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On the equilibrium properties of locational sorting models

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Abstract

Important to many models of location choice is the role of *local interactions* or *spillovers*, whereby the payoffs from choosing a location depend in part on the number or attributes of other individuals or firms that choose the same or nearby locations in equilibrium. This paper develops the equilibrium properties of a broadly applicable and readily estimable class of sorting models that allow location decisions to depend on both fixed local attributes (including unobserved attributes) and local interactions, describes the conditions under which equilibria exist and are unique, and provides a test for uniqueness in empirical analyses of sorting equilibrium.

Keywords: Local spillovers; Social interactions; Economic geography; Endogenous sorting; Discrete choice models; Agglomeration; Congestion; Natural advantage; Random utility

1. Introduction

Models of location choice—whether of firms or households, within or across cities—have long been central in urban and public economics. From the inter-jurisdictional sorting models of Tiebout [23] to the models of segregation developed by Schelling [19,20] to the "new economic geography" of Fujita et al. [10], a central feature of these models has

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been the role of *local interactions* or *spillovers*, whereby the payoffs from choosing a location depend in part on the number or attributes of other individuals or firms that choose the same or nearby locations in equilibrium. In some cases, these local spillovers operate through anonymous channels, with payoffs depending upon simply the number of other individuals or firms selecting the same location. In other circumstances, the attributes of one's neighbors (e.g., race, income, or education in the case of individuals, and industry classification in the case of firms) might matter as well.

It is the interplay between these sorts of spillovers and the *natural advantages* embedded in the landscape of alternative locations that can explain, at a regional level, the geographic and size distribution of cities, and at an urban level, the stratification of households across communities on the basis of income, education, and race, neighborhood density patterns, ethnic enclaves, ghettos, and problems of inner-city decay and suburban sprawl. Recent empirical work aimed at distinguishing the magnitude of local interactions has focused on subjects as diverse as crime in cities [12], racial segregation [2], inter-jurisdictional sorting related to schooling [1,9], human capital spillovers in the labor market [17], the general equilibrium effects of environmental policy [22,24], welfare participation [5], unemployment spells [25], growth and development [8,15] and agglomeration economies in firm locations and investment [13], among many others.

In light of this growing empirical literature, this paper develops the equilibrium properties of a broadly applicable and readily estimable class of sorting models that includes local spillovers of the sort found in the contexts described above. Building on the random utility framework of McFadden [16], the key feature of this class of models is that it allows the location decision to depend on both fixed local attributes (including unobserved attributes) and local interactions or spillovers. In related work [3], we develop an estimator for this class of models that identifies local spillovers even in the presence of unobservable local attributes. In this paper, we first define and prove the existence of an equilibrium for this class of sorting models and explore the uniqueness properties of that equilibrium. The estimation strategy we develop in [3] is based on first-order conditions derived from agents' optimal location decisions, and at no point requires a unique equilibrium for estimation purposes.[1]

Uniqueness becomes a valuable feature of the sorting problem, however, when the goal is to simulate the equilibrium responses of individuals to a large policy change. In this context, the presence of multiple potential equilibria (while an unavoidable feature of reality in many problems) makes it difficult to draw strong policy or welfare conclusions. When the choice environment in question contains more than two alternatives, we show that uniqueness can only be guaranteed analytically in the presence of a congestion effect. In the presence of an agglomeration effect, whether or not a unique equilibrium obtains depends explicitly on the particular data environment in a manner that is not possible to characterize analytically. This result is not new to the literature. Brock and Durlauf [7], for example, describe the equilibrium properties of a similar class of models, focusing on a few extreme cases where simple conclusions regarding the number of equilibria can be drawn analyti-

[1] This is in contrast to pure likelihood-based algorithms for estimating the utility effects of spillovers, which require uniqueness or an arbitrary equilibrium selection rule when there are more than one.

cally, but providing only limited intuition for more general cases.[2] To extend an analysis of the equilibrium properties of this class of models to more general cases, a simulation-based approach is needed. In particular, given the model's parameters and a characterization of the process that generates the exogenous data, we calculate the likelihood that a unique sorting equilibrium arises. Using such simulations, we extend the intuition in [7] to more general cases, exploring how the primitives of the economic environment affect the likelihood that the sorting equilibrium is unique.

In demonstrating that the computational algorithm used in these simulations always finds multiple equilibria whenever multiple equilibria exist, this paper makes a second contribution. Namely, having estimated the parameters of the model, this computational algorithm can be used by researchers to test for uniqueness in a particular application. The availability of such a test is of practical importance if one wishes to conduct counterfactual equilibrium simulations, e.g., determining the welfare effects of a non-marginal policy change, as the algorithm that we propose provides a simple method for determining whether the equilibrium is unique in any counterfactual environment.

