International Journal of Game Theory (1993) 21: 395-404

Solvable States in Stochastic Games

NICOLAS VIEILLE
L.S.T.A., Tour 45-55 3ème étage, Université Paris 6, 4, place Jussieu, 75005 Paris, France

Abstract: This paper deals with undiscounted stochastic games.[1] As in Thuijsman-Vrieze [9], we consider specific states, which we call solvable. The existence of such states in every game is proved in a new way. This proof implies the existence of equilibrium payoffs in stochastic games with at most 3 states. On an example, we relate our work to the construction of Thuijsman and Vrieze.

Introduction

This paper addresses the problem of the existence of equilibrium payoffs in non zero-sum undiscounted stochastic games. It is strongly influenced by the work of Thuijsman and Vrieze [9].

The aim is to define a subset of states, such that every game starting from this class has equilibrium payoffs and it is moreover possible to design simple ε-equilibrium strategies. The definition we suggest captures the main ideas of Thuijsman and Vrieze but is more formalized, and expressed only in terms of stationary strategies of both players. It is then shown, in section 2, that, for every stochastic game, the set of solvable states is not empty. Our proof of existence does not proceed along the same lines as the proof of Thuijsman and Vrieze. Some of the tools used are similar, but not used in the same context. Our method yields some corollaries which enable us, in section 3, to deduce the existence of equilibrium payoffs in stochastic games with at most three states. Furthermore, our construction does not induce the same set as the method of Thuijsman and Vrieze, as is shown on an example at the end of the paper.

1 Notations and Definitions

We briefly introduce our framework. We are interested in non-zero sum, two-person undiscounted stochastic games.

[1] I would like to thank Frank Thuijsman, for pointing out errors in an earlier version of the proof, and a referee, for a careful reading.

0020-7276/93/4/395-404 $ 2.50

Setup

K is a finite set of states. We will, throughout the paper, assume that the actions set of each player is finite, in each state. We may thus assume, *w.l.o.g*, that the action set of each player does not depend on the state. These will be denoted by I and J.

$g: K\times I\times J\to \mathbb{R}^2$ is the stage payoff. The first coordinate (*resp.* the second) of g is the payoff of Player 1 (*resp.* Player 2). (g will also denote its mixed extension to the set of mixed moves).

For any finite set S, let us denote by $\Delta(S)$ the set of probability distributions over S.

$P: K\times I\times J\to\Delta(K)$ is the transition function on the set of states: $P_{k,i,j}(k')$ is the probability that the next state will be k' if the actual state is k and if the actions selected by the players are i and j.

Recall that Γ, the inifinitely repeated game, is played as follows. Some initial state k_1 is given.

At any stage $n\geq 1$, the information available to the players is the current state k_n and the whole past history $(k_1, i_1, j_1, \ldots, k_{n-1}, i_{n-1}, j_{n-1})$. Both players choose simultaneously and independently $i_n\in I$ and $j_n\in J$. The payoff received is $g_n = g(k_n, i_n, j_n)$, and k_{n+1} is selected according to $P_{(k_n, i_n, j_n)}$. The game proceeds then to the next stage.

Denote by $H_n = (K\times I\times J)^{n-1}\times K$ the set of histories up to stage n, and H_∞ the set of plays, *i.e.* the set of histories of infinite length. In this framework, a strategy of Player 1 is a map

$$\sigma: \bigcup_{n\geq 1} H_n \to \Delta(I).$$

Similarly, a strategy of Player 2 is a map

$$\tau: \bigcup_{n\geq 1} H_n \to \Delta(J).$$

Given an initial state k, any pair (σ, τ) of strategies induces a probability distribution $\mu_{(k,\sigma,\tau)}$ over H_∞. The expectation with respect to $\mu_{(k,\sigma,\tau)}$ will be denoted by $E_{(k,\sigma,\tau)}$.

A stationary strategy is a strategy that depends on the history h_n only through the actual state k_n.

