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Revising preferences and choices

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Abstract

Preferences and choices are subject to be modified when new information is brought to the knowledge of an agent, such strategy may suddenly appear more reliable than such other, such restaurant has to be retrieved from the list of the ten best tables of the town. We propose here an easy way to perform this revision through a simple modification of the chain of subsets attached to the agent's behavior: it can be shown indeed that, in the rational case, these chains offer an adequate representation of preferences and of choice functions. Thus the revision problem boils down to adding, retracting or modifying some of the links of the original chain, a perspective that enables an effective treatment of the problem of iterated revision.

Keywords: Rationality; Preferences; Preference orders; Choice functions; Arrow conditions

1. Introduction

We showed recently (Freund, 1998a,b, 1999) that it was possible to study the behavior of an agent, be it expressed by a set of preferences or by an inference relation, through some auxiliary subset of the underlying propositional language $\mathcal{L}$. These results easily translate into a pure set-theoretic framework, showing that when the behavior to be studied is of a *rational* type, it can be simply represented by means of a chain of sets embedded in the sample space Ω. This means that if $\succ$ is the strict partial order over $\wp(\Omega)$ that represents the preferences of the agent, there exists a sequence of subsets, $D_0 \subset D_1 \subset \cdots \subset D_n \subset \Omega$, such that the individual preference $A \succ B$ holds if and only if there exists an index i such that D_i intersects A and does not intersect B. The behavior of the agent, which can be represented by the set of his preferences, is thus entirely determined by this subjacent chain. This framework provides a powerful tool to investigate any rational-type behavior, and we

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showed for instance (Freund, 1998b) how it is possible to use this kind of representation to extend any partial information on the agent's preferences.

Representing an agent's preferences by chains of subsets also reveals itself useful when one has to *revise* or to *update* some obsolete information relative to these preferences. It may indeed be the case that a given individual preference $A \succ B$ that does not model accurately enough the behavior of the agent should be retracted from the preference set of this agent, or even that it should be replaced by $B \succ A$. To do so, one has to be aware of all the implications this change may induce on other individual preferences, and choose between several solutions that seem equally good—or equally bad. On which grounds should we take our decision, and which method, if any, should we adopt? To illustrate this problem, let us take a simple example. Let Ω be a set of three samples $\{p, q, r\}$, and suppose that the elements of the power set $\mathcal{S}$ of Ω are given the following utility values:

$$\begin{array}{cc} \{p\}\,\{p,q\}\,\{p,r\}\,\{p,q,r\} & 1 \\ \{q\}\,\{q,r\}\,\{r\} & 0 \\ \emptyset & - \end{array}$$

We have thus $\{p\} \succ \{r\}$, showing that the agent prefers $\{p\}$ to $\{r\}$. But suppose it appears afterwards that this individual preference should be replaced by the inverse one, $\{r\} \succ \{p\}$. How should we change our model ? Clearly, this corresponds to an induced order on Ω in which we should have $r \succ p$, and we have therefore exactly five solutions, corresponding to the extension to $\mathcal{S}$ of the following five elementary ranked orders:

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	utility
	r	r	q		2
r	p	q	r	$r\,q$	1
$p\,q$	q	p	p	p	0

Now, how shall we determine our choice between all these solutions? Note that all these rankings induces the desired preference but they all do it at some expense, inducing changes that do not seem necessary: for instance, in the three first rankings, we get $\{r\} \succ \{q\}$, which we did not have previously, while we only wanted to add $\{r\} \succ \{p\}$; in the two last ones, we get $\{q\} \succ \{p\}$, reversing, without apparent necessity the original $\{p\} \succ \{q\}$. It is clear that none of the above modifications may be considered as *the* (unique) right solution, as they all imply some unnecessary loss of the original information. We see therefore that even in the simplest situation of a sample set with three items, the preference revision problem does not offer quite an obvious solution. As we shall see, the right tool to handle this kind of problem is provided by chains: the whole revision process boils down to a suitable action on the subjacent chain, and this perspective leads to a solution that is by far, the easiest to perform, as well as the most accurate one.

A similar work may be undertaken concerning the theory of social choice. Indeed the choice functions that are classically associated with rational behaviors are those that satisfy the well-known Arrow properties. It turns out that, as in the case of rational orders, such a choice function is fully determined by a chain of embedded subsets. This fact is by no means surprising, as there exists a duality between rational choice functions and rational orders (see for example Lehmann, 2001). As a consequence, it enables to quite easily address the problem of correcting any obsolete information conveyed by this type of functions.

This paper deals with the general problem of revising an agent's behavior, be it represented by a *set of preferences* or by a *choice function*. First, we shall deal with preference orders. We shall provide the translation in the set-theoretic context of the main results established in (Freund, 1998b), giving the analogues in this framework of the representation of preference orders by logical chains. This will enable us to correctly handle the problem of revising rational preference orders: working through the associated chain, we will find a simple solution that furthermore happens to be an *optimal* one, in a sense that will be made precise. We shall then expose the problem of *iterated revision*: is it possible to use this procedure to revise a preference set by more than a single preference? We shall show how to apply the chain representation to this problem, and we shall determine in which cases this problem has a solution. Finally, we will consider choice functions, focusing our attention on those functions that satisfy the Arrow properties. Again, we shall see that one can associate with any such function a chain of embedded subsets, and that there exists a perfect duality between choice functions and chains of subsets. It will be then easy to treat the problem of choice functions revision, which will boil down to that of a simple set-theoretical surgery.

Throughout this paper, we denote by Ω a universal *finite* set, which can be considered as a sample set or as a set of basic alternatives, and we shall work on the power set $\mathcal{S} = \wp(\Omega)$. Elements of $\mathcal{S}$ will be denoted by capital letters. They may be indifferently interpreted as choices, events, strategies or menus.

2. Rational preferences

2.1. Definitions and elementary properties

2.1.1. The notion of rational preference

The central notion of this paper is that of *strict preference*. We will suppose that such a preference is expressed through a strict partial order $\succ$ defined on the power set $\mathcal{S}$ of a (finite) universal set of mutually exclusive options Ω. Thus a relation of the type $A \succ B$ is to be interpreted as "in the agent's mind, A is preferable to B": for this agent, the utility of the item proposed by A is greater than that of the item represented by B. It may also be helpful to keep in mind some other interpretations of the relation $\succ$: thus in a different context we may translate $A \succ B$ by "in the agent's mind the event A is more likely to happen (more plausible) than the event B"; another illustration may be "in the agent's mind, the belief that A is true is more entrenched than the belief that B is true". It is mainly in this latter context that this order was introduced in revision theory in a slightly different form by Gärdenfors and Makinson (1994).

Denote by $\succeq$ the complementary of $\succ$. We have therefore $B \succeq A$ iff one does not have $A \succ B$, that is iff B is at least as desired as A. We will say that the binary relation $\succ$ defined over the power set $\mathcal{S} = \wp(\Omega)$ is a relation of *rational preference* if it satisfies the following five properties:

Pr$_0$: $A \succ \emptyset$ for all subsets A.

$\boldsymbol{Pr}_1$: if $A \succ B$ then $A \succeq B$ for all non-empty subsets A and B.
$\boldsymbol{Pr}_2$: if $B \supseteq A$ and $A \succ C$ then $B \succ C$.
$\boldsymbol{Pr}_3$: if $A \cup B \succ B$, then $A \succ B$.
$\boldsymbol{Pr}_4$: if $C \succ A$ and $A \succeq B$ then $C \succ B$.

The first rule states that anything is preferred to the empty menu. The second one expresses asymmetry, (and hence irreflexivity) for nonempty sets: if A is preferred to B, then B cannot be preferred to A. Note that $\boldsymbol{Pr}_1$ is equivalent to the property of connectedness of the relation $\succeq$: given two subsets A and B, one has either $A \succeq B$ or $B \succeq A$.

The rule $\boldsymbol{Pr}_2$ expresses the fact that one cannot prefer an event A to an event C unless any single consequence of A is itself preferred to C.

The rule $\boldsymbol{Pr}_3$ means that if the set $A \cup B$ is preferred to the set B, so that one may give up B in order to get $A \cup B$, then it must be the case that A is itself preferred to B. In the context where $\succ$ compares the plausibility of two events, this rule means that if $A \cup B$ is more likely to happen than B, then A alone is more likely to happen than B.

Finally, the *modularity rule* $\boldsymbol{Pr}_4$ expresses the natural fact that if C is preferred to A while B is not preferred to A, then it must be the case that C is preferred to B. This rule is equivalent to the transitivity of the relation $\succeq$.

