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Discounting and altruism to future decision-makers☆

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Abstract

Is discounting of future decision-makers' consumption utilities consistent with "pure" altruism toward those decision-makers, that is, a concern that they are better off according to their own, likewise forward-looking, preferences? It turns out that the answer is positive for many but not all discount functions used in the economics literature. In particular, "hyperbolic" discounting of the form used by Phelps and Pollak (Rev. Econ. Studies 35 (1968) 201) and Laibson (Quart. J. Econ. 112 (1997) 443) is consistent with exponential altruism towards future generations. More generally, we establish a one-to-one relationship between discount functions and altruism weight systems, and provide sufficient, as well as necessary, conditions for discount functions to be consistent with pure altruism.

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1. Introduction

Many economics issues concern sequences of decisions. In models of such situations, the successive decisions are usually taken either by one and the same individual or by successive decision-makers, such as the generations in a dynasty. We here analyze a widely used class of time preferences in such analyses, namely those that permit utility representation as a sum of discounted instantaneous utilities. Hence, the decision-maker in each period has some concern for the future—be it his or her own, or that of future decision-makers. One may then ask the question why the current decision-maker is only concerned with the instantaneous utility of these future decision makers, when they, like the present decision-maker, also care about their future? We here ask whether discounting of future instantaneous utilities is consistent with "pure" altruism towards these future decision-makers, that is, a concern that future decision-makers are better off in terms of their own preferences (total utility). [3]

Recognition of the economic importance of altruism goes back at least to Edgeworth [8], who examined the effects of pure altruism on the contract curve in a two-person exchange economy. With *X* and *Y* denoting the two persons in question, Edgeworth wrote that

> "we might suppose that the object which *X* (whose own utility is *P*), tends—in a calm, effective moment—to maximize, is not *P*, but $P + \lambda\Pi$; where λ is a *coefficient of effective sympathy*. And similarly *Y*—not of course while rushing to self-gratification, but in those regnant moments which characterize an ethical 'method'—may propose to himself as an end $\Pi + \mu P$." (op. cit. p. 53). [4]

Many economists have analyzed, in a wide range of settings, the effect of altruism for economic agents' decision-making. For instance, Barro's [4] analysis of Ricardian tax neutrality leans heavily on pure altruism of the type studied here: each generation cares about the next generation's total utility, which in turn depends on the following generation's total utility, in an infinite chain (see [6] for a survey of this literature, and see [2] for a model that allows for both pure and impure altruism). The present analysis identifies conditions under which a concern for future generations' consumption utilities is, or is not, behaviorally equivalent to pure altruism towards these future generations. In his study of pure intergenerational altruism, Ray [19] called for precisely such an investigation. [5]

Our study is closely related to Zeckhauser and Fels [23], Kimball [13], and Bergstrom [5], all of whom analyzed whether systems of altruistically interdependent utility functions, in an intergenerational context, determine utilities as functions of allocations. In a similar vein, Hori [11] investigated the same question in some more generality in the case of finitely

[3] By contrast, a decision-maker A is sometimes called "paternalistically" or "impurely" altruistic if A cares about others' consumption, and/or directly about A's gifts or bequests to others ("warm glow effects"), without full regard to all factors of relevance for others' well-being. Ray [19] and Hori [11] use the term "paternalistic altruism," while Andreoni [2] use the term 1 "impure altruism."

[4] See [7] for an analysis of Edgeworth's treatment of altruism.

[5] "The representation of non-paternalistic functions in paternalistic form has ... been the subject of limited attention .... a systematic analysis of the relationship between these two frameworks is yet to be written, and appears to be quite a challenge, especially for models with an infinite horizon [20, pp. 113–114]."

many decision-makers. However, to the best of our knowledge, the results presented here are all new.

