Reducing False Alarms in the Detection of Human Influence on Data

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Benford's law states that the frequency of first significant digits (FSD) in a random sample decreases as those digits increase. While this curious law is increasingly used to test for human influence on data, including corporate fraud and psychological barriers in financial markets, it often produces frustratingly many false positives. I advance toward the goal of understanding and reducing the false alarm rate by showing that Benford's law is inadequate when data are drawn from various common distributions, including the ubiquitous normal distribution. I also prove that Benford's law is obeyed when the untainted data are lognormally distributed with a high variance parameter. In addition, I explain why data sets expressible as the product of two other sets often conform better to Benford's law than either multiplicand data set. The empirical analysis strongly supports these findings.

1. Introduction

Benford's (1938) law is increasingly used in the accounting, auditing, and finance disciplines as a proxy to test the null hypothesis of absence of human influence on data. For instance, it has been variously used as an aid in uncovering corporate fraud and other types of data tampering, as well as to detect psychological price barriers. This surprising law states that the probability of obtaining a first significant digit (FSD) equal to $k = 1, \ldots, 9$ in a random sample is given by $B(k) = \log[(k + 1)/k]$, where log[ ] denotes base-10 logarithms. Thus, there is a 30.1 percent probability that the FSD is 1, but only a 4.6 percent probability that it is 9.[1]

The underlying logic of these tests is that if the observed FSD distribution does not conform to Benford's law, then the data may possibly contain some human

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1. Although I focus solely on first significant digits in this article, Hill (1995a) proves that Benford's law is valid for any positive integer. For example, the probability that the first and second significant digits of a number are 2 and 3, respectively, is $B(23) = \log(24/23) = 1.85$ percent. Nigrini and Mittermaier (1997) apply Hill's generalization to compute the probabilities for each of the first four significant digits.

influence. For instance, Carslaw (1988) analyzes accounting net income data from New Zealand, finding an excess of 0s and a shortage of 9s for the second significant digit, relative to Benford's law. Thomas (1989) reports similar results for U.S. net income data. These findings suggest that companies may sometimes contrive to report net income above a perceived psychological threshold. Thus, if a company's actual net income were, say, $79,930, it might be reported as $80,020, under the belief that such a small increment produces a disproportionately favorable psychological impact on analysts and investors. Varian (1972) advocates using Benford's law to test the reasonableness of economic forecasting models. Nigrini (1996) uses Benford's law to detect tax evasion, finding that it is more prevalent among low-income taxpayers. Nigrini and Mittermaier (1997) illustrate how departures from Benford's law might be used as an auditing tool to uncover corporate fraud given that (p. 55) "Benford's Law would not apply to numbers that are influenced by human thought." Koedijk and Stork (1994) and De Ceuster, Dhaene, and Schatteman (1998) use Benford's law to investigate the presence of psychological barriers in stock market indices. The latter authors emphasize that any such test must recognize the nonuniformity of significant digits. This recognition casts serious doubts on the validity of previous tests that assumed uniformity, for example, Donaldson and Kim (1993) and Ley and Varian (1994). Mitchell (2001) also relies on Benford's law to search for psychological barriers in the futures market.

Although using Benford's law as a proxy for the null represents a significant improvement over the prior proxy of uniformly distributed FSD, several authors warn that it may produce too many type I errors, or false positives, to which I refer as false alarms. In other words, while Benford's law would not apply in the presence of human influence, a lack of human influence in the data is insufficient to generate Benford's law. For example, Hill (1995b, p. 362) notes that "there are many . . . sampling processes which do not satisfy [Benford's] law," and Nigrini (2000, p. 43) cautions that "We should never assume, dogmatically, that the Law *must* apply." However, this deficiency has received little formal attention in the literature. In particular, rigorously determining which common probability distributions obey Benford's law, and which do not, remains a largely unsolved problem.

To illustrate the dangers of placing a dogmatic belief in Benford's law, consider Ley (1996), who analyzes the FSD distribution from a large sample of daily rates of return for the Dow Jones and Standard and Poor's stock indices. Ley argues that "the series of one-day returns on the DJIA and S&P's indexes can both be reasonably classified to belong to [Benford's law]" (p. 312) and then concludes that "the series of one-day returns on the DJIA follows Benford's Law" (p. 313). In spite of these assertions, standard statistical tests strongly reject Benford's law in all eight data groupings presented by Ley. Although Ley acknowledges this statistical reality, he argues, in effect, that it should be disregarded.

In this article I advance our understanding of the false alarm issue by showing that Benford's law is very likely to be rejected for samples drawn from a host of common probability distributions, including the popular normal distribution and certain lognormal distributions. For example, I show that the probability of rejecting

Benford's law in a random sample of 500 observations drawn from the standard normal distribution is about 96 percent, and that it is rejected with virtual certainty for samples of over 1,000 observations. Clearly, whenever such huge false alarm rates occur, the statistical test is effectively rendered useless.

Although Benford's law sometimes fails, it certainly holds under many circumstances. Indeed, Hill (1995b) proves an existence theorem guaranteeing that various mixtures of distributions will yield Benford's law. In particular, he shows that when data are drawn from a mixture of distributions that are generated randomly in some unbiased way, the FSD from the mixture converge to Benford's law, which explains the appearance of this law in many empirical contexts. While this theorem is a powerful result, Hill does not offer practical examples. However, Benford (1938) gives one such example by proving that a certain mixture of uniform distributions obeys his law. Pinkham (1961) gives a more practical general theorem by proving that if continuous data are multiplied by a nonzero constant, the FSD distribution remains invariant if and only if the FSD obey Benford's law.[2] Thus, if there are reasons for believing that the FSD of the data are scale-invariant, they should obey Benford's law. Real-life illustrations of Pinkham's theorem are given by Hill (1995b, p. 360) and Nigrini (2000, p. 27). Raimi (1976) shows that Benford's law also holds for most geometric sequences. Indeed, the relative FSD frequencies from the first hundred powers of 2 have an insignificant 0.37 chi-square statistic when fitted to Benford's law. In practice, Benford's law should hold even when the sorted data only approximate a geometric sequence.

