theta$ , and with range,  $R_+^J$ . We equip  $\mathbf{C}^+(\mathbf{S})$  with the standard  $C^0$  uniform topology and the standard component wise partial order. By a standard argument,  $\mathbf{C}^+(\mathbf{S})$  is a complete metric space, a normal and minihedral cone of continuous functions, and finally a Banach lattice (under the sup metric topology and when endowed with the pointwise partial order induced by the cone). Define a subset of individually feasible decisions  $\hat{\mathbf{H}}(\mathbf{S}) \subset \mathbf{C}^+(\mathbf{S})$  to be  $\hat{\mathbf{H}}(\mathbf{S}) = \{h | h \in \mathbf{C}^+(\mathbf{S}), 0 \leq h_j \leq g_j\}$ . Then when developing dynamic programs for agents, we assume agents believe the law of motion for the distribution of aggregate capital stocks by  $K = [K_1, \dots, K_J]$  is given by the following recursive representation:

$$K' = h(\mathbf{S}); h \in \hat{\mathbf{H}}(\mathbf{S}).$$

<sup>9</sup> The assumption basically says that the complementarity (or cross elasticity) between capital and labor must majorize pointwise the concavity of the production processes in capital. Not all common production functions satisfy this assumption. Some CES productions work, but the Cobb–Douglas example studied in [Becker and Zilcha \(1997\)](#) does not work even when there are no idiosyncratic risk (e.g., the example in their Section 3). We analyze these cases and other generalizations in Section 6.

<sup>10</sup> This assumption is always satisfied if  $\bar{r}$  does not satisfy an Inada condition. If  $f$  and  $\pi$  are such that  $\bar{r}$  satisfies an Inada condition, the condition is satisfied, for example, if the distortion vector  $\pi \in \Pi$  depends only on the mean aggregate capital  $K_m$  (stated in efficiency units per capita) and the production function is Cobb–Douglas.

We can now state the decision problem for each household of type  $j$  under [Assumptions 1–3](#). Let  $\mathbf{G}(\mathbf{K} \times \mathbf{S})$  be the space of continuous bounded increasing functions  $v(k, S) : \mathbf{K} \times \mathbf{S} \rightarrow \mathbf{R}$  that are concave in  $k$ . Given the state variable  $s_j$  at the beginning of any period, the decision problem for the individual of type  $j$  is represented as the solution of the dynamic programming problem summarized by the Bellman equation for any  $h \in \mathbf{H}(\mathbf{S})$ ,

$$v_j(s_j) = \sup_{(c_j, k'_j) \in \mathcal{T}(s_j)} \{u_j(c_j) + \beta_j \int_{\Theta} v_j(s'_j) \chi(\theta, d\theta')\}. \quad (4)$$

Appealing to the contraction mapping theorem, the existence of a function  $v_j \in \mathbf{G}$  that satisfies the functional Eq. (4) can be established; also, the function  $v_j(s; h)$  is increasing in  $(k_j, K)$  and strictly concave in its first argument (as we have unbounded returns, see [Duran, 2000](#)). From [Mirman and Zilcha \(1975\)](#), the strict concavity of  $v_j$  in its first argument implies that an envelope theorem is available, and  $v_j$  is at least once differentiable in  $k_j$  for each  $j$ .

We now define equilibrium.

**Definition 1.** A Markovian (or recursive) competitive equilibrium decision process (MEDP) consists of initial states  $(k_0, K_0) \in \mathbf{K}^J \times \mathbf{K}^J$  such that  $k_0 = K_0$ ; a collection of function  $h^*(S)$  and value functions  $v_j(K_j, S)$  with associated individual decisions  $c_j^*(K_j, S) = y_j(K_j, S) - h_j^*(S)$  and  $k_j^*(K_j, S) = h_j^*(S)$  are such that: (i) given  $h^*$ ,  $v_j(K_j, S)$  is the value function for each household type  $j$  that satisfies the household's Bellman equation for each type of agent  $j = \{1, 2, \dots, J\}$ ; (ii) for each  $(K, \theta)$ , given the input prices,  $w$  and  $r$  in Eqs. (2) and (3), the firms solve  $\max_{k \geq 0, n \in [0, 1]} f(k, n, K_m, N, \theta_a) - rk - wn$ ; (iii) all markets clear, that is, by Walras' law  $k^*(s) = h^*(S) = K' \in \mathbf{H}$ ,  $c^*(s) = C(S) = g - h^*$  with  $c_j^*(K_j, S) + k_j^*(K_j, S) = g_j(K_j, K, \theta)$  for all  $j$ ; and, (iv) the government budget is balanced, that is,  $J(S) = \pi_n(S)w(S) + \pi_k(S)r(S)$ .

#### 4. Computing isotone MEDPs

Prior to considering the existence of an MEDP for economies satisfying [Assumptions 1–3](#), we define some terms that are used in subsequent discussion.

- I. Ordered sets and lattices: Consider a set  $X$  ordered with a reflexive, transitive, antisymmetric relation denoted by “ $\geq$ ”.  $X$  is a *partially ordered set* (or *Poset*). An *upper (lower) bound* for a set  $B \subset X$  is an element  $x^u (x^l) \in B$ , such that,  $\forall x \in B, x \leq x^u (x^l \leq x)$ . If the set of upper bounds (lower bounds) possesses a least (greatest) element, then we say this element is a *least upper bound* or *supremum* (*greatest lower bound* or *infimum*). A set  $X$  is a *lattice* if for any two elements  $x$  and  $x'$  in  $X$ , the pair has a least upper bound and a greatest lower bound (denoted, respectively, by  $x \vee x'$  and  $x \wedge x'$ ). A subset of a lattice, say  $B \subset X$ , is a *sublattice* of  $X$  if it contains the sup and the inf (with respect to  $X$ ) of any pair of points in  $B$ . A lattice is *complete* if for any subset  $B \subset X$ , the least upper bound ( $\sup(X)$ ) and a greatest lower bound ( $\inf(X)$ ) exist in  $X$ . A *chain*  $C \subset X$  is a totally ordered subset of  $X$  (i.e., all pairs of elements are ordered).

- II. Vector lattices: A *partially ordered vector space* (or *semi-ordered linear space*)  $X$  is a real vector space equipped with a partial order  $\geq$  that is compatible with the following algebraic structure: (i) if  $x \geq x'$ , then  $x + z \geq x' + z$ , for all  $z \in X$ ; (ii) if  $x \geq x'$ , then  $\alpha x \geq \alpha x'$  for all  $\alpha \geq 0$ . Any partially ordered vector space that is also a lattice is referred to as a *vector lattice*. If the space  $X$  has a norm  $\|x\|_X$  such that  $|x| \geq |x'|$  in  $X$ , implies  $\|x\| \geq \|x'\|$ , then we say  $X$  has a *lattice norm*. A complete normed vector space is a *Banach space*. A *normed vector lattice* is a vector lattice equipped with a lattice norm. A normed vector lattice  $X$  that is complete in the Cauchy sense, and is endowed with a lattice norm is referred to as a *Banach lattice*. In much of our work we use compact subsets of Banach lattices.
- III. Cones and order intervals in banach spaces: Let  $X$  be a topological space. Then set  $X^+ = \{x \in X, x \geq 0\}$  is the *cone* of  $X$ . We assume additionally that  $X$  is a real Banach space  $X$ ; therefore the set  $X^+ = \{x \in X, x \geq 0\}$  will be an *order cone* of  $X$ . In particular,  $X^+$  is a nonempty, convex, closed set that has the following two properties: (i)  $x \in X^+ \implies \alpha x \in X^+$  for  $\alpha \geq 0$ ; (ii) if  $x$  and  $-x$  in  $X^+$ ,  $x = 0$ , where  $0$  denote the zero of the cone. The partial order induced by the cone structure of  $X^+$  has  $x_1 \geq x_2$  if  $x_1 - x_2 \in X^+$ . A cone is *normal* if there exists a constant  $m$  such that for any  $x_1, x_2 \in X^+$ ,  $\|x_1 + x_2\| \geq m, \|x_i\| = 1$  for  $i = 1, 2$ . Intuitively, a normal cone can be interpreted geometrically as bounding the angle between any two unit vectors away from the number  $\pi$ . Therefore, a normal cone cannot become “too large”. A cone is *regular* if every increasing and bounded order sequence in  $X^+$  has a limit in  $X^+$ . A *fully regular* cone has every increasing and norm bounded sequence in  $X^+$  having a limit in  $X^+$ . A fully regular cone is also regular. A regular cone is also normal. (Guo and Lakshmikantham, 1988, Theorem 1.2.1). As we work on compact subset of normal cones, the compactness compensates for the lack of full regularity on the entire cone. A cone  $X^+$  is *solid* if the interior of  $X^+$  is nonempty.
- Let  $[a] = \{x | x \in X, x \geq a\}$  be the *upper set* of  $a$ , and  $(b) = \{x | x \in X, x \leq b\}$  the *lower set* of  $b$ . We say  $X$  is an *ordered topological space* if  $[a]$  and  $(b)$  are closed in the topology on  $X$ . An order interval is defined to be  $[a, b] = [a] \cap (b)$ ,  $a \leq b$ . The spaces we study are compact subsets of partially ordered topological spaces. Therefore, if  $[a, b] \subset X$ , and  $X$  is an ordered topological space, all order intervals are closed. In this paper, we study fixed point problems that are set on compact order intervals in normal cones of continuous functions defined on compact sets equipped with the standard uniform topology. Therefore, we need additional structure so that sequences that converge in order and/or norm, converge in the set. We compensate for the loss of regularity (of the cone) with the compactness of the order interval.
- IV. Mappings on partially ordered sets: Let  $(X, \geq_X)$  and  $(Y, \geq_Y)$  be partially ordered sets. A *mapping* is a relational statement between two spaces, say  $X$  and  $Y$ . By referring to the relation as a “mapping”, we consider both “point-to-point” or “point-to-set” relations. In the case of a “point-to-point” mapping, we refer to the mapping as a *function* or an *operator*. A function or operator  $m : X \rightarrow Y$  is defined to be *isotone* on  $X$  if  $m(x') \geq_Y m(x)$ , when  $x' \geq_X x$ , for  $x, x' \in X$ . If  $m(x') >_Y$  (respectively,  $<_Y$ )  $m(x)$  when  $x' >_X x$  for  $x, x' \in X$ , then we say  $m$  is *increasing* (respectively, *decreasing*). If  $m(x') >_Y$  (respectively,  $<_Y$ )  $m(x)$  when  $x' \geq_X x$ ,  $x' \neq x$ , then we say  $m$  is *strictly*

increasing (respectively, strictly decreasing). We say  $m(x)$  is *antitone* if  $m(x) \geq_Y m(x')$  if  $x' \geq_X x$ . If a function is either isotone or antitone, we say it is *monotone*.

Let  $X^+$  be an order cone of a real Banach space, and consider an operator  $A : X^+ \rightarrow X^+$ . We say an operator  $A$  is *e-concave* if there exists non-zero  $e \in X^+$ , such that (i) for an arbitrary non-zero  $x \in X^+$  the inequalities  $\alpha e \leq Ax \leq \beta e$ , where  $\alpha$  and  $\beta$  are positive, are valid and (ii) for every  $x \in X^+$  such that  $\alpha_1(x)e \leq x \leq \beta_1(x)e$  with  $(\alpha_1(x), \beta_1(x)) \gg 0$ , and there is a number  $\eta(x, t) > 0$  such that  $Atx \geq (1 + \eta)tAx$  for any  $t \in (0, 1)$ . An operator is said to be *pseudo-concave* on  $X^+$  if for all  $t \in (0, 1)$ ,  $x \in X^+$ ,  $x > 0$ ,  $Atx \gg tAx$ . Let  $X^+$  be a solid cone, the operator  $A : X \rightarrow X$  is *strongly sublinear* if  $Atx \gg tAx$  for all non-zero  $x \in X^+$  and  $0 < t < 1$  (see Guo and Lakshmikantham, 1988, Definition 2.2.2). A pseudo-concave operator on a solid cone is strongly sublinear on the interior of the cone. An operator  $Ah : \mathbf{P} \rightarrow \mathbf{P}$  is a *cone compression* of a normal cone  $\mathbf{P}$  if there exists an  $R, r > 0$  such that (i)  $Ah \leq rh$ , for  $h \in \mathbf{P}$ ,  $\|h\| < r$ , and (ii)  $Ah \geq rh$ , for  $h \in \mathbf{P}$ ,  $\|h\| > R$ ,  $h \neq 0$ .

We can generalize the notion of a monotone function (or an operator) to a correspondence (or multifunction). A *correspondence or multifunction* is a nonempty-valued mapping,  $M : X \rightarrow 2^Y$ , e.g., a nonempty-valued “point-to-set” mapping. We say a correspondence or multifunction is *ascending* in the set relation  $S$  (denoted by  $\geq_S$ ) if  $M(x') \geq_S M(x)$ , when  $x' \geq_X x$  where  $(X, \geq_X)$  is a partially ordered space. If the set relation for the *powersets of Y* (i.e., the collection of all subsets of  $Y$ , denoted by  $2^Y$  or  $P(Y)$ ) based upon the set relation  $\geq_S$  induces a partial order on the powersets on  $2^Y$  (or perhaps  $2^Y \setminus \emptyset$ ), we refer to the ascending correspondence also as an *isotone correspondence*. An *antitone* correspondence from  $X$  to  $2^Y$  (or perhaps  $2^Y \setminus \emptyset$ ) is isotone in the dual order in its domain  $X$ .

To define an ascending and/or isotone correspondence, we consider the following set relations  $\geq_S$  on the powersets  $2^Y$  (or perhaps  $2^Y \setminus \emptyset$ ).<sup>11</sup> Smithson (1971) and Veinott (1992) have developed set relations for correspondences that admit isotone selections. In this paper, we focus on four set relations. Let  $Y$  be a set, and  $A, B \in 2^Y$ . We define (i) the *Veinott-Weak Set relation*  $\geq_w$  on  $2^Y : A \geq_w B$ , if for any  $a \in A, b \in B$ , either  $a \wedge b \in B$  or  $a \vee b \in A$ ; (ii) the *Veinott-Strong Set Order*  $\geq_s$  on  $2^Y \setminus \emptyset : A \geq_s B$ , if for any  $a \in A, b \in B, a \wedge b \in B$  and  $a \vee b \in A$ ; (iii) The *Smithson-Weak Set relation*  $\geq_{as}$  on  $2^Y \setminus \emptyset : A \geq_{as} B$  if (C1) for any  $b \in B$ , there exists an  $a \in A$  such that  $a \geq b$ ; and (C2) for any  $a \in A$ , there exists a  $b \in B$  such that  $a \geq b$ ; and (iv) the *Pointwise Strong Set Order*  $\geq_{ss}$  on  $2^Y : A \geq_{ss} B$  if and only if  $a \in A, b \in B$ , then  $a \geq b$  in the partial order structure on  $A$ , for all  $a, b$ .<sup>12</sup> Notice that if  $Y$  is a lattice with  $2^Y \setminus \emptyset$  ordered under the induced set order implied by Veinott’s strong set order  $\geq_s$  on  $Y$ ,  $(2^Y \setminus \emptyset, \geq_s)$  is a partially ordered set (actually, a lattice). Also, notice that the Veinott’s strong set order and Smithson’s weak set relations are both compatible with set comparisons where the sets are singletons (and therefore our notations of ascending correspondences based

<sup>11</sup> Some classic references for set relations are Smithson (1971) and Veinott (1992). For additional generalizations of “ascending” correspondences that prove useful in economics are found in Heikkilä and Reffett (2004, 2005).