The paper proceeds as follows. In Section 2, we describe the equilibrium model of locational sorting and demonstrate the existence of an equilibrium with a simple application of Brouwer's fixed-point theorem. In Section 3, we describe a set of sufficiency conditions for uniqueness of that equilibrium, and illustrate that the determination of uniqueness for a particular application will depend upon the size of any agglomeration effect and the specific features of the data set. In Section 4, we describe a computational algorithm for determining the presence of multiple equilibria in our class of sorting models and use it to illustrate how the maximum agglomeration effect that can sustain a unique equilibrium (or "uniqueness threshold") varies with observable features of simulated data. Section 5 extends the intuition of that discussion to broader forms of local spillovers in which individuals care about the type (as opposed to just the number) of their neighbors. Section 6 concludes.

2. Equilibrium in a model of locational sorting

This section sets out a model of locational sorting and demonstrates that an equilibrium exists. To help fix ideas, we restrict attention to a model of residential location choice with anonymous local spillovers that can have a positive (agglomeration) or negative (congestion) effect on utility.[3] A straightforward extension of this framework could be used

[2] The principal difference between the models employed by Brock and Durlauf [7] and those we use here and in [3] is in the treatment of unobservable choice attributes that are correlated across individuals. We find such unobservables to be an important reality in most empirical analyses, and determine in [3] that failing to control for them when they are in fact present will lead to biased estimates of preferences. For the purposes of discussing the equilibrium properties of these models, the distinction between observable and unobservable choice attributes is not important. Their presence does, however, mean that simplifying assumptions of the sort used by [7] to discuss uniqueness (e.g., that all choices are identical in their attributes not determined by sorting) limit the application of their results.

[3] The model could also easily be adapted to study the spatial sorting of firms, where a profit rather than utility function would govern behavior.

to study spillovers that depend upon both the number and attributes of other households choosing the same location, as would arise in a model of racial segregation or sorting due to differences in local public goods provision across communities. In describing the uniqueness properties of the model below, we provide intuition for how the results presented here extend to this more general case.

Consider a setting in which each individual i chooses a location (indexed by j) in order to maximize utility, $U_{i,j}$ given by:[4]

$$U_{i,j} = X_j'\beta_i + \alpha\sigma_j + \varepsilon_{i,j} \tag{2.1}$$

where each location j is described by a vector of attributes (X_j) that may be observed or unobserved, and the share of individuals who choose this location $j(\sigma_j)$.[5] The taste parameters in Eq. (2.1) may vary with individual characteristics (Z_i):[6]

$$\beta_i = \gamma + Z_i\phi \tag{2.2}$$

and individuals can have unobserved idiosyncratic preferences for location j, $\varepsilon_{i,j}$.

The inclusion of σ_j allows for anonymous local spillovers. Such spillovers must ultimately derive from some underlying mechanism. For example, households may desire to live in large metropolitan areas because of the size and scope of the labor market or the urban amenities that large cities provide. At the same time, the increased congestion may detract from the utility provided by large versus small cities. When such mechanisms are observed in the data, they can be included directly in the utility function. In many empirical settings, however, the mechanisms through which local spillovers operate are more numerous, more difficult to characterize, or less easily measured, and the inclusion of σ_j in the utility function distinguishes the collective magnitude of these local spillovers.

Finally, it is important to note three simplifying assumptions that we maintain throughout this paper. First, we assume that an individual's utility from selecting location j is affected only by the characteristics of that location, including the share of individuals who also choose it. In general, the model can be extended to account for the possibility of spillovers across locations (i.e., where the attributes of nearby alternatives enter directly into the utility received from choosing location j). Second, while it is possible to include

[4] The basic form of this utility function is based on the random utility model developed in McFadden [16] and the specification of Berry et al. [4], which includes choice-specific unobservable characteristics. We use the linear form for utility to make examining the equilibrium properties of the model as clear as possible.

[5] This utility function could be written in terms of the number rather than the share of individuals that select the same alternative. Such a specification would be equivalent to the one used here—α would just be rescaled by the total number of individuals sorting across locations. Moreover, for the purposes of proving existence and describing the conditions under which a unique equilibrium obtains, differentiating between observable and unobservable (i.e., to the econometrician) but exogenous (i.e., not determined by the equilibrium sorting process) attributes is unimportant. In other work, we find that recognition of such unobservables is crucial to the recovery of unbiased estimates of congestion and (especially) agglomeration effects, in addition to other utility parameters [3].

[6] Similarly to the case of location attributes, the distinction between observable and unobservable individual attributes is unimportant for the discussion of existence and uniqueness, but matters for estimation. In the context of our model, observable individual heterogeneity yields a heterogeneous parameters logit, while unobservable attributes imply a random parameters logit specification. Either is consistent with the results contained in the remainder of this paper.

other endogenous variables in the analysis (the most important of which is a price associated with choosing each location), we ignore the role of prices and other endogenous variables (e.g., wages, if we consider equilibrium in local labor markets) in order to focus attention on the key issues concerning the equilibrium properties of the sorting model related to local spillovers.[7] Finally, we assume that the coefficient on the share of the population that selects alternative j, σ_j, is constant across individuals. This assumption makes distinguishing models with agglomeration and congestion interactions a simple matter of determining whether α is greater than or less than zero, which is helpful in characterizing the equilibrium properties of the model.