In this article I extend these positive findings by proving that a lognormal distribution with a sufficiently high variance parameter generates a first significant digit distribution that conforms to Benford's law. Thus, if there are compelling arguments for believing that the data are lognormally distributed in the absence of human influence, Benford's law may indeed be a suitable proxy for the null hypothesis of no human influence. Various theoretical considerations may lead to this belief. For example, the common compound process $X_t = X_{t-1}(1 + R_t)$ is asymptotically lognormal under mild conditions, as a consequence of the multiplicative analogue for the additive central limit theorem. Crow and Shimizu (1988, pp. 4–9) discuss other lognormal-generating processes.

On a further positive note, I explain how data sets that are the product of two other data sets may be more likely to conform to Benford's law than either multiplicand data set. Consequently, in the absence of human intervention the product data set may produce fewer false alarms than the multiplicand data sets. As an example, consider data sets for the market value of equity (E), the price per share (P), and number of shares outstanding (N). Because $E = P \times N$, the equity data set may be more likely to conform to Benford's law than either the price or number of shares data sets.

2. Pinkham's (1961, p. 1226) emphasis on the continuity of the data may be understood by considering the set $S = \{1, 2, 3, \ldots, 9\}$ with corresponding probabilities $\{\log[2/1], \log[3/2], \log[4/3], \ldots, \log[10/9]\}$. While Benford's law applies exactly to S, it clearly does not apply to the scaled set $2S = \{2, 4, 6, \ldots, 18\}$ because it has no 3, 5, 7, or 9.

The empirical results are fully consistent with the findings just discussed. Indeed, all the data sets that conform to Benford's law also fit a lognormal distribution with a relatively high variance parameter, while those that do not conform fit either a lognormal with a relatively small variance parameter or are simply not lognormally distributed.

The article proceeds as follows. In Section 2, I discuss the FSD distributions that arise when data are drawn from some common distributions. While Benford's law is often rejected, a notable exception occurs for lognormal distributions with a high variance parameter. I provide a theoretical explanation for this finding in Section 3. In Section 4, I analyze the empirical distribution of FSD for various data sets and discuss their goodness of fit to Benford's law. I summarize in Section 5.

2. First Significant Digit Probabilities from Some Common Distributions

In this section, I examine the FSD probabilities derived from various common distributions and test their conformance to Benford's law. This is an important empirical question because recent research assumes that this law approximates the actual FSD probabilities for a broad spectrum of data, under the null hypothesis of no human influence. However, inasmuch as this assumption is unwarranted, the conclusions may be invalid.

The FSD probabilities for any continuous random variable X with distribution function $F(x)$ and density $f(x)$ can be calculated by noting that the FSD of any observation x equals $k = 1, \ldots, 9$ if and only if $x \in (-\{k+1\}10^i, -k10^i] \cup [k10^i, \{k+1\}10^i)$ for some integer i. This means that the probability that x has k as its first significant digit is found by integrating $f(x)$ over the infinite possible intervals of this form; that is,

$$P(\text{FSD} = k) = \sum_{i=-\infty}^{\infty}\left[\int_{-\{k+1\}10^i}^{-k10^i} f(x)dx + \int_{k10^i}^{\{k+1\}10^i} f(x)dx\right]. \tag{1}$$

For example, to compute $P(\text{FSD} = 2)$ from eq. (1), integrate the density $f(x)$ over the set $\{\ldots \cup (-3 \times 10^{-1}, -2 \times 10^{-1}] \cup [2 \times 10^{-1}, 3 \times 10^{-1}) \cup (-3 \times 10^0, -2 \times 10^0] \cup [2 \times 10^0, 3 \times 10^0) \cup (-3 \times 10^1, -2 \times 10^1] \cup [2 \times 10^1, 3 \times 10^1) \cup \ldots\}$.

The general expression for the FSD probability in eq. (1) can also be written in terms of the distribution function, $F(x)$, as follows:[3]

$$P(\text{FSD} = k) = \sum_{i=-\infty}^{\infty}\Big[F(-k10^i) - F(-\{k+1\}10^i) + F(\{k+1\}10^i) - F(k10^i)\Big]. \tag{2}$$

3. Equation (2) is consistent with eq. (1) in Pinkham (1961, p. 1225).

For some distributions, the FSD probabilities can be evaluated exactly. A simple example is the uniform distribution $U(0, 1)$, for which $F(x) = f(x) = 0$ for $x \in (-\infty, 0)$, $F(x) = x$ and $f(x) = 1$ for $x \in [0, 1)$, and $F(x) = 1$ and $f(x) = 0$ for $x \in [1, \infty)$. Thus, for $i \geq 0$ the summands in eqs. (1) and (2) are zero, and for $i < 0$ the summands reduce to 10^i. Consequently, the summations become infinite geometric series that converge to $P(\text{FSD} = k) = 1/9$ for all $k \in \{1, 2, \ldots, 9\}$.[4] When an exact evaluation of eqs. (1) and (2) is not feasible because a closed-form expression for $F(x)$ is unavailable, for example, for normal distributions, $P(\text{FSD} = k)$ can be computed numerically to the desired accuracy.

I use eq. (2) to calculate the set of nine FSD probabilities, $\{P(\text{FSD} = k)\}$, shown in Table 1 for a variety of uniform, normal, exponential, Cauchy, and lognormal distributions. As a reference, the first row of entries shows Benford's law probabilities, $B(k)$. To assess the goodness of fit between Benford's law and the FSD probabilities derived from each distribution, the last column of Table 1 shows the so-called J-divergence, J_D, between them. Smaller J_D values indicate a better fit to Benford's law, with $J_D = 0$ signifying a perfect fit.

