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ELSEVIER Journal of Economic Dynamics & Control 29 (2005) 1025–1041

JOURNAL OF Economic Dynamics & Control

www.elsevier.com/locate/econbase

Repeated real options: optimal investment behaviour and a good rule of thumb

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Received 14 March 2002; accepted 30 June 2004
Available online 18 October 2004

Abstract

This paper extends the standard investment-under-uncertainty set-up with a single investment option to the case of infinitely repeated options. Analytical solutions are derived, and it is shown that repeated options not only imply a smaller value of waiting than in the case of a single option, but also that the optimal stopping rule is affected differently by changes in underlying parameters. This is shown to allow for the use of a simple hurdle-rate rule as a good and robust approximation to optimal behaviour when investment options are repeated – something which is unlikely in the single-option case.

JEL classification: D81; O33

Keywords: Repeated real options; Rule of thumb; Geometric Brownian motion

1. Introduction

During the last two decades, a large number of studies have analysed the real-option approach to investment decisions. See Dixit and Pindyck (1994) for a unified account of this approach. The object of analysis is optimal investment behaviour

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0165-1889/$ - see front matter
doi:10.1016/j.jedc.2004.08.002

when: (i) investments are irreversible and indivisible; (ii) there is uncertainty about the future, for example, about the cost and profitability of an investment; and (iii) there is an option to postpone the investment. It is argued that incorporating these features into the optimisation problem often provides a more appropriate description of the investment decision than more orthodox and static formulations using a simple net-present-value (NPV) criterion. Furthermore, since (i)–(iii) usually create a (considerable) value of waiting, the derived decision rules are often significantly different from those implied by the simple NPV criterion.

The core model in Dixit and Pindyck (1994), and in much of the related literature, is a single-option model first developed by McDonald and Siegel (1986). In this model, the option to invest is "killed" when the investment is undertaken. Hence, no future re-investment can take place. While this type of model has rightly received considerable attention in the literature, there exist situations where it is less appropriate.

As an example, think of an individual or a firm considering when to invest in IT equipment. New and more productive equipment is continuously introduced at the market and prices are constantly changing. Furthermore, an investment of this type is indivisible – at least to some degree – and also irreversible in the sense that the resale value decreases rapidly. Hence, the features (i)–(iii) above are clearly present in this decision problem. However, the agent knows that the current IT investment is not an isolated event. In a few years, it will be optimal for him to buy an even bigger IT system. Therefore – being rational – he cannot consider the current investment option as independent of future options. When he buys the IT equipment, he "kills" the current option, but he immediately receives a new option: The option to buy even newer IT equipment. The optimal timing of his current investment is therefore likely to depend on and influence the pattern of future investments.

The aim of the present paper is to extend the methodology of the real-option literature to include this repeated-options aspect, and to analyse the implications for optimal investment behaviour. By allowing for the fact that many real-investment options are repeated, the irreversibility feature of the real-options literature is relaxed somewhat. Still, however, the formulation retains most of the features common to the literature. A central question therefore becomes: Do the insights derived from the well-known single-option model carry over to the case of repeated options, or does the single-option model yield invalid descriptions of optimal behaviour when investment options are indeed repeated?

In this paper, the intertemporal decision problem is modelled as follows: In each period, production is assumed to depend only on the productivity of installed technology. There is no effort or input of other factors. Productivity may be increased by investing in the currently available technology in the economy. All investments are irreversible and indivisible, in the sense that the agent has to pay for the productivity *level* of the exogenous technology, not just the *difference* in productivity levels between installed and exogenous technology. While focusing on the case where only the productivity of exogenous technology evolves stochastically, the paper derives a general analytical solution in the case where both the productivity of installed and the productivity of exogenous technology evolve according to (possibly correlated) geometric Brownian motions.

The analysis shows that when investment options are repeated, the value of waiting is reduced significantly compared to the single-option case. In addition, the effects of changes in underlying parameters are shown to be very different when using the repeated-options approach instead of the single-option approach. In fact, allowing for repeated options makes the optimal stopping rule less sensitive to changes in these parameters – a behaviour which resembles the one found using the simple NPV rule. Therefore, a simple rule of thumb is more likely to be successful in the case of repeated options than in the case of a single option. This is illustrated by showing how the use of a simple "hurdle-rate" criterion can serve as a good and robust approximation to optimal behaviour. It clearly outperforms the use of the single-option criterion when investment options are indeed repeated.

