Nonlinear Endogenous Fluctuations with Free Entry and Variable Markups

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Abstract

This paper explores the effects of markup variability, associated with free entry of Cournotian firms, on the occurrence of local indeterminacy, local Hopf bifurcations and multiplicity of steady states. It uses an overlapping generations model with capital, in which differentiated intermediate goods are produced with positive fixed plus constant marginal costs. If the markup factor were constant (the case of no free entry or the case of zero fixed costs leading to the perfectly competitive outcome) no local endogenous fluctuations or multiplicity of steady states can emerge, unless factors of production are complementary. On the contrary, with any positive (variable) market power, indeterminacy and multiplicity of steady states is possible for any value of the elasticity of technological substitution, and local Hopf bifurcations may also emerge, provided market power is sufficiently increased.

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1 Introduction

Recent papers have been showing that the emergence of endogenous fluctuations, i.e., equilibria exhibiting fluctuations driven by self fulfilling volatile expectations, is more likely in imperfect competitive environments. This is essentially due to the dynamic effects caused by non-standard behavior of output supply and labor demand, stemming from imperfect competition. Indeed, the variability of the markup factor on marginal cost may reverse the relation between price and output, and so do global economies of scale that are internal to the firm and hence incompatible with competitive equilibrium (decreasing marginal or average costs).

Markup variability may be the consequence of changes in expectations held by oligopolistic firms facing kinked demand curves (Woodford, 1991). It may also prevail because the optimal level of sustainable collusive agreements varies with the attractiveness for any potential deviator of cutting prices, itself dependent on aggregate demand shocks (Rotemberg and Woodford, 1992). Or it may result, in monopolistic competition, from changes in the composition of aggregate demand, for instance between investment and consumption (Gali, 1994), or between two classes or generations of consumers (d'Aspremont et al., 1995). This list is not exhaustive. Changing market shares, due to entry and exit of Cournotian firms, are another natural source of markup variability, that is easily combined, under free entry, with internal economies of scale originating from fixed costs (Portier, 1995, Weder, 2000). Besides contributing to the emergence of endogenous fluctuations, this source of markup variability also accounts for two often referred stylized facts characterizing the business cycle: procyclicality of business formation and countercyclicality of markups. Indeed, an expanding (contracting) aggregate demand diminishes (increases) average costs, induces entry (exit) and squeezes (swells) markups.

In this paper we explore the last mentioned way to markup variability. Our aim is to study the role of markup variability as a mechanism driving the existence of deterministic and stochastic endogenous fluctuations. We consider a simple overlapping generations model where households live for two periods[1], but our results would also apply if households lived for

[1]Seegmuller (2001) studies the role of markup variability in a overlapping generations economy but, contrary to our work, he only considered the case of an infinitely elastic labour supply. Kuhry (2001), by considering a model of a representative agent with an infinite horizon facing a binding cash in advance constraint, introduces markup variability

any arbitrarily high number of periods, provided they face binding financial constraints. The final output is competitively produced under constant returns to scale, out of differentiated intermediate goods. These are in turn produced out of competitively supplied labor[2] and capital, with a constant marginal cost and overheads expressed in terms of wasted output, and sold in Cournotian markets, where free entry prevails. The equilibrium number of firms, obtained under the usual zero profit condition, depends on the ratio of variable costs to fixed costs, and varies therefore according to the activity level.[3]

Our model embraces the situation of perfect competition in all markets as the limit case of zero overheads (thereby constant returns to scale leading to the appearance of an infinite number of firms). This case has already been explored in Reichlin (1986), in Lloyd-Braga (1995) as a particular case of an economy with Cournot competition and free entry but where decreasing marginal costs lead to a constant markup factor, and more recently also analyzed in Cazzavillan (2001), where the existence of perfect competition and capital externalities are further considered. It was shown that, in such context of constant returns to scale and perfect competition, endogenous fluctuations only emerge for small values of the elasticity of substitution between capital and labor (always lower than the capital share of output, which in turn must be smaller than 1/2), which seems to be empirically implausible.[4] Indeed, as it will become clear later on, these are also the results obtained if in our model we considered instead positive overheads (hence, economies of scale/increasing returns to scale) but with a fixed positive finite number of firms, and thereby a constant markup. Therefore, the reason for the difference of our results with respect to those obtained in Reichlin is to be found on the existence of markup variability, and not on the existence of a markup factor or increasing returns to scale by itself.

in the same way we do, but only analyzed the case where the elasticity of substitution between factors is constant and identical to 1.

[2]Endogenous fluctuations may also be approached by assuming imperfectly competitive labour markets: see for instance Jacobsen (2000).

[3]Some previous studies of similar economies, assuming an isoelastic (instead of a linear affine) cost function, together with an isoelastic demand function, obtain an endogenous but constant number of firms under the zero profit condition: Cazzavillan *et al.* (1998), Aloi *et al.* (2000), Lloyd-Braga (2000). As a consequence, they do not display markup variability.

[4]Empirical estimations of the elasticity of substitution between capital and labour point to values between 0.4 and 2 (see Hamermesh, 1993).

From a more technical point of view, we proceed to a local analysis of the dynamic properties of our model around a steady state. As well known, local indeterminacy allows existence of stationary stochastic fluctuations in which uncertainty is entirely extrinsic, that is, concerns agents' expectations but not the fundamentals ("sunspots"). This is the only way to obtain persistent endogenous fluctuations in linear models.[5] However, by resorting to nonlinearity, one can ensure existence of bounded fluctuating equilibria, both deterministic and stochastic, associated with the occurrence of local bifurcations, even when the steady state under analysis is determinate (for more details see Grandmont et al., 1998). In particular, when a local Hopf bifurcation occurs, periodic or quasi-periodic trajectories on an invariant closed curve exist near the steady state. Also, when a transcritical bifurcation occurs there is another close steady state which may be locally indeterminate. Using a geometrical method as developed in Grandmont et al. (1998) we are able to characterize in detail the local dynamic properties and occurrence of bifurcations in terms of relevant parameters[6], assuming that the capital share of output is lower than 1/2.[7] Since the emergence of endogenous fluctuations is also associated with multiplicity of steady states, possible in nonlinear models, we analyze the existence of multiplicity versus uniqueness of the steady state from a global point of view, by considering a parameterized version of the model (in particular technologies with constant elasticity of substitution) and ensuring that at least one interior steady state exists. Using the results on the local stability properties we further link the conditions for multiplicity/uniqueness with the local stability properties of the steady states involved.

Our main results can be summarized as follows. From a local point of view, the introduction of any positive (and variable) degree of market power, through product differentiation and the presence of overheads, allows us to obtain indeterminacy for any value of the elasticity of technological substi-

[5] Gali (1994) and Weder (2000), using models with an infinitely lived agent, exploit markup variability in precisely this context.

[6] As it will be seen, the local dynamic properties in our model only depend on the capital share of output, the elasticity of substitution between factors, the elasticity of labour supply and on the markup factor, all evaluated at the steady state solution under analysis. We emphasize however that the markup factor only becomes relevant for local dynamics when there is markup variability.

[7] This seems to be a reasonable assumption, in accordance with empirical evidence at least for industrialized countries. Besides, as it will be shown, local indeterminacy in our economy can only occur in this case.

tution higher than 1/2, in contrast with Reichlin results. Also, for a high enough (variable) markup factor (which may however be kept empirically plausible[8]), local Hopf bifurcations can occur in similar technological conditions. Cazzavillan (2001) obtained similar results in the presence of social increasing returns to scale, associated to decreasing social marginal costs due to capital externalities. Our work shows that, even with constant marginal costs, markup variability associated free entry and fixed costs is yet another reason for the emergence of endogenous fluctuations with high substitution between factors in production. Our analysis on multiplicity/uniqueness of the steady state shows that if the elasticity of demand for intermediate goods is low (more precisely r than 1/2) then two steady states generically exist. However, for higher values of the elasticity of demand for intermediate good we cannot exclude uniqueness of the steady state. We further show that if uniqueness holds then an elasticity of substitution between factors higher than 1/2 implies local determinacy. Hence, in our model, local indeterminacy is associated with multiplicity of steady states when the elasticity of substitution between factors is high.

In the following, we present the model in section 2, and proceed to the local analysis of persistent endogenous fluctuations in section 3. In section 4 we study the existence of multiplicity versus uniqueness of the steady state. We conclude in Section 5.

2 The Model

2.1 Consumers

We consider an economy in which two overlapping generations of identical consumers coexist in any period $t = 0, 1, ..., \infty$, each one forming a continuum of unit mass. Consumers live for two periods, work when young, and consume when old. There is a single final (both consumer and capital) good. The representative young consumer chooses to supply z^0 units of labor at real wage ω, and to buy ωz^0 units of capital. He/she plans to consume, when

[8]The markup factor should take values not much higher than 1. On this issue see for instance Rotemberg and Woodford (1991) or the more recent work of Basu and Fernald (1997).

old, $c^1 = \text{þ}\omega z^0$ units of final good,[9] on the basis of a given expectation of the future real net return factor on capital þ. The markets for the final good and for the two factor services (labor and capital) are perfectly competitive. The young consumer maximizes utility $c^1 - \delta V(z^0)$, where $\delta \in (0, \infty)$ is a scale parameter reflecting the degree of present labor disutility in terms of discounted future consumption, and $V : (0, \overline{z}) \to (0, \infty)$, with $\overline{z} \in (1, \infty]$, is a function expressing the way this disutility evolves as the consumer supplies more and more labor. This function is of class C^2, with positive first and second derivatives, such that $\lim_{z \to 0} V'(z) = 0$ and $\lim_{z \to \overline{z}} V'(z) = \infty$, and normalized to: $V'(1) = 1$. Accordingly, labor supply satisfies:

$$\delta V'\left(z^0\right) = \text{þ}\omega. \tag{1}$$

In the following, we will denote $v(z) \equiv \epsilon V'(z) \equiv V''(z) z / V'(z) > 0$ the elasticity of the function V' (the degree of convexity of V), which is also the inverse of labor supply elasticity.

