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SCIENCE DIRECT
Games and Economic Behavior 52 (2005) 1–19
GAMES and Economic Behavior
www.elsevier.com/locate/geb

Renegotiation in the repeated Cournot model

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Received 14 April 2000

Available online 25 September 2004

Abstract

We explore a new concept of renegotiation proofness in the symmetric repeated Cournot model with several players. We show that this concept significantly limits the cooperative outcomes that can be sustained in equilibrium. In particular, the symmetric monopoly outcome cannot be sustained when the number of players is high enough (9 in the case of the linear demand function). When the number of players tends to infinity, (i) the collusive benefits that could be sustained are at most four times the Cournot benefits, and (ii) the *reasonable* price that can be sustained in equilibrium tends to the Cournot price.

JEL classification: C70; C72

Keywords: Infinitely repeated games; Renegotiation; Cournot model

1. Introduction

It is well known that repeating a game may lead to the emergence of new equilibria which can sustain many cooperative outcomes. In Fudenberg and Maskin (1986) it was shown that, under simple conditions, any feasible and individually rational payoff of the one-shot game can be sustained as a subgame perfect equilibrium in the corresponding supergame with discounting provided that the players are sufficiently patient (i.e., that

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0899-8256/$ – see front matter
doi:10.1016/j.geb.2004.06.008

the discount factor is sufficiently close to 1). Recently, many authors have suggested that, in an infinitely repeated game, the players should have the option of renegotiating their prescribed strategies at the beginning of each period.

With the aim of avoiding subgame perfect equilibria which are likely to be renegotiated different concepts of renegotiation-proofness have been introduced. Probably the best-known are the (Weak) Renegotiation Proof Equilibrium (WRP) of Farrell and Maskin (1989)[1] and the Consistent Bargaining Equilibrium (CBE) of Abreu et al. (1993). In the symmetric Cournot supergame with perfect monitoring one has the unsatisfactory result that the symmetric monopoly outcome (and therefore the monopoly price) can be sustained as a subgame perfect equilibrium by means of the trigger strategy, Friedman (1971), for any finite number of players provided that they are sufficiently patient. But the trigger strategy is not renegotiation proof under any definition of this concept. So a natural question arises: Is it still possible to sustain the monopoly outcome as a renegotiation proof equilibrium when the number of players is high enough? As we will see the answer to this question is basically affirmative for both the CBE and the WRP concepts. So these two concepts do not significantly limit the cooperative outcomes that can be sustained in equilibrium.

Moreover, van Damme (1989) in the final remark of his paper (pp. 217), conjectured: "*The Renegotiation Proof Equilibrium concept leads to the conclusion that* (*even for* δ *arbitrarily close to* 1) *the monopoly price can be sustained only when there are at most* 3 *players and that, as the number of players tends to infinity, only prices close to the Cournot price can be sustained*." This comment refers to the WRP definition of Farrell and Maskin. In Section 3 we will prove that this conjecture is wrong. However, in Section 4, we will show that the intuition of van Damme is in some sense correct when we impose an additional requirement of symmetry on the original WRP definition.

As we will see, WRP may lead to rather asymmetric payoffs during the punishment path of a player. This seems unreasonable. Note that all authors agree that equal division of the surplus should be used as a natural solution when the game is symmetric. Then, why should non-deviating symmetric players produce highly asymmetric quantities during a punishment phase? At the other extreme, CBE leads, in the Cournot model, to strongly symmetric strategy profiles. This means that all players have to produce exactly the same in all periods, even after any deviation. As a consequence all the players suffer the punishment of the cheater equally. Neither do we find reasonable the claim of the cheater for same payoffs in the continuation of the game, since when a player cheats he obtains more benefits than the others and also inflicts losses on them.

The above comments led us to define a new renegotiation proof concept, keeping some of the ideas underlying both CBE and WRP. This new definition is only suitable for symmetric games and is closer to that of Farrell and Maskin. The WRP concept requires that, in equilibrium, no continuation payoff be Pareto dominated by any other continuation payoff. In addition we require a kind of reasonable symmetry to exclude those equilibria in which planned punishments hurt any of the punishers. We will call this concept Partially

[1] In an independent paper Bernheim and Ray (1989) proposed an equivalent concept, which they called (Internally) Consistent Equilibrium.

Symmetric Weak Renegotiation Proof equilibrium (PSWRP). We first introduce necessary and sufficient conditions for a symmetric payoff vector to be PSWRP (payoffs associated with PSWRP-equilibria are called PSWRP-payoffs). We will see that the PSWRP concept significantly limits the cooperative outcomes that can be sustained. In particular, for any discount factor, the symmetric monopoly outcome cannot be sustained by a PSWRP equilibrium when the number of players is high enough (9 in the linear demand case). Furthermore, when the number of players tends to infinity (even for a discount factor arbitrarily close to 1),

(i) the collusive benefits that could be sustained as a PSWRP equilibrium are, at most, four times the Cournot benefits;
(ii) the "*reasonable collusive price*" that can be sustained in equilibrium tends to the Cournot price.

We write "*reasonable price*" because technically it is possible to sustain a price significantly higher than the Cournot price, but this is not reasonable in a sense which will be specified further on. For a clearer understanding of this issue we refer the reader to the end of Section 4.

After we had submitted this paper to this journal, we learned about the related, but independent work of Farrell (2000). Farrell proposes the concept of quasi-symmetrically weakly renegotiation proof equilibrium (QSWRP), which he applies to repeated Bertrand competition and to one numerical Cournot example with linear cost, in which case he obtains a similar result as we do. We will further discuss Farrell's paper in the conclusion of this paper.

2. Preliminaries

Let $G = (Q_1, \dots, Q_n; \Pi_1, \dots, \Pi_n)$ be an n-player game where $N = \{1, \dots, n\}$ is the set of players, Q_i is the set of actions q_i of player i and $\Pi_i : Q_1 \times \cdots \times Q_n \longrightarrow R$ is player i's payoff function. The associated infinitely repeated game with discounting is denoted by $G^\infty(\delta)$ where $\delta \in (0,1)$ is the discount factor. If $q(t) = (q_1(t), \dots, q_n(t))$ is the vector of actions played in period t, then $\{q(1), \dots, q(t)\}$ is a history h of length t. A pure strategy σ_i of player i in $G^\infty(\delta)$ is a sequence of functions σ_i^t (or $\sigma_i(t)$) from the set of all histories of length $t-1$ to Q_i, so $\sigma_i^1 \in Q_i$ is the initial action of player i. A stream of action profiles $\{q(t)\}_{t=1}^\infty$ is referred to as an outcome path and is denoted by S. Any strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$ generates an outcome path $S(\sigma) = \{q(\sigma)(t)\}_{t=1}^\infty$ defined inductively by:

$$q(\sigma)(1) = \sigma^1$$
$$q(\sigma)(t) = \sigma^t\big(q(\sigma)(1), \dots, q(\sigma)(t-1)\big), \quad \text{if } t > 1.$$

The value $\Pi_i(q(t))$ denotes the payoff of player i in period t when the outcome in this period is $q(t)$. And $\Pi_i^\delta(S)$ denotes the average discounted payoff of player i for the outcome path $S = \{q(t)\}_{t=1}^{\infty}$:

$$\Pi_i^\delta(S) = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\Pi_i\big(q(t)\big).$$

Then, the average discounted payoff of player i in $G^\infty(\delta)$ obtained with the strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$ is:

$$\Pi_i^\delta(\sigma) = \Pi_i^\delta\big(S(\sigma)\big).$$

A strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$ is a Nash equilibrium in $G^\infty(\delta)$ if for all $i \in N$, σ_i is a best response to $(\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n)$. And it is a subgame perfect equilibrium (SPE) in $G^\infty(\delta)$ if after every history h, σ_h (i.e., the continuation of σ after h) is a Nash equilibrium in the corresponding subgame.

In this paper, the specific game we consider is the Cournot Model with perfect monitoring: n firms produce a homogeneous product at a constant marginal cost of $c > 0$. The industry inverse demand function is denoted by $p(z)$ and the payoffs are $\Pi_i(q_1, \dots, q_n) = (p(q_1 + \dots + q_n) - c)q_i$, where q_i is the output of firm i.

