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The uniform price auction with endogenous supply

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Abstract

This paper shows that low-price equilibria in the uniform price auction with endogenous supply do not exist if the seller employs the proportional rationing rule and is consistent when selecting among profit-maximizing quantities. In a (consistent) subgame perfect equilibrium the Walrasian quantities are traded at the Walrasian price.

Keywords: Endogenous supply; Proportional rationing; Consistency

JEL classification: D44; D41; D42

1. Introduction

The uniform price divisible good auction with fixed supply is known to possess low-price equilibria in which the stopout price lies far below the competitive one.[1] The low-price equilibrium problem can be partially resolved if the seller does not commit to a supply quantity ex ante, but rather retains the right to determine it after the bidding depending on the received bids. Back and Zender (2001) show that the seller's right to adjust the supply quantity after the bidding eliminates the worst outcomes for the seller. They provide a lower bound on the stopout price in a symmetric subgame perfect equilibrium, which increases with the number of bidders, but is always lower than the competitive price.[2] A special feature

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[1] Wilson (1979) first discusses the issue in the context of a share auction; Back and Zender (1993) construct such equilibria in a Treasury auction model with incomplete information.

[2] In their model the auctioned asset has a common value of v for all bidders. The competitive price thus equals v.

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of their model is that bids are rationed according to the "pro rata on the margin" rule. According to this rationing rule, demand above the stopout price is fully served and only the quantities demanded at the stopout price are rationed proportionally.

In contrast, we consider the "pro rata" (or proportional) rationing rule.[3] This rationing rule does not give priority to high demand and each bidder is granted an amount proportional to his quantity at the clearing price. In a model of fixed supply divisible-good auctions, Kremer and Nyborg (2004) study the effect of different rationing rules on the set of equilibrium prices.

This note demonstrates that if the seller chooses supply ex post and uses the "pro rata" rationing rule, then he will sell the competitive quantity at the competitive price in the uniform price auction. The proportional rationing rule allows a bidder with excess demand to buy a larger quantity without the need to bid a higher price by simply magnifying his quantity announcement at the stopout price. This property eliminates all low-price equilibria in the auction game. Moreover, it leads to a unique equilibrium allocation, which is the efficient allocation of the Walrasian mechanism. The model enables one to examine equilibrium bidding of any number of two or more buyers with possibly asymmetric demand curves, but as in Back and Zender (2001) there is no uncertainty in this auction. The note shows that a popular trade mechanism, which is often employed for the sale of financial assets,[4] generates Walrasian trades in a strategic bidding model.

2. The model and the result

There are $n \geq 2$ buyers and a single seller. The demand function $d_i(p)$ of each bidder $i \in \{1,2,\ldots,n\}$ is continuous and monotonically decreasing. The marginal cost function of the seller $c'(\boldsymbol{q})$ is continuous and monotonically increasing. Let us denote the aggregate demand function by $D(p) := \sum_{i=1}^{n} d_i(p)$ and the supply function[5] of the seller by $S(p) := c'^{-1}(\cdot)$. The market-clearing price is denoted by $\bar{p}^{\mathrm{w}} : D(\bar{p}^{\mathrm{w}}) = S(\bar{p}^{\mathrm{w}})$ and it is assumed that at least two consumers demand positive quantities at that price.

Trade is conducted via a uniform price auction with endogenous supply, in which each buyer i submits a bid $\boldsymbol{q}_i$ to the auctioneer in the form of a demand schedule. The demand schedule is a nonincreasing left-continuous function $\boldsymbol{q}_i : [0, \infty) \rightarrow \mathbb{R}_+$. Let us further denote the vector of submitted demand schedules by $\boldsymbol{q} = (\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n)$ and the vector of all bids except that of bidder i by $\boldsymbol{q}_{-i}$. After collecting the bids, the seller determines a sale quantity ϕ depending on the bidding. With the convention that the maximum of the empty set is zero, the stopout price is defined as follows:

$$p_S = \max\left\{ p \;\middle|\; \sum_{j=1}^{n} \boldsymbol{q}_j(p) \geq \phi \right\}.$$

[3] This rule is used for example to ration the bids in the IPO auctions in France (see Biais and Faugeron-Crouzet, 2002). In the terminology of Kremer and Nyborg (2004, p. 150, definition 3) such rationing rule is called "independent of irrelevant demand".

[4] Umlauf (1993), Nyborg et al. (2002) and Heller and Lengwiler (2001) describe the Treasury auctions in Mexico, Sweden and Switzerland (respectively) as being of the multi-unit endogenous supply format. See also Brisley and Busaba (2003), Dunbar and Foerster (2002) and McAdams (2000) for evidence of endogenous supply on IPO markets.

