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# An inventory model under inflation for deteriorating items with stock-dependent consumption rate and partial backlogging shortages

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ABSTRACT

In this paper, we extend Teng, J.T., Chang, H.J., Dye, C.Y., Hung, C.H. [2002. An optimal replenishment policy for deteriorating items with time-varying demand and partial backlogging. Operations Research Letters 30(6), 387–393.] and Hou, K.L. [2006. An inventory model for deteriorating items with stock-dependent consumption rate and shortages under inflation and time discounting. European Journal of Operational Research 168(2), 463–474.] by considering an inventory lot-size model under inflation for deteriorating items with stock-dependent consumption rate when shortages are partial backlogging. The proposed model allows for (1) partial backlogging, (2) time-varying replenishment cycles, and (3) time-varying shortage intervals. Consequently, the proposed model is in a general framework that includes numerous previous models as special cases. We then prove that the optimal replenishment schedule exists uniquely, and provide a good estimate for finding the optimal replenishment number. Furthermore, we briefly discuss some special cases of the proposed model related to previous models. Finally, numerical examples to illustrate the solution process and some managerial implications are provided.

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## 1. Introduction

In many real-life situations, for certain types of consumer goods (e.g., fruits, vegetables, donuts, and others), the consumption rate is sometimes influenced by the stock-level. It is usually observed that a large pile of goods on shelf in a supermarket will lead the customer to buy more and then generate higher demand. The consumption rate may go up or down with the on-hand stock level. These phenomena attract many marketing researchers to investigate inventory models related to stock-level. The related analysis on such inventory system with stock-dependent consumption rate was studied by Levin et al. (1972), Baker and Urban (1988), Mandal and Maiti (1997, 1999), Balkhi and Benkherouf (2004), etc. Recently, Alfares (2007) proposed the inventory model with stock-level dependent demand rate and variable holding cost.

As a matter of fact, some products (e.g., fruits, vegetables, pharmaceuticals, volatile liquids, and others) deteriorate continuously due to evaporation, obsolescence, spoilage, etc. Ghare and Schrader (1963) first derived an economic order quantity (EOQ) model by assuming exponential decay. Next, Covert and Philip (1973) extended Ghare and Schrader's constant deterioration rate to a two-parameter Weibull distribution. Shah

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and Jaiswal (1977) and Aggarwal (1978) then discussed the EOQ model with a constant rate of deterioration. Thereafter, Dave and Patel (1981) considered an inventory model for deteriorating items with time-proportional demand when shortages were prohibited. Sachan (1984) further extended the model to allow for shortages. Later, Hariga (1996) generalized the demand pattern to any log-concave function. Teng et al. (1999) and Yang et al. (2001) further generalized the demand function to include any non-negative, continuous function that fluctuates with time. Recently, Goyal and Giri (2001) wrote an excellent survey on the recent trends in modeling of deteriorating inventory since early 1990s.

The characteristic of all of the above articles is that the unsatisfied demand (due to shortages) is completely backlogged. However, in reality, demands for foods, medicines, etc. are usually lost during the shortage period. Montgomery et al. (1973) studied both deterministic and stochastic demand inventory models with a mixture of backorder and lost sales. Later, Rosenberg (1979) provided a new analysis of partial backorders. Park (1982) reformulated the cost function and established the solution. Mak (1987) modified the model by incorporating a uniform replenishment rate to determine the optimal production-inventory control policies. For fashionable commodities and high-tech products with short product life cycle, the willingness for a customer to wait for backlogging during a shortage period is diminishing with the length of the waiting time. Hence, the longer the waiting time, the smaller the backlogging rate. To reflect this phenomenon, Chang and Dye (1999) developed an inventory model in which the proportion of customers who would like to accept backlogging is the reciprocal of a linear function of the waiting time. Concurrently, Papachristos and Skouri (2000) established a partially backlogged inventory model in which the backlogging rate decreases exponentially as the waiting time increases. Teng et al. (2002, 2003) then extended the fraction of unsatisfied demand backordered to any decreasing function of the waiting time up to the next replenishment. Teng and Yang (2004) further generalized the partial backlogging EOQ model to allow for time-varying purchase cost. Yang (2005) made a comparison among various partial backlogging inventory lot-size models for deteriorating items on the basis of maximum profit. Lately, Hou (2006) developed an inflation model for deteriorating items with stock-dependent consumption rate and completely backordered shortages by assuming a constant length of replenishment cycles and a constant fraction of the shortage length with respect to the cycle

**Table 1**
Major characteristics of inventory models on selected articles.

| Author(s) and published (year) | Demand rate | Deterioration rate | Allow for shortages | With partial backlogging | Under inflation |
|---|---|---|---|---|---|
| Aggarwal (1978) | Order level | Constant | No | No | No |
| Alfares (2007) | Stock-dependent | No | No | No | No |
| Baker and Urban (1988) | Stock-dependent | No | No | No | No |
| Balkhi and Benkherouf (2004) | Stock-dependent | Constant | No | No | No |
| Bierman and Thomas (1977) | Constant | No | No | No | Yes |
| Buzacott (1975) | Constant | No | No | No | Yes |
| Chang and Dye (1999) | Time varying (logconcave) | Constant | Yes | Yes | No |
| Chern et al. (2005) | Time varying | Constant | Yes | Yes | No |
| Chern et al. (2008) | Time varying | Time varying | Yes | Yes | Yes |
| Covert and Philip (1973) | Constant | Weibull distribution | No | No | No |
| Dave and Patel (1981) | Time proportional | Constant | No | No | No |
| Ghare and Schrader (1963) | Constant | Constant | No | No | No |
| Hariga (1996) | Time varying (logconcave) | Constant | Yes | No | No |
| Hou (2006) | Stock-dependent | Constant | Yes | No | Yes |
| Mak (1987) | Constant | No | Yes | Yes | No |
| Mandal and Maiti (1997) | Stock-dependent | Stock-dependent | Yes | No | No |
| Mandal and Maiti (1999) | Stock-dependent | Stock-dependent | No | No | No |
| Misra (1975) | Constant | No | No | No | Yes |
| Misra (1979) | Constant | No | Yes | No | Yes |
| Montgomery et al. (1973) | Constant | No | Yes | Yes | No |
| Papachristos and Skouri (2000) | Time varying (logconcave) | Constant | Yes | Yes | No |
| Park (1982) | Constant | No | Yes | Yes | No |
| Rosenberg (1979) | Constant | No | Yes | Yes | No |
| Sachan (1984) | Time proportional | Constant | Yes | No | No |
| San Jose et al. (2006) | Constant | No | Yes | Yes | No |
| Shah and Jaiswal (1977) | Order level | Constant | No | No | No |
| Teng et al. (2002) | Time varying (logconcave) | Constant | Yes | Yes | No |
| Teng et al. (1999) | Time varying | Constant | Yes | No | No |
| Teng et al. (2007) | Price dependent | No | Yes | Yes | No |
| Teng et al. (2003) | Time varying | Constant | Yes | Yes | No |
| Teng and Yang (2004) | Time varying | Constant | Yes | Yes | No |
| Yang (2005) | Time varying | Constant | Yes | Yes | No |
| Yang et al. (2001) | Time varying | Constant | Yes | No | Yes |
| Present paper | Stock-dependent | Constant | Yes | Yes | Yes |

length. San Jose et al. (2006) proposed an inventory system with exponential partial backordering. Recently, Teng et al. (2007) compared two pricing and lot-sizing models for deteriorating items with partial backlogging.

Moreover, the effects of inflation and time value of money are vital in practical environment, especially in the developing countries with large scale inflation. Therefore, the effect of inflation and time value of money cannot be ignored in real situations. To relax the assumption of no inflationary effects on costs, Buzacott (1975) and Misra (1975) simultaneously developed an EOQ model with a constant inflation rate for all associated costs. Bierman and Thomas (1977) then proposed an EOQ model under inflation that also incorporated the discount rate. Misra (1979) then extended the EOQ model with different inflation rates for various associated costs. Later, Yang et al. (2001) established various inventory models with time varying demand patterns under inflation. Recently, Chern et al. (2008) proposed partial backlogging inventory lot-size models for deteriorating items with fluctuating demand under inflation. The major assumptions used in the above research articles are summarized in Table 1.

