On the number and size of cities*

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Abstract

We study the effects of a decrease in inter-city transport costs on the spatial distribution of population in a multi-regional economy, when a rise in the regional population generates higher urban costs. Holding the number of cities constant, as transport costs are reduced gradually from a very high level to a very low level, there is a first phase in which large cities grow while small cities shrink, a second phase in which both large and small cities grow while medium size cities shrink, and a third phase in which large cities shrink while small cities grow. Furthermore, when the number of cities is allowed to vary as transport costs are reduced from large to small values, it first decreases and then increases.

1. Introduction

There is a rich and vast literature in economic history showing how the secular decline in communication and transport costs has fostered the development of cities (Bairoch, 1988). As all the impediments to the exchange of goods and the movement of people have decreased over a long enough period of time, one has witnessed a massive concentration of economic activities within a fairly small number of urban regions (Pollard, 1981). Yet, it has been argued that the continuous rise of congestion within such regions has triggered a process of counterurbanization (Berry, 1976). This paper addresses these issues by using a simple spatial general equilibrium model in which *the interplay between the same agglomeration and dispersion forces yield early concentration and later dispersion of population in cities*.

More precisely, the primary purpose of this paper is to study the impact of falling inter-city transport costs on the spatial distribution of population when the number of regions, hence of potential cities, is arbitrary. Indeed, models of economic geography have mainly focussed on the special case of two regions (Krugman, 1991; Fujita et al., 1999;

Ottaviano et al., 2002). Yet, it is not clear what the main result of new economic geography, namely the existence of a 'core-periphery divide', becomes when there are more than two regions. This is because a multi-regional economy is able to sustain much richer spatial distributions of population. In addition, the 2-region setting makes the dynamic analysis very simple because moving away from one region automatically implies that workers and firms necessarily go to the other. On the contrary, we may expect the stability analysis of a multi-regional economy to be much more involved.

Our secondary purpose is to allow for housing and commuting costs to be paid by workers when residing in a particular region. Housing and commuting costs, which we call 'urban costs', typically arise when each region has a central business district (CBD) together with a land market that governs the residential allocation of workers around the CBD (Alonso, 1964). In other words, when firms and workers agglomerate within a region, they form a *city* with land consumption and commuting costs. We would like to argue that introducing urban costs is both reasonable and meaningful. It is reasonable because a concentration of workers and firms within a region generates housing and commuting costs that account for more than one third of consumers' expenditures (U.S. Department of Labor, 2003). It is meaningful because, in the absence of such costs, the economy exhibits some bang-bang behavior without passing through intermediate stages when transport costs decrease, a result that strikes us as being implausible.

In this paper, we use a simple model that captures the main forces at work in a spatial economy, that is, market size effects in an imperfectly competitive industry with increasing returns to scale and positive transport costs, as well as urban costs borne by the inhabitants of regional agglomerations. Because a comprehensive general equilibrium model with imperfect competition has so far been out of reach, specific models have been developed in the 1990s. This paper is no exception and develops another 'clarifying example'. More precisely, in order to achieve the above-mentioned two objectives in a simple manner, we extend the 2-region model proposed by Ottaviano et al. (2002) to study the impact of falling transport costs on the equilibrium distribution(s) of firms and workers in the case of – regions, while permitting each city to have specific urban costs. When the number of regions exceeds two, determining the equilibrium prices, wages, and (indirect) utilities in each region becomes a hard task because the corresponding expressions depend on the *whole* distribution of the population across all regions.

In the general case of asymmetric transport costs, one can hardly obtain meaningful analytical results. This is because 'specific geographies' tend to blur the interplay between transport and urban costs. In order to have a tractable model, we thus control for the relative position of the various regions by using a simple transport geometry in which regions are pairwise equidistant: the transport cost of one unit of good is the same regardless of the origin and destination. This assumption may also be partially justified by the fact that distance-related shipping costs have become very low, while distance-unrelated costs such as insurance, loading and unloading are still relatively high (Boyer, 1997). Likewise, the IT revolution has rendered communication costs almost independent of distance.[1]

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[1] In a sense, that assumption is the 'geographical' counterpart of the often assumed symmetry between any pair of varieties in monopolistic competition models such as Dixit-Stiglitz'.

Regarding urban and transport costs, our modeling strategy is as follows. Although we acknowledge the fact that both transport and commuting costs have been decreasing since the beginning of the Industrial Revolution, we assume that inter-city transport costs decrease while intra-city costs do not exhibit such a trend. This assumption aims at capturing the idea that, in modern economies such as the EU or NAFTA, transport costs of manufactured goods keep decreasing at a fast pace, while the decrease in commuting costs tends to slow down (or, maybe, to rise) due to crowding of the land market and higher opportunity time cost for urban residents.[2]

Previewing our main results, when the number of cities is unaffected as transport costs fall, we study in Section 4 how the *size* of cities having different urban costs changes. Specifically, we show that, as transport costs are reduced gradually from a very high level to a very low level, there is a first phase in which *large cities grow while small cities shrink*, a second phase in which *both large and small cities grow while medium size cities shrink*, and a third phase in which *large cities shrink while small cities grow*. This implies that agglomeration takes place in the early stages of the integration process, whereas re-dispersion occurs in the late stages. Note that, for a given value of transport costs, different cities are at different phases, whereas the evolution of the size of cities need not reflect the relative efficiency of the urban infrastructure. All of this suffices to show that the multi-regional setting is indeed much richer than the 2-region setting.

This spatial pattern agrees with Alonso's (1980) hypothesis according to which the evolution of the spatial distribution of population and industry is strongly correlated to the various stages of economic development. In particular, a high degree of urban concentration arises during the early phases of economic growth; as development proceeds, spatial deconcentration occurs. This hypothesis appears to be confirmed by various empirical studies, which use either time-series data for long periods, such as Alperovich (1993) in Israel, Dehghan and Uribe (1999) in Mexico, and Chatterjee and Carlino (2001) as well as Carlino and Chatterjee (2002) in the United States, or international cross-section data, such as Wheaton and Shishido (1981), MacKellar and Vining (1995), and Gallup et al. (1999).

In Section 5, we consider the case in which the *number* of cities changes as transport costs fall. This makes the analysis much more difficult because stable equilibria may vanish while new stable equilibria emerge for a marginal decrease in transport costs. In the special, but meaningful, case of identical urban costs, we prove that *the number of cities initially decreases and then increases*. Hence, in the early stages of the integration process, the core of the economy is made of a shrinking number of cities (agglomeration phase). However, once transport costs are sufficiently low, the market solves the congestion problem induced by the agglomeration of population by redistributing firms and workers among a larger number of cities (dispersion phase). Because the size of the population is fixed, the agglomeration phase (resp., dispersion phase) is one in which the existing cities grow (resp., shrink).

The remainder of the paper is organized as follows. The model is presented in Section 2, while existence and stability of a spatial equilibrium are dealt with in Section 3.

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2   Note that the existence of falling transport costs per unit distance is a well-documented fact (see, e.g. Glaeser and Kohlhase, 2004). By contrast, a large fraction of commuting costs is now due to commuting time, which has not been much reduced since cars have became the main transport mode used by commuters (see, e.g. Mokyr, 2002, 155–156).

Sections 4 and 5 contain our main results. In these two sections, we will appeal to lemmas that are all stated and proven in Appendix A. Section 6 concludes.[3]

2. The model

The space-economy is made of $n \geq 2$ regions $(i = 1, \ldots, n)$. Each region has one city that has a given center but a variable size. As in urban economics, the city center stands for the CBD in which all firms locate once they have chosen to set up in the corresponding region. The CBDs are given by points of the location space.

There are two production factors, denoted A and L. Factor A is evenly distributed across regions (A/n) and is spatially immobile. The assumption of a uniform distribution of A is made in order to focus on the impact of differential urban costs on the distribution of activities. Factor L is mobile between any two regions. Let $\lambda_i \in [0, 1]$ denotes its share in region i and let

$$\Lambda \equiv \left\{ \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n); \quad \sum_{i=1}^{n} \lambda_i = 1 \quad \text{and} \quad \lambda_i \geq 0 \right\}$$

For expositional purposes, we refer to the first sector as 'agriculture' and to the second sector as 'manufacturing'. Accordingly, we call 'farmers' the immobile factor A and 'workers' the mobile factor L.[4] In the same spirit, we will refer to regions accommodating workers $(\lambda_i > 0)$ as *cities*, while regions with no workers $(\lambda_i = 0)$ are called *rural regions*.

(i) The first good is homogeneous and is produced in the agricultural sector using factor A as the only input under constant returns to scale and perfect competition. Technology in agriculture requires one unit of A in order to produce one unit of output. Consumers also have a positive initial endowment of this good. We assume that this good can be traded freely between regions so that its price is the same across regions. Hence this good may be chosen as the numéraire. As a result, farmers' income is equal to one in all regions.

(ii) The second good is a horizontally differentiated product; it is supplied under increasing returns to scale and monopolistic competition. We assume that there is a continuum of potential firms. Technology in manufacturing is such that producing

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3   Before proceeding, we would like to clarify how our paper relates to, yet differs from, existing contributions. Our model is related to Henderson (1974) in that we deal with the existence of an urban system within the interregional economy. However, here, cities are specialized in the production of differentiated varieties, which are sold under monopolistic competition and shipped between cities at a positive cost. In Henderson, cities supply different goods, which are sold under perfect competition and shipped at zero cost. Our model is also connected to those discussed by Fujita et al. (1999) who use monopolistic competition of the Dixit-Stiglitz genre to explain the emergence and location of cities in a continuous space-economy. The industrial good is produced in cities that have no spatial extension and the agricultural good in the country by means of land and labor. It should be clear that these two groups of models do not allow for the study of the question we address in this paper: transport costs are zero in Henderson-like models, whereas there are no urban costs in the models studied by Fujita et al. Our model combines several of the ideas contained in these two settings. It also throws a bridge between the two polar cases analyzed so far in the literature, which involve respectively two regions and a continuum of locations.

4   However, we want to stress the fact that the role of factor A is to capture the idea that some inputs (such as land or some services) are nontradable while some others have a very low spatial mobility (such as low-skilled workers). For example, the first sector could be reinterpreted as the traditional one and the second sector as the modern one.

