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DOXASTIC INFORMATION PARTITIONS, AND AN ANTI-NO-TRADE THEOREM (*)

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ABSTRACT

Partitions of the set of states of nature are the traditional information structures in decision theory. In a multi-person context, common knowledge of the choices, such as it occurs in a trade, implies common knowledge of relevant information: hence a well-known impossibility theorem for trade based on differing belief can be obtained. A new set theoretic axiomatization of the beliefs held by decision makers allows here for the construction of doxastic, i.e. possibly incorrect instruments of measurement of the «outside» world, and allow traders to «agree to disagree».

1. INTRODUCTION

The possibility of trade based on differing information is denied by a consolidated literature. Some of the earlier works are by Aumann (1976), Milgrom and Stokey (1982), Geanakoplos and Sebenius (1983), Bacharach (1985). The core of this impossibility result can be found in the fact that in the process of tentatively agreeing to a transaction, such as a small bet, the parties generally reveal to each other their private information, until neither is any longer prepared to accept the opponent's offer. It can also be argued in a more *ex post* and static fashion that the very prospect of having one's own offer accepted is sufficient to make one withdraw it. Hence common knowledge of a bet

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on the occurence of an event makes the bet impossible (other than as a risk distributing transaction), unless different priors are attributed to the parties. But if one adheres to a non-personalistic view of neo-Bayesian decision theory, differing beliefs are always to be considered as the result of differing information (Aumann, 1987), which should be modeled explicitly, while common priors should be assumed (this is the so-called «Harsanyi's Doctrine»).

Agents' information structures have been traditionally formalized in a set theoretical framework as partitions of the set of states of the world (Nermuth, 1982), and it is this set-up which Aumann (1976) employed to formalize Lewis's (1969) linguistic notion of common knowledge. The no-trade result has more recently been shown to hold under weaker information structures by Samet (1987), Shin (1987), Brown and Geanakoplos (1988). In the wake of the work by Milgrom (1981) and Bacharach (1985), this literature formalizes explicitly the knowledge held by the agents by the means of «knowledge operators» taken from epistemically interpreted models for modal logics (see Hughes and Cresswell, 1968; Hintikka, 1962; Rescher, 1984). While information structures are not partitions in these models, they maintain the assumption that information is *veridical*, i.e. that the actual state of the world is included in the events in the occurrence of which the agents believe. This ends up being sufficient for the exchage of information, which is implicit in a speculative trade, to make the latter impossible. Neo-Bayesian decision theory consists essentially of a decision function from a field of events – to represent the circumstances of the choice – to a set of actions. The information partition is made of the elements in the field which represent the decision-maker's maximal pieces of knowledge; the decision function is defined by taking the maximizer (in the set of actions) of a cardinal utility function (defined over the product of the actions with the states of nature) combined with the *ex post* probabilistic measure on the subset of states which is the decision maker's maximal piece of information. By observing the opponent's choice, an agent may infer the other's information by inverting the decision function of the latter: hence all information becomes common knowledge whenever it is used for action. (If the decision function cannot be inverted, information cannot be completely deduced from choices; but in the case of a decision function which is common to many players – essentially, «Harsanyi's Doctrine» – this is not relevant: it does not matter to me if I cannot find out some piece of information which you have, since this information would not alter either my or your choice anyway).

On the other hand, it could be argued that people engage in bets because they believe their opponent to be wrong, i.e., in one interpretation, to base his or her beliefs on incorrect information. Also, information partitions are often interpreted as formal representations of instruments of measurement of the outside world. Actual instruments are usually less than perfectly fine in discriminating the actual states of things, and this traditionally corresponds to a similar feature in information partitions; actual instruments, on the other hand, may also be incorrect, but this is not allowed for in the traditional theory of information partitions. In this paper we exploit the fact that by using epistemic operators one is able to distinguish the event whose occurrence an agent believes in, from the event that such is the case, and to allow for the possibility that the two do not coincide. We then show that while retaining the assumption that the agents' information structures are partitions, trade can take place without breaching «Harsanyi's Doctrine» of common priors or decision functions, so long as at least one agent's information partition is incorrect.