In this paper, we develop an economic order quantity (EOQ) model, in which (1) shortages are partial backlogging to reflect the fact that longer the waiting time; the smaller the backlogging rate, (2) the effects of inflation and time value of money are relevant or vital, and (3) the replenishment cycles and the shortage intervals are time-varying. As a result, our proposed model is in a general framework that includes numerous previous models as special cases such as Hou (2006), Teng et al. (2002), and others. We then prove that the optimal replenishment schedule uniquely exists and the total profit associated with the inventory system is a concave function of the number of replenishments. Hence, the search for the optimal number of replenishments is simplified to finding a local maximum. In addition, we discuss some special cases and provide numerical examples for illustration. A sensitivity analysis with respect to parameters of the system is also made and some managerial phenomena are presented. Finally, we make a summary and provide suggestions for future research.

## 2. Assumptions and notation

The mathematical model of the inventory replenishment problem is based on the following assumptions:

1. The replenishment rate is infinite and lead time is zero.
2. For deteriorating items, a constant fraction of the on-hand inventory deteriorates per unit of time and there is no repair or replacement of the deteriorated inventory during the planning period.
3. Shortages are allowed. Unsatisfied demand is partially backlogged. The fraction of shortages backordered is a differentiable and decreasing function of time $t$, denoted by $\delta(t)$, where $t$ is the waiting time up to the next replenishment, and $0 \le \delta(t) \le 1$ with $\delta(0) = 1$. Note that if $\delta(t) = 1$ (or 0) for all $t$, then shortages are completely backlogged (or lost).
4. When the objective is maximizing profit, the cost of lost sales is the cost of lost goodwill and future profit, which varies with products and firms and ranges from zero to the estimated future profit per customer.

For convenience, the following notation is used throughout the entire paper.

| | |
|---|---|
| $H$ | the time horizon under consideration |
| $I(t)$ | the inventory level at time $t$ |
| $f(t)$ | $\begin{cases} \alpha+\beta I(t), & I(t)>0 \\ \alpha, & I(t)\le 0 \end{cases}$, the demand rate at time $t$, where $\alpha>0$, $\beta$ is the stock-dependent consumption rate parameter, $0 \le \beta \le 1$ |
| $\delta(t)$ | the backlogging rate which is a decreasing function of the waiting time $t$, WLOG, we here assume that $\delta(t)=e^{-\sigma t}$, where $\sigma \ge 0$, and $t$ is the waiting time |
| $\theta$ | the deterioration rate per unit per unit time |
| $r$ | the discount rate |
| $i$ | the inflation rate, which is varied by the social economical situations (e.g., consumer price index (CPI) and producer price index (PPI)), and the company operation status (e.g., operation cost index, and productivity index) |
| $R$ | $r-i$, the discount rate minus the inflation rate |
| $p$ | selling price per unit |
| $c_o$ | the internal fixed order cost per order |
| $c_p$ | the external variable purchasing cost per unit |
| $c_h$ | the inventory holding cost per unit per unit time |
| $c_b$ | the backlogging cost per unit per unit time |
| $c_l$ | the cost of lost sales per unit |
| $n$ | the number of replenishments over [0, $H$] (a decision variable) |
| $t_i$ | the $i$th replenishment time (a decision variable), $i=1,2,\ldots,n$ |
| $s_i$ | the time at which the inventory level reaches zero after $t_i$ (a decision variable), $i=1,2,\ldots,n$ |

## 3. Mathematical model

The $i$th replenishment is made at time $t_i$. The quantity received at $t_i$ is used partly to meet the accumulated backorders in the previous cycle from time $s_{i-1}$ to $t_i (s_{i-1} < t_i)$. The inventory at $t_i$ gradually reduces to zero at $s_i (s_i > t_i)$. Consequently, based on whether the inventory is permitted to start and/or end with shortages, we have four possible inventory shortage models as shown in Teng et al. (1999), which is depicted graphically in Figs. 1–4. In fact, the model by Hou (2006) is the same as Model 2. Since Model 3 is the most profitable model among these four shortage models (e.g., see Chern et al. (2005) for detailed proof), we discuss Model 3 only. The reader can obtain similar results for the other three shortage models.

During the time interval $[t_i, s_i]$, the inventory is depleted by the combined effect of stock-dependent consumption rate and deterioration, the inventory level at time $t$ during the $i$th replenishment cycle is governed by the following differential equation:

$$\frac{dI(t)}{dt} = -[\alpha+\beta I(t)] - \theta I(t), \quad t_i \le t \le s_i, \tag{1}$$

with the boundary condition $I(s_i)=0$. Solving the differential Eq. (1), we have

$$I(t) = \frac{\alpha}{\beta+\theta}(e^{(\beta+\theta)(s_i-t)} - 1), \quad t_i \le t \le s_i. \tag{2}$$

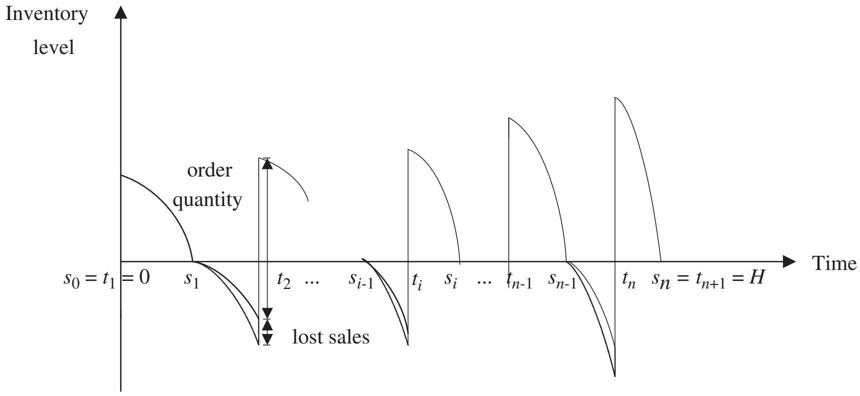


**Fig. 1.** Graphical representation of inventory Model 1.

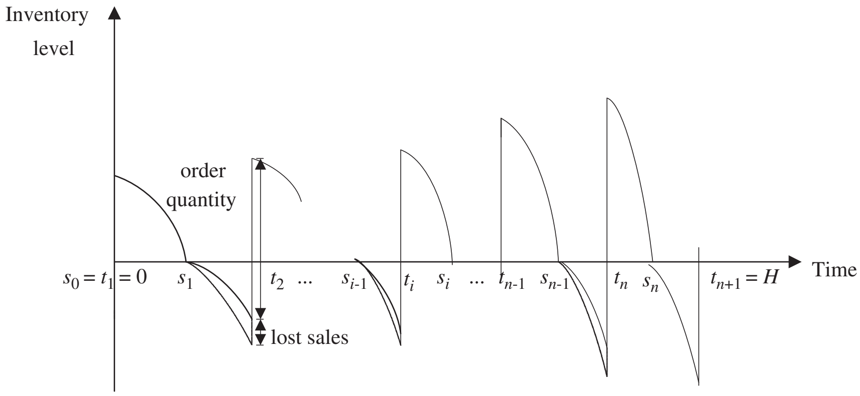


**Fig. 2.** Graphical representation of inventory Model 2.

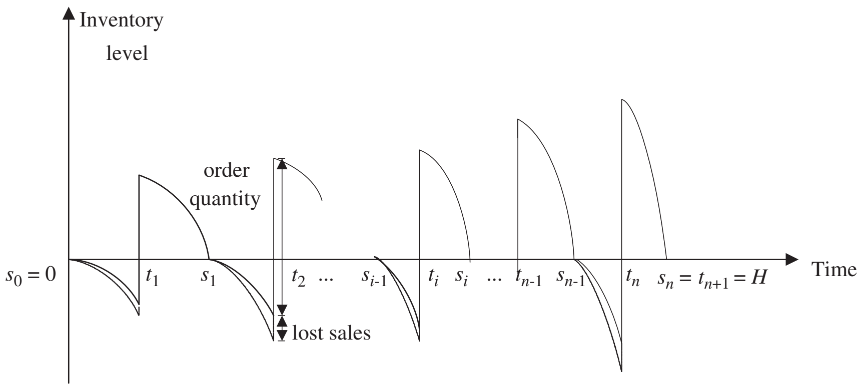


**Fig. 3.** Graphical representation of inventory Model 3.

As a result, we obtain the present value of the inventory holding cost during the $i$th replenishment cycle as

$$I_i = c_h \int_{t_i}^{s_i} e^{-Rt} I(t)dt = \frac{\alpha c_h e^{-Rt_i}}{\beta+\theta}\left[\frac{(e^{(\beta+\theta)(s_i-t_i)} - e^{-R(s_i-t_i)})}{\beta+\theta+R} - \frac{(1-e^{-R(s_i-t_i)})}{R}\right], \quad i = 1,2,\ldots,n. \tag{3}$$