$q(i)$ units of variety i requires ϕ units of L and $vq(i)$ units of A. Given the demand structure described below, without loss of generality we may assume that the marginal cost of production of a variety is equal to zero ($v = 0$). There are no scope economies so that, due to increasing returns to scale, there is a one-to-one relationship between firms and varieties. Clearly, regardless of the distribution of firms and the value of transport costs, the total number of firms in the whole economy is given by $N = L/\phi$. Although this might seem restrictive at first sight, this allows us to focus on the spatial distribution of population without accounting for the possible variations in the number of firms.

Because each firm sells a differentiated variety, it faces a downward sloping demand. Since there is a continuum of potential firms, each one is negligible and the interaction between any two firms is zero. However, as will be seen below, aggregate market conditions of some kind affect any single firm. This provides a setting in which individual firms are not competitive in the classic economic sense of having infinite demand elasticity but, at the same time, have no strategic interactions with one another.

Finally, inter-city trade flows go from one CBD to another. As discussed in the introduction, the corresponding transport costs per unit shipped are assumed to be identical between any two regions:

$$\tau_{ij} = \begin{cases} \tau > 0 & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases}$$

In words, one unit of any variety can be traded at a positive cost of τ units of the numéraire between any pair of regions. The underlying geography is simple: the n regions are located along a circumference, while shipping a good from one region to another involves going through the center of the circumference.

(iii) The third good is land (or housing). When they live in a city, workers consume land and commute to the regional CBD where they work. In order to keep things simple, all urban costs borne by a worker who chooses to reside in city i are subsumed in a cost function $\theta_i(\lambda)_i$, which enters the budget constraint (2) given below. As shown by Fujita (1989, Property 5.5), the aggregate land rent and commuting costs in city i are increasing and convex in city size $\lambda_i L$, thus implying that the urban costs $\theta_i(\lambda_i)$ are increasing with respect to λ_i.[5] This function is assumed to be three times continuously differentiable and to satisfy the following properties:

$$\theta_i(0) = 0 \quad \theta_i(1) < \infty \quad \theta_i'(\lambda_i) \geq 0 \quad i = 1, \ldots, n \quad \text{and} \quad \lambda_i \in [0, 1]$$

For example, in the case of a one-dimensional space, when the lot size is fixed and commuting costs are linear in distance, we have $\theta_i(\lambda_i) = \theta_i \lambda_i L/2$ (see, e.g. Ottaviano et al., 2002).

Urban costs are region-specific, reflecting the fact that living conditions may vastly differ across cities for the same population size due to differences in the amount of natural amenities, the quality of transport facilities, the supply of local public services, and/or the quantity of land available for housing. For example, data show that commuting times are not just determined by the population size (U.S. Department of Transportation, 2004). Note also that this assumption breaks the symmetry of the model, thus allowing us to

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5 Observe that urban costs are increasing in city size provided that the set-up costs of a city are negligible.

study how city size may change along some continuous paths involving the same cities when transport costs fall. Asymmetry ensures that 'large' and 'small' cities exist. By contrast, when the equilibrium is fully symmetric ($\lambda_i^* = \lambda_j^*$ holds for all $i, j = 1, \ldots, n$), we will see in Section 5 that city sizes do not change continuously as τ falls.

If land rents are given to landlords or equally distributed among farmers, nothing changes in our results. If land rents are uniformly redistributed among a city's workers (local public ownership), the corresponding additional income is accounted for in the function $\theta_i(\lambda_i)$. For example, in the case of a one-dimensional city with fixed lot sizes and linear commuting costs, we have $\theta_i(\lambda_i) = \theta_i \lambda_i L/4$.

Preferences over the first two goods are identical across individuals and described by a quasi-linear utility with a quadratic subutility, which is supposed to be symmetric in all varieties:

$$U(q_0; q(x), x \in [0, N]) = \alpha \int_0^N q(x)dx - \frac{\beta - \gamma}{2} \int_0^N [q(x)]^2 dx - \frac{\gamma}{2} \left[\int_0^N q(x)dx \right]^2 + q_0 \quad (1)$$

where $q(x)$ is the quantity of variety $x \in [0, N]$ and q_0 the quantity of the numéraire. The parameters in (1) are such that $\alpha > 0, \beta > \gamma > 0$. In this expression, α expresses the intensity of preferences for the differentiated product, whereas $\beta > \gamma$ means that consumers are biased toward a dispersed consumption of varieties.[6]

If the consumption of the homogeneous good is positive, maximizing (1) under the budget constraint

$$\int_0^N p(x)q(x)dx + q_0 = w_i + \overline{q}_0 - \theta_i(\lambda_i) \quad (2)$$

(where w_i denotes the wage prevailing in city i and $\overline{q}_0$ is the initial endowment of the numéraire) yields the following first-order conditions:

$$\alpha - (\beta - \gamma)q(x) - \gamma \int_0^N q(y)dy = p(x) \quad x \in [0, N]$$

or, alternatively,

$$q(x) = a - bp(x) + c \int_0^N [p(y) - p(x)]dy \qquad x \in [0, N] \quad (3)$$

where

$$a \equiv \frac{\alpha}{\beta + (N-1)\gamma} \qquad b \equiv \frac{1}{\beta + (N-1)\gamma} \qquad c \equiv \frac{\gamma}{(\beta - \gamma)[\beta + (N-1)\gamma]}$$

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6 Observe that our results hold true when varieties are complements ($\gamma < 0$) as long as $\beta + (N-1)\gamma > 0$ remains valid. This condition amounts to assuming that varieties are 'weak complements'.

Substituting (2) and (3) into (1), we obtain the indirect utility of a worker residing in this city:

$$V_i = \frac{a^2 N}{2b} - a \int_0^N p(x) dx + \frac{b + cN}{2} \int_0^N [p(x)]^2 dx - \frac{c}{2} \left[\int_0^N p(x) dx \right]^2$$
$$+ \overline{q}_0 + w_i - \theta_i(\lambda_i)$$

As empirical evidence suggests that firms practice some form of spatial discrimination in pricing (Greenhut, 1981; Haskel and Wolf, 2001), we assume that markets are regionally segmented so that each firm chooses a delivered price that is specific to the region in which its variety is sold. Let $p_{ij}(x)$ be the price of variety x produced in region i and sold in region j, and $q_{ij}(x)$ the demand in region j for variety x produced in region i. Because varieties are symmetric, we may ease the burden of notation by dropping x hereafter. Consequently, operating profits of a firm established in region i are given by

$$\Pi_i(\boldsymbol{\lambda}) = \sum_j (p_{ij} - \delta_{ij}\tau) q_{ij} \left(\frac{A}{n} + \lambda_j L \right)$$

where $\delta_{ij} = 1$ when $i \neq j$ and 0 otherwise.

The equilibrium prices and wages are determined as follows. First, by maximizing firms' profits with respect to prices, we obtain:[7]

$$p_{ii}^* = \frac{2a + c\tau(1 - \lambda_i)N}{2(2b + cN)}$$
$$p_{ji}^* = p_{ii}^*(\lambda_i) + \frac{\tau}{2} \quad \text{for } i \neq j$$
$$q_{ii}^* = (b + cN) p_{ii}^*$$
$$q_{ji}^* = (b + cN)(p_{ji}^* - \tau) \quad \text{for } i \neq j$$

Because firms' prices net of transport costs are to be positive for any distribution of workers, we assume throughout this paper that

$$\tau < \tau_{trade} \equiv \frac{2a\phi}{2b\phi + cL}$$

This condition also guarantees that it is always profitable for a firm to export to any other region.

Second, due to free entry and exit, profits net of fixed costs are zero in equilibrium. As in Krugman (1991), the equilibrium wages are determined by a bidding process between firms for workers, which ends when no firm can earn strictly positive profits at the equilibrium market prices. In other words, all operating profits are absorbed by the wage bills. Hence, the wage prevailing in region i is determined as follows:

$$w_i^*(\boldsymbol{\lambda}) = \frac{\Pi_i}{\phi} = \frac{1}{\phi} \sum_j (p_{ij}^* - \delta_{ij}\tau) q_{ij}^* \left(\frac{A}{n} + \lambda_j L \right)$$

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7   Note that arbitrage is never profitable in equilibrium because $p_{jj}^* + \tau - p_{ji}^* > 0$.

Accordingly, the indirect utility of a worker living in region i can be shown to be as follows:

$$V_i(\boldsymbol{\lambda}) = \frac{a^2 N}{2b} - a \sum_j \lambda_j N p_{ji}^* + \frac{b+cN}{2} \sum_j \lambda_j N \left(p_{ji}^*\right)^2$$
$$- \frac{c}{2} \left(\sum_j \lambda_j N p_{ji}^*\right)^2 + \overline{q}_0 + w_i - \theta_i(\lambda_i) \tag{4}$$

As expected, the indirect utility depends on the whole distribution $\boldsymbol{\lambda}$.

3. Existence and stability of a spatial equilibrium

The distribution $\boldsymbol{\lambda}^* \in \Lambda$ is a *spatial equilibrium* when no worker may get a higher utility level by moving to another region. Formally, a distribution $\boldsymbol{\lambda}^*$ is an equilibrium if V^* exists such that

$$\begin{aligned} V_i(\boldsymbol{\lambda}^*) &= V^* \quad \text{if } \lambda_i^* > 0 \\ V_i(\boldsymbol{\lambda}^*) &\leq V^* \quad \text{if } \lambda_i^* = 0 \end{aligned} \tag{5}$$

Hence, a spatial equilibrium is such that workers' utility in cities is (weakly) higher than in rural regions, while the utility level is constant across cities. Since $V_i(\boldsymbol{\lambda})$ is continuous in $\boldsymbol{\lambda} \in \Lambda$ as shown by (4), Proposition 1 of Ginsburgh et al. (1985) implies that a spatial equilibrium always exists.

However, proving the existence of a *stable* spatial equilibrium when there are more than two regions may be a problematic issue. This is because characterizing the eigenvalues of a nonnumerical system is often a formidable task. Interestingly, our model displays some nice features that allow us to apply recent stability theorems without having to compute eigenvalues. In this paper, we use the concept of "positive definite dynamics" to study the stability of spatial equilibria, which in the aggregate has been shown to be compatible with various types of individual behaviors in large populations (Hopkins, 1999).

