Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER Journal of Economic Dynamics & Control 29 (2005) 1237–1266
JOURNAL OF Economic Dynamics & Control
www.elsevier.com/locate/econbase

Explicit solutions to an optimal portfolio choice problem with stochastic income

Vicky Henderson

ORFE and Bendheim Center for Finance, E-Quad, Princeton University, Princeton, NJ 08544, USA

Received 15 October 2003; accepted 23 July 2004

Abstract

This paper solves in closed form the optimal portfolio choice problem for an investor with negative exponential utility over terminal wealth facing imperfectly hedgeable stochastic income. The returns on income and the stock are imperfectly correlated, so the market is incomplete. We identify how an investor adjusts their [Merton, Rev. Econ. Statist. 51 (1969) 247] portfolio of the stock and riskless asset via an intertemporal hedging demand, in reaction to the stochastic income. Under general assumptions on the process governing the income, the sign of the hedging demand is the opposite of the sign of the correlation between the income and stock. The optimal portfolio in the stock is long stock if the risk premium is positive and correlation negative, and short if these signs are reversed. Specializing in turn to normally distributed income and lognormally distributed income with or without mean-reversion, the effect of a number of parameters on the optimal portfolio in the stock can be studied. When the risk premium on the stock and correlation have opposite signs, the optimal portfolio decreases in magnitude with risk aversion, unhedgeable variance of income and time, and increases in magnitude with the magnitude of correlation under both the lognormal (without mean-reversion) and normal models for income. These relationships do not necessarily hold if the signs on the risk premium and correlation are the same, in particular the optimal portfolio is not necessarily monotone in risk aversion, violating well-known static results in models without income. When income follows a lognormal model with mean-reversion, more complicated behavior can occur. For instance, the optimal

E-mail address: vhenders@princeton.edu (V. Henderson).

doi:10.1016/j.jedc.2004.07.004

portfolio does not have to be monotonic in time, regardless of the signs of correlation and the risk premium.

JEL classification: D91; G11; J26; J30

Keywords: Incomplete markets; Stochastic income; Labor income; Optimal portfolio choice; Asset allocation; Wage income; Mean-reversion

1. Introduction

This paper examines a dynamic portfolio choice problem where, in addition to an initial wealth endowment, the investor receives a stream of income over a finite investment horizon. The income rate process is stochastic and correlated with the stock. If this correlation is perfect, the income risk can be perfectly hedged via the stock and the investor faces a familiar complete market situation. We allow correlation between the stochastic income and stock to be imperfect, implying the investor cannot replicate income risk with the stock and riskless assets alone. In such an incomplete market, we can study the effect of unhedgeable income risk on the investor's portfolio choice problem. That is, how the non-tradability of income affects the optimal allocation of wealth between the stock and riskless asset.

Our partial equilibrium model is adapted from the classic model of Merton (1969, 1971) by incorporating stochastic income of a general form. The investor seeks to maximize expected utility of terminal wealth, where wealth is generated by holdings in the stock and riskless asset, and the stream of stochastic income. Our formulation includes a Markovian model for income, and we later specialize to income which is normally distributed, or lognormally distributed, with or without mean-reversion.

In a growing literature, the stream of income received by the investor is thought of as labor income. For many investors, human capital represents a major fraction of their wealth. Labor income, the yield on human capital is uncertain, and investors cannot sell contracts contingent on the future value of their labor income for moral hazard reasons. In their data set, Heaton and Lucas (2000) find wage income is the most important source of wealth for the typical household. Campbell and Viceira (2002, Table 7.1) find labor income is positively correlated with stock returns, using PSID data and CRSP data on NYSE value weighted stock returns relative to the Treasury bill rate. The question of interest from our perspective is how the labor income affects the asset allocation decision of the investor, and how correlation between labor income and stock returns alters this decision.

Early research on related problems assumed income was known with certainty. Merton (1971) derived optimal policies when an investor receives a deterministic income stream by adding the capitalized discounted lifetime income flow to initial wealth. Bodie et al. (1992) extend the model to allow the investor to choose their labor supply in each period. Svensson and Werner (1993) survey these early complete market models in more detail.

A number of papers remove the spanning condition, but usually make little progress analytically. Merton (1977) and Weil (1994) examine a two-period model, whilst a series of papers including Duffie and Zariphopoulou (1993), Duffie et al. (1997), Koo (1998) and Munk (2000) consider models with imperfect correlation between stock returns and income in an infinite horizon model with CRRA preferences.

Incomplete models where more progress has been made analytically are rare. A number of papers consider models where income is effectively received at the investment horizon, rather than as a stream over time. Svensson and Werner (1993) study an infinite horizon investment and consumption model with stochastic income. Under CARA, they obtain explicit results in the special case of normally distributed income. For a finite horizon problem with utility of terminal wealth, CARA and normally distributed income, Duffie and Jackson (1990, Case 2) and Teplá (2000) obtain a portfolio of risky assets corresponding to Svensson and Werner (1993) with the appropriate adjustments for the fixed horizon.

Once we move away from CARA, even fewer analytical solutions exist. In addition to the papers mentioned earlier, Chan and Viceira (2000) extend Bodie et al. (1992) to incomplete markets and consider log-linear approximations to the budget constraint and first-order conditions resulting in an approximate solution. Viceira (2001) examines the effect of labor income and retirement on consumption and portfolio choice, under CRRA. Keppo and Sullivan (2003) consider a model with stochastic income, treating the special case of log utility but restricting to zero correlation between the stock and income. Other related models employ numerical solution techniques. Franke et al. (2002) study a similar problem to Viceira (2001), giving numerical solutions. In the context of a pension plan, Cairns et al. (2003) investigates a finite horizon model where contributions at a rate proportional to salary are paid into the fund and salary is not fully hedgeable.

We can also relate problems of stochastic income to the literature on the effect of background risk on an investor's behavior with respect to financial risk. This literature includes Kimball (1993), Gollier and Pratt (1996), Eeckhoudt et al. (1996) and Franke et al. (2001) and show for various classes of utility functions that investors reduce their holdings of risky assets when there is an increase in background risk.

Our model enables us to analyze the effect of stochastic income on the asset allocation decision whilst obtaining closed form expressions for the quantities of interest. The model is formulated initially with general investor preferences and a general specification for stochastic income. The investor's optimal portfolio in the stock is comprised of a Merton (1969) term, followed in the absence of stochastic income, and a hedging demand term. Specializing preferences to CARA, closed form solutions to the investor's portfolio choice problem are given, and are valid for general distributions of income. In both the complete and incomplete case, the sign of the hedge demand depends on the sign of the correlation between the stock and the income state variable and how the state variable affects income. We find that if income is increasing in the state variable and correlation is positive, the hedge demand is negative, as income partly replaces stock. Previous research identified a similar effect but required a specific model (usually lognormal) for income. Additionally, if the risk premium on the stock and correlation between stock and income state variable have opposite signs, the investor should be long (if risk premium positive) or short (if risk premium negative) the stock.

When we specialize to lognormally distributed income (with or without mean-reversion) and normally distributed income in turn, the investor's optimal portfolio, value function and certainty equivalent of future income can be computed explicitly. In the case of normally distributed income, the optimal portfolio choice is found to be related to that previously obtained by Duffie and Jackson (1990) and Teplá (2000). The results vary due to the differing assumptions made on when the income is received, which introduces a discounting term. There are no obvious comparisons for the optimal portfolio under the lognormal models either in the case with mean-reversion or without.

Under our models for income, the effect of various parameters on the optimal portfolio choice can be observed via comparative statics. Common to both models is the criteria that if correlation and the risk premium on the stock have opposite signs, then the effect of various parameters on the optimal portfolio is known. For instance, in both the lognormal and normal models, the optimal portfolio in the stock is decreasing in magnitude with risk aversion, provided the correlation between stock and income state variable and risk premium have opposite sign. Otherwise, the optimal portfolio may not be monotonic in risk aversion, violating the simple static results of Huang and Litzenberger (1988). We cannot deduce a similar relationship when income is lognormal with mean-reversion as the effects are more complex in this case. A second conclusion is that our model gives a precise criteria under which unhedgeable variance of income reduces the magnitude of the optimal investment in the stock, that is, when the risk premium and correlation have opposite signs. We also find the optimal portfolio (in units of T money) decreases in magnitude over time, under the same condition. This is consistent with the advice of investment professionals who advise investors to reduce their allocation to stocks over time. It is also consistent with the findings of Campbell and Viceira (1999) and with the background risk literature. However, outside this special case, for instance if the risk premium and correlation have the same sign, or there is mean-reversion, these conclusions do not hold. In these cases, the hedge demand may not be monotonic in time, and we cannot draw simple conclusions concerning the behavior of the optimal portfolio in the stock.

This paper contributes to and extends the literature in several directions. First, it extends the known set of portfolio problems for which exact solutions may be found. Analytical solutions are rarely available and allow for robust comparative static analysis in a way that numerical solutions cannot.

Second, we use a different solution technique to other papers concerning stochastic income. The approach used in this paper is to transform the non-linear HJB equation and obtain a linear pde which can be solved explicitly.[1] This is in

[1]This approach has been used previously by Liu (2001a), Henderson (2002), Henderson and Hobson (2002) and Zariphopoulou (1999, 2001), none of which are stochastic income models.

contrast to the martingale approach developed for incomplete markets in Karatzas et al. (1991) and He and Pearson (1991). Keppo and Sullivan (2003), Svensson and Werner (1993) and Teplá (2000) use this martingale approach in stochastic income problems. As far as the author is aware, the HJB technique has not been used to solve explicitly an incomplete stochastic income model.[2]

Thirdly, our framework allows for general distributions for stochastic income, which includes the special cases where income is normally distributed, or is lognormal with or without mean-reversion. Thus it generalizes the assumption of normally distributed income made in Svensson and Werner (1993), Duffie and Jackson (1990) and Teplá (2000), whilst retaining the explicit nature of solutions under CARA. Although other papers have examined the impact of mean-reverting stock returns on the optimal portfolio (see Kim and Omberg, 1996; Wachter, 2002), none have included mean-reverting stochastic income.

Additionally, the investor in our model receives a stream of stochastic income over time. This is in contrast to the papers of Svensson and Werner (1993), Duffie and Jackson (1990) and Teplá (2000) who use identical preferences, but assume income makes an uncertain contribution to terminal wealth at the horizon.

There are also links between our paper and the literature on pricing options on non-traded assets. Davis (2000), Henderson (2002), Henderson and Hobson (2002) and Teplá (2000) all study this problem under CARA. All except the latter assume a lognormally distributed non-traded asset whilst Teplá (2000) assumes the non-traded asset is normally distributed. Henderson (2002), Henderson and Hobson (2002) and Kahl et al. (2003) study the same problem under CRRA, the first two papers obtain expansions for the solution whilst the third paper employs numerical methods, but also allows for intermediate consumption.

2. The investor's problem

Consider a financial market consisting of a single risky asset (which we refer to as stock for the remainder of the paper) with price P, and a riskless bond with price $R_t = R_0 e^{rt}$ at time t where r is a constant rate of interest. The stock price follows

$$dP = P(\mu\, dt + \sigma\, dB), \tag{1}$$

where μ, σ are constants and B is a standard Brownian motion. The portfolio choice problem of an investor in this simple market is well studied, beginning with Merton (1969, 1971).