We now introduce some reasonable assumptions about the Cournot model.

Assumption A$_1$. $p : R_+ \longrightarrow R_+$ is continuous, differentiable and with $p'(z) < 0$ for all $z > 0$ such that $p(z) > 0$, $\lim_{z\to\infty} p(z) = 0$, and $p(0) > c$.

Let $q_i^D(q_{-i})$ be a single period best response to $q_{-i} = (q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n)$, that is, $q_i^D(q_{-i})$ satisfies $\Pi_i(q_1, \dots, q_i^D(q_{-i}), \dots, q_n) \geqslant \Pi_i(q_1, \dots, q_i, \dots, q_n)$ for all $q_i \in Q_i$.

Assumption A$_2$. $q_i^D(q_{-i})$ is well defined, unique, and $q^D(z) = q_1^D(q_2, \dots, q_n)$, where $z = q_2 + \dots + q_n$, is a continuous, non-increasing function.

Let $\Pi_i^D(q)$ be player i's best response payoff when other players play according to q_{-i}, that is, $\Pi_i^D(q) = \Pi_i(q_1, \dots, q_i^D(q_{-i}), \dots, q_n)$. Let nq^M be the monopoly output level, that is, q^M holds $\Pi_1(nq^M, 0, \dots, 0) \geqslant \Pi_1(q, 0, \dots, 0)$ for all $q \in Q_1$.

Assumption A$_3$. q^M is unique, strictly positive, $\Pi_1(q, \dots, q)$ declines strictly monotonically as output increases beyond q^M or falls bellow q^M.

We also introduce a capacity constraint in Q_i in order to make this set compact. Formally, $Q_i = [0, \bar{q}(\delta)]$ for all $i = 1, \dots, n$, where $\bar{q}(\delta)$ is such that

$$-\Pi_1(\bar{q}(\delta), 0, \dots, 0) > \left(\frac{\delta}{1-\delta}\right)\sup_q \Pi_1(q, 0, \dots, 0).$$

Note that this capacity constraint $\bar{q}(\delta)$ is not at all restrictive, since the loss to a firm from producing an output greater than $\bar{q}(\delta)$ cannot be recouped by any possible future gain (Abreu, 1986).

Assumptions A_1, A_2, A_3 are equivalent to the assumptions used in Segerstrom (1988) and Abreu (1986). Segerstrom (1988) proved in Lemma 1 of his paper "Demons and Repentance" that, given the assumptions A_1, A_2, A_3, the game G with two players has exactly one Cournot–Nash equilibrium, which has to be symmetric. For a game G with n players the generalization of this result is immediate.

We denote this unique Cournot–Nash equilibrium by $(q^C, \dots, q^C)$ and its payoff by $\Pi^C = \Pi_i(q^C, \dots, q^C)$. Also $\Pi^M = \Pi_i(q^M, \dots, q^M)$. For the sake of simplicity, we will use q^C instead of $(q^C, \dots, q^C)$ and q^M instead of $(q^M, \dots, q^M)$, whenever it is clear which is which from the context.

To obtain some of our results we will also use the following weak assumption.

Assumption $\mathbf{A_4}$. The monopoly profit function $U(z) = (p(z) - c)z$ is concave on $[0, \bar{z}]$, where $\bar{z}$ is the unique total quantity satisfying $p(\bar{z}) = c$.

This assumption is a little stronger than $\mathbf{A_3}$. It was also used in the work of Abreu et al. (1993). It is not really necessary but we prefer to include it in order to make the proofs more intelligible.

Throughout the paper we will use simple strategy profiles with a two-phase punishment. A simple strategy profile, Abreu (1988), is determined by $n + 1$ outcome paths $(S^0, S^1, \dots, S^n)$, where $S^i = \{q^i(t)\}_{t=1}^{\infty}$, for $i = 0, 1, \dots, n$, that induce the following strategy profile defined inductively:

(i) Play S^0 until a player deviates singly from S^0.
(ii) For any $j \in N$, play S^j if the jth player deviates singly from S^i, $i = 0, 1, \dots, n$, where S^i is an ongoing previously specified path. Continue with S^i if no deviations occur or two or more players deviate simultaneously.

Simple strategy profiles with a two-phase punishment are $\sigma \equiv (S^0, S^1, \dots, S^n)$ such that: $S^0 = \{q^0, \dots\}$; $S^1 = \{q^1, \dots, q^1, q^2, \dots\}$; $S^i = S^{1(i/1)}$ $(i = 2, \dots, n)$, where the normal (cooperative) path S^0 takes the action q^0 for every t, the punishment path S^1 takes T times the action q^1 and then q^2 forever, and $S^{1(i/1)}$ is identical to S^1 except that the roles of player 1 and player i are interchanged. In the proof of our results we will use the following lemma, which establishes sufficient conditions for σ to be SPE when the discount factor δ is sufficiently close to 1.

Lemma 1. *The above strategy profile σ is SPE for a suitable T and δ sufficiently close to* 1 *if the following conditions are met*:

(a) $\Pi_1^D(q^1) < \Pi_1(q^2) \leqslant \Pi_1(q^0)$.
(b) $\Pi_1(q^2) \leqslant \Pi_i(q^2)$ *for* $i = 2, \dots, n$, *and if* $\Pi_1(q^2) = \Pi_i(q^2)$ *for some* $i \neq 1$ *then* $\Pi_1(q^1) \leqslant \Pi_i(q^1)$.
(c) *If* $\Pi_i(q^1) = \Pi_1(q^1)$ *and* $\Pi_i(q^2) = \Pi_1(q^2)$ *for some* $i \neq 1$, *then* $\Pi_i^D(q^1) < \Pi_1(q^2)$.

Proof. See Appendix A. □

3. Weakly renegotiation proof equilibrium

Farrell and Maskin (1989) introduced the concept of weakly renegotiation proof equilibrium (WRP) in the context of two-player repeated games. A straightforward generalization of their definition to the n-players case is:

Definition 2. An SPE strategy profile σ is weakly renegotiation proof if no continuation payoff of σ is strictly Pareto dominated by any other continuation payoff of σ. If an equilibrium σ is WRP, then we also say that the payoffs $(\Pi_1^\delta(\sigma), \ldots, \Pi_n^\delta(\sigma))$are WRP.

Before discussing this definition, we present a sufficient condition that allows to sustain the symmetric monopoly outcome as a WRP equilibrium for any number of players, when δ is close enough to 1.

Lemma 3. *The symmetric monopoly outcome* $(\Pi^M, \ldots, \Pi^M)$ *is WRP for* δ *sufficiently close to* 1, *if there exists* q^* *such that*

$$\Pi_j(0, \ldots, 0, q^*, 0, \ldots, 0) \geqslant \Pi^M \quad \text{and} \quad \Pi_1^D(0, \ldots, 0, q^*, 0, \ldots, 0) < \Pi^M.$$

Proof. Consider the simple strategy profile $\sigma^* \equiv (S^0, S^1, \ldots, S^n)$ defined by $S^0 = \{q^M, \ldots\}$, $S^1 = \{q^1, \ldots, q^1, q^M, \ldots\}$, and $S^i = S^{1(i/1)}$ for $i = 2, \ldots, n$; where $q^1 = (0, \ldots, 0, q^*, 0, \ldots, 0)$ and q^* satisfies the conditions given above. Note that all the punishing players but one (who may be randomly chosen) suffer equally while the cheating player is punished. It is easy to check that the conditions of Lemma 1 hold, so σ is SPE for δ sufficiently close to 1. The number of periods that q^1 has to be repeated in S^1 is

$$T = 1 + \text{IntegerPart}\left[\max\left\{\frac{\Pi_1^D(q^M) - \Pi^M}{\Pi^M}; \frac{\Pi_j^D(q^1) - \Pi^M}{\Pi_j(q^1)}\right\}\right].$$

Furthermore, it is clear that no continuation payoff of σ is Pareto dominated by any other continuation payoff of σ. Hence σ is WRP. □

We use this sufficient condition in the proof of the following theorem. But first we introduce some useful notation. Set $\Pi^D(z) = \max\{(p(q+z) - c)q \mid q \in Q_1\}$. Recall that $\bar{z}$ is the unique total quantity satisfying $p(\bar{z}) = c$. It follows that $U(\bar{z}) = 0$ and $\Pi^D(\bar{z}) = 0$. Abreu (1986) proved that $\Pi^D(z)$ is strictly decreasing and convex on $[0, \bar{z}]$. It can be easily proved that it is also strictly convex and one-to-one.