The paper is organized as follows. In the next section, we present a version of Bacharach's axiomatic system of epistemic information partition. In the third section, a weakening of Bacharach's system is presented, to form «doxastic», i.e. belief information partitions, where room is made for mistakes in information perception. In the fourth section, the notions of common knowledge and common belief are defined by using the systems in the two previous sections, and it is shown that our doxastic partitions allow for common beliefs of beliefs to be consistent with differing beliefs. In the fifth section, we state the anti-no-trade result, with some comments. The last section concludes the paper.

2. EPISTEMIC INFORMATION PARTITIONS

We present in this section a version of Bacharach's (1985) axiomatization of the agents' information partitions by the means of algebraic models of epistemically interpreted modal operators, defined on a σ-field of events.

Let $\mathscr{F}$ be a σ-field of sets containing the singletons, and whose universal event Ω has countably many elements, and is to be interpreted as the set of all possible states of the world. Since $\mathscr{F}$ is σ-complete (i.e. closed under numerable intersection), and Ω is at most numerable, then $\mathscr{F}$ is complete (i.e. closed under any intersection) (see, e.g. Sikorski, 1964, p. 72). Let K_i be an epistemic operator defined on $\mathscr{F}$, in such a way that if E belongs to $\mathscr{F}$, then K_iE (possibly empty) also belongs to $\mathscr{F}$, and is to be interpreted as the epistemic event that agent i has knowledge of the occurrence of the event E. We shall omit the index i in the rest of this section.

Assume that, for E, E_1, E_2, in $\mathscr{F}$,

$$K(E_1 \cap E_2 \cap \ldots) = KE_1 \cap KE_2 \cap \ldots, \tag{P1}$$

$$KE \subseteq E, \tag{P2}$$

$$-KE \subseteq K - KE. \tag{P3}$$

It is immediate to see that the converse of (P3) is a special case of (P2); hence,

$$-KE = K - KE \tag{1}$$

(P1) and (1) guarantee that the set of epistemic events are closed under σ-intersection and complementation; hence they form a σ-field $\mathscr{G}$ which is a sub-field of $\mathscr{F}$. $\mathscr{G}$ is complete for the same resons as $\mathscr{F}$, hence it is set-theoretically atomic (see, e.g. Rasiowa and Sikorsky, 1963, p. 90).

The atoms of $\mathscr{G}$ form a partition of Ω, but this is not yet the agent's information partition, which is to be made of inverse images of the epistemic operator. On the way to prove that, some lemmas will be useful.

Lemma 1: $KE = KKE$.

Proof. Let $KF = -KE$. By (1), $K - KF = -KF$. Hence, $KE = KKE$. ■

Lemma 2: *i)* $K\varnothing = \varnothing$; *ii)* $K\Omega = \Omega$

Proof. *i)* $K\varnothing = K(KE \cap -KE) = KKE \cap K - KE = KE \cap -KE = \varnothing$.

(The second equality holds by (P1); the third equality holds by Lemma 1 and (1)); *ii)* $K\Omega = K - \varnothing = K - K\varnothing = -K\varnothing = -\varnothing = \Omega$, by *i)* and (1). ■

Lemma 3: If $E \subseteq F$, then $KE \subseteq KF$, and $-K-E \subseteq -K-F$.

Proof. $E \subseteq F$ iff $E \cap F = E$. Then, $K(E \cap F) = KE$, and $KE \cap KF = KE$, by (P1). Hence, $KE \subseteq KF$. Also, $E \subseteq F$ iff $-F \subseteq -E$; hence, $K-F \subseteq K-E$, and, by taking complements on both sides, $-K-E \subseteq -K-F$. ■

Lemma 4: (P2) is equivalent to: *i)* $E \subseteq -K-E$, and to: *ii)* $\{\omega\} \subseteq -K-\{\omega\}$.