Following a well-established tradition in migration modeling, we focus on adjustment processes in which workers spread themselves among several cities, being attracted (resp., repulsed) by cities providing high (resp., low) utility levels. In particular, we assume that the net migration from city j to city i during the infinitesimal time interval dt at time t is as follows:

$$\frac{d\lambda_{ji}}{dt} = f_{ij}(\boldsymbol{\lambda})[V_i(\boldsymbol{\lambda}) - V_j(\boldsymbol{\lambda})] \quad \text{for } i, j = 1, \dots, n$$

This dynamics may be justified by the assumption that migration decisions are made on the basis of pairwise comparisons between cities so that the net migration from city j to city i is proportional to their utility differential $V_i - V_j$. In addition, the speed $f_{ij}(\boldsymbol{\lambda}) > 0$ of adjustment may vary according to the cities of origin and destination as well as with the distribution $\boldsymbol{\lambda}$. In order to keep things simple, we assume that $f_{ij}(\boldsymbol{\lambda}) = f_{ji}(\boldsymbol{\lambda})$ for all $i, j = 1, \dots, n$. Consequently, the dynamic migration system is such as:

$$\frac{d\lambda_i}{dt} \equiv \sum_j f_{ij}(\boldsymbol{\lambda})[V_i(\boldsymbol{\lambda}) - V_j(\boldsymbol{\lambda})] \quad \text{for } i = 1, \dots, n$$

Because the total population of workers remains constant during the adjustment process, it must be that $\sum_i d\lambda_i / dt = 0$.

Using the vector form, we may rewrite the foregoing system as follows:

$$\frac{d\boldsymbol{\lambda}}{dt} = \mathbf{F}(\boldsymbol{\lambda}) \cdot \mathbf{V}(\boldsymbol{\lambda}) \tag{6}$$

where

$$\mathbf{F}(\boldsymbol{\lambda}) \equiv \begin{pmatrix} \sum_{j\neq 1}^{n} f_{1j}(\boldsymbol{\lambda}) & -f_{12}(\boldsymbol{\lambda}) & \cdots & -f_{1n}(\boldsymbol{\lambda}) \\ -f_{21}(\boldsymbol{\lambda}) & \sum_{j\neq 2}^{n} f_{2j}(\boldsymbol{\lambda}) & \cdots & -f_{2n}(\boldsymbol{\lambda}) \\ \vdots & \vdots & \ddots & \vdots \\ -f_{n1}(\boldsymbol{\lambda}) & -f_{n2}(\boldsymbol{\lambda}) & \cdots & \sum_{j\neq n}^{n} f_{nj}(\boldsymbol{\lambda}) \end{pmatrix} \tag{7}$$

and

$$\mathbf{V}(\boldsymbol{\lambda}) \equiv (V_1(\boldsymbol{\lambda}), V_2(\boldsymbol{\lambda}), \ldots, V_n(\boldsymbol{\lambda}))$$

Note that (6) corresponds to what is called a *positive definite dynamics*,[8] which includes several special, but meaningful, cases. A first example is given by the replicator dynamics in which $f_{ij}(\boldsymbol{\lambda}) = \lambda_i \lambda_j$ (Fujita et al., 1999), whereas a second one is given by the dynamics

$$\frac{d\lambda_i}{dt} = V_i(\boldsymbol{\lambda}) - \frac{1}{n}\sum_j V_j(\boldsymbol{\lambda})$$

where $f_{ij}(\boldsymbol{\lambda}) = 1/n$, used by Ginsburgh et al. (1985), Tabuchi (1986) and Zeng (2002).

Some tedious, but standard, calculations show that (4) may be rewritten as follows:[9]

$$\mathbf{V}(\boldsymbol{\lambda}) = \begin{pmatrix} V_1(\boldsymbol{\lambda}) \\ \vdots \\ V_n(\boldsymbol{\lambda}) \end{pmatrix} = \begin{pmatrix} S_1(\lambda_1) \\ \vdots \\ S_n(\lambda_n) \end{pmatrix} + R(\boldsymbol{\lambda}) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \equiv \mathbf{S}(\boldsymbol{\lambda}) + R(\boldsymbol{\lambda})\,\mathbf{1}$$

where

$$\begin{aligned} S_i(\lambda_i) &\equiv (C_1\tau - C_2\tau^2)\lambda_i - C_3\tau^2\lambda_i^2 - \theta_i(\lambda_i) \\ \mathbf{1} &\equiv (1, 1, \ldots, 1) \\ C_1 &\equiv \frac{aN(b+cN)(3b+2cN)}{(2b+cN)^2} \\ C_2 &\equiv \frac{N(b+cN)}{8(2b+cN)^2}\left[4(2b+cN)\frac{cNA}{nL} + 12b^2 + 4bcN - 3c^2N^2\right] \\ C_3 &\equiv \frac{cN^2(b+cN)(8b+5cN)}{8(2b+cN)^2} \end{aligned}$$

and $R(\boldsymbol{\lambda})$ is a function of $\boldsymbol{\lambda}$. Clearly, $S_i(0) = 0$ whereas we have $C_1 > 0, C_3 > 0$ and $C_2 + C_3 > 0$ because $\beta > \gamma$.

By definition of (7), it must be that $\mathbf{F}(\boldsymbol{\lambda}) \cdot \mathbf{1} = \mathbf{0}$, so that we have:

$$\mathbf{F}(\boldsymbol{\lambda}) \cdot \mathbf{V}(\boldsymbol{\lambda}) = \mathbf{F}(\boldsymbol{\lambda}) \cdot [\mathbf{S}(\boldsymbol{\lambda}) + R(\boldsymbol{\lambda})\,\mathbf{1}] = \mathbf{F}(\boldsymbol{\lambda}) \cdot \mathbf{S}(\boldsymbol{\lambda})$$

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8   Additional conditions for a positive definite dynamics are: (i) every element of $\mathbf{F}(\boldsymbol{\lambda})$ is continuously differentiable in $\boldsymbol{\lambda}$ and (ii) $\boldsymbol{\lambda} \cdot \mathbf{F}(\boldsymbol{\lambda})\boldsymbol{\lambda} > 0$ for all $\boldsymbol{\lambda} \in \mathbb{R}^n$ that is not a multiple of $\mathbf{1} = (1, \ldots, 1)$. Note that some $f_{ij}(\boldsymbol{\lambda})$ may be negative insofar as (ii) is satisfied.

9   For a detailed proof, see Tabuchi et al. (2002).

Hence, $V_i(\boldsymbol{\lambda})$ may be replaced by $S_i(\lambda_i)$ in (6). This leads to a major simplification because $S_i(\lambda_i)$ depends only upon the size of city i, unlike $V_i(\boldsymbol{\lambda})$ that depends on the whole distribution $\boldsymbol{\lambda}$. From now on, we follow the tradition of urban economics and refer to $S_i(\lambda_i)$ as the 'surplus' of region i.

Because the number n of regions may be arbitrarily large, the number m of cities may be strictly smaller than n. If $\boldsymbol{\lambda}^*$ is a spatial equilibrium with $m \leq n$ cities, denoted i_j with $j = 1, \ldots, m$, then $S^* \geq 0$ exists such that

$$S_{i_1}(\lambda^*_{i_1}) = \cdots = S_{i_m}(\lambda^*_{i_m}) = S^* \tag{8}$$

$$S_j(\lambda^*_j) = 0 \leq S^* \quad \text{for all } j \neq i_1, \ldots, i_m \tag{9}$$

which amounts to (5). When $m = n$, there may exist a spatial equilibrium in which all the surpluses are negative and equal. However, (9) shows that, if the equilibrium involves at least one rural region ($m < n$), the surpluses of all cities are nonnegative and equal.

Let

$$S'_i \equiv \left. \frac{dS_i(\lambda_i)}{d\lambda_i} \right|_{\lambda_i = \lambda^*_i}$$

and consider an equilibrium with $m \leq n$ cities indexed in a way such that

$$S'_{i_1} \leq \cdots \leq S'_{i_{m-1}} \leq S'_{i_m}$$

Then, Tabuchi and Zeng (2004) have shown that the spatial equilibrium $\boldsymbol{\lambda}^*$ is (locally) *stable* if

$$S'_{i_{m-1}} < 0 \quad \text{and} \quad \sum_{j=1}^{m-1} \frac{S'_{i_m}}{S'_{i_j}} > -1 \tag{10}$$

whereas $\boldsymbol{\lambda}^*$ is unstable if one of the inequalities in (10) is reversed.[10] We also know from Tabuchi and Zeng (2004) that, under the dynamic system (6), a stable equilibrium generically exists.[11]

Finally, when the whole population is concentrated into a single region ($m = 1$), the stability condition (10) ceases to hold and the equilibrium conditions (8) and (9) boil down to $S_1(1) \geq 0$.

4. The size of cities

In this section, we consider the evolution of a stable equilibrium such that the set of cities does not change whatever the values of τ and study how their size is affected by a marginal decrease in transport costs. Without loss of generality, we may thus assume that the number of cities is such that $m = n$. Denoting the corresponding equilibrium by

$$\boldsymbol{\lambda}^*(\tau) = (\lambda^*_1(\tau), \ldots, \lambda^*_n(\tau))$$

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10   When either one of the two inequalities in (10) becomes an equality, the equilibrium may be stable or unstable. We will return to this case in Section 5.

11   For more details, see Mas-Collel (1985, ch. 8).

with $\lambda_i^*(\tau) > 0$, our objective is therefore to study the sign of $d\lambda_i^*(\tau)/d\tau$. Because $S_i(\lambda_i)$ also depends on τ, we denote it as $S_i(\lambda_i; \tau)$ whenever ambiguity may arise.

In equilibrium, it must be that

$$\sum_j [S_i(\lambda_i^*(\tau); \tau) - S_j(\lambda_j^*(\tau); \tau)] = 0$$

Differentiating this equation yields the following system of linear equations whose unknowns are precisely $d\lambda_i^*(\tau)/d\tau$:

$$-(n-1)S_i' \frac{d\lambda_i^*(\tau)}{d\tau} + \sum_{j \neq i} S_j' \frac{d\lambda_j^*(\tau)}{d\tau} = z_i \quad i = 1, \ldots, n \tag{11}$$

where

$$z_i \equiv \sum_j \frac{\partial(S_i - S_j)}{\partial \tau}\bigg|_{\boldsymbol{\lambda} = \boldsymbol{\lambda}^*} = (C_1 - 2C_2\tau)(n\lambda_i^* - 1) - 2C_3\tau[n(\lambda_i^*)^2 - \sum_j (\lambda_j^*)^2] \tag{12}$$

Hence, *we must solve the system (11) to determine how the size of cities changes when transport costs fall*. This is the formal counterpart of the general interdependence that arises within the urban system.