More exactly, we consider a sequence of decision-makers, one in each time period. These decision-makers could be successive generations or the successive incarnations of one and the same individual. The decision-makers are assumed to have preferences that can be represented as the discounted sum of instantaneous utilities (for example from consumption). As in most macroeconomic models, the instantaneous utility function is the same in all periods, and all decision-makers use the same discount function over their respective futures. Let thus $f(t) \in [0, 1]$ be the *discount factor* that a decision-maker attaches to the instantaneous utility t periods ahead. Such representations of time preferences are commonplace in the economics literature. For example, in the seminal paper by Samuelson [22], $f(t) = \delta^t$ for some $\delta \in (0, 1)$, while $f(t) = \beta\delta^t$ for some $\beta \in (0, 1]$ and $\delta \in (0, 1)$ in [14,18]. We will refer to the first case as *exponential discounting* and call the second, more general case, *quasi-exponential discounting*. [6]

We ask whether utility functions of this form are consistent with pure altruism towards future decision-makers in the sense that each decision-maker's utility can be written as the sum of own instantaneous utility (from, say, current consumption), and some weighted sum of all future decision-makers' total utilities. Let $a(t)$ be the *altruism weight* that the current decision-maker implicitly places on the total utility of his or her t:th successor. [7] We find that such a function a always exists and that its values can be determined from a relatively simple recursive equation (Proposition 1). However, there is no guarantee, a priori, that all function values are non-negative. A negative function value $a(t)$ means that the decision-maker is "spiteful" to his or her t:s successor, that is, the decision-maker prefers allocations that make that later decision-maker *worse* off. It is thus desirable to identify conditions on the discount function f that guarantee that all function values $a(t)$ be nonnegative. It turns out that a sufficient condition for this is that f be positive and that the ratio $g(t) = f(t)/f(t-1)$ between successive discount factors be non-decreasing in t (Proposition 2). This ratio reflects the decision-maker's *patience* concerning events t periods ahead: it expresses the decision maker's dislike of a one-period postponement as a function of how many periods ahead this delay is to occur. The condition thus requires decision-makers to be more patient with postponements that are further away in the future—a property that seems to conform with all available empirical evidence (see for example [9]). This condition is trivially met by exponential discounting—since then g is constant—and, more generally, by all quasi-exponential discount functions. The condition is also met by some, but not all, hyperbolic discount functions discussed in the psychology literature on time preferences (see for example [1]). We also show that our "patience" condition is closely related to, but distinct from, convexity of the discount function f (Proposition 4).

[6] Such discount functions are frequently called hyperbolic or quasi-hyperbolic. We prefer the present terminology since this class of functions contain exponential (but not hyperbolic) functions as special cases.

[7] Altruistic concern for *earlier selves* seems bizarre since earlier selves, by definition, do not exist at the time of the decision in question. The same holds for *earlier generations*, to the extent that these are not alive when the current generation makes its decisions. Kimball [13] analyses such a model.

Note, however, that if a decision-maker derives utility from memories of (his or his ancestors) past consumption, then even a strictly forward-looking altruistic decision-maker may rationally "invest" in future memories. However, that falls outside the scope of the present study—see [13,20] for discussions of these issues.

As is well-known, exponential discounting—the canonical model in the economics literature—corresponds to one-period pure altruism: each decision-maker attaches a positive weight to his or her successor's utility and zero weight to all other decision-makers. We show that exponential discounting is a boundary case in the following sense: a necessary condition for altruism towards future decision-makers is that future instantaneous utilities should not be more heavily discounted than what is obtained by exponential extrapolation of the discounting from the present to the next period (Proposition 3). Also this necessary condition seems to agree with empirical evidence. However, the condition is evidently violated by certain discount functions used in the economics literature, namely those that place positive weight on instantaneous utility in some nearby period, but zero weight on instantaneous utility in some more distant period (as, for example, when each generation only cares about its own and the next generation's consumption). We also find that quasi-exponential discounting in the Laibson–Phelps–Pollak (β, δ)-form corresponds to *exponential altruism*: the implied altruism weights decline exponentially over all future decision-makers. Hence, while the special case $\beta = 1$ of pure exponential discounting corresponds to altruism to the next generation only, $\beta < 1$ corresponds to exponential altruism to *all* future generations.

The remainder of the paper is organized as follows. The model is set up in Section 2, and Section 3 presents our results. Section 4 analyzes a few examples, and Section 5 concludes. Mathematical proofs are provided in an Appendix A.

