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Joint pricing and inventory policies for make-to-stock products with deterministic price-sensitive demand

Saibal Ray[a], Yigal Gerchak[b], Elizabeth M. Jewkes[c,*]

[a]*Faculty of Management, McGill University, Montreal, Que., Canada H3A 1G5*
[b]*Department of Industrial Engineering, Tel-Aviv University, Ramat-Aviv, Tel Aviv 69978, Israel*
[c]*Department of Management Sciences, University of Waterloo, Waterloo, ON N2L 3G1, Canada*

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Abstract

In this paper, we focus on a firm selling a single make-to-stock product to price-sensitive end customers. We develop an integrated operations–marketing model that can help determine the relevant profit-maximizing decision variable values for two pricing policies that the firm might follow—price as a decision variable, which is advocated by academicians, and mark-up pricing, used by most practitioners. We first consider an EOQ-based model with price and order quantity as independent decision variables. We then develop an analogous model where price is a mark-up over operating costs per unit, and order quantity becomes the sole decision variable. We are able to ascertain the optimal decision variable values for each model for log-linear *and* linear demand functions. We prove that for such profit-maximizing models, the optimal batch size is *not* necessarily monotone increasing in set-up cost. Interestingly, our numerical/analytical evidence suggests that from a profit perspective it is better for managers to be *aggressive on price rather than reducing price too much*, especially for highly price-sensitive and non-linear demand. Moreover, we establish that, in general, the profit penalty *for not including inventory costs in determining the optimal batch size, or ignoring the batch size optimization issue* in a mark-up price model is not significant. Only when the *set-up cost is quite high and/or the firm faces non-linear demand from highly price-sensitive end consumers* does it become crucial for managers to determine the "exact" optimal batch size and base the mark-up price on the entire unit operating cost, not only the unit (variable) production cost.

Keywords: Operations–marketing integration; Price-sensitive demand; Mark-up pricing; EOQ model

1. Introduction

Most traditional models in the economics/marketing literature postulate that a firm should assume price as a decision variable and set its

*Corresponding author.
E-mail address: emjewkes@engmail.uwaterloo.ca (E.M. Jewkes).

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optimal value based on equating the marginal cost to the marginal revenue. The only operational costs in these pricing models appear in the form of unit (variable) production/purchase cost. However, in most practical cases, inventory costs account for as much as 15–30% of the operational costs. In fact, it can be argued that among the many decisions made by a manager, the two most generic and strategic ones are those of setting prices and planning for how much inventory to hold (Stern and Al-Ansary, 1992). Keeping these facts in mind, recent operations management literature has started focussing on developing integrated models that can simultaneously optimize the relevant inventory (operations) *and* pricing (marketing) decisions (refer to Viswanathan and Wang, 2003, and Section 2.1).

Though optimal pricing models are popular among academicians, empirical studies dating back to Hall and Hitch (1939) have revealed that many managers work in terms of mark-ups as their basis for pricing. Mark-up pricing set prices based on the unit operating cost plus a constant percentage (or amount) mark-up, which can depend on factors such as the industry and the product type (Hay and Morris, 1991). Recent studies by Hall (1988), Morrison (1994), Park and Lohr (1996), Bloch and Olive (1997) and Dobrinsky et al. (2001) have confirmed the continued popularity of mark-up pricing. These studies indicate that almost three-quarters of firms use some form of cost-based mark-up pricing. In fact, it is the most common pricing method employed by retailers such as Wal-Mart, the biggest retailer in the world (www.profitablepricing.com, www.walmartstores.com).

There are a number of instances when mark-up might be the preferred pricing strategy for managers. For example, when managers are not clear about the exact market condition facing the firm, they sometimes resort to mark-up pricing policy based on past experience that the current mark-up will generate sufficient demand. From an analytical perspective, the optimal mark-up for a product should be inversely proportional to its demand elasticity. There is ample evidence that, in practice, firms do vary their mark-ups with the price elasticity for a product's demand, but do so based on empirical observation, rather than optimization (Hay and Morris, 1991). Another instance when mark-up pricing might be used is when the demand is fixed, and such pricing policy is "required" by customers. The classic examples are the defence department or construction contracts, where the final price for the contract is decided as a mark-up over the actual/expected costs. However, note that in recent times such mark-up pricing strategies are becoming popular even for consumer products. Some authors (e.g., Rao et al., 2000; Sinha, 2000) have postulated that because of the cost transparency in the internet era, customers will only pay the seller's actual costs and a "reasonable" premium, i.e., that the price should be based on a realistic mark-up over cost. Clearly, mark-up pricing reduces the decision space for managers to only cost, i.e., inventory-related issues. Hence, this pricing paradigm makes it imperative for managers to understand the effects of optimal inventory policies on the operating costs (and ultimately profits) of the firm.

In this paper, we model a firm selling a single make-to-stock product to end customers. The demand faced by the firm is deterministic, but price-sensitive. Moreover, the firm can set the product price either assuming it to be a decision variable or as a mark-up over its operating costs per unit. Our aim is to determine the relevant operations *and* marketing decision variable values that will maximize the firm's profit for both pricing policies.

The paper is organized as follows: Related literature is presented in Section 2. Section 3 deals with an initial profit-maximizing model with managers deciding on both optimal price and inventory (batch size) levels, while Section 4 investigates a model where price is a fixed mark-up over the unit operating cost, resulting in batch size being the sole decision variable. Our conclusions and avenues for future research are provided in Section 5.

2. Related literature

In this section, we discuss the related literature, which can be divided into two main groups: (i)

Research concerning issues at the operations-marketing interface, and (ii) Literature related to mark-up pricing. We will end this section by indicating in what way our research differs from the existing literature.

2.1. Integrated operations-marketing models

There is a considerable volume of literature that deals with integrated operations-marketing models. Eliashberg and Steinberg (1993) give a useful argument for why such models are important, and provide a comprehensive literature review up to the late 1980s. We are mostly concerned with continuous time concave-cost models (since set-up cost is present) with static pricing. The first model of this kind was formulated by Whitin (1955) who incorporated pricing into the traditional EOQ model through a linear price–demand relation for the end consumers. The objective was to determine the profit-maximizing price for the firm to charge and the optimal order quantity. This problem was later explicitly solved by Porteus (1985). Abad (1988, 1996) addresses a related problem for a more general demand function where the retailer is offered an all-unit quantity discount by its supplier, or the goods being sold are perishable in nature. Parlar and Wang (1994) and Weng (1995) also uses a similar framework, but focus on understanding the impact of the "game" between the firm (retailer) and its supplier in the presence of a quantity discount, and how a properly designed discount schedule should be able to increase profits of both parties involved. Recently, Lau and Lau (2003) have investigated a joint pricing-inventory model, but in the *absence* of set-up cost. One of their major findings is that the nature of the price-demand relationship (whether linear or non-linear) might have a significant effect on model results. Another recent research in this area closely related to ours is by Viswanathan and Wang (2003) who model a single-retailer, single-vendor distribution channel. The profit function for the retailer facing price-sensitive deterministic demand in their paper is similar to our price-as-a-decision-variable model of Section 3. However, the focus in Viswanathan and Wang is to show how a combination of discount mechanisms, volume and quantity, can coordinate the channel, while we concentrate on identifying managerial insights regarding the behaviour of the optimal decision variable values.

There is also a small stream of integrated decision-making literature which deals with mark-up pricing. Ladany and Sternlieb (1974) set price as a fixed mark-up over the purchase/production cost in a profit-maximizing EOQ model where the demand rate is decreasing in price. Lee and Rosenblatt (1986) also consider a model similar to that of Ladany and Sternlieb but include advertising investments and quality issues.