<sup>12</sup> The  $S$ -weak set relation in (iii) was studied extensively in Smithson (1971). It has been subsequently referred to under the name “weak induced set order” and is characterized in Topkis (1998, p38). For a set  $Y$ , the weak induced set relation does not induce a valid partial order on the powersets of  $Y$  (i.e., the collection  $2^Y$ ).

upon these set relations will be compatible with the standard definition of an isotone function)

Finally, a sequence  $\{h_n \rightarrow h\}$  in  $H$  is *order convergent on sequences* if there exists two monotonic sequences (e.g., countable chains) of elements from  $H$ , one decreasing  $\{h_{\downarrow n}\}$ , and one increasing  $\{h_{\uparrow n}\}$ , such that  $h = \inf h_{\downarrow n} = \sup h_{\uparrow n}$  and  $h_{\uparrow n} \leq h_n \leq h_{\downarrow n}$ . A necessary and sufficient condition for an increasing sequence  $h_n \rightarrow h$  to be order convergent is  $h = \sup h_n$ . An operator  $Ah$  is *order continuous on sequences* on  $H$  if whenever  $h_n \rightarrow h$ , in order,  $Ah_n \rightarrow Ah$ , in order. For example, an operator  $A$  is order continuous along directed sequences  $\{h_n\}$  if  $\vee A(h_n) = A(\vee h_n)$  (see Davey and Priestley, 1990, Chapters 2 and 3 for a discussion of order continuity and related notions).

With this investment in terminology, we are now ready to study the equilibrium problem.

#### 4.1. Existence and computation of isotone MEDPs

Appealing to standard arguments, the Euler inequalities associated with the right side of the Bellman Eq. (4) can be written for each agent  $j \in \mathbf{J}$  as follows,

$$u'_j(c_j(s_j)) \geq \beta_j \int_{\Theta} u'_j(c_j(s'_j)) \bar{r}(h(K, \theta), \theta'_a) \chi(\theta, d\theta'), \quad \text{if } k'_j > 0, \quad (5)$$

where  $s'_j$ ,  $\theta'_a$  and  $k'_j$  refer to next period's state, aggregate shock and capital, respectively; and  $u'_j$  is the marginal utility (for agent  $j$ ). In a recursive competitive equilibrium, a candidate consumption function for the household of type  $j \in \mathbf{J}$  necessarily satisfies  $c_j(s_j) = C_j(S)$  pointwise when  $k = K$  where  $C_j(S)$  depends only on the aggregate state. Further, recall the income process for household  $j \in \mathbf{J}$  is given as  $g_j(K_j, K, \theta) = g_j(K, \theta)$  when  $k = K$ .

Given these equilibrium restrictions, we can rewrite the set of  $J$  Euler inequalities in (5), which are written in terms of household *consumption*, equivalently, as a system of  $J$  Euler inequalities in equilibrium written in terms of *investment*. Given the concavity of each agent's value function in  $k_j$  under Assumptions 1–3, a solution to the following system of equilibrium Euler system of inequalities is necessary and sufficient condition for the existence of an MEDP; namely finding a vector of investment functions  $i^* = i^*(K, \theta)$ , such that the consumption vector,  $C = g - i^* > 0$ , when  $K \geq 0$  and  $K \neq 0$ , in the expression,

$$u'_j(g_j - i_j^*) \geq \beta \int_{\Theta} u'_j(g_j(i^*, \theta') - i_j^*(i^*, \theta')) \bar{r}(i^*, \theta'_a) \chi(\theta, d\theta'),$$

if  $i_j^* > 0$  for  $j = 1, \dots, J$ , (6)

where  $g - i^* = [g_1 - i_1^*, \dots, g_J - i_J^*]$  and  $g_j = g_j(K, \theta)$ .

We now prove the existence of a complete lattice of functions  $i^*(K, \theta)$  that satisfy (6). To do this, we use an equilibrium version of the system of Euler inequalities in (6) to define implicitly a nonlinear operator  $Ah$  on the space of candidate equilibrium investment decisions  $\mathbf{H} \subset \bar{\mathbf{H}}$  for which a fixed point of  $Ah$ , say  $h^* \in \mathbf{H}$ , can be shown to be an MEDP. Recalling that the space  $\bar{\mathbf{H}}$  is a subset of the space of positive continuous functions,  $\mathbf{C}^+(\mathbf{S})$ , that is socially feasible, define the closed subset  $\mathbf{H} \subset \bar{\mathbf{H}}$  of candidate equilibrium investment equilibrium functions (that have stronger isotonicity properties) as follows:

- (i)  $h = [h_1, \dots, h_J] : \mathbf{K}^J \times \Theta \rightarrow \mathbf{K}^J$  is continuous,
- (ii)  $h_j \in [0, g_j]$  for all  $(K, \theta)$ ,<sup>13</sup>
- (iii)  $0 \leq h_j(K', \theta) - h_j(K, \theta) \leq g_j(K', \theta) - g_j(K, \theta)$  for all  $j$  when  $K' \geq K$  for each  $\theta \in \Theta$ .

We prove now the existence of MEDPs within this subset of candidate equilibrium investment functions. Notice the space  $\mathbf{H}$  consists of continuous functions such that consumption and savings for each agent  $j \in \mathbf{J}$ , are both isotone in distribution of wealth for the aggregate economy (condition (iii) implies that). We remark that under [Assumptions 2 and 3](#), as  $g_j$  is a continuous and differentiable function for all  $K \geq 0$ ,  $K \neq 0$ , each  $h_j$  for  $j \in \mathbf{J}$  defined on a compact set  $\mathbf{S}$ ,  $g_j$  is locally Lipschitz for each agent  $j$  (and, therefore, Clarke differentiable) on the interior of  $\mathbf{K}^J$ . This implies each element of  $h \in \mathbf{H}$  inherits this local Lipschitz property.<sup>14</sup>

In [Lemma 1](#), as  $\mathbf{H}$  is an equicontinuous subset of functions in the normal and solid cone of continuous functions  $\mathbf{C}^+(\mathbf{S})$  when ordered by the pointwise Euclidean order induced by the cone, then by Arzela-Ascoli's theorem  $\mathbf{H}$  is relatively compact in  $\mathbf{C}^+(\mathbf{S})$ . As  $\mathbf{H}$  is also closed in  $\mathbf{C}^+(\mathbf{S})$ ,  $\mathbf{H}$  is therefore compact in  $\mathbf{C}^+(\mathbf{S})$ . As  $\mathbf{H}$  is compact in a topology finer than the interval topology and a sublattice of  $\mathbf{C}^+(\mathbf{S})$ , by Frink's theorem (e.g., see [Birkhoff, 1967](#), Theorem 20, p. 250) we concluded  $\mathbf{H}$  is subcomplete in the relative order on  $\mathbf{C}^+(\mathbf{S})$ . We summarize these observations in [Lemma 1](#).<sup>15</sup>

**Lemma 1.**  $\mathbf{H}$  is a convex, subcomplete, order interval in  $\mathbf{C}^+(\mathbf{S})$ .  $\mathbf{H}$  is also a compact in  $\mathbf{C}^+(\mathbf{S})$  such that each  $h \in \mathbf{H}$ ,  $h$  is locally Lipschitz on the interior of  $\mathbf{K}_+ \times \Theta$ .

Let  $u'(g - i) = [u'_1(g_1 - i_1), \dots, u'_J(g_J - i_J)]^T$  denote the vector of marginal utilities. Then the  $J$  Euler equations in [\(6\)](#) can be written more compactly as,

$$u'(g - i) = \beta \int_{\Theta} u'(g(i, \theta') - i(i, \theta')) \bar{r}(i, \theta') \chi(\theta, d\theta').$$

As, in general,  $u'(c)$  is extended real valued for  $c \in \mathbf{K}^J$ , define (using the obvious vector notation) the extended real valued mapping  $Z : \mathbf{K}^J \times \mathbf{K}_+^J \times [0, g) \times \Theta \rightarrow \bar{\mathbf{R}}^J = \mathbf{R}^J + \{-\infty, \infty\}$ ,<sup>16</sup>

$$\begin{aligned} Z(x, K, h, \theta) &= \Psi_2(x, K, \theta) - \Psi_1(x, K, h, \theta), \\ \Psi_1(x, K, h, \theta) &= \beta \int_{\Theta} u'(g(x, \theta') - h(x, \theta')) \bar{r}(x, \theta') \chi(\theta, d\theta'), \\ \Psi_2(x, K, \theta) &= u'(g(K, \theta) - x). \end{aligned} \tag{7}$$

<sup>13</sup> We note that this restriction under [Assumption 3\(ii\)](#) implies  $h(0, \theta) = 0$  for each  $\theta \in \Theta$ .

<sup>14</sup> By proving equilibrium of MEDPs within this class, we extend a result concerning Clarke differentiability of equilibrium policies functions due to [Montrucchio \(1987\)](#) to our environment with many agents, production nonconvexities, and incomplete markets.

<sup>15</sup> Proofs of the lemma, propositions, and theorems are in [Appendix A](#).

<sup>16</sup> Given that  $Z$  is extended real valued, notice we are careful to avoid any comparison of  $\infty$  and  $-\infty$  in the definition of  $Z$ . That is, we only define the correspondence  $x_{Zj}^*(K, h, \theta)$  using the equation  $Z$  only at points where this comparison is not made.



Further define the nonlinear operator  $A : \mathbf{H} \rightarrow \mathbf{H}'$  implicitly in  $Z$  as follows (here let  $\mathbf{H}'$  be a subset of the set of space of positive bounded functions on  $\mathbf{K} \times \Theta$ ):

$$Ah_j(K, \theta) = \{x_j(K, h, \theta) \in x_{Z_j}^*(K, h, \theta) : (K, h, \theta) \in \mathbf{K} \times \mathbf{H} \times \Theta;$$

$$x_j(K, \theta, h) \text{ isotone in } (K, h),$$

where the correspondence  $x_{Z_j}^*(K, h, \theta)$  is given on  $\mathbf{K}_+^J \times [0, g] \times \Theta$  as:

$$x_{Z_j}^*(K, h, \theta) = \{x | Z(x, K, h, \theta) \leq 0, \text{ and } Z_j(x, K, h, \theta) = 0 \text{ if } x_j > 0\}. \quad (8)$$

and extended to all of  $\mathbf{K} \times \mathbf{H} \times \Theta$  as  $x_{Z_j}^*(K, h, \theta) = g_j(K, \theta)$  if  $h_j(K, \theta) = g_j$  for  $(K, \theta) \in \mathbf{K}^J \times \Theta$ ,  $x_{Z_j}^*(K, h, \theta) = 0$  if  $K = \mathbf{0}$ ,  $\theta \in \Theta$ .

Recalling that a *compact operator* is an operator that maps bounded subsets into relatively compact subsets, and a *completely continuous operator* is a compact operator that is also continuous in the relevant topologies, we can now prove a key proposition useful in the subsequent proofs of the main theorems of the paper.<sup>17</sup>

**Proposition 2.** Under Assumptions 1–3, (i) for any  $h \in \mathbf{H}$  and  $(K, \theta) \in \mathbf{S}$ ,  $Ah(K, \theta) : \mathbf{H} \rightarrow \mathbf{H}'$  is well-defined; (ii)  $Ah(K, \theta) \subset \mathbf{H} \subset \mathbf{H}'$ ; (iii)  $Ah$  is a continuous and compact operator (e.g., a completely continuous operator) on  $\mathbf{H}$  in the uniform topology; (iv)  $Ah$  is isotone on  $\mathbf{H}$ ; (v)  $Ah$  is order-continuous along sequences (i.e., countable chains) in  $\mathbf{H}$ .

Since  $\mathbf{H}$  is a nonempty, compact and convex order interval in the space of continuous functions on  $\mathbf{K} \times \Theta$ , and  $A$  is a completely continuous operator, a simple topological argument for the existence of Markovian equilibrium might be available. For example, Schauder's theorem guarantees that existence of a fixed point for such an operator  $A$  (see, [Hutson and Pym, 1980](#), p. 208, Theorem 8.2.3.). In our case, this approach is not useful from the point of establishing existence of equilibrium since we already know  $A$  has a trivial fixed point (namely the zero consumption plan), and in order to get interior fixed points, it proves difficult to associate a (strictly concave) value function.

An alternative approach is to follow [Becker and Zilcha \(1997\)](#) and [Miao \(2002\)](#). In their approach, one defines a Markov (or adjoint) operator on the space of probability measures on  $\mathbf{S}$  that aggregate the optimal policy functions of each agent  $j \in \mathbf{J}$ , where the policy functions are given as the solution to each agents (strictly concave) dynamic program. Given that  $\mathbf{S}$  is compact, the space of probability measures is weakly compact; assuming a Feller property for the shocks in  $\chi$  and noting the strict concavity to the agent dynamic program in (4), this Markov operator is weakly continuous. Existence then follows from a standard application of a local convexity fixed point (e.g. Schauder's theorem, or Fan-Glicksberg). Although latter approach is very powerful for existence, it is unfortunately very difficult to relate such an approach to applied work using numerical methods based on constructing policy functions. Therefore, we also do not follow this approach.

<sup>17</sup> As an implication of Proposition 2(i) and (ii), we can provide a generalized version of an implicit function theorem for multifunctions. Examining our proof, the argument is very related to [Robinson \(1982\)](#). His proof uses a Karash–Kuhn–Tucker approach to obtaining a locally Lipschitz selection. Estimates of the Clarke-gradient of this selection can be constructed directly using Robinson's results. Additional related generalizations of Robinson's arguments are discussed in the monograph by [Rockafellar and Wets \(1997\)](#).

We follow instead an approach that is somewhat in the spirit of Coleman (1991). The implementation of such an approach in our multiagent setting (relative to Coleman's representative agent setting) is much more complicated. First, we have equilibrium inequality constraints. Proposition 2 shows that one can systematically deal with these considerations in a class of nonlinear fixed point problem associated with Euler equations. This proposition is, therefore, of independent interest for other dynamic, general equilibrium environments. A second consideration is that our operator  $Ah$  has a much more complicated geometric structure than the operator in a single agent economy with no borrowing constraints (and strict interiority). As in Coleman (1991), we are able to exploit the monotonicity of  $A$  to construct fixed points; also like Coleman (1991), we are able to combine this underlying monotonicity with the results on order continuity for our operator to apply a stronger version of Tarski's theorem that proves useful for the computation of extremal fixed points via successive approximation off the least and greatest points of the transformation space. We defer a discussion of this issue to Section 5, where we make explicit how nonlinear operators in heterogeneous agent models differ radically from their homogeneous agent counterparts.

For existence and computation of MEDPs, we consider the application of the Knaster–Tarski fixed-point theorem to the operator  $A$  relative to the space  $\mathbf{H}$ . We denote the fixed points of the operator  $Ah$  by  $E(\pi)$ , and, as we are interested in how this equilibrium correspondence changes with perturbations of  $\pi$ , we make explicit the dependence of  $E$  on the policy distortions  $\pi$ . The application of the Knaster–Tarski theorem we use on our setting is due to Kantorovich (for a proof, see Vulikh, 1967, Lemma XXII.2.1; Dugundji and Granas, 1982, Theorem 4.2). The theorem states that, for an operator  $A$  that is order continuous (and therefore necessarily isotone) on sequences (i.e., countable chains), that transforms a nonempty complete lattice  $\mathbf{H}$  into itself, for which there exists some element  $\bar{h} \in \mathbf{H}$  with the property  $A\bar{h} \leq \bar{h}$ , has a nonempty set of fixed points. In addition, the sequence  $\{A^n \bar{h}\}$  converges, by successive approximation, to the maximal fixed point  $h_u^* = \sup E(\pi)$ .<sup>18</sup> Using a dual argument, we can obtain similar results for the minimal fixed point  $h_l^*$ . In our problem,  $h_u^* = g$  is a (trivial) maximal fixed point, and cannot be decentralized as a competitive equilibrium. Hence, we focus on the minimal fixed point. As long as  $h_l^* \in E(\pi)$  is such that  $g - h_l^* \gg 0$ , then  $h_l^*$  is an MEDP.