In this paper we assume the investor also receives income over time, the income rate at t is $a(Y_t, t)$ where Y_t is an income state variable following

$$dY = \nu(Y_t, t)\, dt + \eta(Y_t, t)\, dZ. \tag{2}$$

[2] In fact, Zariphopoulou (2001) mentions this style of problem but says "the scaling properties and transformation employed herein cannot be applied".

The correlation between $\mathrm{d}B$ and $\mathrm{d}Z$ is $\rho\,\mathrm{d}t$, where $-1 \leqslant \rho \leqslant 1$. It is also convenient to write Z as $Z_t = \rho B_t + \sqrt{1-\rho^2}\,W_t$ where W is a standard Brownian motion independent of B. We assume $v(y,t), \eta(y,t)$ are continuous and satisfy Lipschitz and growth conditions in y. These are sufficient to ensure a unique solution. For $|\rho| < 1$, the presence of the second Brownian motion W means the income cannot be perfectly hedged via the stock P and the market faced by the investor is incomplete.

Wealth, denoted by X, is generated by the investor holding cash amount θ in the stock P, the remainder in the riskless bond, and by the inflow of stochastic income $a(Y_t,t)$.[3] The dynamics of wealth can be written as

$$\mathrm{d}X = \theta\,\frac{\mathrm{d}P}{P} + r(X-\theta)\,\mathrm{d}t + a(Y_t,t)\,\mathrm{d}t,$$

where $a(y,t)$ is non-negative and continuous. By taking $a(y,t) = y$, the investor simply receives the value of the income state variable Y_t itself over time. Here θ is taken to be adapted and satisfies sufficient regularity conditions to rule out doubling strategies. A sufficient condition for this is that discounted wealth (without the income flow a) be a supermartingale under equivalent martingale measures. For example, this is achieved by assuming $\int_t^T \theta_u^2\,\mathrm{d}u$ has finite expectation under each equivalent martingale measure.

Note the use of the function $a(y,t)$ allows flexibility in modeling but also introduces some indeterminancy as there can be many equivalent characterizations of the same model. However, this is not a drawback as the most convenient parameterization can be chosen. For instance, in Section 4 we will consider the special case where income is lognormally distributed. This is obtained by either taking $a(y,t) = y$ and $v(y,t) = vy, \eta(y,t) = \eta y$ where v, η constants, or $a(y,t) = \mathrm{e}^y$ and $v(y,t) = v, \eta(y,t) = \eta$. Likewise, we will also specialize to normally distributed income obtained via $a(y,t) = y$ and $v(y,t) = v, \eta(y,t) = \eta$ or $a(y,t) = \ln y$ and $v(y,t) = vy, \eta(y,t) = \eta y$.

We consider the problem of an investor with utility over terminal wealth which can be maximized by the selection of the portfolio θ. Define the value function

$$V(t, X_t, Y_t) = \sup_{\theta_s, s\in(t,T]} \mathbb{E}_t U(X_T), \tag{4}$$

which is the utility attained by the investor if the optimal portfolio policy is followed.

Since (X, Y) are jointly Markov, the value function $V(t, X_t, Y_t)$ satisfies the non-linear HJB equation

$$\begin{aligned}&\sup_\theta[\dot V + V_x(\theta\mu + a(y,t) + r(x-\theta)) + V_y v(y,t) + \tfrac12 V_{xx}\theta^2\sigma^2 \\ &\quad + \tfrac12 V_{yy}\eta(y,t)^2 + V_{xy}\eta(y,t)\rho\theta\sigma] = 0.\end{aligned} \tag{5}$$

[3] Applying Itô's formula gives the dynamics for income $a(Y_t,t)$ as

$$\mathrm{d}a(Y_t,t) = (\dot a + a_y v(Y_t,t) + \tfrac12 a_{yy}\eta^2(Y_t,t))\,\mathrm{d}t + a_y\eta(Y_t,t)\,\mathrm{d}Z. \tag{3}$$

Differentiating with respect to θ gives the first-order condition

$$\theta_t^* = \frac{-V_x(\mu - r)}{V_{xx}\sigma^2} - \frac{V_{xy}\eta(Y_t, t)\rho}{V_{xx}\sigma}. \tag{6}$$

We write

$$\theta_{\mathrm{M}}^* = \frac{-V_x(\mu - r)}{V_{xx}\sigma^2} \tag{7}$$

and

$$\theta_{\mathrm{H}}^* = -\frac{V_{xy}\eta(Y_t, t)\rho}{V_{xx}\sigma} \tag{8}$$

both of which vary over time. Incorporating the first-order condition in the HJB equation gives the pde

$$\dot{V} + V_x a(y,t) + V_x rx + V_y v(y,t) + \frac{1}{2} V_{yy}\eta(y,t)^2 - \frac{(V_x(\mu - r) + V_{yx}\eta(y,t)\rho\sigma)^2}{2V_{xx}\sigma^2} = 0, \tag{9}$$

with $V(T, x, y) = U(x)$.

In general, the optimal portfolio in the stock, θ^* (expressed as a cash amount) given in (6) is comprised of two components denoted θ_{M}^* and θ_{H}^* given in (7) and (8). The first we call the Merton investment strategy as it is of the form of the Merton (1969) strategy followed in the absence of stochastic income. Note subtly, that at this level of generality, θ_{M}^* is not necessarily identical to the Merton (1969) strategy since the value function V may be different in the two problems. However, as we will see, they are identical when we choose the negative exponential utility function. The Merton strategy is myopic because it is the portfolio choice for an investor who only has a single period objective. When the correlation and the volatility of the income state variable are non-zero, the optimal portfolio given in (6) also includes a hedging component θ_{H}^*. This can be interpreted as in Merton (1971) as intertemporal hedging demand and is hedging the inflow of stochastic income. It depends on the volatilities and correlation and the value function V. The term V_{xy} measures the sensitivity of the marginal utility of wealth to the stochastic income, or the attitude toward changes in stochastic income. Note also that each of these two terms appear in Svensson and Werner (1993) as the tangency and state variable hedge portfolio.

However, if ρ or $\eta(y, t)$ are zero, the hedging term θ_{H}^* disappears and the existence of nontraded income has no effect on the stock portfolio. The stochastic income cannot be hedged against in the case where the stock returns and income are uncorrelated. Income is certain if $\eta(Y_t, t) = 0$ and therefore no hedging need take place. Even in these cases, however, the existence of the income still impacts on the solution via the value function.

Although the expression for θ^* characterizes the optimal strategy for the portfolio choice problem (4), it is expressed in terms of derivatives of the value function solving HJB equation. In general, since the value function can depend on investor preferences and the investment horizon, the portfolio choice can depend in a complex way on these variables and the market parameters. To gain more interpretable results we need to specialize to a particular utility, and for the remainder of the paper we will assume $U(x) = -(1/\gamma)e^{-\gamma x}$, $\gamma > 0$, the negative exponential utility function. We chose these preferences for tractability but also note that Bliss and Panigirtzoglou (2004) find evidence from option prices that exponential utility provides a better representation of preferences than CRRA.

3. The solution: negative exponential utility and general stochastic income

The objective of this section is to obtain an explicit solution of the portfolio choice problem in (4) under CARA. Merton (1969) showed when utility is given by $U(x) = -(1/\gamma)e^{-\gamma x}$, if wealth dynamics are given by $dX = r(X - \theta)\,dt + \theta(dP/P)$ (where θ is the cash amount invested in the stock P) and the investor seeks to maximize expected utility of terminal wealth $V_0(t, X_t) = \sup_\theta \mathbb{E}_t U(X_T)$, the value function is given by

$$V_0(t, X_t) = -\frac{1}{\gamma} e^{-\gamma X_t e^{r(T-t)}} e^{-((\mu-r)^2/2\sigma^2)(T-t)}. \tag{10}$$

It is well known that the optimal portfolio in the stock (expressed in units of cash) for this problem is

$$\frac{(\mu - r)}{\gamma\sigma^2 e^{r(T-t)}}. \tag{11}$$

Since this can also be derived from the general expression given in (7) (using V_0 above) we conclude that $\theta^*_{\mathrm{M}} = (\mu - r)/(\gamma\sigma^2 e^{r(T-t)})$ represents the Merton (1969) optimal strategy under CARA. The Merton portfolio θ^*_{M} when expressed in units of time T money is constant over time. We can write $\hat{\theta}^*_{\mathrm{M}} = \theta^*_{\mathrm{M}} e^{r(T-t)} = (\mu - r)/\gamma\sigma^2$. The optimal cash amount invested in the stock is proportional to the risk premium and reciprocal of the coefficient of absolute risk aversion. This myopic strategy corresponds to that followed in a static problem. If the risk premium is positive, the optimal portfolio (11) invests a positive amount into the stock, and this amount decreases with risk aversion. If the risk premium is negative, the investor should short the stock, shorting less as risk aversion increases. In fact, Huang and Litzenberger (1988) showed that these conclusions hold in a static set-up with the weaker assumption that the investor is risk averse. Liu (2001b) calls these the static participation and calibration theorems. We will show that these conclusions may be violated in the model with stochastic income.

3.1. Perfect correlation between stock returns and income state variable

In the special case when the correlation between the stock and income state variable is perfect, we can characterize the value function and optimal portfolio choice without the complications of incompleteness. The dynamics in (2) can be written as

$$\mathrm{d}Y = v(Y,t)\,\mathrm{d}t + \eta(Y,t)\rho\,\mathrm{d}B,$$

with $|\rho| = 1$. Since the market is complete, and income is spanned by the stock P, a certainty equivalent value for the stream of future stochastic income can be found. In a complete market, the value of future uncertain cash flows is obtained by discounting their expected value under the unique risk neutral measure. This value is given by

$$C(t, Y_t) = \mathbb{E}\left(\int_t^T \pi_u^C a(Y_u, u)\,\mathrm{d}u\right), \tag{12}$$

where

$$\pi_u^C = \mathrm{e}^{-r(u-t)}\xi_u^C = \mathrm{e}^{-r(u-t)}\exp\left(-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(u-t) - \left(\frac{\mu-r}{\sigma}\right)(B_u - B_t)\right) \tag{13}$$

is the unique state price density.[4]

Since $C(t, Y_t)$ is a certainty equivalent value, we can now think of the optimization problem as one without stochastic income, but with a modified initial wealth given by $\tilde{X}_t = X_t + C(t, Y_t)$. The value function when correlation is perfect is given by the Merton value function in (10), with modified initial wealth $\tilde{X}_t = X_t + C(t, Y_t)$ giving

$$V(t, X_t) = V_0(t, \tilde{X}_t) = -\frac{1}{\gamma}\mathrm{e}^{-\gamma\tilde{X}_t\mathrm{e}^{r(T-t)}}\mathrm{e}^{-((\mu-r)^2/2\sigma^2)(T-t)}. \tag{14}$$

This value function must satisfy the "complete version" of the HJB in (9) with $|\rho| = 1$. A verification argument shows $C(t, Y_t)$ must satisfy

$$-\dot{C} + rC - \left(v(t,y) - \frac{(\mu-r)\rho\eta(t,y)}{\sigma}\right)C_y - \frac{1}{2}\eta(t,y)^2 C_{yy} = a(y,t), \tag{15}$$

with $C(T,y) = 0$ for this to be true. This equation can also be explained by thinking of the value of the claim on income, $C(t, Y_t)$, where the income $a(t, Y_t)$ and expected capital gain are being forced to grow at the riskless rate under the risk neutral measure. We use (15) in determining the optimal portfolio below.