2.2. Mark-up pricing models

In a classic study, Hall and Hitch (1939) found that many firms decide on prices relative to some notion of average cost and a reasonable mark-up to cover profits. As we have indicated in Section 1 there might be a number of reasons that mark-up pricing is used in practice. While traditionally this particular pricing policy is associated with government/make-to-order contracts, recent evidence suggests its growing popularity even in the retail (make-to-stock) sector. Empirical studies by Skinner (1970), Hall (1988), Morrison (1994), Bloch and Olive (1997) and Dobrinsky et al. (2001) have shown that, in spite of the knowledge that mark-up pricing might not be optimal, it is the preferred pricing strategy for most practitioners due to the fact it is easy to understand and apply. There is an extensive literature in industrial economics that consistently shows that for most firms (almost 75%) changes in price and cost are proportional which suggests the use of fixed mark-ups. Rather than going through the details, we refer the reader to the excellent reviews by Hay and Morris (1991) and Bloch and Olive (1997). The application of mark-up pricing is prevalent in sectors like wholesale petroleum products, organic foods, textiles, telecommunications equipment and retailing. Mark-up pricing has also been studied in the marketing literature (refer to Monroe, 1990).

Based on our review of previous research, we observe that:

- Most theoretical pricing models assume price as a decision variable ignoring the fact that a large

segment of make-to-stock firms decide on price as a mark-up over the unit cost.
- The models that deal with mark-up pricing assume that the mark-up is based on unit variable (production/purchase) costs only (e.g., Ladany and Sternlieb, 1974) and hence disregard a potentially significant portion of operating costs such as inventory (holding and set-up) costs.

In this paper we analyse how profit-maximizing make-to-stock firms should optimally decide on their relevant operational and marketing decision variable values for both mark-up and price-as-a-decision-variable policies. Our focus will be on generating managerial insights regarding the behaviour of the decision variables, as well as on understanding the efficacy of approximations like using mark-up over the unit variable, rather than total operating, cost.

3. An EOQ model with price and order quantity as decision variables

In this section, we consider a monopolist price-setting firm buying/producing a single make-to-stock product and selling it directly to end consumers. We assume demand to be deterministic but price-sensitive and hence make the usual EOQ assumptions with the exception that the demand rate λ (units/time) is a decreasing function of price per unit p (\$/unit) charged to the end consumers (Abad, 1988; Parlar and Wang, 1994; Viswanathan and Wang, 2003). While the two most commonly studied demand functions in the literature are log-linear and linear (Lau and Lau, 2003), we will initially focus on a log-linear demand function. Subsequently, we discuss how the results are affected if the demand function takes a linear form. Hence, we begin with

$$\lambda = ap^{-b}, \quad a, b > 0, \tag{1}$$

where higher values of a correspond to higher overall potential demand for the product and b denotes the price-sensitivity of customers (demand elasticity). The firm's unit operating cost, m (\$/unit), includes a set-up cost of K (\$) per order, a holding cost of h (\$/unit/unit time) and a constant[1] purchase/production cost per unit, c (\$/unit). The holding cost is assumed to consist primarily of cost-of-capital invested in the inventory. Each order is for a batch of Q units. Clearly, the firm's ordering cost per unit time is $(K\lambda/Q)$, the holding cost per unit time is $(hQ/2)$, and the purchase/production cost per unit time is $c\lambda$. Hence m is given by

$$m = \frac{1}{\lambda}\left(\frac{K\lambda}{Q} + \frac{hQ}{2} + c\lambda\right),$$
$$\text{where } h = ic \ (i = \text{interest rate (\$/\$/unit time)}). \tag{2}$$

The firm wishes to maximize its profit per unit time, π, through the optimal choice of price p, and order quantity Q, i.e.,

$$\text{(P1)} \quad \underset{Q,\,p}{\text{Maximize}} \quad \pi(p, Q) = (p - m)\lambda$$
$$\text{subject to:} \quad p > m > 0 \text{ and } \lambda \geq 0, \tag{3}$$

and where m is given by (2) and λ by (1). We note here that Viswanathan and Wang (2003) deal with a problem similar to (P1), and hence, some of the following results (specifically Proposition 1) are similar to theirs. They base their analysis on identifying the optimal-order quantity and demand, rather than price (obviously there is a one-to-one relationship between price and demand). The first and second derivatives[2] of π with respect to Q are

$$\pi(Q,p)_Q = (\lambda K/Q^2) - (h/2), \text{ and}$$
$$\pi(Q,p)_{QQ} = (-2\lambda K/Q^3) \tag{4}$$

from which we obtain the optimal order quantity $Q^*(p) = \sqrt{2K\lambda/h}$. Clearly, the optimal batch size is decreasing in p. Substituting $Q^*(p)$ in (3), we have the profit as a function of p:

$$\pi(Q^*(p), p) = \pi(p) = \lambda(p - c) - \sqrt{2\lambda hK}. \tag{5}$$

[1]Ray et al. (2003) shows that most of our subsequent analysis holds true when c is a non-increasing function of Q (e.g., when there are economies of batch-size scale).

[2]For this paper, Z_y will represent the first derivative of Z with respect to y, Z_{yy} will represent the second derivative, $\rightarrow$ will denote tends towards, $\Rightarrow$ will denote imply, and $\uparrow$ and $\downarrow$ will stand for increases and decreases, respectively.

Note that we require $p \geqslant c$ for $\pi \geqslant 0$. Differentiating $\pi(p)$ with respect to p we have

$$\pi(p)_p = -\frac{\lambda}{2p\sqrt{kh\lambda}} \times \left[2\sqrt{\frac{aKh}{p^b}}(-p + b(p-c)) - \sqrt{2}bKh\right], \quad (6)$$

$$\pi(p)_{pp} = \frac{-b\lambda}{4p^2\sqrt{kh\lambda}} \times \left[4\sqrt{\frac{aKh}{p^b}}(p - b(p-c) + c) + \sqrt{2}bKh(b+2)\right]. \quad (7)$$

From (7) it is evident that $\pi(p)$ is, in general, not necessarily concave in p. However, for $0<b\leqslant 1$, we can show that the profit function is increasing concave $\forall\ p\geqslant c$, implying that the optimal price $p^* = \propto$, a rather uninteresting scenario. Hence, we focus on $b>1$, in which case we can show that there is *only one contiguous region of p for which the profit* function *is decreasing* (see Appendix). This allows us to characterize the behaviour of the firm's profit.

Proposition 1. *For $b>1$, assuming potential demand for the product (a) is sufficiently high and the business is profitable for some p, $\pi(p)$ attains its maximum at the smallest value of p for which* $\pi(p)_p \leqslant 0$.

Proof. See Appendix. □

Let us denote the optimal price by p^*, which can be obtained from solving (6) for p and the optimal batch size is then obtained from $Q^*(p^*)$. For $b\leqslant 2$ there will be a unique solution to the first-order condition of (6). In fact, for $b=2$ this solution can be expressed explicitly (see the Appendix). On the other hand, for $b>2$ there will be exactly two positive solutions for (6), the smaller of which would be p^* (see Appendix). It is also possible to perform analytical comparative statics of the optimal decision variable values for some of the parameters.