2. Model

Consider finitely or infinitely many decision-makers $\tau = 0, 1, 2, \ldots$. In order to save on notation, and also to treat the most challenging case, we henceforth presume an infinite sequence of decision-makers.[8] Suppose, thus, that in each time period $t \in \mathbb{N} = \{0, 1, 2, \ldots\}$ there is a single decision-maker who takes some action $x_t \in X$, where X is the set of alternatives available in each period t (for example, X may be a set of relevant consumption bundles). For the sake of concreteness, we will call action x_t *consumption* in period t, and by a *consumption stream* (or *allocation*) x we mean the infinite sequence of consumption vectors x_t, $x = (x_0, x_1, \ldots) \in X^\infty$.

Each decision-maker τ has preferences $\succcurlyeq_\tau$ over consumption streams $x \in X^\infty$. A *preference profile* $\succcurlyeq$ for the sequence of decision-makers is thus a sequence $\langle \succcurlyeq_\tau \rangle_{\tau \in \mathbb{N}}$ of preferences, one for each decision-maker τ. We here focus on preference profiles $\langle \succcurlyeq_\tau \rangle_{\tau \in \mathbb{N}}$ for which there exist functions $U_\tau : X^\infty \to \mathbb{R}$, one for each decision-maker τ, such that $x \succcurlyeq_\tau y$ if and only if $U_\tau(x) \geqslant U_\tau(y)$, where

$$U_\tau(x) = \sum_{t=0}^{\infty} f(t) u(x_{\tau+t}) \tag{1}$$

for some $u : X \to \mathbb{R}$ and $f : \mathbb{N} \to \mathbb{R}_+$ with $f(0) = 1$. We will call $u(x_s)$ the *instantaneous* (*sub*)*utility* from consumption in period s, and $f(t)$ the *discount factor* that each decision-

[8] All results are easily adapted to the case of a finite number of decision makers.

maker attaches to the instantaneous utility t periods later. Hence, each decision-maker uses the same instantaneous subutility function u and discount function f.

We will say that a sequence $\langle U_\tau \rangle_{\tau \in \mathbb{N}}$ of utility functions (1) is *consistent with* (additively separable) *pure altruism* if for all $\tau \in \mathbb{N}$ and $x \in X^\infty$,

$$U_\tau(x) = u(x_\tau) + \sum_{t=1}^{\infty} a(t) U_{\tau+t}(x) \tag{2}$$

for some $a : \mathbb{N}_+ \to \mathbb{R}_+$, where $\mathbb{N}_+ = \{1, 2, \ldots\}$. Here $a(t)$ will be called the *altruism weight* that each decision-maker places on the welfare or total utility of the decision-maker t periods later. [9]

3. Results

Under what conditions is a sequence $\langle U_\tau \rangle_{\tau \in \mathbb{N}}$ of utility functions, defined in Eq. (1), consistent with pure altruism, and if it is, what are the implied altruism weights? A key result for answering this and related questions is the following observation:

Proposition 1. *If* $\langle U_\tau \rangle$ *is a sequence of real-valued utility functions satisfying Eq.* (1) *for some* $u : X \to \mathbb{R}$ *and* $f : \mathbb{N} \to \mathbb{R}$, *then* $\langle U_\tau \rangle$ *also satisfies Eq.* (2), *where* $a : \mathbb{N}_+ \to \mathbb{R}$ *is the unique solution to*

$$a(t) = \begin{cases} f(1) & \text{if } t = 1, \\ f(t) - \sum_{s=1}^{t-1} f(t-s) a(s) & \text{if } t > 1. \end{cases} \tag{3}$$

Clearly, the discount function f may be recovered from the altruism function a from Eq. (3):

$$f(t) = \begin{cases} a(1) & \text{if } t = 1, \\ \sum_{s=0}^{t-1} a(t-s) f(s) & \text{if } t > 1. \end{cases} \tag{4}$$

This recursive equation, which determines f from a (recall the normalization $f(0) = 1$), essentially states that the discount factor $f(t)$ attached to the instantaneous utility t periods later equals that period's contribution to the utility of all interim decision-makers, weighted by their respective altruism weights. For example, the discount factor $f(2)$ equals the altruism weight placed on the decision-maker two periods ahead, plus the altruism weight placed on the decision-maker one period ahead, multiplied by that decision-maker's one-period discounting: $f(2) = a(2) + a(1) f(1) = a(2) + a^2(1)$.