A common objection to simple overlapping generations models, where consumers live for only two periods, concerns the inadequate generational timing when used to analyze business cycles. It is however possible to extend the model to encompass an arbitrarily low period length, while keeping the same aggregate behavior, provided consumers are assumed to face binding financial constraints, in the spirit of Woodford (1988).[10]

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[9] Old consumers at period $t = 0$, are endowed with all the capital stock K_{-1} existing at the outset of that period and use the real net return $\rho_0 K_{-1}$ to buy consumption goods.

[10] Assume indeed that each generation is formed by a continuum, of mass normalized to $1/T$, of identical consumers living for $T + 1$ periods. Also, assume that the representative consumer of age $\theta \in \{0, 1, ..., T\}$, at period t_0, has preferences represented by: $\sum_{t=t_0}^{t_0+T-\theta} \beta^{t-t_0} \left(c_t^\theta - bV\left(z_t^\theta\right)\right)$, with $0 < \beta < 1$ and $b > 0$. Individual real expenditure in each period t (on consumption c_t^θ and productive capital K_t^θ) is constrained by current real income (from labor z_t^θ, at real wage rate ω_t, and from capital K_{t-1}^θ, at real net return factor ρ_t), according to the budget constraint: $c_t^\tau + K_t^\tau \leq \omega_t z_t^\tau + \rho_t K_{t-1}^\tau$. Besides, assume that productive capital is alone acceptable as collateral to secure a loan (contrary to human capital), so that consumption spending must be financed by capital income, according to the financial constraint: $c_t^\theta \leq \rho_t K_{t-1}^\theta$. Clearly, all budget constraints are binding for an optimizing choice, and so is the financial constraint in any period t provided $\beta \rho_{t+1} < 1$, which may be ensured if ρ_{t+1} is bounded and if time preference is strong enough. We then obtain the first order condition (for $0 < z_t^\theta < \overline{z}$): $(b/\beta) V'\left(z_t^\theta\right) = \omega_t \rho_{t+1}$, leading to the same z_t^θ whatever the age, up to $T - 1$ (for $\theta = T$, we clearly have $z_{t_0}^T = 0$). Thus, multiplying individual quantities by T (the number of generations involved) times $1/T$ (the mass of each generation), we get an aggregate which coincides with individual decisions.

2.2 Producers

Final output Y is competitively produced from a continuum of intermediate goods, according to a symmetric production function with constant elasticity of substitution $\tau \in (0,\infty)$: $Y^{1-1/\tau} = \int_0^1 Y(i)^{1-1/\tau} di$. Cost minimization leads to demand for the intermediate good i:

$$Y(i) = [P(i)/P]^{-\tau} Y, \text{ with } P^{1-\tau} = \int_0^1 P(i')^{1-\tau} di'. \tag{2}$$

Notice that perfect competition in the final good market requires equality of price and marginal cost P.

Each intermediate good i is assumed to be produced by a set $J(i)$ of $n(i)$ Cournotian firms ($n(i) \geq 2$), with identical technologies. These are represented by a positive fixed cost ϕ in terms of wasted output, and by a neoclassical production function F, homogeneous of degree one in capital k and labor l, such that the firm's output is at most equal to $\pi[F(k,l) - \phi] \equiv \pi[lf(a) - \phi]$, with $a \equiv k/l$ and $f(1) = 1$ by convention, $\pi \in]0,\infty[$ being a scale parameter reflecting the productivity level. We shall use in the following the notation $\sigma(a)$ for the elasticity of substitution between the two factors. By definition, and denoting ϵ the elasticity operator:

$$\frac{1}{\sigma(a)} \equiv \epsilon\left(\frac{f(a) - f'(a)a}{f'(a)}\right) = 1 + \epsilon\left(\frac{1}{\epsilon f(a)} - 1\right) = \frac{-\epsilon f'(a)}{1 - \epsilon f(a)} > 0. \tag{3}$$

Each firm j in industry i takes as given the inverse demand function implicit in equation (2) and the quantities $y_{j'}$ supplied by other firms j' in $J(i)$. It also takes as given the quantity Y of final output produced by the competitive sector, the aggregate price P of the monopolistic sector (on which the influence of its own choice is negligible) and the prices w and r in the factor markets. This leads to the profit maximization problem:

$$\max_{(y_j,l_j,k_j)\in\Re^3_+} P\left[\frac{y_j + \sum_{j'\in J(i)\setminus\{j\}} y_{j'}}{Y}\right]^{-1/\tau} y_j - wl_j - rk_j \tag{4}$$

$$\text{s.t. } y_j \leq \pi[l_j f(k_j/l_j) - \phi]. \tag{5}$$

The same argument applies for consumption (with $c^\theta_{t_0} = \rho_{t_0} K^\theta_{t_0-1}$, if $\theta > 0$, and $c^0_{t_0} = 0$), and capital (with $K^\theta_{t_0} = \omega_{t_0} z^\theta_{t_0}$, if $\theta > 0$, and $K^0_{t_0} = 0$), extending the aggregate behavior obtained in the case $T = 1$ to an arbitrary T.

A straightforward calculation gives the necessary and sufficient first order conditions:

$$\frac{w}{P(i)} = \pi \frac{f(a_j) - f'(a_j)\, a_j}{\mu_j}, \tag{6}$$

$$\frac{r}{P(i)} = \pi \frac{f'(a_j)}{\mu_j}, \tag{7}$$

$$y_j = \pi \left[l_j f(a_j) - \phi \right], \text{ where } a_j = k_j / l_j, \tag{8}$$

$$\text{with } \frac{1}{\mu_j} = 1 - \frac{1}{\tau} \frac{y_j}{y_j + \sum_{j' \in J(i) \setminus \{j\}} y_{j'}}. \tag{9}$$

The markup factor μ_j of price $P(i)$ over marginal cost $w/\pi \left[f(a_j) - f'(a_j)\, a_j \right] = r/\pi f'(a_j)$ may be taken as a measure of the market power of firm j (the Lerner index of degree of monopoly is: $1 - 1/\mu_j$).

2.3 Equilibria

We now exploit the symmetry of the economy, and restrain our analysis to symmetric free-entry equilibria, which satisfy the zero profit condition, neglecting at the same time the discrete nature of the variable $n(i)$. Using relations (4) to (8) and omitting indices, given symmetry, we get:

$$\pi \left[l f(a) - \phi \right] = y = \pi l f(a) / \mu, \tag{10}$$

so that, using also (9), we obtain:

$$l f(a) = \phi \mu / (\mu - 1) = n \tau \phi. \tag{11}$$

Notice that these equations show that the number n of firms (resp. the markup factor μ) is positively (resp. negatively) related to individual output y (and hence to aggregate output ny), leading to procyclicality (resp. countercyclicality) of these variables, as expected.[11]

[11] Indeed an increase in output $(lf(a))$ implies an increase in profits $(\pi \left[lf(a)\mu / (\mu - 1) - \phi \right])$ which induces entry of new firms and thereby a reduction on the markup factor. This cyclical behaviour of the number of firms and markup factor is, from a qualitative point of view, in accordance with that obtained in other works where similar productive structure is considered as well as with empirical evidence, as refereed in the Introduction.