**Theorem 3.** *Under Assumptions 1–3 for any given  $\pi \in \Pi$ , (i) the set of fixed points of  $Ah$ , namely  $E(\pi)$ , is nonempty complete lattice such that any  $h^* \in E(\pi)$ ; (ii) among the set of fixed points of  $A$ , there exists a greatest fixed point  $h_u^*(\pi) = \sup E(\pi)$  (respectively, least fixed point  $h_l^*(\pi) = \inf E(\pi)$ ) such that  $\lim_n A^n(g) = h_u^*(\pi) (= g)$  (respectively,  $\lim_{n \rightarrow \infty} A^n(0) = h_l^*(\pi) \geq 0$ ) and the convergence is uniform; (iii) the least fixed point  $h_l^*(\pi) = \inf E(\pi)$  is such that for all  $j \in J$ ,  $g_j - h_{jl}^*(\pi) \gg 0$ ; (iv)  $E(\pi_k)$  is isotone on  $(\Pi_k, \leq_d)$  in the strong set order  $\geq_s$  in  $\pi_k$ , where  $\Pi_k = \{\pi_k | (\pi_k, \pi_n) \in \Pi\}$  and  $\leq_d$  is the dual standard pointwise partial order on  $\Pi_k$ , and  $\sup E(\pi)$  and  $\inf E(\pi)$  form isotone selections.*

<sup>18</sup> Note that, there are other fixed point arguments that deliver this successive approximation result. For excellent discussions, see Amann (1977) and Heikkilä and Lakshmikantham (1994), Chapter 1.

Our work can be related to the findings on determinacy of equilibrium of Kehoe et al. (1990) and Santos (1992). A so-called Negishi problem is constructed, which is a modified planning problem that can be used to characterize the equilibrium manifold, and a sufficient condition for establishing comparative analysis results is that the value function be  $C^2$ .<sup>19</sup> In contrast, we take an order based approach to comparative analysis, and consider the question of global uniqueness within the class of equicontinuous Markovian equilibrium where both investment and consumption decisions are monotone.

## 5. On the stability of successive approximation algorithms

When applying numerical methods to the models that satisfy Assumptions 1–3, the applied researcher first develops an approximation scheme (e.g., a set of basis functions for a standard projection method a la Krasnosel'skii et al., 1972, Chapters 1 and 4) that delivers asymptotically consistent numerical approximations for an unknown element of the equicontinuous collection functions  $h \in \mathbf{H}$  using successive approximations on an isotone operator.<sup>20</sup> Therefore, when applying such a scheme to the operator  $Ah$ , one obtains an iterative projection method as in Kurpeí (1976), Chapter 1. The existence of such approximation schemes for the class  $\mathbf{H}$  is well-known. In this section, our concern is the stability of successive approximations on  $Ah$  from any initial  $h_0 \in [0, g]$ .

To address the issue of stability of successive approximations, we develop sufficient conditions under which we can characterize arbitrary trajectories of iterative procedures based upon the operator  $A$ . In Theorem 3, we provide a characterization concerning the computation of extremal fixed points. That is, the theorem proves that the limit of monotone iterations,  $\lim_n A^n(0)$ , is stable for the minimal fixed point. This fixed point can be associated with strictly positive consumption functions. In this section, we identify additional conditions that sharpen the stability property of  $Ah$  from any initial  $h_0 \in \mathbf{H}$ ,  $h_0 < g$  (i.e.,  $h_0 \in [0, g)$ ). That is, we characterize the limit,  $\lim_n A^n(h_0)$ , for any  $h_0 \in [0, g)$ . For this, we require an additional interiority condition, namely  $A(0) > 0$ . We refer to this as the “strong” interiority case. We note that the condition is *very strong* relative to the question of global stability. One should not expect uniqueness of MEDPs relative to the class  $\mathbf{H}$ .

In Theorem 4, we provide a characterization of strictly interior Markovian equilibrium within the class  $\mathbf{H}$ . We develop a new proof of strict positivity within the class  $\mathbf{H}$  under the stated hypotheses using the ideas of a cone compression found in Krasnosel'skii and Zabreiko (1984). The argument can be viewed as a multidimensional extension of an argument in Coleman (1991).

<sup>19</sup> Santos (1992) provides a set of sufficient conditions for smooth equilibrium policy functions.

<sup>20</sup> That is, we assume that the basis elements are chosen such that there exists a sequence of approximations  $\hat{h}_n \in \mathbf{H}_n$  such that  $\hat{h}_n \rightarrow h \in \mathbf{H}$ , uniformly as  $n \rightarrow \infty$ ,  $n \in N$ , where  $N$  the set of natural numbers. The nonlinear fixed point operator  $A$  defined in (7), is both topologically and order continuous. Relative to topological continuity, in particular, the trajectory  $\{A^n(0)\}$  forms an equicontinuous sequence in  $\mathbf{H}$ . Therefore, an iterative projection method based upon  $A\hat{h}_n \rightarrow Ah$ , for suitable  $\mathbf{H}_n$  (e.g., piecewise linear as in Coleman, 1990) has the desired convergence properties from an initial function  $h_{n0} = 0$ . Asymptotic convergence issues for successive approximations of extremal fixed points are, thus, straightforward, and can be addressed by various approaches (e.g., Judd, 1992).

**Theorem 4.** Under Assumptions 1–3, for  $\pi \in \Pi$ ,  $h_0 \in [0, g)$ , if  $A(0) > 0$ , then  $g > \min E(\pi) = h_1^* = \lim_n A^n h_0 > 0$  is the unique strictly interior Markovian equilibrium within the class of equicontinuous Markovian equilibrium in  $\mathbf{H}$ .

To prove this theorem, we first consider rewriting the operator  $Ah$  as a new operator  $\hat{A}m$  that can be studied more readily using geometric arguments. To do this, define the space  $\mathbf{M}$  of functions  $m : S \rightarrow R_+^J$  satisfying the following conditions:

- (i)  $m(K, \theta) = [m_1(K, \theta), \dots, m_J(K, \theta)]$  with  $m_j : \mathbf{K}^J \times \Theta \rightarrow \mathbf{K}^J$  continuous for all  $j$ ;
- (ii) for  $K > 0$ ,  $0 \leq m_j(K, \theta) \leq 1/u'_j(g_j(K, K, \theta)) =$  for all  $j$ ;
- (iii)  $m_j(0, \theta) = 0$  for all  $j$  and all  $\theta$ ;
- (iv)  $\bar{r}(K', \theta)/m_j(K', \theta) \leq \bar{r}(K, \theta)/m_j(K, \theta)$  when  $K' \geq K$ .

For any  $m \in \mathbf{M}$ , we define the function  $H(m, K, \theta) = [H_1(m, K, \theta), \dots, H_J(m, K, \theta)]$  implicitly in the following system of  $n$  equations:

$$\begin{aligned} u'(H_j(m(K, \theta))) &= \frac{1}{m_j(K, \theta)} \text{ for } m > 0 \\ H_j(m, K, \theta) &= 0 \text{ for } m_j(K, \theta) = 0. \end{aligned} \quad (9)$$

which under our concavity, continuity, and monotonicity assumptions embodied in Assumptions 1–3 can be solved globally for a unique inverse (using the inverse function theorem noting that properness follows from compactness of the domain  $\mathbf{S}$ ). Notice, this mapping  $H_j$  can be defined if a particular agent  $j$  is borrowing constrained, i.e.,  $H_j = g_j$ . Finally note that if  $A(0) > 0$  implies all agents invest in equilibrium, therefore defining  $m^u = \sup \mathbf{M}$ , we must have  $\hat{A}m^u < m^u$ . Therefore, we have for the indicator variables for investment  $I_x(x = 0) = \mathbf{0}$  in the definition of  $Ah$  in (7) and (8); thus we can use (7)–(9) to define the operator  $\hat{A}m$  without any inequality constraints.

Now, under the strict concavity and boundary assumptions on  $u_i(c)$  in Assumption 1, by a standard inverse function argument relative to the definition of  $H$  defined above in (9), we have  $H$  well defined for each  $m(K, \theta) \in \mathbf{M}$ . In particular, using the definition of  $Ah$  (which is a monotone selection in the correspondence  $x^*(t)$  as discussed in Proof of Proposition 2 in Appendix A), noting that “today’s” consumption in (6) is now defined using  $H = g - Ah$ . Further noting that “tomorrow’s” consumption in (6) is  $H' = g - h$ , we can rewrite  $\Psi_1$  and  $\Psi_2$  in (6) using (9) as follows:

$$\begin{aligned} \Psi_1 &= \beta \int_{\Theta} u'(g(x, \theta') - h(x, \theta')) \bar{r}(x, \theta') \chi(\theta, d\theta') \\ &= \beta \int_{\Theta} \frac{\bar{r}(g - H(\tilde{m}(K, \theta)), \theta')}{m(g - H(\tilde{m}(K, \theta)), \theta')} \chi(\theta, d\theta'), \quad \Psi_2 = \frac{1}{\tilde{m}}. \end{aligned}$$

$Z(x, K, \theta, h)$  in (6) then becomes with equality in all states:

$$\hat{Z}(m, x, K, \theta) = \frac{1}{x} - \beta \int_{\Theta} \frac{\bar{r}(g - H(x), \theta')}{m(g - H(x), \theta')} \chi(\theta, d\theta'), \quad (10)$$

where the new operator  $\hat{A}$  is defined implicitly as follows:

$$\hat{A}m(K, \theta) = \{x(K, \theta) \in x^*(K, \theta) \mid \hat{Z}(m, x^*(K, \theta, K, \theta)) = 0, \\ x(K, \theta) \text{ isotone}, K \neq \mathbf{0}, m > 0; 0 \text{ elsewhere}\}.$$

Noting (10), observe must have  $\hat{A}m = x(K, \theta) = 1/u'(g - Ah)$  for the isotone selection in (6) used to define  $Ah$  for this case when  $h < g$ . Further under Assumptions 1–3, noting  $m \in \mathbf{M}$ , using an argument similar to the one in the Proof of Proposition 2, we have  $\hat{Z}$  is strictly increasing in  $m$ , and strictly decreasing in  $x$ . In addition, for fixed  $m > 0$ ,  $K \neq \mathbf{0}$  and  $\theta$ ,  $\hat{A}m = 1/(u'(g(K, \theta) - Ah(K, \theta))) \rightarrow 0$  implies that  $\hat{Z} \rightarrow +\infty$ , and  $\hat{A}m \rightarrow 1/u'(g(K, \theta))$ , implies that  $\hat{Z} \rightarrow \infty$ . Consequently, again following arguments in Proposition 2,  $\hat{A}m$  is well defined using an isotone selection in the correspondence  $x^*(K, \theta)$  for each  $m > 0$ ,  $K \neq \mathbf{0}$ , and  $\theta$ . Note that  $\hat{A}m = 0$ , elsewhere.

The operator  $\hat{A}$  is related to the mapping  $Ah$  via the mapping  $Bh = g - Ah$  in the following manner (where  $Bh$  simply defines the operator  $Ah$  equivalently in terms of consumption).<sup>21</sup> To any orbit of the operator  $Bh = g - Ah$  (i.e., any trajectory  $B^n h_0$  for  $h_0 \in \mathbf{H}$  with proposition) corresponds to an orbit of the operator  $\hat{A}$ , and can be obtained through the following construction. Given any  $h_0 \in \mathbf{H}$ , we have  $m = 1/u'(g - h) \in \mathbf{M}$ . (Equivalently,  $m$  satisfies  $H(m) = g - h$ ). Then by construction, there exists an  $\hat{A}m$  associated with any  $Ah$  that satisfies  $\hat{Z}(m_0, \hat{A}m_0, K, \theta) = 0$ , that is,

$$\frac{1}{\hat{A}m(K, \theta)} = \beta E_\theta \frac{\bar{r}(g(K, \theta) - H(\hat{A}m(K, \theta)), \theta')}{m(g(K, \theta) - H(\hat{A}m(K, \theta)), \theta')};$$

Therefore, by the definition of  $\hat{A}m$  it must be that  $1/\hat{A}m = u'(Bh)$  (or, equivalently, that  $H(\hat{A}m) = Bh = g - Ah$ ). Then starting at  $h_0 \in \mathbf{H}$ , by induction, we have for all  $n = 1, 2, \dots$ ,  $Bh_0 = g - H(\hat{A}^n m)$ . It is also true that any fixed point of  $B$  (and therefore  $Ah$ ) corresponds a fixed point of  $\hat{A}$ . Indeed, for any  $h^*$  fixed point of  $A$ , define  $m^* = 1/u'(g - h^*)$  (or, equivalently,  $H(m^*) = h^*$ ). By definition  $h^*$  satisfies (using the definition that  $B = g - Ah$ )

$$u'(g - h^*(K, \theta)) = \beta E_\theta u'(g(h^*(K, \theta), \theta')) - h^*(h^*(K, \theta), \theta')\bar{r}(h^*(K, \theta), \theta'),$$

or, equivalently,

$$\frac{1}{m^*(K, \theta)} = \beta E_\theta \frac{\bar{r}(g(K, \theta) - H(m^*(K, \theta)), \theta')}{m^*(g(K, \theta) - H(m^*(K, \theta)), \theta')},$$

using the fact that  $Bh = g - Ah$ , with  $m^*$  a fixed point of  $\hat{A}$ .

<sup>21</sup> In this proof, it will become clear that from a notational perspective it is easier to translate arguments involving  $Ah$  (e.g., investment) in terms of  $Bh$  (e.g., consumption). That is why here we introduce the notation  $Bh = g - Ah$ , where  $Ah$  is our original operator defined in (7).

We have several remarks on this theorem. First in the case of CES preferences, we can work directly with the operator  $Ah$  defined in (7). That is, if  $A(0) > 0$ , as  $u'(tm) = u'(t)u'(m)$ , then as in Coleman (1991) and Datta et al. (2002), one can show directly  $Ah$  is a strongly sublinear operator on the interior of  $\mathbf{H}$  (that argument follow for example from Coleman, 1991; Theorem 11). Second, note that the hypotheses to obtain the stability result in Theorem 4 are very strong. They require preferences and technologies to be such that *all* agents in a MEDP are not borrowing constrained.

Third, we note the sense in which the condition  $A(0) > 0$  is necessary for our operator to be a cone compression in Theorem 4. First, consider the mapping  $\hat{Z}(m, \hat{A}m, K, \theta)$  defined in (10) under Assumptions 1–3. Then in general we have  $A(0) \geq 0$ . One can easily see that now  $\hat{A}m$  is not a cone compression (or strongly sublinear on all of  $\mathbf{M}$ ). Consider  $\hat{A}m$  when  $A(0) \geq 0$  (and therefore  $\hat{A}m^u \leq m^u$  with strict inequality for only for one  $j \in J$ ). In this case, we modify the definition of  $\hat{Z}$  to include the borrowing constraints as in Proof of Proposition 2 for the mapping  $\hat{Z}$ , and we obtain that  $\hat{A}m$  is not strongly sublinear on all of  $\mathbf{M}$ ; i.e.,

$$\begin{aligned}\hat{Z}(m, \hat{A}m, K, \theta) &= \frac{1}{\hat{A}m} - \beta \int_{\Theta} \frac{\bar{r}(g - H(\hat{A}m), \theta')}{m(g - H(\hat{A}m), \theta')} \chi(\theta, d\theta') - \phi I(\hat{A}m = m^u) \\ &= \frac{1}{t\hat{A}m} - \beta \int_{\Theta} \frac{\bar{r}(g - H(\hat{A}m), \theta')}{tm(g - H(\hat{A}m), \theta')} \chi(\theta, d\theta') - \frac{\phi I(\hat{A}m = m^u)}{t} = 0 \\ &\leq \frac{1}{t\hat{A}m} - \beta \int_{\Theta} \frac{\bar{r}(g - H(\hat{A}tm), \theta')}{tm(g - H(\hat{A}tm), \theta')} \chi(\theta, d\theta') + \phi I(\hat{A}tm = m^u),\end{aligned}$$

where the last (weak) inequality follows from  $\hat{A}tm \leq \hat{A}m$  even though  $tm < m$ . This lack of strict local isotonicity on the boundary is the result of the presence of occasionally binding equilibrium inequality constraints that affect the shape of  $\hat{A}$  over the entire space  $\mathbf{M}$ . In previous work on uniqueness, without borrowing constraints (e.g., the homogeneous agent economy in Coleman, 2000), interiority conditions are strong enough to guarantee the needed strict form of isotonicity on or near the boundary. The borrowing constraint, therefore, implies when  $A(0) \geq 0$  (and not  $A(0) > 0$ ), the fixed point operator  $\hat{A}m$  is only sublinear, not strongly sublinear. Thus, we do not have sufficient geometric structure to guarantee globally unique fixed points for  $\hat{A}m$ .