Proof. i) Let $F = -E$. (If) From $-F \subseteq -KF$ follows that $KF \subseteq F$. (Only if) $K-F \subseteq -F$, so $F \subseteq -K-F$. *ii)* If *i*), then *ii*), trivially. Also, for any $E \in \mathscr{F}$, $E = \bigcap_{\omega \in -E} -\{\omega\}$. So, $KE = K\left(\bigcap_{\omega \in -E} -\{\omega\}\right) =$

$= \bigcap_{\omega \in -E} K-\{\omega\} \subseteq \bigcap_{\omega \in -E} -\{\omega\} = E$, by the completeness of $\mathscr{F}$ and $\mathscr{G}$, (P1), and *ii)* in the form of $K-\{\omega\} \subseteq -\{\omega\}$ (notice that the universal intersections are allowed for by the completeness of $\mathscr{G}$, which in turn is granted by the restriction on the cardinality of Ω). ■

By (P2) and Lemma 4 *i*), we may immediately see that the epistemic events are «weakly consistent», that is, $KE \subseteq -K-E$.

Let Ke be an atom of $\mathscr{G}$. If $\{\omega\} \subseteq -e$, i.e. $e \subseteq -\{\omega\}$, then, by Lemma 3, $Ke \subseteq K-\{\omega\}$, and not $Ke \subseteq -K-\{\omega\}$, since Ke is an atom, therefore is not empty, and only one of the two relations may hold. In the interpretation, if the agent has knowledge of the occurence of an event e, and a state ω is not a member of e, then the agent can also exclude the possibility that ω is the actual state. On the other hand, there may be some state $\omega' \in e$ such that $Ke \subseteq K-\{\omega'\}$, i.e. such that the agent knows it is not the actual one. If this is the case, then e does not constitute the agent's maximal piece of information. Define now $e^* := \{\omega : Ke \subseteq -K-\{\omega\},\ \omega \in e\}$; e^* will constistute the agent's maximal piece of information, and Ke^* will be said to be «strictly consistent».

We can now prove the following.

Lemma 5: $Ke^* = Ke$.

Proof. On the one hand, $e^* \subseteq e$ entails $Ke^* \subseteq Ke$, by Lemma 3. On the other hand, for all $\omega' \in -e^*$, $Ke \subseteq K-\{\omega'\}$, by Lemma 3 again; therefore $Ke \subseteq \bigcap_{\omega' \in -e^*} K-\{w'\} = K \bigcap_{\omega' \in -e^*} -\{\omega'\} = Ke^*$, by (P1). ■

For the agent's maximal pieces of information e^**'s* to be a partition of Ω, two requirements must be fulfilled: (*a*) that they cover the whole of Ω; (*b*) that they do not overlap.

By Lemma 4 *ii*), for any $\omega \in \Omega$, $-K-\{\omega\}$ is not empty. Thus there will be an atom Ke^* of $\mathscr{G}$ such that $Ke^* \subseteq -K-\{\omega\}$, and $\omega \in e^*$, by the construction of e^*, which takes care of requirement (*a*).

Newt we prove the following:

Proposition 1: $Ke^* = e^*$.

Proof. In the light of (P2), we need only prove that $e^* \subseteq Ke^*$. Assume not. Then there will be an $\omega \in e^*$ such that $\{\omega\} \subseteq -Ke^*$. By the construction of e^*, $Ke^* \subseteq -K-\{\omega\}$; on the other hand, $Ke^* \subseteq -\{\omega\}$; hence, by Lemmas 1 and 3, $Ke^* = KKe^* \subseteq K-\{\omega\}$, a contradiction. ■

In the light of Proposition 1, requirement (*b*) above is also fulfilled, the agent's maximal pieces of information are fixed points of the epistemic operator as seen as a function from $\mathscr{F}$ to $\mathscr{F}$, and they coincide with the atoms of $\mathscr{G}$; they are therefore elements of a partition of Ω, which is the agent's information partition.

3. DOXASTIC INFORMATION PARTITIONS

We now intend to construct a weaker axiomatic system for a belief operator which induces an information partition that is not necessarily correct, that is, where the «axiom of correctness» (P2) does not hold. On the same field $\mathscr{F}$ we therefore instead define the doxastic operator B_i, characterizing agent i's beliefs, and such that if $E \in \mathscr{F}$, the doxastic event B_iE also belongs to $\mathscr{F}$, and it constitutes the event that agent *i* believes that event E occurs. We shall drop the index *i* for the rest of this section.