The set of stationary strategies of Player 1 (*resp.* Player 2), is then $\Delta(I)^K$ (resp. $\Delta(J)^K$). Note that a profile (x, y) of stationary strategies induce a Markov chain over K.

Let u be any real-valued function on K, and Q be a probability distribution on K. Throughout the paper, we will denote by Qu the expectation of u under Q.

Payoffs and Equilibrium payoffs

We now define two ways of evaluating a stream $(g_n)_{n\in\mathbb{N}}$ of stage payoffs.

Discounted games: given a initial state k, the λ-discounted payoff induced by a pair (σ, τ) of strategies is

$$\gamma_\lambda(k, \sigma, \tau) = E_{(k,\sigma,\tau)}\left(\sum_{n=1}^{\infty} \lambda(1-\lambda)^n g_n\right).$$

This is a well-defined function over the product of the strategy spaces of the players. For this criteria, the existence of stationary equilibrium strategies is well-known. This was proved by Shapley [7] in the zero-sum case, then by Fink [2] and Takahashi [8] in the undiscounted case.

Undiscounted games: let $\bar{\gamma}_n$ denote the average payoff in the first n stages:

$$\bar{\gamma}_n(k, \sigma, \tau) = E_{(k,\sigma,\tau)}\left(\frac{1}{n}\sum_{l=1}^{n} g_l\right).$$

Let us denote by

$$\bar{\gamma}_n(\sigma, \tau) = \bar{\gamma}_n(k, \sigma, \tau)_{k \in K}$$

the vector of these payoffs. This kind of notation will be used repeatedly in various contexts. For instance, for any profile (x, y) of stationary strategies, $g(x, y)$ will stand for $(g^1(k, x, y), g^2(k, x, y))_{k \in K}$, and $P_{(x,y)}$ will denote the transition function on K induced by (x, y).

One may wish to associate to any couple (σ, τ) the limit of $\bar{\gamma}_n(\sigma, \tau)$. This is not possible, since this sequence does not converge in general.

We thus define the set of ε-equilibrium payoffs of the game starting from k as follows:

$$\begin{aligned} E_\varepsilon^k = \{ & d \in \mathbb{R}^2,\ \exists(\sigma_\varepsilon, \tau_\varepsilon),\ \exists N,\ \forall n \geq N, \\ & d - \varepsilon \leq \bar{\gamma}_n(k, \sigma_\varepsilon, \tau_\varepsilon) \leq d + \varepsilon \end{aligned}$$

and

$$\begin{aligned} & \forall \sigma,\ \bar{\gamma}_n^1(k, \sigma, \tau_\varepsilon) \leq \bar{\gamma}_n^1(k, \sigma_\varepsilon, \tau_\varepsilon) + \varepsilon \\ & \forall \tau,\ \bar{\gamma}_n^2(k, \sigma_\varepsilon, \tau) \leq \bar{\gamma}_n^2(k, \sigma_\varepsilon, \tau_\varepsilon) + \varepsilon \}. \end{aligned}$$

The set of equilibrium payoffs is defined as $E^k = \bigcap_{\varepsilon > 0} E_\varepsilon^k$ (see Mertens-Sorin-Zamir [6]).

Solvable subsets

We need a few more comments, in order to be able to introduce formally the solvable subsets.

Let (x, y) be stationary strategies. Then the sequence $\bar{\gamma}_n(x, y)$ has a limit, when n goes to infinity (this is a standard ergodic theorem for finite Markov chains). Thus, given any initial state k, the undiscounted payoff $\gamma(k, x, y)$ is well-defined. Furthermore, we have the following property.

Lemma 1: let R by an ergodic subset of the Markov chain induced by (x, y). Then $\gamma(k, x, y)$ does not depend on $k \in R$.

This common payoff will be written $\gamma_R(x, y)$.