The J-divergence between the distributions $\{P(\text{FSD} = k)\}$ and $\{B(k)\}$ is defined as the sum of their two possible Kullback-Leibler distances; that is, $J_D = \text{KL}(P, B) + \text{KL}(B, P)$. The first of these distances is given by

$$\text{KL}(P, B) = \sum_{k=1}^{9} P(\text{FSD} = k) \times \log_2 \left[\frac{P(\text{FSD} = k)}{B(k)} \right], \tag{3}$$

where $\log_2[\;]$ denotes base-2 logarithms. The other distance exchanges the role of P and B. The desirability of measuring the goodness of fit between two probability distributions by their J-divergence, and not by eq. (3), arises from the fact that $\text{KL}(P, B) \neq \text{KL}(B, P)$ whenever $P \neq B$. Therefore, the Kullback-Leibler distance, also known as relative entropy, is not a true metric. Nevertheless, it has several desirable measure-theoretic properties, including being nonnegative and equaling zero if and only if P and B are identical distributions.[5]

Table 1 indicates that the FSD from uniformly distributed data fit Benford's law poorly, as these distributions have the highest J-divergence values, $J_D \approx 0.50$, in the table. The table also shows that the FSD from the normal and exponential distributions provide an increasingly better fit to Benford's law. For example, $J_D = 0.0741$ for the normal distribution $N(0, 1)$, and $J_D = 0.0089$ for the exponential distribution exp(1.0). The fit further improves for the Cauchy distributions, with $J_D \approx 0.002$. The best fit is obtained for the lognormal distributions in the table, for which $J_D \leq 0.0015$. In fact, with $J_D = 0$, the lognormal LN(2, 9) fits Benford's law exactly, up to 4-decimal accuracy.

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4. For the general case of $U(a, b)$, $b > a$, eq. (1) or (2) can be used to obtain explicit formulas for the FSD probabilities. Benford (1938, p. 567) and Raimi (1976, p. 526) give such formulas for some uniform distributions other than $U(0, 1)$.

5. For details, see Jeffreys (1946) and Kullback and Leibler (1951). I thank the referee for suggesting the use of this quasi-measure.

TABLE 1

First Significant Digit Probabilities Derived from Some Common Probability Distributions

	First Significant Digit (k)									
Distribution	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	J-divergence
$B(k)$	30.1030%	17.6091%	12.4939%	9.6910%	7.9181%	6.6947%	5.7992%	5.1153%	4.5757%	
Uniform (0, 10)	11.1111	11.1111	11.1111	11.1111	11.1111	11.1111	11.1111	11.1111	11.1111	0.5699
Uniform (0, 43)	25.8398	25.8398	25.8398	9.5607	2.5840	2.5840	2.5840	2.5840	2.5840	0.4164
Uniform (0, 76)	14.6199	14.6199	14.6199	14.6199	14.6199	14.6199	9.3567	1.4620	1.4620	0.4938
Normal (0, 1)	35.9538	12.8966	8.6518	8.1002	7.7423	7.3427	6.9059	6.4427	5.9639	0.0741
Normal (13, 400)	30.7887	23.4939	15.6506	9.4973	5.8638	4.2223	3.6385	3.4573	3.3875	0.0894
Exponential (0.2)	28.7056	15.8582	12.2482	10.2223	8.6842	7.4361	6.4108	5.5659	4.8687	0.0079
Exponential (1.0)	32.9657	17.4322	11.2733	8.6034	7.2549	6.4268	5.8214	5.3259	4.8963	0.0089
Cauchy (0.5, 0)	28.7577	17.2632	12.8513	10.1975	8.3411	6.9732	5.9404	5.1472	4.5284	0.0020
Cauchy (1, 0)	30.9303	16.9056	11.8521	9.3817	7.8815	6.8291	6.0222	5.3702	4.8273	0.0019
Lognormal (0, 1)	30.8329	16.9894	11.9237	9.4212	7.8909	6.8159	5.9951	5.3370	4.7938	0.0015
Lognormal (2, 1)	29.8545	16.8077	12.2137	9.7829	8.1773	6.9954	6.0777	5.3440	4.7468	0.0014
Lognormal (2, 9)	30.1030	17.6091	12.4939	9.6910	7.9181	6.6947	5.7992	5.1153	4.5757	0.0000

The entries in each row indicate the theoretical probability of finding a first significant digit $k = 1, \ldots, 9$, for data drawn from the distribution shown in the first column. All probabilities are computed from eq. (2) of the text, but limiting the summation to the number of terms needed for four-decimal accuracy. For comparison, Benford's law probabilities, $B(k)$, are shown in the first row of entries. The last column assesses the closeness between each FSD distribution and Benford's law by showing the so-called J-divergence value between them. A smaller J-divergence indicates a better fit to Benford's law, with zero J-divergence signifying a perfect fit. See the text surrounding eq. (3) for a discussion of J-divergence.

TABLE 2

Rejection Rates of Benford's Law for Samples Drawn from Various Distributions

	Sample Size (N)					
Distribution	100	500	1,000	2,000	5,000	10,000
Uniform (0, 10)	99.7	100.0	100.0	100.0	100.0	100.0
Uniform (0, 43)	97.9	100.0	100.0	100.0	100.0	100.0
Uniform (0, 76)	99.8	100.0	100.0	100.0	100.0	100.0
Normal (0, 1)	30.0	96.5	100.0	100.0	100.0	100.0
Normal (13, 400)	26.9	98.2	100.0	100.0	100.0	100.0
Exponential (0.2)	9.3	21.6	33.2	65.6	98.0	100.0
Exponential (1.0)	5.9	19.1	35.5	69.4	99.5	100.0
Cauchy (0.5, 0)	6.6	8.0	10.8	19.3	47.9	79.5
Cauchy (1, 0)	6.0	7.2	11.2	17.1	46.0	73.9
Lognormal (0, 1)	6.4	6.7	8.9	14.6	33.7	58.3
Lognormal (2, 1)	6.8	7.4	9.1	13.1	29.8	59.3
Lognormal (2, 9)	6.3	5.6	5.0	4.6	5.3	6.1

Each of the entries reports the percentage of rejections of Benford's law in 1,000 simulations, based on the chi-square test. Each simulated sample randomly draws N observations from a probability distribution in the first column. The $\chi^2(8)$ statistic is then calculated for each sample, with rejection occurring if $\chi^2(8) > 15.507$, the critical value at 5 percent significance. The number of rejections is tallied and divided by 1,000 to find the rejection rate. The columns labeled 100, 500, . . . , 10,000 indicate the number of observations in each of the 1,000 simulations. For example, when 1,000 samples of $N = 500$ observations were generated from the exponential (1.0) distribution, Benford's law was rejected in 19.1 percent of the samples. With the notable exception of the lognormal (2, 9) distribution, rejection rates tend to increase toward 100 percent as the number of observations increases.