If the events which constitute the agent's maximal pieces of information are to be a partition of Ω, then the set of doxastic events also must partition Ω. Let e_1^* and e_2^* be two elements of the assumed doxastic partition, so that $e_1^* \cap e_2^* = \emptyset$; then we will want $Be_1^* \subseteq B - e_2^*$, that is, the doxastic events to be «counterfactually consistent» (counterfactual consistency is guaranteed to the epistemic atomic events: since $\emptyset = Ke_1^* \cap Ke_2^* = Ke_1^* \cap e_2^*$, or $Ke_2^* \subseteq -e_2^*$, then, by Lemma 1, $Ke_1^* = KKe_1^* \subseteq K - e_2^*$). Otherwise, in some states, a belief in the occurence of e_1^* will not rule out a belief that the different element of the partition

e_2^* may also be occuring. On the other hand, if the doxastic events do not form a partition, that is, $Be_1^* \cap Be_2^* \neq \varnothing$, there will be some $\{\omega\}$ such that, by counterfactual consistency, $\{\omega\} \subseteq Be_1^* \subseteq B - e_1^*$, while by a clearly needed requirement of weak consistency, $\{\omega\} \subseteq Be_2^* \subseteq -B - e_2^*$, a contradiction.

We must therefore close the set of doxastic events under σ-intersection and complementation, so that the atoms of the resulting sub-field $\mathscr{H}$ can be a partition of Ω.

Assume then the following postulates, with $E, E_1, E_2, \ldots \in \mathscr{F}$.

$$B(E_1 \cap E_2 \cap \ldots) = BE_1 \cap BE_2 \cap \ldots, \tag{P4}$$

$$-BE = B(-BE \cup -E);\text{(}^1\text{)} \tag{P5}$$

(P4) and (P5) provide for the required closures. Notice that while (P4) is equivalent to (P1), (P5) is a weakening of (1), in the sense that it is only when $BE \subseteq E$, i.e. when beliefs are correct, that (P5) is equivalent to (1).

Some lemmas can be proved, and are obviously sometimes weaker than their analogue with K. Let $E, F \in \mathscr{F}$.

Lemma 6: If $E \subseteq F$, then $BE \subseteq BF$, and $-B-E \subseteq -B-F$.

Proof. The same as in Lemma 3, by substitution of B to K throughout. ■

Lemma 7: $B-BE \subseteq -BE$.

Proof. $-BE \subseteq -BE \cup -E$; so $B-BE \subseteq B(-BE \cup -E) = -BE$, by Lemma 6 and (P5). ■

Lemma 8: $BBE \subseteq BE$.

Proof. Let $BE = -BF$, and substitute into Lemma 7. ■

Lemma 9: i) $B\varnothing = \varnothing$; *ii)* $B\Omega = \Omega$.

Proof. i) $B\varnothing = B(BE \cap -BE) = BBE \cap B-BE \subseteq BE \cap -BE = \varnothing$, by

(¹) (P5) can be transcribed into propositional modal logic as $\neg bp \equiv b(bp \rightarrow \neg p)$, where p is a generic proposition, and b a modal operator, and read: not to believe that p is equivalent to believing that p is not correct. This can be compared with an analogous transcription of (1): $\neg k \equiv kp \rightarrow \neg p$, which can be read as: not knowing that p is equivalent to the fact of knowing that p is incorrect.

(P4), and Lemmas 7 and 6. *ii)* $\Omega = -B\varnothing = B(-B\varnothing \cup \varnothing) = B\Omega$, by *i)* and (P5). ■

Even though no analogue to the axiom of correctness (P2) has been introduced, it can be shown that the doxastic events are «weakly consistent» (which also proves that the same holds for epistemic events indepentently of (P2)).

Lemma 10: $BE \subseteq -B-E$.

Proof. $-E \subseteq -BE \cup -E$; so, $B-E \subseteq B(-BE \cup -E) = -BE$, by Lemma 6 and (P5). Hence, $BE \subseteq -B-E$, by taking complements on both sides. ■

$\mathscr{H}$ is atomic, for the same reasons as $\mathscr{G}$, and for any $\omega \in \Omega$ there will be some atom Be which covers ω. On the other hand, there may be some $\omega \in \Omega$ such that for no $\omega' \in \Omega$, $\{\omega'\} \subseteq -B-\{\omega\}$; that is, for all $\omega' \in \Omega$, $\{\omega'\} \subseteq B-\{\omega\}$: the agent would under all circumstances believe ω not to be the case. Such a possibility is obviously ruled out by the axiom of correctness in the form of Lemma 4 *ii)*, but it must be ruled out now in a weaker form, and as follows.