Proof: Let $\tilde{P}_{(x,y)}$ denote the restriction of $P_{(x,y)}$ to R, and

$$Q = \lim_{n\to\infty} \frac{1}{n} \sum_{n=1}^{n} \tilde{P}^m_{(x,y)}.$$

Then $\tilde{P}_{(x,y)}Q = Q\tilde{P}_{(x,y)} = Q$. Thus $\forall k \in R$, $\gamma(k, x, y) = Q_k g(x, y)$, where Q_k is the kth row of Q. Q_k is a measure over R, invariant *w.r.t* to $\tilde{P}_{(x,y)}$. Since R is ergodic, such a measure is unique. Hence $\gamma(k, x, y)$ is independent of $k \in R$. □

We associate to Γ two sequences of zero-sum games, by retaining only the payoff function of Player 1 or Player 2: G^1 (resp. G^1_λ) is the undiscounted (resp. discounted) zero-sum game where Player 1 maximizes his own payoff (and Player 2 minimizes the payoff of Player 1). G^2 and G^2_λ are the zero-sum games where Player 2 maximizes his payoff. Denote by v^i (*resp.* v^i_λ) the value of G^i (*resp.* G^i_λ).

Recall that $v^i = \lim_{\lambda\to 0} v^i_\lambda$ (see Mertens-Neyman [5]).

Definition 1: A subset R of K is solvable if there exists stationary strategies (x, y) such that:

- *R is an ergodic subset of the Markov chain induced by (x, y);*
- *$\forall k \in R$, $\forall j \in J$, $P_{(k,x,j)} v^2 \le \gamma^2_R(x, y)$;*
- *$\forall k \in R$, $\forall i \in I$, $P_{(k,i,y)} v^1 \le \gamma^1_R(x, y)$.*

A state is solvable if it belongs to some solvable subset of K. The set of solvable states is denoted by S.

2 Existence of Solvable States

In this section, we prove the following property.

Proposition 1: Every finite stochastic game has solvable states.

Proof: there is a simple proof (see Remark 2 below). However, we give here a more intricate one that enables us to deduce Remark 1 below. This remark is a key property to get existence results for equilibria in "small" games.

Let $K^1 = \mathrm{Argmax}_K v^1$ be the subset of K where the value for Player 1 is maximal. Take $\varepsilon > 0$ such that $v^1(k) + \varepsilon < v^1(k')$ as soon as k is in $(K^1)^c$ and k' is in K^1, and replace, in Γ, each state k outside K^1 with an absorbing state with payoff $(v^1(k) + \varepsilon, v^2(k))$. We thus get a well-defined stochastic game $\overline{\Gamma}$. All quantities in $\overline{\Gamma}$ will be overlined, in order not to get confused between Γ and $\overline{\Gamma}$.

Let $x_\lambda \in \Delta(I)^K$ be an optimal (discounted) stationary strategy of Player 1 in G^1_λ. Thus,

$$\forall y \in \Delta(J)^K,\ \lambda g^1(x_\lambda, y) + (1-\lambda) P_{(x_\lambda, y)} v^1_\lambda \geq v^1_\lambda. \tag{1}$$

The inequality (1) has to be interpreted as a vector inequality, that holds for each component.

Let x be a limit point of x_λ, as λ goes to zero, and let $y \in \Delta(J)^K$ be a (undiscounted) best response of Player 2 to x in Γ (for the existence of y, see Blackwell [1]). It is clear that, for every state k, and every possible action j of Player 2, $P_{(k,x,j)} \gamma^2(x, y) \leq \gamma^2(k, x, y)$.