Theorem 4. *The symmetric monopoly outcome can be sustained as a WRP equilibrium for any number of players provided that the discount factor is sufficiently close to* 1.

Proof. Set $F(z) = \Pi^D(z) - U(z)$ for all $z \in [0, \bar{z}]$. Note that $F(z)$ is strictly convex on $[0, \bar{z}]$ since $\Pi^D(z)$ is strictly convex and $U(z)$ is concave. We have $F(0) = n\Pi^M > 0$, and $F(nq^M) = \Pi^D(nq^M) - U(nq^M) = \Pi^D(nq^M) - n\Pi^M = \Pi^D(nq^M) - \Pi^D(0) < 0$ since Π^D is strictly decreasing. Then, there exists a unique $\hat{z}$, $(\hat{z} \neq \bar{z})$ and $\hat{z} \in (0, nq^M)$, such that $F(\hat{z}) = 0$, and furthermore $F(z) < 0$, that is, $\Pi^D(z) < U(z)$, if and only if, $z \in (\hat{z}, \bar{z})$.

Set $q^* = \max\{z \mid U(z) = \Pi^M\}$. Then, $q^* > nq^M > \hat{z}$. Therefore, $\Pi^D(q^*) < U(q^*) = \Pi^M$. Then, the sufficient condition of the previous lemma holds. □

This theorem proves that the conjecture of van Damme was wrong. The monopoly price, which is independent of the number of players n, can be sustained for any n and does not come close to the Cournot price.

We can describe the strategy profile that allows us to obtain this result in terms of payoffs as follows: $\sigma^* \equiv (S^0, S^1, \ldots, S^n)$ with $S^0 = \{(\Pi^M, \ldots, \Pi^M), \ldots\}$ and $S^1 = \{(0, \ldots, 0, \Pi^M, 0, \ldots, 0), \ldots, (0, \ldots, 0, \Pi^M, 0, \ldots, 0), (\Pi^M, \ldots, \Pi^M), \ldots\}$.

In view of this strategy, the reader may wonder whether the above result would continue to hold if the WRP concept is strengthened to "no continuation payoff is weakly Pareto dominated by any other continuation payoff". Note that by continuity there exists ε such that $q^* - \varepsilon$ is still bigger than nq^M, $\Pi_j(0, \ldots, 0, q^* - \varepsilon, 0, \ldots, 0) > \Pi^M$ and $\Pi_1^D(0, \ldots, 0, q^* - \varepsilon, 0, \ldots, 0) < \Pi^M$, which implies that the theorem holds with the weakly Pareto dominated concept.

The above strategy has the following drawbacks:

(i) All the punishing players but one suffer equally while the cheating player is punished. Note that Farrell and Maskin introduced their WRP concept in a two-player context. In this setting, a nice consequence of their definition is that the punishing player always profits from carrying out the punishment. This makes the punishment more credible since the punishing player has no temptation to forgive the cheater on the prospect of improving his benefits if both return to the cooperative path. However, in the strategy profile σ^* there is only one player who is not interested in skipping the punishment phase. We now recall the abstract of their work: "*In repeated games, subgame perfect equilibria involving threats of punishment may be implausible if punishing one player hurts the other(s). If players can renegotiate after a defection, such a punishment may not be carried out*". We do not think that the strategy σ fulfills the spirit of this comment (despite its being WRP).

(ii) Related to the above remark we have the question of symmetry. If all authors agree that the equal division of the surplus should be used as a natural solution when the game is symmetric, then why should non-deviating players produce such asymmetric quantities during a punishment phase?

Abreu et al. (1993) proposed a new concept of renegotiation proofness for symmetric repeated games. They called it Consistent Bargaining Equilibrium (CBE). It is based on a very different philosophy. The idea underlying their definition is that, when the game is symmetric, all the players (even the cheater) have the same bargaining power in every subgame. Then a player can impose renegotiation in a certain subgame if he is the worst-off player in that subgame and there exists another strategy profile where in any subgame the worst-off player is better-off. Although their definition, in principle, does not impose symmetry in every subgame, the authors proved that in the perfect monitoring Cournot model (with assumptions stronger than A_1, A_2, A_3) any CBE strategy profile is strongly symmetric in the sense that all the players have to play the same quantity in every period of every subgame.[2]

[2] In their work the authors proved that this concept does not limit the prospects of cooperation when δ is sufficiently close to 1.

We think that this symmetry condition is too strong. Note that when player i cheats he obtains more benefits than the others. He also inflicts losses on the rest of the players. Then, we do not find it reasonable for the cheater to claim the same payoffs in the continuation of the game. Symmetry seems equitable for the rest of the players but not for the cheater.

4. Partially symmetric weak renegotiation proof equilibrium

We believe that some reasonable requirements of symmetry are included in the following

Definition 5. A strategy profile σ is partially symmetric if:

(i) all the players produce the same on the cooperative path;
(ii) during the punishment phase of any player all the punishing players produce the same quantities;
(iii) the continuation actions of σ when player i deviates are identical to the continuation actions of σ when player j deviates, except that the roles of players i and j are interchanged.

That is, same payoffs for all players on the cooperative path, same payoffs for all punishing players on any punishment path, and same punishments for all cheaters. We now propose a second definition of renegotiation-proofness which is suitable for symmetric games, and is closer to that of Farrell and Maskin but more restrictive. Recall that the WRP concept requires that no continuation payoff be strictly Pareto dominated by any other continuation payoff. In addition we ask for partial symmetry. Note that in this way we also ensure that the punishing players profit from carrying out the punishment.

Definition 6. A strategy profile σ is a partially symmetric weak renegotiation proof equilibrium (PSWRP) if it is partially symmetric and WRP. If an equilibrium σ is PSWRP, then we also say that the payoffs $(\Pi_1^\delta(\sigma), \ldots, \Pi_n^\delta(\sigma))$ are PSWRP.

We now characterize PSWRP payoffs.

Lemma 7. *The symmetric payoff vector $(v, \ldots, v)$ is PSWRP for δ sufficiently close to 1 if there exists an action profile $q^* = (q', q, \ldots, q)$ such that $\Pi_1^D(q^*) < v \leqslant \Pi_2(q^*)$.*

Proof. Consider the partially symmetric strategy profile $\sigma \equiv (S^0, S^1, \ldots, S^n)$ defined by $S^0 = \{q^v\}_{t=1}^\infty$, $S^1 = \{q^*, \ldots_T, q^*, q^v, q^v, \ldots\}$, and $S^i = S^{1(i/1)}$ for $i = 2, \ldots, n$; where $q^v = (q^v, \ldots, q^v)$ is such that $\Pi_i(q^v) = v$ and $q^* = (q', q, \ldots, q)$ meets the above conditions. It is easy to check that the conditions of Lemma 1 are satisfied. Then σ is SPE for δ sufficiently close to 1, and the number of periods for which the action q^* is specified before returning to the normal phase is

$$T = 1 + \text{IntegerPart}\left[\max\left\{\frac{\Pi_1^D(q^v) - v}{v - \Pi_1(q^*)}; \frac{\Pi_2^D(q^*) - v}{\Pi_2(q^*) - \Pi_1(q^*)}\right\}\right].$$

Also, the relations $\Pi_1(q^*) \leqslant \Pi_1^D(q^*) < v \leqslant \Pi_2(q^*)$ clearly show that no continuation payoff of σ is strictly Pareto dominated by any other continuation payoff of σ. Consequently σ is WRP. □

Lemma 8. *If the symmetric payoff vector $(v, \dots, v)$, where $v > \Pi^C$, is PSWRP for some $\delta < 1$, then there exists an action profile $q^* = (q', q, \dots, q)$ such that $\Pi_1^D(q^*) < v \leqslant \Pi_2(q^*)$.*

Proof. See Appendix A. □

We remark that the condition of the above lemma is necessary for any δ. However, the condition given in Lemma 7 is sufficient only for δ close enough to 1.