It is clear that any rational preference relation $\succ$ is transitive, and therefore a strict partial order when restricted to the set $\mathcal{S} - \{\emptyset\}$. For short, we will refer to $\succ$ as a *rational order* on $\mathcal{S}$, although, strictly speaking, it is not an order.

Let us give a simple example of such a rational order: we denote by Δ a fixed subset of Ω, which we may consider as the set of its *best* elements, and we set, for any subsets A and B of Ω: $A \succ_\Delta B$ iff $A \cap \Delta \neq \emptyset$ and $B \cap \Delta = \emptyset$. The subset A is therefore preferred to the subset B iff A includes some best elements of Ω whilst B includes none of them. It is straightforward to check that properties $\boldsymbol{Pr}_0$ to $\boldsymbol{Pr}_4$ are satisfied, so that $\succ_\Delta$ is a rational order.

We close this section with five derived rules that are satisfied by the restriction of a rational order $\succ$ to the set $\mathcal{S} - \{\emptyset\}$ (we denote by $\bar{X}$ the complementary set of X):

1. $A \succ (B \cup C)$ iff $A \succ B$ and $A \succ C$.
2. $(B \cup C) \succ A$ iff $B \succ A$ or $C \succ A$.
3. $A \succ (A \cap \bar{B})$ iff $(A \cap B) \succ (A \cap \bar{B})$.
4. $\Omega \succ A$ iff $\bar{A} \succ A$.
5. If $B \supseteq A$ then $B \succeq A$.

The first rule may be translated as: a menu A is preferred to a union of menus $B \cup C$ if and only if it is both preferred to B and to C. For the proof, note that if one has $A \succ (B \cup C)$ and, for instance, $B \succeq A$, then it follows from $\boldsymbol{Pr}_4$ that $B \succ (B \cup C)$; this together with the fact that $B \subseteq (B \cup C)$ contradicts $\boldsymbol{Pr}_2$. For the converse, suppose we have $A \succ B$ and $A \succ C$ and $(B \cup C) \succeq A$.Then by $\boldsymbol{Pr}_4$ we get $(B \cap C) \succ B$ and $B \cup C \succ C$, that is, using $\boldsymbol{Pr}_3$, $C \succ B$ and $B \succ C$, contradicting $\boldsymbol{Pr}_1$.

Rule 2, which means that a union of menus $B \cup C$ is preferred to a menu A iff one at least of B or C is itself preferred to A, is a direct consequence of $\boldsymbol{Pr}_4$.

Rule 3 is immediate writing $A = (A \cap \bar{B}) \cup (A \cap B)$.

Similarly, the fourth rule is a direct consequence of $\Omega = A \cup \bar{A}$, and the last one is an immediate application of $\boldsymbol{Pr}_2$.

2.1.2. The ranking function associated with a rational preference

Given a rational preference $\succ$ on $\mathcal{S}$, one easily checks that the *indifference relation* $\sim$ defined by: $A \sim B$ *iff* $A \succeq B$ *and* $B \succeq A$ is an equivalence relation. The set of equivalence classes $[A]$ (A not empty) is totally ordered through the relation $\succ$ defined by $[A] \succ [B]$ iff $A \succ B$. The map that associates with every nonempty subset its equivalence class therefore provides a function κ from the set $\mathcal{S} - \{\emptyset\}$ onto a finite totally ordered set, which can be normalized to be the interval $\{0, 1, 2, \cdots, h-1\}$. This function κ will be referred to as the *utility function* or the *ranking function* associated with the preference $\succ$, and the integer $\kappa(A)$ as the *rank of* A. As for the integer h, we will indifferently refer to it as the *height* of κ, or the *height* of $\succ$. We have $\kappa(A) \succ \kappa(B)$ iff $A \succ B$. Thus the menu A will be preferred to the menu B iff its utility is greater than that of B. If a is an element of Ω, we shall write $\kappa(a)$ for $\kappa(\{a\})$.

Lemma 1. *For nonempty sets A and B one has*

$$\kappa(A \cup B) = \max(\kappa(A), \kappa(B))$$

Proof. Suppose for instance that $\max(\kappa(A), \kappa(B)) = \kappa(A)$. We have then $A \succeq B$, hence, by $\boldsymbol{Pr}_3$, $A \succeq (A \cup B)$. But it also follows from $\boldsymbol{Pr}_1$ and $\boldsymbol{Pr}_2$ that $(A \cup B) \succeq A$. This shows that $[A] = [A \cup B]$, so that these formulas have same rank. $\square$

The utility of a union is thus the greatest of the utility of its components. This important property in fact *characterizes* the class of utility functions that stem from rational preference orders: indeed, let κ be a function from $\mathcal{S} - \{\emptyset\}$ onto $\{0, 1, 2, \dots, h-1\}$ that satisfies $\kappa(A \cup B) = \max(\kappa(A), \kappa(B))$. Define the relation $\succ$ on $\mathcal{S}$ by: $A \succ B$ iff $A = \emptyset$ or $\kappa(A) > \kappa(B)$. Then one easily checks that $\succ$ is a rational preference relation, the associated ranking function of which is equal to κ.

This result together with the above lemma shows that there is a one-to-one mapping between the family of rational preference orders and that of the integral functions κ that are defined on the power set of a finite sample set Ω and satisfy the equality $\kappa(A \cup B) = \max(\kappa(A), \kappa(B))$.

Corollary 1. *One has $A \succ B$ iff there exists an element a of A such that $\{a\} \succ \{b\}$ for all elements b of B.*

Proof. By the above lemma, we have $\kappa(A) = \max_{a \in A} \kappa(a)$. We have therefore $A \succ B$ iff $\max_{a \in A} \kappa(a) > \max_{b \in B} \kappa(b)$, whence the result. $\square$

This corollary can be directly retrieved from the first two derived rules. It shows that the *set of ordered pairs of elements of $\mathcal{S}$ is fully determined by the subset of ordered pairs of elements of Ω*. We shall refer to this latter subset as the *elementary set of preferences* of the relation $\succ$. If we identify elements x of Ω with their corresponding singleton $\{x\}$, we see that the elementary set of preferences is just the restriction to Ω of the order relation $\succ$. We shall still denote by $\succ$ this restriction, writing $x \succ y$ for $\{x\} \succ \{y\}$. Note that $\succ$ is a *strict partial modular order* on Ω: besides irreflexivity and transitivity, it satisfies

$\boldsymbol{Pr}'_4$: if $z \succ x$ and $x \succeq y$ then $z \succ y$.

By what precedes, any rational preference order on $\mathcal{S}$ induces by restriction a strict partial modular order on Ω. The following theorem shows that conversely every strict partial rational order on Ω may be extended into a rational preference order on $\mathcal{S}$.

Theorem 1. *Let $\succ$ be a strict partial modular order on Ω. Still denote by $\succ$ the following relation on the power set $\mathcal{S}$ of Ω defined by*

$A \succ B$ iff there exists an element a of A such that $a \succ b$ for all elements b of B.

Then $\succ$ is a rational preference order that extends the original modular order $\succ$. □

Proof. The extension of the original modular order may be interpreted as: menu A is preferred to menu B iff there exists an item in menu A that is preferred to any item of B. The proof of the theorem is straightforward. Note that this result would still be valid if Ω were not supposed to be finite.

2.2. Rational preference orders and logical chains

An *Ω-chain* Δ is a sequence of strictly embedded subsets of Ω

$$D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_{h-1}.$$

In the sequel, we will always suppose that the last term of the chain is equal to Ω. The *length* of a chain of the above form with $D_{h-1} = \Omega$ is the integer h. Such a chain gives rise to a function κ_Δ from $\mathcal{S} - \{\emptyset\}$ to $\{0, 1, 2, \ldots, h-1\}$ defined by

$$\kappa_\Delta(A) = h - 1 - r_\Delta(A),$$

where $r_\Delta(A)$ is the first index i such that D_i intersects A: the subsets of highest rank $h-1$ are the ones that intersect D_0, while the subsets of rank 0 are those that have empty intersection with all the links of the chain but the last one. One has readily $\kappa_\Delta(A \cup B) = \max(\kappa_\Delta(A), \kappa_\Delta(B))$, so, by what precedes, $\kappa_\Delta$is a ranking function. Its associated rational preference order $\succ_\Delta$ is defined by

$$A \succ_\Delta B \text{ if and only if } \kappa_\Delta(A) > \kappa_\Delta(B).$$

or, equivalently

$$A \succ_\Delta B \text{ if and only if } r(A) < r(B).$$

For nonempty sets, one has therefore

$$A \succ_\Delta B \text{ if and only if there exists a link } D_i \text{ that intersects } A \text{ and not } B.$$

We will refer to the order $\succ_\Delta$ as to the order *induced* or *entailed* by the chain Δ. The example of the rational order $\succ_\Delta$ given in the preceding paragraph ($A \succ_\Delta B$ iff $A \cap \Delta \neq \emptyset$ and $B \cap \Delta = \emptyset$) is just the order induced by a chain of length 2 with first link equal to D. In the general case of a chain of length h, the induced order may be interpreted as follows: elements of D_i are preferred to elements of $D_{i+1} - D_i$ (that is, any element of D_i is preferred to any element of $\overline{D_i}$). For arbitrary subsets, the preference relation is as follows: a nonempty subset set A of Ω will be preferred to a subset B if its contains at least one element 'better' than any element of B.