[5] As the market agents will not act as price takers it makes little sense to talk about supply and demand curves. Nonetheless, we use this terminology in order to introduce the competitive equilibrium and use it as a reference concept.

Each bidder i is assigned the quantity

$$Q_i = \frac{\boldsymbol{q}_i(p_S)\cdot\phi}{\sum_{j=1}^{n}\boldsymbol{q}_j(p_S)}$$

and obtains the payoff

$$V_i(\boldsymbol{q};\phi) = \int_0^{\infty}\min\{d_i(p),Q_i\}dp - Q_i\cdot p_S.$$

The payoff of the seller is $R(\boldsymbol{q};\phi)=p_S\cdot\phi-c(\phi)$. Let $M(\boldsymbol{q}) = \arg_{\phi\geq 0}\max R(\boldsymbol{q};\phi)$ denote the set of the profit maximizing quantities of the seller for every bid vector $\boldsymbol{q}$.

Definition. The bid profile $\boldsymbol{q}^*$ and the seller's supply function $\phi^*:\mathfrak{Q}^n\rightarrow\mathbb{R}_+$ constitute a consistent subgame perfect equilibrium, if they satisfy the following three conditions:

(P) Profit maximization: the seller chooses a profit maximizing quantity for each bid vector:

$$\phi^*(\boldsymbol{q})\in M(\boldsymbol{q}), \qquad \forall \boldsymbol{q}\in\mathfrak{Q}^n.$$

(C) Consistency: for any two bid profiles $\bar{\boldsymbol{q}}$ and $\tilde{\boldsymbol{q}}$ for which $M(\tilde{\boldsymbol{q}})\subseteq M(\bar{\boldsymbol{q}})$ the function $\phi^*(\cdot)$ satisfies:[6]

$$\phi^*(\bar{\boldsymbol{q}})\in M(\tilde{\boldsymbol{q}})\Rightarrow\phi^*(\tilde{\boldsymbol{q}}) = \phi^*(\bar{\boldsymbol{q}}).$$

(N) Nash play on the bidding stage:

$$V_i(\boldsymbol{q}_i^*,\boldsymbol{q}_{-i}^*;\phi^*(\cdot))\geq V_i(\boldsymbol{q}_i,\boldsymbol{q}_{-i}^*;\phi^*(\cdot)), \qquad \forall\boldsymbol{q}_i\in\mathfrak{Q},\forall_i.$$

Theorem. *In a consistent subgame perfect equilibrium the competitive quantities are traded at the competitive price*:

$$p_S(\boldsymbol{q}^*;\phi^*(\cdot)) = \bar{p}^{\mathrm{w}},$$

$$Q_i(\boldsymbol{q}^*;\phi^*(\cdot)) = d_i(\bar{p}^{\mathrm{w}}),\forall_i.$$

Proof. For the sake of brevity let us denote the equilibrium stopout price by p_S^* and the seller's equilibrium supply quantity by ϕ^*. We show that no equilibria exist in which $p_S^*>\bar{p}^{\mathrm{w}}$ or $p_S^*<\bar{p}^{\mathrm{w}}$. For each of these cases, we construct a profitable deviation for one of the bidders, thereby reaching a

[6] The consistency condition requires that if the seller chooses an element from a maximizer set $M(\bar{\boldsymbol{q}})$, he chooses the same element in all subsets of $M(\bar{\boldsymbol{q}})$ in which this element is available. Simple examples of consistent supply functions are:

$$\phi^*(\boldsymbol{q}) = \max\{\phi:\phi\in M(\boldsymbol{q})\};$$

$$\phi^*(\boldsymbol{q}) = \min\{\phi:\phi\in M(\boldsymbol{q})\}.$$

Supplying consistently, the seller precludes situations in which a winning bidder, who deviates by extending his bidding quantity, will no longer be served (see Case 1 in the proof).

contradiction to the equilibrium condition (N). The deviations are characterized by a single price-quantity pair $\langle p^j, x^j \rangle$, which defines the demand schedule

$$\boldsymbol{q}_j(p) = \begin{cases} 0 & \text{for } p > p^j, \\ x^j & \text{for } 0 \leq p \leq p^j. \end{cases}$$

Case 1. $p_S^* < \bar{p}^{\mathrm{w}}$. As the seller supplies a quantity not higher than $S(p_S^*)$, there is excess demand at the stopout price and hence there exists a bidder j who in equilibrium receives less than he is willing to buy at that price. Consider the following deviation of that bidder: $\langle p_S^*, q_j^*(p_S) + \varepsilon \rangle$, where $\varepsilon > 0$. If ε is chosen sufficiently small the optimal response of the seller to the deviation is $\phi^{*D} = \phi^* + \varepsilon$ for $\phi^* < S(p_S^*)$ (see condition (P)) and $\phi^{*D} = \phi^*$ otherwise (see condition (C)). Bidder j is assigned the quantity $q_j^*(p_S) + \varepsilon$ for $\phi^* < S(p_S^*)$ and $(q_j^*(p_S) + \varepsilon)/[(\sum_{i=1}^n \boldsymbol{q}_i^*(p_S) + \varepsilon) \cdot \phi^*]$ otherwise. For ε sufficiently small the deviation is profitable.