From now on, we will use the subscript n, to clarify the notation when there are variations on the number of players.

The existence of a PSWRP payoff is trivially guaranteed since the strategy profile consisting of always playing q_n^C is PSWRP. This strategy sustains the value $v = \Pi_n^C$. Now, in order to find the most collusive payoff that can be sustained by means of PSWRP strategy profiles, we define

$$W_n = \left\{ v \mid v > \Pi_n^C, (v, \dots, v) \text{ is PSWRP for some } \delta < 1 \right\}.$$

Formally our goal is to obtain $\max\{v \mid v \in W_n\}$.

The following lemma is a different characterization of PSWRP payoffs. It will be useful to understand how the PSWRP concept restricts the cooperative outcomes that can be sustained when we progressively increase the number of players.

Lemma 9. *$v \in W_n$ if and only if there exists an action profile q^v such that $\Pi_i(q^v) = v$ and there exists z such that*

$$\Pi^D(z) < v \leqslant \frac{1}{n-1} U(z).$$

Proof. Since $\Pi_1^D(q', q, \dots, q) = \Pi_1^D(0, q, \dots, q)$ and $\Pi_2(q', q, \dots, q) \leqslant \Pi_2(0, q, \dots, q)$, we may take $q' = 0$ with no loss of generality in Lemmas 7 and 8. So $v \in W_n$ if and only if there exists q such that

$$\Pi_1^D(0, q, \dots, q) < v \leqslant \Pi_2(0, q, \dots, q). \tag{1}$$

Now,

$$\begin{aligned} \Pi_1^D(0, q, \dots, q) = \Pi^D\big((n-1)q\big) \quad \text{and} \quad & \Pi_2(0, q, \dots, q) \\ & = \frac{1}{n-1}\big(p\big((n-1)q\big) - c\big)(n-1)q = \frac{1}{n-1} U\big((n-1)q\big). \end{aligned}$$

Then, taking $z = (n-1)q$ in (1) the result follows. □

We will see further on that, when n is big enough, we only have to solve the following equation to obtain $\sup\{v \mid v \in W_n\}$:

$$\Pi^D(z) = \frac{1}{n-1} U(z).$$

The following lemma proves that this equation has a unique solution.

Lemma 10. *There exists a unique* $\bar{z}_n \in [0, \bar{z})$ *such that*

$$\Pi^D(\bar{z}_n) = \frac{1}{n-1} U(\bar{z}_n)$$

for all $n \geqslant 2$. *Moreover* $\Pi^D(z) < \frac{1}{n-1} U(z)$ *iff* $z \in (\bar{z}_n, \bar{z})$, *the sequence* $\{\bar{z}_n\}_{n=2}^{\infty}$ *is strictly increasing, and* $\lim_{n\to\infty} \bar{z}_n = \bar{z}$.

Proof. See Appendix A. □

Note that by A_4 the function $\frac{1}{n-1} U(z)$ is continuous and concave on $[0, \bar{z}]$. Therefore, the equation $\frac{1}{n-1} U(z) = v$, with $v \in (\Pi_n^C, \Pi_n^M]$, has two solutions.[3] We consider the bigger one, which is on the interval $(nq_n^M, \bar{z}]$, where the function declines. Formally

$$z_n^{v+} = \max\left\{ z \,\middle|\, \frac{1}{n-1} U(z) = v \right\}.$$

When $v = \Pi_n^M$ we simply write z_n^{M+}.

Lemma 11. *Set* $v \in (\Pi_n^C, \Pi_n^M]$. *Then,* $v \in W_n$ *if and only if* $\Pi^D(z_n^{v+}) < v$.

Proof. We only need to see that if $v \in W_n$ then $\Pi^D(z_n^{v+}) < v$. From the previous lemma there exists z such that $\Pi^D(z) < v \leqslant \frac{1}{n-1} U(z)$. If $z \leqslant nq_n^M$ then $\Pi^D(z_n^{v+}) < \Pi^D(z) < v$. But if $z > nq_n^M$ and since $\frac{1}{n-1} U(z_n^{v+}) = v \leqslant \frac{1}{n-1} U(z)$, we have $z_n^{v+} \geqslant z$. Hence, $\Pi^D(z_n^{v+}) \leqslant \Pi^D(z) < v$. □

The next theorem shows that the symmetric monopoly outcome cannot be sustained by a PSWRP equilibrium when the number of players is high enough.

Theorem 12. *There exists a number of players* $\bar{n}$ *such that* $\Pi_n^M \notin W_n$ *for all* $n \geqslant \bar{n}$. *Moreover,*

$$W_n = \begin{cases} (\Pi_n^C, \Pi_n^M] & \text{for all } n < \bar{n}, \\ (\Pi_n^C, \frac{1}{n-1} U(\bar{z}_n)) & \text{for all } n \geqslant \bar{n}. \end{cases}$$

Proof. Set $v \in (\Pi_n^C, \Pi_n^M]$. From Lemmas 10 and 11, we have

$$v \in W_n \quad \text{if and only if} \quad z_n^{v+} \in (\bar{z}_n, \bar{z}). \tag{2}$$

It is clear that $n\Pi_n^M = U(nq_n^M) = \max\{U(z) \mid z \geqslant 0\}$ (constant) for all n. As $n\Pi_n^M = \frac{n}{n-1} U(z_n^{M+})$, then $\lim_{n\to\infty} \frac{n}{n-1} U(z_n^{M+}) = n\Pi_n^M$, hence $\lim_{n\to\infty} U(z_n^{M+}) = n\Pi_n^M$. But as the function $U(z)$ is continuous and one-to-one on $[nq_n^M, \bar{z}]$, it follows that $\lim_{n\to\infty} z_n^{M+} =$

[3] If the function is not concave the equation could have more than two solutions. This is not a problem since we can also take the biggest one.

nq_n^M. Furthermore, the sequence $\{z_n^{M+}\}_{n=2}^{\infty}$ is strictly decreasing, since $U(z_n^{M+}) = (n-1)\Pi_n^M < n\Pi_{n+1}^M = U(z_{n+1}^{M+})$. Therefore $z_n^{M+} > z_{n+1}^{M+}$.

On the other hand, recall that $\{\bar{z}_n\}_{n=2}^{\infty}$ is strictly increasing and $\lim_{n\to\infty} \bar{z}_n = \bar{z}$. Taking

$$\bar{n} = \min\{n \mid z_n^{M+} \leqslant \bar{z}_n\}$$

it is easy to see that $z_n^{M+} \leqslant \bar{z}_n$ iff $n \geqslant \bar{n}$. And from (2) $\Pi_n^M \notin W_n$ iff $n \geqslant \bar{n}$.

Now, if $n \geqslant \bar{n}$ we have $\bar{z}_n \geqslant z_n^{M+} > nq_n^M$. Since, by (2), $v = \frac{1}{n-1}U(z_n^{v+}) \in W_n$ if and only if $z_n^{v+} \in (\bar{z}_n, \bar{z})$. Then by the continuity of $\frac{1}{n-1}U(z)$ on $(nq_n^M, \bar{z}]$ where this function declines, the result follows directly. Clearly, for all $n < \bar{n}$, $W_n = (\Pi_n^C, \Pi_n^M]$. □

As an example we consider the linear version of the Cournot model, where the inverse demand function is given by

$$p(z) = \begin{cases} D - az & \text{if } z \leqslant D/a, \\ 0 & \text{otherwise,} \end{cases}$$

where $a > 0$ and $D > c$. With simple computations we obtain

$$\Pi_n^C = \frac{(D-c)^2}{a(n+1)^2}, \qquad \Pi_n^M = \frac{(D-c)^2}{4an}, \quad \text{and} \quad \Pi^D(z) = \frac{(D-c-az)^2}{4a}$$
$$\text{for all } z \in [0, \bar{z}].$$

We now describe how to compute $\bar{n}$ and W_n in this model. Recall that, by Lemma 11, $\Pi_n^M \in W_n$ iff $\Pi^D(z_n^{M+}) < \Pi_n^M$, where $z_n^{M+} > nq_n^M$ is such that $\frac{1}{n-1}U(z_n^{M+}) = \Pi_n^M$. In this case,

$$z_n^{M+} = \frac{(D-c)(1+\sqrt{n})}{2a\sqrt{n}}.$$

It can be easily checked that

$$\Pi^D(z_n^{M+}) = \frac{(D-c)^2(\sqrt{n}-1)^2}{16an} < \Pi_n^M \quad \text{iff } n < 9.$$

Therefore $\bar{n} = 9$.