We make a few final remarks on the limitations of these methods. First, for the numerical implementation of the operator  $\hat{A}m$  or  $Ah$ , one needs to develop algorithms for constructing continuous isotone selections. That we leave for future work. Second, Theorem 4 only provides the basis for pointwise strong set order comparative statics for MEDPs in  $\mathbf{H}$ . It is important to realize that our uniqueness argument does not rule out existence of Markov equilibrium outside the class  $\mathbf{H}$  (or  $\mathbf{M}$ ). Given the way selections are formed in the definition of the operator  $Ah$ , we believe, for example, there could be other MEDPs within the class of bounded functions. So uniqueness here (even in the case of Theorem 4) is, in principle, a “weak” result.

Finally, the assumptions on the boundedness of the state space can be relaxed, allowing for models with unbounded growth, for example, in the two sector incomplete markets economy studied by Krebs (2004a). Using the methods in Morand and Refett (2003), one

can use pure lattice theoretic fixed point constructions to study the operator  $Ah$ , and all the main results hold. In future work, we will address the issues of long-run stationary Markovian equilibrium. To use order theoretic limit theory of [Hopenhayn and Prescott \(1992\)](#) for characterizing the long-run ergodic measure for the capital stocks and idiosyncratic shocks, one requires joint monotonicity of equilibrium in endogenous states and random shocks.

## 6. Continuity of MEDPs with non-monotonic income processes

We now discuss the question of existence and computation of Lipschitz (but not necessarily isotone) MEDPs. For this discussion, we can relax [Assumption 3](#), and study income processes associated with general concave production processes. To do this, we develop an new approach to applying isotone recursive methods to economies that does not necessarily require on to have isotone MEDPs. For this new approach (the so-called “expanded state space” approach to isotone recursive methods), we expand our representation of the “aggregate” state space in the equilibrium system of functional equations. In this method, we only require that  $j$ th household’s income process in equilibrium (which is  $g_j(K_j, K, \theta)$ ) be re-written as an isotone mapping on an “expanded state space” of  $(K_j, K, \hat{K}, \theta)$  (where there is an antitone response of the income process in  $\hat{K}$  in the “natural” componentwise order that is modeled as isotone in a dual order for this component of the expanded aggregate state space). So, for example, we can rewrite the household  $j$ ’s income as  $g_j(K_j, K, \hat{K}, \theta) = r(\hat{K}, \theta_a)K_j + w(K, \theta_a)\theta_j + J(K, \hat{K}, \theta)$  where in the natural order,  $g_j$  is isotone in  $(K, \theta)$  for each  $\hat{K}$  and antitone in  $\hat{K}$  for each  $(K, \theta)$  (where  $K$  and  $\hat{K}$  here denote the mean capital stock).<sup>22</sup> Using the dual order for the space  $\hat{K} \in \mathbf{K}^J$  (and denoted the dual ordered space by  $\hat{\mathbf{K}}^J$ ), intuitively one can view this expanded aggregate state space method for studying functional equations finding a state where one can break the effects of perturbations on the aggregate state vector  $K$  into an isotone part ( $K$ ) and an antitone part ( $\hat{K}$ ). This “decomposition” proves helpful as it allows one to find a function space (and a transformation of this space) where we can consider the existence of continuous (but not necessarily isotone) MEDPs that satisfy equilibrium system (6). Then, when one solves a resulting functional equation on the expanded aggregate state space, and then imposes  $K = \hat{K}$ , general equilibrium is restored, and we arrive at a MEDP. As these fixed points exist on their expanded aggregate state spaces as *mixed monotone* functions, when

<sup>22</sup> For example, in the classical production case with no public policy or production nonconvexity in social returns, assuming  $f(k, n, \theta) = \theta k^\alpha n^\beta$ , with  $(\alpha + \beta) = 1$ , and  $\alpha, \beta \geq 0$ . With  $n = 1$ , the income process for household  $j$  on the expanded state space is given by

$$g_j(k_j, K, \hat{K}, \theta) = rk_j + w\theta_j = \alpha\theta\hat{K}^{\alpha-1} \cdot k_j + \beta\theta K^\alpha\theta_j.$$

Therefore, in this Cobb–Douglas production example, household income processes have a *mixed-monotone* structure on the expanded state space: this allows the total variation in household income to be “decomposed” into an isotone variation in  $K$  and an antitone variation in  $\hat{K}$ . In equilibrium, of course, we require  $K = \hat{K}$ .



we impose  $K = \hat{K}$ , one cannot claim the MEDP is isotone in  $K$ .<sup>23</sup> Most importantly though, the MEDP remains continuous (and actually Lipschitz if each income process is assumed to be Lipschitz on the expanded state space). This proves useful for studying stationarity of the Markovian equilibrium limiting distribution, as well as allowing one to obtain standard uniform approximation theory of Krasnosel'skii et al. (1972) can be applied to our isotone operator, and in principle, error bounds can be studied. One set of uniform error bounds then available (for example) can be derived from the approximation schemes presented in Santos and Vigo-Aguiar (1998) and Santos (2000) (as we also have a collection of dynamic programming algorithms that can be tied to particular orbits of our resulting fixed point operator). Others are also available (e.g., see Reffett, 2005a,b).

To begin our description of the mixed monotone method, we first state the following assumption on the primitives of the environment:

**Assumption 4.** The vector of distortions  $\pi = [\pi_k, \pi_n]$  and  $f$  are such that the distorted wage  $\bar{w}$  and the distorted rental rate  $\bar{r}$  are as in Assumption 3 (now relative to  $K$  and  $\hat{K}$  as defined below) except instead of Assumption 3(ii), we assume  $(\theta_1, \dots, \theta_J) \subset R_+^J$  with  $f, \theta$ , and  $\pi$  are such that the distorted income in equilibrium for the household  $g_j(k_j, K, \theta)$  can be written as  $g_j(k_j, K, \hat{K}, \theta) = \theta_j \bar{w}(K, \theta_a) + \bar{r}(\hat{K}, \theta_a) k_j + J(K, \hat{K}, \theta)$  where using the standard componentwise orders  $g(k_j, K, \hat{K}, \theta)$  is isotone in  $(K, \theta)$  when  $k_j = K_j$  for each  $\hat{K}$ , and antitone in  $\hat{K}$  for each  $(K, \theta)$ ;  $g(0, 0, 0, \theta) = 0$  for each  $\theta$ .

Most production functions used in applied work in incomplete market models (with or without public policy and production nonconvexities) are consistent with Assumption 4.

To construct MEDPs, we first expand the state-space for the equilibrium system of Euler equations in (6) as in Reffett (2005a,b), Datta and Reffett (2005a), and Mirman et al. (2004b). Under Assumptions 1, 2 and 4, consider studying the solutions to (6) and (8) where we change the space of parameters for the fixed point mapping appropriately. Let  $\hat{K} \in \hat{\mathbf{K}}^J$ . Given  $\hat{\mathbf{K}}^J$  the dual componentwise Euclidean order. Define the following set of functions to be using in (6) and (8) (and the dynamic program (4) also) to be:

$$\begin{aligned} h \in \hat{\mathbf{H}} &= \{h(K, \hat{K}, \theta) = (h_1, \dots, h_J) | h_j(K, \hat{K}, \theta) : \mathbf{K}^J \times \hat{\mathbf{K}}^J \times \Theta \rightarrow \mathbf{K} \\ &\text{is continuous, isotone in } (K, \hat{K}); g_j(K, \hat{K}, \theta) - h_j(K, \hat{K}, \theta) \\ &\text{isotone in } (K, \hat{K})\}. \end{aligned} \quad (11)$$

We make a few observations about the space  $\hat{\mathbf{H}}$ . First,  $\hat{\mathbf{H}}$  is an compact order interval in the normal cone of positive continuous functions on the compact set  $\mathbf{K}^J \times \hat{\mathbf{K}}^J \times \Theta$  (when the space of continuous functions on  $\mathbf{K}^J \times \hat{\mathbf{K}}^J \times \Theta$  is endowed with the uniform metric topology. The cone of positive continuous functions is denoted by  $\mathbf{C}^+(\mathbf{K}^J \times \hat{\mathbf{K}}^J \times \Theta)$ ). Second for any  $h \in \hat{\mathbf{H}}$ , as  $h$  is isotone in  $(K, \hat{K})$  for fixed  $\theta \in \Theta$ ,  $\hat{\mathbf{H}}$  is a nonempty sublattice

<sup>23</sup> We shall say a function  $f(x, y) : X \times X \rightarrow X$  is *mixed-monotone*, if  $f$  is isotone in  $x$  for each  $y$ , and antitone in  $y$  for each  $x$ .

of  $\mathbf{C}^+(\mathbf{K}^J \times \hat{\mathbf{K}}^J \times \Theta)$  consisting of a collection of mixed-monotone functions on  $\mathbf{K}^J \times \mathbf{K}^J$  for each  $\theta \in \Theta$ . We summarize these observations in the following lemma (the proof follows similar to Lemma 1):

**Lemma 5.**  $\hat{\mathbf{H}}$  is (i) a compact order interval in  $\mathbf{C}^+(\mathbf{K}^J \times \hat{\mathbf{K}}^J \times \Theta)$ ; (ii) a complete sublattice in  $\mathbf{C}^+(\mathbf{K}^J \times \hat{\mathbf{K}}^J \times \Theta)$  under the induced partial order associated with the normal cone  $\mathbf{C}^+(\mathbf{K}^J \times \hat{\mathbf{K}}^J \times \Theta)$ .

Next, following the ideas of Section 4, consider the modified version of the extend-real valued mapping  $Z(x, K, \theta, h)$  studied in Eq. (7), which changes the domain of  $Z$  to accommodate the expanded state space  $(K, \hat{K}, \theta, h)$  used to model the mixed-monotone structure of equilibrium decision rules. Consider the new extend-real valued mapping, denoted by  $Z_c : \mathbf{K}^J \times \mathbf{K}_+^J \times \hat{\mathbf{K}}_+^J \times [0, g) \times \Theta \rightarrow \bar{\mathbf{R}}^J = \mathbf{R}^n + \{-\infty, \infty\}$  (where now for this section, we remind the reader we take the interval  $[0, g) \subset \hat{\mathbf{H}}$ ):

$$Z_c(x, K, \hat{K}, h, \theta) = \Psi_2(x, K, \hat{K}, \theta) - \Psi_1(x, K, \hat{K}, h, \theta), \quad (12)$$

where for  $j \in J$ ,

$$\Psi_{1jc} = \beta \int_{\Theta} u'_j(g_j(x, h(K, \hat{K}, \theta), \theta') - h_j(x, h(K, \hat{K}, \theta), \theta')) \bar{r}(x, \theta') \chi(\theta, d\theta') \quad (13)$$

and

$$\Psi_{2jc} = u'_j(g_j(K, \hat{K}, \theta') - x). \quad (14)$$

Notice that for fixed  $(\hat{K}, \theta)$ ,  $Z_c$  has the same qualitative structure in  $(x, K, h)$  as the mapping  $Z(x, K, h, \theta)$  before (see Lemma 7 in Appendix A). That is,  $Z_c(x, K, \hat{K}, h, \theta)$  is (i) antitone in  $h$  for fixed  $(x, K, \hat{K}, \theta) \in \mathbf{K}^J \times \mathbf{K}_+^J \times \hat{\mathbf{K}}_+^J \times \Theta$  (as for  $h \in \hat{\mathbf{H}}$ , we additionally note that  $g - h$  decreasing in  $\hat{K}$ , and therefore  $\Psi_2$  is isotone in  $h$ , and  $Z_c$  is antitone in  $h$ ); (ii) strictly antitone in  $K$  for each  $(x, \hat{K}, h, \theta) \in \mathbf{K}^J \times \hat{\mathbf{K}}_+^J \times [0, g) \times \Theta$  (using similar argument as for  $h$  except that  $h \in \hat{\mathbf{H}}$  has  $h$  increasing in  $K$ ); and (iii) strictly isotone in  $x$  for each  $(K, \hat{K}, h, \theta) \in \mathbf{K}_+^J \times \hat{\mathbf{K}}_+^J \times [0, g) \times \Theta$ . As for the comparative statics in  $\hat{K}$  for each  $(K, h, \theta)$ , as  $\hat{Z}_c$  has the dual comparative structure in  $\hat{K}$  (for the usual componentwise order on  $\hat{K}^J$ ) as for  $K$  by construction,  $Z_c(x, K, \hat{K}, h, \theta)$  is antitone in  $\hat{K}$  when using the dual order for  $\hat{K}^J$  for each  $(x, K, h, \theta) \in \mathbf{K}^J \times \mathbf{K}_+^J \times [0, g) \times \Theta$ . Additionally, for a fixed  $(x, K, h, \theta)$ , it can be shown that  $Z_c(x, K, \hat{K}, h, \theta)$  is actually strictly antitone (again, using the dual order) in  $\hat{K}$ .

Define  $\hat{\mathbf{T}}_+ = \mathbf{K}_+^J \times \hat{\mathbf{K}}_+^J \times [0, g)$  where  $(K, \hat{K}, h)$  is a typical element of  $\hat{\mathbf{T}}_+$ , where we give the subspace  $\hat{K} \in \hat{\mathbf{K}}_+^J \subset \mathbf{R}_+^J$  the dual componentwise order on  $\mathbf{R}_+^J$ . Consider the nonlinear operator  $A^c : \hat{\mathbf{H}} \rightarrow \mathbf{H}'$  (where  $\mathbf{H}'$  is at least a subset of the set of bounded functions on  $\mathbf{K}^J \times \hat{\mathbf{K}}^J \times \Theta$ ) for  $j \in J$ ,

$$A^c h_j(K, \hat{K}, \theta) = x_j(K, \hat{K}, h, \theta) \in x_{Z_{jc}}^*(K, \hat{K}, h, \theta) \text{ for } (K, \hat{K}, h) \in \hat{\mathbf{T}}_+, \quad (15)$$

$$\theta \in \Theta, x_j(K, \hat{K}, \theta, h) \text{ isotone in } (K, \hat{K}, h) \text{ given } \theta \in \Theta,$$

where the correspondence  $x_{Zc}^*(K, \hat{K}, h, \theta)$  is given on  $\hat{\mathbf{T}}_+ \times \Theta$  as

$$x_{Zc}^*(K, \hat{K}, h, \theta) = \{x | Z_c(x, K, \hat{K}, h, \theta) \leq 0, \text{ and } Z_{jc}(x, K, \hat{K}, h, \theta) = 0 \text{ if } x_j > 0\}. \quad (16)$$

and  $x_{Zc}^*(K, h, \theta) = g_j(K, \hat{K}, \theta) = x_{Zjc}^*(K, \hat{K}, h, \theta)$  when  $h_j(K, \hat{K}, \theta) = g_j$  for  $(K, \hat{K}, \theta) \in \mathbf{K}^J \times \hat{\mathbf{K}}^J \times \Theta$ ; and  $x_{Zc}^*(K, h, \theta) = 0 = x_{Zjc}^*(K, \hat{K}, h, \theta)$  when  $K$  and/or  $\hat{K} = \mathbf{0}$ ,  $\theta \in \Theta$  (i.e., we extend  $x_{Zc}^*(K, \hat{K}, h, \theta)$  to all of  $\mathbf{K}^J \times \hat{\mathbf{K}}^J \times \mathbf{H} \times \Theta$ ). Remember as the space  $\hat{K} \in \hat{\mathbf{K}}_+^J$  is given the dual order,  $A^c h$  is antitone in  $\hat{K}$  in the standard componentwise partial order on  $\mathbf{R}_+^J$ .