2. Related work

Grenadier and Weiss (1997) analyse the adoption of technological innovations in a real-options framework. They analyse a situation where an agent has the option to adopt an existing technology, and later to upgrade once to an improved technology, which arrives stochastically at a later point in time. The agent can also choose to forego the first option and only adopt the improved technology. Given that only one innovation can arrive, the model of Grenadier and Weiss (1997) is basically a single-option model: Once the improved technology arrives, the decision problem stops. The methodological similarities to the present study are therefore limited. The paper by Farzin et al. (1998) and the associated comment by Doraszelski (2001) consider technological adoptions in a set-up more closely related to that of the present paper. Still, however, they assume only a finite number of investment options, thereby preventing an analytical solution.

As in these papers, the model of the present paper is interpreted as a model of optimal technology adoption, where focus is on the uncertainty of the exogenous technology. If focus is instead on the stochastic depreciation of installed technology, the model might alternatively be interpreted as a model of optimal capital replacement. Related replacement problems have previously been analysed in the literature.

Ye (1990) analyses a replacement problem where maintenance and operation costs of installed equipment follow an Ito process. By investing in new equipment, the process is returned to its initial state without affecting the productivity of the equipment. With everything else constant, Ye (1990) derives the closed-form decision rule for the case of a simple Brownian motion. However, having a different focus, Ye (1990) does not analyse the model from an option-pricing perspective. More recently, Mauer and Ott (1995) have analysed a related replacement problem where maintenance and operation costs follow a geometric Brownian motion. As in Ye (1990), the process is reset to its initial value whenever an investment in new equipment is made. While Mauer and Ott (1995) apply an option-pricing perspective, they impose a restriction on the level of the stochastic process which prevents an analytical solution.

The general model of this paper shows a close analytical correspondence between replacement and technology-adoption models. However, focusing on the uncertainty of the exogenous investment options instead of the uncertainty related to the cost of maintenance is considered to be of strong empirical relevance. The IT example above is just one example illustrating that with modern technology, operation and maintenance costs are no longer of major importance when considering replacement. It is the productivity of installed technology relative to newer available technology which is at the core of modern replacement decisions.

Compared to the existing literature, the present paper presents a new and simple analytical solution to the problem of truly infinitely-repeated investment options in the case of a geometric Brownian motion.[1] Furthermore, by providing a comparison of the derived decision rule with that of the single-option approach and the simple NPV rule, the focus of the paper differs strongly from earlier papers.

The rest of the paper is organised as follows: Section 3 presents the set-up of the model. The model is solved and the optimal decision rule is derived in Section 4. Section 5 contains a comparison of the derived decision rule with those of the simple NPV approach and the single-option approach. In Section 6, numerical illustrations are provided, and the performance of a rule of thumb is discussed. Section 7 concludes the paper.

3. The model

As in the related real-options literature, the model of this paper is a continuous-time model where the agent is assumed to maximise the net present value of all future income streams. Income is derived from production, and the only costs are those associated with new investments in technology. Thus, the model abstracts from decisions regarding other factor inputs. This is done in order to keep it analytically tractable and to focus on the dynamics of investment.

For now, assume that production at time t is given by the non-stochastic productivity level, Θ_t, of installed technology.[2] The agent can improve this productivity level by investing in exogenous technology. By doing this, he obtains the current productivity level, θ_t, of the exogenous technology process. His production is then given non-stochastically by this level until the next date of investment. Meanwhile, the exogenous productivity level is assumed to evolve according to a geometric Brownian motion:

$$\mathrm{d}\theta = \alpha\theta\,\mathrm{d}t + \sigma\theta\,\mathrm{d}z, \tag{1}$$

[1]The idea of chains of options has also been discussed in the world of financial assets and derivatives. As an example, Carr (1988) presents an approach to the valuation of compound options, where one asset (a European option) can be exchanged for another asset at exogenously fixed dates over a finite time period.

[2]This assumption is relaxed below.

where α and σ are the parameters of the process, and $\mathrm{d}z$ is the increment of a Wiener process. The cost of investing in the new technology level is given by $c\theta_t$, where c is a positive constant.[3] Hence, at each point in time, the agent must decide whether to *continue* producing with productivity level Θ_t, or to *stop* and adopt the current productivity level, θ_t, of the exogenous technology process.