This obviously yields

$$P_{(k,x,j)} v^2 \leq \gamma^2(k, x, y). \tag{2}$$

We will use the following well-known fact. Consider any stochastic game. Let u be a real-valued function defined on K, and assume, for some $\lambda > 0$, that the (vector) inequality

$$\lambda g^1(x, y) + (1-\lambda) P_{(x,y)} u \geq u$$

holds for a given profile (x, y). Then

$$\lambda^1_\lambda(x, y) \geq u.$$

Consider $\bar{\Gamma}$, and let u be defined as follows: $u(k) = v^1_\lambda(k)$ if $k \in K^1$, $u(k) = v^1(k) + \varepsilon$ otherwise. For $k \notin K^1$, we obviously have

$$\lambda \bar{g}^1(k, x_\lambda, y) + (1-\lambda) \bar{P}_{(k, x_\lambda, y)} u = u(k),$$

since k is absorbing. On the other hand, $u \geq v^1_\lambda$, as soon as λ is small enough. We thus deduce, from (1),

$$\lambda \bar{g}^1(x_\lambda, y) + (1-\lambda) \bar{P}_{(x_\lambda, y)} u \geq u.$$

This yields the vector inequality $\bar{\gamma}^1_\lambda(x_\lambda, y) \geq u \geq v^1_\lambda$.

Let $\mathscr{R}$ denote the set of the subsets of K, which are ergodic *w.r.t* $\bar{P}_{(x,y)}$. An ergodic set $R \in \mathscr{R}$ consists either of a state $k \notin K^1$, or is an ergodic set for $P_{(x,y)}$ (and is then a subset of K^1).

We now prove that there is an ergodic set $R \subset K^1$ where the payoff for Player 1 is individually rational.

Lemma 2: $\exists R \in \mathscr{R}$, $R \subset K^1$, *such that* $\gamma^1_R(x, y) \geq Max_K v^1$.

Proof: the proof proceeds as in Vrieze and Thuijsman ([9])

$$\bar{\gamma}^1_\lambda(x_\lambda, y) = \lambda \bar{g}^1(x_\lambda, y) + (1-\lambda) \bar{P}_{(x_\lambda, y)} \bar{\gamma}^1_\lambda(x_\lambda, y).$$

Thus $\bar{\gamma}^1_\lambda(x_\lambda, y) = Q(\lambda)\bar{g}^1(x_\lambda, y)$, where

$$Q(\lambda) = \lambda(I - (1-\lambda)\bar{P}_{(x_\lambda, y)})^{-1}. \tag{3}$$

We obviously have

$$(I - (1-\lambda)\bar{P}_{(x_\lambda, y)})Q(\lambda) = \lambda I. \tag{4}$$

Let Q denote a limit point of $Q(\lambda)$, as λ goes to 0. We thus have $Q = \lim_{n\to\infty} Q(\lambda_n)$, for a specific sequence $(\lambda_n)_{n\in\mathbb{N}}$. Furthermore, $\bar{P}_{(x,y)} = \lim_{\lambda\to 0}\bar{P}_{(x_\lambda, y)}$. From the equality 4, we deduce $Q = \bar{P}_{(x,y)}Q$. For any $k\in K$,

$$\lim_{n\to\infty} \bar{\gamma}^1_{\lambda_n}(k, x_{\lambda_n}, y) = Q_k\bar{g}^1(x, y),$$

and Q_k is invariant *w.r.t* to $\bar{P}_{(x,y)}$.

For any $R\in\mathscr{R}$, let μ_R denote the unique distribution on R, invariant *w.r.t* to $\bar{P}_{(x,y)}$.

It is well-known that the set of invariant measures is the convex hull of $(\mu_R)_{R\in\mathscr{R}}$, thus

$$Q_k = \sum_{R\in\mathscr{R}} \alpha^k_R \mu_R,$$

where $\alpha^k_R \geq 0$ and $\sum_R \alpha^k_R = 1$ (see Kemeny-Snell [4]).