It is immediate from Eq. (4) that if the function a is nonnegative, so is f. However, as pointed out above, and seen in Eq. (3), a may well take negative values even when f is positive. In particular, $a(2) = f(2) - a(1) f(1) = f(2) - f^2(1)$. Hence, in order for $a(2)$ to

[9] Proposition 1 below shows that the restriction to additively separable altruism is not binding.

be negative it suffices that $f(2) < f^2(1)$.[10] This is the case when $f(1) > 0$ and $f(2) = 0$, as in models where each generation's welfare is a function of its own consumption and that of its immediate descendant. Another example of negative utility weights $a(t)$ is when $f(0) = 1$ and $f(t) = 1/(0.5 + t)$ for all $t > 0$; again $f^2(1) > f(2)$. A third example is $f(t) = 1/(1 + t^2)$ for all t. A fourth example is when the parameter β in the quasi-exponential (β, δ)-representation kicks in with one period's delay, that is, when $f(0) = 1$, $f(1) = \delta$ and $f(t) = \beta\delta^t$ for all $t \geqslant 2$. Then $a(2) = (\beta - 1)\delta^2 < 0$ for all $\beta < 1$.

For what class of discount functions f can one then guarantee that all welfare weights $a(t)$ are nonnegative? It turns out that a sufficient condition for this is that f be everywhere positive and that the associated "patience" function $g : \mathbb{N}_+ \to \mathbb{R}$, defined by $g(t) = f(t)/f(t-1)$, be non-decreasing. This condition is clearly met by all quasi-exponential discount functions with $\beta \leqslant 1$.

Proposition 2. *Suppose $f > 0$, and let g be the associated patience function. If g is non-decreasing, then $a \geqslant 0$. If g is strictly increasing, then $a > 0$.*

We note that if both f and a are non-negative—as under the hypothesis of Proposition 2—then the altruistic weight $a(t)$ attached to the decision-maker in any period t cannot exceed the discount factor $f(t)$ attached to consumption in that period; by Eq. (3) we have $0 \leqslant a(t) \leqslant f(t)$ for all $t > 0$. In particular, if $f(t)$ goes to zero as t goes to infinity, then so does $a(t)$.

Another observation is that the hypothesis in Proposition 2 is closely related to, but distinct from, the condition that f be convex, where we call a function $f : \mathbb{N} \to \mathbb{R}$ *convex* if its piece-wise affine extension to $\mathbb{R}_+$ is convex.[11] It is easy to see that convexity is not sufficient for the altruism weights to be nonnegative. For example, any convex function with $f(0) = 1$, $f(1) = 0.2$ and $f(2) = 0.02$ has $a(2) = -0.02$. However, a seemingly slight strengthening of the hypothesis in Proposition 2 effectively requires f to be convex. To see this, let $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}$ be twice differentiable with $\tilde{f}(0) = 1$, $\tilde{f} > 0$ and $\tilde{f}' \leqslant 0$, and let $G : \mathbb{R}^2_+ \to \mathbb{R}$ be defined by

$$G(t, s) = \frac{\tilde{f}(t+s)}{\tilde{f}(t)}. \tag{5}$$

Clearly, the restriction of $\tilde{f}$ to $\mathbb{N}$ is a discount function f, and the associated patience function g satisfies $g(t+1) = G(t, 1)$ for all $t \in \mathbb{N}$. In particular, g is non-decreasing—as required in Proposition 2—if G'_1, the partial derivative of G with respect to its first argument, is nonnegative. Under the latter, somewhat more stringent hypothesis, $\tilde{f}$, and hence also f, are convex (and thus, by Proposition 2, a is nonnegative):

[10] More generally, it is easy to verify that Eq. (3) gives $a(t) = (-1)^{t+1} f(1)^t$ for all $t \geqslant 1$ if $f(t) = 0$ for all $t > 1$.