3.1.1. The optimal portfolio choice

We construct directly the optimal portfolio for the investor by exploiting an equivalence between the original problem with stochastic income and a modified

[4] The state price density has the property

$$\mathbb{E}_t[\pi_u^C P_u] = P_t$$

and $\pi_u^C P_u$ is a $\mathbb{P}$-martingale. It also characterizes the Radon–Nikodym derivative that defines the change of probability measure from objective $\mathbb{P}$ to risk neutral measure $\mathbb{P}^C$ via

$$\mathrm{e}^{r(u-t)}\pi_u^C = \xi_u^C = \mathbb{E}\left[\frac{\mathrm{d}\mathbb{P}^C}{\mathrm{d}\mathbb{P}}\middle|\mathscr{F}_u\right].$$

Under $\mathbb{P}^C$, $B_u^C = B_u + [(\mu-r)/\sigma]u$ is standard Brownian motion and $\mathrm{d}Y = (v(Y,t) - [(\mu-r)/\sigma]\rho\eta(Y,t))\,\mathrm{d}t + \eta(Y,t)\rho\,\mathrm{d}B^C$ with $|\rho| = 1$.

problem where there is no stochastic income but the investor receives the certainty equivalent value of the income as an initial endowment.

In the original problem with stochastic income, terminal wealth can be expressed as

$$\mathrm{e}^{-r(T-t)}X_T - X_t = \int_t^T \mathrm{e}^{-r(u-t)}\theta\left(\frac{\mathrm{d}P}{P} - r\,\mathrm{d}u\right) + \int_t^T \mathrm{e}^{-r(u-t)}a(Y_u, u)\,\mathrm{d}u. \tag{16}$$

We also have, via Itô on $C(u, Y_u)\mathrm{e}^{-r(u-t)}$,

$$\int_t^T \mathrm{e}^{-r(u-t)}\left[rC - \dot{C} - C_y\left(v(Y_t, t) - \frac{(\mu - r)\eta(Y_t, u)\rho}{\sigma}\right)\right.$$
$$\left. - \frac{1}{2}C_{yy}\eta(Y_t, u)^2\right]\mathrm{d}u = C(t, Y_t) + \int_t^T \mathrm{e}^{-r(u-t)}C_y\eta(Y_u, u)\rho\,\mathrm{d}B^C, \tag{17}$$

which becomes

$$\int_t^T \mathrm{e}^{-r(u-t)}a(Y_u, u)\,\mathrm{d}u = C(t, Y_t) + \int_t^T \mathrm{e}^{-r(u-t)}C_y\eta(Y_u, u)\rho\,\mathrm{d}B^C \tag{18}$$

using (15).

Returning to (16) and incorporating (18) gives

$$\mathrm{e}^{-r(T-t)}X_T = X_t + C(t, Y_t) + \int_t^T \mathrm{e}^{-r(u-t)}\left(\frac{\mathrm{d}P}{P} - r\,\mathrm{d}u\right)\left[\theta + \frac{C_y\eta(Y_t, t)\rho}{\sigma}\right]. \tag{19}$$

Now consider a new problem where initial endowment is $\tilde{X}_t = X_t + C(t, Y_t)$ and there is no stochastic income. Terminal wealth in this problem, denoted by $\tilde{X}_T$ can be written

$$\mathrm{e}^{-r(T-t)}\tilde{X}_T = X_t + C(t, Y_t) + \int_t^T \mathrm{e}^{-r(u-t)}\tilde{\theta}\left(\frac{\mathrm{d}P}{P} - r\,\mathrm{d}u\right), \tag{20}$$

where $\tilde{\theta}$ is a trading strategy in the stock. This is the Merton (1969) problem with solution $\tilde{\theta}^* = \theta^*_{\mathrm{M}}$. Note there is an equivalence between the two problems generating wealth in (19) and (20), namely taking $\tilde{\theta} = \theta + (C_y\eta(Y_t, t)\rho)/\sigma$ would give identical terminal wealths. Thus the solution to our original stochastic income problem with perfect correlation can be identified as

$$\theta^* = \theta^*_{\mathrm{M}} - \frac{C_y\eta(Y_t, t)\rho}{\sigma}.$$

We have proved the first part of the following result. The second part is proved in Appendix A.1.

Proposition 3.1. *In the case of perfect correlation between the stock and the income state variable, the hedge demand is given by*

$$\theta^*_{\mathrm{H}} = -\frac{\rho C_y\eta(Y_t, t)}{\sigma};\ |\rho| = 1. \tag{21}$$

The sign of θ_{H}^{*} depends on the sign of ρa_y. In the natural case where $a_y>0$, if $\rho = 1$ then hedge demand is negative, and if $\rho = -1$ hedge demand is positive.

The hedge demand in a complete market only depends on the certainty equivalent value of income, level of volatilities and the sign of correlation. These are not investor specific, and the income hedge is the same for all investors. The hedge demand does not depend on the investor's risk aversion γ as there is a unique state price density, and risk preferences do not enter the portfolio. This contrasts to the Merton (1969) term which depends on the stock's risk premium and investor's risk aversion. The investor's overall optimal portfolio is given by the addition of the Merton component θ_{M}^{*} and hedge demand θ_{H}^{*} in (21).

Proposition 3.1 tells us that the sign of the correlation between stock and income state variable returns as well as how the state variable affects income (via a_y), determine the sign of the hedge demand. The condition $a_y>0$ means income is positively related to the state variable Y. If this is the case, and $\rho = 1$ then the stock and income itself are perfectly positively correlated. Intuitively, when correlation is positive, the investor holds less of the stock overall, since income replaces holdings of the stock. The hedge demand in (21) is exactly the amount needed to perfectly offset the income received over time.

3.2. Imperfect correlation between stock returns and income state variable

Now consider the case where the correlation between the stock and income state variable is not perfect. In the case of constant absolute risk aversion $U(x) = -(1/\gamma)\mathrm{e}^{-\gamma x}$, wealth factors out of the problem and the value function can be written as

$$\begin{aligned} V(t, X_t, Y_t) &= \sup_{\theta_s, s\in(t,T]} -\frac{1}{\gamma}\mathbb{E}_t \mathrm{e}^{-\gamma[X_t \mathrm{e}^{r(T-t)} + \int_t^T \theta\, \mathrm{d}P/P + \int_t^T a(Y_u,u)\,\mathrm{d}u]} \\ &= -\frac{1}{\gamma}\,\mathrm{e}^{-\gamma X_t \mathrm{e}^{r(T-t)}} g(T-t, Y_t), \end{aligned}$$

with $g(0, y) = 1$.

Applying Itô's formula to V and using the fact that V is a supermartingale for any strategy and a martingale for the optimal strategy.

Using this form of V in the general HJB equation (9) gives g solves the pde[5]

$$\begin{aligned} &\dot{g} - g_y\left(v(y,t) - \frac{(\mu - r)\eta(y,t)\rho}{\sigma}\right) - \frac{1}{2}\, g_{yy}\eta(y,t)^2 + g\mathrm{e}^{r(T-t)}a(y,t)\gamma \\ &\quad + \frac{(\mu - r)^2}{2\sigma^2} g + \frac{g_y^2\eta(y,t)^2\rho^2}{2g} = 0. \end{aligned} \tag{22}$$

[5]If there was no stochastic income, the pde in (22) would collapse to the Merton (1969) equation involving only the time derivative and $[(\mu - r)^2/2\sigma^2]g$ term. If stochastic income was certain, the $v(Y_t, t)g_y$ and income term would also be present. Finally, the case of stochastic income introduces the remaining terms.

Our goal is to reduce the pde for g to a linear one, at which point we can call upon the Feynman–Kac theorem (see Duffie, 1996, Theorem and Condition 2 on p. 296) to give a stochastic representation for g. Setting $g(\tau, y) = v(\tau, y)^b e^{\alpha\tau}$ and then choosing α and b appropriately will allow us to reduce the pde to a linear one. We find v solves

$$b\dot{v} + \left[\alpha + \gamma e^{r(T-t)} a(y,t) + \frac{(\mu - r)^2}{2\sigma^2}\right] v + \left[\frac{(\mu - r)\eta(y,t)\rho}{\sigma} - v(y,t)\right] b v_y$$
$$- \frac{1}{2}\eta(y,t)^2 b v_{yy} - \frac{1}{2}\eta^2 [b(b-1) - \rho^2 b^2] \frac{v_y^2}{v} = 0,$$

with $v(0, y) = 1$. Choosing $\alpha = -(\mu - r)^2/2\sigma^2$ and $b = 1/(1 - \rho^2)$ and defining $\tilde{v}(t, y) = v(\tau, y)$ gives

$$\dot{\tilde{v}} - (1 - \rho^2)\gamma e^{r(T-t)} a(y,t)\tilde{v} + \left(v(y,t) - \frac{(\mu - r)\eta(y,t)\rho}{\sigma}\right)\tilde{v}_y$$
$$+ \frac{1}{2}\eta(y,t)^2 \tilde{v}_{yy} = 0, \tag{23}$$

with $\tilde{v}(T, y) = 1$. In addition to the regularity assumptions already made, we assume: $a(y,t)$ satisfies a Lipschitz and growth condition in y, $a_y(y,t)$, $a_{yy}(y,t)$, $v_y(y,t)$, $v_{yy}(y,t)$, $\eta_y(y,t)$, $\eta_{yy}(y,t)$ continuous and satisfy a growth condition in y. Under these conditions the Feynman–Kac theorem gives the unique solution of the Cauchy problem (23) to be

$$\tilde{v}(t, y) = \mathbb{E}[e^{r(T-t)} \pi_T^I e^{\tilde{\alpha}\Phi}], \tag{24}$$

where $\tilde{\alpha} = (1 - \rho^2)$, $\Phi = -\gamma \int_t^T e^{r(T-u)} a(Y_u, u)\, du$ and[6] $\pi_u^I = e^{-r(u-t)} \xi_u^I$ where

$$\xi_u^I = \exp\left\{-\frac{1}{2}\left(\frac{(\mu - r)\rho}{\sigma}\right)^2 (u - t) - \left(\frac{(\mu - r)\rho}{\sigma}\right)(Z_u - Z_t)\right\}. \tag{25}$$

We can also see

$$g(\tau, y) = e^{-[(\mu - r)^2/2\sigma^2](T-t)} (\mathbb{E}[e^{r(T-t)} \pi_T^I e^{\tilde{\alpha}\Phi}])^{1/\tilde{\alpha}} \tag{26}$$

leading to the following result.

Proposition 3.2. *The investor's value function for the optimal portfolio choice problem* (4) *with exponential utility, is given by*

$$V(t, X_t, Y_t) = -\frac{1}{\gamma} e^{-\gamma X_t e^{r(T-t)}} e^{-[(\mu - r)^2/2\sigma^2](T-t)} (\mathbb{E}[e^{r(T-t)} \pi_T^I e^{\tilde{\alpha}\Phi}])^{1/\tilde{\alpha}}. \tag{27}$$

It can be verified that the value function in (27) satisfies the general HJB in (9) and therefore is indeed the solution. Notice in (27) the state price density π_T^I appears. The

[6] Under $\mathbb{P}^I$, $Z_u^I = Z_u + [(\mu - r)\rho/\sigma]u$ is standard Brownian motion and $dY = (v(Y_t, t) - [((\mu - r)\rho\eta(Y_t, t))/\sigma])\,dt + \eta(Y_t, t)\,dZ^I$. $\mathbb{P}^I$ is called the minimal martingale measure.

value function depends on the expectation under measure $\mathbb{P}^I$, which makes P a martingale and leaves orthogonal risk W unchanged.