During the time interval $[s_{i-1}, t_i)$, the demand rate $f(t) = \alpha$ and the backlogging rate $\delta(t_i - t) = e^{-\sigma(t_i-t)}$. Hence, the amount of backorders $B(t)$ is governed by the following differential equation:

$$\frac{dB(t)}{dt} = \alpha e^{-\sigma(t_i-t)}, \quad s_{i-1} \le t < t_i, \tag{4}$$

with the boundary condition $B(s_{i-1}) = 0$. Solving the differential Eq. (4), we have

$$B(t) = \frac{\alpha}{\sigma}(e^{-\sigma(t_i-t)} - e^{-\sigma(t_i-s_{i-1})}), \quad s_{i-1} \le t < t_i, \tag{5}$$

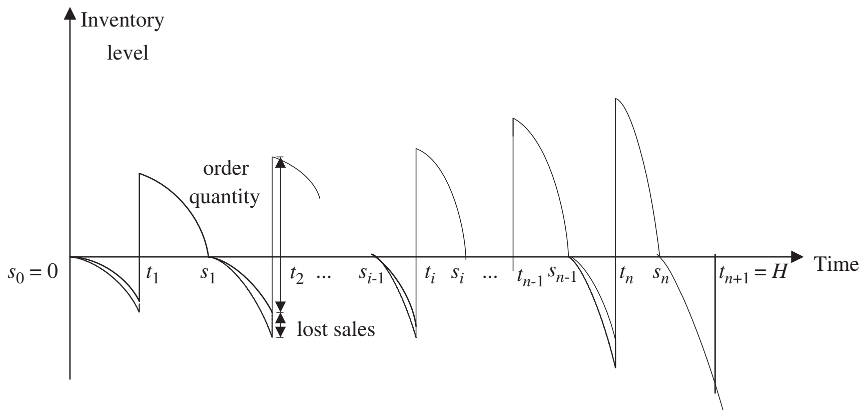


**Fig. 4.** Graphical representation of inventory Model 4.

and the amount of lost sales at time $t$ during $[s_{i-1}, t_i)$ is

$$L(t) = \alpha \int_{s_{i-1}}^{t} [1 - e^{-\sigma(t_i-u)}]\,du, \quad s_{i-1} \le t < t_i. \tag{6}$$

Consequently, the present value of the backlogging cost during $[s_{i-1}, t_i)$ is

$$\begin{aligned} B_i &= c_b \int_{s_{i-1}}^{t_i} e^{-Rt} B(t)\,dt \\ &= \frac{\alpha c_b e^{-Rt_i}}{R}\left(\frac{e^{(R-\sigma)(t_i-s_{i-1})}-1}{R-\sigma} - \frac{1-e^{-\sigma(t_i-s_{i-1})}}{\sigma}\right), \quad \text{if } R \neq \sigma, \\ &= \frac{\alpha c_b}{\sigma}\left[e^{-\sigma t_i}(t_i - s_{i-1}) - \frac{e^{-Rt_i}(e^{(R-\sigma)(t_i-s_{i-1})} - e^{-\sigma(t_i-s_{i-1})})}{R}\right], \\ &\quad \text{if } R = \sigma \end{aligned} \tag{7}$$

and the present value of the cost of lost sales during $[s_{i-1}, t_i)$ is

$$\begin{aligned} L_i &= \alpha c_l \int_{s_{i-1}}^{t_i} e^{-Rt}[1 - e^{-\sigma(t_i-t)}]\,dt \\ &= \alpha c_l e^{-Rt_i}\left(\frac{e^{R(t_i-s_{i-1})}-1}{R} - \frac{e^{(R-\sigma)(t_i-s_{i-1})}-1}{R-\sigma}\right), \quad \text{if } R \neq \sigma, \\ &= \alpha c_l \left[\frac{e^{-Rs_{i-1}} - e^{-Rt_i}}{R} - e^{-\sigma t_i}(t_i - s_{i-1})\right], \quad \text{if } R = \sigma. \end{aligned} \tag{8}$$

From (2) and (5), we have the order quantity at $t_i$ in the $i$th replenishment cycle as

$$Q_i = B(t_i) + I(t_i) = \frac{\alpha}{\sigma}(1 - e^{-\sigma(t_i-s_{i-1})}) + \frac{\alpha}{\beta+\theta}(e^{(\beta+\theta)(s_i-t_i)} - 1). \tag{9}$$

Therefore, the present value of the purchase cost during the $i$th replenishment cycle is

$$P_i = c_o e^{-Rt_i} + c_p e^{-Rt_i} Q_i. \tag{10}$$

The present value of revenue during the $i$th replenishment cycle is

$$\begin{aligned} R_i &= p\left[e^{-Rt_i}B(t_i) + \int_{t_i}^{s_i} e^{-Rt} f(t)\,dt\right] \\ &= p\left[e^{-Rt_i}B(t_i) + \int_{t_i}^{s_i} e^{-Rt}(\alpha + \beta I(t))\,dt\right]. \end{aligned} \tag{11}$$

For simplicity, we assume that $R \neq \sigma$ throughout the rest of the paper. By using a similar analogous argument, the reader can obtain the results for the case of $R = \sigma$. Hence, if $n$ replenishment orders are placed in [0, $H$], then the present value of the total profit during the planning horizon from 0 to $H$ is as follows:

$$\begin{aligned} TP(n, \{s_i\}, \{t_i\}) &= \sum_{i=1}^{n}(R_i - P_i - I_i - B_i - L_i) \\ &= \sum_{i=1}^{n} \frac{\alpha(p\beta - c_h)e^{-Rt_i}}{\beta+\theta+R}\left(\frac{e^{(\beta+\theta)(s_i-t_i)}-1}{\beta+\theta} - \frac{1-e^{-R(s_i-t_i)}}{R}\right) \\ &+ \sum_{i=1}^{n} p\alpha e^{-Rt_i}\left(\frac{1-e^{-R(s_i-t_i)}}{R} + \frac{1-e^{-\sigma(t_i-s_{i-1})}}{\sigma}\right) \\ &- \sum_{i=1}^{n} c_o e^{-Rt_i} - \sum_{i=1}^{n} \alpha c_p e^{-Rt_i}\left(\frac{1-e^{-\sigma(t_i-s_{i-1})}}{\sigma} + \frac{e^{(\beta+\theta)(s_i-t_i)}-1}{\beta+\theta}\right) \\ &- \sum_{i=1}^{n} \frac{\alpha c_b e^{-Rt_i}}{R}\left(\frac{e^{(R-\sigma)(t_i-s_{i-1})}-1}{R-\sigma} - \frac{1-e^{-\sigma(t_i-s_{i-1})}}{\sigma}\right) \\ &- \sum_{i=1}^{n} \alpha c_l e^{-Rt_i}\left(\frac{e^{R(t_i-s_{i-1})}-1}{R} - \frac{e^{(R-\sigma)(t_i-s_{i-1})}-1}{R-\sigma}\right), \end{aligned} \tag{12}$$

with $0 = s_0 < t_1$, $s_{i-1} < t_i < s_i$, $i = 1, 2, \ldots, n$, and $s_n = H$. The objective of the problem here is to determine $n$, $\{s_i\}$ and $\{t_i\}$ such that $TP(n, \{s_i\}, \{t_i\})$ in (12) is maximized.

## 4. Theoretical results and solution

$TP(n, \{s_i\}, \{t_i\})$ is a continuous (and differentiable) function maximized over the compact set $[0, H]^{2n}$. Hence, there exists an absolute maximum. The optimal value of $t_i$ (*i.e.*, $t_i^*$) can not be on the boundary since $TP(n, \{s_i\}, \{t_i\})$ decreases when any one of the $t_i$s is shifted to the end points 0 or $H$. Consequently, there exists at least an inner optimal solution for each $t_i$. In addition, since the shortage cost is not infinity, we know from the previous research result that $s_i$ must be an inner point between $t_i$ and $t_{i+1}$. Consequently, we obtain from Fig. 3 that

$$0 = s_0 < t_1 < s_1 < t_2 < s_2 < t_3 < \ldots < s_{n-1} < t_n < s_n = H.$$

As a result, for a fixed value of $n$, the necessary conditions for $TP(n, \{s_i\}, \{t_i\})$ to be maximized are: $\partial TP(n, \{s_i\}, \{t_i\})/\partial s_i = 0$, for $i = 1, 2, \ldots, n-1$, and $\partial TP(n, \{s_i\}, \{t_i\})/\partial t_i = 0$, for $i = 1, 2, \ldots, n$. Consequently, we obtain