The chosen form of the stochastic process implies that the productivity of exogenous technology can get below the productivity of installed technology. This may at first glance seem unreasonable, but does increase analytical tractability. However, it might also reflect that although the newest technology in itself has improved, the potential productivity from the newest technology in a given firm might not have improved. This could be due to lack of knowledge about the new technology or lack of supportive infrastructure leading to firm specific operational costs. Alternatively, it could be the case that the newest technology represents such a different technology that some firms may actually face productivity decreases if they install it, because of the need to enter a costly process of learning. Not all inventions are equally profitable to everyone in the short run.

An every-day example, which to some extent supports this line of reasoning, is the evolution within text-editors. WordPerfect 5.1, a DOS-based text-editor, was a widely used and excellent text-editor which, however, at some point was outdated by, e.g., the Windows-based Microsoft Word-editor. The user interface (with its WYSIWYG-principle) and whole functioning of this program was very different from WordPerfect 5.1. Therefore, even if the new technology was in many ways superior, it was evident that users had to undergo a process of learning before their productivity would be as good with the new technology as with the old. Consequently, many organisations, firms and people stayed with WordPerfect 5.1 for a long period of time until learning was picked up in other software types or acquired by hiring new people, or until the technology had developed so far that even learning costs and delays could not hamper the potential productivity increases.

The above specification of the model implies that the problem is *stationary*, i.e., optimal decisions and values do not depend on time per se. All relevant information is captured in the variables θ and Θ, together with the parameters of the problem. Thus, drop time subscripts and let $V(\theta,\Theta)$ be the value function of the agent when the productivity of installed technology is Θ and that of the exogenous technology is θ. The value function is then given by

$$\begin{aligned} V(\theta,\Theta) = \; & \max\{\Theta\,\mathrm{d}t + (1 + r\,\mathrm{d}t)^{-1}E[V(\theta + \mathrm{d}\theta,\Theta)|\theta], \\ & \theta\,\mathrm{d}t - c\theta + (1 + r\,\mathrm{d}t)^{-1}E[V(\theta + \mathrm{d}\theta,\theta)|\theta]\}. \end{aligned} \tag{2}$$

[3]The fact that the cost of investing depends on the level of exogenous technology, and not just the difference between exogenous and installed technology, is what causes investments to be indivisible. Letting the cost depend only on the difference would make continuous (or incremental) investments optimal.

The first argument on the right-hand side is the value of continuing with the given technology for at least one more "short" time interval, $\mathrm{d}t$. This will yield immediate output, $\Theta\,\mathrm{d}t$, and result in a change of productivity of exogenous technology to $\theta + \mathrm{d}\theta$. This in turn changes the expectation of the value function, which is discounted by $(1 + r\,\mathrm{d}t)^{-1}$. Likewise, the second term on the right-hand side is the value of stopping, i.e., switching to the technology producing θ. This immediately costs $c\theta$ and yields current output $\theta\,\mathrm{d}t$ in the next short interval of time. After that interval, the value function will be $V(\theta + \mathrm{d}\theta, \theta)$, since the productivity of exogenous technology changes as above, whereas the productivity of installed technology now equals θ.

It is assumed that the parameters α, c, and r satisfy: $\infty > \frac{1}{\alpha} > \frac{1}{r} > c > 0$. These assumptions are imposed to ensure that the problem is well defined and that investments are profitable. If $r \leqslant \alpha$, the value function becomes infinite, which is not considered an interesting case. If, on the other hand, $c \geqslant \frac{1}{r}$, it will never be profitable to undertake an investment.

Note that the choice of a simple proportional cost structure, $c\theta$, is needed to obtain the simple analytical solution. Using the personal computer as an example, it might be argued that costs do not evolve proportionally to the level of technology but rather stay constant. However, thinking of the cost of an IT investment more broadly, including the cost of software and the hours spent installing and getting used to the new equipment, a proportional cost structure does not at all seem unreasonable. Furthermore, using a more complicated cost structure is not likely to yield additional insight. As an example, consider a cost function of the type: $c_1 + c_2\theta$, i.e., a fixed cost in addition to the proportional cost (or just a fixed cost if $c_2 = 0$). As θ increases, the behaviour implied by this cost function converges to the solution obtained below, since the fixed cost becomes unimportant. In the case of $c_2 = 0$, this implies continuous investments in the end, which seems absurd for the type of investments analysed in this paper.