Without loss of generality, we may assume that cities are indexed such that:

$$\lambda_i^* \geq \lambda_{i+1}^* > 0 \quad i = 1, \ldots, n-1 \tag{13}$$

The urban cost functions $\theta_i(\cdot)$ being city-specific, the ranking of cities by population size may change as τ falls. When this is so, we re-index them for (13) to hold.

Let s be the city such that $S_s'(\lambda_s^*) = \max_i S_i'(\lambda_i^*)$. The stability condition (10) implies that $S_i'(\lambda_i^*) < 0$ for all $i \neq s$. A stable spatial equilibrium $\boldsymbol{\lambda}^*$ is said to be *regular* when $S_s'(\lambda_s^*) \leq 0$; otherwise, it is called *irregular*. Regular equilibria in which $S_s' < 0$ and $S_s' = 0$ are represented in Figures 1a and 1b, respectively. The case of an irregular equilibrium ($S_s' > 0$) is depicted in Figure 1c. In what follows, the two cases of regular and irregular equilibria are considered.

As discussed in the foregoing, in this section we restrict our attention to the case of asymmetric equilibria in that $\lambda_i^* \neq \lambda_j^*$ holds for some i, j. In addition, we make the following technical assumption:

$$\lambda_i^* \neq \lambda_s^* \quad \text{for all } i \neq s \quad \text{when } S_s' = 0 \tag{14}$$

Because $S_i' < 0$ holds for all $i \neq s$ when $S_s' = 0$, this condition does not entail much loss of generality.

Consider any two different cities i and j. Lemma 3 shows that the sign of $z_i - z_j$ is important to determine the sign of $d\lambda_i^*(\tau)/d\tau$. Using (12), it is readily verified that

$$z_i - z_j = n\frac{\partial(S_i - S_j)}{\partial \tau}\bigg|_{\boldsymbol{\lambda} = \boldsymbol{\lambda}^*}$$
$$= n(\lambda_i^* - \lambda_j^*)\{C_1 - 2[C_2 + C_3(\lambda_i^* + \lambda_j^*)]\tau\} \qquad \text{for } i \neq j \tag{15}$$

Because $\lambda_i^* - \lambda_j^* \geq 0$ for $i < j$, $z_i - z_j$ is positive (resp., negative) when the term in brackets is positive (resp., negative). To determine the sign of $z_i - z_j$, we thus consider the two polar cases in which (i) the whole population is agglomerated in cities i and j ($\lambda_i^* + \lambda_j^* = 1$)
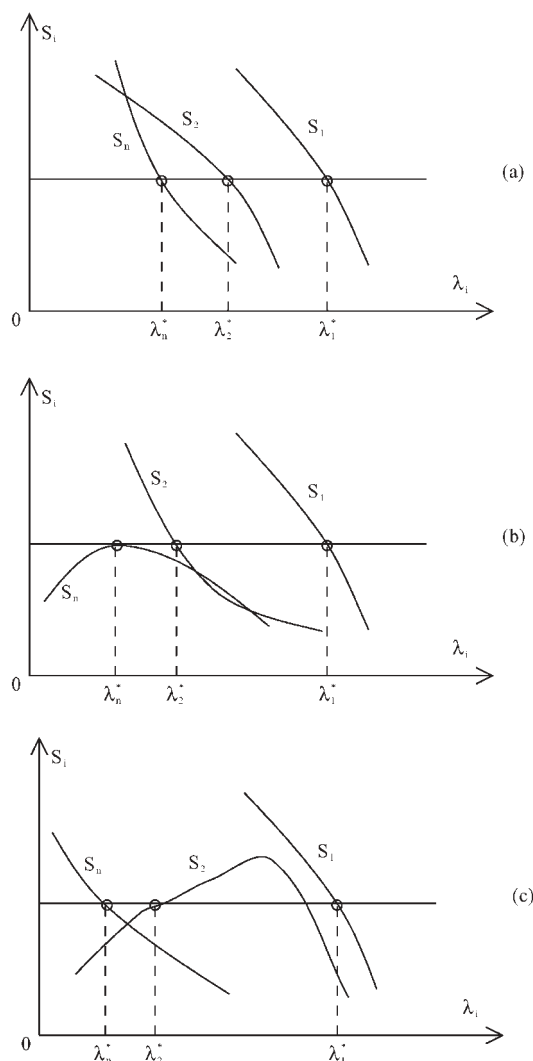
Figure 1. Regular and irregular equilibra.

and (ii) regions i and j are rural ($\lambda_i^* + \lambda_j^* = 0$). The corresponding solutions of the equations

$$C_1 - 2(C_2 + C_3)\tau = 0 \qquad C_1 - 2C_2\tau = 0$$

are independent of i and j, and respectively given by

$$\tau_1^* \equiv C_1/2(C_2 + C_3) \qquad \tau_2^* \equiv C_1/2C_2$$

Clearly, $0 < \tau_1^* < \tau_2^*$ (resp., $\tau_2^* < 0 < \tau_1^*$) when $C_2 > 0$ (resp., $C_2 < 0$). In words, $0 < \tau_1^* < \tau_2^*$ is likely to hold when varieties are sufficiently differentiated, or when the number of farmers is not too low, or both.

In order to shed light on the nature of the difficulties encountered when studying the impact of economic integration in the n-region case, we now give a heuristic argument for

a regular equilibrium with three cities. Suppose that τ exceeds τ_2^* so that the bracketed term in (15) is always negative. Because $\lambda_1^* > \lambda_2^* > \lambda_3^*$, it must be that $z_1 < z_2 < z_3$. In this case, Lemma 3 implies that $d\lambda_1^*(\tau)/d\tau < 0$ because z_1 is strictly smaller than the weighted mean of the z_i's, whereas $d\lambda_3^*(\tau)/d\tau > 0$ because z_3 is strictly larger than the weighted mean of the z_i's. Stated differently, the large city grows and the small city shrinks when transport costs decrease. It is more difficult to figure out what happens in the interval $\tau_1^* < \tau < \tau_2^*$ because the sign of the bracketed term in (15) changes. Indeed, there is a priori no indication about the ranking of the z_i's. Nevertheless, as τ keeps falling below τ_2^* while remaining above τ_1^*, (15) implies that we get, first, the ranking $z_1 < z_3 < z_2$ and, then, $z_3 < z_1 < z_2$. In both cases, Lemma 3 implies that $d\lambda_2^*(\tau)/d\tau > 0$, which means that the medium size city 2 shrinks when transport costs decline. Finally, when τ is smaller than τ_1^*, the bracketed term in (15) is always positive so that we now have the ranking $z_3 < z_2 < z_1$. As above, it follows from Lemma 3 that $d\lambda_1^*(\tau)/d\tau > 0$ and $d\lambda_3^*(\tau)/d\tau < 0$: the large city shrinks while the small city expands. Our results below show that the general tendency uncovered in this example carries over to the case of any city number.

The total population of workers being fixed, when some cities become larger due to a decrease in τ, some others must become smaller. It is shown in Lemma 4 that the large city experiences a population increase in the first stage of the integration process ($\tau > \tau_2^*$), whereas the small city experiences a population increase in the second stage ($\tau_1^* < \tau < \tau_2^*$), a result which agrees with the 2-region case studied by Ottaviano *et al.* (2002). However, in the case of several cities with asymmetric urban costs, the peaks of the large cities and the troughs of the small ones do not necessarily appear at the same value of τ. Nonetheless, we have been able to show the following result.

Proposition 1. Assume that $\tau_2^* \in (0, \tau_{trade})$, the equilibrium is regular and the set of cities is the same as τ falls. Then, (i) the smallest city switches first from decline to growth, then so does the second smallest city, and so on among all the small cities; and (ii) the largest city switches last from growth to decline, the second largest city switches right before it, and so forth among all the large cities.

This proposition follows from Lemma 4 and is illustrated in Figure 2.[12] Starting from large values of τ ($\tau \geq \tau_2^*$), we see that all large (resp., small) cities become larger (resp., smaller) as τ decreases. In the interval (τ_1^*, τ_2^*) of intermediate values of τ, as transport costs fall, net migration changes sign and, for each city, the evolution of size is reversed one city after the other. For low transport costs ($\tau \leq \tau_1^*$), all large (resp., small) cities lose (resp., gain) workers and firms. Accordingly, urban inequalities are largest for intermediate values of transport costs.

In the case of an irregular equilibrium ($S_s' > 0$), we show in Lemma 5 that Lemma 4 still holds except for city s. Putting together Lemmas 4 and 5, we obtain our main result.

Proposition 2. Assume that $\tau_2^* \in (0, \tau_{trade})$ and the set of cities does not change as τ falls. If $S_s' \leq 0$, then large cities become larger and small cities become smaller as long as $\tau \geq \tau_2^*$; large and small cities become larger and medium size cities become smaller when $\tau_1^* < \tau < \tau_2^*$; finally, large cities become smaller and small cities become larger once $\tau \leq \tau_1^*$. Furthermore, when $S_s' > 0$, the direction of migration is reverse for at most one city.