[11] For all reals t between any two integers k and $k+1$, let

$$f^*(t) = (t-k)f(k+1) + (k+1-t)f(k),$$

thus defining a piece-wise affine extension f^* of f.

Proposition 3. *If* $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}$ *is twice differentiable with* $\tilde{f}(0) = 1$, $\tilde{f} > 0$, $\tilde{f}' \leqslant 0$, *and* $G_1' \geqslant 0$, *then f is convex and g non-decreasing.*

We next turn to the task of identifying a *necessary* condition for all altruism weights to be nonnegative. For this purpose, let us first briefly consider the classical case when the discount function f is exponential: $f(t) = \delta^t$ for all t, for some $\delta \in (0, 1)$. Eq. (3) then gives $a(1) = \delta$ and $a(t) = 0$ for all integers $t > 1$. In other words, exponential discounting is equivalent to altruism only to the next decision-maker.

Conversely, one-period altruism implies exponential discounting: if $a(1) = \alpha \geqslant 0$ and $a(t) = 0$ for all $t > 1$, then $f(t) = \alpha^t$ for all t (see Eq. (4)). This is not surprising: if each decision-maker attaches an altruistic weight α to the next decision-maker, and zero weight to all others, then the contribution to current welfare from the instantaneous utility t periods ahead should be the product of how much the current decision-maker cares about the next decision-maker, how much the next decision-maker cares about his successor, and so on, up to the t'th decision maker.

These observations concerning exponential discounting can be used to establish a necessary condition for (nonnegative) altruism in general, namely, that the discount function should not decline faster than exponentially, as compared with its decline from the current period to the next. In this sense, exponential discounting is a boundary case from the viewpoint of altruism.

Proposition 4. *If* $a \geqslant 0$, *then* $f(t) \geqslant [f(1)]^t$ *for all t.*

4. Examples

4.1. Quasi-exponential discounting

What altruism weights correspond to the quasi-exponential discounting in the Laibson–Phelps–Pollak model? Suppose, thus, that $f(0) = 1$ and $f(t) = \beta\delta^t$ for all positive integers t, for some $\beta \in (0, 1]$ and $\delta \in (0, 1)$. Then Proposition 1 gives $a(1) = \beta\delta$ and $a(2) = \beta(1-\beta)\delta^2$, and, by induction in t: [12]

$$a(t) = \beta(1-\beta)^{t-1}\delta^t \quad \forall t \geqslant 1. \tag{6}$$

Not surprisingly, *one-period altruism* is obtained when $\beta = 1$: then $a(1) = \delta$ and $a(t) = 0$ for all $t > 1$. By contrast, when $\beta < 1$, then one obtains *exponential altruism*:

$$U_\tau(x) = u(x_\tau) + \alpha \sum_{t=1}^{\infty} \gamma^t U_{\tau+t}(x), \tag{7}$$

[12] Suppose $a(s) = \beta(1-\beta)^{s-1}\delta^s$ for $s \leqslant t$. Then (3) gives

$$a(t+1) = \beta\delta^{t+1} - \beta^2\delta^{t+1} \sum_{s=1}^{t} (1-\beta)^{s-1},$$

which boils down to $a(t+1) = \beta(1-\beta)^t\delta^{t+1}$.

where $\alpha = \beta/(1-\beta)$ and $\gamma = (1-\beta)\delta$. Here γ is the constant factor by which altruism declines over the infinite sequence of future decision-makers. The total utility weight that (7) places on the aggregate of all future decision-makers is $A = \alpha\gamma/(1-\gamma)$. We note that $\gamma \to 0$ and $\alpha\gamma \to \delta$ as $\beta \to 1$. Hence, the representation is continuous at $\beta = 1$–the boundary case of classical exponential discounting.

Conversely, if the sequence $\langle U_\tau \rangle$ of utility functions satisfy (7) for some $\alpha > 0$ and $\gamma \in (0, 1)$, then these are behaviorally equivalent with quasi-exponential discounting with $\beta = \alpha/(\alpha+1)$ and $\delta = (\alpha+1)\gamma$. In particular, the induced discount function f is summable iff $\gamma < 1/(\alpha+1)$, that is, iff the altruism weights do not taper off to slowly, given α. [13] We also note that if decision makers place altruism weight $1/2^t$ on its t:th successor—the genetic kinship factor between parent and child—then $\alpha = 1$, $\gamma = \frac{1}{2}$, $A = 1$, $\beta = \frac{1}{2}$ and $\delta = 1$. In this border-line case, f is not summable—the utility functions U_τ are then not well-defined for constant consumption streams.