Remark 1. Recalling that, for any element $a \in \Omega$, we write $\kappa_\Delta(a)$ for $\kappa_\Delta(\{a\})$, we see that D_i may be retrieved from the ranking function κ as the set of all elements $a \in \Omega$ such that $\kappa_\Delta(a) \geq h - 1 - i$.

The main interest in introducing Ω-chains is that any rational preference order is induced by such a chain. More precisely we have the

Theorem 2. *Let $\succ$ be a rational preference order on $\mathcal{S}$. Then there exists a unique chain Δ such that $\succ$ agrees with $\succ_\Delta$.*

Proof. Let κ be the normalized ranking function associated with the preference $\succ$ order, and h its height. For each integer $i \in \{0, 1, \ldots, h-1\}$ denote by D_i the set of all the elements of Δ with rank $\geq h-1-i$ (the D_i's are the *better-than sets* associated with κ). This yields a chain Δ

$$D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_{h-1}$$

of length h, with last term equal to Ω. We have to prove that, for any nonempty subset A of $\Omega, \kappa_\Delta(A) = \kappa(A)$. Set $j = r_\Delta(A)$. The set A therefore does not intersect D_{j-1} but intersects D_j. This means that all elements of A have rank $< h - j$ and that there exists an element of A with rank $\geq h - j - 1$. We have therefore $\kappa(A) = h - j - 1 = h - 1 - r_\Delta(A) = \kappa_\Delta(A)$ as desired. The uniqueness of Δ is an immediate consequence of Remark 1. □

The above theorem shows that in a finite environment the preferences, or the behavior, of a rational agent are always determined by a subjacent chain of subsets. If we restrict our attention to the elementary set of preferences of this agent, we see that an item a is preferred to an item b iff there exists a link D_i of Δ such that $a \in D_i$ and $b \notin D_i$. This amounts to saying that a is preferred to b iff

1. For every index k, if $b \in D_k$ then $a \in D_k$.
2. There exists an index j such that $b \notin D_j$ and $a \in D_j$.

Under this form, it is possible to generalize this representation theorem in the non-rational case: this was done in the particular framework of finite propositional languages in (Freund, 1998a), showing an analogue result for preference orders that do not satisfy the property of *modularity* expressed by $\boldsymbol{Pr}_4$. In the set-theory framework we are interested in, this generalization is almost trivial:

Theorem 3. *Let $\succ$ be an strict partial order on a* (*not necessarily finite*) *set Ω. Then there exists a subset Δ of the power set $\mathcal{S}$ of Ω such that, for all elements a and b of Ω, one has $a \succ b$ iff*

1. *For every subset $D \in \Delta$, if $b \in D$ then $a \in D$.*
2. *There exists a subset $X \in \Delta$ such that $b \notin X$ and $a \in X$.*

Proof. For any element x of Ω, let D_x be the set of $y \in \Omega$ such that $y = x$ or $y \succ x$. Let Δ be the set of all these D_x's. We have to check that the preference $a \succ b$ holds iff the two above conditions are satisfied. Suppose first that we have $a \succ b$. By the transitivity of $\succ$, we see that if b is a member of a set D_x, so must be a, so the first condition is satisfied. Moreover, by irreflexivity, we have clearly $a \in D_a$ and $b \notin D_a$, showing that the second condition also holds. Conversely, if we do not have $a \succ b$, either $a = b$, in which case the second condition is not satisfied, or we have $b \in D_b$ and $a \notin D_b$, contradicting therefore the first condition. □

When Ω is a finite set, there is a one-to-one mapping between the strict partial orders on Ω and the ordering relations on $\mathcal{S}$ that satisfy conditions $\boldsymbol{Pr}_0$ to $\boldsymbol{Pr}_3$. The above theorem may be used to show that these relations are exactly those that are induced by arbitrary subsets Δ of $\mathcal{S}$.

2.3. *Revising rational preferences*

2.3.1. *How to handle the problem of preferences revision*

Classically, a *revision* problem occurs when, disposing of a set of formulas (*a knowledge base*) that is supposed to represent all the information at disposal, it appears necessary to modify this set in order to take into account a new piece of information. A revision operation then consists in retracting from or adding to this basis one or several formulas. The famous A.G.M. postulates (Alchourròn et al., 1985) have been the guiding line in A.I. for an attempt to find optimal solutions. We have to point of, though, that this problem of classical revision does not fall within the scope of the present work: even if the reader finds some formal analogy with the classical formalism—e.g. similar terms and definitions—he should be aware that we are working now in a different perspective. Indeed our problem is that of *preference revising* which can be defined as the following one: we suppose the behavior of a given agent is represented by the set of his rational preferences, and we decide to modify this behavior; in this purpose, we want to withdraw, add, or replace one—or several—given individual preferences. Clearly, we always dispose of several ways to do so, and our problem is: on which grounds should we decide, and which method, if any, should we adopt ? It is possible to solve this problem by transposing it in the framework of *subset chains*: indeed, by what we saw in the preceding section, any rational order may be represented by a well-determined chain, and the operation of transforming this preference into another one therefore boils down to a chain-transformation problem, which may be stated as: *given the subset chain Δ inducing the individual preference $A \succ B$, transform it in a reasonable fashion into a chain Δ' that will no more induce $A \succ B$, (or that will induce $A' \succ B'$).* Of course, we have to be a bit more precise concerning this 'reasonable' fashion and examine more closely what properties we should expect from an ideal transformation. For this reason, it will be useful to make a distinction between two different problems, that of the *contraction* of a rational preference set by an individual preference $A \succ B$ and that of its *revision* by this preference:

(a) The *contraction* problem occurs when one wants to withdraw an individual preference $A \succ B$ from the given set of rational preferences. This problem amounts, given, a chain

Δ that entails $A \succ B$, to building a new chain $\Delta \div (A \succ B)$ that no longer entails this particular preference.

(b) We shall talk of the *revision* of a set of rational preferences by an individual preference $A \succ B$ when we deal with the problem of adding this individual preference to the given set. In other words, starting from a chain Δ that does not entail $A \succ B$, we are looking for a chain $\Delta\bigstar(A \succ B)$ that will entail this individual preference.

Considering both operations, there exists a natural and elementary principle that we should observe, which is that of *minimal change*: we want to change as little as possible from the behavior of the agent, in as much as this behavior is represented by his set of preferences. More precisely we shall work with the following constraints:

(1) If Δ does not entail $A \succ B$, then $\Delta \div (A \succ B) = D$.
(2) If Δ entails $A \succ B$, then $\Delta\bigstar(A \succ B) = D$.
(3) Contracting by $A \succ B$ does not add any new preferences.
(4) Contracting by $A \succ B$ eliminates only the preferences that necessarily implied $A \succ B$.
(5) Revising by $A \succ B$ eliminates only the preferences that were incompatible with $A \succ B$.
(6) Revising by $A \succ B$ doesn't unnecessarily add new preferences.

The two first constraints, translating the principles of *minimal change* and *success*, are self-explanatory. The third one states that if one desires to withdraw a particular preference from the set of preferences of an agent, this should be done without adding any new preference. Indeed, the introduction of a new preference is supported by no justification, and would be thus quite arbitrary. The fourth rule recalls that the aim of the contraction is, if possible to withdraw $A \succ B$ and nothing else, when possible. If this is not possible, then only a minimal number of preferences have to be withdrawn. Similarly the two last constraints define a principle of *minimal change* for revision: it may well be the case that a new preference cannot be added alone in the agent's preference set, and that it implies moreover to withdraw some of the initial preferences. But in any case, the operation of revision should be performed in such a way that no unnecessary changes will be operated.