Let us finally introduce dated variables, and formulate conditions for an intertemporal (perfect foresight) equilibrium. At period t, aggregate demand for labor $n_t l_t$ must be equal to labor supply z_t, as given by equation (1). Assuming, for simplicity, total capital depreciation[12], we obtain from equations (6) and (7), and since, by symmetry, $\omega_t = w_t/P_t = w_t/P_t(i)$ and $\rho_t = r_t/P_t = r_t/P_t(i)$:

$$\delta V'(n_t l_t) = \pi^2 \left[f(a_t) - f'(a_t)\,a_t\right](1/\mu_t)\, f'(a_{t+1}) \left(1/\mu_{t+1}\right). \tag{12}$$

Similarly, aggregate demand for capital services at period t, $n_t a_t l_t$ must be equal to the corresponding aggregate supply:

$$n_t a_t l_t = K_{t-1}, \tag{13}$$

fixed at period $t-1$ by the savings of the old generation $n_{t-1} l_{t-1} \omega_{t-1}$:

$$K_{t-1} = \pi n_{t-1} l_{t-1} \left[f(a_{t-1}) - f'(a_{t-1})\,a_{t-1}\right]/\mu_{t-1}. \tag{14}$$

Finally, using equations (11) and (13), we can determine:

$$n_t = \sqrt{\frac{K_{t-1} f(a_t)}{\tau\phi a_t}} \equiv n(a_t, K_{t-1}) \geq 2, \tag{15}$$

leading to the restriction:

$$K_{t-1} f(a_t)/a_t \geq 4\tau\phi. \tag{16}$$

We can further determine that at equilibrium we must have:

$$\mu_t = \frac{\tau n_t}{\tau n_t - 1} = \frac{\sqrt{(\tau/\phi)\,K_{t-1} f(a_t)/a_t}}{\sqrt{(\tau/\phi)\,K_{t-1} f(a_t)/a_t} - 1} \equiv \mu(a_t, K_{t-1}; \tau/\phi) > 1. \tag{17}$$

Equations (12) to (17) can be straightforwardly transformed into a two-dimensional dynamic system in the variables a_t and K_{t-1}, of which the second is pre-determined:

$$K_t = \pi K_{t-1}\left[1/\mu(a_t, K_{t-1}; \tau/\phi)\right] f'(a_t)\left[1/\epsilon f(a_t) - 1\right] \tag{18}$$

$$\pi K_t\left[1/\mu(a_{t+1}, K_t; \tau/\phi)\right] f'(a_{t+1}) = \delta V'(K_{t-1}/a_t)\, K_{t-1}/a_t. \tag{19}$$

This system, together with the restrictions on n and μ, characterizes a symmetric free-entry equilibrium with perfect foresight:

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[12] This simplifying assumption does not modify qualitative results, but it would of course be inappropriate in view of quantitative assessments concerning parameter values, in particular under the interpretation of the model suggested in footnote 6.

Definition 1 A symmetric free-entry equilibrium with perfect foresight is a sequence $(a_t, K_{t-1})_t \in \left(\mathbb{R}^2_{++}\right)^{\mathbb{N}}$, such that equations (18) and (19), together with restrictions (16) and (17), are satisfied.

From (17), notice that μ is a decreasing function of τ/ϕ. Market power decreases when either the elasticity of substitution τ between intermediate goods increases, or fixed costs ϕ diminish, inducing a higher number of competing firms. In the limit, as τ/ϕ tends to infinity (which, because of the restriction (16), is only possible if ϕ tends to zero), μ_t converges to 1 (n converging to ∞), which corresponds to perfect competition. We shall also use in the following section the partial elasticities $\epsilon_a\mu$ and $\epsilon_K\mu$ with respect to a and K, respectively:π

$$\epsilon_a\mu\left(a_t, K_{t-1}\right) = \left(1 - \epsilon f\left(a_t\right)\right)\left(\mu\left(a_t, K_{t-1}\right) - 1\right)/2 \text{ and} \tag{20}$$

$$\epsilon_K\mu\left(a_t, K_{t-1}\right) = -\left(\mu\left(a_t, K_{t-1}\right) - 1\right)/2. \tag{21}$$

The absolute value of each one of these two elasticities, expressing the response of the markup factor to changes in a and K, is an increasing function of the markup factor itself. As μ tends to 1, that is, as we approach perfect competition, the markup factor ceases to be sensitive to changes in the relevant variables. For the sake of interpretation of our results in next sections, we also consider the case of an equilibrium with an exogenous number n of firms ($n \geq 2$, $n > 1/\tau$), also leading to a constant markup factor $\mu = n\tau/\left(n\tau - 1\right) > 1$ (where by definition $\epsilon_a\mu\left(a_t, K_{t-1}\right) = \epsilon_K\mu\left(a_t, K_{t-1}\right) \equiv 0$). Non-negativity of profits imposes: $K_{t-1}f\left(a_t\right)/a_t \geq n^2\tau\phi$. By assuming lump sum distribution of profits to old consumers, we obtain the same dynamic system characterizing an equilibrium with perfect foresight (but now with a constant μ). Indeed, capital accumulation by young consumers is independent of profit income, and so is labor supply, because of additive separability of the utility function, with affine linearity in consumption.

In order to proceed, in the next section, to a local analysis of the dynamic system defined by equations (18) and (19), with the function μ as defined in (17), we need to ensure existence of a steady state solution $(a_t, K_{t-1}) = (a^*, K^*) \in \mathbb{R}^2_{++}$, for any t in $\mathbb{N}$. Using (18), we obtain for stationary values a^* and K^*:

$$\mu\left(a^*, K^*; \tau/\phi\right) = \pi f'\left(a^*\right)\left[1/\epsilon f\left(a^*\right) - 1\right]. \tag{22}$$

Also, by dividing both sides of equation (19) by the opposite sides of equation (18), we obtain:

$$\frac{a^*}{1/\epsilon f(a^*) - 1} = \delta V'\left(\frac{K^*}{a^*}\right). \tag{23}$$

Existence of a steady state is clearly ensured by assuming that the scale parameters π and δ verify equations (22) and (23), for any configuration (a^*, K^*) satisfying restrictions (16) and (17). Notice that these restrictions are both satisfied[13] if $K^* f(a^*)/a^* > \max\{4\tau\phi, \phi/\tau\}$. This motivates the following statement:

Lemma 2 Consider $(a^*, K^*) \in \mathbb{R}^2_{++}$ such that $K^* f(a^*)/a^* > \max\{4\tau\phi, \phi/\tau\}$. Let $\pi = \frac{\mu(a^*, K^*; \tau/\phi)}{f'(a^*)[1/\epsilon f(a^*) - 1]}$ and $\delta = \frac{a^*}{[1/\epsilon f(a^*) - 1]V'(K^*/a^*)}$. Then (a^*, K^*) is a steady state equilibrium.

3 Local Dynamics and Bifurcation Analysis

Assuming existence of a steady state equilibrium, according to the former lemma, we now examine its local dynamic properties. In particular, we analyze how the occurrence of local indeterminacy and local bifurcations, and ultimately the emergence of deterministic or stochastic fluctuations driven by self fulfilling expectations, depend on some relevant parameters of the model.

It is known since Reichlin (1986) that local endogenous fluctuations can emerge in OLG economies with constant returns to scale and perfectly competitive markets only when high complementarity between factors is assumed. More precisely, local indeterminacy may only occur when the elasticity σ of factor substitution is smaller than the capital share of output, which must in turn be lower than$1/2$, thus requiring an implausible value of σ if we refer to empirical estimations. Cazzavillan (2001) has shown that local indeterminacy may appear for values of $\sigma > 1/2$, by assuming decreasing social marginal costs due to positive capital externalities. Our main purpose is to establish that, even with constant marginal costs, markup variability associated with free entry and fixed costs is also able to generate endogenous fluctuations without resorting to small values of σ.

[13] We take here restriction (16) as a strict inequality, implying $n^* > 2$, in order to preserve admissibility of non-stationary trajectories in some neighborhood of the steady state.

3.1 The general framework

In order to study local stability properties we follow the usual procedure of considering a (log)linearized version of the dynamic system (18)-(19) around some steady state (a^*, K^*), and we accordingly obtain (using (3)) the matrix equation for (percentage) deviations of a and K from the steady state:

$$\begin{pmatrix} 0 & 1 \\ -\epsilon_a\mu^* + \epsilon f'(a^*) & 1-\epsilon_K\mu^* \end{pmatrix} \begin{pmatrix} da_{t+1}/a^* \\ dK_t/K^* \end{pmatrix}$$

$$= \begin{pmatrix} -\epsilon_a\mu^* + \epsilon f'(a^*) + 1/\sigma^* - 1 & 1-\epsilon_K\mu^* \\ -(1+v^*) & 1+v^* \end{pmatrix} \begin{pmatrix} da_t/a^* \\ dK_{t-1}/K^* \end{pmatrix}, \quad (24)$$

where $\epsilon_x\mu^*$ denotes the partial elasticity with respect to the variable x ($x = a, K$) of the function μ at (a^*, K^*), $\sigma^* \in (0,\infty)$ and $v^* \in (0,\infty)$ replacing $\sigma(a^*)$ and $v(K^*/a^*)$, respectively, for simplicity of notation. We can now premultiply the matrix of the right-hand side of this equation by the inverse of the left-hand side matrix, in order to get the Jacobian matrix J, evaluated at the steady state under consideration, with the following elements:

$$\begin{aligned} J_{aa} &= \frac{(1-\epsilon_K\mu^*)(\alpha^*/\sigma^* - \epsilon_a\mu^*) + \epsilon_K\mu^* + v^*}{\epsilon_a\mu^* + (1-\alpha^*)/\sigma^*} \\ J_{aK} &= \frac{(\epsilon_K\mu^* - 2)\epsilon_K\mu^* - v^*}{\epsilon_a\mu^* + (1-\alpha^*)/\sigma^*} \\ J_{Ka} &= -\epsilon_a\mu^* + \alpha^*/\sigma^* - 1 \\ J_{KK} &= 1-\epsilon_K\mu^*, \end{aligned} \quad (25)$$

where we denote, for simplicity, $\alpha^* \equiv \epsilon f(a^*) \in (0,1)$, equal to the capital share of income[14] (as can be seen from (7) and (10)), and use again (3) to get: $\epsilon f'(a^*) = -(1-\alpha^*)/\sigma^*$. Using (6) and (7), we observe that α^*/σ^* is the inverse of the Marshallian elasticity of the labor demand function.