$$e^{-Rt_{i+1}}\left[\left(p+\frac{c_b}{R}-c_p\right)e^{-\sigma(t_{i+1}-s_i)}+\left(c_l-\frac{c_b}{R}\right)e^{(R-\sigma)(t_{i+1}-s_i)}-c_l e^{R(t_{i+1}-s_i)}\right]$$
$$=e^{-Rt_i}\left[\frac{p\beta-c_h}{\beta+\theta+R}(e^{(\beta+\theta)(s_i-t_i)}-e^{-R(s_i-t_i)})-c_p e^{(\beta+\theta)(s_i-t_i)}+pe^{-R(s_i-t_i)}\right], \tag{13}$$

and

$$\alpha\left[\frac{c_h-p\beta+(\beta+\theta+R)c_p}{\beta+\theta}(e^{(\beta+\theta)(s_i-t_i)}-1)\right]$$
$$=\alpha\left[\frac{\sigma}{R-\sigma}\left(c_l-\frac{c_b}{R}\right)(e^{(R-\sigma)(t_i-s_{i-1})}-1)+\frac{R+\sigma}{\sigma}\left(p+\frac{c_b}{R}-c_p\right)(1-e^{-\sigma(t_i-s_{i-1})})\right]-Rc_o, \tag{14}$$

respectively. From (14), we assume that the left-hand side is greater than zero. Consequently, the problem here has a solution only if the right-hand side of (14) is also greater than zero. For simplicity, let the right-hand side of (14) be

$$RH(s_{i-1},t_i)=\alpha\left[\frac{\sigma}{R-\sigma}\left(c_l-\frac{c_b}{R}\right)(e^{(R-\sigma)(t_i-s_{i-1})}-1)+\frac{R+\sigma}{\sigma}\left(p+\frac{c_b}{R}-c_p\right)(1-e^{-\sigma(t_i-s_{i-1})})\right]-Rc_o, \tag{15}$$

and the partial derivative of $RH(s_{i-1},t_i)$ with respect to $t_i$ is

$$RH_{t_i}(s_{i-1},t_i)=\alpha\left[\sigma\left(c_l-\frac{c_b}{R}\right)e^{(R-\sigma)(t_i-s_{i-1})}+(R+\sigma)\left(p+\frac{c_b}{R}-c_p\right)e^{-\sigma(t_i-s_{i-1})}\right]. \tag{16}$$

Note that $RH(s_{i-1},t_i)\le 0$, only if the ordering cost, $c_o$, is huge. However, in today's e-procurements, the ordering cost is small in general. Thus, from now on, we assume WLOG that $RH(s_{i-1},t_i)>0$. Besides, in order to guarantee the solution exists, we here also assume that $RH_{t_i}(s_{i-1},t_i)>0$ for all $t_i>s_{i-1}$, $i=1,2,\dots,n$.

**Theorem 1.** *For a fixed value of $n$, if $RH(s_{i-1},t_i)>0$ and $RH_{t_i}(s_{i-1},t_i)>0$ for all $t_i>s_{i-1}$, $i=1,2,\dots,n$, then the solution to Eqs.* (13) *and* (14) *uniquely exists* (*i.e., the optimal values of $\{s_i^*\}$ and $\{t_i^*\}$ are uniquely determined by* (13) *and* (14)).

**Proof.** See Appendix A.

Note that Theorem 1 reduces the $2n$-dimensional problem of finding $\{s_i^*\}$ and $\{t_i^*\}$ to a one-dimensional problem. Since $s_0=0$, we only need to find $t_1^*$ to generate $s_1^*$ by (14), $t_2^*$ by (13), and then the rest of $\{s_i^*\}$ and $\{t_i^*\}$ uniquely by repeatedly using (14) and (13). For any chosen $t_1^*$, if $s_n^*=H$, then $t_1^*$ is chosen correctly. Otherwise, we can easily find the optimal $t_1^*$ by standard search techniques. For any given value of $n$, the solution procedure for finding $\{s_i^*\}$ and $\{t_i^*\}$ can be obtained by the algorithm in Yang et al. (2001) with $L=H/(4n)$ and $U=H/n$ or any standard search method.

**Theorem 2.** *For a fixed value of $n$, if $RH(s_{i-1},t_i)>0$ and $RH_{t_i}(s_{i-1},t_i)>0$ for all $t_i>s_{i-1}$, $i=1,2,\dots,n$, then the solution to Eqs.* (13) *and* (14) *is a global maximum.*

**Proof.** See Appendix B.

Next, we show that the present value of the total profit $TP(n,\{s_i^*\},\{t_i^*\})$ is a concave function of the number of replenishments. As a result, the search for the optimal replenishment number, $n^*$, is reduced to find a local maximum. For simplicity, let

$TP(n)=TP(n,\{s_i^*\},\{t_i^*\})$.

By applying Bellman's principle of optimality (1957), we have the following theorem:

**Theorem 3.** *$TP(n)$ is concave in $n$.*

**Proof.** See Appendix C.

## 5. Estimation of the replenishment number

To avoid using a brute force enumeration for finding $n^*$ with starting value $n=1$, we further simplify the search process by providing an intuitively good starting value for $n^*$. In fact, we obtained an estimate of the optimal number of replenishments in Teng et al. (2002) as shown below.

$$n_1=\text{round integer of}\left[\frac{\alpha(c_h+\theta c_p)\{c_b\delta(1)+(c_l-c_p)[1-\delta(1)]\}H^2}{2c_o\{c_h+\theta c_p+c_b\delta(1)+(c_l-c_p)[1-\delta(1)]\}}\right]^{1/2}. \tag{17}$$

Since $TP(n)$ is concave in $n$, the search for the optimal $n^*$ by using a good starting value such as $n_1$ in (17) will reduce the computational complexity significantly, comparing to the brute force enumeration starting with $n=1$ (such as in Chang and Dye (1999) or Papachristos and Skouri (2000)). Judging from numerous numerical examples, our proposed $n_1$ in (17) is significantly closer to the optimal number of replenishments than $n=1$ for all realistic examples, as shown in Section 7.

## 6. Special cases

In this section, we will discuss some special cases that influence the total profit.

*Case* 1. The inflationary effect is not considered, i.e., $R=0$. In the proposed model, the total profit is given by

$$\begin{aligned}TP_1(n,\{s_i\},\{t_i\})&=\sum_{i=1}^{n}(R_i-P_i-I_i-B_i-L_i)\\&=\sum_{i=1}^{n}\alpha\left(\frac{p\beta-c_h}{\beta+\theta}-c_p\right)\left(\frac{e^{(\beta+\theta)(s_i-t_i)}-1}{\beta+\theta}\right)\\&\quad+\sum_{i=1}^{n}\alpha\left[\frac{c_h-p\beta}{\beta+\theta}+p\right](s_i-t_i)-nc_o\\&\quad+\sum_{i=1}^{n}\alpha\left(p+c_l-\frac{c_b}{\sigma}-c_p\right)\left(\frac{1-e^{-\sigma(t_i-s_{i-1})}}{\sigma}\right)\\&\quad-\sum_{i=1}^{n}\alpha\left(c_l-\frac{c_b}{\sigma}e^{-\sigma(t_i-s_{i-1})}\right)(t_i-s_{i-1}).\end{aligned} \tag{18}$$

For a fixed value of $n$, the necessary conditions for $TP_1(n,\{s_i\},\{t_i\})$ to be maximized are: $\partial TP_1(n,\{s_i\},\{t_i\})/\partial s_i=0$, for $i=1,2,\dots,n-1$, and $\partial TP_1(n,\{s_i\},\{t_i\})/\partial t_i=0$, for $i=1,2,\dots,n$. Consequently, we obtain

$$[c_b(t_{i+1}-s_i)+c_p-p-c_l]e^{-\sigma(t_{i+1}-s_i)}+c_l = \frac{c_h-p\beta}{\beta+\theta}(e^{(\beta+\theta)(s_i-t_i)}-1)+c_p e^{(\beta+\theta)(s_i-t_i)}-p, \tag{19}$$

and

$$\frac{c_h-p\beta}{\beta+\theta}(e^{(\beta+\theta)(s_i-t_i)}-1)+c_p e^{(\beta+\theta)(s_i-t_i)}-p = [c_b(t_i-s_{i-1})+c_p-p-c_l]e^{-\sigma(t_i-s_{i-1})}+c_l. \tag{20}$$