To gain additional insight into the appropriateness of using Benford's law as a proxy for the null hypothesis of absence of human influence on data, Table 2 depicts the rejection rates of Benford's law for simulated samples of various sizes drawn from the same distributions of Table 1. Because these distributions clearly lack any human influence, false rejections of the true null hypothesis are expected to occur in 5 percent of the samples, the chosen significance level, if Benford's law is an appropriate proxy for the null. Instead, Table 2 shows that the likelihood of rejecting Benford's law generally increases toward 100 percent as the sample size grows. To take an example, for the exponential (1.0) distribution, the null is rejected in 5.9 percent of the 1,000 samples with $N = 100$ observations, with the rejection rate increasing to 69.4 percent and 100 percent for samples of $N = 2{,}000$ and $N = 10{,}000$ observations, respectively. This pattern of increasing rejection rates is observed for 11 of the 12 distributions in Table 2, confirming that Benford's law may produce too many false alarms, especially for large samples.

In sharp contrast to the pattern of too many false alarms just discussed, Table 2 shows that the rejection rates for samples drawn from the lognormal distribution

LN(2, 9) are essentially stable, meandering between 4.6 and 6.3 percent. A rejection rate commensurate with the significance level of the test suggests that Benford's law is a suitable proxy for the null hypothesis when the data follow this lognormal distribution in the absence of human influence. I formally confirm this conjecture in the next section.

3. Benford's Law and the Variance Parameter of the Lognormal Distribution

Recall that of the 12 distributions in Tables 1 and 2, Benford's law is obeyed solely by the lognormal distribution LN(2, 9). It is not coincidental that this is the lognormal with the greatest variance parameter in those tables. The following proposition fully explains this finding by proving that a lognormal distribution's conformance to Benford's law improves as the variance parameter increases.

Proposition (Convergence of first significant digit probabilities): For the family of lognormal distributions LN(μ, σ^2), the absolute differences $|P(\text{FSD} = k) - B(k)|$, $k = 1, 2, \ldots, 9$, decrease monotonically toward zero as the variance parameter, σ^2, increases without limit.

Proof: See the appendix.

While this proposition guarantees that the FSD from a lognormal distribution with a sufficiently high variance parameter obeys Benford's law, it does not indicate what is "sufficiently high." In other words, the proposition says nothing about the speed of convergence. This is an important empirical issue because if convergence is slow, too many false alarms may still occur even for samples drawn from lognormals with seemingly high variance parameters.

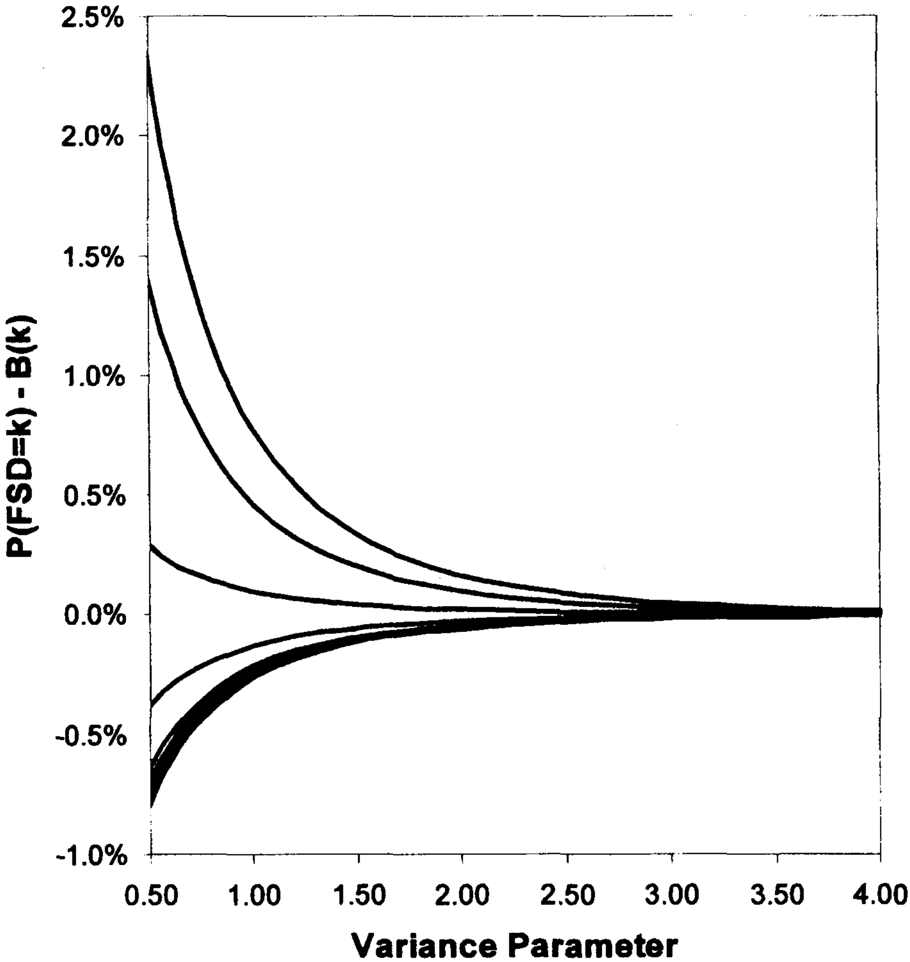
To investigate how fast the FSD probabilities derived from a lognormal distribution converge to Benford's law as the variance parameter increases, Figure 1 graphs the differences $P(\text{FSD} = k) - B(k)$ for the family of lognormals LN(1, σ^2). The figure shows that all nine differences rapidly decrease to zero, having practically vanished for $\sigma^2 = 4$. Therefore, data drawn from lognormals with relatively low variance parameters may conform well to Benford's law.[6]

The preceding proposition is consistent with Nigrini's (2000, p. 38) experience-based rule stating that "in a Benford set the mean value is always larger than the median, and the skewness value is always positive. The larger the ratio of mean to median, the higher the degree of conformity." This practical rule applies exactly to lognormal random variables $X \sim$ LN(μ, σ^2), which have a mean of $E(X) =$

6. The convergence pattern for other values of μ is similar. It is interesting to note, however, that the order of the nine curves may change with μ. For example, the curve for $k = 1$ is the highest when $\mu = 0$, whereas it is the lowest when $\mu = 1$, as shown in Figure 1.