We now extend Theorem 3 to economies under Assumptions 1, 2 and 4.

**Theorem 6.** Under Assumptions 1, 2 and 4, for fixed  $\pi \in \Pi$ , (i) the set of fixed points of  $Ah$ , namely  $E(\pi) \subset \hat{\mathbf{H}}$ , is a nonempty complete lattice where the set of MEDPs is given along the restriction  $K = \hat{K}$ ; (ii) among the set of fixed points of  $A$ , there exists a greatest (respectively, least) fixed point  $h_u^*(\pi) = \sup E(\pi)$  (respectively,  $h_l^*(\pi) = \inf E(\pi)$ ) such that  $\lim_{n \rightarrow \infty} A^n(g) = h_u^*(\pi) = g$  (respectively,  $\lim_{n \rightarrow \infty} A^n(0) = h_l^*(\pi) \geq 0$ ) and the convergence is uniform. (iii) The least fixed point  $h_l^*(\pi) = \sup E(\pi)$  is such that at least for one index  $j \in J$ ,  $g_j - h_{jl}^*(\pi) \gg 0$ ; (iv)  $E(\pi_k)$  is isotone on  $(\Pi_k, \leq_d)$  in the strong set order  $\geq_s$  in  $\pi_k$  where  $\Pi_k = \{\pi_k | (\pi_k, \pi_n) \in \Pi\}$  and  $\leq_d$  is the dual standard pointwise partial order on  $\Pi_k$ , and  $\sup E(\pi)$  and  $\inf E(\pi)$  form isotone selections.

## 7. Conclusion

This paper develops an isotone recursive fixed point approach to the study of MEDPs in multiagent incomplete markets models with production. Our methods are based on the existence of Euler equations (which are needed to provide sharp geometric characterizations of the nonlinear operators used to construct Markovian equilibrium), and concavity of the underlying production and preference primitives. One advantage of our method is that they are developed using a single valued continuous operator that maps an equicontinuous set of functions into itself. This latter fact (concerning continuity and compactness) is critical for numerical approaches to computing the equilibrium. In particular, versions of the Stone–Weierstrass theorem can be used to prove that projection methods ala Krasnosel'skii et al. (1972) (e.g. piecewise linear methods based upon Coleman (1990) or more general projection methods based on splines and orthogonal polynomials as in Judd (1992)) are asymptotically consistent. That is, as the dimension of the basis set used to approximate both the elements of the domain as well as the operator, tend toward infinity, the projection converges to the actual equilibrium fixed point.

The policies in our equilibrium set are locally Lipschitz continuous functions, and each agent's value function (in equilibrium when  $k = K$ ) is concave. These are key properties that could be exploited to achieve the error bounds on numerical solutions via dynamic programming arguments, as in Santos and Vigo-Aguiar (1998), or more general non-parametric error bound methods as presented in Santos (2000). Since trajectories of operators can be tied explicitly to a collection of value functions (one for each type of agent), one should,

in principle, be able to apply Santos (2000) to construct asymptotic error bounds to numerical methods based upon our monotone operator  $A$  following a version of the argument in Mirman et al. (2004, Section 4). We leave this work for the future.

Additionally, the issue of existence of long-run stationary Markovian equilibrium remains, as do the characterization of the equilibrium pricing functional. One can change the assumptions on the primitive data of the model to obtain sufficient conditions under which Markovian equilibrium are jointly monotone in  $(K, \theta)$ . For example, adapting a condition studied in Mirman et al. (2004a) to our setting, one might assume  $u$  and  $f$  are such that  $u''(c)f_1f_2 + u'(c)f_{12} \geq 0$ . In our setting, this condition is sufficient to show that the Markovian equilibrium in Theorem 3 is actually jointly monotone in  $(K, \theta)$ . In this case, using the limiting distribution theory of Hopenhayn and Prescott (1992) and Danthine and Donaldson (1981), one can construct a stationary Markovian equilibrium in the sense of Duffie et al. (1994) using the fixed point theorems for correspondences in Heikkilä and Reffett (2005, 2004). We leave further work along this line to future research.

In this economy, we study MEDPs in situations where the production processes are bounded. Morand and Reffett (2003) show how to relax this condition for representative agent economies. A similar argument is available in this paper. If one assumes that the income processes  $g_j(k_j, K, \theta)$  associated with the distorted production function  $f(k, n, K_m, N, \theta)$  are uniformly continuous jointly in  $(k_j, K)$  for each  $\theta$ , then the order intervals  $\mathbf{H}$  in Lemma 1 and  $\hat{\mathbf{H}}$  in Lemma 5 are closed equicontinuous subsets of the space of continuous functions on the unbounded state space  $\mathbf{K} \times \Theta$ . Although not compact, these spaces are complete lattices. In this case, the arguments of Theorem 3 and 5 carry through with little modification. Of course, in such a setting, one must also verify that the functional equation describing agent preferences in (4) can be solved on a space of bounded value functions. But, in principle, allowing for asymptotic growth in two sector models (for example, Krebs, 2004a) is not a problem.

Finally, it might be possible to develop a method of monotone comparative analysis for the Markovian equilibrium set for this economy under more general assumptions on the primitive data of taste, technology, and the class of equilibrium distortions considered. In many situations concavity of the value function is lost, and, therefore, single valued operators on spaces of continuous functions are no longer available. However, it is often possible to apply generalizations of Tarski's theorem to such problems by exploiting the fact that equilibrium best responses (under weak supermodularity assumptions along an equilibrium path) are uniformly bounded and monotone functions. Spaces of uniformly bounded, monotone functions form complete lattices, and thus, existence and characterization using order based fixed point theory are available. Mirman et al. (2004a) show how such methods can be used to construct monotone Markovian equilibrium for a large class of nonoptimal representative agent models. Further, using these methods, it should be possible to study with monotone methods multisector, multiagent economies with incomplete markets such as those presented in a series of papers by Krebs (2004a, 2003). His results concern a symmetric class with identical CES preferences, constant returns to scale technologies, and no public policy. Our methods could be used to extend these results. One might develop a set of tools that would make general versions of the Krebs environment tractable for applied researchers interested in incorporating human capital and physical capital into a

richer model of economic fluctuations and growth that also allow for incomplete markets and trade: perhaps, mixed-monotone methods can be used to obtain analytical results.

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## Appendix A. Proofs

In the appendix, we discuss all the proofs of the Lemma, propositions, and theorems in the paper. We begin with [Lemma 1](#):

**Proof of Lemma 1.** Convexity and the fact that  $\mathbf{H}$  is a closed sublattice of the Banach lattice of continuous functions, equipped with the standard pointwise partial order on  $\mathbf{C}^+(\mathbf{S})$ , is obvious.  $\mathbf{H}$  is, therefore, an order interval.

For compactness, simply note that  $\mathbf{H}$  is an equicontinuous set of functions defined on a compact space under the restriction that  $g - h$  is increasing in  $K$ ,  $h$  increasing in  $K$ ,  $\Theta$  is discrete, and  $g$  and  $h$  both continuous on a compact set (and the variation in  $h$  is uniformly bounded by a uniformly continuous, actually locally Lipschitz function  $g$ ). Therefore, and by a standard application of the Arzela-Ascoli theorem,  $\mathbf{H}$  is relatively compact in  $\mathbf{C}^+(\mathbf{S})$ . As  $\mathbf{H}$  is closed in  $\mathbf{C}^+(\mathbf{S})$ ,  $\mathbf{H}$  is compact in  $\mathbf{C}^+(\mathbf{S})$ . As  $g$  is Lipschitz in  $K$  on  $\mathbf{K}_+$  and  $\Theta$  is denumerable (and therefore Lipschitz in  $\theta$  in the discrete topology), we conclude that  $\mathbf{H}$  consists of Lipschitz continuous functions on  $\mathbf{K}_+ \times \Theta$ .

Finally as  $\mathbf{H}$  is a closed sublattice in  $\mathbf{C}^+(\mathbf{S})$  that is also equicontinuous in the  $C^0$  uniform topology,  $\mathbf{S}$  is compact, therefore  $\mathbf{H}$  is compact in the interval topology. We conclude that  $\mathbf{H}$  is a complete (sub)lattice by the complete characterization of complete lattices in the order topology by Frink–Birkhoff Theorem (Birkhoff, 1967, Chapter 10, Theorem 20). See also the discussion in Frink (1942).

So for any subset of  $\mathbf{H}$ , say  $H \subset \mathbf{H}$ ,  $\sup_{K,\theta} H = h^u(K, \theta)$  and  $\inf_{K,\theta} H = h^l(K, \theta)$  are in  $\mathbf{H}$ .  $\square$

We next consider the [Proof of Proposition 2](#). It is useful to prove two lemma before we get to the proposition. In [Lemma 7](#), we note some properties of the extended-real mapping  $Z(x, K, h, \theta) : \mathbf{K}^J \times \mathbf{K}_+^J \times [0, g] \times \Theta \rightarrow \bar{\mathbf{R}}^J$  as defined in (7) of Section 4 that prove useful when characterize the mapping  $x_Z^*(K, h, \theta)$  in (7) on  $\mathbf{K}_+^J \times [0, g] \times \Theta$ .

**Lemma 7.** Under Assumptions 1–3, for fixed  $\theta \in \Theta$ , (i)  $Z(x, K, h, \theta)$  is antitone in  $h$  with each  $Z_j(x, K, h, \theta)$  is strictly decreasing in  $h_j$ ; (ii)  $Z(x, K, h, \theta)$  is strictly decreasing in  $K$ ; and (iii)  $Z_j(x, K, h, \theta)$  is strictly increasing in  $x$ .

**Proof.**

- (i) Fix  $(x, K, \theta)$ , and let  $h' \geq h$  (for  $h, h' \in [0, g]$ ). Under the concavity assumptions in Assumption 1,  $\Psi_1$  is increasing in  $h$  (as the vector  $u(c)$  is concave); therefore  $Z$  is antitone in  $h$ .  
Now, take  $h'_j \geq h_j, h', h \in [0, g]$ . Then by Assumption 1,  $\Psi_{1j}$  is increasing as each  $u_j(c_j)$  for  $j \in \mathbf{J}$  is strictly concave, therefore we have for each  $Z_j(x, K, h'_j, \theta) \leq Z_j(x, K, h_j, \theta)$ ; if  $h'_j \geq h_j$ , and  $h'_j \neq h_j$ , then the inequality is strict. Therefore,  $Z_j$  is strictly antitone in  $h$ .
- (ii) Fix  $(x, h, \theta)$ , and take  $K' \geq K$ . A similar argument to the above for  $h$  shows the system  $\Psi_1$  is increasing in  $K$ . Now, as  $g$  is strictly increasing in  $K$  by Assumptions 2 and 3, we have  $\Psi_2$  decreasing in  $K$ . So  $Z$  is antitone in  $K$ . Again, if  $K' \neq K$ , and  $K' \geq K$ , all the inequalities are strict. Therefore,  $Z$  is strictly antitone in  $K$ .
- (iii) For each  $(K, h, \theta)$ , when  $x' \geq x$ ,  $\Psi_2$  decreases by Assumptions 1–3 and the fact  $h \in [0, g]$ , while  $\Psi_1$  increases by Assumption 1; therefore  $Z(x', K, h, \theta) \geq Z(x, K, h, \theta)$ . If  $x' \geq x, x' \neq x$  then the inequality is strict by the strict concavity of  $u(c)$  in Assumption 1 and the fact that  $\bar{r}$  is decreasing in Assumption 3.  $\square$

We now prove the following lemma concerning sufficient conditions for the existence of an isotone selection in a nonempty-valued correspondence  $y(x) : X \rightarrow 2^Y \setminus \emptyset$  where both  $\mathbf{X}$  and  $\mathbf{Y}$  are complete lattices. Consider an additional regularity condition for the correspondence  $y(x)$ . Let  $C \subset \mathbf{X}$  be a chain and  $g(x) : C \rightarrow \mathbf{Y}$  an isotone map such that  $g(x) \in y(x)$  (e.g., an isotone selection),  $x \in C$ . Denote  $\vee C = x_{0\vee}$  (respectively,  $\wedge C = x_{0\wedge}$ ). We say a nonempty valued correspondence  $y(x)$  satisfies condition C (respectively, Condition C(d)), if for all  $x \in C$ : there exists a  $y_0 \in y(x_{0\vee})$  such that  $g(x) \leq y_0$ ; (respectively, there exists a  $y_0 \in y(x_{0\wedge}) \leq g(x)$ ). We remark that as  $\mathbf{X}$  and  $\mathbf{Y}$  are complete lattices (and therefore are chain complete Posets), by Höft (1987, Lemma 2.1), a nonempty correspondence  $y(x)$  that is ascending in Smithson's weak set relation (C1) and satisfies condition C is equivalent to a correspondence  $y(x)$  satisfying the following *chain faithfulness condition*: if  $x_0 \geq x$  for all  $x \in C$ , we say  $y(x)$  is *chain faithful upward* if there exists  $y_0 \in y(x_0)$  such that for an isotone mapping  $g(x) : C \rightarrow \mathbf{Y}, g(x) \in y(x)$ , we have  $g(x) \leq y_0$  for all  $x \in C$ . We can define *chain faithful downward* using the obvious dual to the definition of chain faithful upward. Höft arrives at a characterization of  $y(x)$  ascending in (C2) and satisfying condition C(d). Finally, we say a correspondence is *chain faithful* if it is both chain faithful upward and chain faithful downward.

We have the following result (which actually follows as a corollary to Höft, 1987, Theorem 2.2) that we use extensively in the sequel and include a proof for completeness.<sup>24</sup>

<sup>24</sup> Our Lemma is obtained as a corollary of Höft's theorem on the fixed points of a composition correspondence  $G \circ F$  by (a) setting in her theorem  $G$  equal to the identity map, and (b) noting that the existence of a fixed point for the resulting composition (now just  $F$ ) is proven in her proof by showing her correspondence  $G \circ F$  admits an isotone selection. We note that in our argument concerning existence of an isotone selection, the requirement that

**Lemma 8.** Let  $(\mathbf{X}, \geq_X)$  and  $(\mathbf{Y}, \geq_Y)$  be complete lattices,  $y^*(x) : \mathbf{X} \rightarrow 2^{\mathbf{Y}} \setminus \emptyset$  be nonempty, chain subcomplete-valued, and chain-faithful such that (a)  $\bigwedge \mathbf{Y} \in y(\bigwedge \mathbf{X})$  and (b)  $\bigvee \mathbf{Y} \in y(\bigvee \mathbf{X})$ . Then  $y^*(x)$  admits an isotone selection.

**Proof.** First note that by hypothesis,  $y^*(x)$  is chain faithful; therefore  $y^*(x)$  is necessarily Smithson weak set relation ascending in (C1) and (C2) and it satisfies Condition C and C(d) (Höft, 1987, Lemma 2.1).