Since $\bar{\gamma}^1_{\lambda_n}(k, x_{\lambda_n}, y) \geq v^1_{\lambda_n}(k)$, we get $\lim_{n\to\infty}\bar{\gamma}^1_{\lambda_n}(k, x_{\lambda_n}, y) \geq v^1(k)$. Furthermore, $\bar{\gamma}_R(x, y) = \mu_R\bar{g}(x, y)$. Thus

$$\lim_{\lambda\to\infty} \bar{\gamma}^1_\lambda(k, x_\lambda, y) = \sum_{R\in\mathscr{R}} \alpha^k_R \bar{\gamma}^1_R(x, y).$$

Therefore,

$$\exists R_0\in\mathscr{R}, \bar{\gamma}^1_{R_0}(x, y) \geq v^1(k).$$

Whenever $R\in\mathscr{R}$ is an ergodic set of $P_{(x,y)}$, then $\gamma^1_R(x, y) = \bar{\gamma}^1_R(x, y)$. If R is a singleton $\{k'\}$, where $k'\notin K^1$, and $k\in K^1$, then $\bar{\gamma}^1_R(x, y) = u(k')$. By assumption, $u(k') = v^1(k') + \varepsilon < v^1(k)$.

Thus, $R_0 \subset K^1$. □

We are now able to get the result. Take a initial state k in K^1, and any ergodic set such that $\gamma^1_R(x, y) \geq v^1(k) = \mathrm{Max}_K v^1$. We claim that R is solvable.

We have already noticed that, for any $k\in R$, and any $j\in J$, $P_{(k,x,j)}v^2 \leq \gamma^2_R(x, y)$.

Since $\gamma^1_R(x, y) \geq \mathrm{Max}_K v^1$, we have $P_{(k,i,y)}v^1 \leq \gamma^1_R(x, y)$, for any $k\in R$ and any $i\in I$. In addition, we have inequality (2).

Thus R is solvable. □

Remark 1: $\mathrm{Max}_S v^j = \mathrm{Max}_K v^j$, $j = 1, 2$.

Indeed, the equality $\mathrm{Max}_S v^1 = \mathrm{Max}_K v^1$ follows from the previous proof. The equality for Player 2 may be proved similarly.

Remark 2: if one is only interested in the existence of solvable states, one may give a more direct proof. There is no need to introduce the auxiliary game $\bar{\Gamma}$.

Let (x, y) be as above: $x = \lim_{\lambda \to 0} x_\lambda$, and y is a best response of Player 2 to x.

Let $\mathscr{R}$ be the set of ergodic sets for (x, y), and let k be a state such that $v^1(k) = \mathrm{Max}_K v^1$.

Just as above, $\lim_{\lambda \to 0} \gamma^1(k, x_\lambda, y)$ is in the convex hull of $\gamma_R^1(x, y)_{R \in \mathscr{R}}$.

Since $\gamma_\lambda^1(k, x_\lambda, y) \geq v_\lambda^1(k)$, this yields the existence of an ergodic set $R \in \mathscr{R}$ such that

$$\gamma_R^1(x, y) \geq \mathrm{Max}_K v^1.$$

It is straightforward to check that R is indeed solvable.

However, I see no reason why we should have $\mathrm{Max}_R v^1 = \mathrm{Max}_K v^1$, nor $\mathrm{Max}_R v^2 = \mathrm{Max}_K v^2$. Thus it does not seem possible to deduce the former Remark 1 from this shorter proof.

3 Solvable States and Equilibrium Payoffs

Proposition 2: If k is solvable, then $E^k \neq \emptyset$.

Proof: Let R denote a solvable class that contains k, and (x, y) be the associated strategies. Let $d = \gamma_R(x, y)$. We shall give an informal proof that $d \in E^k$. For a complete one, we refer the reader to Thuijsman and Vrieze [9]. This is achieved by constructing ε-equilibrium strategies (σ, τ), such that with high probability under (σ, τ), the moves are selected at each stage according to (x, y): on "the equilibrium path", (x, y) is played.

Design σ and τ as follows: Player 1 (resp. Player 2) plays according to x (resp. to y) unless he suspects "a deviation of Player 2 from y" (resp. of Player 1 from x), in which case he plays a punishment strategy.