*Case* 2. The backlogging is complete, i.e., $\sigma=0$. The total profit is given by

$$\begin{aligned} TP_2(n,\{s_i\},\{t_i\}) &= \sum_{i=1}^{n}(R_i-P_i-I_i-B_i) \\ &= \sum_{i=1}^{n}\frac{\alpha(p\beta-c_h)e^{-Rt_i}}{\beta+\theta+R}\left(\frac{e^{(\beta+\theta)(s_i-t_i)}-1}{\beta+\theta}-\frac{1-e^{-R(s_i-t_i)}}{R}\right) \\ &\quad+\sum_{i=1}^{n}p\alpha e^{-Rt_i}\left[\frac{1-e^{-R(s_i-t_i)}}{R}+(t_i-s_{i-1})\right] \\ &\quad-\sum_{i=1}^{n}c_o e^{-Rt_i}-\sum_{i=1}^{n}\alpha c_p e^{-Rt_i}\left[(t_i-s_{i-1})+\frac{e^{(\beta+\theta)(s_i-t_i)}-1}{\beta+\theta}\right] \\ &\quad-\sum_{i=1}^{n}\frac{\alpha c_b e^{-Rt_i}}{R}\left[\frac{e^{R(t_i-s_{i-1})}-1}{R}-(t_i-s_{i-1})\right]. \end{aligned} \tag{21}$$

For a fixed value of $n$, the necessary conditions for $TP_2(n,\{s_i\},\{t_i\})$ to be maximized are: $\partial TP_2(n,\{s_i\},\{t_i\})/\partial s_i=0$, for $i=1,2,\ldots,n-1$, and $\partial TP_2(n,\{s_i\},\{t_i\})/\partial t_i=0$, for $i=1,2,\ldots,n$. Consequently, we obtain

$$\begin{aligned} e^{-Rt_{i+1}}\left[\frac{c_b}{R}(e^{R(t_{i+1}-s_i)}-1)+c_p-p\right] &= e^{-Rt_i}\left[\frac{c_h-p\beta}{\beta+\theta+R}(e^{(\beta+\theta)(s_i-t_i)}-e^{-R(s_i-t_i)}) \right.\\ &\quad \left. +c_p e^{(\beta+\theta)(s_i-t_i)}-pe^{-R(s_i-t_i)}\right], \end{aligned} \tag{22}$$

and

$$\alpha\left[\frac{c_h-p\beta+(\beta+\theta+R)c_p}{\beta+\theta}(e^{(\beta+\theta)(s_i-t_i)}-1)\right] = \alpha(c_b+Rp-Rc_p)(t_i-s_{i-1})-Rc_o. \tag{23}$$

*Case* 3. The inflationary effect is not considered, and the backlogging is complete, i.e., $R=0$ and $\sigma=0$. The total profit is given by

$$\begin{aligned} TP_3(n,\{s_i\},\{t_i\}) &= \sum_{i=1}^{n}(R_i-P_i-I_i-B_i) \\ &= \sum_{i=1}^{n}\alpha\left(\frac{p\beta-c_h}{\beta+\theta}-c_p\right)\left(\frac{e^{(\beta+\theta)(s_i-t_i)}-1}{\beta+\theta}\right) \\ &\quad+\sum_{i=1}^{n}\alpha\left[\frac{c_h-p\beta}{\beta+\theta}+p\right](s_i-t_i)-nc_o \\ &\quad-\sum_{i=1}^{n}\alpha\left[\frac{c_b}{2}(t_i-s_{i-1})^2+(c_p-p)(t_i-s_{i-1})\right]. \end{aligned} \tag{24}$$

For a fixed value of $n$, the necessary conditions for $TP_3(n,\{s_i\},\{t_i\})$ to be maximized are: $\partial TP_3(n,\{s_i\},\{t_i\})/\partial s_i=0$, for $i=1,2,\ldots,n-1$, and $\partial TP_3(n,\{s_i\},\{t_i\})/\partial t_i=0$, for $i=1,2,\ldots,n$. Consequently, we obtain

$$c_b(t_{i+1}-s_i)=\left(\frac{c_h-p\beta}{\beta+\theta}+c_p\right)[e^{(\beta+\theta)(s_i-t_i)}-1], \tag{25}$$

and

$$\left(\frac{c_h-p\beta}{\beta+\theta}+c_p\right)[e^{(\beta+\theta)(s_i-t_i)}-1]=c_b(t_i-s_{i-1}). \tag{26}$$

**Theorem 4.** *If $R=0$ (Cases 1 and 3), the cycle length and the no-shortage fraction in each cycle are equal.*

**Proof.** From (19) to (20) and (25) to (26), we have the following results:

$$[c_b(t_{i+1}-s_i)+c_p-p-c_l]e^{-\sigma(t_{i+1}-s_i)} = [c_b(t_i-s_{i-1})+c_p-p-c_l]e^{-\sigma(t_i-s_{i-1})},$$

$$\frac{c_h-p\beta}{\beta+\theta}(e^{(\beta+\theta)(s_i-t_i)}-1)+c_p e^{(\beta+\theta)(s_i-t_i)} = \frac{c_h-p\beta}{\beta+\theta}(e^{(\beta+\theta)(s_{i-1}-t_{i-1})}-1)+c_p e^{(\beta+\theta)(s_{i-1}-t_{i-1})},$$

and

$$c_b(t_{i+1}-s_i)=c_b(t_i-s_{i-1}),$$

$$\left(\frac{c_h-p\beta}{\beta+\theta}+c_p\right)[e^{(\beta+\theta)(s_i-t_i)}-1]=\left(\frac{c_h-p\beta}{\beta+\theta}+c_p\right)[e^{(\beta+\theta)(s_{i-1}-t_{i-1})}-1].$$

Thus, we know that $t_{i+1}-s_i=t_i-s_{i-1}$, $i=1,2,\ldots,n$ and $s_i-t_i=s_{i-1}-t_{i-1}$, for $i=2,\ldots,n$. □

## 7. Numerical examples

**Example 1.** Let $\alpha=600$, $\beta=0.25$, $c_o=250$, $c_h=1.75$, $c_b=3$, $c_l=7$, $c_p=5$, $p=10$ in appropriate unit, and $\theta=0.2$, $\sigma=0.02$, $R=0.06$, $H=10$ years. By using (17) and after some mathematical manipulations, we get $\delta(1)\approx 0.98$ and the estimate number of replenishments as $n_1=13$. From (12), we obtain $TP(12)=17920.06$, $TP(13)=17922.80$, and $TP(14)=17898.05$. Thus, the optimal replenishment number during the planning horizon $H$ is 13 and the optimal profit is 17922.80. The optimal replenishment schedule is shown in Table 2. In addition, if $\beta=0$, then we have the optimal replenishment number and profit are 14 and 17252.49, respectively. From Table 2, we know that the replenishment cycle intervals and the fractions of shortages are not equal.

**Example 2.** (*Case of no inflationary effect*) Using the same numerical values as in Example 1 except $R=0$. From (17), we obtain the estimate number of replenishments as $n_1=13$. By (18), we obtain $TP_1(11)=24279.65$, $TP_1(12)=24290.38$, and $TP_1(13)=24259.14$. Thus, the optimal replenishment number during the planning horizon $H$ is 12, the optimal profit is 24290.38, the no-shortage fraction is 0.6546, and each replenishment cycle is 0.8333. The optimal replenishment schedule is shown as in Table 3. In addition, if $\beta=0$, then we have the optimal replenishment number and the profit are 14 and 23275.03, respectively.

**Example 3.** (*Case of completely backordered*) Using the same numerical values as in Example 1 except $\sigma = 0$ and $c_l = 0$. By (21), we obtain $TP_2(11) = 17949.45$, $TP_2(12) = 17981.89$, and $TP_2(13) = 17979.72$. Thus, the optimal replenishment number during the planning horizon $H$ is 12, and the optimal profit is 17981.89. In addition, if $\beta = 0$, then we have the optimal replenishment number and the profit are 14 and 17339.65, respectively.

**Example 4.** (*Case of no inflationary effect and completely backordered*) Using the same numerical values as in Example 1 except $R = 0$, $\sigma = 0$ and $c_l = 0$. By (24), we obtain $TP_3(11) = 24357.83$, $TP_3(12) = 24361.39$, and $TP_3(13) = 24324.17$. Thus, the optimal replenishment number during the planning horizon $H$ is 12, and the optimal profit is 24361.39. In addition, if $\beta = 0$, then we have the optimal replenishment number and the profit are 13 and 23393.15, respectively.

In order to understand the effects of the changes in parameters on the solution, we use the numerical values in Example 1 to perform the sensitivity analysis, and the numerical results obtained are shown in Table 4.

Based on the computational results as shown in Table 4, we obtain the following managerial phenomena.

(1) The present value of the total profit increases, if $\alpha$, $\beta$, $p$, $H$ or $\delta(t)$ increases. However, it decreases as $c_p$, $c_o$, $c_h$, $c_b$, $c_l$, $\theta$ or $R$ increases.