FIGURE 1

Deviation from Benford's Law of the FSD Probabilities Derived from the Lognormal Distribution LN(1, σ^2)



Each of the nine curves depicts the difference $P(\text{FSD} = k) - B(k)$ between the probability of obtaining a first significant digit equal to $k = 1, \ldots, 9$ from the lognormal distribution LN(1, σ^2) and the corresponding Benford's law probability, as a function of the variance parameter, σ^2. The three curves with positive deviations correspond, from highest to lowest, to $k = 2$, $k = 3$, and $k = 4$. The lowest curve corresponds to $k = 1$. Some curves may be hard to distinguish from others in the graph. For all curves, the absolute deviation decreases monotonically toward zero as the variance parameter increases.

$\exp(\mu + \sigma^2/2)$ and a median of $\text{med}(X) = \exp(\mu)$ (Crow and Shimizu [1988, p. 9]). Therefore, the mean/median ratio for lognormal random variables is

$$\frac{E(X)}{\text{med}(X)} = \exp(\sigma^2/2), \tag{4}$$

confirming that $E(X) > \text{med}(X)$ and thus skew > 0 for all lognormals. Furthermore, eq. (4) shows that the mean/median ratio increases monotonically with the variance parameter, σ^2, so the proposition also applies if the variance parameter is replaced by the mean/median ratio.

Given the success in deriving Benford's law from lognormal distributions with sufficient dispersion, it might be conjectured that a similar asymptotic result may hold in general, but this is not the case. Figure 2 provides a dramatic counterexample illustrating that the deviations of $P(\text{FSD} = k)$ from $B(k)$ arising from the family of normal distributions $N(0, \sigma^2)$ show no sign of converging to zero, even for very high standard deviations, σ. Rather, the deviations oscillate without perceptible dampening. This explains the extreme rejection rates observed in Table 2 for samples drawn from the standard normal distribution, $N(0, 1)$.[7]

4. First Significant Digit Frequencies in Accounting and Financial Data

In this section I analyze the empirical behavior of first significant digits for select accounting and financial data sets. Based on the proposition, I expect to find that among data sets that are statistically consistent with a lognormal distribution, only those with relatively high variance parameters obey Benford's law. Also, the results for the nonlognormal distributions in Tables 1 and 2 suggest that nonlognormal data sets may be less likely to conform to Benford's law.

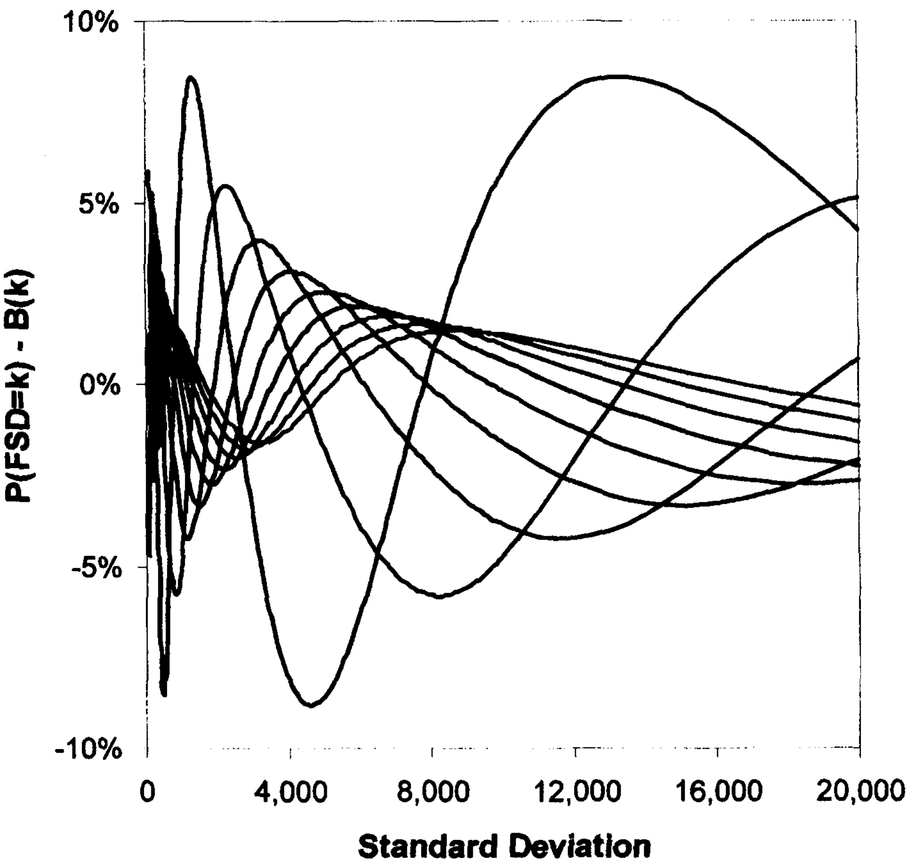
The results are shown in Table 3. The data for net income (NI) and betas were taken from the Disclosure Global Researcher SEC database; the annual market rates of return (Mkt Return) were obtained from Ibbotson Associates' Stocks, Bonds, Bills, and Inflation yearbooks; the gross national product (GNP) data were obtained from the 1998 World Bank Atlas; the group of initial public offering (IPO) data sets are the initial price (IPO Price), number of shares (IPO Shares), and total dollar value (IPO Value) offered to the public by a group of firms studied by Schultz and Zaman (1994); and the daily Dow Jones Industrial Average (DJ) index values were obtained from America Online's internet portal, from which I derived the daily rates of return (ΔDJ/DJ) and the daily changes of the index (ΔDJ).