---

12 Figure 2 is numerically computed for the parameter values: $n = 6$, $a = 9$, $b = 1$, $c = 1$, $\phi = 1$, $L = 100$, $A = 1200$, $\theta_1(\lambda) = 100\lambda$, $\theta_2(\lambda) = 102\lambda$, $\theta_3(\lambda) = 104\lambda$, $\theta_4(\lambda) = 106\lambda$, $\theta_5(\lambda) = 108\lambda$, $\theta_6(\lambda) = 110\lambda$.
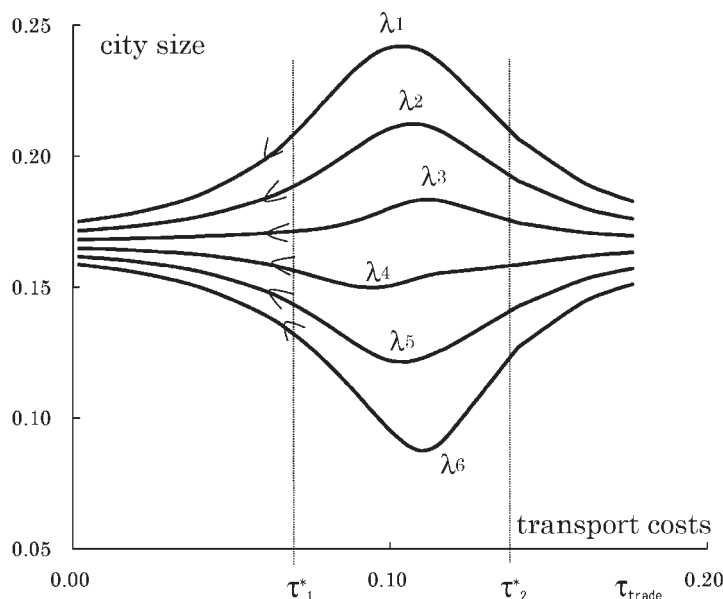
Figure 2. Evolution of city size distributions.

This proposition has several important implications. First, when transport costs are high ($\tau \geq \tau_2^*$), their decrease triggers an agglomeration process in which large cities attract workers and firms from the small cities, which shrink. By contrast, when transport costs are small ($\tau \leq \tau_1^*$), large cities lose workers and firms while small cities grow. Hence, agglomeration takes place in the early stages of economic integration, while re-dispersion occurs in the late stages. Second, when transport costs take intermediate values ($\tau_1^* < \tau < \tau_2^*$), medium size cities lose workers and firms while large and small cities grow. We thus obtain what is probably the main distinctive property of the multi-regional model: *small cities shrink in the early stages, medium size cities shrink in the next stages, and large cities shrink in the late stages of the process of economic integration*. During this process, each city experiences variations in the size, but also in the sign, of its net migrations.

This may be understood as follows. Transport costs are harmful to all cities. However, when they are high, small cities are relatively more affected because they supply less varieties than others. Urban costs are similarly harmful to all cities. However, when transport costs are low, the burden of urban costs is relatively higher for large cities because they accommodate bigger populations. Because the ranking of the slopes of city surpluses is monotone, medium size cities are the most affected ones when transport costs take intermediate values. We may then conclude that *falling transport costs have changing impacts on migration flows* through the interplay between the relative equilibrium sizes of cities and the relative equilibrium values of urban costs.

Finally, when there are few farmers, or when the varieties are close substitutes, or both, it must be that $\tau_2^* < 0$. Therefore, the early stages of agglomeration do not arise. Furthermore, when the number of farmers is very low, or when the degree of differentiation between varieties is very low, or both we have $\tau_1^* \geq \tau_{trade}$. The early and intermediate stages no longer arise. In this case, *the integration process triggers a dispersion of population*

from the large cities to the small ones, through the medium size cities, thus generalizing Helpman's (1998) numerical results.

5. The number of cities

So far, we have assumed that the number of cities was not affected by a decrease in inter-city transport costs. This may be justified when τ varies within a fairly small domain. However, it is important to figure out how the number of cities is affected when the decrease in transport costs is large. In order to achieve this goal, we impose a minor restriction on the urban cost functions: we assume that $S_i(\lambda_i)$ is *strictly quasi-concave* in the population size λ_i. This is satisfied when the urban cost functions $\theta_i(\lambda_i)$ are convex ($S_i(\lambda_i)$ is then strictly concave), a condition which often holds in urban economics (Fujita, 1989, p.145).

Despite its simplicity, our model exhibits a high degree of complexity in that a stable equilibrium may become unstable or may even disappear, whereas a new stable equilibrium may emerge, for a marginal decrease in τ. In order to illustrate the nature of the difficulty, consider the following example. There are three regions ($n = 3$) and the surplus function is identical across regions. Assume that the initial equilibrium involves the 3 cities and is such that $\boldsymbol{\lambda} = (1/3, 1/3, 1/3)$, as shown in Figure 3(a). This equilibrium becomes unstable for a marginal decrease in τ. When the surplus function at $\lambda = 1/6$ is flatter than at $\lambda = 5/6$, then the equilibrium $\boldsymbol{\lambda} = (5/6, 1/6, 0)$ with 2 cities is stable as shown in Figure 3(b); note that other stable equilibria might exist. This equilibrium, in turn, becomes unstable for a further decrease in τ. Indeed, there are now three stable equilibria given by $(2/5, 2/5, 1/5)$, $(1/2, 1/2, 0)$ and $(1, 0, 0)$, which are all displayed in Figure 3(c). Hence, there exist several stable equilibria, some of them having a larger number of cities, while others involve a smaller number. In particular, falling transport costs may lead to dispersion ($m = 3$) or to full agglomeration ($m = 1$). Thus, monotonicity in the number of cities does not necessarily holds.

In the multi-regional case, unlike the 2-region case, the multiplicity of stable equilibria does no longer allow one to tell nice stories about the evolution of the spatial economy. Rather, it becomes a nightmare that prevents us from deriving any general result.[13]

In order to gain some insights about the possible evolution of the urban system, we consider the case of identical urban costs: $\theta_i(\lambda) = \theta(\lambda)$ for all i with $\theta''(\lambda) \geq 0$ and $\theta'''(\lambda) \leq 0$—the latter condition means that urban costs are not 'too convex'. The convexity of $\theta(\lambda)$ implies that $S(\lambda)$ is strictly concave (and not strictly quasi-concave as in Figure 3). In this case, irregular equilibria never arise as shown by Lemma 6. Consequently, we can fully characterize the number and size of cities in terms of τ. Indeed, it can readily be shown that stable equilibria are always symmetric $\lambda_i^* = 1/m$ ($m \leq n$) because $\theta_i(\lambda) = \theta(\lambda)$, so that the spatial equilibrium condition becomes $S_i(1/m) \geq 0$ whereas the stability condition is $S_i'(1/m) < 0$. Therefore, assuming that a stable equilibrium describes the actual evolution of the economy until it becomes unstable, we have the following result (the proof is contained in Appendix B).

Proposition 3. Assume that urban costs are identical, convex and $\theta'''(\lambda) \leq 0$. Then, there exists a threshold value τ^* such that (i) the number of cities decreases as transport costs

13 In this respect, the numerical results presented by Krugman (1993) as well as by Brakman et al. (2001, ch. 4) are somewhat misleading in that they do not make it clear how the transition from full dispersion to partial agglomeration occurs.
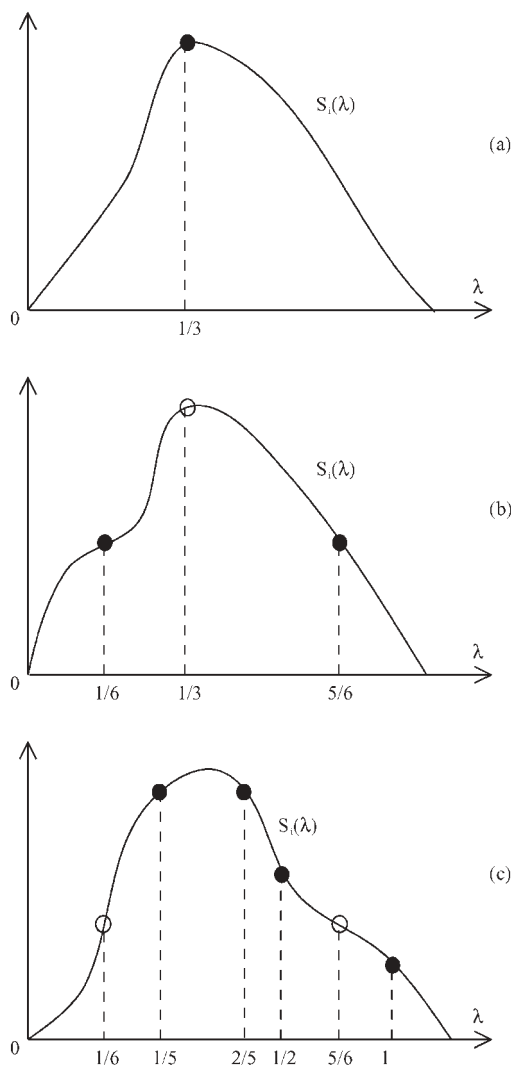
Figure 3. Multiple equilibra in the number of cites.

decrease for $\tau \geq \tau^*$; and (ii) the number of cities increases as transport costs decrease for $\tau < \tau^*$.

This shows that *urban concentration first arises while re-dispersion comes afterwards*. Stated differently, as transport costs decrease, workers and firms, first, move from small to large cities (not to say, from rural to urban areas) and, then, from large to small cities. As discussed in the introduction, the first phase is historically well documented while there are on-going debates about whether or not the second phase has already started.

In addition, for sufficiently large or sufficiently small transport costs, dispersion is likely to arise, whereas the highest degree of agglomeration within the economy may involve more than one city. In other words, *the presence of urban costs may prevent the full agglomeration into a single core region*. On the other hand, if the urban costs in region 1

are sufficiently low, then all workers are agglomerated into a single city for intermediate values of the transport costs. Consequently, when transport costs decrease while urban costs do not, *the economy moves from dispersion to the emergence of some urban giants and, then, displays gradual deconcentration.*

Note that τ^* may exceed τ_{trade}, and hence case (i) in the proposition may not arise. For example, when there are very few farmers, the mere existence of urban costs is sufficient for the decrease in transport costs to induce right away a gradual dispersion of population over a growing number of cities, a result which extends Helpman (1998) to an arbitrary number of regions.

Finally, even in the special case of identical urban costs, the number of cities does not necessarily decrease (resp., increase) one by one when transport costs fall. Indeed, when $m > 2$, there always exist several stable equilibria for any value of τ. Insofar as there is no reason to choose a particular equilibrium, the evolutionary processes simulated by Krugman (1993) and Brakman et al. (2001) might well be the outcome of some arbitrary selections implicitly used when several regions get more integrated.

6. Concluding remarks

Our model has proven to be useful in shedding light on a major debate for economies involved in an integration process, namely the impact of falling transport costs on the size and number of cities. We have seen that such a fall should lead to a (possibly strong) concentration of mobile activities, which will eventually be followed by a re-dispersion of these activities within a multi-regional economy. In other words, *the general pattern of activities as transport costs decrease is likely to exhibit a bell-shaped pattern*, as suggested by Puga (1999) and Tabuchi (1998).