Proposition 2. *For log-linear demand and $b>1$: As $a\uparrow$, $p^*\downarrow$, $Q^*\uparrow$; as $K\uparrow$, $p^*\uparrow$, $Q^*\uparrow$ until $K=(2ac^2(b-1)^{2b-2})/(h((b^2c)^b)$ (say K_b) and then $\downarrow$; as $h\uparrow$, $p^*\uparrow$, $Q^*\downarrow$; as $c\uparrow$, $p^*\uparrow$, $Q^*\downarrow$.*

Proof. See Appendix. □

While difficult to prove unequivocally in an analytical fashion, extensive numerical experiments clearly indicate the following:

Claim 1. For log-linear demand and $b>1$: As $b\uparrow$, p^* mostly $\downarrow$, Q^* initially $\uparrow$ and then $\downarrow$.

While most results in Proposition 2 and Claim 1 are intuitive, the effect of set-up cost is interesting. In all types of cost-minimizing inventory models we are aware of, the optimal batch size is monotone increasing in the set-up cost K. We observe slightly different behaviour. In our model, as set-up cost (K) increases from a low value, the cost and price increase, causing a reduction in demand. The set-up cost effect initially dominates the decrease in demand, and we observe increasing batch sizes. For very high values of K, however, the effect of demand reduction becomes stronger, and the firm will then *reduce* its batch size with further increase in K (p^* still increases to counterbalance the higher set-up cost). Thus we observe that the optimal batch size is not necessarily monotone in the set-up cost K. We do note that if the primary decision of interest is the optimal replenishment interval or days of production, i.e., Q^*/λ, rather than the absolute batch size, then this value is still increasing in the set-up cost K.

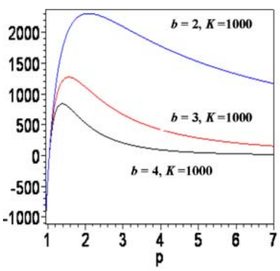
The proof of Proposition 1 in the appendix provides us with some interesting insights about the behaviour of the profit function. First of all, for $b>1$: as $p\to 0$, $\pi(p)\to -\infty$, as $p\to$, $\pi(p)\to 0$, and for $p=c$, $\pi(p)<0$, $\pi_p(p)>0$, $\pi_{pp}(p)<0$. In fact, it is possible to characterize $\pi(p)$ in considerably more detail. Specifically, when $b\leqslant 2$, as p increases from 0, the profit function increases from $-\infty$, reaches its maximum at p^*, and then decreases to 0 as $p\to\infty$. In that case there is a threshold value of $p(>c, <p^*)$ above which $\pi(p)>0\ \forall\ p$. On the other hand, for $b>2$, the profit function will increase from $-\infty$ to its maximum at p^*, i.e., the smaller solution to (6), then decrease (in the process becoming negative) until the larger solution to (6) (say p'), and subsequently increase again tending towards 0 as

$p \to \infty$. This implies that for $1 < b \leqslant 2$, the profit function is unimodal (not necessarily concave), while for $b > 2$ the smaller solution of the FOC maximizes the profit and the larger one is the local minimum. Clearly, for $b > 2$, the profit function attains its minimum value for $p > p^*$ at $p = p'$ ($\pi(p') < 0$). However, note that $\pi(p')$ must be finitely negative, since otherwise $\pi(p)$ cannot be 0 as $p \to \infty$ (refer to Appendix). Contrasting this with the scenario that $\pi(p) \to -\infty$ as $p \to 0$, we can conclude that *for an aggressive pricing strategy ($p > p^*$), there is a bound on the maximum losses, whereas for prices lower than optimal, the losses are unbounded for very low prices*. In fact, we can prove the following even stronger result:

Proposition 3. *For log-linear demand and $b > 1$, the loss in profit due to setting a below-optimal price is greater than that associated with setting an above-optimal price, provided the price deviations (from the optimal) are equal.*

Proof. See Appendix. □

The above proposition essentially implies that the *profit* function *is relatively flat for prices higher than optimal, while it decreases more sharply for lower prices* ($\to -\infty$ as $p \to 0$). We illustrate this result in Fig. 1 which depicts $\pi(p)$ for different values of b and K. Clearly, in all cases the profit function is quite flat for $p > p^*$, but reduces rapidly for $p < p^*$. This implies that an aggressive pricing strategy might be less harmful (from a profit perspective) than a conservative one. It might be argued that $p < c$ does not make sense, so a fair comparison should be based on $\pi(c)$ (< 0) and $\pi(p')$. Obviously, $\pi(p') > 0 > \pi(c)$ when $b \leqslant 2$ since the profit function is positive for $p > p^*$ Our extensive numerical experiments have shown that even if $b > 2$, $\pi(c)$ is considerably more negative than $\pi(p')$, regardless of the parameter values (refer to Table 3 below). Hence, we can say that whenever in doubt, *it is better for managers to be aggressive in terms of their pricing strategy*. We provide some numerical examples below to provide further evidence of the above results.



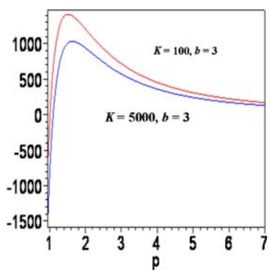


Fig. 1. Profit function ($\pi(p)$) for different values of b and K, log-linear demand ($a = 10000$, $c = 1$, $h = 0.0077$).

Example 1. Consider the following parameter set: $c = \$1$/unit, $i = \$0.0077/\$/$week, $h = i^*c = \$0.0077/$unit/week, $a = 10{,}000$. Based on these values and various values of b and/or K, we obtain the results summarized in Tables 1–4.

The above tables illustrate most of our propositions for realistic parameter values (we have kept the set-up + holding costs to less than 20% of the purchase/production costs in all the examples). The non-monotone nature of Q^* as b changes is evident from Table 1. As far as K_b is concerned, Table 2 shows that for low values of b, K_b is very large, and for all practical purposes, Q^* is increasing in K. Only for very price sensitive customers ($b = 8$) and quite high, but realistic ($Q^*/\lambda^* = 0.7$ year and $K_b/\lambda^* c = 0.10$ year) values of K do we see that for $K > K_b$ the firm might have

Table 1
Optimal decision variable values, demand rate and profit for log-linear demand

b ($K = 400$)	P^* (\$/unit)	Q^* (units/batch)	λ^* (units/week)	π^* (\$/week)
1.5	3.08	13,841	1844	3741.27
2	2.05	15,716	2377	2377.44
4	1.36	17,322	2888	918.18
6	1.23	17,425	2923	530.85
8	1.17	17,238	2860	351.68

Table 2
Turning point for the set-up cost (K_b) for log-linear demand

b	1.5	2	4	6	8
K_b (\$/order)	384,800	162,338	28,893	11,652	6258

Table 3
Comparison of aggressive ($p=p'$) and conservative ($p=c$) pricing strategy profits

Log-linear	$b=3$	$b=4$	$b=5$
K (\$/order)	p^*, p' $\pi(p')$, $\pi(c)$[a]	p^*, p' $\pi(p')$, $\pi(c)$	p^*, p' $\pi(p')$, $\pi(c)$
400	1.53, 2883.00	1.36, 59.07	1.28, 15.18
	−0.0004, −248.19	−0.02, −248.19	−0.10, −248.19
4000	1.62, 285.00	1.44, 15.64	1.35, 6.47
	−0.04, −784.86	−1.14, −784.86	−2.54, −784.86
Linear	$b=1000$	$b=2000$	$b=4000$
K (\$/order)	p^*, p' $\pi(p')$, $\pi(c)$	p^*, p' $\pi(p')$, $\pi(c)$	p^*, p' $\pi(p')$, $\pi(c)$
400	5.51, 9.9999	3.01, 4.9999	1.76, 2.4998
	−0.1711, −235.45	−0.39, −222.00	−1.03, −192.25
4000	5.53, 9.9998	3.03, 4.9995	1.79, 2.4983
	−1.71, −744.58	−3.85, −702.00	−10.28, −607.95

[a] $\pi(c)=\sqrt{2Kh(a-bc)}$ for linear demand, and $\pi(c)=\sqrt{2Khac^{-b}}$ for log-linear demand.

