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Duality, Income and Substitution Effects for the Competitive Firm Under Price Uncertainty

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This paper uses duality theory to decompose the total effect on the competitive firm's output of an increase in the riskiness of output price into income and substitution effects. Properties of preferences that control the sign of each effect are identified. The analysis extends to the general class of quasi-linear decision models in which the payoff is linear in the random variable. Copyright © 2005 John Wiley & Sons, Ltd.

INTRODUCTION

Following Sandmo's (1970, 1971) expected utility maximization formulations of a decision maker's choice under uncertainty, a substantial literature developed concerning choice in a variety of uncertainty settings. Many of the models in this literature have a common underlying formal structure; that is, the payoffs are linear in the random variable. Early examples include consumption and savings decision under technological uncertainty (Mirman, 1971) and under interest rate uncertainty (Hanson and Menezes, 1978), the portfolio problem (Rothschild and Stiglitz, 1971), the demand for labor under wage rate uncertainty (Batra and Ullah, 1974) and the supply of labor under wage rate uncertainty (Coyte, 1986), among many others. In these models, the decision makers are competitive agents; that is, their decisions cannot affect the exogenously given random variable. Following Bigelow and Menezes (1995), we call these models quasi-linear decision models. Because of the ubiquity of quasi-linear problems in economics, considerable attention has been focused on the effect of increases in the riskiness of the random variable on optimal decisions. Most of the literature concluded that there is no a priori rationale to determine the effect of increased risk on the optimal decision due to the possibility of two opposing tendencies. These tendencies were first formally identified and necessary and sufficient conditions on preferences presented to sign the combined (total) effect by Bigelow and Menezes (1995).

While there has been considerable interest in the Slutsky decomposition of an increase in the riskiness of the random variable into income and substitution effects, the literature has yet to derive such a decomposition. The two opposing tendencies mentioned above do not yield a Slutsky decomposition. While the literature has successfully derived the Slutsky equation for an increase in the expected value of the random variable (e.g. Sandmo, 1971; Pope and Chavas, 1985), it has only obtained 'Slutsky-like' equations for increases in risk (e.g. Sandmo, 1971; Block and Heineke, 1973).[1] The problem encountered in obtaining the

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Slutsky decomposition for an increase in risk has been the difficulty in identifying an analytically based method of compensation (see Davis, 1989; Hadar and Seo, 1992).

This paper shows that duality theory can be used to overcome this difficulty. Specifically, we formulate the uncertainty counterpart of the expenditure function under certainty which in turn provides the basis for deriving the Slutsky equation decomposing the total effect of an increase in risk into an income effect and a substitution effect. The income effect is due solely to the reduction in certainty equivalent wealth arising from the increased riskiness of the random variable. The substitution effect is the effect on optimal choice after compensation. It represents solely the effect of increased riskiness on choice. We also identify the properties of preference that govern the sign of both the income effect and the substitution effect.

Income and substitution effects play a central role in the theory of choice under certainty. They provide analytical as well as intuitive insights about the decision maker's response to a change in a parameter in the model. The significance of the Slutsky decomposition follows also from the fact that compensated choices provide the theoretical underpinnings for the analysis of the welfare effects of parameter changes. Since compensated choices are generally unobservable, they must be inferred from uncompensated choices. The Slutsky equation provides the bridge between compensated and uncompensated choices. For example, compensated (Hicksian) demand elasticities can be deduced from uncompensated (Marshallian) demand elasticities using the Slutsky equation.

In the next section, we present the general quasi-linear decision problem and formulate its dual. For concreteness, the rest of the paper focuses on the competitive firm's output choice problem under price uncertainty as a particular application of the general quasi-linear model. In the following section, we provide the necessary and sufficient conditions for the firm's primal and dual problems. In the later section, we derive the Slutsky equation for a general Rothschild and Stiglitz (1970) increase in price uncertainty decomposing the total effect on output into an income effect and a substitution effect. Both effects are signed in terms of plausible properties of preferences in the penultimate section. The last section contains extensions of our dual approach to changes in non-stochastic parameters in the competitive firm's output choice problem under price uncertainty.

THE GENERAL QUASI-LINEAR MODEL AND ITS DUAL

Consider a decision maker with a von Neumann–Morgenstern utility function of wealth $u(\bullet)$. The decision maker's wealth is given by the random variable

$$\tilde{w} = w_0 + g(x) + \tilde{z}x. \tag{1}$$

In (1), w_0 is initial wealth, x is the decision variable, $g(x)$ is the non-random component of wealth that is dependent on x and $\tilde{z}$ is a random variable with distribution function $F(z, \beta)$ and support $[0,\infty)$, where $\beta \in [0, 1]$ is an index of Rothschild–Stiglitz (RS) risk. The RS riskiness of $\tilde{z}$ increases as β increases. The decision maker chooses x to

$$\mathrm{Max}_x\, Eu(\tilde{w}) \equiv \int_0^{\infty} u(w_0 + g(x) + \tilde{z}x)\,\mathrm{d}F(\tilde{z}, \beta). \tag{2}$$

Problem (2) is called a quasi-linear problem because random wealth $\tilde{w}$ is linear in the random variable $\tilde{z}$.