However, much work remains to be done in order to understand how cities evolve when transport costs take intermediate values, while it is also important to figure out how the medium size cities react to decreasing transport costs. Our analysis has also dismissed the fact that commuting costs have decreased together with transport costs. Finally, we consider a given and fixed set of economic activities. This should be recasted within a dynamic framework in order to better understand the evolution of urban systems.

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Appendix A

Lemmas

Let D be the matrix of the coefficients of the system (11), D_{ij} be the submatrix of D obtained by deleting the i-th row and j-th column, and set

$$E \equiv \begin{cases} \dfrac{1}{n^3|D|} \displaystyle\prod_{\ell=1}^{n}(-nS'_\ell) \sum_j \dfrac{1}{(-S'_j)} & \text{for } S'_s \neq 0 \\[3ex] \dfrac{1}{n^2|D|} \displaystyle\prod_{\ell \neq s}(-nS'_\ell) & \text{for } S'_s = 0 \end{cases}$$

Lemma 1 If (10) holds at the equilibrium $\boldsymbol{\lambda}^*$, then

$$|D| > 0$$

$$|D_{ii}| = \frac{1}{m}\left(1 + \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{S'_m}{S'_j}\right) \prod_{\substack{\ell=1 \\ \ell \neq i}}^{m-1}(-mS'_\ell) \qquad i = 1,\ldots, m-1 \tag{16}$$

$$|D_{ij}| = (-1)^{i-j+1}(S'_i - S'_m) \prod_{\substack{\ell=1 \\ \ell \neq i, \ell \neq j}}^{m-1}(-mS'_\ell) \quad i, j = 1,\ldots, m-1, i \neq j \tag{17}$$

Proof. (i) Using some basic properties of determinants, we obtain

$$|D| = \begin{vmatrix} -(m-1)S'_1 - S'_m & S'_2 - S'_m & S'_3 - S'_m & \cdots & S'_{m-1} - S'_m \\ mS'_1 & -mS'_2 & 0 & \cdots & 0 \\ mS'_1 & 0 & -mS'_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ mS'_1 & 0 & 0 & \cdots & mS'_{m-1} \end{vmatrix}$$

$$= \left[-(m-1)S'_1 - S'_m + \sum_{\ell=2}^{m-1} \frac{S'_1(S'_\ell - S'_m)}{S'_\ell}\right] \prod_{\ell=2}^{m-1}(-mS'_\ell)$$

$$= \frac{1}{m}\left(1 + \sum_{\ell=1}^{m-1} \frac{S'_m}{S'_\ell}\right) \prod_{\ell=1}^{m-1}(-mS'_\ell) > 0,$$

where the inequality follows from the stability condition (10).

(ii) We first consider the case of $i > 1$. By definition of D_{ii} and some properties of determinants, we have

$$
|D_{ii}| = \begin{vmatrix} -(m-1)S'_1 - S'_m & \cdots & S'_{i-1} - S'_m & S'_{i+1} - S'_m & \cdots & S'_{m-1} - S'_m \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S'_1 - S'_m & \cdots & -(m-1)S'_{i-1} - S'_m & S'_{i+1} - S'_m & \cdots & S'_{m-1} - S'_m \\ S'_1 - S'_m & \cdots & S'_{i-1} - S'_m & -(m-1)S'_{i+1} - S'_m & \cdots & S'_{m-1} - S'_m \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S'_1 - S'_m & \cdots & S'_{i-1} - S'_m & S'_{i+1} - S'_m & \cdots & -(m-1)S'_{m-1} - S'_m \end{vmatrix}
$$

$$
= \begin{vmatrix} -(m-1)S'_1 - S'_m & S'_2 - S'_m & \cdots & S'_{i-1} - S'_m & S'_{i+1} - S'_m & \cdots & S'_{m-1} - S'_m \\ mS'_1 & -mS'_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ mS'_1 & 0 & \cdots & -mS'_{i-1} & 0 & \cdots & 0 \\ mS'_1 & 0 & \cdots & 0 & -mS'_{i+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ mS'_1 & 0 & \cdots & 0 & 0 & \cdots & -mS'_{m-1} \end{vmatrix}
$$

$$
= \left[-(m-1)S'_1 - S'_m + \sum_{\substack{j=2 \\ j \neq i}}^{m-1} \frac{S'_1(S'_j - S'_m)}{S'_j} \right] \prod_{\substack{\ell=2 \\ \ell \neq i}}^{m-1} (-mS'_\ell)
$$

$$
= \frac{1}{m} \left(1 + \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{S'_m}{S'_j} \right) \prod_{\substack{\ell=1 \\ \ell \neq i}}^{m-1} (-mS'_\ell).
$$

Next, for $i = 1$, we have

$$
|D_{11}| = \begin{vmatrix} -(m-1)S'_2 - S'_m & S'_3 - S'_m & S'_4 - S'_m & \cdots & S'_{m-1} - S'_m \\ mS'_2 & -mS'_3 & 0 & \cdots & 0 \\ mS'_2 & 0 & -mS'_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ mS'_2 & 0 & 0 & \cdots & -mS'_{m-1} \end{vmatrix}
$$

$$
= \left[-(m-1)S'_2 - S'_m + \sum_{j=3}^{m-1} \frac{S'_j - S'_m}{S'_j} S'_2 \right] \prod_{\ell=3}^{m-1} (-mS'_\ell)
$$

$$
= \frac{1}{m} \left(1 + \sum_{j=2}^{m} \frac{S'_m}{S'_j} \right) \prod_{\ell=2}^{m-1} (-mS'_\ell)
$$

(iii) We consider only the case where $j < i$. By straightforward calculation, we know

$$|D_{ij}|$$

$$= \begin{vmatrix} -(m-1)S_1' - S_m' & S_2' - S_m' & \cdots & S_{j-1}' - S_m' & S_{j+1}' - S_m' & \cdots & S_{i-1}' - S_m' & S_i' - S_m' & S_{i+1}' - S_m' & \cdots & S_{m-1}' - S_m' \\ mS_1' & -mS_2' & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ mS_1' & 0 & \cdots & -mS_{j-1}' & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ mS_1' & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ mS_1' & 0 & \cdots & 0 & -mS_{j+1}' & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ mS_1' & 0 & \cdots & 0 & 0 & \cdots & -mS_{i-1}' & 0 & 0 & \cdots & 0 \\ mS_1' & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -mS_{i+1}' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ mS_1' & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -mS_{m-1}' \end{vmatrix}$$

$$= (-1)^{i-j+1}(S_i' - S_m') \prod_{\substack{\ell=1 \\ \ell \neq i, \ell \neq j}}^{m-1} (-mS_\ell') \qquad \square$$

It follows from Lemma 1 that $|D| > 0$ for any stable equilibrium. This in turn implies that $E > 0$. The argument is in three steps. (i) If the equilibrium is regular with $S_s' < 0$, it must be that $-nS_\ell' > 0$ and $-S_j' > 0$. (ii) If the equilibrium is regular with $S_s' = 0$, then we have $S_\ell' < 0$ for all $\ell \neq s$. (iii) If the equilibrium is irregular, we have $S_s' > 0 > S_\ell'$ for all $\ell \neq s$. In this case, the second stability condition of (10) implies that $\sum_j 1/(-S_j') < 0$ whereas $(-nS_\ell') > 0$ for all $\ell \neq s$ and $(-nS_s') < 0$, which imply $E > 0$.

Lemma 2 For any τ, there exists a city k such that

$$z_i \geq z_{i-1} \quad \text{for } i < k$$
$$z_i \geq z_{i+1} \quad \text{for } i \geq k$$

Proof. If $C_1 - 2[C_2 + C_3(\lambda_i^* + \lambda_{i+1}^*)]\tau < 0$ for all $i = 1, \ldots, n-1$, then

$$z_i - z_{i+1} = n(\lambda_i^* - \lambda_{i+1}^*)\{C_1 - 2[C_2 + C_3(\lambda_i^* + \lambda_{i+1}^*)]\tau\}$$

is nonnegative for all i since $\lambda_i^* \geq \lambda_{i+1}^*$. In this case, we let $k = n$. Otherwise, let

$$k = \min\{i | C_1 - 2[C_2 + C_3(\lambda_i^* + \lambda_{i+1}^*)]\tau \geq 0\}$$

If $i < k$, then $C_1 - 2[C_2 + C_3(\lambda_i^* + \lambda_{i+1}^*)\tau < 0$, which implies $z_i \leq z_{i+1}$. On the other hand, if $i \geq k$, then $\lambda_i^* + \lambda_{i+1}^* \leq \lambda_k^* + \lambda_{k+1}^*$, and hence

$$C_1 - 2[C_2 + C_3(\lambda_i^* + \lambda_{i+1}^*)]\tau \geq C_1 - 2[C_2 + C_3(\lambda_k^* + \lambda_{k+1}^*)]\tau \geq 0$$

which means $z_i \geq z_{i+1}$. $\qquad \square$

Lemma 2 says that z_i is 'single-peaked' in i, that is, $1 \leq k \leq n$ exists such that $z_i \geq z_{i-1}$ for $i < k$ and $z_i \geq z_{i+1}$ for $i \geq k$. In particular, if $k = 1$ (resp. n), then z_i is decreasing (resp., increasing) in i. If $k \neq 1$ and n, then z_i is increasing in i for $i < k$ and decreasing for $i \geq k$.