Throughout our analysis, we assume that individual i's vector of unobserved preferences $\bar{\varepsilon}_i$ is observed by all of the other individuals in the model, and that agents play a static simultaneous-move game according to a Nash equilibrium concept. Moreover, we assume that a continuum of individuals with different unobserved preferences exists for each vector of observed characteristics Z_i that occurs in the world. This assumption (which is essentially that the number of agents is sufficiently large to avoid integer problems) ensures that the unobserved components of preferences can be integrated out.[8] The resulting choice probabilities depict the distribution of location decisions that would result from a continuum of individuals with a given set of observed characteristics Z_i, each responding to his particular unobserved preferences.[9]

Given the utility specification described in Eq. (2.1), the probability $P_{i,j}$ that individual i chooses alternative j can be written as a function of the vectors of choice characteristics (both exogenous X_js and σ_js, which are determined in equilibrium), individual i's observed characteristics Z_i, and a vector of parameters, θ:

$$P_{i,j} = g_{i,j}\big(Z_i, \overline{X}, \bar{\sigma}; \theta\big) \quad \forall i, j. \tag{2.3}$$

Aggregating these probabilities over all individuals yields the share choosing location j:

$$\sigma_j = \sum_i g_{i,j}\big(Z_i, \overline{X}, \bar{\sigma}; \theta\big) \quad \forall j \tag{2.4}$$

which, rewritten in vector notation, is given by:

$$\bar{\sigma} = g\big(\overline{Z}, \overline{X}, \bar{\sigma}; \theta\big). \tag{2.5}$$

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7 See [2,24] for explicit analyses of sorting models with local spillovers and endogenous prices.

8 Note that it is possible to incorporate other assumptions concerning the nature of idiosyncratic preferences and the equilibrium concept within this framework. We could, for example, treat each household's idiosyncratic preferences as private information and relax the assumption that each household observed in the data represents a continuum of other households. In this case, the choice probabilities correspond to the expected decisions of other agents (possibly masking important elements of strategic interactions between households). Seim [21] uses this interpretation of the error structure, along with a Bayesian–Nash equilibrium concept, in estimating a model of entry in retail markets. In developing the theoretical properties of the equilibrium, the estimation procedure, and the identification strategy, we work with the interpretation of $\bar{\varepsilon}_i$ specified above.

9 It is worth noting that the use of choice probabilities does not affect the attractive properties of the choice framework related to self-selection. Among the continuum of individuals, those individuals that choose a particular alternative j will be those that get a relatively high draw of $\varepsilon_{i,j}$ relative to the other choices. In this way, the individuals predicted to choose an alternative are those that place the highest value on it, as governed by both their characteristics and idiosyncratic preferences.

This system of equations implicitly defines the vector of population shares $\bar{\sigma}$ and maps $[0,1]^J$ into itself, where J is the total number of alternatives in the discrete choice set. We define a *sorting equilibrium* to be a set of individual location decisions that are each optimal given the location decisions of all other individuals in the population. Because any fixed point of (2.5) is associated with a set of location decisions that satisfy the conditions for a sorting equilibrium, we are now in position to prove the following proposition.[10]

Proposition 1. *If $\bar{\varepsilon}_i$ is drawn from a continuous, well-defined distribution function and $U_{i,j}$ is defined as in Eq.* (2.1) $\forall i, j$, *a sorting equilibrium exists.*

Proof. The assumptions imply that Eq. (2.5) implicitly defines $\bar{\sigma}$ as a continuous mapping of a closed and bounded interval into itself. The existence of a fixed point of this mapping, $\bar{\sigma}^*$, follows directly from Brouwer's fixed-point theorem. Any fixed point, $\bar{\sigma}^*$, is associated with a unique set of choice probabilities given in Eq. (2.3) that satisfy the conditions for a sorting equilibrium. Consequently, the existence of a fixed point, $\bar{\sigma}^*$, implies the existence of a sorting equilibrium. □

Under the simple set of continuity assumptions described in Proposition 1, which are reasonable in many empirical settings, an equilibrium will always exist for this class of models. We now characterize the conditions under which this equilibrium is unique.

3. Uniqueness

In general, whether a unique equilibrium arises is related to four features of the choice problem:

(i) the sign and magnitude of the social interaction, α;
(ii) the meaningful variation in household tastes;
(iii) the meaningful variation in fixed attributes across choices; and
(iv) the total number of choices, J.[11]

This section of the paper derives two analytical results that mirror those found in the previous literature.

To facilitate this analysis consider the function that implicitly defines the share of individuals that choose each alternative:

$$\sigma_j = \sum_i P_{i,j} = \sum_i g_{i,j}\big(Z_i, \overline{X}, \bar{\sigma}; \theta\big) = g_j\big(\overline{Z}, \overline{X}, \bar{\sigma}; \theta\big) \tag{3.1}$$

[10] In fact, in this perfect information setting, all potential equilibria must be associated with a distinct fixed point of the mapping defined in Eq. (2.5).