To formulate the dual approach, we now transform the maximization problem (2) into a constrained maximization problem. Let $\bar{z}$ denote the expected value of $\tilde{z}$ and let $z^* = \tilde{z} - \bar{z}$. Note that the distribution of z^* is the same as that of $\tilde{z}$ but its support is that of $\tilde{z}$ shifted leftward. The decision maker's expected wealth and total wealth are, respectively, $w = w_0 + g(x) + \bar{z}x$ and $\tilde{w} = w + z^*x$, where $E(z^*x) = 0$ for all x. The decision problem (2) is equivalent to

$$\begin{aligned} \mathrm{Max}\quad v(x, w, \beta) &\equiv Eu(w + z^*x) \\ &= \int_{-\bar{z}}^{\infty} u(w + z^*x)\,\mathrm{d}F(z^*, \beta) \end{aligned}$$

$$\{x, w\}$$

$$\text{s.t.}\quad w = w_0 + g(x) + \bar{z}x. \tag{3}$$

In (3), v is the derived utility function that induces a preference ordering over triples (x, w, β).

The dual to the maximization problem (3) is to choose x and w so as to minimize the amount of initial wealth required to attain a given level of expected utility v^0, i.e.

$$\begin{aligned}&\underset{\{x,w\}}{\text{Min}}\ \ w-[g(x)+\bar{z}x]\\ &\text{s.t.}\quad v(x,w,\beta)=v^0.\end{aligned}\tag{4}$$

As noted in the introduction, a variety of decision problems are quasi-linear. For concreteness, in the remainder of this paper, we apply the general model to the competitive firm's output choice problem under price uncertainty.

THE COMPETITIVE FIRM UNDER PRICE UNCERTAINTY

In the theory of the competitive firm's output decision under price uncertainty, the firm chooses its output q before market price $\tilde{p}$ is known so as to maximize the expected utility of wealth $\tilde{w}$. To apply the general model developed in the previous section to the firm's problem, we replace x by $q, \tilde{z}$ by $\tilde{p}$ and $g(x)$ by $-c(q)$, where $c(q)$ is the firm's total cost function. We assume that the firm's utility function is increasing and concave and that its cost function is increasing and convex.

The firm's decision problem (corresponding to (3)) is

$$\begin{aligned}&\underset{\{q,w\}}{\text{Max}}\ \ v(q,w,\beta)\equiv\int_{-\bar{p}}^{\infty}u(w+p^*q)\,\mathrm{d}F(p^*,\beta)\\ &\text{s.t.}\quad w=w_0+\bar{p}q-c(q).\end{aligned}\tag{5}$$

From the properties of u, the derived utility function $v(q,w,\beta)$ is decreasing in q and β, increasing in w, and concave in q and w.[2] Concavity of $v(q,w,\beta)$ and convexity of $c(q)$ ensure that the maximization problem (5) is a standard concave programming problem. It has a unique solution. In particular, the second-order sufficient condition is satisfied. We assume that $\bar{p} > c'(0)$ so that an interior solution for (5) exists.[3]

The solution to (5), $(q^{\mathrm{U}}(\bar{p},w_0,\beta),\ \ w^{\mathrm{U}}(\bar{p},w_0,\beta))$, satisfies the necessary conditions [4]

$$\begin{aligned}&-\frac{v_q(q,w,\beta)}{v_w(q,w,\beta)}=\bar{p}-c'(q),\\ &w_0+\bar{p}q-c(q)-w=0.\end{aligned}\tag{6}$$

To provide an interpretation of (6), note that the left-hand side of the first equation in (6) is the slope of an indifference curve $v=\text{constant}$,[5] i.e. the marginal rate of substitution between risky revenue and expected wealth. We denote it by

$$\mathrm{MRS}(q,w,\beta)=-\frac{v_q(q,w,\beta)}{v_w(q,w,\beta)}.\tag{7}$$

Since $v_q<0$ and $v_w>0$, MRS is positive. MRS gives the minimum amount of wealth (w) required to compensate the firm for the increase in risky revenue p^*q induced by a unit increase in output.[6] The first equation in (6) stipulates that, at the optimum, the amount of wealth that the firm requires to compensate it for the increase in risk induced by the increment in output is equal to the expected marginal profit $(\bar{p}-c'(q))$ it receives from producing the increment in output.