For the sake of illustration of this one-to-one relationship between discount factors and altruism weights, we note that Angeletos et al. [3] made the following estimate of the parameter pair (β, δ) in the Laibson–Phelps–Pollak model, based on annual US data: $\beta = 0.55$ and $\delta = 0.96$. The associated altruism parameters (for annual decision makers) are thus $\alpha \approx 1.22$ and $\gamma \approx 0.43$, which yields total altruism $A \approx 0.92$ towards the aggregate of future decision makers. This is not far from the divergent case of $\alpha = 1$, $\gamma = \frac{1}{2}$ and thus $A = 1$: unit total utility weight on the aggregate of all future (annual) selves and altruism halved from each future self to the next.

4.2. Hyperbolic discounting

Psychologists who have studied temporal preferences of human and animal subjects suggest that the discount function f be hyperbolic, see [1,10,17]. In this vein, Loewenstein and Prelec [16] suggested discount functions f of the form $f(t) = (1+\mu t)^{-\gamma}$ for $\mu, \gamma > 0$. It is easily verified that the associated patience function g is increasing. Hence, all discount functions of the Loewenstein and Prelec variety are consistent with altruism towards future decision-makers. [14] By contrast, we saw in Section 3 that some other, closely related, discount functions are not. For example, if $f(0) = 1$ and $f(t) = (\lambda + \mu t)^{-\gamma}$ for all $t > 0$, then the monotonicity condition for g is violated for $\lambda = 0.5$ and $\mu = \gamma = 1$. (Note that f is non-increasing if $\lambda, \mu \geqslant 0$ and $\lambda + \mu \geqslant 1$, which we now assume.) In this class of discount functions, the sufficient condition in Proposition 2 is in fact also necessary: if $(\lambda+\mu)^2 < \lambda+2\mu$, then $a(2) < 0$. That proposition is thus sharp within this class of discount functions. Actually, Proposition 4 is sharp as well: the necessary condition for nonnegative altruism in that proposition is equivalent with the condition that $(\lambda+\mu)^t \geqslant \lambda + \mu t$ for all $t > 0$, which, in turn, is equivalent with $(\lambda+\mu)^2 \geqslant \lambda + 2\mu$. [15]

[13] A function $f : \mathbb{N} \to \mathbb{R}$ is *summable* if $\sum_{t=0}^{T} f(t)$ converges to some real number as $T \to \infty$. Summability is necessary for the utility functions U_τ to be real-valued for constant consumption streams.

[14] We have been unable to obtain a closed-form representation of the altruistic weights corresponding to hyperbolic discounting.

[15] To see this, note that the derivative of $(\lambda+\mu)^t - \lambda - \mu t$ with respect to t is nonnegative for all $t \geqslant 2$.

5. Related literature

The closest work we have found are in [5,13,23], to which we now turn. Zeckhauser and Fels [23] studied a class of altruistic intergenerational preferences, and, like here, derived the corresponding representation based on each generation's utility from its own consumption. In our notation, they studied preferences represented by utility functions $\langle U_\tau \rangle_{\tau \in \mathbb{N}}$ satisfying the following recursive equation:

$$U_\tau(x) = u(x_\tau) + bU_\tau(x) + d \sum_{t=1}^{\infty} a^t U_{\tau+t}(x) \tag{8}$$

for some $a, b, d \geqslant 0$. Kimball [13] analyzed the utility interdependence that may arise when the current generation cares not only about future generations' consumption but also about that of past generations. [16] In our notation:

$$U_\tau(x) = \sum_{t \in \mathbb{Z}} f(t) u(x_{\tau+t}), \tag{9}$$

where $\mathbb{Z}$ denotes the set of integers, and $f : \mathbb{Z} \to \mathbb{R}_+$. He posed the question whether this is consistent with pure altruism towards one's immediate successor and descendant, in the sense that there exist scalars $a(1), a(-1) > 0$ such that, for all consumption streams x and generations τ,

$$U_\tau(x) = u(x_\tau) + a(-1)U_{\tau-1}(x) + a(1)U_{\tau+1}(x). \tag{10}$$

Kimball [13] identified necessary and sufficient conditions on f for this. Our results are complementary to those of [13], and agree with his along the boundary between the parametric domains of the two models.