While focus in this paper is on analysing the implications of the above model where the productivity of installed technology, Θ, is fixed, we also provide the solution to a more generalised version where both θ and Θ evolve stochastictally. More specifically, in the generalised version, it is assumed that θ and Θ evolve according to the following geometric Brownian motions:

$$\mathrm{d}\theta = \alpha_1\theta\,\mathrm{d}t + \sigma_1\theta\,\mathrm{d}z_1, \tag{3}$$

$$\mathrm{d}\Theta = \alpha_2\Theta\,\mathrm{d}t + \sigma_2\Theta\,\mathrm{d}z_2, \tag{4}$$

where α_1, α_2, σ_1, and σ_2 are the parameters of the respective processes, and $\mathrm{d}z_1$ and $\mathrm{d}z_2$ are increments of Wiener processes with $E(\mathrm{d}z_1\,\mathrm{d}z_2) = \rho$. Furthermore, in this case, it is assumed that $r > \alpha_1$, $\alpha_2 \leqslant 0$, and $c < (r - \alpha_2)^{-1}$. The value function can then be expressed as:

$$\begin{aligned} V(\theta, \Theta) = \max\{&\Theta\,\mathrm{d}t + (1 + r\,\mathrm{d}t)^{-1}E[V(\theta + \mathrm{d}\theta, \Theta + \mathrm{d}\Theta)|\theta, \Theta], \\ &\theta\,\mathrm{d}t - c\theta + (1 + r\,\mathrm{d}t)^{-1}E[V(\theta + \mathrm{d}\theta, \theta + \mathrm{d}\Theta)|\theta, \Theta = \theta]\}. \end{aligned} \tag{5}$$

4. Solving the model

The simple structure of the problem implies that there exists a unique stopping line, $\theta(\Theta)$, which separates the stopping region from the continuation region. For values of θ above this line, it will be optimal to stop (invest), and for values below, it will be optimal to continue with the installed technology level. Furthermore, the stationarity of the problem implies that only relative values of θ and Θ are relevant for the optimal decisions. Scaling θ and Θ with the same factor will not change the decision problem of the agent, but will merely scale the optimal value by the same factor. Thus, $V(\theta, \Theta)$ is homogenous of degree one in θ and Θ, and the stopping line must be given by $\theta(\Theta) = \lambda\Theta$ for some constant $\lambda > 1$, i.e., λ is the unique proportionality parameter that characterises every optimal stopping decision. At any future point in time, if the installed technology is Θ, it will be optimal to invest when θ reaches $\lambda\Theta$, and subsequently to invest again when θ reaches $\lambda\Theta' = \lambda^2\Theta$ and so on, where $\Theta' = \lambda\Theta$ is the installed technology after the first investment.

In the following, we first derive the solution to the more general version of the model presented in Section 3, where both the productivity of installed technology, Θ, and the productivity of exogenous technology, θ, evolve according to (possibly correlated) geometric Brownian motions. We then focus on the simpler model where only θ evolves stochastically, as it turns out that there is a close correspondence between the solutions to the two models.

4.1. The generalised version

Using Ito's Lemma, it follows from (5) that in the continuation region, the following second-order partial differential equation applies:

$$rV = \Theta + \theta\alpha_1 V'_\theta + \frac{1}{2}\theta^2\sigma_1^2 V''_{\theta\theta} + \Theta\alpha_2 V'_\Theta + \frac{1}{2}\Theta^2\sigma_2^2 V''_{\Theta\Theta} + \rho\theta\Theta\alpha_1\alpha_2 V''_{\theta\Theta}, \tag{6}$$

where the arguments of the value function have been suppressed, and where V'_θ is the first-order partial derivative with respect to θ, etc. Now, the homogeneity of $V(\theta, \Theta)$ implies that a normalised value function, $v(w)$, can be defined as

$$v(w) = \Theta^{-1} V(\theta, \Theta), \tag{7}$$

where $w = \theta/\Theta$, and

$$\begin{aligned} &V'_\theta = v', \qquad V'_\Theta = v - wv' \\ &V''_{\theta\theta} = \Theta^{-1}v'', \quad V''_{\Theta\Theta} = \Theta^{-1}w^2v'', \quad V''_{\theta\Theta} = -\Theta^{-1}wv''. \end{aligned} \tag{8}$$