**Table 2**
The optimal replenishment schedule of Example 1.

| $i$ | $t_i$ | $s_i$ | $s_i-t_i$ | $t_{i+1}-t_i$ | $k_i$ |
|---|---|---|---|---|---|
| 0 | – | 0.0000 | 0.0000 | – | |
| 1 | 0.2867 | 0.7759 | 0.4892 | 0.7755 | 0.6308 |
| 2 | 1.0622 | 1.5508 | 0.4886 | 0.7746 | 0.6308 |
| 3 | 1.8368 | 2.3248 | 0.4880 | 0.7736 | 0.6308 |
| 4 | 2.6104 | 3.0978 | 0.4874 | 0.7726 | 0.6309 |
| 5 | 3.3829 | 3.8679 | 0.4867 | 0.7715 | 0.6309 |
| 6 | 4.1544 | 4.6405 | 0.4861 | 0.7703 | 0.6310 |
| 7 | 4.9247 | 5.4101 | 0.4853 | 0.7691 | 0.6310 |
| 8 | 5.6939 | 6.1785 | 0.4846 | 0.7679 | 0.6311 |
| 9 | 6.4618 | 6.9456 | 0.4838 | 0.7666 | 0.6311 |
| 10 | 7.2284 | 7.7114 | 0.4830 | 0.7652 | 0.6312 |
| 11 | 7.9936 | 8.4757 | 0.4821 | 0.7638 | 0.6312 |
| 12 | 8.7574 | 9.2386 | 0.4812 | 0.7623 | 0.6313 |
| 13 | 9.5197 | 10.0000 | 0.4803 | – | – |

*Note*: $k_i = (s_i - t_i)/(t_{i+1} - t_i)$, $i = 1,\ldots,n-1$.

**Table 3**
The optimal replenishment schedule of Example 2.

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $t_i$ | – | 0.2878 | 1.1212 | 1.7545 | 2.7878 | 3.6212 | 4.4545 |
| $s_i$ | 0.0000 | 0.8333 | 1.6667 | 2.5000 | 3.3333 | 4.1667 | 5.0000 |
| $i$ | 7 | 8 | 9 | 10 | 11 | 12 | |
| $t_i$ | 5.2878 | 6.1212 | 6.9545 | 7.7878 | 8.6212 | 9.4545 | |
| $s_i$ | 5.8333 | 6.6667 | 7.5000 | 8.3333 | 9.1667 | 10.000 | |

**Table 4**
Sensitivity analysis on the total profit.

| $\alpha$ | $n^*$ | $TP^*$ | $\beta$ | $n^*$ | $TP^*$ |
|---|---|---|---|---|---|
| 450 | 11 | 12889.92 | 0.20 | 13 | 17757.54 |
| 600 | 13 | 17922.80 | 0.25 | 13 | 17922.80 |
| 750 | 14 | 23033.09 | 0.30 | 12 | 18118.15 |
| $p$ | $n^*$ | $TP^*$ | $H$ | $n^*$ | $TP^*$ |
| 8 | 13 | 8614.65 | 8 | 10 | 15145.39 |
| 10 | 13 | 17922.80 | 10 | 13 | 17922.80 |
| 12 | 12 | 27364.09 | 12 | 15 | 20390.88 |
| $\delta(t)$ | $n^*$ | $TP^*$ | $c_p$ | $n^*$ | $TP^*$ |
| $e^{-0.03t}$ | 13 | 17896.00 | 4 | 11 | 22876.86 |
| $e^{-0.02t}$ | 13 | 17922.80 | 5 | 13 | 17922.80 |
| $e^{-0.01t}$ | 13 | 17950.69 | 6 | 13 | 13094.33 |
| $c_o$ | $n^*$ | $TP^*$ | $c_h$ | $n^*$ | $TP^*$ |
| 200 | 14 | 18426.47 | 1.4 | 12 | 18222.60 |
| 250 | 13 | 17922.80 | 1.75 | 13 | 17922.80 |
| 300 | 12 | 17465.99 | 2.1 | 13 | 17680.74 |
| $c_b$ | $n^*$ | $TP^*$ | $c_l$ | $n^*$ | $TP^*$ |
| 2 | 12 | 18234.91 | 5 | 13 | 17932.50 |
| 3 | 13 | 17922.80 | 7 | 13 | 17922.80 |
| 4 | 13 | 17719.03 | 9 | 13 | 17913.24 |
| $\theta$ | $n^*$ | $TP^*$ | $R$ | $n^*$ | $TP^*$ |
| 0.1 | 11 | 18400.42 | 0.04 | 12 | 19765.67 |
| 0.2 | 13 | 17922.80 | 0.06 | 13 | 17922.80 |
| 0.3 | 14 | 17566.57 | 0.08 | 13 | 16310.82 |

(2) The present value of the total profit is more sensitive on the change in $\alpha$, $p$, $H$, $c_p$ or $R$. It implies that the effect of these parameters on the total profit is significant.
(3) The total number of replenishments increases as $\alpha$, $H$, $c_p$, $c_h$, $c_b$, $\theta$ or $R$ increases, while it decreases as $p$, $\beta$, $c_o$ increases.
(4) The total number of replenishments is insensitive to the change in $c_l$ or $\delta(t)$, while the others are not.

## 8. Conclusions

In this paper, a partial backlogging inventory lot-size model for deteriorating items with stock-dependent demand has been proposed. We have shown that not only the optimal replenishment schedule exists uniquely, but also the total profit associated with the inventory system is a concave function of the number of replenishments. We also have simplified the search process by establishing an intuitively good starting value for the optimal number of replenishments. From the numerical results and sensitivity analysis, we have provided several managerial phenomena.

The proposed model can be further extended in several ways. For example, we may add pricing strategy into consideration. Also, we could extend the deterministic model into a stochastic model. Finally, we could generalize the model to allow for quantity discounts, trade credits, or others.


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## Appendix A. Proof of Theorem 1

From the first paragraph of Section 4, we know that there exists an optimal solution such that

$0=s_0<t_1<s_1<t_2<s_2<t_3<\ldots<s_{n-1}<t_n<s_n=H.$

From (13), let

$$\begin{aligned} &G(t_i,s_i;x)\\ &=e^{-Rx}\left[\left(p+\frac{c_b}{R}-c_p\right)e^{-\sigma(x-s_i)}+\left(c_l-\frac{c_b}{R}\right)e^{(R-\sigma)(x-s_i)}-c_le^{R(x-s_i)}\right]\\ &\quad-e^{-Rt_i}\left[\frac{p\beta-c_h}{\beta+\theta+R}(e^{(\beta+\theta)(s_i-t_i)}-e^{-R(s_i-t_i)})\right.\\ &\quad\left.-c_pe^{(\beta+\theta)(s_i-t_i)}+pe^{-R(s_i-t_i)}\right], \qquad (A.1)\end{aligned}$$

with $x\geq s_i$. Differentiating (A.1) with respect to $x$, we know from the assumption on (16) that

$$\begin{aligned} dG(t_i,s_i;x)/dx=&-e^{-Rx}\left[(R+\sigma)\left(p+\frac{c_b}{R}-c_p\right)e^{-\sigma(x-s_i)}\right.\\ &\left.+\sigma\left(c_l-\frac{c_b}{R}\right)e^{(R-\sigma)(x-s_i)}\right]<0. \qquad (A.2)\end{aligned}$$

In addition, we obtain $G(t_i,s_i;\,s_i)>0$, $G(t_i,s_i;\,\infty)=\lim_{x\to 0}G(t_i,s_i;\,x)<0$. Note that if $G(t_i,s_i,\,\infty)\geq 0$, then $\partial\, TP(n,\{s_i\},\{t_i\})/\partial\, s_i=-G(t_i,s_i;t_{i+1})<0$, which in turn implies that $s_i=t_i$. By using (A.1), we have $G(t_i,s_i;\,\infty)=-e^{-Rt_i}(p-c_p)<0$ if $s_i=t_i$. This leads to a contradiction. As a result, for any given $t_i$ and $s_i$, there exists a unique $t_{i+1}$ ($>s_i$ and $<H$) such that $G(t_i,s_i;\,t_{i+1})=0$. (i.e., (13) implies a unique interior solution of $t_{i+1}$).