In Figure 1, the optimal point $(q^{\mathrm{U}},w^{\mathrm{U}})$ is where the indifference curve $v=v^0$, and the opportunity constraint, $w=w_0+\bar{p}q-c(q)$, are tangent, i.e. $\mathrm{MRS}=\bar{p}-c'(q)$. The first component of the solution $q^{\mathrm{U}}(\bar{p},w_0,\beta)$ is the firm's (uncompensated) output supply, and the second component $w^{\mathrm{U}}(\bar{p},w_0,\beta)$ is its expected wealth.

The dual to the firm's maximization problem (5) (corresponding to (4)) is

$$\begin{aligned}&\underset{\{q,w\}}{\text{Min}}\ \ w-[\bar{p}q-c(q)]\\ &\text{s.t.}\quad v(q,w,\beta)=v^0.\end{aligned}\tag{8}$$

The solution to (8), $(q^{\mathrm{C}}(\bar{p},v^0,\beta),w^{\mathrm{C}}(\bar{p},v^0,\beta))$, satisfies the necessary conditions

$$\begin{aligned}&-\frac{v_q(q,w,\beta)}{v_w(q,w,\beta)}=\bar{p}-c'(q),\\ &\quad v(q,w,\beta)=v^0.\end{aligned}\tag{9}$$

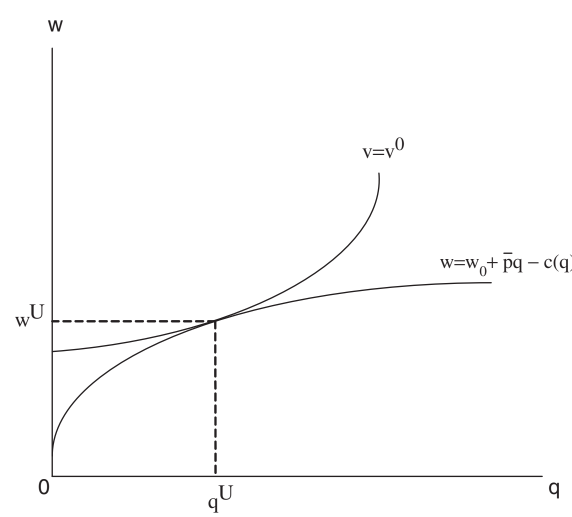

Figure 1. Production decision under price uncertainty.

$q^{\mathrm{C}}(\bar{p}, v^0, \beta)$ is the firm's compensated output supply. Like the original problem (5), the assumption that $\bar{p} > c'(0)$ ensures that an interior solution for (8) exists and it occurs where the indifference curve is tangent to the opportunity constraint as shown in Figure 1.

THE SLUTSKY EQUATION

The Slutsky equation for an increase in price uncertainty may be derived from the relationship between the firm's primal and dual problems. Consider the function

$$\varphi(\bar{p}, v^0, \beta) \equiv w^{\mathrm{C}}(\bar{p}, v^0, \beta) - [\,\bar{p}q^{\mathrm{C}}(\bar{p}, v^0, \beta) - c(q^{\mathrm{C}}(\bar{p}, v^0, \beta))].$$

It gives the minimum initial wealth required for the firm to attain expected utility v^0. Accordingly, we call φ the *wealth requirement function*. The firm's supply and compensated supply coincide when its initial wealth is $w_0 = \varphi(\bar{p}, v^0, \beta)$, i.e.

$$q^{\mathrm{U}}(\bar{p}, \varphi(\bar{p}, v^0, \beta), \beta) \equiv q^{\mathrm{C}}(\bar{p}, v^0, \beta).$$

Note that the minimum amount of initial wealth required to attain expected utility v^0 is w_0 when v^0 is the utility of the optimal uncompensated decision $(q^{\mathrm{U}}, w^{\mathrm{U}})$, i.e. $v^0 = v(q^{\mathrm{U}}(\bar{p}, w_0, \beta), w^{\mathrm{U}}(\bar{p}, w_0, \beta), \beta)$. Using this fact and differentiating the above equality gives immediately the following result.[7]

Proposition 1:
The Slutsky equation for an increase in price uncertainty is

$$\frac{\partial q^{\mathrm{U}}}{\partial \beta} = \frac{\partial q^{\mathrm{C}}}{\partial \beta} - \frac{\partial q^{\mathrm{U}}}{\partial w_0} \times \frac{\partial \varphi}{\partial \beta}. \tag{10}$$

In (10), the term $-[\partial q^{\mathrm{U}}/\partial w_0] \times [\partial \varphi/\partial \beta]$ is the income effect of an increase in price uncertainty and the compensated term $\partial q^{\mathrm{C}}/\partial \beta$ is the substitution effect of an increase in price uncertainty.