The determinant and the trace of this Jacobian matrix are given by the following expressions:

$$D = \frac{(1+v^*)(\alpha^*/\sigma^* - \epsilon_a\mu^* - \epsilon_K\mu^*)}{\epsilon_a\mu^* + (1-\alpha^*)/\sigma^*} \quad (26)$$

$$T = \frac{(1-\epsilon_K\mu^*)/\sigma^* + \epsilon_K\mu^* + v^*}{\epsilon_a\mu^* + (1-\alpha^*)/\sigma^*}. \quad (27)$$

[14] Or the capital share of non-profit income, in the case of exogenous n, where profits are positive.

Letting $\mu^* \in (1, \infty)$ be the markup level at the steady state, and referring to equations (20) and (21), we can further calculate:

$$D = \frac{\alpha^*}{1-\alpha^*}\left(1+v^*\right) > 0 \tag{28}$$

$$T = \frac{\left(\mu^*-1\right)\left(1-\sigma^*\right)+2\left(1+\sigma^* v^*\right)}{\left[\left(\mu^*-1\right)\sigma^*+2\right]\left(1-\alpha^*\right)}. \tag{29}$$

We may already observe at this stage that, as $\phi \to 0$ and consequently $n^* \to \infty$ and $\mu^* \to 1$ (implying $\epsilon_a \mu^* \to 0$ and $\epsilon_K \mu^* \to 0$), that is, as fixed costs vanish and competition tends to perfection, the determinant keeps the same expression, whereas the trace tends to $\left(1+\sigma^* v^*\right) / \left(1-\alpha^*\right)$. Also, if we take the number n of firms to be exogenous, and the markup factor μ to be accordingly constant (so that $\epsilon_a \mu^* = \epsilon_K \mu^* = 0$ in (26) and (27)), we get precisely the same expressions for the determinant $\left(\left(1+v^*\right)\alpha^* / \left(1-\alpha^*\right)\right)$ and the trace $\left(\left(1+\sigma^* v^*\right) / \left(1-\alpha^*\right)\right)$. In other words, in spite of increasing returns due to the presence of fixed costs and of the existence of market power, the model with a constant positive markup rate behaves dynamically as if the economy were perfectly competitive with constant returns (and without externalities). It is markup variability ($\epsilon_a \mu^* \neq 0 \neq \epsilon_K \mu^*$), not markup positivity ($\mu - 1 > 0$) nor decreasing average costs ($\phi > 0$), that makes the difference.

Local stability analysis of the steady state (a^*, K^*) is now effected by considering the eigenvalues of J, i.e., the roots of the characteristic polynomial $\mathcal{P}(\lambda) \equiv \lambda^2 - T\lambda + D$. We first observe that

$$\mathcal{P}(-1) = 1 + T + D = \frac{\left(\mu^*-1\right)\left(1+\alpha^*\sigma^* v^*\right)+2\left[2+\left(\alpha^*+\sigma^*\right)v^*\right]}{\left[\left(\mu^*-1\right)\sigma^*+2\right]\left(1-\alpha^*\right)} > 0 \tag{30}$$

$$\mathcal{P}(1) = 1 - T + D = \frac{\left(\mu^*-1\right)\left[2\sigma^*-1+\alpha^*\sigma^* v^*\right]+2\left(\alpha^*-\sigma^*\right)v^*}{\left[\left(\mu^*-1\right)\sigma^*+2\right]\left(1-\alpha^*\right)} \equiv H^*. \tag{31}$$

As $\mathcal{P}(-1) > 0$, the characteristic polynomial will have a root in the interval $(-1, 1)$ and another in the interval $(1, \infty)$ if $\mathcal{P}(1) < 0$, implying that the steady state is a saddle. If $\mathcal{P}(1) > 0$, the modulus of both roots lies generically either in the interval $(0, 1)$, implying that the steady state is a sink or in the interval $(1, \infty)$, implying that it is a source, depending upon the determinant being smaller or larger than one. We summarize these facts in the following proposition.

Proposition 3 A steady state (a^*, K^*) is a saddle if and only if $H^* < 0$. If $H^* > 0$, it is a sink if and only if $v^* < 1/\alpha^* - 2$, and it is a source if and only if $v^* > 1/\alpha^* - 2$.

Notice that, since $v > 0$, local indeterminacy (sink) may only arise when the capital share of (non-profit) income is lower than 1/2. We may further notice that, in the limit case $\mu^* \to 1$ (or, equivalently, in the case of exogenous n), $H^* > 0$ if and only if $\sigma^* < \alpha^*$: the steady state is always a saddle, unless the Marshallian elasticity of labor demand σ^*/α^* is smaller than one, in conformity with Reichlin (1986) and Cazzavillan (2001).[15]. Moreover, the existence of endogenous fluctuations, linked with the possibility of a locally indeterminate steady state, requires in this limit case that $\sigma^* < \alpha^* < 1/2$, i.e., high complementarity between factors.

3.2 Bifurcation analysis

We want next to consider how the qualitative properties of the dynamic system in the neighborhood of the steady state change by perturbing some bifurcation parameter, given particular configurations of the other parameter values. The choice of v^* as the bifurcation parameter seems natural when referring to Proposition 3. More precisely, we shall admit that the steady state remains fixed, normalized to $(a^*, K^*) = (1, 1)$ (through an appropriate choice of labor and capital units). To simplify notations, we will henceforth drop the stars for steady state values, and denote $\alpha \equiv \epsilon f(1)$, $\mu \equiv \mu(1, 1; \tau/\phi)$, $\sigma \equiv \sigma(1)$ and $v \equiv v(1)$, treated as parameters. Of course, when considering particular configurations of parameters α and μ, we must admit that, in order for the steady state to be kept equal to $(1, 1)$, and according to Lemma 2,

[15] In fact, our Jacobian matrix J coincides both in the limit case of zero fixed costs and markups, and in the case of an exogenous n and a constant markup factor μ, with the Jacobian matrix in Proposition 3.1.1 in Cazzavillan (2001), corresponding to perfect competition and constant returns to scale (with no externalities).

the scale parameters δ and π automatically take the following values[16]:

$$\delta = \frac{\alpha}{1-\alpha} \text{ and } \pi = \frac{\mu}{1-\alpha}. \tag{32}$$

Also, it is clear from (17) that considering particular values for the markup factor μ, as if it were exogenous, is only a short cut for perturbing the fixed cost ϕ and/or the elasticity τ of substitution between intermediate goods, which are the genuine parameters. Thus, when choosing a value of μ, we are implicitly assuming that ϕ and τ automatically take values such that:

$$\frac{\phi}{\tau} = \left(1 - \frac{1}{\mu}\right)^2, \tag{33}$$

with the restriction $1 > \max\{4\tau\phi, \phi/\tau\}$ as assumed in Lemma 2 (implying $\phi < 1/2$). Besides, by (17) and since $n \geq 2$ as in (15), the admissible interval for μ can be $(1, \infty)$ only if $\tau \leq 1/2$, and $(1, 2\tau/(2\tau-1))$ otherwise. Finally, by Proposition 3, local indeterminacy can only appear for $\alpha < 1/2$, which corresponds indeed to empirical observations. Accordingly, we shall assume throughout in our analysis that α is constant taking a fixed value in $(0, 1/2)$.

We will proceed by using a geometrical method, developed in Grandmont et al. (1998), which focus on how the determinant and the trace of the Jacobian matrix evolve in the space (T, D) as the bifurcation parameter v changes continuously in its admissible range, for different fixed configurations of relevant parameters (i.e, μ and σ). From (28) and (29), one can easily obtain the locus of points (T_v, D_v) defining the values of trace and determinant as v changes, through the relation:

$$D = \alpha\left(\frac{\mu-1}{2} + \frac{1}{\sigma}\right)T - \frac{\alpha(1-\sigma)}{(1-\alpha)\sigma}\frac{\mu+1}{2}, \tag{34}$$

which, for $v \in (0, \infty)$, describes a half-line $\Delta(\mu, \sigma)$, in the positive orthant of space (T, D) with the translated origin (T_0, D_0) for $v = 0$[17]:

$$(T_0(\mu, \sigma), D_0) = \left(\frac{(\mu-1)(1-\sigma)+2}{[(\mu-1)\sigma+2](1-\alpha)}, \frac{\alpha}{1-\alpha}\right), \tag{35}$$

[16] By Lemma 2, the scale parameters do not have to change, while perturbing the degree of convexity of V, and keeping the steady state fixed at $(1,1)$, since we have normalized the function V so as to keep its derivative at 1 equal to 1, so that: $V'(K^*/a^*) = V'(1) = 1$. Thus the degree of convexity of V is made to vary continuously in $(0, \infty)$ by considering that $v^* = v(1) = V''(1)$ runs the interval $(0, \infty)$.

