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Source: *Econometric Theory*, Vol. 22, No. 2 (Apr., 2006), pp. 338-344

Published by: Cambridge University Press

Stable URL: <http://www.jstor.org/stable/4093229>

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ON THE PRODUCT AND RATIO OF GAMMA AND WEIBULL RANDOM VARIABLES

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The distributions of products and ratios of random variables are of interest in many areas of the sciences. In this note, the exact distributions of the product XY and the ratio X/Y are derived when X and Y are gamma and Weibull random variables distributed independently of each other.

1. INTRODUCTION

For given random variables X and Y , the distributions of the product XY and the ratio X/Y are of interest in many areas of the sciences.

In traditional portfolio selection models certain cases involve the product of random variables. The best examples of this are in the case of investment in a number of different overseas markets. In portfolio diversification models (see, e.g., Grubel, 1968) not only are prices of shares in local markets uncertain, but also the exchange rates are uncertain, and so the value of the portfolio in domestic currency is related to a product of random variables. Similarly in models of diversified production by multinationals (see, e.g., Rugman, 1979) there are local production uncertainty and exchange rate uncertainty, and so profits in home currency are again related to a product of random variables. An entirely different example is drawn from the econometric literature. In making a forecast from an estimated equation Feldstein (1971) pointed out that both the parameter and the value of the exogenous variable in the forecast period could be considered random variables. Hence the forecast was proportional to a product of random variables.

An important example of ratios of random variables is the stress-strength model in the context of reliability. It describes the life of a component that has a random strength Y and is subjected to random stress X . The component fails at the instant that the stress applied to it exceeds the strength, and the compo-

The authors thank the referee and the editor for carefully reading the paper and for their help in improving the paper. Address correspondence to Saralees Nadarajah, Department of Statistics, University of Nebraska, Lincoln, NE 68583, USA; e-mail: snadaraj@unlserve.unl.edu.

nent will function satisfactorily whenever $Y > X$. Thus, $\Pr(X < Y)$ is a measure of component reliability. It has many applications, especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures, and the aging of concrete pressure vessels.

The distributions of XY and X/Y have been studied by several authors, especially when X and Y are independent random variables and come from the same family. With respect to products of random variables, see Sakamoto (1943) for uniform family, Harter (1951) and Wallgren (1980) for Student's t family, Springer and Thompson (1970) for normal family, Stuart (1962) and Podolski (1972) for gamma family, Steece (1976), Bhargava and Khatri (1981), and Tang and Gupta (1984) for beta family, AbuSalih (1983) for power function family, and Malik and Trudel (1986) for exponential family (for a comprehensive review of known results, see also Rathie and Rohrer, 1987). With respect to ratios of random variables, see Marsaglia (1965) and Korhonen and Narula (1989) for normal family, Press (1969) for Student's t family, Basu and Lochner (1971) for Weibull family, Shcolnick (1985) for stable family, Hawkins and Han (1986) for noncentral chi-squared family, Provost (1989) for gamma family, and Pham-Gia (2000) for beta family.

There is relatively little work of the preceding kind when X and Y belong to different families (the only work known to the authors is that by Nadarajah and Ali, 2004, and Nadarajah and Gupta, 2005). In the applications mentioned before, it is quite possible that X and Y could arise from different but similar distributions. In this note, we study the exact distributions of XY and X/Y when X and Y are independent random variables having the gamma and Weibull distributions with probability density functions (p.d.f.s)

$$f_X(x) = \frac{\mu^\alpha x^{\alpha-1} \exp(-\mu x)}{\Gamma(\alpha)} \quad (1)$$

and

$$f_Y(y) = \beta \lambda^\beta y^{\beta-1} \exp\{-(\lambda y)^\beta\}, \quad (2)$$

respectively, for $x > 0$, $y > 0$, $\alpha > 0$, $\beta > 0$, $\lambda > 0$, and $\mu > 0$. The calculations involve the incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^\infty \exp(-t) t^{a-1} dt$$

and the Gauss hypergeometric function defined by

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1)\dots(e+k-1)$ denotes the ascending factorial (also known as Pochhammer's symbol). Further properties of the preceding special functions can be found in Prudnikov, Brychkov, and Marichev (1986) and Gradshteyn and Ryzhik (2000).