Bergstrom [5] asks the question when a system of interdependent utility functions, in which each individual's utility depends on his or her own consumption as well as on the utilities of all other members of society, uniquely determines all individuals' utility functions over allocations. Interpreted in terms of successive generations, and using our notation, he considers additively separable and stationary interdependent utility functions, with ancestors running back to an infinite past and descendants running forward to an infinite future:

$$U_\tau(x) = u(x_\tau) + \sum_{t=1}^{+\infty} a(-t)U_{\tau-t}(x) + \sum_{t=1}^{+\infty} a(t)U_{\tau+t}(x) \tag{11}$$

with $a : \mathbb{Z} \to \mathbb{R}_+$. This system of functional equations can be written in matrix form as $U = u + AU$, where U and u are the (infinite) vectors of total and instantaneous utilities, respectively, and A is the (infinite) matrix whose ij-th entry is $a(j-i)$ for all $i \neq j$, and where $a_{ii} = 0$ for all i. Bergstrom shows that if $I - A$ is a dominant-diagonal matrix,

[16] However, unlike us, he did not allow for altruism towards future generations beyond the first, see below.

then there exists a unique bounded matrix F such that $U = Fu$, where all entries f_{ij} of F are nonnegative. [17] Our model is the special case of Bergstrom's when altruism is directed only to descendants, that is, when $a(t) = 0$ for all $t < 0$. In this case, the matrix F is the triangular matrix that has $f_{ii} = 1$ for all i, $f_{ij} = f(j-i)$ for all $j > i$, and $f_{ij} = 0$ for all $j < i$, where f is the function uniquely determined by the recursive equation system (4).

6. Conclusions

We started out by asking if discounting of future instantaneous utilities is consistent with "pure" altruism towards future decision-makers. We identified a recursive functional equation which establishes a one-to-one relationship between discount factors and altruistic weights attached to future generations or future selves. We saw that some discount functions used in the literature are consistent with altruism towards one's future selves or future generations, while others are not. We also established a sufficient condition, and a necessary condition, for consistency in this respect. These conditions are met by the quasi-exponential discounting models currently under investigation in the macroeconomics literature (see, for example, [3,4,14,15]), as well as by some of the hyperbolic discounting models in the psychology literature (see, for example, [16]).

From a behavioral viewpoint, however, separable discounting models seem quite restrictive as representations of intertemporal consumer preferences. See, for example, [9,12,21] for alternative approaches to intertemporal choice. We hope, though, that our study has shed some light on a somewhat wider path than the well-trodden but narrow path of exponential discounting.

Appendix A.

A.1. Proof of Proposition 1

Suppose $\langle U_\tau \rangle$ satisfies Eq. (1) for some $u : X \to \mathbb{R}$ and $f : \mathbb{N} \to \mathbb{R}$ with $f(0) = 1$. Let $a : \mathbb{N}_+ \to \mathbb{R}$ be defined by (3). Then

$$f(t) = \sum_{s=1}^{t} a(s) f(t-s) \quad \forall t \in \mathbb{N}_+. \tag{12}$$

Hence,

$$U_\tau(x) = u(x_\tau) + \sum_{t=1}^{\infty} \sum_{s=1}^{t} a(s) f(t-s) u(x_{\tau+t})$$

[17] Bergstrom [5] defines a (finite or countably infinite) matrix $I - A$, where $A \geqslant 0$, as *dominant diagonal* if there exists a bounded diagonal matrix D such that the infimum of the row sums of the matrix $(I - A)\,D$ is positive. (A matrix is *bounded* if its entries constitute a bounded set in $\mathbb{R}$.)