Since the information available to the players is only the past history, we have to make clear what we mean by deviation. First remark that if, at some stage, Player 2 selects a move that is not in the carrier of y, then Player 1 knows for sure that Player 2 has deviated from y.

Otherwise, the play does not leave R. Then Player 1 may compute the empirical mean of the moves selected by Player 2. This distribution must become close to y, as time goes (strong law of large numbers). If, far from the beginning of the game, this is not the case, then Player 1 may reasonably infer that Player 2 has deviated.

The fact that (σ, τ) is a couple of ε-equilibrium strategies is deduced from the following insights:

– with high probability, the average payoff induced by (σ, τ) is asymptotically close to d (under (σ, τ), punishment may occur, but only with a low probability).

– a detectable deviation of Player 2 is punished and, since

$$\gamma_R^2(x, y) \geq P_{(k,x,j)} v^2,$$

not profitable.

– a non-detectable deviation induces a payoff close to d and, therefore, can only be a little profitable. □

We now define as "reduction" of Γ. For each solvable state k, let $d(k) \in E^k$. Let Γ^* denote the stochastic game deduced from Γ by replacing each solvable state k with an absorbing state with payoff $d(k)$. Denote by E^* the set of equilibrium payoffs of Γ^*.

Proposition 3: $\forall k \in K$, $E^{*k} \subset E^k$.

Remark: this kind of result is well-known in other contexts (see [3]).

Proof: For any solvable state k_0 of Γ (*i.e.* absorbing state in Γ^*), let $(\bar{\sigma}_{k_0}, \bar{\tau}_{k_0})$ be ε-equilibrium strategies in Γ which induce a payoff ε-close to $d(k_0)$.

Let now $d \in E^{*k}$, and (σ^*, τ^*) ε-equilibrium strategies in Γ^* with payoff ε-close to d. We define a couple (σ, τ) of strategies in Γ as follows:

- as long as the play doesn't reach any solvable state, $\sigma = \sigma^*$ and $\tau = \tau^*$.
- as soon as the play reaches a solvable state k_0, play according to $\bar{\sigma}_{k_0}$ and $\bar{\tau}_{k_0}$.

Given the properties of (σ^*, τ^*) and $(\bar{\sigma}, \bar{\tau})$, it is easy to check that (σ, τ) are ε-equilibrium strategies in Γ (when k is the initial state), and induce an average payoff 2ε-close to d. Thus $d \in E^k$. □

Corollary 1: $\# \mathscr{S} = 1 \Rightarrow \forall k, E^k \neq \emptyset$.
$\# \mathscr{S} = \# K - 1 \Rightarrow \forall k, E^k \neq \emptyset$.

Proof: we shall use the previous result and consider the game Γ^*. We distinguish the two cases.

$\# \mathscr{S} = 1$: then Γ^* has only one absorbing state k_0. Furthermore, the absorbing payoff $d(k_0)$ is such that $d(k_0) \geq v(k_0) \geq \mathrm{Max}_K v$. We need the following lemma

Lemma 3: If (x, y) are completely mixed strategies, then the Markov chain induced by (x, y) is absorbing, in Γ^*.

Proof: otherwise there exists a subset $K_0 \subset K \setminus \{k_0\}$, such that K_0 does not communicate with the other states:

$$\forall k \in K_0, \ \forall (i, j) \in I_k \times J_k, \ P_{k,i,j}(K_0) = 1.$$

The restriction of P to K_0 defines a stochastic game. That has at least one solvable set. Thus there exists solvable subsets of K_0. This is in contradiction with the assumption that k_0 is the only solvable state of Γ. □

Fix a initial state k, and take (x, y) completely mixed. Define

$$\theta = \inf\{n, k_n = k_0\}.$$

$\theta < \infty$, $\mu_{(k,x,y)}$- p.s. Thus $\exists N$, $\mu_{(k,x,y)}(\theta \leq N) \geq 1 - \varepsilon$. Define now σ as follows: play according to x until stage N, and punish Player 2 after stage N (if the absorption did not occur).