Set $\overline{\Pi}_n = \sup\{v \mid v \in W_n\}$. By Theorem 12, to obtain $\overline{\Pi}_n$ (when $\Pi_n^M \notin W_n$) we have to solve $\Pi^D(z) = \frac{1}{n-1}U(z)$, that is $(D-c-az)^2/(4a) = (D-c-az)z/(n-1)$. With simple computations we obtain $\bar{z}_n = (D-c)(n-1)/(a(n+3))$. Then, $\overline{\Pi}_n = \frac{1}{n-1}U(\bar{z}_n) = 4(D-c)^2/(a(n+3)^2)$ for all $n \geqslant \bar{n} = 9$. Therefore, $W_n = (\Pi_n^C, \Pi_n^M]$, for $n = 2, \ldots, 8$. And

$$W_n = \left(\Pi_n^C, \frac{4(D-c)^2}{a(n+3)^2}\right) \quad \text{for } n = 9, 10, \ldots.$$

Note that, when $n > 8$, there is no PSWRP strategy profile that sustains $\overline{\Pi}_n$ for any $\delta < 1$, but we can get as close as we want to $\overline{\Pi}_n$. It can also be stated that $\bar{n}$ depends on the inverse demand function, that is, $\bar{n}$ is not 9 in general. For instance, take $p(z) = 110 - z^2$ (if $z^2 \leqslant 110$ and 0 otherwise) and $c = 10$, following the above steps we obtain $\bar{n} = 11$.

Again, we may wonder whether the set W_n would be reduced if we use the weak (instead of the strong) Pareto dominance in the comparison of the continuation payoffs. Note that by continuity there exists ε such that $z_n^{v+} - \varepsilon$ is still bigger than nq^M, $\frac{1}{n-1}U(z_n^{v+} - \varepsilon) > v$ and $\Pi^D(z_n^{v+} - \varepsilon) < v$, which implies that the above results hold and the set W_n does not diminish with the weakly Pareto dominated concept.

Next we prove that when n tends to infinity, the Cournot price tends to the marginal cost c.

Proposition 13. $\lim_{n\to\infty} p(nq_n^C) = c$ *and* $\lim_{n\to\infty} nq_n^C = \bar{z}$.

Proof. As $q^C = (q_n^C, \dots, q_n^C)$ is a Cournot–Nash equilibrium, $\Pi_i^D(q^C) = \Pi_n^C$ for all $i = 1, \dots, n$. Set $z_n^C = nq_n^C$. Note that

$$\Pi^D\left(\frac{(n-1)z_n^C}{n}\right) = \frac{1}{n}U\left(z_n^C\right).$$

Since the sequence $\{U(z_n^C)\}_{n=2}^{\infty}$ is bounded, $\lim_{n\to\infty} \frac{1}{n}U(z_n^C) = 0$, and therefore

$$\lim_{n\to\infty} \Pi^D\left(\frac{(n-1)z_n^C}{n}\right) = 0.$$

Now as $\Pi^D(z)$ is continuous and one-to-one on $[0, \bar{z}]$, it follows that $\lim_{n\to\infty} \frac{(n-1)z_n^C}{n} = \bar{z}$. Hence, $\lim_{n\to\infty} z_n^C = \bar{z}$ and from the continuity of $p(z)$ we conclude that $\lim_{n\to\infty} p(z_n^C) = p(\bar{z}) = c$. $\square$

In the linear Cournot model, the bound for the maximal collusion $\overline{\Pi}_n$ comes close to four times the Cournot benefits:

$$\lim_{n\to\infty} \frac{\overline{\Pi}_n}{\Pi_n^C} = \lim_{n\to\infty} \frac{4(D-c)^2}{a(n+3)^2} \Big/ \frac{(D-c)^2}{a(n+1)^2} = 4.$$

It can be easily checked that this is also true for the nonlinear example $p(z) = 110 - z^2$. The following theorem shows that this conclusion is general. In the proof we assume that $q^D(z)$ is differentiable on $(0, \bar{z})$, and $p'(z)$ is differentiable for all $z > 0$ such that $p(z) > 0$.

Theorem 14.

$$\lim_{n\to\infty} \frac{\overline{\Pi}_n}{\Pi_n^C} = 4.$$

Proof. See Appendix A. $\square$

We can only sustain almost four times the Cournot payoff when the discount factor is sufficiently close to 1. When the discount factor is significantly smaller than 1, then we can only say that the above proportion is less than 4. So, the general conclusion is that the collusive benefits that could be sustained as a PSWRP equilibrium are, at most, four times the Cournot benefits.

Note that the restriction in the collusive payoffs that imposes this notion of renegotiation is extremely important because, $\lim_{n\to\infty} \Pi_n^M/\Pi_n^C = \lim_{n\to\infty} n\Pi_n^M/n\Pi_n^C = \infty$ (since $n\Pi_n^M$ is constant and the sequence $\{n\Pi_n^C\}_{n=2}^{\infty}$ tends to 0 by Proposition 13).

We now study the evolution of the equilibrium price when the number of players tends to infinity. First note that by A_4 the equation $\Pi(q,\dots,q) = \overline{\Pi}_n$ has exactly two solutions (whenever $\overline{\Pi}_n < \Pi_n^M$).[4] We will denote the bigger as q_n^+ and the smaller as q_n^-. The following theorem shows that the difference between these two quantities can be enormous when n grows, and the difference between the corresponding prices, $p(nq_n^+)$ and $p(nq_n^-)$, can be also very large.

Theorem 15.

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{p(nq_n^+)}{p(nq_n^C)} = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{q_n^+}{q_n^C} = 1.$$

$$\text{(ii)} \qquad \lim_{n\to\infty} \frac{p(nq_n^-)}{p(nq_n^C)} = \frac{p(0)}{c} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{q_n^-}{q_n^C} = 0.$$

Proof. We first prove that $\lim_{n\to\infty} n\overline{\Pi}_n = 0$. Note that $n\overline{\Pi}_n = \frac{n}{n-1}U(\bar{z}_n)$ for all $n \geqslant \bar{n}$. Then, $\lim_{n\to\infty} \frac{n}{n-1}U(\bar{z}_n) = \lim_{n\to\infty} U(\bar{z}_n) = U(\bar{z}) = 0$ since by Lemma 10 $\lim_{n\to\infty} \bar{z}_n = \bar{z}$. Therefore,

$$\lim_{n\to\infty} n\overline{\Pi}_n = 0. \tag{3}$$

As $(p(nq_n^+) - c)nq_n^+ = n\overline{\Pi}_n$, by (3) it follows that $\lim_{n\to\infty}(p(nq_n^+) - c)nq_n^+ = 0$. Now, note that (for all $n \geqslant \bar{n}$) $nq_n^+ > nq_n^M$ which is constant. Hence, $\lim_{n\to\infty} p(nq_n^+) = c$. Then, taking into account that by Proposition 13, $\lim_{n\to\infty} p(nq_n^C) = c$, (i) holds since $p(z)$ is continuous and one-to-one on $[0, \bar{z}]$.