2. PRODUCT

Theorem 1 expresses the cumulative distribution function (c.d.f.) of XY as an infinite sum of elementary functions.

THEOREM 1. *Suppose X and Y are distributed according to (1) and (2), respectively. The c.d.f. of $Z = XY$ can be expressed as*

$$F(z) = \frac{\lambda^\alpha (\mu z)^\alpha}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{1}{k!(k+\alpha)} \Gamma\left(-\frac{\alpha-\beta+k}{\beta}\right) (-\mu z \lambda)^k \\ + \frac{(\mu z \lambda)^\beta}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha-\beta-\beta k)}{(k+1)!} (-\mu^\beta z^\beta \lambda^\beta)^k \quad (3)$$

for $z > 0$.

Proof. The c.d.f. corresponding to (1) is $1 - \Gamma(\alpha, \mu x)/\Gamma(\alpha)$. Thus, one can write the c.d.f. of XY as

$$\Pr(XY \leq z) = \int_0^\infty F_X(z/y) f_Y(y) dy \\ = 1 - \frac{\beta \lambda^\beta}{\Gamma(\alpha)} \int_0^\infty \Gamma(\alpha, \mu z/y) y^{\beta-1} \exp\{-(\lambda y)^\beta\} dy \\ = 1 - \frac{\beta \lambda^\beta}{\Gamma(\alpha)} \int_0^\infty \Gamma(\alpha, \mu z w) w^{-(1+\beta)} \exp(-\lambda^\beta w^{-\beta}) dw \\ = 1 - \frac{\beta \lambda^\beta}{\Gamma(\alpha)} I, \quad (4)$$

where we have set $w = 1/y$. Application of equation (2.10.1.12) in Prudnikov et al. (1986, vol. 2) shows that the integral I can be calculated as

$$I = -\frac{(\mu z)^\alpha \lambda^{\alpha-\beta}}{\beta} \sum_{k=0}^{\infty} \frac{1}{k!(k+\alpha)} \Gamma\left(-\frac{\alpha-\beta+k}{\beta}\right) (-\mu z \lambda)^k \\ + \frac{\Gamma(\alpha)}{\beta \lambda^\beta} - \frac{(\mu z)^\beta}{\beta} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha-\beta-\beta k)}{(k+1)!} (-\mu^\beta z^\beta \lambda^\beta)^k. \quad (5)$$