This principles being stated, it will be easy to apply them in both cases of contraction and revision, making simple use of the following

Lemma 2. *Let Δ and D' be two Ω-chains, with induced preference orders $\succ_\Delta$ and $\succ_{\Delta'}$. Then $\succ_\Delta \subseteq \succ_{\Delta'}$ iff Δ is a sub-chain of Δ', that is iff every link of Δ is a link of Δ'.*

Proof. Suppose first that Δ is a sub-chain of Δ'. If we have $A \succ_\Delta B$ for two menus A and B, this implies by definition that there exists a link of Δ that intersects A and does not intersect B. Since this link is also a member of the chain Δ', we have readily $A \succ_{\Delta'} B$.

Conversely, let us show that if $\succ_\Delta \subseteq \succ_{\Delta'}$ implies that Δ is a sub-chain of Δ'. Recall that, as shown in the proof of Theorem 2, the i-th link D_i of the chain Δ is the set of all elementary items $x \in \Omega$ such that $\kappa_\Delta(x) \geq h-1-i$, where h is the length of Δ. Let z be an elementary item with minimal $\kappa_{\Delta'}$-rank among the samples that have κ_Δ rank equal to $h-1-i$. The link D_i is then the set of all items x such that $\kappa_\Delta(x) \geq \kappa_\Delta(z)$. Set $j = h' - 1 - \kappa_{\Delta'}(z)$. We claim that D_i is precisely equal to D'_j, and is thus a link of Δ'. By the choice of z, and the fact that $\succ_\Delta \subseteq \succ_{\Delta'}$, we have indeed $h' - 1 - j = \kappa_{\Delta'}(z) \leq \kappa_{\Delta'}(x)$ for any sample x with

κ_Δ rank $\geq h-1-i$. Conversely, if x is a sample such that $\kappa_{\Delta'}(x) \geq h'-1-j$, we have $\kappa_{\Delta'}(z) \leq \kappa_{\Delta'}(x)$, and it follows that $z \preceq_{\Delta'} x$ and therefore, by our hypothesis, that $z \preceq_\Delta x$. This means that $h-1-i \leq \kappa_\Delta(x)$. We have therefore proven that, given an elementary item x, one has $\kappa_\Delta(x) \geq h-1-i$ iff $\kappa_{\Delta'}(x) \geq h'-1-\kappa_{\Delta'}(z)$. This shows that $D_i = D'_j$, and the proof of the lemma is complete. □

2.3.2. Chain contraction

The preferences of an agent are supposed to be represented by a chain Δ:

$$D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_{h-1}$$

with last link equal to Ω. We want to build a chain $\Delta' = \Delta \div (A \succ B)$ that does not induce $A \succ_{\Delta'} B$. Clearly, for this to be possible, we have to suppose that $B \neq \emptyset$. Taking into account the principles exposed in the precedent paragraph, it follows from Lemma 2 that the chain Δ' we are looking for should be *a maximal sub-chain of Δ that does not entail $A \succ B$.*

Denote by $i = r(A)$ the first index such that D_i intersects A, and by $j = r(B)$ the first index such that D_j intersects B. Recall that a chain Γ entails the individual preference $A \succ B$ iff $r_\Gamma(A) < r_\Gamma(B)$. If $j \leq i$, Δ does not entail the preference $A \succ B$, and we just set $\Delta' = \Delta$. If $i < j$, any sub-chain of Δ that contains a link D_s with $i \leq s < j$ will still induce $A \succ B$. We have therefore to remove of the chain Δ all its links D_s such that $r(\alpha) \leq s < r(\beta)$. It follows that the (unique) solution to the contraction problem is the chain:

$$D_0 \subset D_1 \subset \cdots \subset D_{i-1} \subset D_j \subset \cdots \subset D_{h-1} = \Omega.$$

Let us compute the new rank $\kappa' = \kappa_{\Delta \div (A \succ B)}$ induced by the contracted chain $\Delta' = \Delta \div (A \succ B)$. We have, for any subset X, $\kappa(X) = h-1-r(X)$ and $\kappa'(X) = h'-1-r'(X)$. Note first that we have $h' = h-(j-i)$, that is $h' = h-((\kappa(A)-\kappa(B))$. We get therefore

$r'(X) = r(X)$ if $r(X) \leq r(A)$, that is if $\kappa(A) \leq \kappa(X)$,
$r'(X) = r(A)$ if $r(A) < r(X) \leq r(B)$, that is if $\kappa(B) \leq \kappa(X) < \kappa(A)$, and
$r'(X) = r(X) - (\kappa(A) - \kappa(B))$ if $\kappa(X) < \kappa(B)$.

It follows that the new ranking κ' is given by

$$\kappa'(X) = \begin{cases} \kappa(X) - [\kappa(A) - \kappa(B)], & \text{if } \kappa(A) \leq \kappa(X) \\ \kappa(B), & \text{if } \kappa(B) \leq \kappa(X) < \kappa(A) \\ \kappa(X), & \text{if } \kappa(X) < \kappa(B) \end{cases}$$

Clearly the contraction operation thus defined fully meets the constraints of *success* and of *minimal change* that were required for any 'reasonable' contraction operation.

Example 1. The preferences of the agent are given by the rational order $\succ$

$\{p\}\,\{p,q\}\,\{p,r\}\,\{p,q,r\}$	2
$\{q\}\,\{q,r\}$	1
$\{r\}$	0
$\emptyset$	–

Suppose we want to contract $\succ$ by $\{p\} \succ \{r\}$. We first have to determine the underlying chain Δ. The original associated elementary modular order is given by

$$\begin{array}{cc} p & 2 \\ q & 1 \\ r & 0 \end{array}$$

and the chain Δ is just $\{p\} \subset \{p, q\} \subset \{p, q, r\}$. With the preceding notations, we have have $i = 0$ and $j = 2$, and the contracted chain therefore boils down to the trivial chain $\{p, q, r\}$. The modular order on Ω is the trivial one, where all three items have rank 0, and the associated rational order is just: $A \succ B$ iff $B = \emptyset$. Note that two other choices were possible in order to give equal rank to p and q: in the first one, we could give rank 0 at p and r and put q at rank 1, whilst, in the second one, p and r could get rank 1 and q rank 0. The chain contraction we chose disregards these solutions as not *economical*: indeed, in the first one, the original preference $p \succ q$ would be unnecessarily reversed, and, in the second one, it is the preference $q \succ r$ that would be reversed.

Remark 2. It is possible and usually easier, to perform a contraction by simply working on the elementary set of preferences on the sample set Ω. Suppose indeed we have to contract by $A \succ B$. Let $i = r(A)$ and $j = r(B)$. There exist elements $a \in A$ and $b \in B$ such that $r(a) = i$ and $r(b) = j$, and we have $a \succ b$. Recall that the k-th link D_k of the chain Δ that induces $\succ$ is the set of all elementary samples x such that $r(x) \leq k$. After the contraction has been performed, we see that we do not have anymore $a \succ b$ in the resulting elementary preference set, exactly as if we had performed in this set a contraction by $a \succ b$. It is straightforward to check that it is equivalent to contract a rational preference order by the individual preference $A \succ B$, or to contract the corresponding elementary modular order by $a \succ b$, where a and b are maximal elements of A and B.

2.3.3. *Chain revision*

Our problem is now to add a given individual preference, say $A \succ B$, to the set of rational preferences that reflects the behavior of an agent. Clearly, the procedure will differ, depending on whether or not this individual preference stands in *contradiction* with the original set of preferences. This intuitive notion of contradiction can be made more precise: clearly, such a contradiction would result of the presence of $B \succ A$ among the given set of preferences of the agent, and there would also be a contradiction if the original set included the individual preference $A \succ \bar{B}$, since, by the first derived rule, $A \succ B$ and $A \succ \bar{B}$ lead to $A \succ \Omega$, contradiction the fifth derived rule. We would again have a contradiction, if the set of preferences included the individual preference $B \succ A \cap \bar{B}$, for this inequality together with $A \succ B$ implies $(A \cup B) \succ A \cup B$. As a matter of facts, it will turn out that only this latter case only poses a problem. We shall therefore first examine the situation where $B \succ A \cap \bar{B}$ is *not* induced by the chain Δ

$$D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_{h-1}$$

that is supposed to represent the agent's behavior. Note that this requires that A is not a subset of B.