Table 4
Profit loss due to deviations from the optimal price

Log-linear (b), $K=400$	Δ	$\pi(p^*+\Delta)-\pi(p^*-\Delta)$	Linear (b), $K=400$	Δ	$\pi(p^*+\Delta)-\pi(p^*-\Delta)$
2	0.05	\$0.14	1000	0.1	\$0
	0.10	\$1.10		0.3	\$0.01
4	0.05	\$1.87	2000	0.1	\$0
	0.10	\$15.23		0.3	\$0.07
8	0.05	\$10.36	4000	0.1	\$0.04
	0.10	\$88.13		0.3	\$1.22

to reduce the optimal batch size with an increase in the set-up cost.

Table 4 and Fig. 1 demonstrate Proposition 3 where the profit function for $p>p^*$ is seen to be much "flatter" compared to that for $p<p^*$. Furthermore, from Table 3 we can see that the profit losses can be substantial if the price is set below the optimal (obviously it will be unbounded if price is set very low), but setting prices significantly above the optimal does not result in high losses, even in the worst case scenario. Note from Table 4 that the profit penalty for deviating from the optimal increases with the elasticity of the demand function, i.e., for highly price-sensitive customers.

It is also worthwhile to consider the case where demand is linear in price, $\lambda=a-bp$ (refer to Parlar and Wang (1994) for a related model). In the interest of space we will focus on the similarities and differences in results compared to log-linear demand, rather than providing a detailed analysis. With a linear demand function, profits behave in a manner very similar to log-linear demand with $b>2$ (see Appendix). That is, for a linear demand function, the profit-maximizing price p^* is given by the smaller of the two solutions to $\pi(Q^*(p),p)_p=0$, while the larger solution (p') results in the minimum value of $\pi(p)$ for $p>p^*$. However, for linear demand, *the profit* function *is strictly concave for almost the entire range of feasible p,* except near the upper feasible limit, a/b (refer to Fig. 2). As evident from Table 3, p' is typically quite close to (a/b). Another difference is the fact that as $p\to 0$, $\pi(p)\to -(ac+\sqrt{2aKh})$, which is significantly negative, but finite. On the other hand, $\pi(p')$, the minimum

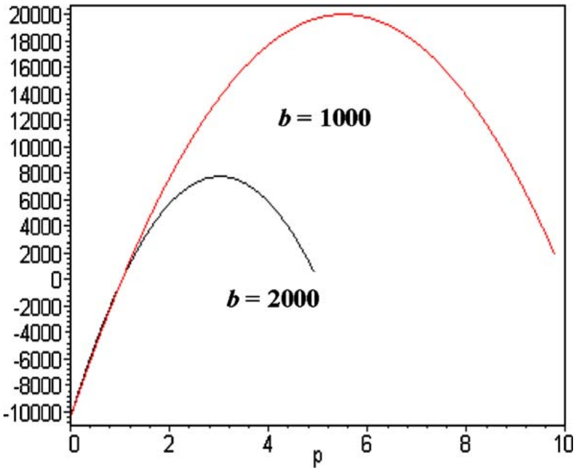


Fig. 2. Profit function ($\pi(p)$) for linear demand ($a = 10000$, $c = 1$, $h = 0.0077$, $K = 1000$).

possible profit for $p > p^*$, is in general only slightly negative. In fact, our extensive numerical experiments (Table 3) have shown that, as with log-linear demand, $\pi(c)\,(<0) << \pi(p')\,(<0)$. So, our managerial insight regarding *aggressive pricing being more desirable than being cautious* still holds true. However, it is worthwhile to mention that while Proposition 3 is still valid, the penalty for upward/downward deviations from p^* are almost the same for equal deviations, unless the extent of deviations and/or price sensitivity is large (Table 4). Moreover, we also noted that for linear demand, the optimal batch size (Q^*) is strictly increasing in K and decreasing in b for realistic parameter values. *The non-intuitive managerial insights are thus less important if demand is linearly dependent on price*.

4. A mark-up model with order quantity as the only decision variable

Though academics prefer to model price as a decision variable, practitioners frequently use a mark-up pricing strategy for decision-making. In this section we analyse the scenario where a firm has already decided on a mark-up rate η (>1). Once again we first concentrate on log-linear demand, and subsequently show how a linear demand model affects our results. The price, p, is set as a percentage mark-up over the total operating cost per unit, m, i.e.,

$$p = \eta m, \tag{8}$$

where $(\eta-1)\,100\%$ is the desired per unit contribution margin. With a mark-up model, the choice of a mark-up rate η effectively links price and quantity decisions. Thus the firm has to decide only on the batch size Q, in order to be able to maximize its profit per unit time π, i.e.,

(P2) $\underset{Q}{\text{Maximize}}$

$$\pi(Q) = (p-m)\lambda = a\eta^{-b}(\eta-1)m^{(1-b)}$$
$$\text{subject to :}\quad p > m > 0 \text{ and } \lambda \geq 0, \tag{9}$$

where m is given by (2), p by (8) and λ by (1). Since price is a function of the operating cost, and demand is a function of price, demand is also a function of the operating cost. Also note from (2) that demand itself affects the unit operating cost.

We will initially prove some results assuming $b = 2$ for log-linear demand. Substituting $\lambda = a\eta^{-2}(m^{-2})$ into (2) and solving for m we obtain

$$m = \left(1 \pm \sqrt{1-4uv}\right)/(2u),$$
$$\text{where } u = (cQ)(i/2a\eta^{-2})$$
$$\text{and } v = (K/Q) + c. \tag{10}$$

We can show that the root corresponding to the minus sign in m will always give a higher profit and thus there is a single relevant m for a given Q. Substituting the root corresponding to the minus sign of m into (10), the firm's problem can be written as:

(P3) $\underset{Q}{\text{Maximize}}$

$$\pi(Q) = (2a\eta^{-2}(\eta-1)u)/\left(1-\sqrt{1-4uv}\right)$$
$$\text{subject to :}\quad 0 < 4uv \leq 1, \tag{11}$$

where the feasible range of K is $0 < K \leqslant [(a\eta^{-2}/2ic) - Qc]$.

Proposition 4. *$\pi(Q)$ is concave in Q for feasible Q.*

Proof. See Appendix. □

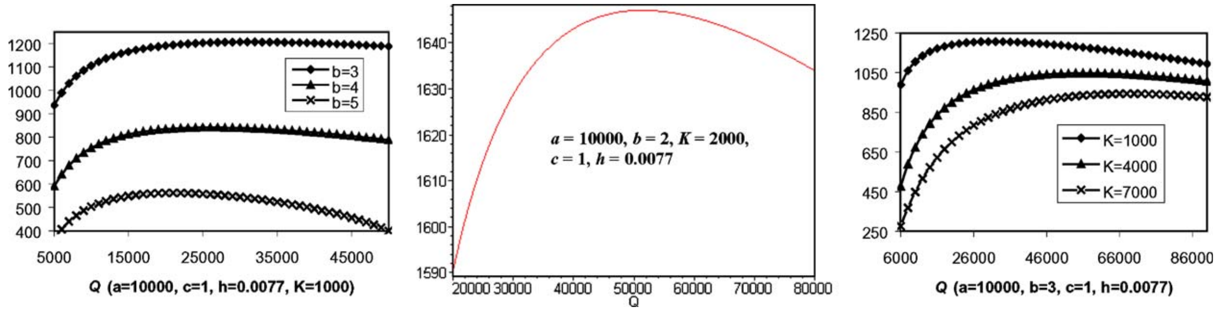


Fig. 3. Mark-up profit function ($\pi(Q)$) for different values of K and b, log-linear demand.