The result of the theorem follows by substituting (5) and (4). ■

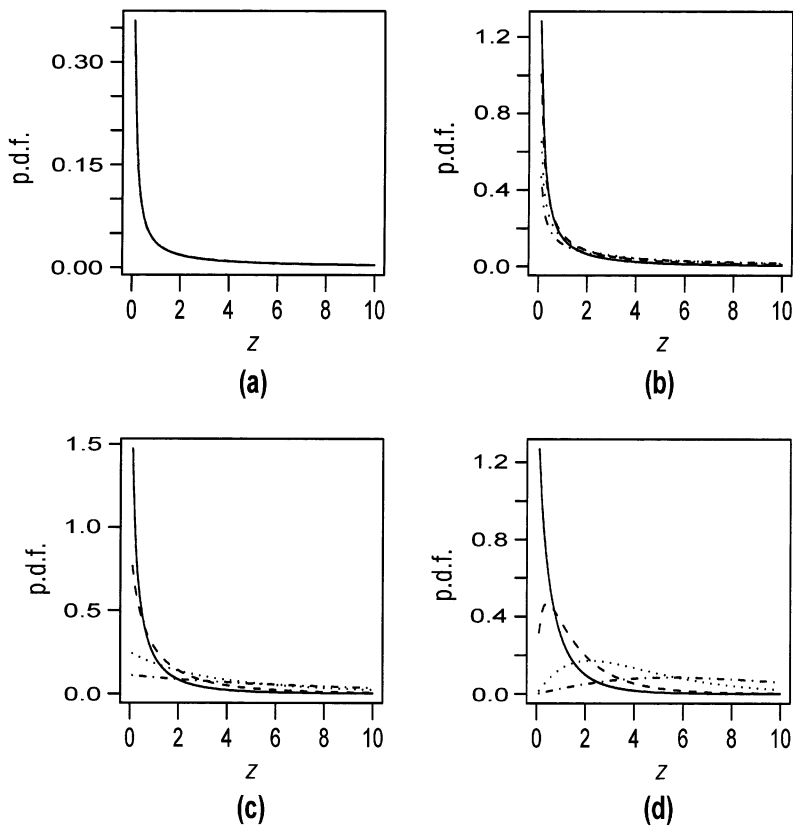


FIGURE 1. Plots of the p.d.f. of (3) for $\lambda = 1$, $\mu = 1$, $\alpha = 1, 2, 5, 10$, and (a) $\beta = 0.1$, (b) $\beta = 0.5$, (c) $\beta = 1$, and (d) $\beta = 2$. The four curves in each plot are the solid curve ($\alpha = 1$), the curve of dots ($\alpha = 2$), the curve of lines ($\alpha = 5$), and the curve of dots and lines ($\alpha = 10$).

Figure 1 illustrates possible shapes of the p.d.f. of XY for selected values of α and β . The four curves in each plot correspond to selected values of α . The effect of the parameters is evident.

3. RATIO

Theorem 2 expresses the c.d.f. of X/Y as an infinite sum of elementary functions.

THEOREM 2. Suppose X and Y are distributed according to (1) and (2), respectively. The c.d.f. of $Z = X/Y$ can be expressed as

$$F(z) = \frac{\beta \mu^\alpha z^\alpha}{\lambda^\alpha \Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{1}{k!(k+\alpha)} \Gamma\left(\frac{\alpha + \beta + k}{\beta}\right) \left(-\frac{\mu z}{\lambda}\right)^k \quad (6)$$

for $\beta > 1$,

$$F(z) = -\frac{\lambda^\beta}{(\mu z)^\beta \Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + \beta + \beta k)}{(k+1)!} \left(-\frac{\lambda^\beta}{\mu^\beta z^\beta}\right)^k \quad (7)$$

for $\beta < 1$, and

$$F(z) = \frac{(\mu z)^\alpha}{\lambda^\alpha} \left(1 + \frac{\mu z}{\lambda}\right)^{-\alpha} \quad (8)$$

for $\beta = 1$.

Proof. Because $1 - \Gamma(\alpha, \mu x)/\Gamma(\alpha)$ is the c.d.f. corresponding to (1), one can write the c.d.f. of X/Y as

$$\begin{aligned} \Pr(X/Y \leq z) &= \int_0^\infty F_X(zy) f_Y(y) dy \\ &= 1 - \frac{\beta \lambda^\beta}{\Gamma(\alpha)} \int_0^\infty \Gamma(\alpha, \mu y z) y^{\beta-1} \exp\{-(\lambda y)^\beta\} dy \\ &= 1 - \frac{\beta \lambda^\beta}{\Gamma(\alpha)} I. \end{aligned} \quad (9)$$