Substituting for V and its derivatives in the partial differential equation in (6), it can be written as the following ordinary differential equation:

$$v(r - \alpha_2) + w(\alpha_2 - \alpha_1)v' + \frac{1}{2}w^2(2\rho\alpha_1\alpha_2 - \sigma_1^2 - \sigma_2^2)v'' = 1 \tag{9}$$

which has the following general solution:

$$v = A_1 w^{a_1} + A_2 w^{a_2} + K, \tag{10}$$

where a_1, a_2, A_1, A_2, and K are constants to be determined. Substituting the solution in (10) back into the differential equation in (9) yields $K = (r - \alpha_2)^{-1}$, and a_1 and a_2 as the roots of the following quadratic equation:

$$a^2\left(\rho\alpha_1\alpha_2 - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2}\right) + a\left(\alpha_2 - \alpha_1 - \rho\alpha_1\alpha_2 + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}\right) + r - \alpha_2 = 0. \tag{11}$$

Hence

$$\begin{aligned} a_1 = {} & \frac{1}{2} - \frac{\alpha_1 - \alpha_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\alpha_1\alpha_2} \\ & + \sqrt{\left(\frac{1}{2} - \frac{\alpha_1 - \alpha_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\alpha_1\alpha_2}\right)^2 + \frac{2(r - \alpha_2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\alpha_1\alpha_2}} > 1 \end{aligned} \tag{12}$$

and

$$\begin{aligned} a_2 = {} & \frac{1}{2} - \frac{\alpha_1 - \alpha_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\alpha_1\alpha_2} \\ & - \sqrt{\left(\frac{1}{2} + \frac{\alpha_1 - \alpha_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\alpha_1\alpha_2}\right)^2 + \frac{2(r - \alpha_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho\alpha_1\alpha_2}} < 0 \end{aligned} \tag{13}$$

As a consequence, A_2 must equal zero. Otherwise, V would approach infinity as w becomes small. Hence, the value function for the continuation region can be expressed more compactly as

$$v = A_1 w^{a_1} + (r - \alpha_2)^{-1}, \tag{14}$$

where A_1 has yet to be determined together with the optimal stopping ryle, λ. For this purpose, the value-matching and smooth-pasting conditions are used. The value-matching condition

$$v(\lambda) = -c\lambda + \lambda v(1) \tag{15}$$

says that at the stopping line, where $w = \lambda$, the normalised value of continuing (the left-hand side) should equal the normalised value of stopping (the right-hand side). Furthermore, these values should meet smoothly, resulting in the following smooth-pasting condition:

$$v'(\lambda) = -c + v(1). \tag{16}$$

Using the smooth-pasting condition, (16), to substitute for $v(1)$ in the value-matching condition, (15), and also inserting the solution for v from (14), the result – after reduction – becomes

$$A_1 = -\frac{\lambda^{-a_1}}{(r - \alpha_2)(1 - a_1)} \tag{17}$$

which can be inserted back into the value-matching condition in (15) to get

$$\lambda^{-a_1} - a_1\lambda^{-1} = (1 - a_1)(1 - (r - \alpha_2)c). \tag{18}$$

The latter expression indirectly determines a unique optimal stopping rule, λ.[4]

4.2. The simpler version

We now focus on the simpler version of the model where only θ evolves stochastically according to (1). From the solution of the generalised model, it follows that in this case, where $\alpha_2 = \sigma_2^2 = 0$, the optimal stopping rule, λ, is given by

$$\lambda^{-a_1} - a_1\lambda^{-1} = (1 - a_1)(1 - rc), \tag{19}$$

where

$$a_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\alpha}{\sigma^2}\right)^2 + \frac{1}{4} + \frac{2r - \alpha}{\sigma^2}} > 1. \tag{20}$$

Furthermore, the value function, $V(\theta, \Theta)$, can in this case be expressed as:

$$V(\theta, \Theta) = -\frac{\lambda^{-a_1}}{r(1 - a_1)}\Theta^{1-a_1}\theta^{a_1} + \frac{\Theta}{r} \tag{21}$$

with λ and a_1 given by (19) and (20), respectively. The last term in (21) is the net present value of the technology in place, whereas the first term expresses the value of the option to invest in new technology. Note that the latter is decreasing in the productivity of the technology in place and increasing in the productivity of the exogenous technology process.