$$= u(x_\tau) + \sum_{s=1}^{\infty} a(s) \left[\sum_{t=s}^{\infty} f(t-s) u(x_{\tau+t}) \right]$$

$$= u(x_\tau) + \sum_{s=1}^{\infty} a(s) \left[\sum_{k=0}^{\infty} f(k) u(x_{\tau+s+k}) \right] = u(x_\tau) + \sum_{s=1}^{\infty} a(s) U_{\tau+s}(x). \tag{13}$$

Since the resulting equation holds for all τ, this proves the claim. [18]

A.2. Proof of Proposition 2

Suppose first that g is non-decreasing. Since $a(1) = f(1)$, we have $a(1) > 0$. Suppose $a(s) \geqslant 0 \; \forall s < t$. By Eq. (3):

$$\begin{aligned} a(t) &= f(t) - f(1)a(t-1) - \sum_{s=1}^{t-2} a(s) f(t-s) \\ &= g(t) f(t-1) - f(1)a(t-1) - \sum_{s=1}^{t-2} g(t-s) a(s) f(t-s-1) \\ &\geqslant g(t) \left[f(t-1) - \sum_{s=1}^{t-2} a(s) f(t-s-1) \right] - f(1)a(t-1) \\ &= g(t)a(t-1) - f(1)a(t-1) \\ &= [g(t) - f(1)]a(t-1) \geqslant 0, \end{aligned} \tag{14}$$

where Eq. (3) was used again for $a(t-1)$, and where the last inequality follows from the fact that, by assumption, g is non-decreasing with $g(1) = f(1)$.

Secondly, suppose that g is strictly increasing. If $a(s) > 0 \; \forall s \leqslant t$, then the same reasoning as above leads to $a(t) > [g(t) - g(1)]\, a(t-1) > 0$.

A.3. Proof of Proposition 3

By differentiation,

$$G_1'(t,s) \geqslant 0 \quad \Leftrightarrow \quad \tilde{f}'(t+s)\tilde{f}(t) \geqslant \tilde{f}(t+s)\tilde{f}'(t). \tag{15}$$

[18] It is easily verified that the change of order of summation in this derivation is justified. For by assumption U_τ is a real-valued function and hence the sum in the first line converges to some real number λ. Hence, for every $\varepsilon > 0$ there exists a T_ε such that summation from $t = 1$ up to T_ε brings this partial sum within ε from λ. Given T_ε, the order of summation may be changed, and the final expression is obtained by letting $\varepsilon \to 0$.

Suppose that $G_1'(t,s) \geqslant 0$ for all $s > 0$. If $\tilde{f}'(t) = 0$, then $\tilde{f}'(t+s) \geqslant 0$ for all $s > 0$, and hence $\tilde{f}''(t) \geqslant 0$. If instead $\tilde{f}'(t) < 0$, then

$$\tilde{f}''(t) = \lim_{s \downarrow 0} \frac{\tilde{f}'(t+s) - \tilde{f}'(t)}{s} \geqslant \lim_{s \downarrow 0} \frac{1}{s} \left[\frac{\tilde{f}(t+s)}{\tilde{f}(t)} - 1 \right] \tilde{f}'(t)$$

$$= \lim_{s \downarrow 0} \left[\frac{\tilde{f}(t+s) - \tilde{f}(t)}{s} \right] \frac{\tilde{f}'(t)}{\tilde{f}(t)} = \frac{\left[\tilde{f}'(t) \right]^2}{\tilde{f}(t)} > 0. \tag{16}$$

A.4. Proof of Proposition 4

Suppose $a \geqslant 0$, and let f be the associated discount function, as defined in (4). Let $\alpha = a(1)$, and let a^* be the altruism-weight function defined by $a^*(1) = \alpha$ and $a^*(t) = 0$ for all $t > 1$. We know from the above observation that the discount function f^* associated with a^* is $f^*(t) = \alpha^t$ for all t. However, it follows from (4) that $f(t) \geqslant f^*(t)$ for all $t > 1$ since $a \geqslant a^*$. Hence, $f(t) \geqslant \alpha^t = [f(1)]^t$ for all $t > 1$.

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