Define τ similarly. Then (σ, τ) is an ε-equilibrium: any absorbing deviation does not change the payoffs received by the players. Any non-absorbing deviation is prevented by the punishment threat.

$\# \mathscr{S} = \# K - 1$: Γ^* is a game with absorbing states (all states but one are absorbing). Denote by k_0 the only non-absorbing state. Thuijsman and Vrieze [10] proved that $E^{*k_0} \neq \emptyset$. This ends the proof of the corollary. □

Corollary 2: If $\# K \leq 3$, *then* $E \neq \emptyset$.

Proof: since $\mathscr{S} \neq \emptyset$, then the assumptions of Corollary 3 are obviously satisfied. □

4 An Example

We recall briefly the method of proof of Thuijsman and Vrieze. Let (x_λ, y_λ) be stationary equilibrium strategies in the discounted game with discount factor λ. Let (x, y) be a limit point of (x_λ, y_λ), as the discount factor goes to zero. Then Thuijsman and Vrieze show that at least one ergodic subset of the Markov chain induced by (x, y) is solvable.

In the following example, the only ergodic state of (x, y) is k_2, for any limit point (x, y) of any sequence of stationary discounted equilibrium. However, both k_1 and k_2 are solvable, and generated by our construction.

Consider the following two-state game, where Player 1 is dummy, and where k_2 is absorbing. The payoffs and the transitions are given by the two matrices:

$$\begin{array}{ll} (1/k_2 & 0/k_1) \\ & (0/k_2) \end{array}$$

Remark: the notation $(1/k_2)$ means that the immediate payoff of Player 2 is 1 and that the next state of the game is k_2.

So that $v_\lambda^2(k_1)=\lambda$ and $v_\lambda^2(k_2)=0$.

Then, obviously, $y_\lambda=(1,0)$: Player 2 plays Left with probability 1. The only ergodic state of the Markov chain induced by y is the absorbing state k_2.

However the stationary strategy $\tilde{y}=(0,1)$ of Player 2 is optimal the undiscounted game G^2. $\{k_1\}$ is ergodic for the Markov chain induced by $\tilde{y}$ and is clearly solvable.

References

[1] D. Blackwell. Discrete dynamic programming. *Annals of Mathematical Statistics,* 331:719–726, 1962.

[2] A. M. Fink. Equilibrium in a stochastic n-person game. *J. Sci. Hiroshima Univ. Ser. A,* 28:89–93, 1964.

[3] F. Forges. Infinitely repeated games of incomplete information: Symmetric case with random signals. *International Journal of Game Theory,* 11:203–213, 1982.

[4] J. G. Kemeny and J. L. Snell. *Finite Markov Chains.* Van Nostrand, Princeton, 1960.

[5] J. F. Mertens and A. Neyman. Stochastic games. *International Journal of Game Theory,* 10:53–66, 1981.

[6] J. F. Mertens, S. Sorin, and S. Zamir. *Repeated Games.* Forthcoming.

[7] L. S. Shapley. Stochastic games. *Proceedings of the NAS,* 39:1095–1100, 1953.

[8] M. Takahashi. Equilibrium points of stochastic non-cooperative n-person games. *Jour. Sci. Hiroshima Univ. Ser A.I,* 28:95–99, 1964.

[9] F. Thuijsman and O. J. Vrieze. Easy initial states in stochastic games. In *Stochastic Games and Related Topics, in Honor of L. S. Shapley,* pages 85–100, Dordrecht, 1991. Kluwer Academic Publishers.

[10] O. J. Vrieze and F. Thuijsman. On equilibria in stochastic games with absorbing states. *International Journal of Game Theory,* 18:293–310, 1989.

Received March 1992
Revised version December 1992