In this simple case, the usual terminology is that of an *expansion* of Δ by $A \succ B$. Similarly to the notation used in classical revision theory, we shall denote by $\Delta + (A \succ B)$ the result of this chain expansion. As follows from the principle of minimal change and Lemma 2, $\Delta + (A \succ B)$ should be the smallest super-chain of Δ that induces $A \succ B$, if such a chain exists.

Let $i = r(\alpha)$. Since we supposed that Δ does not induce the preference $B \succ A \cap \bar{B}$, we do not have $B \succ A$, as results from the last derived rule. It follows that $D_{i-1} \subseteq \bar{B}$. Observe furthermore that D_i intersects the set $A \cap \bar{B}$: otherwise we would have $A \succ (A \cap \bar{B})$, hence $B \succ (A \cap \bar{B})$ as follows from the third and fifth derived rules.

Consider now the following chain Δ'

$$D_0 \subset D_1 \subset \cdots \subset D_{i-1} \subset D_i \cap \bar{B} \subset D_i \subset \cdots \subset D_{h-1} = \Omega.$$

obtained from Δ by simply adding the link $D_i \cap \bar{B}$. We claim that this chain fulfils all the requirements that were expected from the expansion of Δ by $A \succ B$:

- Δ' is indeed a chain since $D_{i-1} \subseteq D_i \cap \bar{B}$.
- Δ' induces $A \succ B$ because $D_i \cap \bar{B}$ intersects A and does not intersect B.
- Δ' is a minimal extension of Δ in the sense that
  1. One has readily $\Delta' = \Delta$ if Δ primitively entailed $A \succ B$.
  2. Only one link was added to he original chain.
  3. The link that was added has minimal strength: if adding a link D to Δ is enough to entail $A \succ B$, then it must be the case that $D \subseteq D_i \cap \bar{B}$).

We shall refer to this chain as *the* expansion of Δ by $A \succ B$, and denote it by $\Delta + (A \succ B)$.

Let us compute the new rank $\kappa' = \kappa_{\Delta+(A \succ B}$ induced by this chain. We have $h' = h + 1$, and, for any menu C:

$r'(C) = r(C)$ if $r(C) < i$.
$r'(C) = r(C) + 1$ if $r(C) > i$.
$r'(C) = i$ if $r(C) = i$ and C intersects $D_i \cap \bar{B}$.
$r'(C) = i + 1$ if $r(C) = i$ and C does not intersect $D_i \cap \bar{B}$.

It follows that

$$\kappa'(C) = \begin{cases} \kappa(C) + 1, & \text{if } \kappa(C) > \kappa(\alpha) \\ \kappa(C), & \text{if } \kappa(\alpha) > \kappa(C) \\ \kappa(A) + 1, & \text{if } \kappa(C) = \kappa(A) \text{ and } \kappa(C \cap \bar{B}) \geq \kappa(A) \\ \kappa(A), & \text{if } \kappa(C) = \kappa(A) \text{ and } \kappa(C \cap \bar{B}) < \kappa(A) \end{cases}$$

The row of the elementary items that had rank equal to $\kappa(A)$ has just split into two rows: the samples that are elements of B form a new rank, one notch downwards. The other ones don't move.

Example 2. We take again $\Omega = \{p, q, r\}$, and suppose we are given the following rational order in $\mathcal{S}$:

$$\begin{array}{cc} \{p\}\,\{p,q\}\,\{q\}\,\{q,r\}\,\{p,r\}\,\{p,q,r\} & 1 \\ \{r\} & 0 \\ \emptyset & - \end{array}$$

This preference order is induced by the chain $\{p, q\} \subset \{p, q, r\}$. Suppose we want to revise it by $\{p, q, r\} \succ \{p, r\}$. Note that $\{p, q, r\} \cap \overline{\{p, r\}} = \{q\}$. As we do not have $\{p, r\} \succ \{q\}$, the expansion by $\{p, q, r\} \succ \{p, r\}$ is possible. The expanded chain is $\{p, q\} \cap \{q\} \subset \{p, q\} \subset \{p, q, r\}$, that is

$$\{q\} \subset \{p, q\} \subset \{p, q, r\}.$$

The revised preference order is therefore

$$\begin{array}{cc} \{p,q\}\,\{q\}\,\{q,r\}\,\{p,q,r\} & 2 \\ \{p\}\quad\{p,r\} & 1 \\ \{r\} & 0 \\ \emptyset & - \end{array}$$

We have $\{p, q, r\} \succ \{p, r\}$ as desired. Note that in the original elementary modular order on Ω, r had rank 0 and p and q had rank 1. The new ranking put q at rank 2, leaving the other elements unchanged.

Remark 3. The condition that one does not have $B \succ A \cap \bar{B}$ is equivalent to the following condition: for every element $b \in B$ with maximal rank, there exists an element $a \in A$ with maximal rank, $a \notin B$, such that $a \succeq b$. When this condition it is satisfied, the expansion by $A \succ B$ may be directly performed through a simple expansion by $a \succ b$ on the elementary preference set. For instance, in the above example, the elementary preference order on Ω is given by $\kappa(p) = \kappa(q) = 1$, and $\kappa(r) = 0$. To perform the expansion by $\{p, q, r\} \succ \{p, r\}$, we only have to expand the original chain $\{p, q\} \subset \{p, q, r\}$ by $q \succ p$. This leads to

$$\{p, q\} \cap \overline{\{p\}} \subset \{p, q\} \subset \{p, q, r\},$$

that is to

$$\{q\} \subset \{p, q\} \subset \{p, q, r\}.$$

We finally turn to the more complicated case where the given chain Δ induces the individual preference $B \succ A \cap \bar{B}$. As we noticed, it is not anymore possible to just make an expansion of Δ by $A \succ B$. The natural way to add this latter preference to our agent's scheme is to first retract the preference $B \succ A \cap \bar{B}$ through our contraction procedure, and then to make an expansion by $A \succ B$. As follows from the fifth derived rule, a necessary condition is that A is not a subset of B. We shall see that this is also a sufficient condition. Using the notations of classical revision theory, we shall denote by $\Delta\bigstar(A \succ B)$ the resulting chain, that is $\Delta\bigstar(A \succ B) = (\Delta \div (B \succ A \cap \bar{B})) + (A \succ B)$.

Set $i = r(B)$ and $j = r(B \succ A \cap \bar{B})$. The contraction by $B \succ A \cap \bar{B}$ gives rise to the chain

$$D_0 \subset \cdots \subset D_{i-1} \subset D_j \subset \cdots \subset \Omega.$$

The first link of this chain that intersects A is now D_j. Expanding by $A \succ B$ therefore provides the revised chain $D\bigstar(A \succ B)$ that is equal to

$$D_0 \cdots \subset D_{i-1} \subset D_j \cap \bar{B} \subset D_j \subset D_{j+1} \cdots \subset \Omega.$$

The new rank $\kappa' = \kappa\bigstar(A \succ B)$ is given, for any elementary item z by

$$\kappa'(z) = \begin{cases} \kappa(z) - \kappa(B) - \kappa(A \cap \bar{B}), & \text{if } \kappa(z) > \kappa(B) \\ \kappa(z), & \text{if } \kappa(z) < \kappa(A \cap \bar{B}) \\ \kappa(A \cap \bar{B}), & \text{if } \kappa(A \cap \bar{B} \leq \kappa(z) \leq \kappa(B) \text{ and } z \notin B \\ \kappa(A \cap \bar{B}) - 1, & \text{if } \kappa(A \cap \bar{B} \leq \kappa(z) \leq \kappa(B) \text{ and } z \in B \end{cases}$$

Example 3. Let us consider again the example of Section (page 2):

$$\begin{array}{cc} \{p\}\,\{p,q\}\,\{p,r\}\,\{p,q,r\} & 1 \\ \{q\}\,\{q,r\}\,\{r\} & 0 \\ \emptyset & - \end{array}$$

Suppose we want to revise by $\{r\} \succ \{p\}$. The underlying chain is

$$\{p\} \subset \{p, q, r\}$$

Using the above notations, we get $A \cap \bar{B} = \{r\}$, $i = 0$ and $j = 1$. It follows that the revised chain is

$$\{q, r\} \subset \{p, q, r\}.$$

The revised elementary modular order is therefore the order $\succ_4$ of Section 1.