Column (1) of Table 3 identifies the data sets and their sample sizes, and the next nine columns show the observed relative FSD frequencies, $\{P(\text{FSD} = k)\}$. Column (11) gives the chi-square statistics assessing the goodness of fit of the actual FSD distribution to Benford's law, $\{B(k)\}$, which is shown in the first row

7. Given the symmetry of normal distributions, their failure to obey Benford's law is implicit in their noncompliance with Nigrini's (2000) practical rule. Indeed, for normals the mean equals the median, there is no skewness, and the mean/median ratio is 1 for all σ.

FIGURE 2

Deviation from Benford's Law of the FSD Probabilities Derived from the Normal Distribution $N(0, \sigma^2)$



Each of the nine curves depicts the difference between the probability, $P(\text{FSD} = k)$, of obtaining a first significant digit equal to $k = 1, \ldots, 9$ from the normal distribution $N(0, \sigma^2)$ and the corresponding probability from Benford's law, $B(k)$, as a function of the standard deviation, σ. The curve with the greatest amplitude corresponds to $k = 1$. All the curves exhibit an oscillatory pattern with no apparent dampening.

of entries. The last three columns, (12) to (14), pertain to the lognormal distribution $LN(\mu, \sigma^2)$ whose derived FSD distribution best fits the observed FSD distribution. Columns (12) and (13) report the parameters of the best-fitting lognormal distributions, and column (14) shows the corresponding lowest chi-square values.[8]

8. Because the lognormal distribution is supported only on the nonnegative half line, for data sets that contain negative values, for example, net income, the lognormal distribution parameters in columns

TABLE 3

The Empirical Relative Frequency of First Significant Digits

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
	First Significant Digit (k)									Fit to $B(k)$	Best Fitting LN(μ, σ^2)		
Data Set	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$\chi^2(8)$	μ	σ^2	$\chi^2(8)$
$B(k)$	30.10%	17.61%	12.49%	9.69%	7.92%	6.69%	5.80%	5.12%	4.58%				
NI (6,364)	30.77	17.22	11.63	9.51	8.31	6.40	6.05	5.41	4.71	9.6*	2.2	1.00	4.9*
Mkt Return (76)	27.63	18.42	19.74	10.53	6.58	3.95	2.63	6.58	3.95	6.2*	−5.3	2.40	6.2*
GNP (157)	28.66	24.20	20.38	8.28	4.46	5.73	1.91	3.18	3.18	20.6	8.2	0.64	15.0*
Betas (1,459)	39.20	7.33	5.69	4.73	7.06	9.87	7.54	10.01	8.57	368.8	−0.2	0.25	83.5
IPO Price (72)	54.17	1.39	1.39	0	5.56	8.33	9.72	8.33	11.11	12,655.9	2.3	0.12	4.4*
IPO Shares (72)	34.72	33.33	13.89	5.56	1.39	2.78	1.39	4.17	2.78	5,253.5	7.8	0.36	7.0*
IPO Value (72)	31.94	20.83	12.50	9.72	2.78	8.33	6.94	2.78	4.17	4.2*	10.1	0.88	3.8*
DJ (18,380)	32.70	13.95	5.92	5.25	4.18	6.13	6.38	14.02	11.46	6,285.0	7.0	0.30	2,181.4
ΔDJ/DJ (17,988)	29.02	14.95	11.68	9.76	8.81	7.89	6.92	5.88	5.10	220.1	−9.9	0.72	160.6
ΔDJ (17,988)	30.74	17.39	12.29	9.76	7.69	6.49	5.93	5.20	4.50	7.0*	0.7	2.40	6.8*

The entries in columns (2)-(10) indicate the observed relative frequency of a first significant digit equal to $k = 1, 2, \ldots, 9$, for the data set shown in column (1), where the numbers in parentheses denote sample size. For comparison, Benford's law probabilities are given in the first row of entries. Column (11) shows the chi-square values obtained under the null hypothesis that the empirical FSD frequencies conform to Benford's law. Columns (12) and (13) give the parameters of the lognormal distribution LN(μ, σ^2) whose FSD best fit the actual FSD distribution of each data set, while column (14) shows the corresponding minimum chi-square value. An asterisk (*) in columns (11) and (14) indicates statistical consistency with Benford's law and with the FSD probabilities derived from the best fitting lognormal distribution, respectively, at 5 percent significance. The critical value is $\chi^2(8) = 15.507$.

Of the 10 empirical data sets in Table 3, seven have FSD probabilities that are statistically consistent with originating from a lognormal distribution, but only four of these seven are also consistent with Benford's law (NI, Mkt Return, IPO Value, and ΔDJ). As expected from the proposition, the three lognormal data sets that are inconsistent with Benford's law (GNP, IPO Price, and IPO Shares) have the smallest variance parameters. For example, the lognormally distributed IPO Price data set has both the smallest variance parameter, $\sigma^2 = 0.12$, and the worst fit to Benford's law, $\chi^2(8) = 12{,}655.9$. Finally, the three data sets that are statistically inconsistent with a lognormal distribution (Betas, DJ, and ΔDJ/DJ) are also inconsistent with Benford's law. The unanimous rejection of Benford's law for these non-lognormal data sets is in line with the generally very high rejection rates observed for the nonlognormal distributions in Table 2.

It might be argued that the strong empirical verification of the theoretical findings may be peculiar to the admittedly arbitrary choice of data sets in Table 3. In this regard, it is reassuring to note that the empirical results in Table 3 closely parallel the behavior of Benford's (1938) own data sets. Indeed, 17 of his 20 data sets have FSD that are statistically consistent with a lognormal distribution, but only 11 of these 17 actually conform to Benford's law. Also paralleling the conclusions from Table 3 is the fact that for the group of 17 lognormal data sets, conformance to Benford's law is overwhelmingly determined by the magnitude of the variance parameter. Indeed, five of the six lognormal sets that do not conform to Benford's law have relatively small variance parameters $\sigma^2 < 0.7$, while 10 of the 11 lognormal sets that conform have $\sigma^2 > 0.7$. Furthermore, as in Table 3, all three of Benford's nonlognormal data sets fail to follow Benford's law. Therefore, the overall empirical confirmation of the theoretical expectations appears to be robust.