As (10) indicates, the total effect on output of an increase in price uncertainty is the sum of the substitution effect and the income effect. The substitution effect is the change in output when wealth is adjusted so that the firm achieves the same expected utility at the higher level of price uncertainty as it did at the original optimum. The income effect is the change in output when wealth is adjusted back to the original level but price uncertainty remains at the new level. That is, the income effect is the change in output due solely to the implicit reduction in wealth when price uncertainty increases. This implicit reduction in wealth arises because the increase in price uncertainty reduces the certainty equivalent wealth of the firm.

We now present the expressions for the income and substitution effects in terms of the firm's derived utility function. By the envelope theorem, the derivative of the wealth requirement function with respect to the risk parameter is (derivations for (11)–(14) are provided in Appendix A)

$$\frac{\partial \varphi}{\partial \beta} = -\frac{v_\beta}{v_w}, \tag{11}$$

which is the minimum amount of wealth required to compensate the firm for the increase in price uncertainty brought about by the increase in β. Totally differentiating (6) gives

$$\frac{\partial q^{\mathrm{U}}}{\partial w_0} = \frac{\mathrm{MRS}_w}{J}, \tag{12}$$

where J is the Jacobian determinant for (6), given by $J = -(\mathrm{MRS}_q + \mathrm{MRS} \cdot \mathrm{MRS}_w) - c''(q)$, it is negative by the second-order sufficient condition for (5). Thus,

$$\text{income effect} = -\frac{\partial q^{\mathrm{U}}}{\partial w_0} \times \frac{\partial \varphi}{\partial \beta} = -\left[\frac{\mathrm{MRS}_w}{J}\right] \times \left[-\frac{v_\beta}{v_w}\right]. \tag{13}$$

Totally differentiating (9) gives

$$\text{substitution effect} = \frac{\partial q^{\mathrm{C}}}{\partial \beta} = \frac{\mathrm{MRS}_\beta v_w - \mathrm{MRS}_w v_\beta}{v_w J}. \tag{14}$$

SIGNING THE INCOME AND SUBSTITUTION EFFECTS

We begin with the income effect. In (13), the term $-v_\beta/v_w$ is the amount of wealth that the firm requires to be compensated for a small increase in β in order to keep expected utility constant. It is positive because expected utility increases with w and decreases with β. Hence, since $J < 0$ the sign of the income effect is the same as the sign of MRS_w. Differentiating the marginal rate of substitution

MRS in (7) with respect to w yields,

$$\mathrm{MRS}_w = -\frac{[Eu'][E(p^*u'')] - [Eu''][E(p^*u')]}{(Eu')^2} = -\frac{E[(\tilde{p} - c'(q))u'']}{Eu'}.$$

$E[(\tilde{p} - c'(q))u'']$ is positive if absolute risk aversion $(-u''/u')$ is decreasing.[8] Thus we have the following result.

Proposition 2:
Suppose that the utility function $u(w)$ exhibits decreasing absolute risk aversion (DARA). Then the income effect of an increase in price uncertainty on output is negative.

Consider next the substitution effect, given by (14). The following lemma relates the sign of the substitution effect to the behavior of $\rho(q, w, \beta)$ with respect to q, where

$$\rho(q, w, \beta) \equiv -\frac{v_\beta(q, w, \beta)}{v_w(q, w, \beta)}. \tag{15}$$

By (11), ρ is equal to the derivative of the wealth requirement function with respect to the risk parameter β. See Appendix B for a geometric interpretation of ρ as the wealth difference between a pair of indifference curves representing the same expected utility but different values of β.

Lemma 1:

$$\frac{\partial q^C}{\partial \beta}\begin{cases} \leqslant 0 \\ \text{or} \\ \geqslant 0 \end{cases} \text{if and only if } \left.\frac{\partial \rho}{\partial q}\right|_{v = v^0} \begin{cases} \geqslant 0 \\ \text{or} \\ \leqslant 0 \end{cases} \tag{16}$$

Proof:
In Appendix A.