2.4. Iterated revision

At this point, we have dealt with the problem of revising a set of preferences $\mathcal{P}$ in order to incorporate in it a single individual preference $A \succ B$. The problem of iterated revision arises when, this having been done, it appears desirable or necessary to revise the new preference set by a second individual preference $C \succ D$. Theoretically, one could proceed naturally by revising, through the method developed above, the set $P\bigstar(A \succ B)$ by $C \succ D$. The solution simply consists in taking a resulting preference set the set $(\mathcal{P}\bigstar(A \succ B))\bigstar(C \succ D)$. If the preference $A \succ B$ then happens to disappear in the process, this is justified by the choice of the sequence in which we decided to make the revisions. Indeed, deciding to revise by $C \succ D$ only after the revision by $A \succ B$ has been performed is interpreted as: the requirement to get the individual preference $C \succ D$ follows that concerning $A \succ B$, and must therefore be considered as more important or more reliable than this latter. In this perspective where former information or constraint may become obsolete, it is natural to give up an individual preference that contradicts a new one, even if the former was itself inserted in the initial preference set by means of a revision process.

Nevertheless, it may well be the case that one wishes to add *both* individual preferences $A \succ B$ and $C \succ D$ to the initial preference set. We may then think of simply performing a revision by $A \cap C \succ B \cup D$, as this individual preference entails both $A \succ B$ and $C \succ D$, but this method may be considered as an excessive one—think for instance to the case where the set $A \cap C$ is the empty set.

To be more precise and show the type of problems we have to deal with, it is useful to consider some elementary examples:

2.4.1. *Fruit puzzles*

2.4.1.1. Puzzle 1. We dispose of a sample set Ω with four elements, an apple a, a banana b, a cherry c and a date d, and we start with the default hypothesis that a given agent has no preference on that set. Then we come to learn that he prefers dates to bananas and bananas to both apples and cherries. We therefore want to build a chain that will induce an order $\succ$ such that one has $d \succ b$ and $b \succ \{a, c\}$. One way to construct such a chain would be to start with the trivial chain Ω and successively revise it by $d \succ b$ and $b \succ \{a, c\}$. But after the first revision has been performed (here an elementary expansion), we obtain the chain $\{a, c, d\} \subset \Omega$, which cannot be revised by $b \succ \{a, c\}$ through a simple expansion: indeed, we have $\{a, c\} \succ (\{b\} \cap \overline{\{a, c\}}$. Using the results of the preceding section, we get as resulting chain the sequence $\{b, d\} \subset \Omega$, and this chain no longer induces the individual preference $d \succ b$, and therefore does not model the agent's preference.

It is nevertheless possible to get this latter individual preference back and integrate both our desired individual preferences into a rational preference set. Observe indeed that, starting from our latter chain $\{b, d\} \subset \Omega$, all we have to do is to proceed to a third revision. As may be seen immediately, a simple expansion by $d \succ b$ does the job and yields the chain $\{d\} \subset \{b, d\} \subset \Omega$ that induces both desired preferences. Note that this chain could have been obtained by two revisions instead of three, had we first begun by $b \succ \{a, c\}$, and only then revised by $d \succ b$.

2.4.1.2. Puzzle 2. Still working with the set $\Omega = \{a, b, c, d\}$, we want now to build a rational preference order such that both $\{a, b\} \succ \{c\}$ and $\{c, d\} \succ \{a, b, d\}$ hold. The expansion of the trivial chain by the $\{a, b\} > \{c\}$ yields the chain

$$\{a, b, d\} \subset \Omega.$$

The subsequent revision by $\{c, d\} > \{a, b, d\}$ yields the chain

$$\{c\} \subset \Omega.$$

Again, the first individual preference has been lost in the process. We may try, like in our preceding puzzle, to perform the revision of this latter chain by $\{a, b\} \succ \{c\}$, but this leads back to the chain

$$\{a, b, d\} \subset \Omega.$$

Clearly, it is impossible to get both desired preferences through iterated revision. In fact it is impossible to find a preference set that would include at the same time $\{a, b\} \succ \{c\}$ and $\{c, d\} \succ \{a, b, d\}$: indeed, it follows from the second inequality that the element of highest rank in $\{c, d\}$ is c, and that this element has greater rank than a, b and d, and this contradicts the first inequality.

2.4.1.3. Puzzle 3. We start now from the non-trivial following chain Δ

$$\{a, b, c\} \subset \Omega$$

that represents the preferences of an agent who would prefer any of the three first fruits to a date. Suppose we want to modify this chain so that both individual preferences $\Omega \succ \{b, c\}$ and $\Omega \succ \{a, c\}$ hold. The expansion by the first individual preference leads to the chain Δ_1:

$$\{a\} \subset \{a, b, c\} \subset \Omega.$$

The revision by $\Omega \succ \{a, c\}$ first requires a contraction by $\{a, c\} \succ \{b, d\}$, leading back to Δ_1, then an expansion, which yields the chain Δ_2

$$\{b\} \subset \{a, b, c\} \subset \Omega.$$

The first individual preference $\Omega \succ \{b, c\}$ is no more entailed by this chain. We may try a revision of Δ_2 by $\Omega \succ \{b, c\}$, but then we get again the chain Δ_1, losing this time our second individual preference. Hence it is clear that we cannot get both preferences from the original chain Δ applying the revision processes which were described in the preceding paragraphs. Note though that these two inequalities may well coexist: they are for instance induced by the chain Δ':

$$\{d\} \subset \{a, b, c\} \subset \Omega.$$

This example shows that the revision process we defined in the preceding paragraphs may not give a satisfactory answer to the problem of iterating a revision under the constraint that a given individual preference has to be preserved. The question we shall address now is: given a rational preference set and four menus A, B, A' and B', under what conditions can we be sure to obtain, through our chain revision procedure, both individual preferences $A \succ B$ and $A' \succ B'$?

2.4.2. Possible and impossible iterated revision

The preference set of an agent being given by the chain Δ

$$D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_{h-1}$$

with last link equal to Ω, our purpose is now to determine whether it is possible to transform this chain by a succession of iterated revisions so that the resulting chain entails both $A \succ B$ and $A' \succ B'$.

We may restrict ourselves to the case where Δ already entails $A \succ B$ because, anyway, this will be the case for the revised chain $\Delta' = \Delta\bigstar(A \succ B)$.

We set $i = r(A) = r(A \cap \bar{B})$, $j = r(B)$, $j' = r(B')$ and $k = r(A' \cap \bar{B}')$. We have $i > j$. We shall adopt the '*impossibility hypothesis*' that no iteration of the procedure described in the preceding section is sufficient to turn Δ into a chain that would induce at the same time $A \succ B$ and $A' \succ B'$.

Under this hypothesis, we see that the original chain Δ necessarily entails $B' \succ (A' \cap \bar{B}')$, otherwise revising by $A' \succ B'$ would be a simple expansion, preserving $A \succ B$. We have therefore $j' < k$ and the revised chain $\Delta \bigstar (A' \succ B')$ is of the form

$$D_0 \subset \cdots \subset D_{j'-1} \subset D_k \cap \bar{B}' \subset D_k \subset \cdots D_{h-1}.$$

By our hypothesis, this chain does not anymore entail $A \succ B$, and this readily implies $j' \leq i < j \leq k$. Moreover, this chain entails $B \succ (A \cap \bar{B})$, since otherwise, making an expansion by $A \succ B$ would yield both preferences. It follows that the link D_k, which intersects B, cannot be the first one that meets B, since it also intersects $A \cap \bar{B}$. Since $D_{j'-1}$ does not intersect B, we must have $(D_k \cap \bar{B}') \cap B \neq \emptyset$. The set D_k is thus the first link of the revised chain that intersects B. Also, as we saw, this link does not intersect $A \cap \bar{B}$.

Revising then by $A \succ B$ yields the chain $(\Delta \bigstar (A' \cap \bar{B}')) \bigstar (A \succ B)$

$$D_0 \subset \cdots \subset D_{j'-1} \subset D_k \cap \bar{B} \subset D_k \subset \cdots D_{h-1}.$$

Applying again our hypothesis concerning the impossibility to get at the same time our two individual preferences, we see that this chain does not entail $A' \succ B'$ but entails $B' \succ (A' \cap \bar{B}')$. Thus $D_k \cap \bar{B}$ does not intersect $A' \cap \bar{B}'$. As a last attempt to revise this chain by $A' \cap \bar{B}'$ leads back to the previous $\Delta \bigstar (A' \cap \bar{B}')$, we see that our impossibility condition is equivalent to the conjunction of the following conditions: $j' \leq i < j \leq k$, D_k intersects $B \cap \bar{B}'$, and D_k has an empty intersection with $A \cap \bar{B} \cap \bar{B}'$ as well as with $A' \cap \bar{B} \cap \bar{B}'$. Translating all this in terms of preference ranking, we obtain the

Theorem 4. *Let $\mathcal{P}$ be a preference set, $A \succ B$ a individual preference of $\mathcal{P}$ and $A' \succ B'$ an arbitrary individual preference. Then it is possible to modify $\mathcal{P}$ through iterated revisions and obtain both individual preferences $A \succ B$ and $A' \succ B'$ iff one at least of the following conditions is satisfied:*

1. $A \succ B'$.
2. $A' \cap \bar{B}' \succ B$.
3. $A' \cap \bar{B}' \preceq (A \cup A') \cap \bar{B} \cap \bar{B}' \quad (A \cup A') \cap \bar{B} \cap \bar{B}' \succeq A' \cap \bar{B}'$.