Note that in the case of zero correlation, $\rho = 0$, the solution to the generalised model with two stochastic processes is equivalent to the solution of the model with just a single stochastic process with parameters $\alpha = \alpha_1 - \alpha_2$ and $\sigma = \sigma_1^2 + \sigma_2^2$, and a modified interest rate of $r - \alpha_2$. Hence, focusing on the case where only θ evolves stochastically can be justified not only by its empirical relevance, but also as a convenient simplification of the more general model.

5. A comparison with related decision rules

In this section, the repeated-options rule given by λ in (19) will be interpreted and compared with two related decision rules in the literature: The simple NPV rule and the single-option rule.

[4]The existence of a unique value of $\lambda > 1$ satisfying (18) can be verified by noting that the left-hand side is monotonically increasing in λ, from $1 - a_1$ to 0, as λ increases from 1 to ∞. The right-hand side, on the other hand, is a constant which is strictly larger than $1 - a_1$ and strictly smaller than 0.

5.1. The simple NPV rule

The simple NPV rule says that an investment should be undertaken if the present value of the excess income associated with the investment exceeds the cost. In terms of the above notation, excess income is given by $r^{-1}(\theta - \Theta)$, whereas the cost is $c\theta$. This implies an optimal stopping value, λ, given by

$$\lambda = \frac{1}{1 - rc}. \tag{22}$$

According to the simple NPV criterion, higher cost, c, and higher interest rate, r, both serve to postpone an investment, since they unambiguously decrease the present value of the net returns from the investment. Note also that λ in (22) is independent of the drift, α, and uncertainty, σ, since this investment criterion does not take future dynamics of θ into account.

5.2. The single-option rule

The core model from Dixit and Pindyck (1994), i.e., the single-option model first analysed by McDonald and Siegel (1986), incorporates the future dynamics of θ, but does not allow for repeated investments. In the notation of the model of the previous sections, the λ of the single-option case is

$$\lambda = \frac{1}{\left(1 - \frac{1}{a_1}\right)(1 - rc)}, \tag{23}$$

where a_1 is given by (20).[5] Note that the value of λ implied by (23) is always larger than the value implied by (22). The difference reflects the value of waiting. Thus, according to the single-option criterion, the simple NPV criterion dictates too rapid investments – a point repeatedly made by the real-options literature.

Using that $\lambda = \theta(\Theta)/\Theta$, the expression in (23) can alternatively be written as

$$\Theta + \frac{r}{a_1}\theta(\Theta)\left(\frac{1}{r} - c\right) = r\theta(\Theta)\left(\frac{1}{r} - c\right). \tag{24}$$

The left-hand side is then the return from holding the option, i.e., from continuing with the installed technology having productivity Θ, whereas the productivity of exogenous technology is $\theta(\Theta)$. This return has two components: (i) a current income flow of Θ; and (ii) an increase in the net present value of stopping. The latter is given by the second term on the left-hand side, where $\theta(\Theta)(\frac{1}{r} - c)$ is the net present value of stopping and obtaining the productivity $\theta(\Theta)$, and r/a_1 is the expected growth rate of this value. Note that r/a_1 increases monotonically from α to r as σ goes from 0 to ∞.[6] Hence, in the deterministic case, the value of stopping grows at the rate of α, but

[5] The stopping rule in (23) can be found by proceeding exactly as in the case with repeated options, replacing the value of stopping with $\theta(\frac{1}{r} - c)$.

[6] This follows immediately from (20) and the fact that $\frac{\mathrm{d}a_1}{\mathrm{d}\sigma} < 0$, where the latter can be derived from differentiation of the quadratic equation in (11).

under uncertainty, the value of stopping is expected to grow at a higher rate. The reason is the standard one that uncertainty has an asymmetric effect, since the agent can, in case of a negative shock to θ, choose to postpone the investment, thereby mitigating the consequences of a negative shock.

The right-hand side in (24) is the cost of keeping the investment option alive when the productivity of exogenous technology is $\theta(\Theta)$. This cost is given by the interests lost on the value of stopping.