Case 2. $p_S^* > \bar{p}^{\mathrm{w}}$. The deviations are given in the Appendix A.

One concludes than in equilibrium (provided that one exists) trade is conducted at the competitive price $\bar{p}^{\mathrm{w}}$. The seller supplies

$$\phi^* = \min\left\{ \sum_{i=1}^n \boldsymbol{q}_i^*(\bar{p}^{\mathrm{w}}), S(\bar{p}^{\mathrm{w}}) \right\}.$$

If $\phi^* < S(\bar{p}^{\mathrm{w}})$, there is excess demand and consequently one of the bidders obtains less than his Walrasian quantity. With the deviation presented in Case 1, this bidder will be able to obtain a larger quantity at the competitive price. It follows that $\phi^* = S(\bar{p}^{\mathrm{w}})$. If it is assumed that in equilibrium not all bidders obtain their Walrasian quantities, then there is a bidder, who obtains less than his Walrasian quantity and the deviation presented in Case 1 will again be profitable. It remains to be shown that an equilibrium with competitive trades exists. □

Claim. *The bidders' strategy profile* $(\langle \bar{p}^{\mathrm{w}}, q^1 \rangle, \langle \bar{p}^{\mathrm{w}}, q^2 \rangle, \ldots, \langle \bar{p}^{\mathrm{w}}, q^n \rangle)$, *where*

$$\boldsymbol{q}^i = \alpha \cdot d_i(\bar{p}^{\mathrm{w}}), \forall_i, \qquad \alpha \geq \frac{\sum_{i=1}^n d_i(\bar{p}^{w})}{\sum_{i=1}^n d_i(\bar{p}^{w}) - \max_{i \in \{1,2,\ldots,n\}} \{d_i(\bar{p}^{\mathrm{w}})\}}$$

is an equilibrium profile.

Proof. Observe that the seller supplies $\phi^* = S(\bar{p}^{\mathrm{w}})$; every bidder is granted his Walrasian quantity and has to pay the competitive price. No player has a profitable deviation, because no deviation leads to a lower stopout price. As the described strategies assign to every bidder his Walrasian quantity, a possible deviation can have only two effects. It can either lead to a higher stopout price or the deviating bidder will not be assigned the Walrasian quantity, both of which is not profitable. The strategy profile is thus an equilibrium. □

3. Conclusion

The analysis shows that low-price equilibria of the endogenous supply uniform price auction do not exist if the seller uses the proportional rationing rule and supplies consistently. The uniform price auction is found to always generate the (Pareto efficient) allocation of the Walrasian mechanism. However, the economic agents in the model are not price takers and the auctioneer is not imaginary and benevolent, but a profit-maximizing player of the studied market game. The model provides a strategic foundation of the competitive equilibrium paradigm.

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Appendix A

Case 2(a). $p_S^* > \bar{p}^{\mathrm{w}}$ and $\sum_{i=1}^{n} \boldsymbol{q}_i^*(p_S^*) > S(p_S^*)$. There is a bidder j who obtained more than he is willing to buy at that price. Playing the deviation $\langle p_S^*, \boldsymbol{q}_j^*(p_S^*) - \varepsilon \rangle$, where $\varepsilon > 0$ is sufficiently small, this bidder will be able to slightly reduce the quantity assigned to him since a consistent seller (see condition (C)) will not change his supply quantity $\phi^* = S(p_S^*)$ when facing such a deviation.

Case 2(b). $p_S^* > \bar{p}^{\mathrm{w}}$ and $\sum_{i=1}^{n} \boldsymbol{q}_i^*(p_S^*) = S(p_S^*)$. Take an arbitrary bidder j who demands a positive quantity at the stopout price defined by the lowest supply quantity from the set $M(\boldsymbol{q}^*)$. Consider the following deviation of that bidder:

$$\left\langle p_S^* - \varepsilon, \quad \frac{\boldsymbol{q}_j^*(p_S^*) \cdot \left(\sum_{i \neq j} \boldsymbol{q}_i^*(p_S^* - \varepsilon) \right)}{S(p_S^* - \varepsilon) - \boldsymbol{q}_j^*(p_S^*)} \right\rangle.$$