[17] From (28) and (29), note that the determinant and trace are increasing in v and that for $v = \infty$ we have $D_\infty = T_\infty = \infty$.

and a positive slope:

$$\Delta'(\mu,\sigma) = \alpha \left(\frac{\mu-1}{2} + \frac{1}{\sigma} \right) \tag{36}$$

We are interested in the position of this half-line with respect to three regions, corresponding to the couples (T, D) such that the steady state $(1,1)$ is a saddle, a sink, and a source. We have established in the former subsection that the determinant is positive and that $\mathcal{P}(-1) > 0$, that is, $D > \max\{0, -T-1\}$ whatever the (admissible) parameter values. By Proposition 3, the steady state is a saddle if and only if $\mathcal{P}(1) < 0$, hence if $0 < D < T-1$, a sink if $\max\{T-1, 0, -T-1\} < D < 1$, and a source if $D > \max\{T-1, 1, -T-1\}$. Figure 1 represents these three regions of the space (T, D) - bounded by two thick lines, AC (defined by $P(1) = 0$, i.e. $D = T-1$) and BC (defined by $D = 1$), and a thin line AB (given by $P(-1) = 0$, i.e. $D = -T-1$)- together with two (thin) examples of possible half-lines Δ.

(Insert Figure 1)

In the example of Δ^a, the steady state is a sink for low values of v, undergoes generically a Hopf bifurcation for the value v_H at which $D_{v_H} = 1$ (Δ crossing the segment BC in its interior), becomes a source for intermediate values of v, undergoes generically a transcritical bifurcation at v_T such that $D_{v_T} = T_{v_T} - 1$ (the line Δ crossing the line AC[18]), and ends up as a saddle for high values of v. By using (28) and (29), the values v_H and v_T are easily determined as functions of the remaining parameters α, μ and σ:

$$v_H = \frac{1}{\alpha} - 2 \tag{37}$$

$$v_T = \frac{(\mu-1)(1-2\sigma)}{(\mu-1)\alpha\sigma + 2(\alpha-\sigma)}. \tag{38}$$

[18] A bifurcation generically occurs when, by slightly changing a parameter of a nonlinear model, an eigen value crosses the unit circle. For more details see for instance Grandmont (1988) or Hale and Koçak (1991). When (T, D) cross the line AC an eigen value is crossing the value 1. Since the steady state $(a^*, K^*) = (1,1)$ is persistent in our analysis, we may expect that a transcritical bifurcation occurs, by which two close steady states exchange stability properties. As shown in Section 4, this is indeed the case for an economy with constant elasticities.

We are next going to examine how this picture changes when μ and σ take different values (while keeping α constant), which amounts to study how the origin and the slope of the half-line Δ move with these parameters.

3.3 The role of the markup factor (variability) and of the elasticity of factor substitution

First notice that, by (35), as μ changes, while keeping α constant, the origin of Δ moves on a horizontal line, with ordinate $D(0) = \alpha/(1-\alpha) \in (0,1)$ (since we assumed $0 < \alpha < 1/2$). Begin with the limit case $\mu = 1$ of perfect competition (which, as already seen, is dynamically equivalent to the case of a constant markup factor). As $T_0 = 1/(1-\alpha) = D_0 + 1$ in this case, the origin of Δ lies on the line AC, given any fixed value of σ. Local indeterminacy then requires the slope of Δ, α/σ, to be larger than one, which is precisely Reichlin's result.

When increasing μ (for σ constant), two cases have to be distinguished according to the value of σ, since the origin of Δ moves to the left if $\sigma > 1/2$, to the right if $\sigma < 1/2$ (resting on line AC for every μ if $\sigma = 1/2$). Take the former case. For given arbitrarily low values of $\mu - 1$, local indeterminacy of the steady state prevails at least for v close to zero, since the origin of Δ lies in the interior of the sink region of the space (T, D). Also, if we assume a large enough μ, hence a sufficiently large slope of Δ (i.e., such that the line Δ crosses AC above point C), the steady state will undergo a Hopf bifurcation as we increase v. See Figure 1, line Δ^a. In the latter case of $\sigma < 1/2$, although the steady state is always a saddle for low values of v (since the origin lies to right of line AC, i.e., in the saddle region), local indeterminacy and existence of a Hopf bifurcation may again result from a large slope of Δ, due to a high value of the markup factor (see line Δ^b). We develop these ideas and make them more precise in the proposition below.

Knowing also how the slope and origin of the half-line Δ moves with σ, the reader may wish to figure out how the Δ line evolves with σ, for other values of μ. When σ decreases the slope of Δ increases (see (36)), taking the maximum value ∞ for $\sigma = 0$, while the origin of Δ is shifting to the right along an horizontal line (see (35) and notice that T_0 is decreasing in σ), from a point on line AB when $\sigma = \infty$ (i.e. $D_0 = -T_0(\mu, \infty) - 1$, for every μ) to a point on the right of line AC for $\sigma = 0$, going through the line AC for $\sigma = 1/2$ (i.e. $D_0 = T_0(\mu, 1/2) - 1$, for every μ).

Take for instance a given value of μ sufficiently high, so that the slope of Δ is higher than 1 for every $\sigma > 0$, as in Figure 2. That is, since the slope $\Delta'(\mu, \sigma)$ decreases with σ, choose a fixed $\mu > \mu_4$ where μ_4 is such that $\Delta'(\mu_4, +\infty) = 1$, i.e. by (36) $\mu_4 \equiv 1 + \frac{2}{\alpha}$. Since the origin of Δ lies to the left of line AC for $\sigma > 1/2$, only Hopf bifurcations occur and transcritical bifurcations cannot occur in this case. However as soon as $\sigma < 1/2$ the occurrence of transcritical bifurcations is a pervasive phenomena, although the occurrence of a Hopf bifurcation may disappear for low values of σ if the origin of the half-line Δ for $\sigma = 0$ is such that $T_0(\mu, 0) > 2$ (which is indeed the case for $\mu > \mu_4$).

(Insert Figure 2)

Fixing $\alpha \in (0, 1/2)$ and considering different values of μ by order of decreasing 'empirical relevance' (i.e. increasing values of μ), the following proposition characterize in detail the local dynamic properties and the occurrence of bifurcations for the different admissible values of σ and v.

Proposition 4 Let $(a^*, K^*) = (1, 1)$ be a steady state of the dynamic system (18)-(19), with scale parameters δ and π adjusted so as to verify the assumptions of Lemma 2. Take the capital share of income $\alpha \in (0, 1/2)$ as constant throughout, and consider admissible values of the markup factor $\mu \in (1, \infty)$ ($\mu \in (1, 2\tau/(2\tau - 1))$ if $\tau > 1/2$), of the elasticity of factor substitution $\sigma \in (0, \infty)$, and of the inverse of the elasticity of labor supply $v \in (0, \infty)$, all evaluated at the steady state. Take v_H and v_T as defined respectively in (37) and (38). Let μ_1, μ_2, μ_3 and μ_4, satisfying $1 < \mu_1 < \mu_2 < \mu_3 < \mu_4$ be respectively defined as: $\mu_1 \equiv 1 + 2(1 - 2\alpha)$; $\mu_2 \equiv 1 + \frac{2(1-2\alpha)}{\alpha(3-2\alpha)}$; $\mu_3 \equiv 1 + \frac{2(1-2\alpha)}{\alpha}$; $\mu_4 \equiv 1 + \frac{2}{\alpha}$. Finally, consider σ_T and σ_H defined as: $\sigma_T \equiv \alpha \frac{\mu_4 - 1}{\mu_4 - \mu}$; $\sigma_H \equiv \alpha \frac{\mu_1 - \mu}{\mu_1 - 1} \frac{\mu_2 - 1}{\mu_2 - \mu}$.
Then the following generically holds:

(i) If $1 < \mu < \mu_1$, then:

· when $\sigma > 1/2$, the steady state is a sink for $v < v_T$, it undergoes a transcritical bifurcation at $v = v_T$, becoming a saddle for $v > v_T$
· when $1/2 \geq \sigma > \sigma_T$, the steady state is always a saddle;
· when $\sigma_T > \sigma > \sigma_H$, the steady state is a saddle for $\nu < \nu_T$, it undergoes a transcritical bifurcation at $\nu = \nu_T$, becoming a source for $\nu > \nu_T$;

· when $\sigma < \sigma_H$, the steady state is a saddle for $\nu < \nu_T$, it undergoes a transcritical bifurcation at $\nu = \nu_T$, becoming a sink for $\nu_T < \nu < \nu_H$; then it undergoes a Hopf bifurcation at $\nu = \nu_H$, becoming a source for $\nu > \nu_H$.

(ii) If $\mu_1 < \mu < \mu_2$, then:
· when $\sigma > 1/2$, the steady state is a sink for $v < v_T$, it undergoes a transcritical bifurcation at $v = v_T$, becoming a saddle for $v > v_T$;
· when $1/2 \geq \sigma > \sigma_T$, the steady state is always a saddle;
· when $\sigma < \sigma_T$, the steady state is a saddle for $\nu < \nu_T$, it undergoes a transcritical bifurcation at $\nu = \nu_T$, becoming a source for $\nu > \nu_T$.

(iii) If $\mu_2 < \mu < \mu_3$, then:
· when $\sigma > \sigma_H$, the steady state is a sink for $v < v_H$, it undergoes a Hopf bifurcation at $v = v_H$, becoming a source for $v_H < v < v_T$; then it undergoes a transcritical bifurcation at $v = v_T$, becoming a saddle for $v > v_T$;
· when $\sigma_H > \sigma > 1/2$, the steady state is a sink for $v < v_T$, it undergoes a transcritical bifurcation at $v = v_T$, becoming a saddle for $v > v_T$;
· when $1/2 \geq \sigma > \sigma_T$, the steady state is always a saddle;
· when $\sigma < \sigma_T$, the steady state is a saddle for $\nu < \nu_T$, it undergoes a transcritical bifurcation at $\nu = \nu_T$, becoming a source for $\nu > \nu_T$.