Let  $X$  be the collection of nonempty subsets of  $\mathbf{X}$  with typical element  $X \in \mathbf{X}$  where each  $X$  satisfies (i)  $x_L = \bigwedge \mathbf{X} \in X$ , and (ii) for  $z \leq x \in \mathbf{X}$ ,  $z \in X$ . Define the collection of isotone functions  $g : X \rightarrow \mathbf{Y} \in \mathbf{G}$  such that  $g(x) \in y^*(x)$  for each  $x \in X$ . Define the following partial order on the pair  $(X, g) \in X \times \mathbf{G}$ :  $(X_1, g_1) \leq (X_2, g_2)$  if and only if  $X_1 \subset X_2$  and  $g_1 = g_2|_{X_1}$  (where  $g|_X$  denotes the restriction of  $g$  to  $X$ ). This is a valid partial order (see, for example, Smithson, 1971, Theorem 1.9). By Zorn's lemma therefore there is a maximal pair, say  $(X_0, g_0) \in X \times \mathbf{G}$ . If  $X_0 = \mathbf{X}$ ,  $g_0(x)$  is the isotone selection.

Therefore, assume  $x \in \mathbf{X} \setminus X_0$ . Define  $C$  to be the maximal chain containing  $x_L$  and  $x$ . Consider  $C \cap X_0 = C_1 \neq \emptyset$  (as at least we have  $x_L \in C_1$ ). By  $y^*(x)$  chain-faithful downward, given hypothesis (a) of the theorem, for any isotone function  $g(x) : C_1 \rightarrow \mathbf{Y}$ ,  $g(x) \in y^*(x)$ , we have  $y_L = \bigwedge \mathbf{Y} \in y^*(x_L) \leq g(x)$  for all  $x \in C_1$ . Let  $x_0 = \sup C_1 \leq x^u = \bigvee \mathbf{X}$ . By  $y^*(x)$  chain-faithful upward, we know  $y^*(x)$  is (necessarily) ascending in Smithson's weak set relation (C1) and satisfies condition C. Further by hypothesis in the theorem,  $y^*(x)$  is chain subcomplete valued. Therefore, by the maximality of  $\mathbf{X}$  (e.g., by the definition of  $X_0$ ), and  $y^*(x)$  being chain faithful and satisfying hypothesis (b), we arrive at the following two facts: (i)  $x_0 \in X_0$ ; (ii) for any  $x_L \leq x \leq x_0$ , we have there exists of  $y_0$  such that  $g_0(x) \leq y_0 \leq y^u \in y^*(x^u)$  where  $y_0 \in y^*(x_0)$ .

Consider the set  $X_1$  where  $X_0 \subset X_1 := X_0 \cup \{z | x_0 \leq z \leq x^u\} \in X$ . We now construct an isotone mapping  $\hat{g}(x) : X_1 \rightarrow \mathbf{Y}$ . The definition shall take place in two steps. First, let  $\hat{g}(x) = g_0|_{X_0}$  for  $x \in X_0$  (which is isotone on  $[x_L, x_0]$  by the fact that  $x_0 \in X_0$ , and maximality of  $(X_0, g_0)$ ). Define the nonempty subset  $S([x_0, x^u]) = \bigcup_{x \in [x_0, x^u]} y^*(x)$ . As  $y^*(x)$  is chain-faithful upward, this implies we have by Condition C for  $y^*(x)$ . Therefore, for any isotone mapping  $g(x) : [x_L, x_0] \rightarrow \mathbf{Y}$ ,  $g(x) \in y^*(x)$ , we have some  $y_0$  such that  $g(x) \leq y_0 \in y^*(x_0)$ . Therefore, the same is true for maximal selection  $g_0(x)$  on  $[x_L, x_0]$ . Therefore, by definition, we have this property for the mapping  $\hat{g}(x)$  restricted to  $[x_L, x_0]$ . Further as  $y^*(x)$  satisfies hypothesis (b) and chain faithful upward, we have for each such  $y_0 \in S(x_0)$ , we have the following:  $S(z) = y^*(z) \cap [y_0, y^u] \neq \emptyset$  for  $x_0 \leq z \leq x^u$ ,  $y^u = \bigvee \mathbf{Y} \in y^*(x^u)$ .

We can now define the extension of  $\hat{g}$  to all of  $X_1$ . As  $y^*(x)$  is nonempty, chain-subcomplete-valued, chain faithful, and satisfies (a) and (b), so does the restriction of

$\bigwedge \mathbf{Y} \in y(\bigwedge \mathbf{X})$  and  $\bigvee \mathbf{Y} \in y(\bigvee \mathbf{X})$  can be dropped completely, and the complete lattice assumptions on the domain and range could be weakened to a relevant chain complete Poset conditions.

We also remark, this lemma is closely related to Veinott's isotone selection theorem (Veinott, 1992) (Veinott, 1992, Chapter 4, Theorem 5). The additional regularity conditions guarantee all subchains in  $\bigcup_x y^*(x)$  are "topped" and "bottomed". We note that statement is true under weaker hypotheses (e.g., when the domain and range are Posets, and the top and bottoms are not necessarily the tops and bottoms of the range), but we simply prove a version that we apply in our paper.



$y^*(x)$  to the domain  $[x_0, x^u]$ , namely the mapping  $S(z) : [x_0, x^u] \rightarrow 2^{\mathbf{Y}}$ . As  $S(z)$  is chain faithful, this implies  $S(z)$  satisfies condition C and C(d). We therefore arrive at the following two statements for any isotone mapping  $g(x) : [x_0, x^u] \rightarrow \mathbf{Y}$  and  $x \in [x_0, x^u]$ : (a) there exists a  $y_0 \in y^*(x_0)$  such that  $g_0(x_0) = y_0 \leq g(x)$ ,  $y_0 \in S(x_0)$ ,  $g(x) \in y^*(x)$  (b) as  $y^u \in y^*(x^u)$ , we have  $g(x) \leq y^u = S(x^u)$  for any isotone mapping  $g(x) : [x_0, x^u] \rightarrow \mathbf{Y}$ . Finally, as  $S(z)$  is chain faithful upward, chain subcomplete-valued, and noting hypothesis (b) of the theorem, we conclude: (i) there exists a subcomplete chain in  $S([x_0, x^u])$ , say  $s([x_0, x^u])$ , and (ii)  $s([x_0, x^u])$  has the following property: there is a  $y_0 \in y^*(x_0)$  such that for all  $\hat{x} \in (x_0) \cap X_0$ , we have  $g_0(\hat{x}) \leq y_0 = \wedge s([x_0, x^u]) \leq \vee s([x_0, x^u]) = y^u \in y^*(x^u)$ ,  $s([x_0, x^u]) \subset S([x_0, x^u])$ .

Let  $s(z)$  be a typical element of the subchain  $s([x_0, x^u])$  for  $z \in [x_0, x^u]$ . Therefore, define the extension of the function  $\hat{g}(x)$  to  $X_1 \setminus X_0$  as the function  $\hat{g}(z) = s(z) \in s([x_0, x^u])$  for each  $z \in [x_0, x^u]$ . Recalling that restricted to the domain  $X_0$ ,  $\hat{g}(x) = g_0|_{X_0}$ , we now arrive at an isotone mapping  $\hat{g}(x) : X_1 \rightarrow \mathbf{Y}$ . But then given the partial order on  $X \times \mathbf{G}$ , as  $X_0 \subset X_1$ , this implies  $(X_1, \hat{g}) > (X_0, g_0)$ . This contradicts the maximality of  $(X_0, g_0)$ .

Therefore, there is an isotone selection in  $y^*(x) : \mathbf{X} \rightarrow 2^{\mathbf{Y}}$ , namely  $g_0(x) : \mathbf{X} \rightarrow \mathbf{Y}$ .  $\square$

The operator  $Ah$  defined in (7) and (8) is any isotone selection in the correspondence  $x_Z^*(K, h, \theta)$ . We now characterize the structure of  $x_Z^*(K, h, \theta)$  so as to check whether the conditions of Lemma 8 are satisfied for  $x_Z^*(K, h, \theta)$ . To do this, we first rewrite the inequalities in (8) used to define  $x_Z^*$  as a collection of Karash–Kuhn–Tucker conditions (KKT) in equilibrium. We then define a new mapping  $\hat{\Psi}(x, K, \theta, h)$  that adds to the mapping  $Z(x, h, K, \theta)$  this new collection of  $J$  equilibrium KKT multipliers, one for each agent who in equilibrium might be borrowing constrained in (6).

We first define the equilibrium KKT multipliers. Fix  $t = (K, h) \in \mathbf{T}_+ \doteq \mathbf{K}_+^J \times [0, g)$ . Recalling the definition of the mapping  $Z : \mathbf{K}^J \times \mathbf{T}_+ \times \Theta \rightarrow \mathbf{R}^J$ , we define a new mapping  $\phi_j(x, t, \theta) : \mathbf{K}^J \times \mathbf{T}_+ \times \Theta \rightarrow \mathbf{R}^J$  defined componentwise for each  $j \in J$  as follows:

$$\phi_j(x, t, \theta) = \Psi_{j2}(x, K, \theta) - \Psi_{j1}(x, t, \theta) \quad \text{for } j = 1, 2, \dots, J.$$

By Lemma 7,  $\phi(x, t, \theta) = [\phi_1, \phi_2, \dots, \phi_J]$  is well-defined (as the mapping  $Z$  is well defined).<sup>25</sup> Further for fixed  $(t, \theta) \in \mathbf{T}_+ \times \Theta$ , we have the following properties for the vector of KKT multipliers  $\phi(x, t, \theta)$ : (a)  $\phi$  is continuous and strictly increasing in  $x \in \Pi_{j \in J} \mathbf{K} \setminus 0$ , and  $\phi$  is upper-semicontinuous and strictly increasing in  $x$  on  $\mathbf{K}^J$ ; (b) for  $x = 0$ ,  $\phi(0, t, \theta) = -\infty$  (as  $h \in \mathbf{H}$  and we have the Inada condition in Assumption 1); and (c)  $\phi_{J_1}(x_{-J_1}, \mathbf{g}_{J_1}, t, \theta) = \infty$  (by Assumption 1 for any subset of indices of agents  $J_1 \subset J$  where  $x_{-J_1}$  is the vector  $x$  with the components for agents  $j \in J_1$  deleted).

Therefore, the KKT multipliers in equilibrium are defined by the vector  $\phi \cdot I_x(x = 0)$  where the  $j$ th component of the KKT is given by  $\phi_j(x, K, h, \theta) \cdot I_{x_j}(x_j = 0)$  where  $I_x(x_j = 0)$  is an indicator variable for the condition  $x_j = 0$ . Define the extended-valued mapping  $\hat{\Psi}$  that embeds  $Z$  and the KKT multipliers  $\phi I_x(x = 0)$  allowing us to study (8) using a

<sup>25</sup> Given the definition of the operator  $Ah$ , we do not need to consider the situation defining  $Ah$  when  $h \in \mathbf{H}$ ,  $K \in \mathbf{K}^J$ , but  $(K, h) \notin \mathbf{T}_+$ . See the definition of the operator  $Ah$  in (7).

single system of functional inequalities: namely, consider extended real valued mapping  $\hat{\Psi}(x, t, \theta) : \mathbf{K}^J \times \mathbf{T}_+ \times \Theta \rightarrow \mathbf{R}^J$ :

$$\begin{aligned}\hat{\Psi}(x, t, \theta) &= \Psi_2(x, K, \theta) - \hat{\Psi}_1(x, t, \theta) \text{ and} \\ \hat{\Psi}_1 &= \beta \int_{\Theta} u'(g(x, \theta') - h(x, \theta')) \bar{r}(x, \theta') \chi(\theta, d\theta') + \phi(x, t, \theta) I_x(x = 0), \\ \Psi_2 &= u'(g(K, \theta) - x),\end{aligned}$$

where  $I_x(x = 0)$  is the indicator variable on vector of equations whose components indicate in each equation  $j \in \mathbf{J}$  whether household  $j$  is borrowing constrained, i.e.,  $I_{x_j} = 1$  when  $x_j = 0$ .

Prior to proving Proposition 2, we make a few observations about the system  $\hat{\Psi}(x, t, \theta)$ . Under Assumptions 1–3, noting the results in Lemma 7 and given the definition of  $\phi$ , we conclude the system  $\hat{\Psi}(x, t, \theta)$  is well-defined. Further observe that the following four statements concerning the structure of the system  $\hat{\Psi}$  are the case: for any fixed  $(t, \theta) \in \mathbf{T}_+ \times \Theta$ , we have (x.i) as  $x_j \rightarrow g_j(K, \theta)$ ,  $\hat{\Psi}_j \rightarrow \infty$  for any  $j \in \mathbf{J}$ ; (x.ii) as  $x \rightarrow g(K, \theta)$ , the entire system  $\hat{\Psi} \rightarrow \infty$  (both of these as  $I_x = \mathbf{0}$ , and  $Z \rightarrow \infty$ ); (x.iii) letting  $x = (x_{-j}, x_j)$  for some component  $j \in J_1 \subset \mathbf{J}$  where  $x_{-j}$  deletes  $x_j$  from  $x$ , when  $x_{-j} \rightarrow \mathbf{0}_{-j}$ ,  $x_j > \mathbf{0}_j$ ,  $\hat{\Psi}_{-j}(x, t, \theta) \leq 0$ ; and finally (x.iv) when additionally  $x_j \rightarrow \mathbf{0}_j$  (implying therefore that vector  $x \rightarrow \mathbf{0}$ ), then by the definition of  $\phi$  (noting Assumption 3 and  $h \in [0, g)$ ), we have  $\lim_{x \rightarrow \mathbf{0}} \hat{\Psi}(x, t, \theta) \leq \mathbf{0}$ .<sup>26</sup>

We are now ready to prove Proposition 2.

### Proof of Proposition 2.

(i) The proof takes place in sequence of steps that check the hypotheses of Lemma 8.

**Claim A.** The correspondence  $x_Z^*(t, \theta) : \mathbf{T} \times \Theta \rightarrow 2^{\mathbf{K}^J} \setminus \emptyset$  is nonempty for each  $(t, \theta) \in \mathbf{T} \times \Theta$ .

**Proof.** We first prove  $x_Z^*(t, \theta)$  is nonempty on  $\mathbf{T}_+ \times \Theta$ . First note that for any  $(t, \theta) \in \mathbf{T}_+ \times \Theta$ , as by Lemma 7(iii) we have  $\hat{\Psi}$  strictly increasing and continuous in  $x$  on  $(0, g(K, \theta)]$ , and nondecreasing and upper-semicontinuous on  $[0, g(K, \theta)]$  with  $\hat{\Psi}_j = 0$  whenever  $x_j = 0$ . Therefore, for each  $(t, \theta) \in \mathbf{T}_+ \times \Theta$  there is a pair of vectors  $(x_l(t, \theta), x^T(t, \theta)) \in [0, g(K, \theta)] \times [0, g(K, \theta)]$  such that (a)  $\hat{\Psi}(x_l(t, \theta), t, \theta) \leq 0$ ; (b)  $\hat{\Psi}(x^T(t, \theta), t, \theta) \geq 0$ , and (c)  $x_l(t, \theta) \leq x^T(t, \theta)$  in the standard component product Euclidean order on  $R^J$ , with  $x^T(t, \theta) < g$ . As  $\hat{\Psi}$  is strictly increasing and continuous jointly in  $x$  on  $(0, g(K, \theta)]$  and nondecreasing and upper-semicontinuous on all of the (connected) order interval  $[0, g(K, \theta)]$ , it is nondecreasing in  $x_j$  on  $[0, g_j(K, \theta)]$  for fixed  $x^j = (x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_J)$ ,  $x = (x_j, x^j)$  each  $j \in J$ . The mapping  $\hat{\Psi}(x, t, \theta)$  therefore satisfies the semi-continuity conditions in the generalized intermediate value theorem on arbitrary product spaces of Guilleme (1995, Theorem 3) on  $[x_L(t, \theta), x^T(t, \theta)]$ . Therefore, we conclude that correspondence  $x_Z^*(t, \theta)$  is

<sup>26</sup> Notice, Assumption 3(i) is not needed for our fixed point construction. Given the definition of  $\phi$ , our operator is well-defined. Assumption 3 guarantees that we have a non-trivial Markov equilibrium with non-zero production.

nonempty on the (connected) suborder interval  $[x_l(t, \theta), x^T(t, \theta)] \subset [0, g(K, \theta)] \subset R_+^J$  with  $x^T(t, \theta) < g$ . Therefore, we conclude that  $x_Z^*(t, \theta)$  is nonempty for each  $(t, \theta) \in \mathbf{T}_+ \times \Theta$ .