To sign the substitution effect, we first consider a special case in which the sign of $\partial\rho/\partial q$ along an indifference curve is readily obtained. Consider the widely used multiplicative increase in price risk given by the form $\tilde{p} = \bar{p} + \beta p^*$.[9] From the definition of v, in this case

$$\rho = -\frac{v_\beta}{v_w} = -\frac{E[u'(w + \beta p^* q)p^* q]}{E[u'(w + \beta p^* q)]} = q\mathrm{MRS}/\beta,$$

which is increasing in q along an indifference curve since the marginal rate of substitution MRS is increasing in q along an indifference curve. Hence, from (16), concavity of u guarantees that the substitution effect is negative when the increase in price risk takes the multiplicative form. In Appendix B, a general hypothesis is advanced to make the argument that it is reasonable to assume that ρ increases in q along an indifference curve for a general RS increase in price risk.

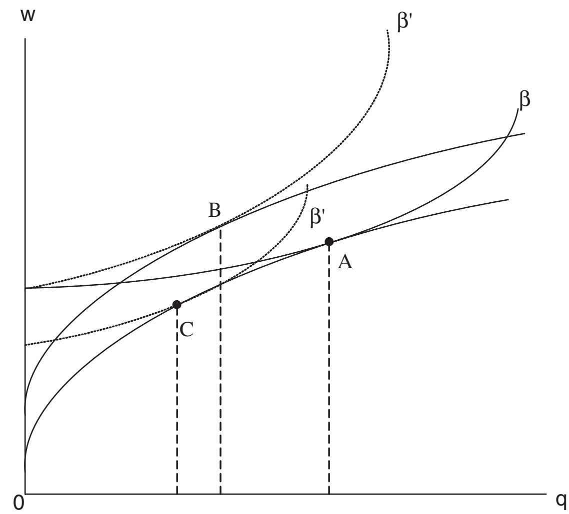

Figure 2. Income and substitution effects of increased price uncertainty.

Proposition 3:
Suppose that ρ, as defined by (15), increases in q along an indifference curve. Then the substitution effect of an increase in price uncertainty on output is negative.

It follows from Propositions 1 and 2 that an RS increase in output price uncertainty causes a firm to decrease its output. Figure 2 presents a geometric illustration of the income and substitution effects for an increase in price uncertainty. In it, the solid β-indifference curve represents the firm's indifference curve prior to an increase in β. The dashed β'-indifference curves represent indifference curves after the increase in β. The firm is initially in equilibrium at point A on the β-indifference curve. *The movement from A to B is the substitution effect of the increase in* β.[10] It is obtained by shifting upward the opportunity constraint so that it is tangent to the higher β'-indifference curve (which has the same expected utility as that of the β-indifference curve). *The movement from B to C is the income effect of the increase in* β.[11] It is obtained by moving from the compensated optimal bundle (B) to the bundle (C) where the lower β'-indifference curve is tangent to the initial opportunity constraint. *The movement from A to C is the total effect of the increase in* β.

EXTENSION TO NON-STOCHASTIC PARAMETERS

The framework developed in this paper can be used to study the comparative static effects of a change in any non-stochastic parameter. These effects depend on whether the parameter enters the derived utility function only or the opportunity constraint only or both. If it only enters the firm's opportunity constraint, the income and substitution effects are analogous to those of conventional analysis under certainty, with the utility function replaced by the derived utility function. For example, an excise tax on output does not affect the derived utility function $Eu(w + p^*q)$, but alters the opportunity constraint which now becomes $w = w_0 + \bar{p}q - c(q) - tq$, where t denotes the excise tax rate. Geometrically, the substitution effect is obtained by shifting the new (flatter) opportunity constraint upward until it is tangent to the original indifference curve; it is negative since the indifference curve is convex. The firm's new output occurs where an indifference curve is tangent to the new opportunity constraint. Decreasing absolute risk aversion guarantees that the income effect is also negative.

A more interesting analysis is the effect of a profit tax on the firm. Here, both derived utility and the opportunity constraint are affected. Let $t \in (0, 1)$ denote the profit tax rate. The firm's profits are $(1 - t)[\bar{p}q - c(q)]$, its opportunity constraint is $w = w_0 + (1 - t)[\bar{p}q - c(q)]$, and its derived utility function is $v(q, w, t) = Eu(w + (1 - t)p^*q)$. The firm's output supply $q^U \times (\bar{p}, w_0, t)$ is the solution to the problem

$$\begin{aligned} &\underset{\{q,w\}}{\text{Max}} \quad v(q, w, t) = Eu(w + (1 - t)p^*q) \\ &\text{s.t.} \quad w = w_0 + (1 - t)(\bar{p}q - c(q)). \end{aligned} \tag{17}$$

and the firm's compensated output supply $q^C(\bar{p}, v^0, t)$ is the solution to the problem

$$\begin{aligned} &\underset{\{q,w\}}{\text{Max}} \quad w - (1 - t)[\bar{p}q - c(q)] \\ &\text{s.t.} \quad v(q, w, t) = v^0. \end{aligned} \tag{18}$$