Similarly, from (14), let

$$\begin{aligned} &F(s_{i-1},t_i;x)\\ &=\alpha\left[\frac{c_h-p\beta+(\beta+\theta+R)c_p}{\beta+\theta}(e^{(\beta+\theta)(x-t_i)}-1)\right]\\ &\quad-\alpha\left[\frac{\sigma}{R-\sigma}\left(c_l-\frac{c_b}{R}\right)(e^{(R-\sigma)(t_i-s_{i-1})}-1)\right.\\ &\quad\left.+\frac{R+\sigma}{\sigma}\left(p+\frac{c_b}{R}-c_p\right)(1-e^{-\sigma(t_i-s_{i-1})})\right]+Rc_o, \qquad (A.3)\end{aligned}$$

with $x\geq t_i$. Differentiating (A.3) with respect to $x$, we have

$$dF(s_{i-1},t_i;x)/dx=\alpha[c_h-p\beta+(\beta+\theta+R)c_p]e^{(\beta+\theta)(x-t_i)}>0. \qquad (A.4)$$

By the assumption that $RH(s_{i-1},t_i)>0$, we know that $F(s_{i-1},t_i;t_i)<0$, and $F(s_{i-1},t_i;\,\infty)=\lim_{x\to\infty}F(s_{i-1},t_i,x)>0$. Therefore, for any given $s_{i-1}$ and $t_i$, there exists a unique $s_i(>t_i$ and $\leq t_{i+1})$ such that $F(s_{i-1},t_i;s_i)=0$. (i.e., (14) implies a unique solution of $s_i$).

## Appendix B. Proof of Theorem 2

Taking second partial derivatives with respect to $t_i$ and $s_i$ on (13) and (14), respectively, we have

$$\begin{aligned}\frac{\partial^2 TP}{\partial t_i^2}=&-\alpha e^{-Rt_i}\left[(R+\sigma)\left(p+\frac{c_b}{R}-c_p\right)e^{-\sigma(t_i-s_{i-1})}+\sigma\left(c_l-\frac{c_b}{R}\right)\right.\\ &\left.\times e^{(R-\sigma)(t_i-s_{i-1})}+(c_h-p\beta+(\beta+\theta+R)c_p)e^{(\beta+\theta)(s_i-t_i)}\right], \qquad (A.5)\end{aligned}$$

$$\frac{\partial^2 TP}{\partial t_i\partial s_i}=\alpha e^{-Rt_i}[c_h-p\beta+(\beta+\theta+R)c_p]e^{(\beta+\theta)(s_i-t_i)}>0, \qquad (A.6)$$

$$\begin{aligned}\frac{\partial^2 TP}{\partial s_i\partial t_{i+1}}=&\alpha e^{-Rt_{i+1}}\left[(R+\sigma)\left(p+\frac{c_b}{R}-c_p\right)e^{-\sigma(t_{i+1}-s_i)}\right.\\ &\left.+\sigma\left(c_l-\frac{c_b}{R}\right)e^{(R-\sigma)(t_{i+1}-s_i)}\right]>0, \qquad (A.7)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 TP}{\partial s_i^2}=&\alpha e^{-Rt_i}\left[(\beta+\theta)\left(\frac{p\beta-c_h}{\beta+\theta+R}-c_p\right)e^{(\beta+\theta)(s_i-t_i)}\right.\\ &\left.+R\left(\frac{p\beta-c_h}{\beta+\theta+R}-p\right)e^{-R(s_i-t_i)}\right]\\ &-\alpha e^{-Rt_{i+1}}\left[\sigma\left(p+\frac{c_b}{R}-c_p\right)e^{-\sigma(t_{i+1}-s_i)}\right.\\ &\left.-(R-\sigma)\left(c_l-\frac{c_b}{R}\right)e^{(R-\sigma)(t_{i+1}-s_i)}+Rc_le^{R(t_{i+1}-s_i)}\right]. \qquad (A.8)\end{aligned}$$

Observing the relations among the second-order partial derivatives, we know that

$$\begin{aligned}&\frac{\partial^2 TP}{\partial t_i^2}=-\left[\frac{\partial^2 TP}{\partial t_i\partial s_{i-1}}+\frac{\partial^2 TP}{\partial t_i\partial s_i}\right]<0 \quad \text{and}\\ &\frac{\partial^2 TP}{\partial s_i^2}\leq-\left[\frac{\partial^2 TP}{\partial s_i\partial t_{i+1}}+\frac{\partial^2 TP}{\partial s_i\partial t_i}\right]<0 \qquad (A.9)\end{aligned}$$

Let $\Delta_k$ be the principal minor of order $k$, it is clear that

$$\begin{aligned}\Delta_1=\frac{\partial^2 TP}{\partial t_1^2}=&-\alpha e^{-Rt_1}\left[(R+\sigma)\left(p+\frac{c_b}{R}-c_p\right)e^{-\sigma(t_1-s_0)}\right.\\ &+\sigma\left(c_l-\frac{c_b}{R}\right)e^{(R-\sigma)(t_1-s_0)}\\ &\left.+(c_h-p\beta+(\beta+\theta+R)c_p)e^{(\beta+\theta)(s_1-t_1)}\right]<0\end{aligned}$$

$$\begin{aligned}\Delta_2=&\frac{\partial^2 TP}{\partial t_1^2}\frac{\partial^2 TP}{\partial s_1^2}-\frac{\partial^2 TP}{\partial s_1\partial t_1}\frac{\partial^2 TP}{\partial s_1\partial t_1}\geq\frac{\partial^2 TP}{\partial t_1\partial s_1}\left[\frac{\partial^2 TP}{\partial s_1\partial t_2}+\frac{\partial^2 TP}{\partial s_1\partial t_1}\right]\\ &-\left[\frac{\partial^2 TP}{\partial s_1\partial t_1}\right]^2\geq\frac{\partial^2 TP}{\partial t_1\partial s_1}\frac{\partial^2 TP}{\partial s_1\partial t_2}>0.\end{aligned}$$

(By (A.6) and (A.7)).

For the principal minor of higher order, $i=2,3,\ldots$, it is not difficult to show that they satisfy the following recursive relation:

$$\Delta_{2i-1}=\frac{\partial^2 TP}{\partial t_i^2}\Delta_{2i-2}-\left[\frac{\partial^2 TP}{\partial t_i \partial s_{i-1}}\right]^2\Delta_{2i-3} \quad \text{and}$$

$$\Delta_{2i}=\frac{\partial^2 TP}{\partial s_i^2}\Delta_{2i-1}-\left[\frac{\partial^2 TP}{\partial s_i \partial t_i}\right]^2\Delta_{2i-2}$$

with the initial $\Delta_0=1$. Consequently, by (A.9), we have

$$\Delta_{2i-1}+\frac{\partial^2 TP}{\partial s_i \partial t_i}\Delta_{2i-2}=-\frac{\partial^2 TP}{\partial t_i \partial s_{i-1}}\left(\Delta_{2i-2}+\frac{\partial^2 TP}{\partial t_i \partial s_{i-1}}\Delta_{2i-3}\right),$$

and

$$\Delta_{2i}+\frac{\partial^2 TP}{\partial s_i \partial t_{i+1}}\Delta_{2i-1}\geq-\frac{\partial^2 TP}{\partial t_i \partial s_i}\left(\Delta_{2i-1}+\frac{\partial^2 TP}{\partial s_i \partial t_i}\Delta_{2i-2}\right).$$

For $i=2$, we obtain

$$\Delta_3+\frac{\partial^2 TP}{\partial s_2 \partial t_2}\Delta_2=-\frac{\partial^2 TP}{\partial t_2 \partial s_1}\left(\Delta_2+\frac{\partial^2 TP}{\partial t_2 \partial s_1}\Delta_1\right)<0,$$

and

$$\Delta_4+\frac{\partial^2 TP}{\partial s_2 \partial t_3}\Delta_3\geq-\frac{\partial^2 TP}{\partial t_2 \partial s_2}\left(\Delta_3+\frac{\partial^2 TP}{\partial s_2 \partial t_2}\Delta_2\right)>0.$$

Thus,

$$\Delta_3<-\frac{\partial^2 TP}{\partial s_2 \partial t_2}\Delta_2 \quad \text{and} \quad \Delta_4>-\frac{\partial^2 TP}{\partial s_2 \partial t_3}\Delta_3>0.$$

Proceeding inductively, we have

$$\Delta_{2i-1}+\frac{\partial^2 TP}{\partial s_i \partial t_i}\Delta_{2i-2}<0 \quad \text{and} \quad \Delta_{2i}+\frac{\partial^2 TP}{\partial s_i \partial t_{i+1}}\Delta_{2i-1}>0.$$

Therefore, $\Delta_{2i-1}<0$ and $\Delta_{2i}>0$, for $i=2,3,\ldots$. This completes the proof.