Similarly it follows that $\lim_{n\to\infty}(p(nq_n^-) - c)nq_n^- = 0$. Now observe that (for all $n \geqslant \bar{n}$) $nq_n^- < nq_n^M$ and then, $p(nq_n^-) > p(nq_n^M)$ which is constant and bigger than c. Therefore $\lim_{n\to\infty} nq_n^- = 0$ and (ii) holds trivially. □

Surprisingly, even with our restrictive concept of renegotiation proofness it is theoretically possible to sustain in equilibrium a price greater than the monopoly price. This happens when the players use q_n^- instead of q_n^+. We now want to convince the reader that using this small quantity q_n^- instead of q_n^+ to obtain the same collusion profits is not at all reasonable. First observe that, formally, if no deviation occurs it does not matter whether all the players play q_n^+ or q_n^- along the cooperative path. However, if a deviation occurs, there is a significant difference: the payoff of the non deviating players decreases slightly if they are playing q_n^+. But if they are playing q_n^- and one player deviates, the payoff for the rest decreases proportionally, by a considerable amount. Therefore, if a player considers that there exists a positive probability of a deviation by any other player, then his expected payoff is substantially higher if all players are playing q_n^+ on the cooperative path rather than q_n^-. Also it has to be taken into account that

[4] Note that with only Assumption A_3 the argument does not change.

$\Pi_i^D(q_n^-,\dots,q_n^-)$ is extremely large in comparison to $\Pi_i^D(q_n^+,\dots,q_n^+)$. In fact, it is easy to see that $\lim_{n\to\infty}\Pi_i^D(q_n^-,\dots,q_n^-)/\Pi_i^D(q_n^+,\dots,q_n^+)=\infty$. This means that q_n^- significantly increases the temptation of all players to deviate. On the other hand, it can be checked easily that although the margin in the discount factor is close to 1 in both cases, it is bigger with q_n^+ than with q_n^-. For all these reasons, we think that it is more sensible to play q_n^+ than q_n^-.

Now from Theorem 15 we can conclude that *when the number of players tends to infinity the* “*reasonable collusive price*” *that can be sustained in equilibrium tends to the Cournot price* (*even for a discount factor arbitrarily close to* 1).

5. Conclusions

This paper addresses the question: “Which cooperative outcomes can be sustained as renegotiation-proof equilibria in a repeated symmetric Cournot oligopoly game with n firms?” In Section 3 we prove that with the WRP concept of renegotiation introduced by Farrell and Maskin, the monopoly profits (and therefore the monopoly price) can be sustained for any number of firms. So this concept does not significantly reduce the set of outcomes that can be sustained. However, the strategy that allows us to obtain this result is not convincing despite its being WRP. In this strategy there is only one player who is not interested in forgiving and skipping the punishment phase. Furthermore, the other innocent players suffer the same punishment as the guilty one. We think that it is not very sensible to give only one player the power to block any renegotiation. To avoid this drawback we introduce the PSWRP concept, which in addition to weakly renegotiation proofness calls for a kind of reasonable symmetry. The idea is to exclude those equilibria in which planned punishments hurt any of the punishers, and to assure that no punishing player will consider the possibility of forgiving and forgetting. We show that the PSWRP concept significantly limits the cooperative outcomes that can be sustained. In particular the symmetric monopoly outcome cannot be sustained by a PSWRP equilibrium, for any discount factor, when the number of players is high enough (9 in the linear demand case). Furthermore, when the number of players tends to infinity (even for a discount factor arbitrarily close to 1),

(i) the collusive benefits that could be sustained as a PSWRP equilibrium are, at most, four times the Cournot benefits;
(ii) the “*reasonable collusive price*” that can be sustained in equilibrium tends to the Cournot price.

To conclude this paper we wish to comment on an independent paper by Farrell. In his paper, Renegotiation in Repeated Oligopoly Interaction, Farrell (2000) argues in a wide-ranging, elegant way that the straightforward generalization of the WRP concept to the n-players case is not adequate. He proposes instead the following definition: “A subgame-perfect equilibrium is quasi-symmetrically weakly renegotiation-proof (QSWRP) if, evaluated at the beginning of the period after player i alone deviates from prescribed play, every other player’s continuation payoff (weakly) exceeds what it would have been had player i not just deviated.” We think that this definition is more general, and probably more natural than the one we have introduced in this paper. Note that the PSWRP concept makes sense only for symmetric games, whereas QSWRP can be used in any game, symmetric or asymmetric. Although the two definitions seem to be different, it can be easily checked that in the symmetric case they are almost equivalent. In fact, if we are only interested in symmetric profits both definitions impose the same restriction on the payoffs that can be sustained in equilibrium. In his work, Farrell studies the implication of this concept in repeated Bertrand competition. He shows that a big enough number of firms cannot sustain full—or, indeed, much—collusion even for a discount factor very close to 1. With respect to the Cournot model, he considers just one numerical example with linear demand, $p(z) = 2 - z$, and he concludes that full collusion is not QSWRP with $n \geqslant 9$ firms, and, for large n, roughly four times the Cournot profit is sustainable in QSWRP equilibrium, even for δ very close to 1. In our work we prove the generality of this result and also study how the price changes as n grows.

Acknowledgments

We greatly appreciate the constructive comments of the associated editor. Financial support from Universidad del Pais Vasco (projects UPV 036.321-HA055/98 and UPV 036.321-HA047/99) and DGES of Ministerio de Educación y Cultura (project PB96-0247) is gratefully acknowledged.

Appendix A

Proof of Lemma 1. According to Abreu (1988), only one-shot deviations must be checked to ensure that σ is a SPE, where a one-shot deviation from a strategy in $G^\infty(\delta)$ consists of a single-period deviation from the strategy, and sticking to it subsequently. In this case only the following one-shot deviations must be checked:

(i) *Deviation of player 1 from path S^0.*

$$(1-\delta)\Pi_1^D\big(q^0\big) + \delta\Pi_1^\delta\big(S^1\big) \leqslant \Pi_1\big(q^0\big). \tag{A.1}$$

Set $R(\delta) = (1-\delta)\Pi_1^D(q^0) + \delta\ \Pi_1^\delta(S^1) - \Pi_1(q^0)$. As $\Pi_1^\delta(S^1) = (1-\delta^T)\Pi_1(q^1) + \delta^T\Pi_1(q^2)$, then $\lim_{\delta\to 1} R(\delta) = \Pi_1(q^2) - \Pi_1(q^0) \leqslant 0$ by condition (a). Now, if $\Pi_1(q^2) < \Pi_1(q^0)$, the inequality (A.1) holds for δ close enough to 1. If $\Pi_1(q^2) = \Pi_1(q^0)$, then $\lim_{\delta\to 1} R(\delta) = 0$ and the limit of the derivative $\lim_{\delta\to 1} R'(\delta) = -\Pi_1^D(q^0) - T\Pi_1(q^1) + (T+1)\Pi_1(q^0)$. Thus taking

$$T > \frac{\Pi_1^D(q^0) - \Pi_1(q^0)}{\Pi_1(q^0) - \Pi_1(q^1)},$$

$\lim_{\delta\to 1} R'(\delta) > 0$ and (A.1) holds for δ close enough to 1.

(ii) *Deviation of player 1 from S^1 in the first period*. Now, the inequality that we must check is:

$$(1-\delta)\Pi_1^D\big(q^1\big) + \delta\Pi_1^\delta\big(S^1\big) \leqslant \Pi_1^\delta\big(S^1\big). \tag{A.2}$$

Set $R(\delta) = (1-\delta)\ \Pi_1^D(q^1) + \delta\ \Pi_1^\delta(S^1) - \Pi_1^\delta(S^1)$. Then $\lim_{\delta\to 1} R(\delta) = 0$ and

$$\lim_{\delta\to 1} R'(\delta) = -\Pi_1^D\bigl(q^1\bigr) + \Pi_1\bigl(q^2\bigr) > 0$$

by condition (a), therefore (A.2) follows directly for δ close enough to 1.

(iii) *Deviation of player 1 from S^1 in the $(T+1)$ period*. Let us consider the inequality:

$$(1-\delta)\Pi_1^D\bigl(q^2\bigr) + \delta\bigl(1-\delta^T\bigr)\Pi_1\bigl(q^1\bigr) + \delta^{T+1}\Pi_1\bigl(q^2\bigr) \leqslant \Pi_1\bigl(q^2\bigr). \tag{A.3}$$

Set $R(\delta) = (1-\delta)\Pi_1^D(q^2) + \delta(1-\delta^T)\Pi_1(q^1) + \delta^{T+1}\Pi_1(q^2) - \Pi_1(q^2)$. Then we have $\lim_{\delta\to 1} R(\delta) = 0$. But $\lim_{\delta\to 1} R'(\delta) = -\Pi_1^D(q^2) - T\Pi_1(q^1) + (T+1)\Pi_1(q^2)$, hence taking

$$T > \frac{\Pi_1^D(q^2) - \Pi_1(q^2)}{\Pi_1(q^2) - \Pi_1(q^1)},$$

$\lim_{\delta\to 1} R'(\delta) > 0$ and (A.3) holds for δ close enough to 1.