For a sufficiently small $\varepsilon > 0$ it will be argued that the seller supplies the quantity $\phi^{*D} = S(p_S^* - \varepsilon)$ at the stopout price of $p_S^* - \varepsilon$. Indeed observe that

$$\lim_{\varepsilon \to 0} R\left(\boldsymbol{q}_j^D, \boldsymbol{q}_{-j}^*; \phi^{*D}\right) = R(\boldsymbol{q}^*; \phi^*)$$

and consider also that if the seller decides not to serve bidder j (which means he reduces the supply quantity to charge a stopout price higher than $p_S^* - \varepsilon$), he will obtain a payoff strictly lower than $R(\boldsymbol{q}^*; \phi^*)$. It follows that for a sufficiently small ε it is not optimal for the seller to supply a quantity leading to a stopout price higher than $\boldsymbol{p}_S^* - \varepsilon$. On the other hand, supplying a quantity leading to a stopout price lower than $\boldsymbol{p}_S^* - \varepsilon$ obviously generates a lower payoff.

As a result of the deviation bidder j obtains the quantity

$$\frac{\boldsymbol{q}_j^*\left(p_S^*\right)\cdot\left(\sum_{i\neq j}\boldsymbol{q}_i^*\left(p_S^*-\varepsilon\right)\right)}{\phi^{*D}-\boldsymbol{q}_j^*\left(p_S^*\right)}\cdot\frac{\phi^{*D}}{\frac{\boldsymbol{q}_j^*\left(p_S^*\right)\cdot\left(\sum_{i\neq j}\boldsymbol{q}_i^*\left(p_S^*-\varepsilon\right)\right)}{\phi^{*D}-\boldsymbol{q}_j^*\left(p_S^*\right)}+\sum_{i\neq j}\boldsymbol{q}_i^*\left(p_S^*-\varepsilon\right)}$$

$$=\frac{\boldsymbol{q}_j^*\left(p_S^*\right)\cdot\left(\sum_{i\neq j}\boldsymbol{q}_i^*\left(p_S^*-\varepsilon\right)\right)}{\phi^{*D}-\boldsymbol{q}_j^*\left(p_S^*\right)}\cdot\frac{\phi^{*D}}{\frac{\phi^{*D}\cdot\left(\sum_{i\neq j}\boldsymbol{q}_i{}^*\left(p_S^*-\varepsilon\right)\right)}{\phi^{*D}-\boldsymbol{q}_j^*\left(p_S^*\right)}}=\boldsymbol{q}_j^*\left(p_S^*\right).$$

Thus, by playing the deviation, bidder j obtains the same quantity at a lower price. The deviation is profitable.

Case 2(c). $\boldsymbol{p}_S^*>\bar{\boldsymbol{p}}^{\mathrm{w}}$ and $\sum_{i=1}^{n}\boldsymbol{q}_i^*(p_S^*)<S(p_S^*)$. Let us denote the highest profit maximizing quantity by

$$\phi^*_{\max}=\max\{\phi:\phi\in M(\boldsymbol{q}^*)\}$$

and the corresponding stopout price by $\boldsymbol{p}^*_{\min}$. In the case $\phi^*_{\max}<\boldsymbol{S}(\boldsymbol{p}^*_{\min})$ consider the following deviation of any bidder j who demands positive quantity at the stopout price defined by the lowest quantity from the set $\boldsymbol{M}(\boldsymbol{q}^*)$:

$$\langle p^*_{\min}-\varepsilon,\boldsymbol{q}_j^*\left(p_S^*\right)\rangle.$$

Analogous to the Case 2(b) it can be argued that for a sufficiently small ε, the seller supplies the quantity $\phi^*_{\max}$ at the stopout price $\boldsymbol{p}^*_{\min}-\varepsilon$. Bidder j obtains the same quantity and pays $\boldsymbol{p}^*_{\min}-\varepsilon<\boldsymbol{p}_S^*$. The deviation is profitable. In the case $\phi^*_{\max}=\boldsymbol{S}(\boldsymbol{p}^*_{\min})$ one considers the deviation

$$\left\langle p^*_{\min},\quad\frac{\boldsymbol{q}_j^*(p^*_{\min})\cdot\left(\sum_{i\neq j}\boldsymbol{q}_i^*(p^*_{\min})\right)}{S(p^*_{\min})-\boldsymbol{q}_j^*(p^*_{\min})}\right\rangle.$$

Once again analogously one can show that the seller supplies $\phi^*_{\max}$ and bidder j is granted the quantity $\boldsymbol{q}_j^*$ at the price $\boldsymbol{p}_{\min}<\boldsymbol{p}_S^*$. The deviation is profitable, which completes the proof.

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