Define the weighted average of z_j as follows:

$$\bar{z} \equiv \begin{cases} \dfrac{\sum_j z_j/(-S'_j)}{\sum_j 1/(-S'_j)} & \text{for } S'_s \neq 0 \\ z_s & \text{for } S'_s = 0. \end{cases}$$

Lemma 3 Assume that the number of cities remains equal to n as τ falls. If $S'_i \neq 0$, then

$$\frac{d\lambda^*_i(\tau)}{d\tau} = E\frac{z_i - \bar{z}}{-S'_i} \quad \text{for } i = 1,\ldots,n \tag{18}$$

If $S'_s = 0$, then

$$\frac{d\lambda^*_i(\tau)}{d\tau} = \begin{cases} E\dfrac{z_i - \bar{z}}{-S'_i} & \text{for } i \neq s \\ E\displaystyle\sum_{j\neq s}\dfrac{\bar{z} - z_j}{-S'_i} & \text{for } i = s \end{cases} \tag{19}$$

Proof. Let D_i be the matrix obtained from D by replacing the i-th column with

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{m-1} \end{pmatrix} = (C_1 - 2C_2\tau)\begin{pmatrix} m\lambda^*_1 - 1 \\ m\lambda^*_2 - 1 \\ \vdots \\ m\lambda^*_{m-1} - 1 \end{pmatrix} - 2C_3\tau\begin{pmatrix} m(\lambda^*_1)^2 - \sum_{j=1}^m\left(\lambda^*_j\right)^2 \\ m(\lambda^*_2)^2 - \sum_{j=1}^m\left(\lambda^*_j\right)^2 \\ \vdots \\ m(\lambda^*_{m-1})^2 - \sum_{j=1}^m\left(\lambda^*_j\right)^2 \end{pmatrix}$$

Since

$$\frac{d\lambda^*_n(\tau)}{d\tau} = -\sum_{\ell=1}^{n-1}\frac{d\lambda^*_i(\tau)}{d\tau}$$

it is readily verified that the solution to the system (11) is given by

$$\frac{d\lambda^*_i(\tau)}{d\tau} = \frac{|D_i|}{|D|} = \frac{(-1)^i}{|D|}\sum_{j=1}^{n-1}(-1)^j z_j|D_{ji}| \qquad i = 1,\ldots,n-1$$

for $i = 1,\ldots,n-1$. Using (16) and (17), this expression becomes

$$\frac{(-1)^i}{|D|}\sum_{\substack{j=1\\j\neq i}}^{m-1} z_j(-1)^{-i+1}(S'_j - S'_m)\prod_{\substack{\ell=1\\\ell\neq i\\\ell\neq j}}^{m-1}(-mS'_\ell) + \frac{z_i}{m|D|}\left(1 + \sum_{\substack{j=1\\j\neq i}}^m\frac{S'_m}{S'_j}\right)\prod_{\substack{\ell=1\\\ell\neq i}}^{m-1}(-mS'_\ell)$$

which is equal to

$$
\frac{1}{m|D|} \prod_{\substack{\ell=1 \\ \ell \neq i}}^{m-1} (-mS'_\ell) \left(\frac{m(-1)^i \sum_{\substack{j=1 \\ j \neq i}}^{m-1} z_j (-1)^{-i+1}(S'_j - S'_m)}{-mS'_j} + z_i \left(1 + \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{S'_m}{S'_j} \right) \right)
$$

$$
= \frac{1}{m|D|} \prod_{\substack{\ell=1 \\ \ell \neq i}}^{m-1} (-mS'_\ell) \left(\sum_{\substack{j=1 \\ j \neq i}}^{m-1} z_j - \sum_{\substack{j=1 \\ j \neq i}}^{m-1} z_j \frac{S'_m}{S'_j} + z_i + z_i \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{S'_m}{S'_j} \right)
$$

$$
= \frac{1}{m|D|} \prod_{\substack{\ell=1 \\ \ell \neq i}}^{m-1} (-mS'_\ell) \left(\sum_{\substack{j=1 \\ j \neq i}}^{m-1} z_j - \sum_{\substack{j=1 \\ j \neq i}}^{m} z_j \frac{S'_m}{S'_j} + z_i + z_m + z_i \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{S'_m}{S'_j} \right)
$$

$$
= \frac{1}{m|D|} \prod_{\substack{\ell=1 \\ \ell \neq i}}^{m-1} (-mS'_\ell) \left[\sum_{j=1}^{m} z_j + \sum_{j=1}^{m} \frac{S'_m}{S'_j}(z_i - z_j) \right]
$$

$$
= \frac{1}{m|D|} \prod_{\substack{\ell=1 \\ \ell \neq i}}^{m-1} (-mS'_\ell) \sum_{j=1}^{m} \frac{S'_m}{S'_j}(z_i - z_j)
$$

If $S'_s \neq 0$, then the above expression is equal to

$$
\frac{d\lambda_i^*(\tau)}{d\tau} = \frac{1}{m} \sum_{j=1}^{m} \left(\frac{d\lambda_i^*(\tau)}{d\tau} - \frac{d\lambda_j^*(\tau)}{d\tau} \right) \tag{20}
$$

$$
= \frac{1}{(-S'_i)m^3|D|} \prod_{\ell=1}^{m} (-mS'_\ell) \sum_{j=1}^{m} \frac{1}{(-S'_j)} \left[z_i - \frac{\sum_{j=1}^{m} z_j/(-S'_j)}{\sum_{j=1}^{m} 1/(-S'_j)} \right]
$$

If $S'_s = 0$, then the expression (20) for region $i \neq s$ is equal to

$$
\frac{d\lambda_i^*(\tau)}{d\tau} = \frac{1}{(-S'_i)m^2|D|} \prod_{\substack{\ell=1 \\ \ell \neq s}}^{m} (-mS'_\ell)(z_i - z_s)
$$

For region s, (20) is

$$
\frac{d\lambda_s^*(\tau)}{d\tau} = - \sum_{\substack{j=1 \\ j \neq s}}^{m} \frac{d\lambda_j^*(\tau)}{d\tau} = \frac{1}{m^2|D|} \prod_{\substack{\ell=1 \\ \ell \neq s}}^{m} (-mS'_\ell) \sum_{\substack{j=1 \\ j \neq s}}^{m} \frac{z_s - z_j}{-S'_j} \qquad \square
$$

Thus, the sign of

$$
z_i - \bar{z} = \frac{\sum_j ((z_i - z_j)/S'_j)}{\sum_j (1/S'_j)}
$$

is critical to determine the sign of $d\lambda_i^*(\tau)/d\tau$, thus implying that we are interested in the sign of $z_i - z_j$. We have seen that

$$
z_i - z_j = n(\lambda_i^* - \lambda_j^*)\{C_1 - 2[C_2 + C_3(\lambda_i^* + \lambda_j^*)]\tau\} \quad \text{for } i,j = 1, \ldots, n \tag{21}
$$

Because $\lambda_i^* \geq \lambda_j^*$ for all $i < j$ from (13), the sign of $z_i - z_j$ is the same as the sign of $C_1 - 2[C_2 + C_3(\lambda_i^* + \lambda_j^*)]\tau$.

Let k_- be the largest city (the city with the smallest number if there are several of them) and k_+ be the smallest one (the one with the largest number if there are several of them). Both k_- and k_+ exist, otherwise there would be no city having a decrease in population; if there is only one city whose population decreases, then $k_- = k_+$.

At a regular equilibrium, some city k must become smaller and, hence, $k_- \leq k \leq k_+$ must hold. Indeed, if $S'_k < 0$, then $d\lambda^*_k(\tau)/d\tau > 0$ from (18) because $z_k = \max_i z_i \geq \bar{z}$, and hence city k becomes smaller. If $S'_k = 0$, then it follows from the first equation of (19) that $d\lambda^*_i(\tau)/d\tau \leq 0$ for all $i \neq k$, implying that all the other cities do not become smaller. Therefore, city k must become smaller. More precisely, the following result is shown.

Lemma 4 Assume that $\tau^*_2 \in (0, \tau_{trade})$, the equilibrium is regular and the set of cities is the same as τ falls. Then we have:

(i)   when $\tau \geq \tau^*_2$, the large cities $1, \ldots, k_- - 1$ become larger, whereas the small cities $k_-, \ldots, n$ become smaller;

(ii) when $\tau^*_1 < \tau < \tau^*_2$, the large cities $1, \ldots, k_- - 1$ and the small cities $k_+ + 1, \ldots, n$ become larger, whereas the medium size cities $k_-, \ldots, k_+$ become smaller.

(iii) when $\tau \leq \tau^*_1$, the large cities $1, \ldots, k_+$ become smaller whereas the small cities $k_+ + 1, \ldots, n$ become larger.

Proof. Consider first the case where $S'_s \neq 0$.

(i)   For $\tau \geq \tau^*_2 > 0$, we have $k = k_+ = n$ and z_i is monotone increasing in i (i.e., $z_i \leq z_{i+1}$). From the assumption of asymmetric equilibrium, there is at least one region whose population increases so that $k_- > 1$. Hence, regions $1, \ldots, k_- - 1$ become larger, while all the other regions become smaller.

(ii) For $\tau^*_1 < \tau < \tau^*_2$, z_i is either monotone or single-peaked. Since there is at least one region with a population increase either $1 < k_-$ or $k_+ < n$ holds. In the former case, $1 < k_- \leq k_+ \leq n$ holds, and in the latter case $1 \leq k_- \leq k_+ < n$ holds. By the definition of regions k_- and k_+, we know that regions $1, \ldots, k_- - 1, k_+ + 1, \ldots, n$ become larger. From Lemma 2 with $k_- \leq k \leq k_+$, all the other regions $k_-, \ldots, k_+$ become smaller.

(iii) For $\tau \leq \tau^*_1$, $k_- = k = 1$, and z_i is monotone decreasing in i. Since $k_+ > 1$ due to the assumption of asymmetric equilibrium, regions $1, \ldots, k_+$ become smaller, while all the other regions become larger.

Consider now the case where $S'_s = 0$.

(i') For $\tau \geq \tau^*_2 > 0$, we have $z_1 \leq \cdots \leq z_{s-1} < z_s \leq z_n$ from (21). From the first equation of (19), regions $1, \ldots, s - 1$ become larger and regions $s + 1, \ldots, n$ become smaller, and hence $k_+ = n$. If region s is becomes smaller, then $k_- = s$ If region s becomes larger, then $k_- = s + 1$.

(ii') For $\tau^*_1 < \tau < \tau^*_2$. If regions $s - 1$ and $s + 1$ are both increasing, then $z_{s-1} > z_s$ and $z_{s+1} > z_s$. Thus,

$$C_1 - 2[C_2 + C_3(\lambda^*_{s-1} + \lambda^*_s)]\tau > 0 \quad \text{and} \quad C_1 - 2[C_2 + C_3(\lambda^*_s + \lambda^*_{s+1})]\tau < 0$$

hold from (14) and (21). By subtraction, we have $2C_3(\lambda^*_{s+1} - \lambda^*_{s-1})\tau > 0$, which contradicts (14). Similarly, we can show that both regions $s + 1$ and $s - 1$ cannot become smaller. Therefore, s should be either one of $k_-, k_- - 1, k_+, k_+ + 1$. If region s becomes larger, then s should be either $k_- - 1$ or $k_+ + 1$. If region s becomes smaller, then s should be either k_- or k_+, and hence (ii) also again.