The above result provides a *criterion* to test whether two given individual preferences may be together incorporated *through chain revision* in the preference set of an agent. This will be impossible when none of the three conditions stated in Theorem 4 is satisfied. It may be the case that these two new individual preferences are mutually exclusive, as happened in Puzzle 2, but, as we saw in Puzzle 3, it may also be the case that the revision procedure we described is inadequate to treat the general problem of *multi-revision*. In its all generality, this problem amounts to the following one: given the preference set $\mathcal{P}$ of an agent together with a consistent set K of individual preferences, find a procedure that enables to build a new preference set $\mathcal{P}'$ which includes K and differ as little as possible from $\mathcal{P}$. For instance, we may take the example where K consists of two elements $A \succ B$ and $A' \succ B'$, which was treated by the above theorem. We can also adopt a slightly different point of view, considering that only *one* individual preference has to be added at each time to $\mathcal{P}$ (here for example $A' \succ B'$), but that we have the *constraint* to preserve some subset of $\mathcal{P}$ (e.g. $A \succ B$).

It is doubtful that a generalization of Theorem 4 may provide practical results in the case where K has more than two or three elements. But, above all, it seems that this *constraint* problem (preserving a given preference base K) does not fall within the context of iterated revision, where the new preferences to be added or retracted are taken one by one in a sequence that is the essential part of the revision process, the last preference to be treated being supposed to be more reliable than the preceding ones. Thus the constraint problem requires a study of its own.

3. Application of chain-revision to rational choice functions

3.1. The representation of choice functions by subsets chains

The technique developed in the preceding sections can be applied when addressing the problem of revising choice functions. In our framework, a choice function f is a function defined on the power set $\wp(\Omega)$ of an arbitrary finite set Ω, which returns to any subset A of Ω the set of its *best elements*. We will say that such a function is *rational* if it satisfies

Inclusion:

$$f(X) \subseteq X$$

Consistency:

$$f(X) \neq \emptyset \quad \text{whenever } X \neq \emptyset$$

and the *Arrow condition*:

$$\text{if } X \subseteq Y \text{ and } X \cap f(Y) \neq \emptyset, \text{ then } X \cap f(Y) = f(X).$$

We first show that *any rational choice function can be represented by an Ω-chain*:

Consider first an Ω-chain of length h, that is a chain Δ of subsets of Ω

$$D_0 \subset D_1 \cdots \subset D_{h-1},$$

with last term $D_{h-1} = \Omega$. Given such a sequence and an arbitrary subset X of Ω, recall that we denote by $r(X)$ the first index i such that $X \cap D_i \neq \emptyset$. For any element $x \in \Omega$, we set $r(x) = r\{x\}$. This chain *generates* the function f on $\wp(\Omega)$ defined by

$$\forall X, \quad f(X) = X \cap D_{r(X)}$$

It is immediate that this function is a rational choice function. Conversely, as expected from the duality between rational preference orders and rational choice functions, it is easy to show that any rational function is generated by a chain of subsets of Ω:

Theorem 5. *Any rational function is induced by a Ω-chain.*

Proof. We define inductively the chain $(D_i)_i$ by $D_0 = f(\Omega)$ and $D_{i+1} = D_i \cup f(\Omega - D_i)$. Note that $(D_i)_i$ is a strictly increasing sequence: indeed, by inclusion and consistency, we have $\emptyset \neq f(\Omega - D_i) \subseteq (\Omega - D_i)$ for all $i < h - 1$, where h is the length of the chain. We have to show that for every subset X of Ω $f(X) = X \cap D_{r(X)}$. If $r(X) = 0$, we have $X \cap f(\Omega) \neq \emptyset$, therefore, by *Arrow* condition, $f(X) = X \cap f(\Omega) = X \cap D_{r(X)}$. If $r(X) > 0$, we have $X \subseteq (\Omega - D_{r(X)-1})$ and $\emptyset \neq X \cap D_{r(X)} = X \cap f(\Omega - D_{r(X)-1})$. By *Arrow* condition again, this implies $f(X) = X \cap f(\Omega - D_{r(X)-1}) = X \cap D_{r(X)}$ as desired. □

Representing rational choice functions by Ω-chains shows how these functions return from any subset of Ω the set of its best elements:

Corollary 2. *Let f be a rational choice function, Δ its corresponding chain and $\succ$ the rational preference order induced on Ω by Δ. Then, for any subset X of Ω, $f(X)$ is the set of $\succ$-maximal elements of X. In particular, for all elements x, y of Ω, one has $x \succ y$ iff $f\{x, y\} = \{x\}$.*

Proof. Straightforward. □

3.2. Revising rational choice functions

The problem of revising a choice function arises when it appears necessary to modify its value on one or several subsets of its domain. Revision therefore occurs when an element x of X that was considered as a best choice does not deserve anymore this qualification and has to be withdrawn from $f(X)$, or, on the contrary when an element y has to be added to the initial list $f(X)$. We will call *contraction* the first operation and *revision* the second one; conforming to the usual abuse of language, we will nevertheless refer to one or the other of these operations as a *revision*.

The representation of rational choice functions by subsets chains will be our main tool to study the problem of revision. Nevertheless, contrary to what happened in the case of complete preferences sets, this representation is not sufficient by itself to provide a unique solution. For instance, suppose we are in the simple situation where the function f is given by a chain of length 3, $D_0 \subset D_1 \subset \Omega$, and that an element x has to be removed from the list of the best elements of a given set X for which $r(X) = 1$. Consider the following transformations of the original chain:

$D_0 \subset D_1 - \{x\} \subset \Omega$.
$D_0 \subset D_1 - \{x\} \subset D_1 \subset \Omega$.
$D_0 \subset D_0 \cup D_1 \cap (X - \{x\}) \subset D_1 \subset \Omega$.

Then any of these modifications gives rise to a rational choice function f' fairly close to f, and for which one has $x \notin f'(X)$, as desired. How can we determine our choice? Even the constraints of the elementary principle of *success* (x should not be an element of $f'(X)$), of *minimal change* (the new set of the best elements of X should be $f(X) - \{x\}$), and of *relative independence* ($f'(Y) = f(Y)$ for all subsets Y such that $Y \cap X = \emptyset$), all properties satisfied by the three modified chains, do not permit by itself to directly choose an optimal solution.

We will make use of the following

Lemma 3. *Let X be a subset of Ω and x a element of X. Then $r(x) = r(X)$ iff $x \in f(X)$.*

Proof. Suppose that x is an element of $f(X)$. Then $x \in f(X) = X \cap D_{r(X)}$, hence $x \in D_{r(X)}$, and we see that $r(x) \leq r(X)$. Since $x \in X \cap D_{r(x)}$, this latter set is not empty, and we have therefore $r(X) \leq r(x)$, whence the equality. The converse is immediate. □

3.2.1. Chain contractions of choice functions

Withdrawing an element x from the set $f(X)$ amounts to defining a new rational choice function $f_{\div(x)}$ such that $x \notin f_{\div(x)}(X)$. This is only possible if $f(X) \neq \{x\}$.

Our guiding line will be the following: if x has to be retracted from the set of best elements of X, this set being otherwise unchanged, this means that x does not match anymore the standards of the other elements of $f(X)$, and consequently that any element $y \neq x$ of $f(X)$ has to be considered as better than x. Thus if f' is the new choice function, we will have $f'\{x, y\} = \{y\}$ for all elements $y \in f(X)$, $y \neq x$. Note that this is equivalent to the condition $f'\{x, y\} = \{y\}$ for *some* $y \in f(X)$, since, by the principle of minimal change, $f'(X)$ is bound to be equal to $f(X) - \{x\}$. It follows that the inequality $y \succ x$ has to be induced by the revised choice function f', and we must therefore look for an *expansion* of the set of preferences induced by the chain Δ associated with f. Observe now that, by the principle of minimal change again, we should preserve all the inequalities already induced by f: if, given arbitrary elements z and u, one had $\{z\} = f\{u, z\}$, so that z was considered as a strictly better choice than u, depreciating x with respect to the elements of $f(X)$ should not change this fact. In other words, all the preferences $z \succ u$ should be maintained. We see therefore that the principle of minimal change in the framework of rational choice functions revision agrees with this same principle as studied in the framework of rational preferences. This justifies the application of the results established in the first part of Section 2.3.3.