Notice that by the definition of the extension of  $x_Z^*(t, \theta)$  in (8),  $x_Z^*$  is well defined for any  $(t, \theta) \in \mathbf{T} \setminus \mathbf{T}_+ \times \Theta$ . Therefore,  $x_Z^*(t, \theta) : \mathbf{T} \times \Theta \rightarrow \mathbf{K}^J$  is well-defined.  $\square$

**Remark.** Before we proceed to Claim B, we make a few additional remarks concerning some additional properties of the correspondence  $x_Z^*(t, \theta)$ . First, observe as  $\hat{\psi}$  strictly increasing in  $x$  on  $[0, g(K, \theta)]$ , and noting the definition of  $x_Z^*(t, \theta)$  when  $t \in \mathbf{T} \setminus \mathbf{T}_+$ , one can easily show that  $x_Z^*(t, \theta)$  necessarily *antichain*-valued for each  $(t, \theta) \in \mathbf{T} \times \Theta$  (where the antichain structure is relative to the componentwise order on  $R_+^J$ ).<sup>27</sup>

We now make an additional remark useful in the sequel. Fix  $\theta \in \Theta$ , and consider  $x(t, \theta) \in x_Z^*(t, \theta)$  for  $t \in \mathbf{T}_+$ . By the definition  $x_Z^*$ , for any  $x(t, \theta) \in x_Z^*(t, \theta)$ , we have  $\hat{\psi}(x(t, \theta), t, \theta) = 0$ . Consider  $t' \geq t, t \neq t', t$  and  $t'$  in  $\mathbf{T}_+$ . By Lemma 7(i), as  $\hat{\psi}$  is strictly antitone in  $t$ , we have  $-\infty < \hat{z} = \hat{\psi}(x(t, \theta), t', \theta) < 0$ . Define the directed net  $G(x(t, \theta)) = \{x | x \geq x(t, \theta), \hat{\psi}(x, t, \theta) = z \geq 0 > \hat{z}, z \text{ finite}\}$ . Compute the point  $x^T = \sup G(x(t, \theta)) \in [0, g(K, \theta)]$ , which exists in  $[0, g(K, \theta)]$  as  $[0, g(K, \theta)]$  is a complete sublattice of  $R_+^J$ . Compute  $z^T = \hat{\psi}(x^T, t', \theta) \geq z \geq 0$  where the inequality  $z^T \geq z$  follows from the definition of join  $\vee G(x(t, \theta))$  and the fact that  $\hat{\psi}$  is strictly increasing in  $x$ . By the isotonicity and continuity of  $\hat{\psi}$  in  $x$ , define  $\hat{G}(x(t, \theta)) = \{x | z^T \geq \hat{\psi}(x, t, \theta) \geq 0 > \hat{z}\}$  where, by construction,  $x^T = \sup \hat{G} \subset [0, g]$ . Also, note that,  $\hat{G}(x(t, \theta))$  is nonempty (as certainly we have  $x(t, \theta) \in G(x(t, \theta))$ ), and therefore, so is  $\hat{G}(x(t, \theta))$ .

**Claim B.** The correspondence  $x_Z^*(t, \theta) : \mathbf{T} \times \Theta \rightarrow 2^{\mathbf{K}^J} \setminus$  is ascending in Smithson's weak set relation (C1) and (C2) and satisfies Conditions C and C(d) (and is therefore chain-faithful).

**Proof.** For  $t \in \mathbf{T}_+$ , by Lemma 7, as  $\hat{\psi}$  is (i) strictly increasing and continuous in  $x$  on  $[0, g)$  and nondecreasing and upper semicontinuous in  $x$  on all of  $R_+^J$ , (ii) the order interval  $[0, g(K, \theta)]$  is convex (and, therefore, connected), (iii) by Lemma 7(i)–(ii), and the argument directly above,  $\hat{\psi}(x^*(t, \theta), t', \theta) \leq 0$  for  $t' \geq t$ , following the argument in Claim A and the argument above, let the lower solution be  $x_L(t', \theta) = x^*(t, \theta)$ , and by Lemma 7(iii) noting there exists an upper solution  $x^T(t', \theta)$  such that  $\hat{\psi}(x^T, t', \theta) \geq 0$ . We can then again apply the theorem of Guilleme (1995, Theorem 3) to  $\hat{\psi}(x, t, \theta)$  relative to the connected suborder interval  $[x_L(t', \theta), x^T(t', \theta)] = [x(t, \theta), x^T(t', \theta)] \subset [0, g(K, \theta)] \subset R_+^J$ . We conclude for any  $x(t, \theta) \in x^*(t, \theta)$ , there exists a element  $x(t', \theta) \in x^*(t', \theta)$  such that  $x(t, \theta) \leq x(t', \theta)$ . This proves  $x^*(t, \theta)$  is ascending in Smithson's weak set relation (C1) on  $\mathbf{T}_+$ . A similar argument shows that for any  $x(t', \theta) \in x^*(t', \theta)$ , there exists and  $x(t, \theta) \in x^*(t, \theta)$  that minorizes  $x(t', \theta)$ . This proves  $x^*(t, \theta)$  is ascending to  $\mathbf{K}^J$  in Smithson's weak set relation (C2) on  $\mathbf{T}_+$ .

Noting the definition of  $x_Z^*(t, \theta)$  in (7) that extends  $x_Z^*(t, \theta)$  onto all of  $\mathbf{T} \times \Theta$ , as we have  $x_Z^* : \mathbf{T}_+ \times \Theta \rightarrow \mathbf{K}^J$  ascending in Smithson's weak set relation (C1), noting the condition is

<sup>27</sup> Let  $X$  be a partially ordered set. We say a set  $X_1 \subset X$  is an *antichain* if no two elements of  $X_1$  are ordered. Antichains are necessarily chain complete.

trivial with both  $t, t' \in \mathbf{T} \setminus \mathbf{T}_+$ , then the last case to consider has for any  $t' \in \mathbf{T}_+$ ,  $t' \geq t$ . Then  $x_Z^*(t, \theta)$  is a singleton with  $x_Z^*(t, \theta) \leq x_Z^*(t', \theta)$  for all  $t \in \mathbf{T} \setminus \mathbf{T}_+$ , fixed  $\theta \in \Theta$ . We conclude  $x_Z^*(t, \theta) : \mathbf{T} \times \Theta \rightarrow \mathbf{K}^J$  is a nonempty correspondence that is ascending in the Smithson weak set relation (C1) on  $\mathbf{T} \times \Theta$ .

A similar (dual) argument shows  $x_Z^* : \mathbf{T} \times \Theta \rightarrow \mathbf{K}^J$  is ascending in Smithson's weak set relation (C2).

We finally consider the question of Condition C (respectively, condition C(d)). Fix  $\theta \in \Theta$ , and let  $C \subset \mathbf{T}$  be a chain. Suppressing the notation for  $\theta$  for the moment, let  $g(t) : C \rightarrow \mathbf{K}^J$  be an isotone map. Let  $t_{0\vee} = \sup C$  (respectively,  $t_{0\wedge} = \inf C$ ). First assume that  $t_{0\vee} \neq \vee \mathbf{T}$  (resp,  $t_{0\wedge} \neq \wedge \mathbf{T}$ ). For such a  $C$ , by  $x_Z^*(t, \theta)$  ascending in Smithson's weak set relation (C1) (respectively, Smithson's weak set relation (C2)), for any  $t \leq t_{0\vee}$  (respectively, any  $t_{0\wedge} \leq t$ ), there exists a  $k_{0\vee} \in x_Z^*(t_{0\vee})$  (respectively,  $k_{0\wedge} \in x_Z^*(t_{0\wedge})$ ) such that  $g(t) \leq k_{0\vee}$  (respectively,  $k_{0\wedge} \leq g(t)$ ) for all  $t \in C \subset \mathbf{T}$  (this claim noting the definition of  $x_Z^*(t, \theta)$  for any  $t \in \mathbf{T} \setminus \mathbf{T}_+$ ). Therefore, the correspondence  $x_Z^*(t, \theta)$  satisfies conditions C and C(d). Now, let  $t_{0\vee} = \vee \mathbf{T}$  (resp,  $t_{0\wedge} = \wedge \mathbf{T}$ ). As  $\vee K^J \in x_Z^*(t_{0\vee})$  (resp,  $\wedge K^J \in x_Z^*(t_{0\wedge})$ ), the correspondence  $x_Z^*(t, \theta)$  satisfies conditions C and C(d). Therefore, by Höft's equivalence Lemma (Lemma 2.1 previously cited), as  $\mathbf{T}$  and  $\mathbf{K}^J$  are both complete lattices (and therefore chain complete),  $x_Z^*(t, \theta)$  is chain faithful.  $\square$

**Claim C.**  $x_Z^*(t, \theta) : \mathbf{T} \times \Theta \rightarrow 2^{\mathbf{K}^J} \setminus$  is anti-chained valued (and therefore, chain subcomplete valued) for each  $(t, \theta)$ .

**Proof.** Recall that by remark above, we first note that  $x_Z^* : \mathbf{T}_+ \times \Theta$  is antichain-valued. It is therefore chain subcomplete-valued on  $\mathbf{T}_+ \times \Theta$ . Given the definition of the extension of  $x_Z^*(t, \theta)$  to  $\mathbf{T} \times \Theta$  provided in (7), we conclude that  $x_Z^* : \mathbf{T} \times \Theta \rightarrow \mathbf{K}^J$  is antichained valued. Therefore  $x_Z^*(t, \theta)$  is chain subcomplete for each  $(t, \theta) \in \mathbf{T} \times \Theta$ .

Then by Claims A–C,  $x_Z^*(t, \theta)$  satisfies the hypotheses of Lemma 8, and therefore  $x_Z^*(t, \theta)$  admits an isotone selection. Therefore,  $Ah$  in (7) and (8) is well-defined.

- (ii) We note that by part (i) of the proposition,  $Ah(K, \theta)$  isotone in  $K$  with  $Ah(0, \theta) = 0$  for all  $\theta$  by definition in (8). We therefore have when  $K' \geq K$  and  $K' \neq 0$

$$-\hat{\Psi}_1(Ah(K', K', \theta, h)) + \Psi_2(Ah(K, \theta)) \geq -\hat{\Psi}_1(Ah(K, K, \theta, h)) + \Psi_2(Ah) = 0$$

As each  $\Psi_{j2}(x)$  is only a function of  $g_j - Ah_i(K, \theta)$  for each  $j$ , by the concavity of the vector of marginal utilities  $u(c)$  we have  $Ah(K, \theta)$  such that for each  $j$ ,  $g_j - Ah_j$  is isotone in  $K$ , each  $\theta$ . As  $\Theta$  is finite, and both  $Ah$  and  $g - Ah$  are isotone in  $K$ , and  $g$  is uniformly continuous in  $K$ ,  $Ah$  is an element of an equicontinuous set of functions on the compact set  $\mathbf{S}$ . Therefore,  $Ah \subset \mathbf{H}$ .

- (iii) Continuity follows from a compact argument for operators on compact subsets of  $\mathbf{C}^+(\mathbf{S})$  where  $\mathbf{S}$  is compact (e.g., Coleman, 1991, Proposition 4). Further, as  $A(H_1) = \{Ah|h \in H_1 \subset \mathbf{H}\} \subset \mathbf{H}$  is relative compact for all bounded subsets  $H_1 \subset \mathbf{H}$ ,  $\mathbf{H}$  compact, the closure of  $A(H_1)$  is compact. Therefore,  $Ah$  is continuous and compact in  $h$ , i.e., completely continuous on  $\mathbf{H}$ .

- (iv) Consider any  $h' \geq h$ , with  $h, h' \in \mathbf{H}$ . Since  $\Psi$  is increasing in  $h$  and  $\Psi(Ah, K, \theta, h) = 0$  by definition of the operator  $A$ , then  $Z(Ah, K, \theta, h) \geq 0$ . Consequently, with  $\Psi$  is strictly increasing in its first argument, by an argument similar to (i) for the existence of monotone selection, it must be that  $Ah'(K, \theta) \geq Ah(K, \theta)$  for all  $(K, \theta)$ .
- (v) Consider an increasing sequence  $h_{i \in \mathbf{N}} \subset \mathbf{H}$ ,  $h^u = \sup(h_{i \in \mathbf{N}})$  where  $\mathbf{N}$  denotes the natural numbers. The point  $h^u$  is well-defined as  $\mathbf{H}$  is a complete lattice and therefore countably chain complete. As  $Ah$  is topologically continuous (in the uniform metric topology), we have additionally for  $h_{i \in \mathbf{N}}$  the result that when  $h_i \rightarrow h$  (uniformly) in  $\mathbf{H}$ ,  $A(h_{i \in \mathbf{N}}) \rightarrow A(h^u)$  (uniformly) in  $\mathbf{H}$ . Further, as  $\mathbf{H}$  is equicontinuous, we have  $\vee h_{i \in \mathbf{N}} = h = h^u$  (see, for example, Heikkilä and Reffett, 2005, Lemma 4.1). Since  $Ah$  isotone,  $A(h_{i \in \mathbf{N}})$  is a countable chain. By topological continuity of  $Ah$ , and the fact that  $\vee h_{i \in \mathbf{N}} = h^u = h$ , we have  $\vee A(h_i) = A(h^u) = A(\lim h_{i \in \mathbf{N}})$ . As  $h^u = \vee(h_{i \in \mathbf{N}})$ , we have  $\vee(A(h_{i \in \mathbf{N}})) = A(\vee h_{i \in \mathbf{N}})$ . As dual argument works for a decreasing chain  $h_{i \in \mathbf{N}} \rightarrow h^u$  for  $A(\wedge h_{i \in \mathbf{N}}) = \wedge A(h_{i \in \mathbf{N}}) = \wedge A(h^u)$ . Therefore,  $Ah$  is order continuous along sequences in  $\mathbf{H}$ .  $\square$

We now use Proposition 2 to prove Theorem 3:

### Proof of Theorem 3.

- (i) The set  $\mathbf{H}$  is a nonempty complete lattice by the first Lemma in Section 4, and by Proposition 2(ii) and (iv), the operator  $A$  is a isotone transformation of  $\mathbf{H}$ . Therefore, by the main theorem in Tarski (1955, Theorem 1),  $E(\pi)$  is nonempty complete lattice for each  $\pi$ .
- (ii) By Proposition 2(iii), we have  $Ah$  completely continuous; by Proposition 2(iv) and (v), we have  $Ah$  isotone and order continuous. Note  $A(g) = g$  by definition, and therefore  $g$  is the greatest fixed point in  $\mathbf{H}$ . By the Lemma in Vulikh (1967) cited above, the successive approximations  $\lim_n A^n 0 = \sup_n A^n 0$  converge to a least fixed point  $h_l^* = \inf E(\pi) \geq 0$ . Given the complete continuity of  $Ah$ , and the equicontinuity of  $\mathbf{H}$ , this convergence is uniform (see Amann, 1976, Corollary 6.2).
- (iii) That the least fixed point  $h_l^* \in E(\pi)$  is such that  $g - h_l^* \gg 0$  follows from a modification of the main theorem in Greenwood and Huffman (1995), where for  $k = K$ , we have for the vector of dynamic program at its  $n$  orbit from the terminal condition,  $A^n 0 < g$  the optimal plan for the economic agents along equilibrium trajectories  $n$  periods from her terminal date (along an equilibrium trajectory where  $k = K$ ),  $T^n J_{i0}$  is the value function associated with the Bellman operator for agent  $i$ , with  $J_{i0} = 0$ ,  $h_0^* = 0$ . Then the Inada condition on each dynamic program (when  $k_i = K_i$  for each program for household  $i$ ) and Assumption 3(i.b) and (i.c) imply that  $\lim_n A^n(0) < g$  at the fixed point of  $J^*$  for the value functions.<sup>28</sup>
- (iv) First we note by Tarski's theorem,  $E(\pi)$  is complete lattice valued for each  $\pi$ . Let  $\pi' \leq \pi$  with  $\pi'_k \leq \pi_k$ ,  $\pi'_n = \pi_n$  (implying the return on capital  $\bar{r}$  falls under  $\pi'_k$  point-