For all $\omega \in \Omega$, there exists an ω' such that $\{\omega'\} \subseteq -B-\{\omega\}$. (P6)

«Strict consistency» of the atomic doxastic events will require a similar construction as for the atoms of $\mathscr{G}$. If Be is an atom of $\mathscr{H}$, let $e^* = \{\omega : Be \subseteq -B-\{\omega\}\}$.

Lemma 11: $Be^* = Be$.

Prooof. The proof is the same as in Lemma 5, by substituting B to K, Lemma 6 to Lemma 3, (P4) to (P1), and $\mathscr{H}$ to $\mathscr{G}$ throughout. ■

For the agent's maximal pieces of doxastic information e^*'s to be a partition of Ω, the usual two requirements must be fulfilled: (a') that they cover the whole of Ω; (b') that they do not overlap.

For any $\omega \in \Omega$, there will be some ω' and some Be^* such that $\{\omega'\} \subseteq Be^* \subseteq -\{\omega\}$, $\omega \in e^*$, by (P6), the construction of e^*, and Lemma 11; this takes care of requirement (a').

Since no equivalent to Proposition 1 can be proved, requirement (b') will need an additional axiom.

If Be_1^* and Be_2^* are two atoms of $\mathscr{H}$, then $e_1^* \cap e_2^* = \varnothing$. (P7)

(P7) has no analogue in Bacharach's system. It can be given an interpretation of an «axiom of independence», as follows. We mentioned in the introduction that the information partition is often taken to formalize the agent's «instruments of measurement» of the outside world. A thermometer can be a paradigmatic example: a position of the mercury column will be a «state of the world», and an element of the information partition will be the smallest segment of the adjacent marked scale which encompasses the level of the mercury. A «wrong» thermometer can be thought of as the result of a distortion of the marked scale; a «doxastic» information partition can be had only if the distortion does not depend on the level of the mercury, but is fixed for all temperatures. Otherwise, given a certain level of temperature, the way a different, and not actual level could have been perceived, would not be how the latter level would in fact be perceived if it were the actual one. This is indeed what (P7) postulates.

The fact that (P4) to (P7) are a weakening of (P1) to (P3) is also proved by the following example of an incorrect information partition which satisfies (P4) to (P7), but fails to satisfy (P1) to (P3).

Example 1. Let a_1, a_2, a_3, be the three atoms of a complete field $\mathscr{F}$. Define B on a_3 and on $a_1 \cup a_2$, in such a way that $Ba_3 = a_2 \cup a_3$, and $B(a_1 \cup a_2) = a_1$. There are then two atoms in the sub-field of doxastic events $\mathscr{H}$. (P4), (P6) and (P7) are clearly satisfied. Let us verify (P5).

Consider Ba_3. (P5) requires that $-Ba_3 = B(-Ba_3 \cup -a_3)$, but since $-Ba_3 \cup -a_3 = (a_1 \cup a_2)$, by the definition of B, then $-(a_2 \cup a_3) = -Ba_3 = B(a_1 \cup a_2) = a_1$, which is indeed the case.

Consider now $B(a_1 \cup a_2)$. (P5) requires that $-B(a_1 \cup a_2) = B(-B(a_1 \cup a_2) \cup -(a_1 \cup a_2))$, but since $-B(a_1 \cup a_2) \cup -(a_1 \cup a_2) = a_2 \cup a_3$, by the definition of B, then $Ba_3 = -B(a_1 \cup a_2) = a_2 \cup a_3$, which is indeed the case.

4. COMMON KNOWLEDGE AND COMMON BELIEF

It was argued in the introduction that in the event of a successful trade there is common knowledge of their actions among the traders; that is, both traders know both actions, they both know that each knows both actions, and so on up to any level of interactive knowledge. Aumann (1976) formalized such a notion of common knowledge by identifying it with the events which both agents simultaneously know, i.e. with events of the finest common coarsenig of the information partitions of the two agents. The modal operators which we introduced in the previous section allow for a distinction between knowledge (or belief) of an event which they may both have, and knowledge (or belief) they may have of their own and of the other's knowledge.