## Appendix C. Proof of Theorem 3

Assuming that $n$ orders are placed in [0, $H$], let

$$TP(n,\{s_i\},\{t_i\})=R(n)+T(n,0,H), \tag{A.10}$$

where $R(n)=-\sum_{i=1}^{n}c_o e^{-rt_i}$ and $T(n,0,H)=\sum_{i=1}^{n}(R_i-P_i-I_i-B_i-L_i+c_o e^{-rt_i})$. Firstly, we know that $R(n)$ is a decreasing concave function of $n$. Next, by Bellman's principle of optimality (1957), we know that the maximum value of $T(n,0,H)$ is

$$T^*(n,0,H)=\underset{t\in[0,H]}{\text{Maximize}}\{T^*(n-1,0,t)=T(1,t,H)\}. \tag{A.11}$$

Let $t=H$, and hence, $T^*(n,0,H)>T^*(n-1,0,H)$. The strict inequality follows because the maximum in (A.11) occurs at an interior point. Thus, $T^*(n,0,H)$ is strictly increasing in $n$. Recursive application of (A.11) yields the following relations:

$$t_i^*(n,0,H)=t_i^*(n-j,0,t_{n-j}^*(n,0,H)), \quad i=1,2,\ldots,n-j-1, \tag{A.12}$$

$$s_i^*(n,0,H)=s_i^*(n-j,0,s_{n-j}^*(n,0,H)), \quad i=1,2,\ldots,n-j-1. \tag{A.13}$$

In order to prove that $T^*(n,0,H)$ is strictly concave in $n$, we choose $H_1$ and $H_2$ such that

$$s_n^*(n+1,0,H_1)=s_{n+1}^*(n+2,0,H_2)=H,$$

and

$$s_0^*(n+1,0,H_1)=s_0^*(n+2,0,H_2)=0. \tag{A.14}$$

Employing the principle of optimality on (A.11) again, we have

$$\begin{aligned}T^*(n+1,0,H_1)&=\underset{t\in[0,H]}{\text{Maximize}}\{T^*(n,0,t)+T(1,t,H_1)\}\\&=T^*(n,0,H)+T(1,H,H_1),\end{aligned} \tag{A.15}$$

and

$$\begin{aligned}T^*(n+2,0,H_2)&=\underset{t\in[0,H]}{\text{Maximize}}\{T^*(n+1,0,t)+T(1,t,H_2)\\&=T^*(n+1,0,H)+T(1,H,H_2),\end{aligned} \tag{A.16}$$

respectively. Since $H$ is an optimal interior point in $T^*(n+1,0,H_1)$ and $T^*(n+2,0,H_2)$, we know that

$$\left.\frac{\partial T^*(n,0,t)}{\partial t}+\frac{\partial T(1,t,H_1)}{\partial t}\right|_{t=H}=0, \tag{A.17}$$

and

$$\left.\frac{\partial T^*(n+1,0,t)}{\partial t}+\frac{\partial T(1,t,H_2)}{\partial t}\right|_{t=H}=0. \tag{A.18}$$

Utilizing the fact that

$$\begin{aligned}T(1,a,b)=&\frac{\alpha(p\beta-c_h)e^{-Rw}}{\beta+\theta+R}\left(\frac{e^{(\beta+\theta)(b-w)}-1}{\beta+\theta}-\frac{1-e^{-R(b-w)}}{R}\right)\\&+p\alpha e^{-Rw}\left(\frac{1-e^{-R(b-w)}}{R}+\frac{1-e^{-\sigma(w-a)}}{\sigma}\right)\\&-\alpha c_p e^{-Rw}\left(\frac{1-e^{-\sigma(w-a)}}{\sigma}+\frac{e^{(\beta+\theta)(b-w)}-1}{\beta+\theta}\right)\\&-\frac{\alpha c_b e^{-Rw}}{R}\left(\frac{e^{(R-\sigma)(w-a)}-1}{R-\sigma}-\frac{1-e^{-\sigma(w-a)}}{\sigma}\right)\\&-\alpha c_l e^{-Rw}\left(\frac{e^{R(w-a)}-1}{R}-\frac{e^{(R-\sigma)(w-a)}-1}{R-\sigma}\right),\end{aligned} \tag{A.19}$$

and

$$\begin{aligned}\frac{\partial T(1,a,b)}{\partial a}=&\alpha e^{-Rw}\left[\left(c_p-p-\frac{c_b}{R}\right)e^{-\sigma(w-a)}\right.\\&\left.+\left(\frac{c_b}{R}-c_l\right)e^{(R-\sigma)(w-a)}+c_l e^{R(w-a)}\right],\end{aligned} \tag{A.20}$$

where $w$ is the replenishment time between ($a$, $b$), and both $a$ and $b$ are the time at which the inventory level drops to zero in the cycle ($a$, $b$). We then obtain

$$\begin{aligned}\left.\frac{\partial T^*(n,0,t)}{\partial t}\right|_{t=H}&=-\left.\frac{\partial T(1,t,H_1)}{\partial t}\right|_{t=H}\\&=-\alpha e^{-Rt_{n+1}^*(n+1,0,H_1)}\left[\left(c_p-p-\frac{c_b}{R}\right)e^{-\sigma(t_{n+1}^*(n+1,0,H_1)-H)}\right.\\&\quad\left.+\left(\frac{c_b}{R}-c_l\right)e^{(R-\sigma)(t_{n+1}^*(n+1,0,H_1)-H)}+c_l e^{R(t_{n+1}^*(n+1,0,H_1)-H)}\right]\\&=-\alpha e^{-Rt_n^*(n,0,H)}\left[\frac{c_h-p\beta}{\beta+\theta+R}(e^{(\beta+\theta)(H-t_n^*(n,0,H))}-e^{-R(H-t_n^*(n,0,H))})\right.\\&\quad\left.+c_p e^{(\beta+\theta)(H-t_n^*(n,0,H))}-pe^{-R(H-t_n^*(n,0,H))}\right], \quad \text{(by (13))}\end{aligned} \tag{A.21}$$

where $t_n^*(n,0,H)$ and $t_{n+1}^*(n+1,0,H_1)$are the corresponding last replenishment time when $n$ orders are placed in

[0, $H$], and $n$+1 orders are placed in [0, $H_1$], respectively. Similarly,

$$\left.\frac{\partial T^*(n+1,0,t)}{\partial t}\right|_{t=H} = -\left.\frac{\partial T(1,t,H_2)}{\partial t}\right|_{t=H}$$
$$= -\alpha e^{-Rt^*_{n+1}(n+1,0,H)}$$
$$\times \left[\frac{c_h - p\beta}{\beta+\theta+R}\left(e^{(\beta+\theta)(H-t^*_{n+1}(n+1,0,H))} - e^{-R(H-t^*_{n+1}(n+1,0,H))}\right)\right.$$
$$\left. + c_p e^{(\beta+\theta)(H-t^*_{n+1}(n+1,0,H))} - p e^{-R(H-t^*_{n+1}(n+1,0,H))}\right]. \quad (A.22)$$

Subtracting (A.21) from (A.22), we have

$$\frac{\partial}{\partial t}[T^*(n,0,t) - T^*(n+1,0,t)]|_{t=H} = R(t^*_{n+1}(n+1,0,H),H) - R(t^*_n(n,0,H),H) < 0, \quad (A.23)$$

where

$$R(t,H) = \alpha e^{-Rt}\left[\frac{c_h - p\beta}{\beta+\theta+R}\left(e^{(\beta+\theta)(H-t)} - e^{-R(H-t)}\right) + c_p e^{(\beta+\theta)(H-t)} - p e^{-R(H-t)}\right]$$

is a decreasing function for all $t < H$, since $R_t(t,H) < 0$. This implies that $T^*(n,0,H) - T^*(n+1,0,H)$ is a strictly decreasing function of $H$. Thus,

$$T^*(n,0,H) - T^*(n+1,0,H) > T^*(n,0,H_1) - T^*(n+1,0,H_1). \quad (A.24)$$

Again, by (A.11) and the principle of optimality, we have

$$T^*(n,0,H_1) - T^*(n+1,0,H_1)$$
$$= \underset{t\in[0,H]}{\text{Maximize}}\{T^*(n-1,0,t) + T(1,t,H_1)\} - T^*(n,0,H) - T(1,H,H_1). \quad (A.25)$$

Let $t = H$ in (A.25), we have

$$T^*(n,0,H_1) - T^*(n+1,0,H_1) > T^*(n-1,0,H) - T^*(n,0,H). \quad (A.26)$$

By (A.24) and (A.26), we have

$$T^*(n,0,H) - T^*(n+1,0,H) > T^*(n-1,0,H) - T^*(n,0,H), \quad (A.27)$$

which implies $T^*(n,0,H)$ is concave in $n$, and, hence, $TP(n) = R(n) + T^*(n,0,H)$ is also concave in $n$.

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