(iii') For $\tau \leq \tau^*_1$, the similar argument as (i') applies. □

In this lemma, the large, medium size and small cities have not been identified because the cities k_- and k_+ may change when τ falls.

By comparison with the regular case, the sign of (18) in Lemma 3 is reversed only in region s since the corresponding denominator $(-S'_s)$ is negative. However, the sign of the other denominators $(-S'_i$ for $i \neq s)$ is the same as before. Therefore, we obtain the same result as in Lemma 4, except for region s.

Lemma 5 Assume that the set of cities does not change as τ falls and that $\tau_2^* \in (0, \tau_{trade})$. At any irregular and interior equilibrium with $S_s' > 0$, Lemma 4 holds for all cities $i \neq s$. When s is the smallest region ($s = n$), the lemma may be not hold as $d\lambda_s^*(\tau)/d\tau > 0$ only in city s for $\tau_1^* < \tau < \tau_2^*$.

Proof. Since the proof of Lemma 4 can be applied to each region $i \neq s$, we prove it only for region $s = n$ excluding the case of $d\lambda_s^*(\tau)/d\tau > 0$ with $\tau_1^* < \tau < \tau_2^*$.

(i) For $\tau \geq \tau_2^* > 0$, $z_n - z_j > 0$ holds for all $j = 1, \ldots, n-1$ from (21). Therefore,

$$z_n - \bar{z} = \frac{1}{\sum_{j=1}^n 1/(-S_j')} \sum_{j=1}^{n-1} \frac{z_n - z_j}{-S_j'} < 0 \tag{22}$$

and hence, $d\lambda_n^*(\tau)/d\tau > 0$. Since $\bar{z} > z_{n-1} \geq \ldots \geq z_1$ holds from $\bar{z} > z_n$, $d\lambda_i^*(\tau)/d\tau < 0$ holds for all $i = 1, \ldots, n-1$. Thus, Lemma 4 (i) holds with $k_- = n$.

(ii) For $\tau_1^* < \tau < \tau_2^*$, $z_n \geq \bar{z}$ holds if $d\lambda_n^*(\tau)/d\tau \leq 0$. Since $z_{k_-} \geq \bar{z}$, by definition of k_-, $z_j \geq \bar{z}$ holds for all $j = k_- + 1, \ldots, n-1$. Otherwise, $z_j < \bar{z}$ holds for some j, which implies $z_j < z_{k_-}$ and $z_j < z_n$. Hence,

$$C_1 - 2[C_2 + C_3(\lambda_n^* + \lambda_j^*)\tau] < 0 \quad \text{and} \quad C_1 - 2[C_2 + C_3(\lambda_{k_-}^* + \lambda_j^*)\tau] > 0.$$

from the fact that $\lambda_n \leq \lambda_j \leq \lambda_{k_-}$. By subtraction, we obtain $C_3(\lambda_{k_-}^* - \lambda_n^*) < 0$ which is a contradiction. Therefore, regions $k_-, \ldots, n-1$ become smaller and (ii) of Lemma 4 holds with $k_+ = n-1$.

(iii) For $\tau \leq \tau_1^*$, $z_n - z_j < 0$ holds for all $j = 1, \ldots, n-1$, and hence $d\lambda_n^*(\tau)/d\tau < 0$ from the equality of (22). Since $\bar{z} < z_n < z_{n-1} \leq \cdots \leq z_1$ holds, we have $d\lambda_i^*(\tau)/d\tau > 0$ for all $i = 1, \ldots, n-1$. Thus, Lemma 4 (iii) holds with $k_+ = n-1$ □

In the very special case covered by Lemma 5, the large cities $1, \ldots, k_- - 1$ and the small cities $k_+ + 1, \ldots, n-1$ become larger, while the medium size cities $k_-, \ldots, k_+$ together with the smallest city n become smaller.

Lemma 6 If the urban costs are identical, convex and $\theta'''(\lambda) \leq 0$, then there is no irregular equilibrium.

Proof. Given that $\theta_i(\cdot) = \theta(\cdot)$, we have $S_i(\lambda) = S(\lambda)$. Because $S(\lambda)$ is concave and $\lim_{\lambda \to \infty} S(\lambda) < 0 = S(0)$, $S(\lambda)$ has a maximizer denoted by λ^p. If $\lambda^p \leq 0$, there is no irregular equilibrium. Thus, consider the case where $\lambda^p > 0$. For all $x \in [0, \lambda^p]$, let

$$f(x) \equiv S(\lambda^p + x) - S(\lambda^p - x)$$

then

$$
\begin{aligned}
f'(x) &= S'(\lambda^p + x) + S'(\lambda^p - x) \\
&= 2(C_1\tau - C_2\tau^2) - 4C_3\tau^2\lambda^p - \theta'(\lambda^p - x) - \theta'(\lambda^p + x) \\
&= 2\theta'(\lambda^p) - \theta'(\lambda^p - x) - \theta'(\lambda^p + x) \\
&= [\theta'(\lambda^p) - \theta'(\lambda^p - x)] - [\theta'(\lambda^p + x) - \theta'(\lambda^p)] \\
&= x[\theta''(\lambda^p - x_1) - \theta''(\lambda^p + x_2)] \\
&\geq 0
\end{aligned}
$$

where $x_1, x_2 \in (0, x)$. The third equality follows from $S'(\lambda^p) = 0$, the fifth equality from the mean value theorem, whereas the inequality holds because $\theta'''(\lambda) \leq 0$. Since $f(0) = 0$, we have $f(x) \geq 0$ for all $x \in [0, \lambda^p]$.

Assume that there exists a 2-city equilibrium with different sizes such that

$$S(\lambda^p + y) = S(\lambda^p - x) \qquad x, y > 0 \tag{23}$$

From $f(x) \geq 0$ and (23), we get

$$S(\lambda^p + x) \geq S(\lambda^p - x) = S(\lambda^p + y)$$

implying that $y \geq x$ holds from $S'(\lambda) \leq 0$ for $\lambda \geq \lambda^p$.

Equation (23) implies that

$$
\begin{aligned}
0 &= \int_0^{\lambda^p + y} S'(\lambda) d\lambda - \int_0^{\lambda^p - x} S'(\lambda) d\lambda \\
&= \int_{\lambda^p}^{\lambda^p + y} S'(\lambda) d\lambda + \int_{\lambda^p - x}^{\lambda^p} S'(\lambda) d\lambda \\
&\leq y[S'(\lambda^p + y) - S'(\lambda^p)]/2 + x[S'(\lambda^p - x) - S'(\lambda^p)]/2 \\
&\leq x[S'(\lambda^p + y) + S'(\lambda^p - x)]/2
\end{aligned}
$$

where the first inequality holds because $S'(\lambda^p + y) < S'(\lambda^p) = 0 < S'(\lambda^p - x)$ and $S'(\lambda)$ is convex, whereas the second inequality follows from $y \geq x$ and $S'(\lambda^p + y) \leq 0$. Thus, we have $S'(\lambda^p + y) + S'(\lambda^p - x) \geq 0$, which violates the stability condition (10). Hence, there is no irregular equilibrium with 2 cities. It can be similarly verified that any equilibrium with more than 2 cities having different sizes violates the stability condition (10). □

APPENDIX B: Proof of Proposition 3

The concavity of $S_i(\lambda_i)$ implies that $S_i(\lambda_i) = 0$ has a solution at $\lambda_i = 0$ and at most one solution in $(0, \infty)$. Denote this one by λ_i^o if it exists. Since urban costs are the same across regions, the surplus function S_i is independent of i so that $\lambda_i^o = \lambda^o$. We first show that λ^o is quasi-concave with respect to τ. From

$$\frac{d\lambda^o}{d\tau} = -\frac{\partial S(\lambda^o; \tau)/\partial \tau}{\partial S(\lambda^o; \tau)/\partial \lambda} = \frac{\lambda^o}{-S'(\lambda^o; \tau)}[C_1 - 2(C_2 + C_3 \lambda^o)\tau]$$

and the fact that $-S'(\lambda^o) > 0$ when $\lambda^o > 0$, it follows that

$$\text{sgn}(d\lambda^o/d\tau) = \text{sgn}[C_1 - 2(C_2 + C_3 \lambda^o)\tau]$$

holds for any $\lambda^o > 0$.

If λ^o is monotone, then it is quasi-concave. If it is not monotone, then there exists τ^* satisfying

$$\left. \frac{d\lambda^o}{d\tau} \right|_{\tau = \tau^*} = 0 \tag{24}$$

The second derivative evaluated at τ^* is given by

$$
\begin{aligned}
\left. \frac{d^2\lambda^o}{d\tau^2} \right|_{\tau = \tau^*} &= \frac{1}{-S'(\lambda^o)} \left. \frac{\partial^2 S(\lambda^o)}{\partial \tau^2} \right|_{\tau = \tau^*} \\
&= \frac{\lambda^o}{-S'(\lambda^o)}[-2(C_2 + C_3 \lambda^o)\lambda^o] < 0
\end{aligned}
$$

Thus, τ^* satisfying (24) must be unique. As a consequence, λ^o is increasing in $(0, \tau^*)$ and decreasing in (τ^*, ∞), and hence quasi-concave.

Because urban costs are identical, it is readily verified that an equilibrium with n cities is stable if $n\lambda^p < 1$, and an equilibrium with $m < n$ cities is stable if and only if $m\lambda^o > 1$ and $m\lambda^p < 1$.[14] Therefore, for decreasing τ with $\tau > \tau^*$ (resp., $\tau < \tau^*$), if an equilibrium with m cities ceases to exist, then $m\lambda^p = 1$ (resp., $m\lambda^o = 1$) so that any equilibrium with $m' > m$ (resp., $m' < m$) cities is unstable. Hence, the number of cities must decrease (resp., increase) as τ falls.

---

14   We exclude a finite number of τ-values for which $m\lambda^o = 1$ or $m\lambda^p = 1$ holds.