Totally differentiating the first order conditions for (17) and solving shows that the sign of the profit-tax income effect is

$$\text{sign}\left\{\frac{\partial q^U}{\partial w} \times \frac{\partial \varphi}{\partial t}\right\} = \text{sign}\{-E[u''(w + (1 - t)p^*q)(\tilde{p} - c'(q))]\},$$

which is negative under decreasing absolute risk aversion. Totally differentiating the first order conditions for (18) and solving shows that the sign of the profit-tax substitution effect is

$$\text{sign}\left\{\frac{\partial q^C}{\partial t}\right\} = \text{sign}\{-qEu'(w + (1 - t)p^*q)E[u''(w + (1 - t)p^*q)(\tilde{p} - c'(q))^2]\},$$

which is positive for all risk averse firms. Thus, the profit-tax income and substitution effects are opposite in direction. The sign of the profit-tax total effect depends on which effect dominates. Totally differentiating the first order conditions for (17) and solving shows that the sign of the profit-tax total effect is

$$\text{sign}\left\{\frac{\partial q^U}{\partial t}\right\} = \text{sign}\{-E[u''(w + (1 - t)p^*q) \times (\tilde{p}q - c(q))(\tilde{p} - c'(q))]\},$$

which is positive under increasing partial risk aversion.[12] Thus, under generally accepted assumptions about preferences, the profit-tax substitution effect dominates the profit-tax income effect.

APPENDIX A

In this appendix, derivations for Equations (11)–(14) and proof of Lemma 1 are provided.

Derivation of (11): The Lagrange function for the minimization problem (8) is

$$L(q, w, \beta) = w - [\bar{p}q - c(q)] + \lambda[v^0 - v(q, w, \beta)],$$

where λ is the Lagrange multiplier. The first-order conditions are

$$L_q = 1 - \lambda v_w = 0,$$
$$L_w = -[\bar{p} - c'(q)] - \lambda v_w = 0,$$
$$L_\lambda = v^0 - v(q, w, \beta) = 0.$$

Differentiating the Lagrange function L with respect to β and applying the above equations yields

$$L_\beta = -\lambda v_\beta.$$

Using $\lambda = 1/v_w$, we have $L_\beta = -v_\beta/v_w$. Substituting the solution $(q^C(\bar{p}, v^0, \beta), w^C(\bar{p}, v^0, \beta))$

into L gives

$$L(q^{\mathrm{C}}(\bar{p}, v^0, \beta), w^{\mathrm{C}}(\bar{p}, v^0, \beta), \beta) = w^{\mathrm{C}}(\bar{p}, v^0, \beta) - [\bar{p}q^{\mathrm{C}}(\bar{p}, v^0, \beta) - c(q^{\mathrm{C}}(\bar{p}, v^0, \beta))]$$

which is identical to $\varphi(\bar{p}, v^0, \beta)$. Hence, $\varphi_\beta = L_\beta = -v_\beta/v_w$, which is (11).

Derivation of (12): Recall that the solution to (5), $(q^{\mathrm{U}}(\bar{p}, w_0, \beta), w^{\mathrm{U}}(\bar{p}, w_0, \beta))$, satisfies the following conditions:

$$\mathrm{MRS}(q, w, \beta) = \bar{p} - c'(q),$$
$$w_0 + \bar{p}q - c(q) - w = 0.$$

Totally differentiating both equations yields

$$\mathrm{MRS}_q \,\mathrm{d}q + \mathrm{MRS}_w \,\mathrm{d}w = -c''(q)\,\mathrm{d}q,$$
$$\mathrm{d}w_0 + \bar{p}\mathrm{d}q - c'(q)\,\mathrm{d}q - \mathrm{d}w = 0,$$

which can be rewritten as

$$\begin{bmatrix} \mathrm{MRS}_q + c''(q) & \mathrm{MRS}_w \\ \bar{p} - c'(q) & -1 \end{bmatrix} \begin{bmatrix} \mathrm{d}q \\ \mathrm{d}w \end{bmatrix} = \begin{bmatrix} 0 \\ \mathrm{d}w_0 \end{bmatrix}.$$

Let J denote the determinant of the coefficient matrix above. Using the condition that $\mathrm{MRS} = \bar{p} - c'(q)$, $J = -(\mathrm{MRS}_q + \mathrm{MRS}\cdot\mathrm{MRS}_w) - c''(q)$. Solving the above equation system yields (12) immediately.

Derivation of (13): The income effect is equal to the second term in (10). Applying (11) and (12) gives (13).