(iv) If $\mu_3 < \mu < \mu_4$, then:
· when $\sigma > \sigma_T$, the steady state is a sink for $v < v_H$, it undergoes a Hopf bifurcation at $v = v_H$, becoming a source for $v_H < v < v_T$; then it undergoes a transcritical bifurcation at $v = v_T$, becoming a saddle for $v > v_T$;
· when $\sigma_T > \sigma \geq 1/2$, the steady state is a sink for $v < v_H$, it undergoes a Hopf bifurcation at $v = v_H$, becoming a source for $v > v_H$:
· when $1/2 > \sigma > \sigma_H$, the steady state is a saddle for $\nu < \nu_T$, it undergoes a transcritical bifurcation at $\nu = \nu_T$, becoming a sink for $\nu_T < \nu < \nu_H$; then it undergoes a Hopf bifurcation at $\nu = \nu_H$, becoming a source for $\nu > \nu_H$;
· when $\sigma < \sigma_H$, the steady state is a saddle for $\nu < \nu_T$, it undergoes a transcritical bifurcation at $\nu = \nu_T$, becoming a source for $\nu > \nu_T$.

(v) If $\mu > \mu_4$, then:
· when $\sigma \geq 1/2$, the steady state is a sink for $v < v_H$, it undergoes a

Hopf bifurcation at $v = v_H$, becoming a source for $v > v_H$;
· when $1/2 > \sigma > \sigma_H$, the steady state is a saddle for $\nu < \nu_T$, it undergoes a transcritical bifurcation at $\nu = \nu_T$, becoming a sink for $\nu_T < \nu < \nu_H$; then it undergoes a Hopf bifurcation at $\nu = \nu_H$, becoming a source for $\nu > \nu_H$;
· when $\sigma < \sigma_H$, the steady state is a saddle for $\nu < \nu_T$, it undergoes a transcritical bifurcation at $\nu = \nu_T$, becoming a source for $\nu > \nu_T$.

Proof. See Appendix. ■

3.4 Discussion of the results

Before commenting upon the results on local dynamics we have established, and in order to give an intuition of the mechanism (markup variability) making endogenous fluctuations compatible with self-fulfilling expectations in our economy, let us first consider the case of an exogenous number of firms and a corresponding constant markup factor. Assume an increase in the expected real net return to capital, from its steady state value, inducing a rightward shift of the labor supply curve (see (1)) in the space (l, ω). Given a decreasing labor demand curve (see (6)), this depresses the real wage (and the more so the less elastic the labor supply), which in turn rises the wage bill and expands capital accumulation if labor demand is elastic ($\sigma/\alpha > 1$). But more available capital in the next period will invalidate the expectation of higher return, unless accompanied by a further shift in labor supply, induced by rising expectations of capital remuneration. Such a course of events is incompatible with non-monotonic self-fulfilling expectations. One way out is then to assume increasing marginal (social) returns to scale, as in Lloyd-Braga (1995) or Cazzavillan (2001), allowing the quantity and the price of capital to rise simultaneously without a further upwards shift in labor supply. Another is to admit the variability of the markup factor, for instance by assuming free entry in the presence of increasing average (internal) returns to scale. In this case, which we are exploring in this paper, the same effect is achieved through a countercyclical decrease in the markup factor.

The first case of the former proposition ($\mu < \mu_1$) is the most relevant from an empirical point of view, since μ is low and possibly arbitrarily close to 1. It shows that, even with a markup factor variable but close to 1, indeterminacy may occur with an elastic labor demand ($\sigma/\alpha > 1$) – contrary to Reichlin's (1986) conclusion for a perfectly competitive economy which behaves dynamically as our own when the markup factor is constant, although larger than 1. For this indeterminacy to occur in our model, high elasticity of labor supply is needed ($1/v > 1/v_T$) when labor and capital are not too complementary ($\sigma > 1/2$). By contrast, under Reichlin's complementarity condition ($\sigma < 1/2$), intermediate values of labor supply elasticity are required for indeterminacy ($1/v_T > 1/v > 1/v_H$), the steady state becoming unstable through a Hopf bifurcation when this elasticity falls below $1/v_H$. It is easily checked that, as the economy tends to perfect competition ($\mu \to 1$), the floor for the elasticity of labor supply tends to infinity if $\sigma > 1/2$, and when $\sigma < 1/2$ (indeterminacy requiring $\sigma < \sigma_H$) the ceiling for the elasticity of labor demand (σ/α) tends to one (since $\sigma_H/\alpha \to 1$) in accordance with Reichlin's conclusion. These results are similar to those obtained by Cazzavillan (2001) for a competitive economy with small[19] external economies of scale, except that he loses the possibility of indeterminacy as the factors of production become highly substitutable.

For values of μ larger than μ_1, increasing the markup factor essentially facilitates indeterminacy and emergence of a Hopf bifurcation. For instance, with $\sigma > 1/2$, as soon as $\mu > \mu_2$, Hopf bifurcations[20] can occur and indeterminacy is always obtained for labor supply elasticity larger than a floor, which falls as μ increases to μ_3, and then remains fixed, for $\mu > \mu_3$, at $\alpha/(1-2\alpha)$ (the value corresponding to a Hopf bifurcation). As well known, deterministic equilibria with endogenous fluctuations generically appear in the neighborhood of a steady state, close to a Hopf bifurcation. Also, stochastic equilibria with endogenous fluctuations exist in a neighborhood of an indeterminate steady state, or possibly in a neighborhood of a source, close to a Hopf bifurcation. Finally, multiplicity of steady states, locally associated with the emergence of transcritical bifurcations and globally analyzed in a simple case in the next section, is yet another source of stochastic endogenous fluctuations.

[19]Cazzavillan assumes a degree of social economies of scale smaller than $1-2\alpha$ (our notations, and assuming full capital depreciation, to make things comparable). In our economy, the local degree of internal economies of scale can be seen as $(\partial y/\partial l)(l/y)-1 = \mu-1$ (by (10) and (11)), and the degree of social economies of scale as $(\partial Y/\partial z)(z/Y)-1 = (\mu-1)/2$ (using (13) and (15)). Cazzavillan's assumption thus exactly corresponds to our $\mu < \mu_1$ (with μ measured at the steady state $(1,1)$).

[20]Note that μ_2 although a bit high if α is low, may still take reasonable low values if α is close to $1/2$.

4 Steady state equilibria in an economy with constant elasticities

We recall that a steady state equilibrium is a pair $(a^*, K^*) \in \mathbb{R}^2_{++}$ satisfying equations (22) and (23), together with the inequality: $K^* f(a^*)/a^* > \max\{4\tau\phi, \phi/\tau\}$, ensuring that the number of firms is larger than 2, and that the markup factor is larger than 1. Equation (23), given strict convexity of function V, and using the notation $\alpha(a) \equiv \epsilon f(a)$, can be written as:

$$K^* = K(a^*) \equiv a^* (V')^{-1} \left(\frac{a^*}{\delta} \frac{\alpha(a^*)}{1 - \alpha(a^*)} \right). \tag{39}$$

Also, equation (22)and the definitions of function n in (15) and of function μ in (17) can be equivalently written as:

$$\begin{aligned} G(a^*) &\equiv \tau \sqrt{N(a^*)} \left[1 - 1/m(a^*)\right] = 1, \text{ with} && (40) \\ N(a) &\equiv \frac{K(a) f(a)}{\tau\phi a}, \text{ and} && (41) \\ m(a) &\equiv \pi f'(a) \left[1/\alpha(a) - 1\right], && (42) \end{aligned}$$

where $N(a^*) = n(a^*, K(a^*))$ and $m(a^*) = \mu(a^*, K(a^*))$ must satisfy $N(a^*) > 2$ and $m(a^*) > 1$. Hence, a steady state value for a is a value a^* such that (40) is satisfied with $N(a^*) > 2$. The corresponding steady state value K^* is then given by (39).

We also recall that, according to Lemma 2, we assume values of the parameters δ and π such that a steady state equilibrium exists, by convenience normalized to $(1, 1)$ (which imposes the condition $\max\{4\tau\phi, \phi/\tau\} < 1$ on the parameters ϕ and τ). Hence $G(1) = 1$. In order to assess the existence of other steady states, we proceed to the study of the properties of function G, in the simple case of constant elasticities of substitution σ and of labor supply $1/v$. In other words, we assume in this section the following specifications:

$$\begin{aligned} f(a) &= \left[\alpha a^{\frac{\sigma-1}{\sigma}} + 1 - \alpha\right]^{\frac{\sigma}{\sigma-1}}, \text{ with } \alpha \in (0,1) \text{ and } \sigma \in (0,\infty), && (43) \\ V(z) &= \frac{z^{1+v}}{1+v}, \text{ with } v \in (0,\infty). && (44) \end{aligned}$$

A straightforward calculation of the elasticity of function G, using equation (3), gives:

$$\epsilon G(a) = \epsilon N(a) + \frac{\epsilon m(a)}{m(a)-1}, \text{ with} \tag{45}$$

$$\epsilon N(a) = \frac{2\sigma-1}{2\sigma v} + \frac{\alpha(a)}{2}, \text{ and} \tag{46}$$

$$\epsilon m(a) = \frac{\alpha(a)}{\sigma} - 1. \tag{47}$$

Also,

$$\epsilon\alpha(a) = (1-\alpha(a))\left(1-\frac{1}{\sigma}\right). \tag{48}$$

The following Lemma is a first step in the determination of the number of steady states:

Lemma 5 The function G has always a unique maximum in $(0,\infty)$, becoming negative when a is large enough. As a consequence, there exists at most two steady states in the economy with constant elasticities.