Using Proposition 4, and solving for Q in $\pi(Q)_Q = 0$ yields:

$$Q^* = (1/c)\left[\sqrt{(2Ka\eta^{-2})/h} - 2K\right]. \tag{12}$$

From (12), we can see that we require $K < (a\eta^{-2}/2ic)$ for Q to be non-negative. This requirement is satisfied by all feasible K. Hence, once the manager has decided on the mark-up rate, she/he can easily calculate the profit-maximizing batch size from (12). In fact, based on (12) and (2), we can show that $Q^* = \sqrt{2aK/(h\eta^2 m^{*2})}$, where $m^* = m(Q^*)$. The explicit expression for Q^* in (12) also allows us to investigate analytically the behaviour of the optimal batch size.

Proposition 5. *Q^* is $\uparrow$ in a, $\downarrow$ in h, $\downarrow$ in η, $\downarrow$ in c and concave in K (not monotone) reaching the maximum at $K = a\eta^{-2}/8h$ (say K_η).*

Proof. Immediate. □

Note the similarity between Propositions 5 and 2 with regards to the impact of K on Q^*. Moreover, as η increases, price increases, causing a reduction in demand. The firm then tries to balance the loss in revenue by reducing the order quantity to save on holding costs.

For log-linear demand with $b \neq 2$, the analysis is difficult. We can say, however, that based on (9) it is clear that Q^* is given by the solution to $\partial m/\partial Q = 0$. In general, there is a unique profit-maximizing $Q^* = \sqrt{2aK/(h\eta^b m^{*b})}$, i.e., $Q^* = \sqrt{2K\lambda/h}$ at $p = \eta m^*$, $\forall b$. Proposition 5 also holds true for all b (in general, $K_\eta = (ac^2\eta^{-b}(b-1)^{b-2})/(h((bc)^b))$. However, a more interesting issue is to study the behaviour of the mark-up profit function with changes in batch size. We illustrate this in Fig. 3 for different values of b and K. The most noteworthy aspect of the figure is the flatness of the profit function as the batch size changes, especially around the optimal batch size for low values of b and K. It is well-known that an EOQ cost curve is flat-bottomed, with the cost increase being only 25% if the batch size is double or half the optimal quantity (Nahmias, 2001). *The mark-up based profit model appears to be particularly insensitive to the batch size.* In our numerical experiments we generally observed that the profit penalty is much lower than 25%, even for a 100% deviation of the batch size from optimal. For example, with $b = 3$ and $K = 1000$, $\pi(Q^*)/\pi(2Q^* = 1.04$ and $\pi(Q^*)/\pi(0.5Q^*) = 1.03$, with $b = 3$ and $K = 4000$, $\pi(Q^*\pi(2Q^* = 1.10$ and $\pi(Q^*)/\pi(0.5Q^*) = 1.07$, and even for $b = 5$ and $K = 1000$, $\pi(Q^*)/\pi(2Q^*) = 1.17$ and $\pi(Q^*)/\pi(0.5Q^*) = 1.10$. Only when the set-up cost (K) and/or the price-sensitivity (b) are substantially high (e.g., $b = 6$ and $K = 4000$) would the firm stand to loose as much as 25% (or more) for doubling or halving the optimal batch size. Also it is clear from Fig. 3, *not only is the profit* function *quite flat, it is even more so for batch sizes above the optimal, compared to batch sizes that are below optimal* (note the similarity with the behaviour of the profit function with respect to changes in p in the previous section).

We have not seen any analysis of this type of mark-up model in the literature. The only model that is somewhat similar is the one suggested by Ladany and Sternlieb (1974) who analysed the optimal batch size of a firm when price is a mark-up over the variable purchase (production) cost, i.e., $p = \eta c$. In practice, mark-ups might be based only on the variable costs, excluding inventory costs, due to several reasons: (a) managers do not understand how the price impacts the demand for the products, and therefore the inventory policies are kept fixed over the short term (implying inventory costs are fixed, rather than variable), and (b) purchase cost might be more easily identifiable than set-up or holding costs (Hay and Morris, 1991). This is especially true when ordering and holding costs are allocated. Obviously, Ladany and Sternlieb's model is equivalent to the traditional EOQ model where λ is the demand corresponding to the price ηc, and the profit-maximizing batch size is then given by (for any b and log-linear demand)

$$Q_{\mathrm{LS}} = \sqrt{\frac{2aK}{(\eta c)^b h}}. \tag{13}$$

If we compare Q^* and Q_{LS}, assuming the same value of η, then it is obvious that since (I) $Q^* = \sqrt{2K\lambda/h}$ at $p = \eta m^*$, while $Q_{\mathrm{LS}} = \sqrt{2K\lambda/h}$ at $p = \eta c$, (II) $m^* > c$, and (III) λ is a decreasing function, $Q_{\mathrm{LS}} > Q^*\ \forall b$. With both demand and price being functions of batch size in our model, we would expect that there might be a difference between Q_{LS} and Q^*. In fact, when K and/or b are high the difference can be as high as 50% (Tables 5 and 6). However, it is important to keep in mind that *when mark-up price is based only on c, rather than m, the firm will most probably use a higher value of η to reflect the higher fixed costs* (due to inventory costs). The monotone decreasing nature of $Q_{\mathrm{LS}}(\eta)$ then implies that there is a particular mark-up value ($>\eta$) that results in $Q_{\mathrm{LS}} = Q^*$.

Managers will be interested in understanding the effect on the profit function from using Q_{LS} instead of Q^*. When comparing $\pi(Q_{\mathrm{LS}})$ and $\pi(Q^*)$ we should keep two things in mind: (I) the profit function is relatively flat with respect to the batch size, especially for low values of b and K and $Q > Q^*$, and (II) inventory costs in most practical cases account for only 10–30% of the operating costs. Since $Q_{\mathrm{LS}} > Q^*$, it is then not surprising that the *profit penalty for not including the inventory costs in the mark-up is, in general, not very significant*. In fact, it might be argued that given

Table 5
Profit penalty from ignoring inventory costs or batch size optimization in mark-up model with log-linear demand (for differing K)

K ($b = 3$)	Q^*	Q_{LS}	$\pi(Q^*)$	$\pi(Q_{\mathrm{LS}})$	$\pi(Q = 45000)$	$\pi(Q^*)/\pi(Q_{\mathrm{LS}})$	$\pi(Q^*)/\pi(Q = 45000)$
1000	31,336	34,384	1206.57	1205.88	1195.63	≈ 1	1.01
4000	56,351	68,768	1047.09	1040.41	1039.36	1.01	1.01
7000	68,939	90,971	943.41	925.08	909.01	1.02	1.04
10000	76,857	108,731	859.80	821.97	799.04	1.05	1.08
13000	82,023	123,973	787.31	715.87	705.29	1.10	1.12

Table 6
Profit penalty from ignoring inventory costs or batch size optimization in the mark-up model with log-linear demand (for differing b)

b ($K = 1300$)	Q^*	Q_{LS}	$\pi(Q^*)$	$\pi(Q_{\mathrm{LS}})$	$\pi(Q = 45000)$	$\pi(Q^*)/\pi(Q_{\mathrm{LS}})$	$\pi(Q^*)/\pi(Q = 45000)$
1.5	45,793	47,729	1968.86	1968.81	1968.85	≈ 1	≈ 1
2	42,099	44,699	1671.89	1671.71	1671.66	≈ 1	≈ 1
3	35,233	39,204	1184.26	1183.21	1178.64	≈ 1	≈ 1
4	28,950	34,384	811.52	807.82	785.28	≈ 1	1.03
5	23,098	30,157	527.35	516.05	431.47	1.02	1.22
6	17,428	26,449	310.10	265.10	-[a]	1.17	-

[a] $b = 6$ results in extremely low profits for $Q = 45,000$, so we do not report it.

that the profit function is quite flat, even *the penalty for not optimizing the batch size altogether in a mark-up price model might be relatively low*. One way to understand the "damaging" effect of ignoring inventory costs or not optimizing Q is to study $\pi(Q^*)/\pi(Q_{\mathrm{LS}})$ and $\pi(Q^*\pi(Q)$, where Q is some arbitrary batch size. We illustrate these ratios in Tables 5 and 6 for different values of K and b.