4.1 Reduction of False Alarms When the Data Are Drawn from the Product of Data Sets

Notice in Table 3 that of the three data sets pertaining to initial public offers (IPO) and the three pertaining to the Dow Jones Industrial Average index (DJ), the first two strongly reject Benford's law, whereas the third data set is consistent with that law. Also notice that the third data set results from multiplying the first two sets in the group; that is, (IPO Value) = (IPO Price) × (IPO Shares) and ΔDJ = DJ × (ΔDJ/DJ). Understanding why the product complies with Benford's law even though the multiplicands fail to do so requires distinct explanations for each group. For the IPO group I provide a complete explanation that relies on the proposition, whereas for the DJ group I propose a plausible explanation. I discuss each group in turn.

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(12) and (13) are interpreted as providing the best fit to the distribution of the absolute value of the data. Although this adaptation changes the distribution of the raw data, it has no effect on the observed FSD distribution because the FSD of z and $|z|$ are equal.

Recall that the lognormal distribution possesses the reproductive property under multiplication. Therefore, because the first two data sets in the IPO group are lognormal, their product is also lognormal. Specifically, if the distributions of IPO Price and IPO Shares are the lognormals $LN(\mu_p, \sigma_p^2)$ and $LN(\mu_s, \sigma_s^2)$, respectively, and their joint distribution has correlation coefficient ρ, then the distribution of their product, IPO Value, is the lognormal $LN(\mu_v, \sigma_v^2)$, where $\mu_v = \mu_p + \mu_s$ and $\sigma_v^2 = \sigma_p^2 + 2\sigma_p\sigma_s\rho + \sigma_s^2$ (Crow and Shimizu [1988, pp. 14–15]). Noting from Table 3 that $LN(\mu_p, \sigma_p^2) = LN(2.3, 0.12)$ and $LN(\mu_s, \sigma_s^2) = LN(7.8, 0.36)$, and using the additional fact that the sample correlation coefficient between IPO Price and IPO Shares is $\rho = 0.40$, the IPO Value data set should follow the lognormal distribution LN(10.1, 0.65). In fact, these data are best fitted by the lognormal distribution LN(10.1, 0.88). The crucial point to notice is that the product data set, IPO Value, is lognormally distributed with a higher variance parameter than either the IPO Price or IPO Shares multiplicands.[9] It follows from the proposition that the product is more likely to follow Benford's law, and thus less likely to produce false alarms, than the multiplicands.

The behavior of the Dow Jones group requires a more tentative explanation because the daily index, DJ, and the daily rate of return, ΔDJ/DJ, are not lognormally distributed, as evidenced by their high chi-square values in column (14) of Table 3. Nevertheless, their product, ΔDJ, is consistent with the lognormal distribution LN(0.7, 2.40). Based on the proposition, it should not be surprising to find that this lognormal set with a high variance parameter conforms to Benford's law.

A partial explanation for the closeness between Benford's law and the FSD distribution of daily changes in the Dow Jones index relies on theorems by Boyle (1994) and Hamming (1970) showing that the FSD distribution of the product of two random variables with support on a tight interval, for example, [0.1, 1], provides a better fit to Benford's law than either multiplicand. Although the narrow support required by these theorems limits their practical application, in fact the same result often applies under much less restrictive conditions, as the following example illustrates.

Assume the independent random variables Y_1 and Y_2 have the uniform distribution $U[1, b)$, $b \in (1, \infty)$, so their distribution function is $F(y) = 0$ for $y \in (-\infty, 1)$, $F(y) = (y - 1)/(b - 1)$ for $y \in [1, b)$, and $F(y) = 1$ for $y \in [b, \infty)$. Letting $W = Y_1Y_2$, it is a straightforward, but tedious, exercise to show that the distribution function of W is given by $H(w) = 0$ for $w \in (-\infty, 1)$, $H(w) = 1$ for $w \in [b^2, \infty)$, and[10]

$$H(w) = \begin{cases} (b - 1)^{-2}\,[w\ln(w) + 1 - w] & \text{for } w \in [1, b) \\ (b - 1)^{-2}\,\{w[1 + \ln(a^2 / w)] + 1 - 2b\} & \text{for } w \in [b, b^2). \end{cases} \tag{5}$$

9. In general, if $\sigma_s > \sigma_p$, the necessary and sufficient condition for $\sigma_v^2 > \sigma_s^2$ is $\rho > \rho^* = -\sigma_p/(2\sigma_s)$. For the IPO group this critical correlation is $\rho^* = -0.35/(2 \times 0.60) = -0.29$.

10. For the methodology used to obtain eq. (5), see Meyer (1970, pp. 109–110).

To obtain the FSD probabilities for the multiplicands and the product, apply eq. (2) to the $F(y)$ and $H(w)$ distribution functions, for all values in the upper bound set $b \in \{10, 11, 12, \ldots, 1{,}000\}$. Using these FSD distributions and Benford's law, calculate both Kullback-Leibler distances using eq. (3), and add these distances to get the J-divergence from Benford's law of the multiplicands and the product. Then compute the ratio of the multiplicands' J-divergence to the product's J-divergence for each upper bound, b. A J-divergence ratio greater than 1 indicates that the product's FSD distribution provides a better fit to Benford's law than the multiplicands' FSD distribution.

All the resulting J-divergence ratios are greater than 1, ranging from $J_D = 5.6$ for $b = 15$ to $J_D = 18.6$ for $b = 400$, with an average J_D of 11.3. Thus, in this illustration the J-divergence of the product's FSD distribution with respect to Benford's law is, on average, 8.9 percent of the J-divergence of each multiplicand, implying that the product is much more likely to conform to Benford's law than the multiplicands, and thus less likely to produce false alarms.

Although the preceding illustration supports the observed behavior of the group of Dow Jones data sets in Table 3, its value is only suggestive. Nevertheless, it is reassuring to find that the product of two random variables may provide a better fit to Benford's law than the multiplicands, under more relaxed assumptions than required by Boyle (1994) and Hamming (1970). Achieving a better understanding of how much more those assumptions can be loosened is left for future research.