Application of equation (2.10.1.5) in Prudnikov et al. (1986, vol. 2) shows that the integral I can be calculated as

$$I = \begin{cases} -\frac{(\mu z)^\alpha}{\lambda^{\alpha+\beta}} \sum_{k=0}^{\infty} \frac{1}{k!(k+\alpha)} \Gamma\left(\frac{\alpha + \beta + k}{\beta}\right) \left(-\frac{\mu z}{\lambda}\right)^k + \frac{\Gamma(\alpha)}{\beta \lambda^\beta}, & \text{if } \beta > 1, \\ \frac{\Gamma(\alpha)}{\lambda} - \frac{(\mu z)^\alpha \Gamma(\alpha)}{\lambda^{1+\alpha}} {}_2F_1\left(\alpha, \alpha + 1; \alpha + 1; -\frac{\mu z}{\lambda}\right), & \text{if } \beta = 1, \\ \frac{(\mu z)^{-\beta}}{\beta} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + \beta + \beta k)}{(k+1)!} \left(-\frac{\lambda^\beta}{\mu^\beta z^\beta}\right)^k + \frac{\Gamma(\alpha)}{\beta \lambda^\beta}, & \text{if } \beta < 1. \end{cases} \quad (10)$$

Note that the hypergeometric term in (10) simplifies as

$${}_2F_1\left(\alpha, \alpha + 1; \alpha + 1; -\frac{\mu z}{\lambda}\right) = \left(1 + \frac{\mu z}{\lambda}\right)^{-\alpha}.$$

The result of the theorem follows by substituting (10) into (9). ■

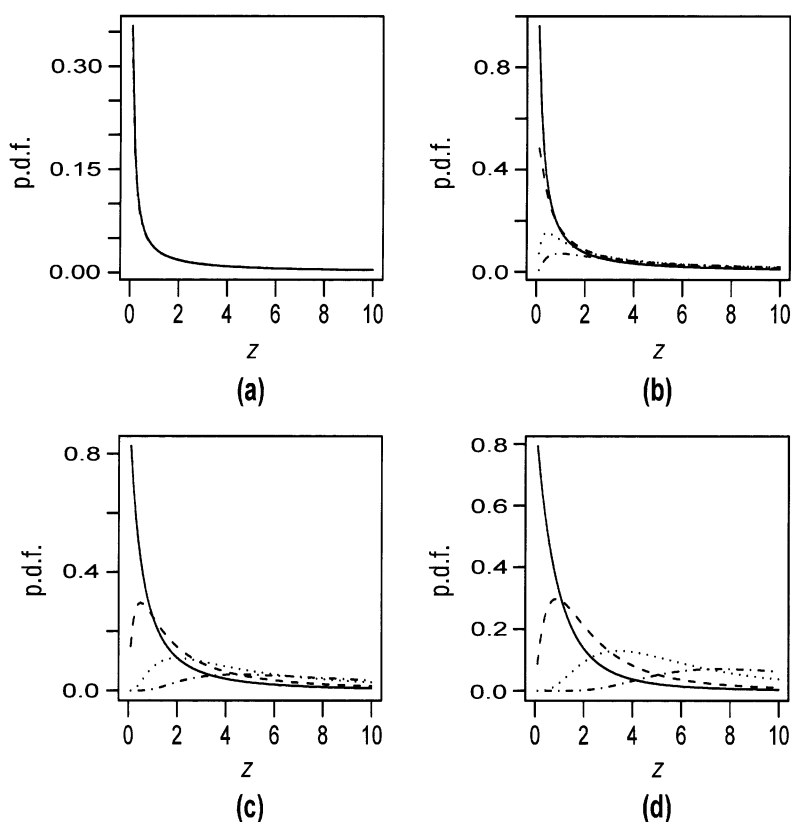


FIGURE 2. Plots of the p.d.f. of (6)–(8) for $\lambda = 1$, $\mu = 1$, $\alpha = 1, 2, 5, 10$, and (a) $\beta = 0.1$, (b) $\beta = 0.5$, (c) $\beta = 1$, and (d) $\beta = 2$. The four curves in each plot are the solid curve ($\alpha = 1$), the curve of dots ($\alpha = 2$), the curve of lines ($\alpha = 5$), and the curve of dots and lines ($\alpha = 10$).

Figure 2 illustrates possible shapes of the p.d.f. of X/Y for selected values of α and β . The four curves in each plot correspond to selected values of α . The effect of the parameters is evident.

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