[11] The expression "meaningful variation" incorporates both the underlying variation in the data and the importance of the given characteristic in the utility function. Thus, a choice attribute can have no meaningful variation even if it has a positive variance if individuals do not value this attribute when selecting an alternative.

which can be rewritten implicitly in vector notation as

$$\Psi\left(\overline{Z}, \overline{X}, \bar{\sigma}; \theta\right) = \bar{\sigma} - g\left(\overline{Z}, \overline{X}, \bar{\sigma}; \theta\right) = 0. \tag{3.2}$$

We begin by considering the case of a congestion interaction (i.e., $\alpha < 0$). In this case, it is possible to prove the following proposition.

Proposition 2. *If $\bar{\varepsilon}_i$ is drawn from a continuous, well-defined distribution function and $U_{i,j}$ is defined as in Eq.* (2.1)*, the sorting equilibrium is unique in the presence of a congestion interaction, $\alpha < 0$.*

Proof. With a congestion effect, the matrix of partial derivatives of Ψ with respect to $\bar{\sigma}$, which we label Ψ_1, has diagonal elements $(1 - \partial g_j/\partial\sigma_j) > 1$. Given that the utility specification is linear in σ_j, increasing all shares by a fixed amount has no effect on individual location decisions. This implies that $\sum_k \partial g_j/\partial\sigma_k = 0$ or $\sum_{k\neq j} \partial g_j/\partial\sigma_k = -\partial g_j/\partial\sigma_j$, which is a positive number. Thus, the diagonal elements of Ψ_1 exceed the sum of the off-diagonal elements by one. Ψ_1 is therefore a matrix with a positive dominant diagonal, and consequently, the equation $\Psi = 0$ has a unique solution. □

In the presence of an agglomeration interaction, the equilibrium is no longer generically unique. In this case, the following proposition holds.

Proposition 3. *If $\bar{\varepsilon}_i$ is drawn from a continuous, well-defined distribution function and $U_{i,j}$ is defined as in Eq.* (2.1)*, the sorting equilibrium is unique in the presence of a moderate agglomeration interaction, $0 < \alpha < T$, where T is a function of the primitives of the model including the distributions of household tastes and exogenous choice attributes.*

Proof. When $\sum_k |\partial g_j/\partial\sigma_k| < 1\ \forall j$, the matrix Ψ_1 has diagonal elements $(1 - \partial g_j/\partial\sigma_j)$ that are each positive and exceed the sum of the off-diagonal elements $\sum_{k\neq j} |\partial g_j/\partial\sigma_k|$. In this case, Ψ_1 is again a matrix with a positive dominant diagonal, and as a consequence, the equation $\Psi = 0$ has a unique solution. That Ψ_1 is a matrix with a positive dominant diagonal for at least some positive threshold T follows from continuity at $\alpha = 0$. When $\alpha = 0$, the diagonal elements of Ψ_1 equal one (i.e., $1 - \partial g_j/\partial\sigma_j = 1$) and the off-diagonal elements of Ψ_1 equal zero (i.e., $\partial g_j/\partial\sigma_k = 0$). Thus, for at least some small region around $\alpha = 0$, Ψ_1 is a positive dominant diagonal matrix and, consequently, there exists a $T > 0$ such that a unique equilibrium exists. □

Propositions 2 and 3 echo the key result concerning uniqueness developed by Brock and Durlauf [6] for the binary choice case. Proposition 3 also echoes the uniqueness result presented by Glaeser and Scheinkman [11] for social multipliers. Intuitively, that multiple equilibria can arise in the presence of a sufficiently strong agglomeration effect but not a congestion effect or a weaker agglomeration effect relates to the power of the social interaction to change the collective rank-order of locations (i.e., the rank-order of equilibrium shares). In the presence of a congestion effect, for example, the social interaction works to diffuse agents across locations but always preserves the rank-order of locations that would prevail in the absence of a social interaction. For a sufficiently strong agglomeration effect, however, the rank-order of locations can be altered. In this case, it is possible for the equilibrium share of the location that would be most popular in the absence of a social interaction to be smaller than that of another location. Fundamentally, it is the power of a sufficiently strong agglomeration effect to switch the equilibrium rank-order of two locations that gives rise to the potential for multiple equilibria.

4. Simulations

Proposition 3 is unsatisfying in that it provides no intuition for how the conditions of the economic environment under study affect the maximum value of α which results in a unique equilibrium. As described in the introduction, [6] provides an analytical solution for this maximum value for the binary choice setting but, as described in [7], obtaining an analytical solution is impossible in a multinomial choice setting except in extremely special cases of the model (e.g., for cases where the private component of utility is constant across both choices and individuals). Moreover, even if one was to fully characterize the distributions that govern the data generating process describing choice characteristics and tastes (X_j, β_i), the maximum value of α for which a unique equilibrium obtains varies with the actual realizations of choice characteristics and tastes in the data.

Consequently, in order to explore how the primitives of the model affect the maximum α such that a unique equilibrium arises for all agglomeration interactions less than that value, we conduct a series of simulations that calculate this threshold value T for a particular draw of a data set given the primitives of the empirical setting. The results of these simulations describe the distribution of T for a given empirical setting and, consequently, by varying the model's primitives we are able to draw conclusions about how these primitives affect the likelihood that a unique equilibrium obtains.