<sup>28</sup> For an alternative proof that does not use Greenwood and Huffman (1995), see the Proof of Theorem 4. We provide an alternative proof, more in the spirit of Coleman (1991), based on fact that  $Ah$  is cone compression and is  $k_0$ -monotone (each true under Assumption 3(i.b)). It is also shown that  $\lim_{n \rightarrow \infty} A^n(0) < g$ .

wise). Notice we have  $Ag = g$  for all  $\pi \in \Pi$ . From the definition of  $Z$  in (7) (and its implied implications on the system  $\hat{\Psi}$  in Proposition 2), when  $\pi'_k \leq \pi_k$ , we have  $Z(\pi'_k; h, Ah(\pi_k; K, \theta), K, \theta) \geq Z(\pi_k; h, Ah(\pi_k; K, \theta), K, \theta) = 0$  where now we make the dependence of  $Ah$  on  $\pi_k$  explicit by adding it as an argument to  $Z$  in (7). Then by Lemma 7(iii),  $Ah(\pi'_k; K, \theta) \leq Ah(\pi_k; K, \theta)$ . By an argument Topkis (1998, Theorem 2.5.2) (and noting the dual order on  $\Pi$ ), we have (i)  $E(\pi)$  is isotone in the strong set order, i.e.,  $E(\pi'_k) \geq_s E(\pi_k)$  when  $\pi' = (\pi'_k, \pi_n) \geq (\pi_k, \pi_n) = \pi$  with the dual order on  $\Pi$ ; and (ii)  $\sup E(\pi)$  and  $\inf E(\pi)$  forming isotone selections.  $\square$

**Proof of Theorem 4.** By an obvious modification of the arguments in Proposition 2,  $\hat{A}m$  is (i) well-defined for each  $(K, \theta, h)$  (e.g., as an isotone selection in  $\mathbf{M}$ ), (ii)  $\hat{A}m' \geq \hat{A}m$ , and (iii)  $\hat{A}m \subset \mathbf{M}$  (each from a simple modification of the argument in Proposition 2).

We now study the geometric structure of  $\hat{A}m$ . We first prove two things: (i)  $\hat{A}m$  is strongly sublinear on the interior of  $\mathbf{M}$  (and therefore pseudo-concave), and (ii)  $\hat{A}m$  is a cone compression on  $\mathbf{M}$ . To see (i) that  $\hat{A}m$  is strongly sublinear on the interior of  $\mathbf{M}$ , notice first that  $\hat{Z}$  define in Eq. (10) above is strictly decreasing in its second argument. Therefore, a sufficient condition for strong sublinearity is  $\hat{A}tm > t\hat{A}m$  for  $t \in (0, 1)$  and  $m \gg 0$ ; i.e.,

$$\hat{Z}(tm, t\hat{A}m, K, \theta) > \hat{Z}(tm, \hat{A}tm, K, \theta) = 0,$$

which is true since  $m \in \mathbf{M}$ ,  $m \gg 0$ ,  $K \neq \mathbf{0}$ ,  $m$  such that  $\bar{r}/m$  is decreasing in  $K$

$$\hat{Z}(tm, t\hat{A}m, K, \theta) = \frac{1}{\bar{m}} - \beta \int_{\Theta} \frac{\bar{r}(g - H(t\bar{m}), \theta')}{m(g - H(t\bar{m}), \theta')} \chi(\theta, d\theta') > 0.$$

To see (ii) that  $\hat{A}m$  is a cone compression, note that the first condition in the definition of a cone compression ( $\hat{A}m \leq m$ , for  $m \in \mathbf{M}$ ,  $\|h\| < r$ ) is implied by the hypothesis  $A(0) > 0$  (as this implies that  $m^u$  is such that  $\hat{A}m^u < m^u$ ). As for the second condition for a cone compression ( $\hat{A}m \leq m$ , for  $m \in \mathbf{M}$ ,  $\|h\| > R > 0$ ,  $h \neq 0$ ), this is guaranteed by the fact that if  $m > 0$ ,  $m \neq 0$  for some  $(K, \theta)$ ,  $\hat{A}m = 0$  and  $\hat{A}m \gg 0$  if  $m \gg 0$ . In particular, given Assumption 3(i.b), there is a constant function  $m_0 \in (0, m^u]$  such the  $m^u > \hat{A}m_0 \geq m_0 \gg 0$  (we known also that  $\hat{A}m^u < m^u$ ). Then we conclude by a theorem in (Krasnosel'skii and Zabreiko, 1984, Theorem 46.4),  $\hat{A}m$  has at least one strictly positive fixed point (and therefore noting the relationship between  $\hat{A}m$  and  $Ah$ , there exists a strictly positive fixed point for  $Ah$ , say  $h^*$ , such that  $g - h^* \gg 0$ ).

Finally, note that under Assumptions 1 and 3(i.c), by a modification of an argument in Coleman (e.g., Coleman, 1991, Lemma 9), we have  $\hat{A}m$  is  $K_0$ -monotone. As under the hypothesis of the theorem that states  $A(0) > 0$ , we conclude the following about  $\hat{A}m$ :  $\hat{A}m$  is such that it is (i) strictly sublinear in the interior of  $\mathbf{M}$  (and therefore pseudo-concave) and (iii)  $K_0$ -monotone. Then by the uniqueness theorem in Coleman (1991, Theorem 8),  $\hat{A}m$  has at most strictly one positive fixed point, say  $m_l^*$ . But as by  $\hat{A}m$  being a cone compression by (ii) above, there we know  $\hat{A}m$  has a strictly positive fixed point in  $(0, m^u] \subset \mathbf{M}$ . Therefore, it must be that  $\lim_n \hat{A}^n m_0 = m_l^*$  for any  $m_0 \in (0, m^u]$ . By noting the relationship between  $\hat{A}m$  and  $Ah$ , we conclude  $h_l^*$  unique in  $[0, g)$ .  $\square$

The proof of Theorem 5 concerning the fixed points of  $A^c h$  in (15) and (16) of Section 6 requires a version of Proposition 2 adapted for the expanded parameter space used to define  $Z_c$  in Eqs. (12) and (14). We remark the arguments here are basically exactly as in the proof of Lemma 7, Proposition 2, and Theorem 3. We first state this version of Proposition 2 as Proposition 9 (which is used in the proof of Theorem 5 just as Proposition 2 is used in Proof of Theorem 3).

**Proposition 9.** *Under Assumptions 1, 2 and 4, (i) for any  $h \in \hat{\mathbf{H}}$  and  $(K, \hat{K}, \theta) \in \mathbf{K}^J \times \hat{\mathbf{K}}^J \times \Theta$ ,  $Ah(K, \hat{K}, \theta)$  is well-defined; (ii)  $Ah(K, \hat{K}, \theta) \subset \hat{\mathbf{H}}$ ; (iii)  $Ah$  is a isotone, completely continuous operator on  $\hat{\mathbf{H}}$  when  $\hat{\mathbf{H}} \subset \mathbf{C}^+(\mathbf{K}^J \times \hat{\mathbf{K}}^J \times \Theta)$  (e.g., space of continuous functions on  $\mathbf{K} \times \mathbf{K} \times \Theta$  equipped with the uniform topology and pointwise order); (iv)  $Ah$  is order-continuous on  $\hat{\mathbf{H}}$ .*

**Proof.** Following Proof of Proposition 2, first rewrite the system of Euler inequalities in (12) and (14) by defining a new mapping  $\hat{Z}_c(x, K, \hat{K}, h, \theta)$  as in Proof of Proposition 2 above (e.g.,  $\hat{Z}_c$  will be the mapping  $Z_c$  defined in (12) and (14) with the addition of a new set of KKT multipliers defined in equilibrium by a new mapping  $\phi_c(x, K, \hat{K}, h, \theta)$  exactly as in Proof of Proposition 2).

For the details, let  $t_c = (K, \hat{K}, h) \in \hat{\mathbf{T}}_+ = \mathbf{K}_+^J \times \hat{\mathbf{K}}_+^J \times [0, g]$  (where recall  $[0, g] \subset \hat{\mathbf{H}}$  in this section, and  $\hat{\mathbf{K}}^J$  is given the dual componentwise Euclidean order), and define extended real valued system  $\hat{\Psi}_c(x, K, \hat{K}, h, \theta): \mathbf{K}^J \times \hat{\mathbf{T}}_+ \times \Theta \rightarrow \bar{\mathbf{R}}^J$  as  $\rightarrow \bar{\mathbf{R}}^J$ :

$$\begin{aligned}\hat{\Psi}_c(x, t_c, \theta) &= \Psi_2(x, K, \theta) - \hat{\Psi}_1(x, t_c, \theta) \text{ and} \\ \hat{\Psi}_{1c} &= \beta \int_{\Theta} u'(g(x, h(K, \hat{K}, \theta), \theta') - h(x, h(K, \hat{K}, \theta), \theta')) \bar{f}(x, \theta') \chi(\theta, d\theta') \\ &\quad + \phi_c(x, t_c, \theta) I_x(x = 0), \\ \Psi_{2c} &= u'(g(K, \theta) - x),\end{aligned}$$

where again we use the shorthand  $I_x(x = 0)$  is the indicator variable on vector of equations whose components indicate in each equation  $j \in \mathbf{J}$  whether household  $j$  is borrowing constrained, i.e.,  $I_{x_j} = 1$  when  $x_j = 0$ . Here  $\phi_c(x, t_c, \theta): \mathbf{K}^J \times \hat{\mathbf{T}}_+ \times \Theta \rightarrow \bar{\mathbf{R}}^J$  defined componentwise for each  $j \in J$  as follows:

$$\phi_{jc}(x, t_c, \theta) = \Psi_{2jc}(x, K, \theta) - \Psi_{1jc}(x, t_c, \theta) \quad \text{for } j = 1, 2, \dots, J.$$

where  $\Psi_{1jc}(x, t_c, \theta)$  and  $\Psi_{2jc}(x, t_c, \theta)$  are given by expressions (13) and (14) of Section 6.

We now follow the exact same construction as Proposition 2: that is,

**Claim A.** *The correspondence  $x_{Zc}^*(t_c, \theta): \hat{\mathbf{T}} \times \Theta \rightarrow 2^{\mathbf{K}^J} \setminus$  is nonempty for each  $(t_c, \theta) \in \hat{\mathbf{T}} \times \Theta$ .*

**Proof.** To prove  $x_{Zc}^*(t_c, \theta)$  is nonempty on  $\hat{\mathbf{T}}_+ \times \Theta$ , we note that we have  $\hat{\Psi}_c$  strictly increasing and continuous in  $x$  on the interior of  $[0, g(K, \hat{K}, \theta)]$ , therefore for each  $(t_c, \theta) \in \hat{\mathbf{T}}_+ \times \Theta$  there is a pair of vectors  $(x_l(t_c, \theta), x^T(t_c, \theta)) \in [0, g(K, \hat{K}, \theta)] \times [0, g(K, \hat{K}, \theta)]$  such that (a)  $\hat{\Psi}(x_l(t_c, \theta), t_c, \theta) \leq 0$ ; (b)  $\hat{\Psi}(x^T(t_c, \theta), t_c, \theta) \geq 0$ , and (c)  $x_l(t_c, \theta) \leq x^T(t_c, \theta)$



in the standard component product Euclidean order on  $R^J$ , and (iii)  $x^T(t_c, \theta) < g$ . As  $\hat{\Psi}$  is strictly increasing and continuous jointly in  $x$  on  $(0, g(K, \hat{K}, \theta))$  and nondecreasing and upper-semicontinuous on all of (connected) order interval  $[0, g(K, \hat{K}, \theta)]$ , it is nondecreasing in  $x_j$  on  $[0, g_j(K, \hat{K}, \theta)]$  for fixed  $x^j = (x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_J)$ ,  $x = (x_j, x^j)$  each  $j \in J$ . The mapping  $\hat{\Psi}(x, t_c, \theta)$  therefore satisfies the semi-continuity conditions in the generalized intermediate value theorem on arbitrary product spaces of Guilleme (1995, Theorem 3) on  $[x_L(t_c, \theta), x^T(t_c, \theta)]$ . Therefore, we conclude that correspondence  $x_{Zc}^*(t_c, \theta)$  is nonempty on the connected suborder interval  $[x_L(t_c, \theta), x^T(t_c, \theta)] \subset [0, g(K, \hat{K}, \theta)] \subset R_+^J$  with  $x_L(t_c, \theta) \neq 0$ . Therefore, we conclude that  $x_{Zc}^*(t_c, \theta)$  is nonempty for each  $(t_c, \theta) \in \hat{\mathbf{T}}_+ \times \Theta$ . Notice that by the definition of  $x_{Zc}^*(t_c, \theta)$  in (15) and (16) is well defined for any  $(t_c, \theta) \in \hat{\mathbf{T}} \setminus \hat{\mathbf{T}}_+ \times \Theta$ . Therefore,  $x_{Zc}^*(t_c, \theta) : \hat{\mathbf{T}} \times \Theta \rightarrow \mathbf{K}^J$  is well-defined.  $\square$

**Claim B.** *The correspondence  $x_{Zc}^*(t_c, \theta) : \hat{\mathbf{T}} \times \Theta \rightarrow 2^{\mathbf{K}^J} \setminus$  is ascending in  $(K, h)$  for each  $(\hat{K}, \theta)$ , and ascending in  $\hat{K}$  for each  $(K, h, \theta)$  in Smithson's weak set relation (C1) and (C2) when  $\hat{K} \in \hat{\mathbf{K}}^J$  (given the dual Euclidean partial order). In addition,  $x_{Zc}^*$  satisfies condition C and C(d), and  $x_{Zc}^*$  is chain faithful.*

**Proof.** The argument follows from proof of Claim B in Proposition 2, noting that one uses the dual order on  $\hat{K} \in \hat{\mathbf{K}}^J$ , using  $t = t_c$ .

**Claim C.**  *$x_{Zc}^*(t_c, \theta) : \hat{\mathbf{T}} \times \Theta \rightarrow 2^{\mathbf{K}^J} \setminus$  is anti-chained valued (and therefore chain subcomplete valued) in  $(t, \theta)$ .*

**Proof.** Follows from  $\hat{\Psi}_c$  is strictly increasing in  $x$  on  $\mathbf{K}^J$  using  $t = t_c$ .

Then by Claims A–C,  $x_{Zc}^*(t_c, \theta)$  satisfies the hypotheses of Lemma 8, and therefore  $x_{Zc}^*(t_c, \theta)$  admits an isotone selection in  $(K, \hat{K}, h) \in \mathbf{K}^J \times \hat{\mathbf{K}}^J \times \hat{\mathbf{H}}$  for each  $\theta \in \Theta$ . Therefore,  $A^c h$  in (15) and (16) is well-defined.

(ii)–(v). See Proof of Proposition 2 for a similar argument noting the expanded aggregate state-space.  $\square$

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