3.2.1. The optimal portfolio choice

In developing (22) we find:

Proposition 3.3. *The optimal investment in the stock for the portfolio choice problem* (4) *with exponential utility is*

$$\theta_t^* = \frac{\mu - r}{\gamma\sigma^2 \mathrm{e}^{r(T-t)}} + \frac{g_y \eta(Y_t, t)\rho}{g\gamma\sigma \mathrm{e}^{r(T-t)}} = \theta_{\mathrm{M}}^* + \theta_{\mathrm{H}}^*. \tag{28}$$

As described earlier, the term θ_{M}^* is exactly the Merton (1969) myopic allocation which would be chosen if changes in income were ignored.

We can now analyze the hedging component θ_{H}^* in more detail. In contrast to the Merton portfolio, θ_{H}^* does depend on the model for stochastic income (through g, g_y). However, since we are assuming CARA, the hedging term is independent of the investor's initial wealth. Unlike the optimal portfolio in the case of perfect correlation in (21), the hedge depends on the investor's risk aversion γ. Manipulation of θ_{H}^* and expanding the exponential terms shows the hedge demand approaches the perfect correlation hedge given in (21) as $\rho^2 \to 1$. [7]

3.2.2. Comparative statics

Of interest is the sign and magnitude of the hedging demand and the effect various model parameters have on this hedge. We can show the following result, which does not depend on the specific model chosen for stochastic income, only that the state variable follows (2) and income is some function $a(Y_t, t)$ of the income state variable Y.

Proposition 3.4. *For the optimal portfolio choice problem in* (4) *where risky asset follows* (1), *income state variable Y follows* (2) *and utility is exponential, the sign of the hedging demand $\theta_{\mathrm{H}}^* = (g_y\eta(Y,t)\rho)/(g\gamma\sigma \mathrm{e}^{r(T-t)})$ depends on the sign of $\rho a_y(y,t)$. In the natural case where $a_y > 0$, if $\rho > 0$, then hedging demand is negative, and if $\rho < 0$, hedging demand is positive.*

The proof of this result is deferred to the appendix. The term $\rho a_y \eta(Y_t, t)$ is the hedgeable part of the diffusion term for income, see (3) and use representation $\mathrm{d}Z = \rho\,\mathrm{d}B + \sqrt{1-\rho^2}\,\mathrm{d}W$. Thus the sign of the hedging demand depends upon the part of income that can be hedged. Recall, the condition $a_y > 0$ simply means that income is positively related to the income state variable Y. If both $a_y > 0$ and $\rho > 0$ then income and the stock are positively correlated. Proposition 3.4 says such positive correlation between the stock and income returns has a negative effect on hedging demand for stocks.

[7] Write $g_y/g = \tilde{\beta}(\mathbb{E}^I(\partial/\partial y)\mathrm{e}^{\Phi/\tilde{\beta}})/\mathbb{E}^I \mathrm{e}^{\Phi/\tilde{\beta}}$ where $\tilde{\beta} = 1/\tilde{\alpha}$. As $\rho^2 \to 1$, $\tilde{\beta} \to \infty$ and $g_y/g \to \mathbb{E}^C(\partial/\partial y)\Phi = -\gamma \mathrm{e}^{r(T-t)} C_y$ since $\xi_u^I \to \xi_u^C$ as $\rho^2 \to 1$.

This relates to the investor taking advantage of hedging properties of the stock arising from correlation, just as in the complete case earlier where the unit correlation allowed for perfect hedging. The intuition is the same in the incomplete case, but now the hedge is only partial. In the case where correlation is positive, the agent is less long (or more short) the stock than under the Merton strategy θ^*_{M} as the income replaces some of the holdings of stock. If correlation is negative, hedging demand is positive. In this case, the agent is longer (or less short) the stock than the Merton portfolio. This is done to create a hedge for the income, as it moves in the opposite direction to the stock.

Note this feature has been identified in other models such as Viceira (2001), however, such models rely on lognormally distributed income. Proposition 3.4 shows the result is true for more general distributions of income, under the assumption of CARA.

The sign of the optimal portfolio θ^* in the stock depends on both the Merton and hedging demand terms. Assuming $a_y > 0$, the investor will clearly be long stocks if the risk premium is positive and correlation negative. Likewise, the investor will be short stocks if there is a negative risk premium and correlation is positive. The other two combinations are less clear and will depend on the relative size of the two terms in a specific model.

Recall, empirical evidence in Campbell and Viceira (2002) suggests that wage income and stock returns are positively correlated. Our model predicts that hedge demand against wage income is negative. If the risk premium is positive, the sign on the optimal portfolio depends on the size of the Merton and hedging terms.

3.2.3. *The certainty equivalent value of future stochastic income*

In our general model, we can calculate the implicit value of future stochastic income. This is the least amount of initial wealth that the investor would exchange for the stream of stochastic income (see Munk, 2000). This can be compared to expression (12) for the case of perfect correlation. To calculate the certainty equivalent value of the future income, we equate the value function with income and an adjustment to initial wealth p, with the value function without income $V_0(t, X_t) = V(t, X_t - p, Y_t)$. Solving for the certainty equivalent value p using (10) and (27) gives the following result.

Proposition 3.5. *The certainty equivalent value of future stochastic income under the optimal portfolio choice model* (4) *with exponential utility is given by*

$$p = -\frac{\mathrm{e}^{-r(T-t)}}{\gamma\tilde{\alpha}} \ln(\mathbb{E}\mathrm{e}^{r(T-t)\pi^I_T \mathrm{e}^{\tilde{\alpha}\Phi}}), \tag{29}$$

where $\tilde{\alpha}, \Phi$ *and* π^I_T *are defined after* (24).

The implicit value the investor attaches to the income stream depends on the expected return on the income state variable $v(Y_t, t)$, the Sharpe ratio of the stock, volatility of the income state variable, the investor's risk aversion and the correlation between the stock and income state variable. Again, the expectation appearing in (29) is taken with respect to $\mathbb{P}^I$. In fact, the expectation is of a non-linear function of the income itself, in contrast to the complete market valuation in (12).

As $|\rho| \to 1$, the implicit value of future stochastic income should approach the value in a complete market $C(t, Y_t)$, defined by (12). We show this by taking a Taylor series expansion in $\tilde{\alpha}$ of the certainty equivalent value p to give

$$p = \left[\int_t^T e^{-r(u-t)} \mathbb{E}_t[\xi_u^I a(Y_u, u)]\,du - \frac{1}{2}\gamma\tilde{\alpha}e^{-r(T-t)} \left(\mathbb{E}^I \left[\int_t^T e^{r(T-u)} a(Y_u, u)\,du \right]^2 - \left[\mathbb{E}^I \int_t^T e^{r(T-u)} a(Y_u, u)\,du \right]^2 \right) + \cdots \right]. \tag{30}$$

As $|\rho| \to 1$, only the first term in the expansion remains. Since $\xi_u^I \to \xi_u^C$ as $|\rho| \to 1$, the first term is $C(t, Y_t)$, the complete value given in (12). Therefore, the certainty equivalent value collapses to the complete value when correlation approaches one.

4. The special cases of normally distributed income, and lognormally distributed income with or without mean-reversion

We have explicit expressions for the investor's value function and optimal portfolio choice which hold for the case of general model for income. We saw that the coefficients $v(Y_t, t)$ and $\eta(Y_t, t)$ enter into the value function and hedge demand, and thus different models for income will imply different optimal choices and value functions.

In this section, we specialize to particular models for income. A lognormal model with mean-reversion can be specified via $a(y,t) = y$, $\eta(y,t) = \eta y$ and $v(y,t) = \lambda(\bar{Y} - y)$. This results in

$$dY = \lambda(\bar{Y} - Y_t)\,dt + \eta Y_t\,dZ, \tag{31}$$

with solution $\forall u \geqslant t$:

$$Y_u = (e^{-(\lambda + (\eta^2/2))(u-t) + \eta(Z_u - Z_t)}) \left(Y_t + \lambda\bar{Y} \int_t^u e^{(\lambda + (\eta^2/2))(s-t) - \eta(Z_s - Z_t)}\,ds \right). \tag{32}$$

The income state variable Y mean-reverts to level $\bar{Y} > 0$ at speed λ. It is natural, although not necessary for the results to think of the case $\lambda > 0$. This process remains non-negative if we assume additionally its initial value $Y_t \geqslant 0$, then $a(Y_u, u)$ is non-negative as desired. Models of this form have appeared in Bhattacharya (1978) and Rocha et al. (2003).

Contained within this model is also the case of lognormally distributed income without mean-reversion, that is, a standard lognormal model. This model is obtained by taking $\bar{Y} = 0$ and $\lambda = -v$ to give

$$dY = vY\,dt + \eta Y\,dZ. \tag{33}$$

We will refer to the model in (33) as the lognormal model without mean-reversion, or simply the lognormal model.

We also consider a model where income is normally distributed. This corresponds to choosing $v(y,t) = v, \eta(y,t) = \eta$ and $a(y,t) = y$ or $v(y,t) = vy, \eta(y,t) = \eta y$ and

$a(y,t) = \ln y$. Both specifications give normally distributed income $a(Y_t, t)$ and result in

$$\mathrm{d}Y = v\,\mathrm{d}t + \eta\,\mathrm{d}Z. \tag{34}$$

Notice that in taking normally distributed income, we violate the assumption made earlier that income $a(Y_t, t)$ is non-negative. This was a technical assumption needed for the existence of solutions in the general case. In the case of normally distributed income, one can derive the value function and other quantities of interest directly, without the need for an application to the general theory.

Prior research by Duffie and Jackson (1990), Svensson and Werner (1993) and Teplá (2000) also treats normally distributed income, differing because their income is received at the investment horizon. We compare our results with theirs and note the differences.

4.1. Perfect correlation between stock returns and income state variable

We can compute explicitly the certainty equivalent value of the future income stream $C(t, Y_t)$ when correlation is perfect, under the lognormal model with and without mean-reversion and the normal model. Note that for the lognormal model with mean-reversion (31), under $\mathbb{P}^C$,

$$\mathrm{d}Y = \lambda^C(\bar{Y}^C - Y_t)\,\mathrm{d}t + \eta Y \rho\,\mathrm{d}B^C,$$

where $\rho^2 = 1$, $\hat{c} = [(\mu - r)/\sigma]\rho\eta$, $\lambda^C = \lambda + \hat{c}$, and $\bar{Y}^C = \lambda\bar{Y}/\lambda^C$. Define $D_m(t) = \int_t^T \mathrm{e}^{-m(T-u)}\,\mathrm{d}u$.