(iv) *Deviation of player i $(i \neq 1)$ from S^1 in the first period*. We show that the following inequality holds:

$$(1-\delta)\Pi_i^D\bigl(q^1\bigr) + \delta\Pi_1^\delta\bigl(S^1\bigr) \leqslant \Pi_i^\delta\bigl(S^1\bigr). \tag{A.4}$$

Set $R(\delta) = (1-\delta)\Pi_i^D(q^1) + \delta\Pi_1^\delta(S^1) - \Pi_i^\delta(S^1)$. So $\lim_{\delta\to 1} R(\delta) = \Pi_1(q^2) - \Pi_i(q^2) \leqslant 0$ by condition (b). Now, if $\Pi_1(q^2) < \Pi_i(q^2)$, (A.4) is satisfied for δ close enough to 1. If $\Pi_1(q^2) = \Pi_i(q^2)$, then $\lim_{\delta\to 1} R'(\delta) = -\Pi_i^D(q^1) - T\Pi_1(q^1) + T\Pi_i(q^1) + \Pi_1(q^2)$. Moreover, the second part of condition (b) implies that $\Pi_i(q^1) \geqslant \Pi_1(q^1)$. Therefore, if $\Pi_i(q^1) > \Pi_1(q^1)$ taking

$$T > \frac{\Pi_i^D(q^1) - \Pi_1(q^2)}{\Pi_i(q^1) - \Pi_1(q^1)},$$

it follows that $\lim_{\delta\to 1} R'(\delta) > 0$. Clearly, if $\Pi_i(q^1) = \Pi_1(q^1)$ then $\lim_{\delta\to 1} R'(\delta) = -\Pi_i^D(q^1) + \Pi_1(q^2) > 0$, by condition (c). So (A.4) is true for δ close enough to 1.

(v) *Deviation of player i $(i \neq 1)$ from S^1 in period $(T+1)$*. In this case we have to check the inequality

$$(1-\delta)\Pi_i^D\bigl(q^2\bigr) + \delta\bigl(1-\delta^T\bigr)\Pi_1\bigl(q^1\bigr) + \delta^{T+1}\Pi_1\bigl(q^2\bigr) \leqslant \Pi_i\bigl(q^2\bigr). \tag{A.5}$$

Now, $R(\delta) = (1-\delta)\Pi_i^D(q^2) + \delta(1-\delta^T)\Pi_1(q^1) + \delta^{T+1}\Pi_1(q^2) - \Pi_i(q^2)$, and $\lim_{\delta\to 1} R(\delta) = \Pi_1(q^2) - \Pi_i(q^2) \leqslant 0$, by condition (b). If $\Pi_1(q^2) < \Pi_i(q^2)$, (A.5) holds for δ close enough to 1. On the other hand, if $\Pi_1(q^2) = \Pi_i(q^2)$, we have $\lim_{\delta\to 1} R(\delta) = 0$ and $\lim_{\delta\to 1} R'(\delta) = -\Pi_i^D(q^2) - T\Pi_1(q^1) + (T+1)\Pi_1(q^2)$. Then, by condition (a), we may take

$$T > \frac{\Pi_i^D(q^2) - \Pi_1(q^2)}{\Pi_1(q^2) - \Pi_1(q^1)},$$

which implies $\lim_{\delta\to 1} R'(\delta) > 0$, and (A.5) holds for δ close enough to 1.

Summing up, the strategy σ is SPE for δ sufficiently close to 1 when the number of periods for which q^1 has to be repeated in path S^1 is at least

$$T = 1 + \text{IntegerPart}\max\left\{\begin{array}{ll} \frac{\Pi_1^D(q^2)-\Pi_1(q^2)}{\Pi_1(q^2)-\Pi_1(q^1)} & \\ \frac{\Pi_1^D(q^0)-\Pi_1(q^0)}{\Pi_1(q^0)-\Pi_1(q^1)}, & \text{if } \Pi_1(q^2)=\Pi_1(q^0) \\ \frac{\Pi_i^D(q^1)-\Pi_1(q^2)}{\Pi_i(q^1)-\Pi_1(q^1)}, & \text{if } \Pi_i(q^2)=\Pi_1(q^2), \Pi_i(q^1)\neq\Pi_1(q^1) \\ \frac{\Pi_i^D(q^2)-\Pi_1(q^2)}{\Pi_1(q^2)-\Pi_1(q^1)}, & \text{if } \Pi_i(q^2)=\Pi_1(q^2) \end{array}\right\}. \qquad \square$$

In the proof of Lemma 8 we will use the following lemma, which is a generalization of Lemma 2, in Farrell and Maskin (1989, p. 356), to the n-player case.

Lemma 16. *Let σ be a PSWRP equilibrium for a discount factor $\delta < 1$. Then, there exists a PSWRP equilibrium $\sigma' \equiv (S^0, S^1, \ldots, S^n)$, for the same discount factor δ, such that $\Pi_i^\delta(\sigma') = \Pi_i^\delta(\sigma)$ and*

$$\begin{aligned}\Pi_i^\delta(S^i) &= \inf\{\Pi_i^\delta(\sigma_h') \mid \sigma_h' \text{ continuation equilibria of } \sigma'\}\\ &= \inf\{\Pi_i^\delta(\sigma_h) \mid \sigma_h \text{ continuation equilibria of } \sigma\}.\end{aligned}$$

Proof. For $i = 1, \ldots, n$, set $m_i = \inf\{\Pi_i^\delta(\sigma_h) \mid \sigma_h \text{ continuation equilibria of } \sigma\}$. Then, there exists a sequence $\{\sigma_n^i\}_{n\in N}$ of continuation equilibria of σ, such that $\lim_{n\to\infty} \Pi_i^\delta(\sigma_n^i) = m_i$.

For $n \in N$, $\sigma_n^i(1)$ (i.e., the first period actions)$\in Q_1 \times \cdots \times Q_n$ which is a compact set. Hence there exists a subsequence $\{\sigma_{n,1}^i\}_{n\in N} \subset \{\sigma_n^i\}_{n\in N}$, and $q^i(1) \in Q_1 \times \cdots \times Q_n$ such that $\lim_{n\to\infty} \sigma_{n,1}^i(1) = q^i(1)$, and since $\Pi : Q_1 \times \cdots \times Q_n \longmapsto R^n$, where $\Pi(q) = (\Pi_1(q), \ldots, \Pi_n(q))$, is continuous, then $\lim_{n\to\infty} \Pi(\sigma_{n,1}^i(1)) = \Pi(q^i(1))$.

Proceeding inductively, there exists a subsequence $\{\sigma_{n,t}^i\}_{n\in N}$, such that $\{\sigma_{n,t}^i\}_{n\in N} \subset \{\sigma_{n,t-1}^i\}_{n\in N} \subset \cdots \subset \{\sigma_n^i\}_{n\in N}$, $\lim_{n\to\infty} \sigma_{n,t}^i(r) = q^i(r)$ and $\lim_{n\to\infty} \Pi(\sigma_{n,t}^i(r)) = \Pi(q^i(r))$ for all $r = 1, \ldots t$.

Now we define the simple strategy profile $\sigma' \equiv (S^0, S^1, \ldots, S^n)$ in the following way: take S^0 as the outcome path of σ and $S^i = \{q^i(t)\}_{t=1}^\infty$ for all $i = 1, \ldots, n$. Following the same steps of Farrell and Maskin in their lemma, we conclude that σ' holds the claims. $\square$

The next proof is similar to the equivalent result of Farrell and Maskin (1989).