Define two knowledge operators, K_1 and K_2, on the same field of events, characterizing agents 1 and 2 respectively, and obeying axioms (P1) to (P3) in section 2. Then if opinions leading to a successful trade are treated as veridical and formalized by such knowledge operators, then in a completed transaction the event

$$K_1 e_1^* \cap K_2 e_2^* \tag{2}$$

is not empty, where e_1 and e_2 are the events which the two agents have knowledge of, respectively, i.e. are elements of their information partitions. In the transaction each of the two events in (2) are common knowledge; that is, both

$$K_2 (K_1 e_1^*) \cap K_1 K_2 (K_1 e_1^*) \cap \ldots \tag{3}$$

and

$$K_1 (K_2 e_2^*) \cap K_2 K_1 (K_2 e_2^*) \cap \ldots \tag{4}$$

are not empty. By Proposition 1, $K_1 e_1^* = e_1^*$, say, so $K_2 K_1 e_1^* = K_2 e_1^*$, that is, knowing that the other knows something is equivalent to knowing it oneself. Hence (3) and (4) are respectively equal to

$$K_2 e_1^* \cap K_1 K_2 e_1^* \cap \ldots, \tag{5}$$

and

$$K_1 e_2^* \cap K_2 K_1 e_2^* \cap \ldots \tag{6}$$

Since events (3), (5), and (6) are assumed to occur, then the event

$$K_1 (e_1^* \cap e_2^*) \cap K_2 (e_1^* \cap e_2^*)$$

also occurs, whereby both agents have knowledge of the same event. If the agents had initially differing information, then the transaction refines both information partitions and brings about the same knowledge;

a transaction based on differing information is thus made impossible. This does not necessarily happen when opinions are treated as beliefs, that is, when information is formalized by the means of the doxastic operator obeying axioms (P4) to (P7) in the preceding section: it then follows that believing that the other agent believes something does not necessarily imply believing it oneself, as shown by the following example.

Example 2. Let a_1, a_2, a_3 be again the three atoms of a complete field $\mathscr{F}$. Define B_1 on $\mathscr{F}$ in the same way as B in example 1, to characterize agent 1's beliefs. Define B_2 on $\mathscr{F}$ in the same way as B_1, but by swapping a_2 and a_3 uniformly, to represent agent 2's beliefs. Clearly B_2 satisfies (P4) to (P7) just in the same way as B_1 did, since nothing changes by changing uniformly the index numbers of the atoms. If a_2, say, realizes itself, then 2 is «wright» and 1 is «wrong», since, in that case, the following event is not empty:

$$B_1 a_3 \cap B_2 a_2 = a_2 \cup a_3. \tag{7}$$

On the other hand, the differing beliefs can be «common belief», since both agents can be informed of the event in the *LHS* of (7). Both $B_1(B_1 a_3 \cap B_2 a_2)$ and $B_2(B_1 a_3 \cap B_2 a_2)$ can be not empty, and occur simultaneously: by substitution, the latter two events are equal to $B_1(a_2 \cup a_3)$ and $B_2(a_2 \cup a_3)$, respectively, and these can be set equal to the atoms $B_1 a_3$ and $B_2 a_2$, respectively, maintaining consistency of the beliefs (the set $a_2 \cup a_3$ is then an inverse image of doxastic atmos, but it is to be refined to guarantee «strict consistency» of the atmos, in the sense of the preceding section). Further iterations of the doxastic operators will also be fixed points, and «common belief» of both agents' beliefs is thus assured.

5. AN «ANTI-NO-TRADE» RESULT

We are now in the position to state the following:

Proposition 2: Let us assume a field $\mathscr{F}$, and agent 1 and 2 have beliefs as defined in Example 2 above. Let the agents have a common decision function δ from the field of events to a common set G of alternative choices, and let the choices be «common knowledge». Then the choices may still differ.