Example 2. Consider the parameters $a = 10{,}000$, $c = \$1/\text{unit}$, $i = \$0.0077/\$/\text{week}$, $h = i^*c = \$0.0077/\text{unit/week}$ and $\eta = 1.3$. In that case, we have the following:

Clearly, in most of the cases using Q_{LS} for Q^* does not result in any significant profit penalty. Especially when K and/or b are low, i.e., low inventory cost situations (e.g., $K = 4000$ and $b = 3$ implies that inventory costs are 14% of c at Q^* the difference between Q_{LS} and Q^* is quite small, and $\pi(Q^*)/\pi(Q_{\mathrm{LS}})$ is very close to 1. Note that in those instances even if the firm does not optimize with respect to batch size and decides to maintain a fixed Q of 45,000, it does not forego much profit (we decided on 45,000 assuming that the firm has some prior knowledge of price-sensitivity being around $b = 3$ and set-up cost around \$2000; however, the essential insight remains valid for quite a large range of Q). Only when the inventory cost becomes substantial due to high set-up cost and/or price-sensitivity, e.g., inventory costs are approximately 31% of c for $K = 13000$ and $b = 3$, the batch size optimization issue becomes important. High values of K and/or b also lead to somewhat higher $(Q_{LS} - Q^*$ and $\pi(Q^*)/\pi(Q_{\mathrm{LS}})$.

However, note that our above results are based on the same η for Q_{LS} and Q^*. If we use a higher η in the Ladany–Sternlieb model to act as a surrogate for inventory costs, the profit penalty will be lower. For example, if we use $\eta = 1.7$ for $K = 13,000$ and $b = 3$, $\eta = 1.5$ for $K = 1300$ and $b = 6$, in place of 1.3 to determine Q_{LS}, the profit penalty will be much lower. In general, a somewhat higher η for Q_{LS} will reduce $\pi(Q^*)/\pi(Q_{\mathrm{LS}})$. When we keep these things in mind, it is clear that in most realistic circumstances, ignoring the inventory costs for mark-up pricing is acceptable (and clearly is better than assuming a fixed Q). However, if K and b are simultaneously high, we would still advise decision makers to determine the optimal batch size and mark-up price taking into account the total operating costs, not only the unit production cost. This will reduce the potential for any profit loss.

There is a close relationship between the models of Sections 3 and 4. We can either assume that p and Q are independent decision variables or equivalently, η and Q are decision variables. For each optimal p there is an equivalent optimal η. For example, with $K = \$400/\text{order}$, $a = 10{,}000$, $h = \$0.0077/\text{unit/week}$, $c = \$1/\text{unit}$ and $b = 2$, at $\eta^* = 1.95$ the mark-up model and the independent price model will lead to the same optimal decision variable values. Evidently, if $\eta \neq \eta^*$, the optimal decision variable values in the mark-up model and the independent-price model might be quite different. Hence, for a given η, managers should use a model like ours to determine the optimal operating decision variable values. Obviously, any $\eta \neq \eta^*$ will lead to some profit loss. As we noted in Section 3, it is better to select a higher than optimal mark-up than a lower one, provided the deviations are equal.

Some of the previous results hold true for linear demand ($\lambda = a - bp$) as well, especially for large values of η. For example, under that assumption also it is possible to characterize the profit function, with $Q^* = \sqrt{2aK/(h\eta^b m^{*b})}$ $(= \sqrt{2K\lambda/h}$ at $p = \eta m^*)$ being given by the unique solution to $\partial m/\partial Q = 0$. Once again we have $Q_{\mathrm{LS}} > Q^*$ (see Table 7), and the difference increases with K and b. However, note that the difference $(Q_{LS} - Q^*)$ is less substantial than for non-linear demand. Another important characteristic of a linear demand model is that the profit function is much flatter in terms of the batch size compared to non-linear demand (Fig. 4). These two properties jointly imply that the penalty for using Q_{LS} in place Q^* is much less severe for linear demand (Table 7), and hence, *managers can safely base their mark-up price only on c*.

Ray et al. (2003) has shown that our models in Sections 3 and 4 can be extended to incorporate process-improving investments, specifically improvements aimed at reducing set-up times (costs) (in the spirit of Porteus, 1985). In these extensions,

Table 7
Profit penalty from ignoring inventory costs in mark-up model for varying K and b with linear demand ($a = 10,000$, $c = 1$, $\eta = 2$, $h = 0.0077$)

	Q^*	Q_{LS}	$\pi(Q^*)$	$\pi(Q_{LS})$	$\pi(Q^*)/\pi(Q_{LS})$
$K = 1000$, $b = 3000$	30,613	32,233	3843.73	3843.47	≈ 1
$K = 3000$, $b = 3000$	50,624	55,829	3678.67	3676.46	≈ 1
$K = 10000$, $b = 3000$	80,830	101,929	3137.80	3083.35	1.02
$K = 3000$, $b = 3500$	38,597	48,350	2209.00	2170.00	1.02

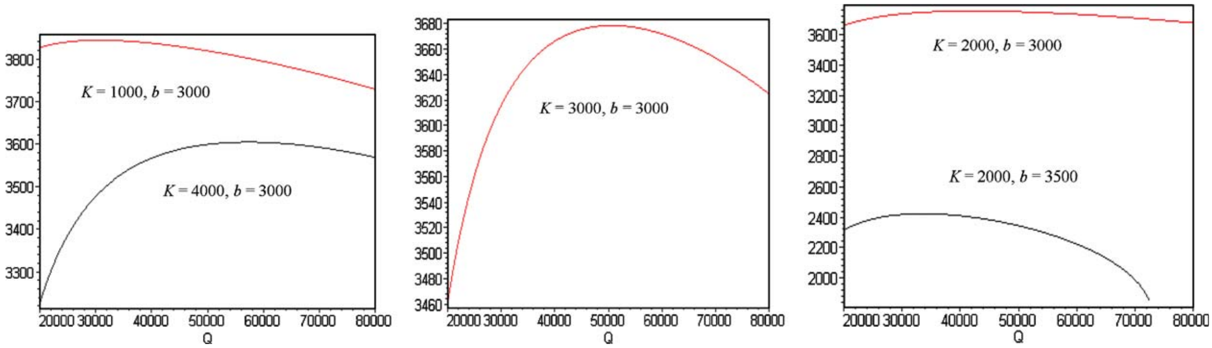


Fig. 4. Mark-up profit function ($\pi(Q)$) for different values of K and b, linear demand.

we were able to integrate the cost-reducing operations research-based models with relevant microeconomic models to establish the significant contribution of optimal selection of the set-up cost level to the profitability of a firm. While most of our previous insights hold true, the analysis becomes considerably more messy. In the interest of space, we do not report the details in this paper.