Proposition 4.1. *Under the lognormal model with mean-reversion* (31), *the certainty equivalent value of future stochastic income when correlation is perfect is*

$$C(t, Y_t) = \mathbb{E}\left(\int_t^T \pi_u^C Y_u\,\mathrm{d}u\right) = (Y_t - \bar{Y}^C)D_{\lambda^C+r}(t) + \bar{Y}^C D_r(t). \tag{35}$$

Under the lognormal model (33), *the certainty equivalent value of future stochastic income when correlation is perfect is*

$$C(t, Y_t) = \mathbb{E}\left(\int_t^T \pi_u^C Y_u\,\mathrm{d}u\right) = D_{-\delta}(t)Y_t, \tag{36}$$

with $\delta = [v - r - [((\mu - r)\rho\eta)/\sigma]]$ *and* $\rho^2 = 1$.

Under normally distributed income, the certainty equivalent value of future stochastic income when correlation is perfect is

$$C(t, Y_t) = \mathbb{E}\left(\int_t^T \pi_u^C Y_u\,\mathrm{d}u\right) = \left[Y_t + \frac{\left(v - \frac{(\mu-r)\rho\eta}{\sigma}\right)}{r}\right]D_r(t) - \left(\frac{v - \frac{(\mu-r)\rho\eta}{\sigma}}{r}\right)(T-t)\mathrm{e}^{-r(T-t)}. \tag{37}$$

By putting $\bar{Y} = 0$ and $\lambda = -v$, the certainty equivalent value for the lognormal model with mean-reversion (35) collapses to (36). As expected, these two values are always positive, increasing in the initial value of income state variable and the remaining time to the investment horizon $T - t$. Note that when the income is normally distributed, Y_t may be negative (at least theoretically) and if so, the certainty equivalent value of income may be negative. This could correspond to a situation where the investor could pay out as well as receive income.

Analogously to the argument in Section 3.1, the value function when correlation is perfect is given by the Merton value function with modified initial endowment.

4.1.1. The optimal portfolio choice

Following Section 3.1.1, we can obtain an expression for the optimal portfolio by comparing our stochastic income problem to one without income but with an adjusted initial endowment. This enables the following result to be shown as a special case of Proposition 3.1.

Proposition 4.2. *Under the lognormal model* (*with or without mean-reversion*) *with perfect correlation between the stock and the income state variable*, *the hedge demand is given by*

$$\theta^*_{\mathrm{H}} = -\frac{\rho C_y \eta Y_t}{\sigma}; \quad |\rho| = 1, \tag{38}$$

where for the model with mean-reversion (31), $C_y = D_{\lambda^C + r}(t)$ *and for the model without mean-reversion* (33), $C_y = D_{-\delta}(t)$. *Under normally distributed income with perfect correlation between the stock and income state variable*, *the hedge demand is given by*

$$\theta^*_{\mathrm{H}} = -\frac{\rho\eta}{\sigma} D_r(t); \quad |\rho| = 1. \tag{39}$$

For each of the models considered, the hedge demand is negative when correlation is one and positive when $\rho = -1$. Less of the stock is held as hedge when the income state variable is perfectly positively correlated with the stock. The hedge demand is larger in magnitude if the initial income state variable is larger, volatility of income larger, volatility of stock smaller, and the investment horizon is further away. The hedge demand under the model with mean-reversion does not depend on the mean-reversion level $\bar{Y}$, but does depend on the reversion speed λ.

4.2. Imperfect correlation between stock returns and income state variable

In the incomplete case under the lognormal model without mean-reversion, the expression for Φ in Section 3.2 simplifies to

$$\Phi = -\gamma Y_t \int_t^T \mathrm{e}^{r(T-u)} \mathrm{e}^{\eta Z_{u-t} + (v - 1/2\eta^2)(u-t)} \, \mathrm{d}u = -\gamma Y_t \psi_t. \tag{40}$$

This is a key observation, as the form of Φ above allows us to perform calculations which could not be done without the lognormality assumption. The solutions for quantities $\tilde{v}(t,y), g(\tau,y)$ and $V(t,X_t,Y_t)$ are given by (24), (26) and (27) with Φ now defined in (40).

In the incomplete case with normally distributed income, the value function in Proposition 3.2 can be calculated to be

$$\begin{aligned} V(t,X_t,Y_t) = & -\frac{1}{\gamma}\mathrm{e}^{-\gamma X_t \mathrm{e}^{r(T-t)}}\mathrm{e}^{-[(\mu-r)^2/2\sigma^2](T-t)}\mathrm{e}^{-\gamma Y_t \mathrm{e}^{r(T-t)}D_r(t)} \\ & \times \mathrm{e}^{\gamma[(\nu-((\mu-r)\rho\eta)/\sigma)r]\left[(T-t)-\mathrm{e}^{r(T-t)}D_r(t)\right]} \\ & \times \mathrm{e}^{1/2(1-\rho^2)(\gamma^2\eta^2/r^2)[-\mathrm{e}^{2r(T-t)}D_{2r}(t)+(T-t)+r\mathrm{e}^{2r(T-t)}D_r^2(t)]}. \end{aligned} \tag{41}$$

4.2.1. Optimal portfolio choice

For the three models, the optimal portfolio choice can be simplified from (28) to be the Merton component plus the hedge demand given as

$$\theta^*_{\mathrm{H}} = \frac{g_y \eta Y_t \rho}{g\gamma\sigma \mathrm{e}^{r(T-t)}} \tag{42}$$

for the lognormal model with and without mean-reversion, and

$$\theta^*_{\mathrm{H}} = -\frac{\eta\rho}{\sigma}D_r(t) \tag{43}$$

for the normal model.

As expected from Proposition 3.4, θ^*_{H} for the normal model has the opposite sign to that of correlation, and for all models is zero if either ρ or η are zero. Of course, the definition of g and hence the hedge demand differs between the two lognormal type specifications for income.

The hedge demand in (43) is very similar to that in Duffie and Jackson (1990) and Teplá (2000). These authors consider normally distributed income which is received at the investment horizon T. In fact, the difference is attributable purely to discounting terms which adjust for the income timing difference.

Using similar arguments to those in Section 3.2.1, we can show each θ^*_{H} approaches the corresponding perfect correlation hedge in Proposition 4.2 as $|\rho| \to 1$.

4.2.2. Comparative statics

Under the assumption of normally or lognormally distributed income with or without mean-reversion, we can make more progress with comparative statics and the economic consequences of the optimal portfolio choice θ^*. Table 1 summarizes the results of this section.

We can examine the effect of risk aversion on the investor's hedging demand. Recall the interpretation of the Merton component: investors hold a smaller amount (long or short) in the risky asset as they become more risk averse.

Proposition 4.3. *In the model with lognormally distributed income without mean-reversion, the magnitude of the intertemporal hedging component decreases with risk aversion. As $\gamma \to \infty$, the hedging demand approaches zero. Conversely, as $\gamma \to 0$, hedging demand approaches ∞ (for negative correlation) or $-\infty$ (positive correlation).*

Table 1
A summary of comparative statics results in Section 4.2.2.

	Normal (34)	Lognormal (33)	Lognormal with mean-reversion (31)
γ	$\uparrow\gamma \rightarrow \lvert\theta^*_{\mathrm{M}}\rvert\downarrow$ θ^*_{H} independent of γ If $\rho, \mu - r$ have opp. sign, $\lvert\theta^*\rvert\downarrow$ Else θ^* not monotonic	$\uparrow\gamma \rightarrow \lvert\theta^*_{\mathrm{M}}\rvert\downarrow, \lvert\theta^*_{\mathrm{H}}\rvert\downarrow$ If $\rho, \mu - r$ have opp. sign, $\lvert\theta^*\rvert\downarrow$ Else θ^* not monotonic	$\uparrow\gamma \rightarrow \lvert\theta^*_{\mathrm{M}}\rvert\downarrow$
ρ	θ^*_{M} : no dependence $\lvert\theta^*_{\mathrm{H}}\rvert\uparrow$ in $\lvert\rho\rvert$ θ^*_{H} proportional to ρ If $\rho, \mu - r$ have opp. sign, $\lvert\theta^*\rvert\uparrow$ in $\lvert\rho\rvert$	Same as for normal, but θ^*_{H} not prop.	θ^*_{M} : no dependence
Unhed. var.	θ^*_{M} : no dependence $\lvert\theta^*_{\mathrm{H}}\rvert\downarrow$ with unhed. var If $\rho, \mu - r$ have opp. sign, $\lvert\theta^*\rvert\downarrow$ with unhed. var Also $\uparrow$ vol $\lvert\theta^*_{\mathrm{H}}\rvert\uparrow$	Same as for normal but no result on vol.	θ^*_{M}: no dependence
Time, t	$\hat{\theta}^*_{\mathrm{M}}$: constant $\lvert\hat{\theta}^*_{\mathrm{H}}\rvert\downarrow$ over time If $\rho, \mu - r$ have opp. sign, $\lvert\hat{\theta}^*\rvert\downarrow$ over time	Same as for normal	$\hat{\theta}^*_{\mathrm{M}}$: constant $\hat{\theta}^*_{\mathrm{H}}$ not monotonic
Inv. hor., T	$\hat{\theta}^*_{\mathrm{M}}$: constant $\lvert\hat{\theta}^*_{\mathrm{H}}\rvert\uparrow$ with T If $\rho, \mu - r$ have opp. sign, $\lvert\hat{\theta}^*\rvert\uparrow$ with T	$\hat{\theta}^*_{\mathrm{M}}$: constant	$\hat{\theta}^*_{\mathrm{M}}$: constant

Interestingly, in the model with normally distributed income, the intertemporal hedging component is independent of risk aversion.

Corollary 4.4. *Assume income is either normally or lognormally distributed without mean-reversion. If the risk premium and correlation have opposite signs, the optimal portfolio in the stock, θ^* is decreasing in magnitude with risk aversion. That is, if correlation between stock returns and the income state variable is negative, and risk premium positive, the optimal portfolio in the stock, θ^* is positive and decreasing with risk aversion. Likewise, if correlation between stock returns and income state variable is positive, and risk premium negative, the optimal portfolio θ^* is negative and decreasing in magnitude with risk aversion.*

The proofs of Proposition 4.3 and Corollary 4.4 are in the appendix. In these special combinations where correlation and the risk premium of the stock have opposite signs, both the Merton component and the hedge demand are of the same sign and behave similarly with change in risk aversion. In this case, investors with higher risk aversion reduce the magnitude of their cash holdings in the risky asset, whether long or short. These special cases are consistent with the static participation and calibration theorem discussed in the beginning of Section 3.

However, if correlation and risk premium have the same sign, either sign could result for the optimal portfolio and the investor may prefer to be long or short the risky asset. Likewise, the optimal portfolio θ^* may not be monotonic in γ. It may be decreasing with γ at low values of risk aversion, and switch to be increasing with γ at high values. We argue this in the appendix, following the proof of Proposition 4.3.

This violates the simple static results of Huang and Litzenberger (1988) and illustrates the more complex behavior that occurs in dynamic models. Non-monotonic behavior of the hedge with respect to risk aversion was also observed by Liu (2001b) in a stochastic volatility model.