We let therefore y be an arbitrary element such that $y \in f(X) - \{x\}$ and consider the auxiliary chain Δ associated with f. By what precedes, we have $r(x) = r(y)$, and we are looking for an *expansion* of Δ by $y \succ x$. As we saw, such an expansion is induced by the chain:

$$D_0 \subset D_1 \subset \cdots \subset D_{r(x)-1} \subset D_{r(x)} - \{x\} \subset D_{r(x)} \cdots \subset D_{h-1}.$$

The revised choice function f' is therefore defined by

$$f'(Y) = \begin{cases} f(Y), & \text{if } x \notin f(Y) \text{or if } f(Y) = \{x\} \\ f(Y) - \{x\}, & \text{otherwise} \end{cases}$$

Note that the set X does not play any role in this contraction process. This comes from the fact that the whole operation amounted to attributing an intermediate greater rank to all elements $t \neq x$ that where initially as good as x or, equivalently, to attributing an intermediate lower grade to the sole element x. For instance, we can suppose that we had $h - 1 = 20$ and $\kappa(x) = 12$. Then changing f to f' is equivalent to raise to 12, 5 the rank of all elements $t \neq x$ that had rank 12, or to keep all initial ranks except for x that is retrograded to 11.5.

We close this paragraph with a remark: the process of contraction we described rests on the fact that we do not know for what reason the element x has to be retracted from the set $f(X)$: *it is this very absence of information that commands the solution we proposed:* indeed if, for instance, one knows from the beginning that the element x, that had initial rank $\kappa(x)$, has to be retrograded to a given lower rank $\kappa(x) - i$, the revision problem boils down to a much simpler one, an evident solution of which is given by the chain

$$\cdots \subset D_{r(X)-1} \subset D_{r(X)} - \{x\} \subset \cdots \subset D_{r(X)+i-1} - \{x\} \subset D_{r(X)+i-1} \subset \cdots$$

3.2.2. Chain expansions of choice functions

The problem is now to add a new element x to a list $f(X)$. The minimal change principle requires this operation be done without unnecessarily adding x to any other list $f(Y)$. It is clear, though, that this principle cannot be applied to the subsets Y of X such that $x \in Y$, since, by *Arrow* condition, we must then have $x \in f'(Y)$. Again, the solution we are now aiming at should be weighed by the fact we only have some *default* information, not knowing on what grounds x should be added to the list $f(X)$; we take for granted that it was decided to reevaluate the original rank of x, bringing it at the level $\kappa(X)$, rather than because the elements of the original list were retrograded to $\kappa(x)$.

With this in mind, since $x \notin f(X)$, we have $z \succ x$ for all elements $z \in f(X)$; after revision none of these inequalities must hold anymore, since both x and z are to be in $f'(X)$. Thus the operation to perform implies a *contraction* by $z \succ x$ for all $z \in f(Z)$. But, by the minimal change principle, we should have $f'(X) = f(X) \cup \{x\}$, and this implies that for all elements $t \neq x$ of $X - f(X)$ and all $z \in f(X)$, the inequalities $z \succ t$ have to be preserved. Therefore we have to perform a contraction with *constraints*, a problem that goes beyond the simple study we did in the preceding sections. The reader may check that a simple chain contraction as in Section 2.3.2 is quite inefficient: it would indeed yield the chain

$$D_0 \subset D_1 \subset \cdots \subset D_{r(X)-1} \subset D_{r(x)} \subset D_{r(x)+1} \subset \cdots \subset D_{h-1},$$

a result that is clearly unacceptable as it amounts to pushing back to $\kappa(x)$ all the elements y of Ω such that $\kappa(x) \leq \kappa(y) \leq \kappa(X)$.

To perform the desired expansion, we cannot therefore use the results established in the framework of rational preferences. A direct study nevertheless provides an immediate and simple solution. Recall indeed that all we want is that x be as good as the best elements of $f(X)$, the rank of these elements being unchanged. Thus we must have $r'(x) = r(X)$, so x must be an element of $W'_{r(X)}$ and therefore an element of all the subsequent links of the chain. This leads to the chain

$$\cdots \subset D_{r(X)-1} \subset D_{r(X)} \cup \{x\} \subset D_{r(X)+1} \cup \{x\} \subset \cdots \subset D_{r(x)} \subset \cdots \subset D_{h-1}.$$

The corresponding choice function f' satisfies $f'(X) = f(X) \cup \{x\}$ as desired. More precisely, the new choice function f' is given by

$$f'(Y) = \begin{cases} f(Y), & \text{if } x \notin Y \text{or } r(Y) < r(X) \\ f(Y) \cup \{x\}, & \text{if } x \in Y \text{and } r(Y) = r(X) \\ \{x\}, & \text{if } x \in Y \text{and } r(Y) > r(X) \end{cases}$$

3.2.3. The revision of choice functions

Our last task is now that of replacing an item of $f(X)$ by another one. We call this a revision, but clearly this problem presents quite a difference with both classical revision and the preference revision that was studied in this paper. Indeed, these two latter domains treated the same problem, which occurred when it was impossible to incorporate a new information to a set of data (a knowledge base or a complete set of rational preferences) without retracting at the same time some item that was not coherent with the new piece of information. In the framework of choice functions where we are now working, the problem is different: as we saw, it is always possible—we even dispose of several methods—to perform an expansion, that is to decide that such or such element of a set X should be added to $f(X)$. We do not therefore have to face a dilemma, debating which item should first be withdrawn from the list. Our problem reduces to the following one: replace the element x of $f(X)$ by the element x' of $X - f(X)$. To do so, we will naturally make use of the contraction and expansion process defined in the preceding paragraphs. Note that we have $r(X) = r(x) < r(x')$ by the choice of x and x'. The original chain is

$$D_0 \subset D_1 \subset \cdots \subset D_{r(x)} \subset \cdots \subset D_{r(x')} \subset \cdots \subset D_{h-1}.$$

The contraction by x first leads to the chain

$$D_0 \subset D_1 \subset \cdots \subset D_{r(x)-1} \subset D_{r(x)} - \{x\} \subset D_{r(x)} \cdots \subset D_{r(x')} \subset \cdots \subset D_{h-1}.$$

For the subsequent expansion by x', we note that we have to distinguish two cases, depending on whether $f(X)$ boils down to $\{x\}$ or not. In the first case, we get as resulting chain

$$\cdots \subset D_{r(x)-1} \subset D_{r(x)} - \{x\} \subset D_{r(x)} \cup \{x'\} \subset D_{r(x)+1} \cup \{x'\} \cdots \subset D_{r(x')} \subset \cdots .$$

In the general case where $f(X) \neq \{x\}$, the resulting chain is

$$\cdots \subset D_{r(x)-1} \subset D_{r(x)} - \{x\} \cup \{x'\} \subset D_{r(x)} \cup \{x'\} \cdots \subset D_{r(x')} \subset \cdots \subset D_{h-1}.$$

As a matter of facts, this latter chain is exactly the one we obtain if, instead of first contracting by $\{x\}$ and then expand by $\{x'\}$, one chooses first to expand by $\{x'\}$ and then to contract by $\{x\}$. Thus, the two operations of expansion and contraction defined in the preceding paragraph commute and give rise to a revision process in the principal case where $f(X) \neq \{x\}$.

In both cases, we have $f'(X) = f(X) - \{x\} \cup \{x'\}$. Note that $f'(Y) = f(Y)$ for all subsets Y's such that $\kappa(Y) > \kappa(x)$ or $\kappa(Y) \leq \kappa(x')$ or $\{x, x'\} \cap Y = \emptyset$.

4. Conclusion

The tool provided by logical or subsets chains reveals itself quite a performing one in the study, be it static or dynamic, of the preference sets or the choice functions that describe an agent behavior in a finite environment. It may be probably carried over to arbitrary uncomplete sets of preference, for we dispose of several methods to complete such sets. It should be also possible to use it in the problem of *choice compromises*: given a finite set of rational choice functions $f_1, f_2, \ldots, f_n$, all represented by their associated Ω-chains, define a resulting choice function f that can be considered as a best compromise between the f_i's.

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