Proof. See Appendix. ■

For a pair $(a^*, K(a^*))$ such that $G(a^*) = 1$ to be an admissible steady state, it should be the case that $N(a^*) > 2$. As N is quasi-concave (by (46) and (48)), this means that a^* should belong to some interval $(\underline{a}, \overline{a})$ containing 1. Now, since G is single peaked, if $G(\underline{a}) > 1$ or if $G(\overline{a}) > 1$, the steady state $(1,1)$ is unique. See an example of the former case in Figure 3. On the contrary, if both $G(\underline{a}) < 1$ and $G(\overline{a}) < 1$, there are generically two steady states. Notice that, by (46), N is increasing if $\sigma \geq 1/2$, so that $\overline{a} = \infty$ and $G(\overline{a}) < 0$ (by Lemma 5) in this case.

(Insert Figure 3)

If $G(\underline{a}) > 1$, the unique steady state $(1,1)$ clearly verifies $\epsilon G(1) < 0$. From equalities (45) to (47) it is easy to calculate:

$$\epsilon G(a^*) = \frac{(\mu^*-1)[2\sigma-1+\alpha^*\sigma v] + 2(\alpha^*-\sigma)v}{2\sigma v(\mu^*-1)} \tag{49}$$

$$= \frac{[(\mu^*-1)\sigma+2](1-\alpha^*)}{2\sigma v(\mu^*-1)}H^*, \tag{50}$$

with H^* as defined in (31). We see that the signs of $\epsilon G(a^*)$ and H^* coincide, and immediately conclude from Proposition 3 that the unique steady state $(1,1)$ is necessarily a saddle when $G(\underline{a}) > 1$. Uniqueness and saddle point stability are in fact linked for not too small values of σ, since the other case entailing uniqueness ($G(\overline{a}) > 1$) can only occur if $\sigma < 1/2$. In this later case, $\epsilon G(1) > 0$ and, by Proposition 3, the steady state $(1,1)$ is a sink or a source.

If both $G(\underline{a}) < 1$ and $G(\overline{a}) < 1$, we obtain generic existence of two steady states, of which one is a saddle and the other a sink or a source. In the singular case $\epsilon G(1) = 0$ the steady state $(1,1)$ is unique, but it undergoes a transcritical bifurcation as soon as $\epsilon G(1)$ crosses the value 0 through perturbation of some parameter, for instance, as v crosses the value v_T in Proposition 4, by which another steady state appears, both exchanging local stability properties.

We summarize the preceding results in the following Proposition:

Proposition 6 Consider an economy with constant elasticities σ and v.

(i) If $G(a_2) = 2\tau[1 - 1/m(a_2)] > 1$ for some a_2 such that $N(a_2) = 2$ (implying $\tau > 1/2$), the steady state $(1,1)$ is the unique stationary equilibrium. It is a saddle if $a_2 < 1$ (which is necessarily the case if $\sigma \geq 1/2$), a sink[21] or a source if $a_2 > 1$.

(ii) If $G(a_2) = 2\tau[1 - 1/m(a_2)] < 1$ for any a_2 such that $N(a_2) = 2$, there are generically two steady states, of which one is a saddle and the other is either a sink or a source. In the singular case $\epsilon G(1) = 0$, the two steady states merge into a single one, the state $(1,1)$ undergoing a transcritical bifurcation as $\epsilon G(1)$ crosses 0.

We end this section with some brief comments. To begin with, we may notice that uniqueness of the steady state is either non-generic, or else results not from intrinsic characteristics of the dynamic system but from side restrictions on the number of firms. This cannot be smaller than 2 (no monopoly equilibrium exists in particular if $\tau \leq 1$) and must be larger than $1/\tau$ (for a positive markup to exist, so that each firm may cover its fixed costs), this last requirement being ensured at any a^* such that $G(a^*) = 1$ (since $G(a^*) > 0 \Rightarrow m(a^*) > 1$). Statement (i) of Proposition 6 shows that

[21] In view of Proposition 3, this case will only be possible if α in (43) is lower than 1/2, since $\alpha(1) = \alpha$.

if $\tau > 1/2$ uniqueness of the steady state may prevail, precisely because $N(a^*) > 2$ may not be ensured for a^* satisfying $G(a^*) = 1$. We have seen in Proposition 4 that when $\sigma \geq 1/2$ local indeterminacy can emerge. However, by statement (i) of Proposition 6 we see that if uniqueness hold then $\sigma \geq 1/2$ implies local determinacy. Even so, if $\tau \leq 1/2$, statement (ii) of Proposition 6 applies, and we obtain generic multiplicity of steady states for any configuration of other parameter values (preserving existence of the steady state $(1, 1)$).

These results contrast with those obtained when taking the number of firms as exogenous, and consequently the markup factor as constant. Indeed, we see from (45) and (47) that function G ceases to be single-peaked (to become decreasing) when $\sigma \geq 1$. We thus obtain in this case a unique steady state, against two if $\sigma < 1$ and $\sigma \neq \alpha$. This variant of the model essentially coincides with the case of perfect competition without externalities, as explored by Cazzavillan (2001) in his Proposition 4.1 (A). It should also be stressed that our results with an endogenous number of firms and variable markups significantly differ, on the issue of uniqueness vs. multiplicity of steady states, from the case of perfect competition with externalities studied by Cazzavillan (2001) in his Proposition 4.1 (B), where multiplicity of steady states is confined to certain configurations of parameter values (σ, α and v, in our notations). Finally, contrary to Cazzavillan's finding, the Cobb-Douglas specification of the production function (corresponding to $\sigma = 1$) does not exhibit any structural instability in our model.[22]

5 Concluding remarks

In this paper, we have analyzed the consequences of markup variability, associated with entry and exit of Cournotian firms, in an otherwise standard model of capital accumulation. Our analysis adds to other recent contributions which emphasize the possible role of imperfect competition (of changes in market power in particular) for the emergence of stochastic and deterministic endogenous fluctuations. The structure of our model is very simple and introduces the crucial elements of imperfect competition, from this point of view, in the most natural way.

[22] This is because in the parameterized version of our model the markup (or the local degree of economies of scale) is variable whereas in Cazzavillan the assumption of a constant elasticity of capital externalities leads to a constant degree of economies of scale.

We have developed a simple overlapping generations model where households live for two periods but our results would also apply if households lived for any arbitritarily high number of periods, provided they face binding financial constraints. Differentiated intermediate goods are produced out of capital and labor inputs with positive fixed costs plus constant marginal costs. Under the usual free entry zero profits condition, the equilibrium number of firms depends procyclically on the economic activity, leading to a countercyclical variable markup. The case of a constant unitary markup (the perfect competition situation) is obtained as a limit case of zero fixed cost (or by considering the intermediate goods as perfect substitutes) while the case of a constant markup factor higher than 1 can be recovered by assuming a constant number of firms, i.e., no free entry.

When the markup factor is constant the occurrence of local indeterminacy or Hopf bifurcations require that factors are highly complementary ($\sigma < 1/2$). Also multiplicity of steady states requires $\sigma < 1$. On the contrary, we found that, for values of the elasticity of substitution between factors of production higher than 1/2 and for a capital share of output smaller than 1/2, the steady state becomes indeterminate as soon as there is any (variable) positive market power, provided that the elasticity of labor supply is sufficiently high. As well known, it is then possible to construct infinitely many equilibrium trajectories, exhibiting stochastic endogenous fluctuations. Also, if we are willing to accept a lower bound for the (variable) markup level higher than 1 deterministic endogenous fluctuations emerge through the occurrence of a Hopf bifurcation. Moreover multiplicity of steady states may be obtained for any value of the elasticity of substitution between factors, in particular even if $\sigma = 1$. Our results show that markup variability is an important mechanism for the occurrence of endogenous fluctuations.

The consideration of other types of market imperfections in addition to those here contemplated is a topic for future development. Another important issue, left open for further research, is the study of possible stabilization policies in the presence of endogenous fluctuations due to markup variability.

A Appendix: Proof of Proposition 4

First note that, using $T_0(\mu, \sigma)$ and $\Delta'(\mu, \sigma)$ in (35) and (36), μ_1, μ_2, μ_3, μ_4 and σ_T, σ_H can be defined as critical values of μ and σ such that: $T_0(\mu_1, 0) = 2$, $\Delta'(\mu_2, \infty) = \frac{1-D_0}{2-T_0(.,\infty)}$, $\Delta'(\mu_3, 1/2) = 1$, $\Delta'(\mu_4, \infty) = 1$, $\Delta'(\mu, \sigma_T) = 1$,

$\Delta'(\mu, \sigma_H) = \frac{1-D_0}{2-T_0(\mu,\sigma_H)}$.