To see why the hedge demand is independent of risk aversion in the case of normally distributed income, we write future income under the normal model as

$$\int_t^T Y_u \,\mathrm{d}u = (T-t)Y_t + \int_t^T (T-u)v \,\mathrm{d}u + \int_t^T (T-u)\eta\rho \,\mathrm{d}B_u + \int_t^T (T-u)\eta\sqrt{1-\rho^2}\,\mathrm{d}W_u.$$

The first two terms are known with certainty today and therefore are not hedged. The third term is a stochastic integral with respect to B, the traded Brownian motion. It can be hedged via the stock P, and this hedge does not depend on risk aversion since it is perfect. The final term is in fact independent of the stock and so cannot be hedged. Combining the effect of all the terms, overall, under a normal model for income, the hedge demand does not depend on risk aversion. This is not true under alternative models for income since the corresponding final term above would not be independent of P and some hedging could be done. This hedging would depend on risk aversion.

Despite the fact that the hedge demand is independent of risk aversion, the static participation and calibration theorems are again violated unless the risk premium and correlation have opposite signs. Otherwise, as in the lognormal model, non-monotonic behavior with respect to risk aversion can result. We now examine the effect of correlation between the stock returns and the income state variable on the optimal portfolio choice. Recall the Merton component of the portfolio choice does not depend on correlation. Also recall the hedge demand is zero if stock returns and the income state variable are uncorrelated.

Proposition 4.5. *Under the lognormal model for income without mean-reversion, the hedge demand θ^*_{H} is decreasing in correlation. Since θ^*_{H} and ρ have opposite signs, the magnitude of the hedge demand is increasing in the magnitude of correlation. If the risk premium and correlation have opposite signs, the magnitude of the optimal portfolio in* *the stock, θ^* is also increasing in the magnitude of correlation. Under the normal model for income, the hedge demand θ^*_{H} is proportional to correlation, ρ.*

The intuition behind Proposition 4.5 is that more changes in income can be hedged with a larger magnitude of correlation. To hedge more changes in income therefore requires a hedging component of greater magnitude. Perfect correlation allows a perfect hedge for income. The proposition is proved in the appendix in the case of lognormal income. For normal income, this is verified by inspection in (43). We can also conclude the magnitude of the optimal portfolio in the stock, θ^* is increasing in the magnitude of correlation when the risk premium and correlation have opposite signs. We can study the effect of increased unhedgeable variance of the income state variable on the investor's portfolio choice. Total variance of income $\eta^2 Y^2$ can be broken up into a hedgeable $\rho^2\eta^2 Y^2$ and unhedgeable part $(1-\rho^2)\eta^2 Y^2$. An increase in the unhedgeable variance can also be thought of as an increase in background risk. We consider the effect of a decrease in the magnitude of correlation $|\rho|$ whilst fixing $\eta^2 Y^2$, total variance. This reduces the hedgeable and increases the unhedgeable variance as a proportion of the total variance. We have the following corollary which uses Proposition 4.5.

Corollary 4.6. *Under the normal or lognormal model for income without mean-reversion, an increase in the unhedgeable variance as a proportion of total variance results in the magnitude of the hedge demand being reduced. If the risk premium and correlation have opposite signs, the magnitude of the optimal portfolio in the stock, θ^* is also decreasing in unhedgeable variance.*

Intuitively, the income becomes more like the risky asset so the investor needs to hold less of the risky asset. Our result is related to that shown by Koo (1995) and Elmendorf and Kimball (2000) for DARA and decreasing absolute prudence and illustrated in Viceira's (2001) stochastic income model. He obtains that increasing idiosyncratic risk reduces stock allocation in a model with CRRA. Note that in our model, the magnitude of the optimal portfolio itself is only decreasing in unhedgeable variance when the risk premium and correlation have opposite signs.

In the special case of normally distributed income, it is straightforward to determine the effect of the volatility of income η on hedge demand.

Proposition 4.7. *Under the normal model for income, an increase in volatility of income results in the magnitude of hedge demand increasing.*

We now investigate properties of the optimal portfolio choice over time. From earlier, the Merton component of the portfolio, when expressed in units of time T money, is constant over time. We can express the hedging demand θ^*_{H} in (42) and (43) for all three models also in terms of time T money, denoted $\hat{\theta}^* = \theta^*_{\mathrm{H}}e^{r(T-t)}$. Note that when the investment horizon T is reached, the hedge (in T money) is zero. This can be seen simply in the lognormal case from the expression for g and its derivative and directly from (43) for the normal model.

Proposition 4.8. *Under normally and lognormally distributed income without mean-reversion, the hedge demand (in terms of T money) decreases in magnitude over time to* *zero at the investment horizon. Under these models, if correlation between stock returns and income is positive,* $\hat{\theta}^*_{\mathrm{H}}$ *is concave in time, and convex if correlation is negative. If the risk premium and correlation have opposite signs, the magnitude of the optimal portfolio* $\hat{\theta}^* = \theta^* e^{r(T-t)}$ *also decreases over time. Under the lognormal model with mean-reversion, the hedge demand (in T money) may not be monotonic in time.*

The proof of Proposition 4.8 for the lognormal models is in the appendix, and can be observed from (43) for the normal model. The model suggests in the case where risk premium and correlation between stock and income have opposite signs, and there is no mean-reversion, investors should reduce the (absolute) cash amount held in the stock over the time horizon of the investment. Outside this case, when either the risk premium and correlation have the same sign and there is no mean-reversion, or when the model includes mean-reversion, hedge demand and the optimal portfolio may not be monotonic in time. In the case of the model with mean-reversion, Appendix A.5 gives the derivative as

$$\frac{\partial}{\partial t}\hat{\theta}^*_{\mathrm{H}} = \frac{\eta Y_t \rho}{\sigma} e^{r(T-t)} + \hat{\theta}^*_{\mathrm{H}}(\lambda + \eta^2/2).$$

The two terms in the above expression have opposite signs. When $\rho < 0$, the hedge demand may decrease with time if the first term dominates, else if λ is very large, the second term may cause the hedge demand to increase with time.

Popular advice given by investment professionals is generally to reduce the proportion of assets in stocks as investors approach retirement. Campbell and Viceira (2002) and importantly, the background risk literature also conclude that investors should buy less stocks over time. This is consistent with the results obtained in our model only if there is no mean-reversion, and if correlation and the risk premium have opposite signs. Otherwise, our model does not give such clear cut conclusions. For instance, specifications of the model with mean-reversion do not lead directly to recommendations that investors reduce the cash amount in the stock over time. We can also obtain results concerning the investment horizon in the case of normally distributed income.

Proposition 4.9. *Under the model with normally distributed income, the magnitude of hedge demand (in terms of T money) increases as T increases. If correlation is positive, hedge demand is concave in T, and convex if correlation is negative. If the risk premium and correlation have opposite signs, the magnitude of the optimal portfolio in the stock,* $\hat{\theta}^* = \theta^* e^{r(T-t)}$ *also increases with investment horizon.*

We see that if the investment horizon is longer, the size of the hedge demand is greater, whether long or short. More hedging is required over a longer horizon. This is consistent with recent research with time-varying investment opportunities resulting in portfolio rules with hedging components whose magnitude depends on the investment horizon. For example, Liu (2001b) finds that the magnitude of hedge demand increases with T under a stochastic volatility model.

However, Proposition 4.9 tells us the magnitude of the optimal portfolio in the stock, θ^* is only increasing in the investment horizon in the case where the risk premium and correlation have opposite sign. This differs from the criteria of Kim and Omberg (1996) who find under a mean-reverting model with CRRA and $R > 1$ that the optimal portfolio increases with T if the risk premium is positive. Here R denotes constant relative risk aversion. Wachter (2002) extends the notion of investment horizon in a model with consumption and finds a related result.

4.2.3. The certainty equivalent value of future stochastic income

Under all three models considered, Proposition 3.5 gives the form of the certainty equivalent value of income. For the lognormal case without mean-reversion, Φ is given in (40) and can be used to simplify the expression. As shown in Section 3.2.3 for the general model, we can verify directly under the lognormal model without mean-reversion, the certainty equivalent value p approaches the perfect correlation value given by $C(t, Y_t)$ in (36) as $|\rho| \to 1$. In the case of normally distributed income, we have the result

Proposition 4.10. *Under normally distributed income, the certainty equivalent value of future stochastic income is given by*

$$p = C(t, Y_t) - \frac{1}{2}(1-\rho^2)\gamma\frac{\eta^2}{r^2}[-e^{2r(T-t)}D_{2r}(t) + (T-t) + re^{2r(T-t)}D_r^2(t)]. \tag{44}$$

where $C(t, Y_t)$ is given in Proposition 4.1.

Immediately we see that as $|\rho| \to 1$, the certainty equivalent value in (44) tends to $C(t, Y_t)$ the perfect correlation value given in (37). In addition, the value in (44) is always lower than $C(t, Y_t)$ the perfect correlation value given in (37). To see this, the term in square brackets in (44) can be shown to be positive for all values of r. Recall we could also observe this in the lognormal model.

We may also compare the certainty equivalent value p to that obtained by Duffie and Jackson (1990) and Teplá (2000). Although their investors receive income only at the investment horizon T, there are similarities between her selling price and our certainty equivalent value.[8]

5. Concluding remarks

In contrast to many recent papers concerning optimal portfolio choice problems which resort to numerical techniques, this paper demonstrates that closed form solutions (via novel transformations of the HJB equation) can be found for a model with stochastic income under CARA. Additionally, a number of interesting comparative statics results were derived as a by-product of this solution. For instance, we can deduce under a Markovian income process, the investor takes advantage of the hedging properties of the stock by choosing a hedge position which is long if correlation is negative, and short if correlation is positive. Their optimal portfolio in the stock is long stock if correlation is negative and the risk premium on

[8] In fact, if we alter our model so income is received only at T, we obtain exactly the selling price of Teplá (2000) under normally distributed income. Details are available upon request.

the stock is positive. Similarly, if correlation is positive and risk premium negative, the investor's optimal portfolio is short stock.

Once we specialize to a number of models for income, stronger conclusions could be drawn on the relationship between various parameters and the investor's optimal portfolio in the stock. The key to making a positive conclusion concerning the effect of risk aversion, unhedgeable variance of income, time and correlation on the optimal portfolio are the signs of the risk premium and correlation. Positive conclusions could only be drawn if these had opposite signs. For example, when income is normally or lognormally distributed, if correlation and the risk premium have the same sign, the optimal portfolio may not be monotonic in risk aversion. This demonstrates behavior outside the simple conclusions of Huang and Litzenberger (1988) that the optimal portfolio of risk averse investors decreased in magnitude with risk aversion.

Under normally or lognormally distributed income, and when correlation and the risk premium have opposite signs, the model predicts investors should reduce the (absolute) cash amount held in the stock over the investment horizon. However, if income is lognormal with mean-reversion or if the sign condition on correlation and the risk premium is not satisfied, then the optimal portfolio is not necessarily monotonic in time. The advice often given by investment professionals to invest less in stocks over time, is backed up by certain special cases of our model but not others, suggesting that this is a somewhat simplified strategy which may not be appropriate when investors have particular income sources.

In this paper, we have assumed the investor cannot consume wealth during the time period $(t, T]$. Such consumption would not change the nature of our results, and its exclusion has enabled us to consider the effect of the investment horizon on portfolio choice within our model, which would be more difficult in a model with consumption, see Wachter (2002) for a discussion. In addition, if we were to incorporate consumption, it is unlikely that the model could be solved analytically. For example, Wachter (2002), Liu (2001a) and Zariphopoulou (1999) all extend their models with mean-reverting stock price dynamics (but no income) to include consumption, at the expense of requiring a complete market to obtain closed form solutions. It seems portfolio choice models incorporating both incompleteness and consumption require numerical solutions (see Viceira, 2001). The explicit nature of our solutions would also be lost if we were to require other distributions for the risky stock P.