For these simulations, we consider the following simplified version of the utility function shown in Eq. (2.1):

$$U_{i,j} = \beta_i X_j + \alpha\sigma_j + \varepsilon_{i,j}. \tag{4.1}$$

We assume that X_j has only a single dimension and that β_i is distributed i.i.d. normal with mean β_0 and variance σ_β^2. We also assume that $\varepsilon_{i,j}$ is distributed according to the Weibull distribution and that the variance of X across choices is given by σ_X^2.

We present a series of simulations that calculate T for a particular draw of a data set given the basic features of the empirical setting:

(i) the number of choices, J;
(ii) the meaningful variation in exogenous choice characteristics, $\beta_0\sigma_x^2$; and
(iii) the heterogeneity in household preferences, σ_β^2.

By repeating this calculation for a large number of simulations, we are able to characterize the distribution of T for a given empirical setting.

To describe the algorithm that we use, it is helpful to introduce some additional notation. In particular, we consider the iterative application of the mapping g defined in Eq. (3.1), which can be written element by element (temporarily ignoring the heterogeneity in β_i in order to simplify the proof) as:

$$\sigma_j^t = g_j\left(\sigma^{t-1}\right) = \frac{\mathrm{EXP}(\beta_0 X_j + \alpha\sigma_j^{t-1})}{\sum_{k=1}^J \mathrm{EXP}(\beta_0 X_k + \alpha\sigma_k^{t-1})}. \tag{4.2}$$

Starting from an initial vector, $\bar{\sigma}^0$, the iterative application of this mapping gives rise to the following sequence which we denote by $h(\bar{\sigma}^0)$:

$$h\left(\bar{\sigma}^0\right) = \left\{\bar{\sigma}^0, g\left(\bar{\sigma}^0\right), g\left(g\left(\bar{\sigma}^0\right)\right), \ldots, g^t\left(\bar{\sigma}^0\right), \ldots\right\}. \tag{4.3}$$

Let the J sequences $\{h^1(\cdot), \ldots, h^J(\cdot)\}$ be defined as those that start from the initial vector $\bar{\sigma}^0$ such that $\sigma_j^0 = 1$, $\sigma_k^0 = 0$ $\forall k \neq j$ for each $j = 1, \ldots, J$, respectively. Given this definition, we can prove the following proposition.

Proposition 4. *Given any β_0 and finite vector $\overline{X} = \{X_1, X_2, \ldots, X_J\}$ and any agglomeration effect (i.e., $\alpha > 0$), the J sequences $\{h^1(\cdot), \ldots, h^J(\cdot)\}$ defined above converge to at least two distinct fixed points whenever multiple fixed points of the mapping defined in* (4.2) *exist.*

Proof. Available from authors upon request.

Proposition 4 implies that for any set of parameters and any specific draw of the exogenous data, it is possible to determine whether a unique equilibrium obtains. The logic of why applying $g(\cdot)$ iteratively starting at the J endpoints leads to multiple fixed points whenever multiple fixed points exist can be seen easily in the two-dimensional case. In this case, it is straightforward to establish two key properties of the mapping g_j defined in Eq. (4.2). First, $0 < g_j < 1$ for all possible vectors $\bar{\sigma}$ and second, g_j is monotonically increasing in σ_j when $\alpha > 0$. Figure 1 illustrates two possible mappings $g_1(\sigma_1)$ for the two-dimensional case. The first property ensures that $g_1(1) < 1$ and thus lies below the 45° line, which when combined with monotonicity assumption ensures that iterating according to Eq. (4.3) leads to the fixed point with the greatest value for σ_1, point A in the

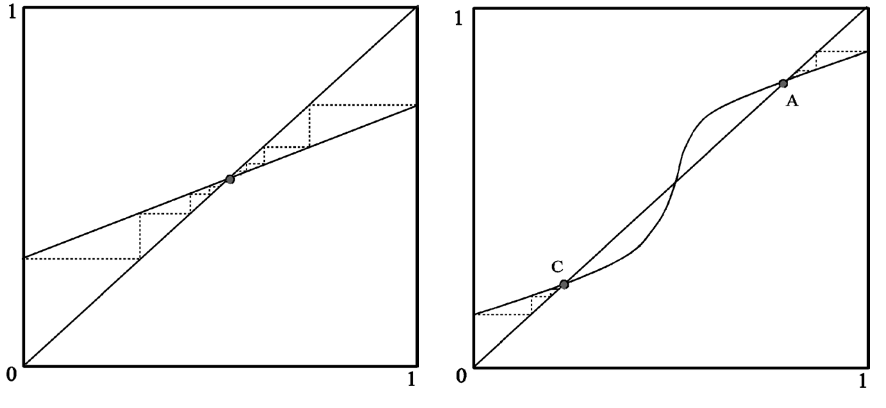

Fig. 1. Unique and multiple equilibria in the two-dimensional case.