Using expressions for μ_1, μ_2, μ_3, μ_4, σ_T and σ_H given in Proposition 4, note also that: $\sigma_T \in (\alpha, 1/2)$ iff $\mu < \mu_3$, $\sigma_T \in (1/2, \infty)$ iff $\mu_3 < \mu < \mu_4$ and $\sigma_T < 0$ iff $\mu > \mu_4$, whereas $\sigma_H \in (0, \alpha)$ iff $\mu < \mu_1$, $\sigma_H < 0$ iff $\mu_1 < \mu < \mu_2$, $\sigma_H \in (1/2, \infty)$ iff $\mu_2 < \mu < \mu_3$ and $\sigma_H \in (\alpha, 1/2)$ iff $\mu > \mu_3$.

One can also check that, by equation (34), the value of the determinant D at $T = 2$ is such that:

$$D - 1 = \frac{\alpha}{1-\alpha}\frac{\mu_1 - \mu}{2}\left(\frac{1}{\sigma} - \frac{1}{\alpha}\frac{\mu_1 - 1}{\mu_1 - \mu}\frac{\mu_2 - \mu}{\mu_2 - 1}\right) = \frac{\alpha}{1-\alpha}\frac{\mu_1 - \mu}{2}\left(\frac{1}{\sigma} - \frac{1}{\sigma_H}\right). \tag{51}$$

We shall divide the proof according to whether $\sigma > 1/2$ or $\sigma \leq 1/2$.

- I. For $\sigma > 1/2$

Recall that the half-line Δ lies in the positive half-space (T, D), and has its origin in the interior of the sink region for $\sigma > 1/2$ and any $\mu > 1$. There are accordingly three possibilities: Δ crosses exclusively the segment AC in Figure 1; it first crosses segment BC (at $v = v_H$) and then the line AC above the point C; it crosses only segment BC.

(i) The half-line Δ crosses exclusively the segment AC whatever the value of σ if its ordinate at $T = 2$ is always below the point C taking a value lower than1, i.e., expressions in (51) take a negative value. This ordinate is clearly a decreasing function of σ if $\mu < \mu_1$ (refer to the R.H.S expression of (51)), taking a value smaller than 1 at $\sigma = 1/2$ when $\mu < \mu_1$ (see above that $\sigma_H < \alpha < 1/2$ in this case). If $\mu > \mu_1$, the same function is increasing and takes a value smaller than 1 at $\sigma = \infty$, if $\mu < \mu_2$ (see definition of μ_2 above and recall that Δ' is increasing in μ).

(ii) If $\mu > \mu_2$, the ordinate of Δ at $T = 2$ is an increasing function of σ, which takes the value 1 when $\sigma = \sigma_H > 1/2$ for $\mu < \mu_3$. When $\sigma > \sigma_H$, Δ crosses the line AC above the point C, after crossing the segment BC.

(iii) If $\mu > \mu_3$, $\sigma_H < 1/2$, and Δ must cross BC for any σ. It will still cross the line AC, above the point C, but only if its slope is smaller than 1, that is, if $\sigma > \sigma_T$.

(iv) If $\mu > \mu_4$, Δ has a slope which is always larger than 1, so that it always crosses the segment BC, and can never cross the line AC.

- II. For $\sigma \leq 1/2$

The proof goes along the same lines as for the former case, with geometric support and based also on expressions above. The case of $\sigma \leq 1/2$ is very simple. The origin of the half-line Δ lies on the line AC. If $\mu < \mu_3$ the slope of line Δ is lower than 1 so that the half-line Δ lies on the right of line AC for every $\nu > 0$ and the steady state is a saddle. If $\mu > \mu_3$ then the slope of Δ is higher than 1 and the half line Δ lies on the left of line AC. Now, with $\sigma < 1/2$, the origin of Δ lies in the interior of the saddle region, leading to three possibilities: Δ first crosses the segment AC and then the segment BC in Figure 1; it crosses only the line AC above the point C; it lies entirely in the saddle region.

(i) As the half-line Δ has a positive slope, it crosses the segment AC and then the segment BC if and only if its ordinate at $T = 2$ is above the point C. By (51), this ordinate is a decreasing function of σ if $\mu < \mu_1$, and it takes the value 1 when $\sigma = \sigma_H < \alpha$. Thus, as σ increases above σ_H the ordinate of Δ at $T = 2$ falls below the point C, yet Δ continues to cross the line AC as long as its slope is larger than 1, that is, as long as $\sigma < \sigma_T$.

(ii) If $\mu_1 < \mu < \mu_3$, the ordinate of Δ at $T = 2$ is increasing in σ, and smaller than 1 for $\sigma = 1/2$, so that Δ can only cross the line AC above the point C, if its slope is larger than 1, that is, if $\sigma < \sigma_T$, and not at all otherwise.

(iii) If $\mu > \mu_3$, the slope of Δ is always larger than 1, so that it always crosses the line AC, below or above the point C depending on whether σ is larger or smaller than σ_H.

B Appendix: Proof of Lemma 5

From equation (43) and after performing standard computations, it is easy to determine that, if $\sigma > 1$, $\lim_{a \to 0} G(a) = 0^+$ and $\lim_{a \to \infty} G(a) = -\infty$. Also, since m is decreasing (by (47)), we see by (40) that function G changes sign only once. If $\sigma < 1$, $\lim_{a\to 0} G(a) = \lim_{a\to\infty} G(a) = -\infty$ and, since α decreases from 1 to 0 as a goes from 0 to infinity, the function m is single-peaked, so that G changes signs twice, from negative becoming positive and then negative again.

Now, from equations (45) to (47), we obtain (as can be checked by referring to the expression of the elasticity of function $G : (0, \infty) \to (-\infty, \infty)$ in (49), with $m(a)$ and $\alpha(a)$ instead of μ^* and α^*, respectively):

$$\operatorname{sign}\{\epsilon G(a)\} = \operatorname{sign}\{\Gamma(a)\}, \text{ with}$$

$$\Gamma(a) \equiv (m(a) - 1)[2\sigma - 1 + \alpha(a)\sigma v] + 2(\alpha(a) - \sigma)v. \tag{52}$$

Imposing the constraint $\epsilon G(a) = 0$, or

$$m(a) - 1 = -\frac{2(\alpha(a) - \sigma)}{(2\sigma - 1)/v + \alpha(a)\sigma} \equiv -\frac{2(\alpha(a) - \sigma)}{y} \tag{53}$$

(with y introduced just to simplify notations), we then obtain (using (47), (48) and (53)):

$$\frac{a\sigma}{v}\Gamma'(a)\big|_{\Gamma(a)=0} = m'(a)a\sigma y + 2\alpha'(a)a\sigma\left(1 - \frac{(\alpha(a) - \sigma)\sigma}{y}\right) \tag{54}$$

$$\begin{aligned} = \; & y(\alpha(a) - \sigma) - 2(\alpha(a) - \sigma)^2 \\ & + 2\alpha(a)(1 - \alpha(a))(\sigma - 1)\left(1 - \frac{(\alpha(a) - \sigma)\sigma}{y}\right). \end{aligned} \tag{55}$$

We shall now prove that the sign of this expression is negative when $m(a) > 1$, i.e., for $G(a) > 0$.

In the last expression, the second term is always negative, and so are the first and third for $m(a) > 1$ (by (53)) if $\sigma \leq 1$. Thus, G has a unique maximum.

In the case $\sigma > 1$ (giving $y > 0$), we can write (with $\tilde{\alpha} = \alpha(a)$ for shortness):

$$\begin{aligned} & \frac{a\sigma y}{v}\Gamma'(a)\big|_{\Gamma(a)=0} \\ = \; & -y^2(\sigma - \tilde{\alpha}) - 2y\left[(\sigma - \tilde{\alpha})^2 - \tilde{\alpha}(1 - \tilde{\alpha})(\sigma - 1)\right] + 2\tilde{\alpha}(1 - \tilde{\alpha})(\sigma - 1)(\sigma - \tilde{\alpha})\sigma. \end{aligned} \tag{56}$$

As the terms in y have negative coefficients, it is clear that by diminishing y we can only increase this expression. Thus, by taking for y its smallest admissible value (with $v = \infty$ and σ given) $y = \mathbf{a}\sigma$, and then by reducing σ to 1 when multiplied by a negative factor, we finally get:

$$
\begin{aligned}
\frac{ay}{\mathbf{a}v}\,\Gamma'(a)\big|_{\Gamma(a)=0} &\leq -\mathbf{a}\sigma(\sigma-\mathbf{a}) - 2(\sigma-\mathbf{a})^2 + 2(1-\mathbf{a})(\sigma-1)\sigma \\
&\leq -\mathbf{a}(\sigma-\mathbf{a}) - 2(\sigma-\mathbf{a})^2 + 2(1-\mathbf{a})(\sigma-1) \\
&= -\ (\sigma-1)^2 - (1-\mathbf{a})^2 - (\sigma-\mathbf{a})\,\sigma < 0, \qquad (57)
\end{aligned}
$$

leading to the same conclusion that the function G has a unique maximum.

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