Acknowledgements

The author would like to thank Takashi Ohmoto, T. Uchiyama (both of Nomura Securities, Tokyo) and Lucie Teplá for useful discussions.

Appendix A

The appendix proves a number of results contained in the body of the paper.

A.1. Proof of Proposition 3.1

The sign of θ^*_{H} in (21) clearly depends on the sign of ρC_y. We show the sign of C_y and a_y are the same. The certainty equivalent value of income is given as

$$C(t, Y_t) = \mathbb{E}\left(\int_t^T \pi_u^C a(Y_u, u)\,\mathrm{d}u\right) = \mathbb{E}^C \int_t^T \mathrm{e}^{-r(u-t)} a(Y_u, u)\,\mathrm{d}u,$$

where Y satisfies $\mathrm{d}Y = (v(t, Y_t) - [(\mu - r)\rho/\sigma]\eta(t, Y_t))\,\mathrm{d}t + \eta(t, Y_t)\rho\,\mathrm{d}B^C$ with $|\rho| = 1$ under $\mathbb{P}^C$.

We assume the stochastic differential equation for Y has a weak solution which is unique in law, and the solution has the strong Markov property. Take two independent realizations of Y started at $Y_t^{(1)}$, $Y_t^{(2)}$ with $Y_t^{(2)} > Y_t^{(1)}$. Define $\tau = \inf_{u \geqslant t}\{u : Y_u^{(2)} = Y_u^{(1)}\}$ and

$$\hat{Y}_r = \begin{cases} Y_r^{(1)}, & r < \tau, \\ Y_r^{(2)}, & r \geqslant \tau. \end{cases}$$

Now $Y_r^{(2)} \geqslant \hat{Y}_r$ by construction and $\hat{Y}_r \overset{\text{law}}{=} Y_r^{(1)}$. Then if $a_y > 0$,

$$\mathbb{E}^c \int_t^T \mathrm{e}^{-r(u-t)} a(Y_u^{(2)}, u)\,\mathrm{d}u \geqslant \mathbb{E}^c \int_t^T \mathrm{e}^{-r(u-t)} a(Y_u^{(1)}, u)\,\mathrm{d}u$$

and so $C(t, Y_t^{(2)}) \geqslant C(t, Y_t^{(1)})$. The inequality is reversed if $a_y < 0$. The sign of θ^*_{H} depends on the sign of ρa_y.

A.2. Proof of Proposition 3.4

Note first that all terms in $\theta^*_{\mathrm{H}} = (g_y \eta(Y, t)\rho)/(g\gamma\sigma \mathrm{e}^{r(T-t)})$ were assumed positive, apart from g, g_y and the correlation which may take either sign. From (26), we know $g > 0$. The derivative term is more complicated. We rewrite (26) as

$$g(T - t, Y_t) = \mathrm{e}^{-((\mu-r)^2/2\sigma^2)(T-t)}[\mathbb{E}^I \mathrm{e}^{-\tilde{\alpha}\gamma \int_t^T \mathrm{e}^{r(T-u)} a(Y_u, u)\,\mathrm{d}u}]^{1/\tilde{\alpha}},$$

where $\mathrm{d}Y = (v(Y, t) - ((\mu - r)\rho\eta(Y, t))/\sigma)\,\mathrm{d}t + \eta(Y, t)\,\mathrm{d}Z^I$ under $\mathbb{P}^I$.

Following similar arguments to the proof of Proposition 3.1, if $a_y > 0$,

$$\mathbb{E}^I \mathrm{e}^{-\tilde{\alpha}\gamma \int_t^T \mathrm{e}^{r(T-u)} a(Y_u^{(1)})\,\mathrm{d}u} \geqslant \mathbb{E}^I \mathrm{e}^{-\tilde{\alpha}\gamma \int_t^T \mathrm{e}^{r(T-u)} a(Y_u^{(2)})\,\mathrm{d}u}$$

and $g(T - t, Y_t^{(1)}) \geqslant g(T - t, Y_t^{(2)})$. The inequality is reversed if $a_y < 0$. We have shown if $a_y > 0$, $g_y < 0$, therefore the sign of θ^*_{H} depends on the sign of ρa_y.

A.3. Proof of Proposition 4.3 and Corollary 4.4

Differentiating θ_{H}^* in (42) with respect to γ gives

$$\frac{\partial}{\partial\gamma}\theta_{\mathrm{H}}^* = \frac{\eta\rho Y_t}{\sigma \mathrm{e}^{r(T-t)}}\frac{1}{g\gamma}\left\{\frac{\partial}{\partial\gamma}(g_y) - \frac{g_y}{g\gamma}\left(\gamma\left(\frac{\partial}{\partial\gamma}g\right)+g\right)\right\},$$

where $g(\tau, y)$ is given in (26) with Φ defined in (40).

The sign of the front term outside the brackets depends only on the sign of the correlation, we investigate the term in the brackets.

First, under lognormally distributed income, (24) is defined by

$$\tilde{v}(t,y) = \mathbb{E}^I \mathrm{e}^{-\tilde{\alpha}\gamma\int_t^T \mathrm{e}^{r(T-u)} Y_u\,\mathrm{d}u}.$$

Define $Z(t,y) = \mathbb{E}^I(\int_t^T \mathrm{e}^{r(T-u)}\frac{Y_u}{Y_t}\mathrm{d}u)\exp(-\tilde{\alpha}\gamma\int_t^T \mathrm{e}^{r(T-u)} Y_u\,\mathrm{d}u)$. Differentiation gives

$$g_y = -\frac{\gamma g}{\tilde{v}(t,y)}Z(t,y),$$
$$\frac{\partial}{\partial\gamma}g = -\frac{gY_t}{\tilde{v}(t,y)}Z(t,y),$$

and

$$\frac{\partial}{\partial\gamma}g_y = \frac{1}{\tilde{v}(t,y)}\left[\frac{\rho^2\gamma g Y_t Z(t,y)^2}{\tilde{v}(t,y)} - gZ(t,y) + \gamma g\tilde{\alpha}Y_t\mathbb{E}^I\left(\int_t^T \mathrm{e}^{r(T-u)}\frac{Y_u}{Y_t}\,\mathrm{d}u\right)^2\exp\left(-\tilde{\alpha}\gamma\int_t^T \mathrm{e}^{r(T-u)}Y_u\,\mathrm{d}u\right)\right].$$

Now

$$\left\{\frac{\partial}{\partial\gamma}(g_y) - \frac{g_y}{g\gamma}\left(\gamma\left(\frac{\partial}{\partial\gamma}g\right)+g\right)\right\} = \frac{\tilde{\alpha}\gamma Y_t g}{\tilde{v}(t,y)}\left[\mathbb{E}^I\left(\int_t^T \mathrm{e}^{r(T-u)}\frac{Y_u}{Y_t}\,\mathrm{d}u\right)^2\exp\left(-\tilde{\alpha}\gamma\int_t^T \mathrm{e}^{r(T-u)}Y_u\,\mathrm{d}u\right) - \frac{Z(t,y)^2}{\tilde{v}(t,y)}\right].$$

Letting $A = \int_t^T \mathrm{e}^{r(T-u)}(Y_u/Y_t)\,\mathrm{d}u$ and $X = \mathrm{e}^{-\tilde{\alpha}\gamma\int_t^T \mathrm{e}^{r(T-u)}Y_u\,\mathrm{d}u}$ then $Z(t,y) = \mathbb{E}^I AX$ and $\tilde{v}(t,y) = \mathbb{E}^I X$ giving

$$\frac{\tilde{\alpha}\gamma Y_t g}{(\mathbb{E}^I X)^2}[\mathbb{E}^I A^2 X\mathbb{E}^I X - (\mathbb{E}^I AX)^2].$$

Letting $U = \sqrt{X}$, $V = A\sqrt{X}$ we see

$$\mathbb{E}^I V^2\mathbb{E}^I U^2 - (\mathbb{E}^I UV)^2 \geqslant 0$$

by the Cauchy–Schwartz inequality, so $(\partial/\partial\gamma)\theta_{\mathrm{H}}^* \geqslant 0$ if $\rho > 0$ and $(\partial/\partial\gamma)\theta_{\mathrm{H}}^* \leqslant 0$ if $\rho < 0$, proving Proposition 4.3.

If we combine the Merton component and the hedge demand, we see

$$\frac{\partial}{\partial\gamma}\,\theta^* = \frac{\eta\rho Y_t}{\sigma e^{r(T-t)}}\,\frac{1}{g\gamma}\left\{-\frac{(\mu-r)g}{\eta\rho Y_t\gamma\sigma} + \frac{\partial}{\partial\gamma}\,(g_y) - \frac{g_y}{g\gamma}\left(\gamma\left(\frac{\partial}{\partial\gamma}\,g\right)+g\right)\right\}.$$

If ρ and $\mu - r$ have opposite signs, the additional term $-((\mu-r)g)/\eta\rho Y_t\gamma\sigma$ contributes positively, and $(\partial/\partial\gamma)\theta^* \geqslant 0$ if $\rho > 0$ and $(\partial/\partial\gamma)\theta^* \leqslant 0$ if $\rho < 0$. This proves Corollary 4.4.

However, if ρ and $\mu - r$ have the same sign, the additional term may be negative, and if negative and large enough could cause the result of Corollary 4.4 to reverse. Whether this occurs will depend on the size of the additional term. For very small γ, the term can be large and the optimal portfolio choice may be decreasing in γ for $\rho > 0$ and $\mu - r > 0$. For large γ, the additional term may be small and the optimal portfolio choice increasing in γ.