Derivation of (14): Recall that the solution to (8), $(q^{\mathrm{C}}(\bar{p}, v^0, \beta), w^{\mathrm{C}}(\bar{p}, v^0, \beta))$, satisfies the following conditions:

$$\mathrm{MRS}(q, w, \beta) = \bar{p} - c'(q),$$
$$v(q, w, \beta) = v^0.$$

Totally differentiating both equations yields

$$\mathrm{MRS}_q \,\mathrm{d}q + \mathrm{MRS}_w \,\mathrm{d}w + \mathrm{MRS}_\beta \,\mathrm{d}\beta = -c''(q)\,\mathrm{d}q,$$
$$v_q \,\mathrm{d}q + v_w \,\mathrm{d}w + v_\beta \,\mathrm{d}\beta = 0,$$

which can be rewritten as

$$\begin{bmatrix} \mathrm{MRS}_q + c''(q) & \mathrm{MRS}_w \\ -v_q & -v_w \end{bmatrix} \begin{bmatrix} \mathrm{d}q \\ \mathrm{d}w \end{bmatrix} = \begin{bmatrix} -\mathrm{MRS}_\beta \,\mathrm{d}\beta \\ v_\beta \,\mathrm{d}\beta \end{bmatrix}.$$

The determinant of the above coefficient matrix is $v_w J$. Solving the above equation system gives

$$\frac{\partial q^{\mathrm{C}}}{\partial \beta} = \frac{\mathrm{MRS}_\beta v_w - \mathrm{MRS}_w v_\beta}{v_w J}.$$

By this is the substitution effect, hence (14).

Proof of Lemma 1:
Differentiating ρ defined by (15) with respect to q along an indifference curve gives

$$\left.\frac{\partial \rho}{\partial q}\right|_{v = v^0} = -\frac{1}{(v_w)^2} \times \left(v_w \cdot \left.\frac{\partial v_\beta}{\partial q}\right|_{v = v^0} - v_\beta \cdot \left.\frac{\partial v_w}{\partial q}\right|_{v = v^0} \right).$$

Applying the fact that the slope of an indifference curve is MRS, we have

$$\left.\frac{\partial v_\beta}{\partial q}\right|_{v = v^0} = v_{q\beta} + v_{w\beta} \cdot \frac{\partial w}{\partial q} = v_{q\beta} + v_{w\beta} \cdot \mathrm{MRS}$$

and

$$\left.\frac{\partial v_w}{\partial q}\right|_{v = v^0} = v_{qw} + v_{ww} \cdot \frac{\partial w}{\partial q} = v_{qw} + v_{ww} \cdot \mathrm{MRS}.$$

By the definition of MRS in (7), its partial derivatives with respect to β and w are, respectively, given by

$$\mathrm{MRS}_\beta = -\frac{v_{q\beta} v_w - v_q v_{w\beta}}{(v_w)^2} = -\frac{1}{v_w}[v_{q\beta} + v_{w\beta} \cdot \mathrm{MRS}]$$

and

$$\mathrm{MRS}_w = -\frac{v_{qw} v_w - v_q v_{ww}}{(v_w)^2} = -\frac{1}{v_w}[v_{qw} + v_{ww} \cdot \mathrm{MRS}].$$

Using all of the above expressions and (14), we have

$$\left.\frac{\partial \rho}{\partial q}\right|_{v = v^0} = \frac{v_w \mathrm{MRS}_\beta - v_\beta \mathrm{MRS}_w}{v_w} = J \cdot \frac{\partial q^{\mathrm{C}}}{\partial \beta}.$$

The lemma follows immediately from this equation and the fact that $J < 0$.

APPENDIX B

The result in (16) indicates that the sign of the substitution effect of a general RS increase in price risk is governed by the behavior of the function $\rho = -v_\beta/v_w$ with respect to q along an indifference curve. In this appendix we advance the argument that it is reasonable to assume that ρ increases in q along an indifference curve.

Consider Figure B1. Both indifference curves in the diagram have the *same* expected utility level v^0, but correspond to two different values of the RS risk parameter, β and $\beta' = \beta + \Delta\beta$. Because the firm is risk averse, the β'-indifference curve is

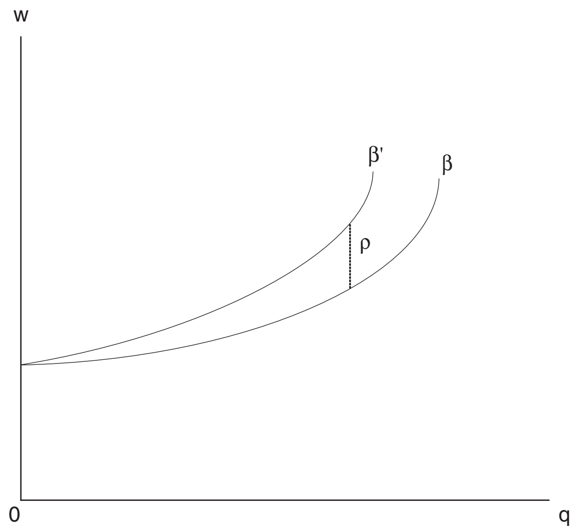

Figure B1. Graphical illustration of ρ.