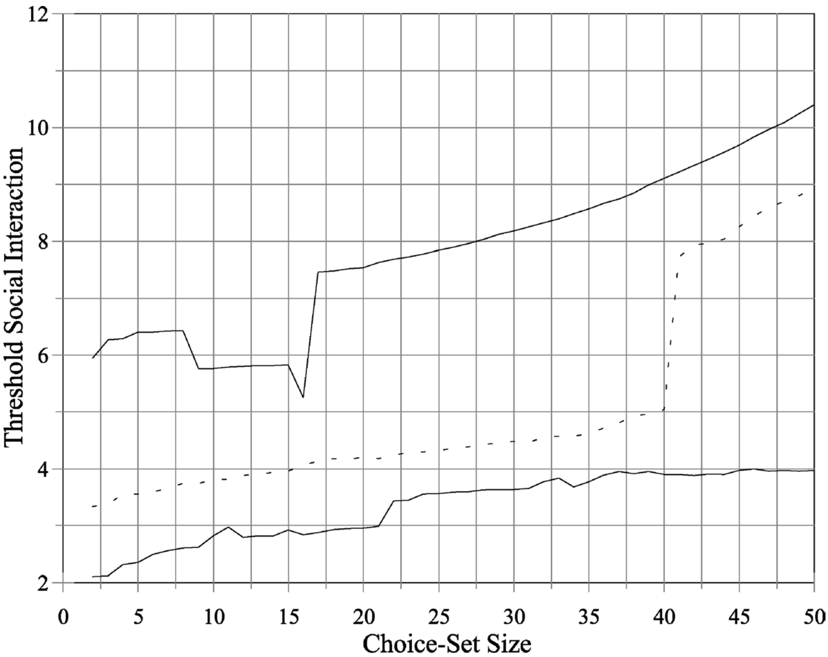

Fig. 2. Effect of choice-set size. Minimum, maximum, and median of the distribution of simulated threshold values of α, constant meaningful variation in X ($\beta_0\sigma_X^2 = 1$), homogeneous preferences ($\sigma_\beta^2 = 0$).

figure on the right. Thus, when multiple fixed points are present, as shown in that figure, iterating Eq. (4.3) starting from the point $\sigma_1 = 1$ will converge to the fixed point A and starting with $\sigma_2 = 1$ (i.e., $\sigma_1 = 0$) will converge to the fixed point C. When a single fixed point is in fact present (i.e., shown in the figure to the left), iterating on Eq. (4.2) will converge to it from either starting point. The proof (available from the authors upon request) extends the logic of this two-dimensional setting to the case of many dimensions.

To solve for the threshold value T, for each simulation, we begin by setting $\alpha = 0.01$, testing whether a unique equilibrium obtains in this case. If we determine that the sorting equilibrium is unique for $\alpha = 0.01$, we set $\alpha = 0.02$ and repeat this procedure, increasing α until we find an α that gives rise to distinct fixed points.[12] The first set of simulations are designed to demonstrate how the threshold agglomeration effect T varies with the size of the choice set. The results of these simulations are described in Fig. 2. We first simulate the maximum agglomeration effect that supports a unique equilibrium for each of 100 different randomly determined choice sets of size 3 to 50, holding $\beta_0\sigma_X^2 = 1$ and $\sigma_\beta^2 = 0$. Figure 2 describes the median, maximum, and minimum of the distribution of maximum agglomeration effects at each choice set size. In order to characterize how meaningful variation in the exogenous characteristics of the choice set affects the maximum α that can sustain a

[12] Allowing the simulation algorithm to consider values of α beyond the threshold (e.g., up to four times the threshold), in no case did we find a reversion to a unique equilibrium.

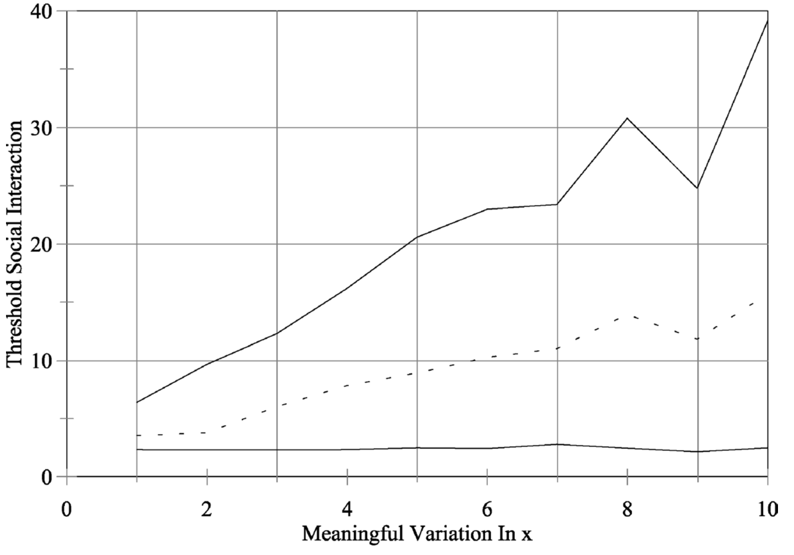

Fig. 3. Effect of meaningful variation in local attributes. Minimum, maximum, and median of the distribution of simulated threshold values of α, constant choice-set size ($J = 5$), homogeneous preferences ($\sigma_\beta^2 = 0$).

unique equilibrium, we conduct additional simulations for integer values of $\beta_0\sigma_x^2$ ranging from 1 to 10. Figure 3 illustrates the results for this case, with $J = 5$ and $\sigma_\beta^2 = 0$.