A.4. Proof of Proposition 4.5

From (26) and (40) we see $g_y/g = 1/\tilde{\alpha}\tilde{v}_y/\tilde{v}$. Now

$$\frac{\partial}{\partial\rho}\,\theta^*_{\mathrm{H}} = \frac{\eta Y_t}{\gamma\sigma e^{r(T-t)}}\left[\frac{\rho}{\tilde{\alpha}}\,\frac{\partial}{\partial\rho}\left(\frac{\tilde{v}_y}{\tilde{v}}\right) + \frac{\tilde{v}_y}{\tilde{v}}\,\frac{\partial}{\partial\rho}\left(\frac{\rho}{\tilde{\alpha}}\right)\right].$$

Using

$$\frac{\partial}{\partial\rho}\left(\frac{\tilde{v}_y}{\tilde{v}}\right) = -\frac{\tilde{v}_y\tilde{v}_\rho}{\tilde{v}^2} + \frac{(\tilde{v}_y)_\rho}{\tilde{v}},$$

$$\tilde{v}_\rho = 2\gamma\rho Y_t Z(t,y),$$

$$(\tilde{v}_y)_\rho = -\gamma[\tilde{\alpha}Z_\rho - 2\rho Z(t,y)],$$

$$Z_\rho = 2\rho\gamma Y_t \mathbb{E}^I\left(\int_t^T e^{r(T-u)}\,\frac{Y_u}{Y_t}\,du\right)^2 \exp\left(-\tilde{\alpha}\gamma\int_t^T e^{r(T-u)}\,Y_u\,du\right)$$

gives

$$\begin{aligned}\frac{\partial}{\partial\rho}\,\theta^*_{\mathrm{H}} &= \frac{\eta Y_t}{\tilde{v}\sigma e^{r(T-t)}}\left\{\frac{2\rho^2\gamma Y_t Z(t,y)^2}{\tilde{v}} - Z(t,y)\right.\\ &\qquad\left. - 2\rho^2\gamma Y_t \mathbb{E}^I\left(\int_t^T e^{r(T-u)}\,\frac{Y_u}{Y_t}\,du\right)^2 \exp\left(-\tilde{\alpha}\gamma\int_t^T e^{r(T-u)}\,Y_u\,du\right)\right\}\\ &= -\frac{\eta Y_t}{\sigma e^{r(T-t)}(\mathbb{E}^I X)^2}\{\mathbb{E}^I AX\mathbb{E}^I X + 2\rho^2\gamma Y_t(\mathbb{E}^I A^2X\mathbb{E}^I X - (\mathbb{E}^I XA)^2)\}.\end{aligned}$$

Again, letting $U = \sqrt{X}$, $V = A\sqrt{X}$ we obtain

$$\frac{\partial}{\partial\rho}\,\theta^*_{\mathrm{H}} = -\frac{\eta Y_t}{\sigma e^{r(T-t)}(\mathbb{E}^I U^2)^2}[\mathbb{E}^I U^2\mathbb{E}^I UV + 2\rho^2\gamma Y_t(\mathbb{E}^I U^2\mathbb{E}^I V^2 - (\mathbb{E}^I UV)^2)].$$

Both terms in square brackets are positive by Cauchy–Schwartz, so

$$\frac{\partial}{\partial\rho}\,\theta^*_{\mathrm{H}} \leqslant 0.$$

A.5. Proof of Proposition 4.8

Recall

$$\hat{\theta}^*_{\mathrm{H}} = \frac{\eta Y_t \rho}{\gamma\sigma}\,\frac{g_y}{g},$$

where $g(\tau, y)$ is defined in (26) for the lognormal model with or without mean-reversion. If there is no mean-reversion, Φ is defined in (40).

Differentiation with respect to t gives

$$\frac{\partial}{\partial t}\,\hat{\theta}^*_{\mathrm{H}} = \frac{\eta Y_t \rho}{\gamma\sigma}\left(g_y\left(\frac{\partial}{\partial t}\frac{1}{g}\right) + \frac{1}{g}\left(\frac{\partial}{\partial t}\,g_y\right)\right).$$

We see

$$\frac{\partial}{\partial t}\,\frac{1}{g} = -\frac{1}{g}\left[\frac{(\mu - r)^2}{2\sigma^2} + \gamma \mathrm{e}^{r(T-t)}\,Y_t\right] \tag{A.1}$$

for both models. Now for the lognormal model,

$$\frac{\partial}{\partial t} g_y = -\gamma\left\{-g\mathrm{e}^{r(T-t)} + \frac{gZ(t,y)}{\tilde{v}}\left(\frac{(\mu - r)^2}{2\sigma^2} + \gamma \mathrm{e}^{r(T-t)}\,Y_t\right)\right\},$$

where $Z(t, y)$ is defined in the proof of Proposition 4.3. These give

$$\frac{\partial}{\partial t}\hat{\theta}^*_{\mathrm{H}} = \frac{\eta Y_t \rho}{\sigma}\mathrm{e}^{r(T-t)}.$$

Its sign depends only on the sign of ρ. Differentiating again with respect to t gives

$$\frac{\partial^2}{\partial t^2}\,\hat{\theta}^*_{\mathrm{H}} = -\frac{\eta Y_t \rho r}{\sigma}\mathrm{e}^{r(T-t)},$$

which is negative if $\rho > 0$ and positive if $\rho < 0$.

Instead, under the mean-reverting lognormal model for income,

$$\frac{\partial}{\partial t} g_y = -\gamma\left\{-g\mathrm{e}^{r(T-t)} + \frac{gZ^M(t,y)}{\tilde{v}}\left(\frac{(\mu - r)^2}{2\sigma^2} + \gamma \mathrm{e}^{r(T-t)}\,Y_t + (\lambda + \eta^2/2)\right)\right\},$$

with

$$Z^M(t,y) = \mathbb{E}^I\left(\int_t^T \mathrm{e}^{r(T-u)}\mathrm{e}^{-(\lambda+\eta^2/2)(u-t)+\eta^2(Z_u - Z_t)}\,\mathrm{d}u\right)\mathrm{e}^{-(1-\rho^2)\gamma\int_t^T \mathrm{e}^{r(T-u)}\,Y_u\,\mathrm{d}u}.$$

These give

$$\frac{\partial}{\partial t}\hat{\theta}^*_{\mathrm{H}} = \frac{\eta Y_t \rho}{\sigma} \mathrm{e}^{r(T-t)} + \hat{\theta}^*_{\mathrm{H}}(\lambda + \eta^2/2).$$

The two terms above have opposite signs, so under the mean-reverting model, the hedge demand may not be monotonic in time.

References

Bhattacharya, S., 1978. Project valuation with mean-reverting cash flow streams. Journal of Finance 33, 1317–1331.

Bliss, R., Panigirtzoglou, N., 2004. Option implied risk aversion estimates. Journal of Finance 59, 407–446.

Bodie, Z., Merton, R.C., Samuelson, W.F., 1992. Labor supply flexibility and portfolio choice in a life cycle model. Journal of Economic Dynamics and Control 16, 427–449.

Campbell, J.Y., Viceira, L.M., 1999. Consumption and portfolio decisions when expected returns are time-varying. Quarterly Journal of Economics 114, 433–495.

Campbell, J.Y., Viceira, L.M., 2002. Strategic Asset Allocation: Portfolio Choice for Long Term Investors. Oxford University Press, Oxford.

Cairns, A.J.G., Blake, D., Dowd, K., 2003. Stochastic lifestyling: optimal dynamic asset allocation for defined-contribution pension plans. Preprint.

Chan, Y.L., Viceira, L.M., 2000. Asset allocation with endogenous labor income: the case of complete markets. Preprint.

Davis, M.H.A., 2000. Optimal Hedging with Basis Risk. Preprint, Imperial College, London.

Duffie, D., 1996. Dynamic Asset Pricing Theory, second ed. Princeton University Press, Princeton.

Duffie, D., Jackson, M.O., 1990. Optimal hedging and equilibrium in a dynamic futures market. Journal of Economic Dynamics and Control 14, 21–33.

Duffie, D., Zariphopoulou, T., 1993. Optimal investment with undiversifiable income risk. Mathematical Finance 3, 135–148.

Duffie, D., Fleming, W., Soner, H.M., Zariphopolou, T., 1997. Hedging in incomplete markets with HARA utility. Journal of Economic Dynamics and Control 21, 753–782.

Eeckhoudt, L., Gollier, C., Schlesinger, H., 1996. Changes in background risk and risk taking behavior. Econometrica 64, 683–689.

Elmendorf, D.W., Kimball, M.S., 2000. Taxation of labor income and the demand for risky assets. International Economic Review 41, 801–832.

Franke, G., Stapleton, R.C, Subrahmanyam, M.G., 2001. Standard risk aversion and the demand for risky assets in the presence of background risk. Preprint.

Franke, G., Peterson, S., Stapleton, R.C., 2002. Long term portfolio choice given uncertain personal savings. Preprint, EFA 2002.

Gollier, C., Pratt, J.W., 1996. Weak proper risk aversion and the tempering effects of background risk. Econometrica 64, 1109–1123.

He, H., Pearson, N.D., 1991. Consumption and portfolio policies with incomplete markets and short sale constraints: the infinite dimensional case. Journal of Economic Theory 54, 259–304.

Heaton, J., Lucas, D., 2000. Portfolio choice and asset prices: the importance of entrepreneurial risk. Journal of Finance 55, 1163–1198.

Henderson, V., 2002. Valuation of claims on nontraded assets using utility maximization. Mathematical Finance 12, 351–373.

Henderson, V., Hobson, D., 2002. Substitute hedging. Risk 15, 71–75.

Huang, C., Litzenberger, R.H., 1988. Foundations for Financial Economics. Elsevier Science Publishing, New York.

Kahl, M., Liu, J., Longstaff, F.A., 2003. Paper millionaires: how valuable is stock to a stockholder who is restricted from selling it? Journal of Financial Economics 67, 385–410.

Karatzas, I., Lehoczky, J.P., Shreve, S.E., Xu, G., 1991. Martingale and duality methods for utility maximization in an incomplete market. SIAM Journal of Control and Optimization 29, 702–720.

Keppo, J.S., Sullivan M.G., 2003. Modeling the optimal strategy in an incomplete market. Preprint, University of Michigan.

Kim, T.S., Omberg, E., 1996. Dynamic nonmyopic portfolio behavior. Review of Financial Studies 9, 141–161.

Kimball, M.S., 1993. Standard risk aversion. Econometrica 61, 564–589.

Koo, H.K., 1995. Consumption and portfolio selection with labor income I: evaluation of human capital. Preprint, Washington University.

Koo, H., 1998. Consumption and portfolio selection with labor income: a continuous time approach. Mathematical Finance 8, 49–65.

Liu, J., 2001a. Portfolio selection in stochastic environments. Preprint, Anderson School of Management, UCLA.

Liu, J., 2001b. Dynamic portfolio choice and risk aversion. Preprint, Anderson School of Management, UCLA.

Merton, R.C., 1969. Lifetime portfolio selection under uncertainty: the continuous time case. The Review of Economics and Statistics 51, 247–257.

Merton, R.C., 1971. Optimum consumption and portfolio rules in a continuous time model. Journal of Economic Theory 3, 373–413.

Merton, R.C., 1977. A reexamination of the capital asset pricing model. In: Friend, I., Bicksler, J. (Eds.), Risk and Return in Finance, Ballinger Press, Cambridge, MA.

Munk, C., 2000. Optimal consumption/investment policies with undiversifiable income risk and liquidity constraints. Journal of Economic Dynamics and Control 24, 1315–1343.

Rocha, K., Dias, M.A.G., Teixeira, J.P., 2003. The timing of development and the optimal production scale: a real option approach to oilfield E&P. Preprint, No. 981, Catholic University of Rio de Janeiro.

Svensson, L.E.O., Werner, I.M., 1993. Non traded assets in incomplete markets. European Economic Review 37, 1149–1168.

Teplá, L., 2000. Optimal hedging and valuation of nontraded assets. European Finance Review 4, 231–251.

Viceira, L.M., 2001. Optimal portfolio choice for long-horizon investors with nontradeable labor income. Journal of Finance 56, 433–470.

Wachter, J.A., 2002. Portfolio and consumption decisions under mean-reverting returns: an exact solution for complete markets. Journal of Financial and Quantitative Analysis 37, 63–91.

Weil, P., 1994. Nontraded assets and the CAPM. European Economic Review 38, 913–922.

Zariphopoulou, T., 1999. Optimal Investment and consumption models with non-linear stock dynamics. Mathematical Methods of Operations Research 50, 271–296.

Zariphopoulou, T., 2001. A solution approach to valuation with unhedgeable risks. Finance and Stochastics 5, 61–82.