everywhere higher than the β-indifference curve except that they are tangent at $q = 0$ (both have slope zero at $q = 0$). For any given q, the vertical distance Δw between the two indifference curves satisfies the equation

$$v(q, w + \Delta w, \beta + \Delta\beta) = v(q, w, \beta).$$

Thus, Δw is the increment in sure wealth required to compensate the firm for an increment in the riskiness of p^* (i.e. $\Delta\beta > 0$). In the limit, as $\Delta\beta$ goes to zero, the ratio $\Delta w/\Delta\beta$ becomes $-v_\beta/v_w$, namely ρ. From (11), ρ is also the derivative of the wealth requirement function with respect to the RS risk parameter, i.e. ρ gives the minimum sure wealth required to compensate the firm for a small increase in RS risk. In Figure B1, ρ corresponds to the vertical distance between the two indifference curves.

Since the firm is risk averse, ρ is increasing immediately to the right of $q = 0$. It follows that ρ is either non-monotone or uniformly increasing in q. Intuition suggests that it is plausible to assume that ρ is increasing in q since an increase in the riskiness of p^* increases the riskiness of random revenue p^*q *more* for larger values of q along the indifference curve $v = Eu(w + p^*q) = v^0$. Hence, the larger the value of q the larger the compensation in w the firm would require to compensate for the increase in the riskiness of p^*. If ρ is increasing, the substitution effect of an increase in price risk is negative. If ρ is non-monotone, the substitution effect does not have a uniform sign. A negative substitution effect is consistent with the finding in the fourth section for the special case where the increase in price risk takes the multiplicative form, it is also consistent with the findings in Davis (1989) and Hadar and Seo (1992).

NOTES

1. The term 'Slutsky-like' equation has been used in the literature to distinguish a decomposition in which the terms in the equation do not correspond to any identifiable derivatives. This is in contrast to the Slutsky equation in which the two terms can be, respectively, identified as an income derivative and a compensated derivative.
2. For any $\alpha \in [0, 1]$, $v(\alpha q_1 + (1-\alpha)q_2, \alpha w_1 + (1-\alpha)w_2, \beta) = Eu(\alpha(w_1 + p^*q_1) + (1-\alpha)(w_2 + p^*q_2)) \leqslant E[\alpha u(w_1 + p^*q_1) + (1-\alpha)u(w_2 + p^*q_2)] = \alpha v(q_1, w_1, \beta) + (1-\alpha) v(q_2, w_2, \beta)$, where the inequality follows from concavity of u. Hence, $v(q, w, \beta)$ is concave in (q, w).
3. Note that an interior solution for (5) exists if and only if $\bar{p} > c'(0)$. This is because the slope of any (convex) indifference curve on the vertical axis ($q = 0$) is equal to zero while the slope of the concave constraint in (5) on the vertical axis is $\bar{p} - c'(0)$ and it is implied that the tangency point must be interior provided that $\bar{p} > c'(0)$.
4. Throughout the paper, subscripts to a function denote partial derivatives.
5. Throughout the paper, the term 'indifference curve' refers to the derived utility function v.
6. Concavity of v guarantees that MRS is increasing in q and w along any indifference curve.
7. When the derivative is with respect to $\bar{p}$, an equation similar to (10) gives the Slutsky equation for an increase in expected price. This case was first derived by Sandmo (1971) using a different technique.
8. See Sandmo (1971, p. 69).
9. This is a special case of an RS increase in risk first popularized by Sandmo (1971). In this representation, β is a multiplicative shift parameter of risk.
10. The direction of change in expected wealth (namely, whether A is above or below B in Figure 2) cannot be determined without further assumptions about preferences.
11. Both effects work to decrease the firm's output. Specifically, the income effect is negative given decreasing absolute risk aversion and the substitution effect is negative assuming that ρ increases in q along an indifference curve.
12. Sandmo (1971) found that a profit tax increases a competitive firm's output under price uncertainty if relative risk aversion is increasing. Partial risk aversion replaces relative risk aversion when initial wealth is included in the model. Sandmo did not consider the income and substitution effects of profit tax. See Menezes and Hanson (1970) or Zeckhauser and Keller (1970) for partial risk aversion.

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