5. Conclusions

We set about to model a make-to-stock firm selling a single product to price-sensitive customers, where the firm might be using either the policy of price as an independent decision variable, or as a mark-up over the operating costs. We first formulated a model where the price and order quantity were independent decision variables, and the demand function was either log-linear or linear in nature. We determined the optimal decision variable values and also performed sensitivity analysis with respect to the different parameters. We showed that when there are interactions between price, demand and operating costs, some of the best-known properties from classical inventory management do not hold true anymore. Specifically, we showed that the optimal order size is not necessarily monotone increasing in the set-up cost for *highly elastic demand and quite high set-up cost*. More importantly, from a profit perspective, we demonstrated that when unsure about the optimal pricing policy, *it is better for managers to be aggressive in terms of pricing* (*by setting a higher than optimal price*), *rather than to set it at a lower value*. In fact, while there is a bound on the losses that a firm can incur by following an aggressive pricing strategy, these losses can be unbounded for low prices. We also developed a rather unique model which can help managers establish optimal decisions when using mark-up pricing. One of the important characteristics of this model is that the mark-up is based on the total operating cost per unit, not just the unit production/purchase cost, as has been done in previous literature. We showed that basing the mark-up only on the unit production/purchase cost would result in above-optimal batch sizes for both log-linear and linear demand, hence leading to a profit penalty. We also illustrated that this penalty is substantial *only* when the demand is non-linear with high price-elasticity, and/or the set-up cost is high. Otherwise, in a mark-up price model the profit penalty for not including inventory costs while determining the optimal batch size, or sometimes even ignoring the batch size optimization issue, might not be significant.

In terms of scope for future research, developing an optimal pricing model of a make-to-stock firm that invests in reducing the variability of supply lead time might be analytically complex but an interesting avenue of research. Most of the process-improvement models in the operations management literature assume demand to be constant or stochastic with the mean demand rate being constant. It might be useful to use modelling technique similar to ours to analyze some of these models with price-sensitive demand.

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Appendix

Proof of Proposition 1. Note from (6) that the term outside [.] is always negative. Hence, the first-order condition reduces to

$$2\sqrt{(aKh)/p^b}(-p + b(p - c)) = \sqrt{2}bKh. \qquad \text{(A.1)}$$

Let us refer to $\sqrt{2}bKh$ as RHS and $2\sqrt{(aKh)/p^b}(-p + b(p - c)$ as LHS. It is clear that RHS is constant positive with respect to (wrt) p. As $p \to 0$, LHS $\to -\infty$ $\forall$ $b>1$. For $p>0$, LHS behaves as follows (Ray et al., 2003): increasing in p and $\to +\infty$ as $p \to \infty$ for $1<b<2$; increasing in p and $\to 2\sqrt{aKh}$ as $p \to \infty$ for $b=2$; increasing until $p = b^2c/(b^2 + 2-3b)$ and then decreasing and $\to 0$ as $p \to \infty$ for $b>2$. Similarly, from (7), $\pi_{pp} = 0$ yields:

$$\sqrt{2}(b + 2)Kh = 4\sqrt{(aKh)/p^b}(-p + b(p - c) - c). \qquad \text{(A.2)}$$

For (A.2), LHS is constant positive wrt p. As far as RHS is concerned, as $p \to 0$, RHS $\to -\infty$. For $p>0$, it behaves exactly like LHS of (A.1), except $\to 4\sqrt{aKh}$ as $p \to \infty$ for $b = 2$, and increasing until $p = (b^2c + bc)/(b^2 + 2-3b)$ and then decreasing for $b>2$. So, there is exactly one change of sign for π_p and π_{pp} when $b \leqslant 2$ and exactly two for $b>2$, assuming that the maximum of LHS of π_p and RHS of π_{pp} are greater than $\sqrt{2}bKh$ and $\sqrt{2}(b + 2)Kh$, respectively. This would necessitate sufficiently large a, which we will assume to hold true. Note that π is increasing concave and negative for $p = c$. Also, if there is no intersection between RHS and LHS it will imply that π is always increasing, and $\pi<0$ for entire feasible range of p (since we know that for $b>1$, $\pi \to 0$ as $p \to \infty$). Since the profit function, as well as its derivatives are continuous, we can then prove Proposition 1.

From the above discussion it is easy to show that (assuming $\pi(p)>0$ for some p), for any value of b, *there is only one contiguous range of p for which π is decreasing*. Specifically, for $1<b\leqslant 2$, there will be a unique solution to $\pi_p = 0$ of (6) (say p^*) until which $\pi_p>0$ and then $\pi_p<0$. Clearly p^* is the smallest p for which $\pi_p \leqslant 0$. Also we can show that for $b \leqslant 2$, if we substitute p^* in π_{pp}, it will always be negative $\Rightarrow$ at p^*, π is concave. For $b>2$, $\pi_p = 0$ will have two solutions. The profit function will be decreasing for all p in between the two solutions (say p^* is the smaller solution and p' is the larger one), and increasing for$p<p^*$, and also for $p>p'$ ($\Rightarrow$ π attains its minimum for $p>p^*$ at $p = p'$). Moreover, it will be concave up to the first solution of $\pi_{pp} = 0$, then convex and again concave after the second solution of $\pi_{pp} = 0$. For $b>2$, as $p \to \infty$, we can show that π is increasing concave in p, $\to 0$ from a negative value. Since both $\pi_p = 0$ and $\pi_{pp} = 0$ has exactly two solutions, π is increasing concave at $p = c$ and as $p \to \infty$ it is increasing concave from the negative side, we can convince ourselves that the profit function attains its maximum at p^* (again p^* is the smallest p for which $\pi_p \leqslant 0$). Since we assume that the business is profitable for some p, it is then trivial to prove that p^* will be the unique profit-maximizing price (recall that for $b > 1$, $\pi \to 0$ as $p \to \infty$). For $b = 2$ we can express p^* and Q^* explicitly as follows:

$$p^* = \frac{2c\sqrt{hKa}}{-Kh\sqrt{2} + \sqrt{hKa}},$$

and

$$Q^* = \frac{1}{\sqrt{2}ch}(-Kh\sqrt{2} + \sqrt{hKa}).$$

Note that while $\pi(p') < 0$, it must be finite. If $\pi(p') = -\infty$, then as long as $\pi(p)$ is continuous we cannot have $\pi(p) \to 0$ as $p \to \infty$. Since we know the latter assertion to be true, $\pi(p')$ must be finite negative. □

Proof of Proposition 2. We only show the proof for the effect of a here for log-linear demand. The others can be determined in a similar fashion. Note that p^* will be obtained from solving (A.1), which clearly implies $(-p^* + b(p^* - c)) > 0$. Total differentiation of (A.1) wrt a yields:

$$\partial p^*/\partial a = -[\partial(\partial\pi/\partial p)/\partial a]/[\partial(\partial\pi/\partial p)/\partial p]\big|_{p=p^*}.$$

Now we can easily show that $\partial(\partial\pi/\partial p)/\partial a\big|_{p=p^*} < 0$, and obviously $[\partial(\partial\pi/\partial p)/\partial p]\big|_{p=p^*} \leqslant 0$, $\Rightarrow$ $\partial p^*/\partial a < 0$. As far as the effect of a on Q^* is concerned,

$$\partial Q^*/\partial p = (\partial Q^*/\partial p)(\partial p/\partial K)\big|_{p=p^*} + (\partial Q^*/\partial a).$$

We know that $\partial p^*/\partial a < 0$ and we can prove that $\partial Q^*/\partial p^* < 0$ and $\partial Q^*/\partial a > 0 \Rightarrow \partial Q^*/\partial a > 0$.