5. Summary

Benford's law has been recently used in the accounting, auditing, and finance disciplines to test for the presence of human influence, including psychological price barriers, data tampering, and outright corporate fraud. The critical underlying assumption is that, in the absence of human intervention, the distribution of first significant digits (FSD) conforms to Benford's law, which assigns decreasing FSD probabilities to succeeding digits. In this article, I strengthen the belief that using Benford's law as a proxy for the null hypothesis of absence of human influence may result in too many type I errors, or false alarms. Indeed, I present evidence showing that the FSD arising from various theoretical distributions fail to obey Benford's law and generate an exceedingly high percentage of false alarms. I also prove that the law fails for lognormal distributions with a small variance parameter.

On a positive note, I prove that a lognormal distribution with a sufficiently high variance parameter is consistent with Benford's law, and thus produces a false alarm rate commensurate with the significance level of the test. I also provide an explanation for the empirical observation that data sets expressible as the product of two other data sets are more likely to conform to Benford's law than either multiplicand. These positive findings constitute an advance toward the goal of reducing the false alarm rate when testing for the presence of human influence on data.

The empirical evidence analyzed in this article strongly supports the findings. Indeed, of the samples that fit a lognormal distribution, those with the highest variance parameter conform to Benford's law, whereas those with the lowest variance parameters do not. In addition, all the nonlognormal samples are inconsistent with Benford's law. These empirical findings appear to be robust, as they closely parallel the behavior of Benford's entirely different choice of 20 data sets. Finally, for the two groups of three data sets where one of them is the product of the other two, I find that the product obeys Benford's law even though both multiplicands do not. I provide a complete explanation for the behavior of one group, and a plausible explanation for the behavior of the other. I conclude that product data sets appear less likely to suffer from false alarms than multiplicand data sets.

Much work remains to be done. Although the proposition and other findings advanced in this article expand the set of general rules that guide researchers in determining which probability distributions obey Benford's law and which do not, additional rules may be needed to reduce even further the likelihood of false alarms when searching for possible human influence on data.

APPENDIX

Proof of the Proposition

Consider an observation z drawn from a nonnegative random variable Z with density $f(z)$. Then, the FSD of z equals $k = 1, \ldots, 9$ if and only if z falls in the interval $[k10^i, \{k + 1\}\ 10^i) \equiv I(i, k)$ for some integer i. Therefore, the probability that the first significant digit equals k is given by the area under $f(z)$ determined by the union of an infinite number of intervals $I(i, k)$. Thus,

$$P(\text{FSD} = k) = \sum_{i=-\infty}^{\infty} \int_{I(i,k)} f(z)dz. \tag{A1}$$

Now recall that if Z is lognormally distributed, its density is given by

$$f(z) = \frac{1}{z\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln(z) - \mu}{\sigma}\right)^2\right\}. \tag{A2}$$

To evaluate each integral in eq. (A1), apply eq. (A2) and make the change of variables $w = [\ln(z) - \mu]/\sigma$. After several algebraic steps we arrive at

$$\int_{I(i,k)} f(z)dz = \frac{1}{\sqrt{2\pi}} \int_{a(i,k)}^{a(i,k+1)} \exp\left\{-\frac{1}{2}w^2\right\}dw = N[a(i, k + 1)] - N[a(i, k)], \tag{A3}$$

where $N[\ ]$ is the standard normal distribution function, and $a(i, x)$, $x = k, (k + 1)$, is defined by

$$a(i,x) \equiv \frac{\ln 10}{\sigma}i + \frac{\ln(x) - \mu}{\sigma}. \tag{A4}$$

Substituting eq. (A3) in eq. (A1) gives the exact probability of obtaining a FSD equal to $k = 1, 2, \ldots, 9$ when an observation is drawn from a lognormal distribution:

$$P(\text{FSD} = k) = \sum_{i=-\infty}^{\infty} \{N[a(i, k + 1)] - N[a(i,k)]\}. \tag{A5}$$

Unfortunately, a closed-form expression for eq. (A5) is not possible. However, by noticing that each term in the summation is equal to the area under the standard normal density curve, $n(y)$, between $y = a(i, k)$ and $y = a(i, k + 1)$, and that $a(i, k + 1) - a(i, k) = \sigma^{-1} \ln[(k + 1)/k]$ from eq. (A4), we can approximate $P(\text{FSD} = k)$ by the following expression:

$$P(\text{FSD} = k) \approx \sum_{i=-\infty}^{\infty} [a(i, k + 1) - a(i, k)]\, n[a(i, k)] = \frac{1}{\sigma} \ln \frac{k + 1}{k} \sum_{i=-\infty}^{\infty} n[a(i, k)]. \tag{A6}$$

Now multiply and divide eq. (A6) by $\ln(10)/\sigma$, recall that $\ln[(k + 1)/k]/\ln(10) = \log[(k + 1)/k]$, and simplify to obtain

$$P(\text{FSD} = k) \approx \log\left(\frac{k + 1}{k}\right) \sum_{i=-\infty}^{\infty} n[a(i,k)] \frac{\ln(10)}{\sigma}. \tag{A7}$$

Recalling that $B(k) \equiv \log[(k + 1)/k]$, the following absolute difference follows from eq. (A7):

$$|P(\text{FSD} = k) - B(k)| \approx \log\left(\frac{k + 1}{k}\right) \left| \sum_{i=-\infty}^{\infty} n[a(i, k)] \frac{\ln(10)}{\sigma} - 1 \right|. \tag{A8}$$

To complete the proof, it suffices to show that the summation in eq. (A8) approaches 1 as the lognormal distribution's standard deviation parameter, σ, increases without limit. To this end, notice from eq. (A4) that $a(i + 1, k) - a(i, k) = \ln(10)/\sigma$, so the terms in the summation can be interpreted as contiguous rectangles of width $\ln(10)/\sigma$ and height $n[a(i, k)]$. Because the width of the rectangles decreases as σ increases, this Riemann approximation to the total area under the normal curve tends to 1 as σ, and therefore σ^2, increases without bound. Q.E.D.

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