Now, for values of θ where the right-hand side is smaller than the left-hand side, i.e., where the cost of keeping the option alive is smaller than the return, it is optimal to keep the option alive and to continue producing Θ with the installed technology. At the stopping line, $\theta(\Theta)$, the two values coincide.

The expression in (24) reveals that the stopping rule, $\lambda = \theta(\Theta)/\Theta$, is increasing in c. Since $r > r/a_1$, a higher c will decrease the cost of holding the option by more than it decreases the return from holding it. Similarly, λ is increasing in α and σ, since α and σ will increase r/a_1 and hence imply a higher return from holding the option.[7] The effect of a change in r is more ambiguous. First, a higher interest rate implies a lower net present value of stopping, $\theta(\Theta)(\frac{1}{r} - c)$, which decreases the cost of holding the option by more than it decreases the return. Second, a higher value of r directly increases the current cost of holding the option through the first factor on the right-hand side in (24). Third, the interest rate has an unclear effect on the expected growth rate, r/a_1, of the value of stopping. This leaves the aggregate effect of a change in r ambiguous.

5.3. The repeated-options rule

Now, turn to the repeated-options approach, where the expression for λ in (19) can be rewritten as:

$$\Theta + \frac{r}{a_1}\theta(\Theta)\left(\frac{1}{r} - c\right) = r\theta(\Theta)\left(\frac{1}{r} - c\right) + \frac{1}{a_1}\theta(\Theta)^{1-a_1}\Theta^{a_1}. \tag{25}$$

The only difference between this expression and the expression in (24) is the last term on the right-hand side. This term can be interpreted as the additional net cost of keeping the option alive which arises from the existence of future investment options. To see this more clearly, note that the term can alternatively be expressed as

$$\frac{1}{a_1}\theta(\Theta)^{1-a_1}\Theta^{a_1} = \left(r - \frac{r}{a_1}\right)\left[V(\theta(\Theta), \theta(\Theta)) - \frac{\theta(\Theta)}{r}\right], \tag{26}$$

where the expression in square brackets is the difference in stopping values under the repeated-options approach and the single-option approach. The factor $r - \frac{r}{a_1}$ gives the net cost, i.e., the cost, r, minus the expected return, r/a_1, of holding this extra option alive.

[7]Again, the signs of $\frac{da_1}{d\alpha}$ and $\frac{da_1}{d\sigma}$ (and $\frac{da_1}{dr}$) can be derived from differentiation of the quadratic equation in (11).

It follows from (25) that λ is smaller under the repeated-options approach than under the single-option approach.[8] Hence, the single-option approach dictates too slow investments when options are in fact repeated. Intuitively, allowing for repeated options makes the current investment "less irreversible". This reduces the value of waiting – or the cost of exercising – compared to a situation with only one investment possibility. Investments are therefore undertaken more rapidly.

However, not only the level of λ deviates from the one from the single-option solution. Changes in the underlying parameters also have different effects on λ when options are repeated.

Compared to the single option case, a stronger drift, α, or a higher uncertainty, σ, will now have the additional effect of increasing the value of future investment options. This has a positive effect on the last term in (25) and therefore on the cost of keeping the current option alive. This must be compared to the increase in the immediate return from holding the option which was also present in the single-option case. The result of the two countervailing effects on λ is, however, still positive although clearly less significant than in the case of a single option.[9]

Similarly, an increase in r will now also decrease the value of future options – in addition to the effects from the single-option set-up. This will make it less costly to keep the current option alive, since the last term in (25) becomes smaller. Hence, in the repeated-options set-up, an increase in r is more likely to cause an increase in λ than in the single-option set-up.

Finally, a higher c has the additional effect of reducing the value of future options and therefore the cost of holding the additional option value given by the last term in (25). Hence, a higher c will have a stronger positive effect on λ compared to the situation with a single option.

The following section contains some numerical illustrations of these differences, and their implications for optimal behaviour are discussed.

6. Illustrations and a rule of thumb

Using $c = 2$, $r = 0.05$, $\alpha = 0.03$, and $\sigma = 0.05$ as baseline parameter values, Figs. 1–3 show examples of how the simple NPV stopping rule, the single-option stopping rule, and the repeated-options stopping rule behave when varying α, σ, and r, respectively. The figures also show a simple rule of thumb computed using a mark-up on the interest rate (a hurdle rate) in the simple NPV formulation. This rule will be discussed in more detail below.