Finally, in order to explore how individual heterogeneity affects the maximum value of α that can sustain a unique equilibrium, we compare the results with $\sigma_\beta^2 = 0, 1$, and 2. Cases with $\sigma_\beta^2 > 0$ are not subject to the Independence of Irrelevant Alternatives (IIA) property at the aggregate level as in the multinomial logit case; i.e., $\sigma_\beta^2 = 0$. In this way, these specifications also demonstrate the robustness of the above results to relaxing the IIA property. The richer individual heterogeneity strengthens the simulation results described above as illustrated in Fig. 4. The figure is constructed for alternative choice set sizes with $\beta_0\sigma_x^2 = 1$; solid lines bound the 25th and 75th percentiles of the distributions of the threshold α, and moving from bottom to top, dashed lines denote the medians of the simulated feedback effects when $\sigma_\beta^2 = 0, 1$, and 2, respectively. A clear increase in the maximum sustainable agglomeration effect accompanies increasing heterogeneity in individual preferences. Beyond a choice-set size of $J = 25$, the three confidence bands are practically disjoint.

These simulation results support the following conclusions. Conditional on the primitives of the sorting model, the mean T defined over a series of randomly drawn data sets is an increasing function of:

(i) the number of alternatives;
(ii) the variation in the contribution to utility made by the exogenous attributes of the choice set; and
(iii) the heterogeneity in individual preferences.

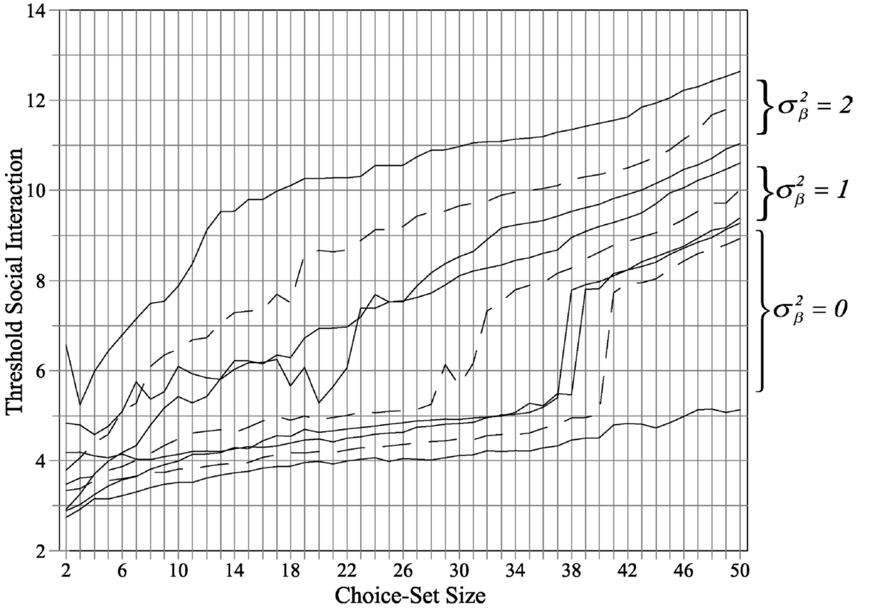

Fig. 4. Effect of heterogeneity in individual preferences. Median, 25th, and 75th percentiles of the distribution of simulated threshold values of α under heterogeneous preferences ($\sigma_\beta^2 = 0, 1, 2$), constant meaningful variation in x ($\beta_0\sigma_X^2 = 1$).

These conclusions echo results found earlier in the network effects and social interactions literatures; Katz and Shapiro [14], for example, write that "consumer heterogeneity and product differentiation tend to limit tipping and sustain multiple networks. If the rival systems have distinct features sought by certain consumers, two or more systems may be able to survive by catering to consumers who care more about product attributes than network size." Likewise, in studying uniqueness in a model of Tiebout sorting, Nechyba [18] points out that when "communities are sufficiently different in their inherent desirability, the partition of households into communities is unique." Finally, Brock and Durlauf [6] state, "one is most likely to observe multiplicity (in equilibria) in those social environments in which private utility renders individuals relatively close to indifferent between choices." In this way, the results of our simulations correspond well to the results and intuition of other authors studying related models.

While useful in ensuring that our simulations are conducted properly, the computational algorithm described in this section also has another potentially valuable application. In particular, given data and the vector of parameter estimates, it is straightforward to apply this algorithm to determine whether the observed equilibrium is unique and, moreover, whether a unique equilibrium obtains in a specified counterfactual environment. The ability to test for uniqueness is especially valuable given the availability of the estimator that we provide in [3], which makes it possible to obtain parameter estimates in many empirical settings without relying on an assumption concerning the uniqueness of the equilibrium.[13] In this way, in addition to providing a method for exploring the general properties of the equilibrium in this class of models as we do in this paper, the computational algorithm described above provides a useful tool for empirical researchers, providing a test for uniqueness that creates a stronger basis for general equilibrium welfare analysis.