Proof of Lemma 8. From Lemma 16, there exists a PSWRP $\sigma' \equiv (S^0, S^1, \ldots, S^n)$ (it is easy to show that if σ is partially symmetric, then so is σ') defined by $S^0 = \{q^v\}_{t=1}^\infty$, $S^1 = \{q^1(t)\}_{t=1}^\infty$, $S^i = S^{1(i/1)}$ for $i = 2, \ldots, n$; where $\Pi_i(q^v) = v$ and

$$\begin{aligned}\Pi_i^\delta(S^i) &= \inf\{\Pi_i^\delta(\sigma_h') \mid \sigma_h' \text{ continuation equilibria of } \sigma'\}\\ &= \inf\{\Pi_i^\delta(\sigma_h) \mid \sigma_h \text{ continuation equilibria of } \sigma\}.\end{aligned}$$

It is clear that $\Pi_1^\delta(S^1) \leqslant v$. Note that this inequality is strict because otherwise, since $(1-\delta)\Pi_1^D(q^v) + \delta\Pi_1^\delta(S^1) \leqslant v$, we have $\Pi_1^D(q^v) = \Pi_1(q^v)$ and $q^v = q^C$, which is a contradiction. So $\Pi_1^\delta(S^1) < v$. Moreover, it is clear that $\Pi_2^\delta(S^1) \geqslant v$ because otherwise S^0 would strictly Pareto-dominate S^1, which cannot be since σ' is PSWRP.

We can assume without loss of generality that $\Pi_2(q^1(1)) \geqslant v$ (if not, we take $\hat{t} = \min\{t \mid \Pi_2(q^1(t)) \geqslant v\}$ and define $S^1 = \{q^1(t)\}_{t=\hat{t}}^\infty$, it is easy to prove that the properties of the former S^1 still hold).

On the other hand, player 1 does not deviate from S^1in the first period, that is, $(1-\delta)\Pi_1^D(q^1(1)) + \delta\Pi_1^\delta(S^1) \leqslant \Pi_1^\delta(S^1)$, so $\Pi_1^D(q^1(1)) \leqslant \Pi_1^\delta(S^1) < v$. Now taking $q^* = q^1(1)$ the result follows. □

Proof of Lemma 10. Set $F_n(z) = \Pi^D(z) - \frac{1}{n-1}U(z)$. This function is continuous and strictly convex on $[0,\bar{z}]$ since $\Pi^D(z)$ is strictly convex on $[0,\bar{z}]$ and $U(z)$ is concave on $[0,\bar{z}]$. We have $F_n(0) = n\Pi_n^M > 0$, $F_n((n-1)q_n^C) = (p(nq_n^C) - c)q_n^C - (p((n-1)q_n^C) - c)q_n^C < 0$ and $F_n(\bar{z}) = 0$. So the first and second results of the lemma follow.

We next see that the sequence $\{\bar{z}_n\}_{n=2}^\infty$ is strictly increasing. Note that $\Pi^D(\bar{z}_{n+1}) = \frac{1}{n}U(\bar{z}_{n+1}) < \frac{1}{n-1}U(\bar{z}_{n+1})$, so $\bar{z}_{n+1} \in (\bar{z}_n, \bar{z})$.

Now, as the sequence $\{U(\bar{z}_n)\}_{n=2}^\infty$ is bounded, we have

$$\lim_{n\to\infty} \Pi^D(\bar{z}_n) = \lim_{n\to\infty} \frac{1}{n-1}U(\bar{z}_n) = 0.$$

But $\Pi^D(z)$ is continuous and one-to-one on $[0,\bar{z}]$, $\Pi^D(z) > 0$ for all $z \in [0,\bar{z})$ and $\Pi^D(\bar{z}) = 0$, so $\lim_{n\to\infty} \bar{z}_n = \bar{z}$. □

Proof of Theorem 14. First note that the total Cournot–Nash quantity z_n^C holds $p'(z_n^C)z_n^C/n + p(z_n^C) - c = 0$, for all $n \geqslant 2$. Then

$$\Pi_n^C = \bigl(p\bigl(z_n^C\bigr) - c\bigr)\frac{z_n^C}{n} = -p'\bigl(z_n^C\bigr)\left(\frac{z_n^C}{n}\right)^2.$$

From Theorem 12, we have $\overline{\Pi}_n = \Pi^D(\bar{z}_n) = \frac{1}{n-1}U(\bar{z}_n)$, for all $n \geqslant \bar{n}$. Hence we can write $n = \frac{\Pi^D(\bar{z}_n)+U(\bar{z}_n)}{\Pi^D(\bar{z}_n)}$. Then,

$$\frac{\overline{\Pi}_n}{\Pi_n^C} = \frac{-n^2\overline{\Pi}_n}{p'(z_n^C)(z_n^C)^2} = \frac{-(\Pi^D(\bar{z}_n) + U(\bar{z}_n))^2/\Pi^D(\bar{z}_n)}{p'(z_n^C)(z_n^C)^2} \tag{A.6}$$

(for all $n \geqslant \bar{n}$). Recall that the sequence $\{\bar{z}_n\}_{n=2}^\infty$ is strictly increasing and $\lim_{n\to\infty} \bar{z}_n = \bar{z}$. Let H be the function

$$H(z) = \frac{(\Pi^D(z) + U(z))^2}{\Pi^D(z)}$$

for all $z \in (0,\bar{z})$. Choosing z on the left side of $\bar{z}$, sufficiently close to $\bar{z}$ ($z \to \bar{z}^-$), then $\lim_{z\to\bar{z}^-} H(z) = 0/0$. In order to apply L'Hôpital's rule, we compute the derivative of $\Pi^D(z) = (p(q^D(z)+z) - c)q^D(z)$ which is $\Pi^{D'}(z) = p'(q^D(z)+z)(q^{D'}(z)+1)q^D(z) + (p(q^D(z)+z) - c)q^{D'}(z)$ for all $z \in (0,\bar{z})$. But $q^D(z)$ satisfies the equation

$$p'(q^D(z)+z)q^D(z) + p(q^D(z)+z) - c = 0, \tag{A.7}$$

hence $\Pi^{D'}(z) = p'(q^D(z) + z)q^D(z)$. It is clear that $\lim_{z\to\bar{z}^-} q^D(z) = 0$, and therefore $\lim_{z\to\bar{z}^-} \Pi^{D'}(z) = 0$. Then, by L'Hôpital's

$$\lim_{z\to\bar{z}^-} H(z) = \lim_{z\to\bar{z}^-} \frac{2(\Pi^D(z) + U(z))(\Pi^{D'}(z) + U^{'}(z))}{\Pi^{D'}(z)} = \frac{0}{0}.$$

So we again apply L'Hôpital's rule and we have $\Pi^{D''}(z) = p''(q^D(z) + z)(q^{D'}(z) + 1)q^D(z) + p'(q^D(z) + z)q^{D'}(z)$, and by (A.7)

$$p''\big(q^D(z) + z\big)\big(q^{D'}(z) + 1\big)q^D(z) + p'\big(q^D(z) + z\big)\big(2q^{D'}(z) + 1\big) = 0.$$

Hence $\Pi^{D''}(z) = -p'(q^D(z) + z)(q^{D'}(z) + 1)$, and $\lim_{z\to\bar{z}^-} q^{D'}(z) = -1/2$.

So $\lim_{z\to\bar{z}^-} \Pi^{D''}(z) = -p'(\bar{z})/2$. Now, as $\lim_{z\to\bar{z}^-} U'(z) = p'(\bar{z})\bar{z}$, then,

$$\begin{aligned}\lim_{z\to\bar{z}^-} H(z) &= \lim_{z\to\bar{z}^-} \frac{2(\Pi^{D'}(z) + U^{'}(z))^2 + 2(\Pi^D(z) + U(z))(\Pi^{D''}(z) + U^{''}(z))}{\Pi^{D''}(z)} \\ &= -4p'(\bar{z})\bar{z}^2.\end{aligned}$$

Finally, since $\lim_{n\to\infty} z_n^C = \bar{z}$, taking limit in (A.6) we have

$$\lim_{n\to\infty} \frac{\overline{\Pi}_n}{\Pi_n^C} = \frac{4p'(\bar{z})\bar{z}^2}{p'(\bar{z})\bar{z}^2} = 4. \qquad \square$$

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