(Before the – trivial – prof, the formulation of the claim itself requires some comment. It is the choices of the agents that in a trade are «common knowledge»; but what our approach (which is common to much of the literature) allows for, is only some formalization of common knowledge – or common belief – of events, i.e. sets of states of nature. It is therefore necessary to be somewhat «sloppy», and admit an informal idea of a «common knowledge of the decision function», which allows the agents to invert the latter and transform information about choices into information about the information from which the choices were derived. This translates the formulation of the claim into one which contains only assertions about knowledge or beliefs of the events).

Proof. Let $C := \{c_{12}, c_{13}, c_2, c_3\}$. It is sufficient to define δ on the four events of the field which constitute the two doxastic partitions, since it is only the four corresponding choices which may matter:

$$\delta(a_2) = c_2;\ \delta(a_3) = c_3;\ \delta(a_1 \cup a_2) = c_{12};\ \delta(a_1 \cup a_3) = c_{13}.$$

If a_2 realizes itself, then a_3 and a_2 are, respectively, 1's and 2's maximal pieces of doxastic information. The agents' choices differ, yet inversion of the decision function leads to belief in the event in the *LHS* of (7) in Example 2; but this is common belief, as was shown in Example 2. ■

The notion of a decision function – into which the more traditional pay-off function and probabilistic measure collapse – is used here to stress a difference in decision theory between the algebraic set theoretical and the measure theoretical structures. In general, an event A may have *ex post* zero probability and may thus be deemed not to be the case by the decision maker, either because it is contained in the complement of an event e – an element of the information partition, say – which is deemed to be the case, or because it as zero probability in the prior distribution (e.g. there are good reasons to assign zero *a priori* probability to the event that an opponent in a game plays a stricly dominated strategy). In the latter case, A may be contained in e, and it could thus be argued that e does not represent the decision maker's maximal information, since e should be refined by subtracting A from it. Yet it could be argued that two different sorts of knowledge are represented in such a formal situation: *a*) The *a priori* knowledge may be construed as a product of the decision maker's rationality (as in the case above of a stricty dominated strategy); this does not easily allow for a notion of a «mistake», without some weakening of the rationality itself; *b*) The information partition may be construed as the result of some perceptive learning of external circumstances. This learning may turn out to be redundant (as was the case with the *a priori* knowledge that excluded the occurence of the event A); this will be represented in our framework by a decision function which cannot be inverted. On the other hand, the set theoretical structure allows here for a notion of a mistake in perception, with no weakening in the rationality of the decision maker.

Our approach envisages the information acquisition process as entirely deterministic, and views the process of appending (probabilistic) truth values to the events as a decision procedure.

6. CONCLUSIONS

The recent literature on neo-Bayesian decision theory puts an emphasis on either the set-theoretic basis or the measure theoretic side of the theory (see for the latter Tan and Werlang, 1985; Mertens and Zamir, 1985, among others). In particular, discussions of the decision-maker possible information structures have been developed within the former approach by Geanakoplos (1989), and Rubinstein and Wolinsky (1989), among others. Information perception is formalized here as a function from the actual state of the world to events which the decision-maker deems to occur, and information structures weaker that partitions are considered, as a result of a weaker rationality in the agent's processing of information. Our modal logic approach gives an account of the relationship between the events which induce beliefs, and the beliefs held; the function which relates the actual state to the events which are deemed to be true by the decision maker, is represented by the set-theoretical notion of inclusion; also, our approach does not weaken the assumption that information structures are partitions, and treats mistakes as of information perception, rather than of rationality.

Within the approach which emphasizes the measure theoretic aspects of neo-Bayesian decision theory, information structures weaker than veridical partitions in a game have been reduced to differences in prior probability distribution among players (Brandenburger, Dekel, and Geanakoplos, 1989). Criticisms to such differences, as mentioned in the introduction to the present paper, may thus be used to criticize weakenings of traditional veridical information partitions. Subjective probability distributions can be considered as components of decision functions from information to choices, which the decision-makers employ: we feel that an emphasis on the set-theoretic basis of the theory may usefully move differences in beliefs between agents away from the rationality embedded in their decision functions, and place them into their information perception. We also conjecture that differences in subjective priors may in their turn be reduced to «unorthodox» information structure.

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