5. Broader forms of social interactions

It is straightforward to extend the model developed above to incorporate broader forms of social interactions that allow, for example, households from a particular group to prefer to live in neighborhoods or communities with other households from the same group. Such a model might be used to describe neighborhood sorting based on racial preferences [19,20] or community sorting governed by concerns over the provision of local public goods [23]. In these cases, preferences are naturally defined in terms of the characteristics, rather than the number, of other households that choose the same location. Because a complete set of results to describe such sorting would be analogous to those presented above and would substantially increase the length of the paper, we instead provide intuition for how the results of the previous section extend to this case.

Consider a simple setting with two types of households (A, B) with preferences defined over the fraction of households of a particular type that choose the same neighborhood. Let the preferences of households of type k for location j be written as:

$$U_{i,j,k} = X_j'\beta_k + \alpha_k\sigma_j^A + \varepsilon_{i,j,k} \tag{5.1}$$

where σ_j^A represents the proportion of individuals in neighborhood j of type A. In this case, self-segregating preferences would be characterized by a positive value of α_A and a negative value of α_B. The intensity of preference for self-segregation could then be measured by $\alpha_A - \alpha_B$.

The uniqueness properties of the model in this setting generally relate to differences in the preferences of the two types in Eq. (5.1). Results analogous to those stated in Proposition 3 would imply that, conditional on the primitives of the sorting model, the maximum strength of preferences for self-segregation $(\alpha_A - \alpha_B)$ defined over a series of randomly drawn data sets would be an increasing function of differences in the meaningful variation in preferences for local attributes, X_j, across households of different types. Consider, for example, the case in which two types of households have preferences for living with neighbors of the same type and must choose between two otherwise identical neighborhoods. In this case, it is easy to see that the model will have multiple equilibria. In particular, two stable equilibria arise with households sorting across neighborhoods by type, with indeterminate matching of households of a particular type to a particular neighborhood. If, on the other hand, households of one type differ from the other in an important attribute

[13] This estimation strategy is based on first-order conditions derived from agents' optimal location decisions, and at no point requires a unique equilibrium for estimation purposes. This is in contrast to pure likelihood-based algorithms for estimating the utility effects of spillovers, which require uniqueness or an arbitrary equilibrium selection rule when there are more than one.

(e.g., have significantly more income), and if the fixed attributes of the neighborhoods are significantly different (e.g., one offers a view of the ocean while the other does not), preferences to segregate will ensure that households will again sort across neighborhoods by type, but the matching of household type to a particular neighborhood will more likely be determined. In particular, when self-segregating preferences are sufficiently weak, a unique equilibrium obtains with richer households always choosing the higher quality neighborhood. In this way, the results of the previous section naturally extend to a broader class of models than those considered above.

6. Conclusion

Models incorporating local spillovers are widespread in economic analysis. In both theoretical and empirical work alike, agents' payoffs are often assumed to be affected by the decisions of other agents in ways not captured by traditional market channels (e.g., prices). In this paper, we describe the equilibrium properties of a model that can be adapted to describe many of these kinds of interactions. Bayer and Timmins [3] show how its structure can be conveniently exploited to estimate the size of spillover effects, even in the presence of unobserved local attributes, and [2,24] apply this approach. In this paper, we demonstrate that an equilibrium will always exist in this class of models under a set of easily satisfied conditions, and that the equilibrium will be unique under conditions encountered in many empirical problems. While uniqueness is not necessary for estimating the size of local spillovers, it is a valuable property of the model if interest is in simulating the new equilibrium that arises in response to a non-marginal policy change. We show that uniqueness will always be sustained under any sized congestion effect (a relevant result, particularly when modeling competition between firms or urban disamenities), but that the maximum agglomeration effect that can sustain a unique equilibrium is a function of the idiosyncratic attributes of the data in any particular application. We can therefore only demonstrate how that maximum agglomeration effect tends to vary with certain features of the empirical context. The computational algorithm used to conduct these simulations, moreover, can be used to test for uniqueness in empirical analyses, both under observed data and estimated parameters, and under counterfactual policy scenarios.

Similar to other authors writing in a variety of modeling frameworks, we find that uniqueness is easier to sustain

(i) the larger the set of choices available to the agent,
(ii) the more "meaningful variation" there is in those choices, and
(iii) the more heterogeneous the agents are themselves.

Conveniently, these also happen to be the empirical characteristics that facilitate the estimation strategy proposed in [3]. This is encouraging for the use of our model to describe the sorting of individuals and firms over geographic space, where the number of choices is usually large. As opposed to models of individuals sorting over clubs whose attributes are defined mainly by the equilibrium membership, there are typically many local attributes—fixed features of infrastructure and the geographic landscape—that will be important to agents in deciding where to live or where to locate their firm. This provides one half of the meaningful variation that makes it increasingly likely that a unique equilibrium will obtain. Moreover, continuing improvements GIS mapping and satellite imaging technology will continue to increase the variety and precision of the spatial data available for analysis in these sorts of applications.

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