The effect of K on p^* can be proved as above. However, the expression for $\partial Q^*(p^*(K))/\partial K$ is quite involved. Nevertheless, it is possible to show that $\partial Q^*(p^*(K))/\partial K = 0$ has only one solution, given by K_b of Proposition 2, with $Q^*(p^*(K))$ increasing until K_b and then decreasing. □

Proof of Proposition 3.

$$\pi(Q^*(p), p)_p = (a/\sqrt{aKh}p^{(b/2)+1})((1-b)\sqrt{aKh}p^{(1-(b/2))} + (b/\sqrt{2})Kh + (bcp^{-b/2})\sqrt{aKh}). \quad \text{(A.3)}$$

Suppose

$$f(p) = (bcp^{-b/2})\sqrt{aKh}) - (b-1)\sqrt{aKh}p^{(1-(b/2))}$$

From (A.3) it is clear that at p^*, $f(p^*)$ must be negative. As $p \to \infty, f(p) \to +\infty$, and is decreasing convex. In fact we can show that $f(p)$ will be decreasing until $p = (b^2c)/[(b-1)(b-2)]$, and then increasing, and also that it is convex until $[(bc)((b/2)+1)/-b)(1-(b/2))]$ $(> (b^2c)/[(b-1)(b-2)])$, and then concave. Basically, it means that for $b \leqslant 2$, $f(p)$ is decreasing convex, while for $b > 2$, it is initially decreasing convex, then increasing convex and then increasing concave. Now suppose, $b = 2$. In that case, $f(p) = (2c/p)(\sqrt{aKh})$, $\pi_p(p^* + \delta) = -(a/(\sqrt{aKh}(p^* + \delta)^2))$ $[f(p^*) - f(p^* + \delta)] < 0$, and $\pi_p(p^* - \delta) = (a/(\sqrt{aKh}(p^* + \delta)^2))$ $[-f(p^*) + f(p^* - \delta)] > 0$. Since $f(p)$ is positive decreasing and convex in p, $0 < f(p^*) - f(p^* + \delta) < f(p^* - \delta) - f(p^*)$, and it is clear that $(p^* + \delta)^2 > (p^* - \delta)^2$. This implies that $\pi_p(p^* - \delta) \geqslant |\pi_p(p^* + \delta)|$, i.e., $\pi(p^*) - \pi(p^* + \delta) \leqslant \pi(p^*) - \pi(p^* - \delta)$, which proves the proposition. The proposition can be proved for any $b \leqslant 2$ in a similar way.

For $b > 2$, let us take for example $b = 4$. In that case $f(p) = (4c/p^2)\sqrt{aKh} - (3/p)\sqrt{aKh}$. This expression is decreasing until from $p = c$ until $p = (8c/3)$ and then increasing, while convex until $p = 4c$ and then concave. Also, we can show that around p^* it is decreasing convex. Now, $\pi_p(p^* + \delta) = -(a/(\sqrt{aKh}(p^* + \delta)^3))$ $[f(p) - f(p^* + \delta)] < 0$, and $\pi_p(p^* - \delta) = (a/(\sqrt{aKh}(p^* + \delta)^3))$ $[-f(p^*) + f(p^* - \delta)] > 0$. Since, $(p^* + \delta)^3 > (p^* - \delta)^3$, $\pi_p(p^* - \delta) \geqslant |\pi_p(p^* + \delta)|$, i.e., $\pi(p^*) - \pi(p^* + \delta) \leqslant \pi(p^*) - \pi(p^* - \delta)$. This holds true until $p = (8c/3)$. After that $f(p)$ starts increasing, so it will still hold true (since $f(p^*) - f(p^* + \delta)$ will decrease while $f(p^* - \delta) - f(p^*)$ will increase). Note that after $f(p^* + \delta) > f(p^*)$, π is increasing and so we are not interested for any p larger than that value. The proof for any $b > 2$ can be accomplished similarly (though the exact values will be different).

For linear demand, $f(p) = -2bp + \sqrt{(Kh/2}(b/\sqrt{(a - bp)})$, which is decreasing convex in p. Also as $p \to 0$, $f(p) > 0$, and at p^*, $f(p^*) < 0$. So the proof will be similar to the log-linear demand with $b = 2$. Note that, for large a, $\partial f/\partial p \sim -2b \Rightarrow$ the profit function will be almost symmetrical, i.e., the profit penalty will be almost equal for equal deviations above and below p^*. $\square$

Proof of optimality for linear demand. For linear demand

$$\pi(Q^*(p), p)_p = a - 2bp + bc + \sqrt{\frac{Kh}{2}\frac{b}{(a-bp)}},$$

$$\pi(Q^*(p), p)_{pp} = -2b + \sqrt{\frac{Kh}{2}}\frac{b^2}{2(a-bp)^{3/2}}.$$

From the above equations and the expression for $\pi(Q^*(p), p)$ it is clear that as $p \to$ its upper feasible limit $= (a/b)$, $\pi \to 0$, $\pi(Q^*(p), p)_p \to \propto$ and $\pi(Q^*(p), p)_{pp} \to \propto$. So, at upper limit the profit function is increasing convex in p. At $p = c$, $\pi < 0$ and increasing concave. Also, analyzing $\pi(Q^*(p), p)_{pp}$ we can see that $\pi(Q^*(p), p)_{pp} = 0$ has only one solution given by $p = (a/b) - (b^2Kh/32)^{1/3}$ (say, p_c) $< (a/b)$. So, π is concave until p_c and then convex. If $p_c < c$, π must be increasing convex for $c \leqslant p \leqslant (a/b)$, and even as $p \to (a/b)$, $\pi \to 0$. So, $\pi < 0$ for feasible p. If $c \leqslant p_c \leqslant (a/b)$, π is initially concave and then convex, and increasing at both feasible limits. Then either π is throughout increasing, in which case once again $\pi < 0$ for feasible p, or $\pi(Q^*(p), p)_p = 0$ will have two solutions, of which the smaller one will be the optimal (it will lie in the concave region) and the larger one will result in minimum π (in the convex region). Hence, p^* might be $= (a/b)$, but then the firm will never make any positive profit. So, we disregard it and only consider p^* to be the smaller solution to $\pi(Q^*(p), p)_p = 0$, i.e., smallest value of p for which $\pi(p)_p \leqslant 0$. Note that p_c will normally be quite close to (a/b), especially if a is sufficiently large $\Rightarrow$ π will normally be concave throughout the feasible p, except very close to (a/b). $\square$

Proof of Proposition 4. With $1 - \sqrt{1 - 4uv} = T$, π in (12) can be written as

$$\pi = (2a\eta^{-2}(\eta - 1)u)/T. \tag{A.4}$$

Differentiating (A.4) twice wrt Q and setting the result to zero gives a cubic equation in Q:

$$u_{QQ}T^2 - uTT_{QQ} - 2TT_Qu_Q + 2u(T_Q)_2 \leq 0. \tag{A.5}$$

We have $u_{QQ} = 0$ and $T_{QQ} \geqslant 0$. Of the three solutions to (A.5), two will be complex and one will be negative. So, for any positive Q (A.5) has the same sign. Note that T is increasing linear in Q, i.e., $T_Q \geqslant 0$ $\forall Q$. It is now easy to show that (A.5) $\leqslant 0$ for any feasible $Q \leqslant Q^*$ (solution to $u_QT - T_Qu = 0$). Since we know that (A.5) will have the same sign for any positive Q, it must be negative for all feasible Q. This implies that π is concave for all feasible Q. $\square$

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