[8] To see this, rewrite (25) as: $\Theta - \theta(\Theta)(r - r/a_1)(r^{-1} - c) = a_1^{-1}\theta(\Theta)^{1-a_1}\Theta^{a_1}$. Given Θ, the value of $\theta(\Theta)$ which causes the left-hand side to be zero corresponds to the stopping point from the single-option set-up. Since the right-hand side is always positive and the left-hand side is monotonically decreasing in $\theta(\Theta)$, the value of $\theta(\Theta)$ that solves the above equation must be strictly smaller than the value that makes the left-hand side equal to zero.

[9] Unfortunately, it has not been possible to show this formally in terms of the derivative, but a careful examination of the parameter space supports the conclusion.

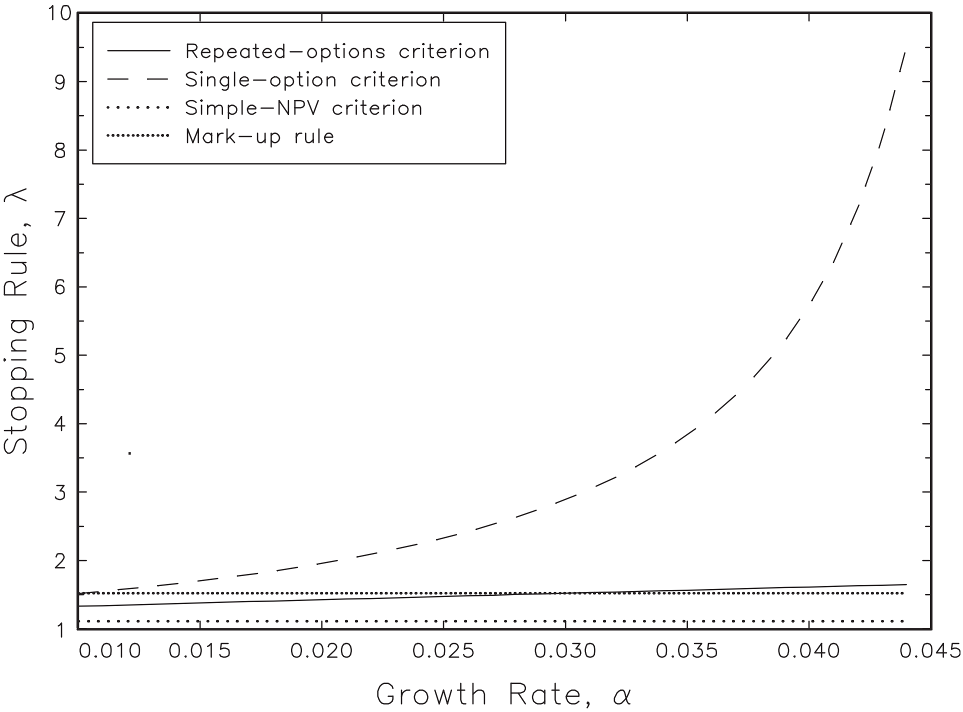

Fig. 1. Sensitivity of stopping rules with respect to the growth rate, α, when $r = 0.05$, $\sigma = 0.05$, and $c = 2$.

The first thing to note is that the value of λ is much smaller according to the repeated-options rule than according to the single-option rule. This confirms the result from the previous section. The value of waiting is smaller when options are repeated.

Secondly, the repeated-options rule is seen to react much differently to changes in the parameters, α, σ, and r. A change in one of these parameters has a much more pronounced effect on the optimal stopping rule under the single-option approach than under the repeated-options approach. Increasing α or σ causes a dramatic increase in the single-option rule whereas it leaves the repeated-options rule almost unaffected. This is due to the additional positive effect on the value of future options from increasing α or σ, as discussed in Section 5.3.

Increasing r is seen to slightly increase λ under the repeated-options approach, whereas it decreases λ significantly according to the single-option approach. Again, this difference is in accordance with the intuition provided in the previous section. Note also that the distance between the optimal stopping rules is most pronounced when r is small (close to α). As r is increased, the future becomes less important, and the repeated-options rule converges to the single-option rule. In the end, as r goes to $\frac{1}{c}$, the stopping rules will converge to infinity.

In sum, it appears that the repeated-options rule is much less sensitive than the single-option rule to changes in the underlying parameters. Actually, the effects

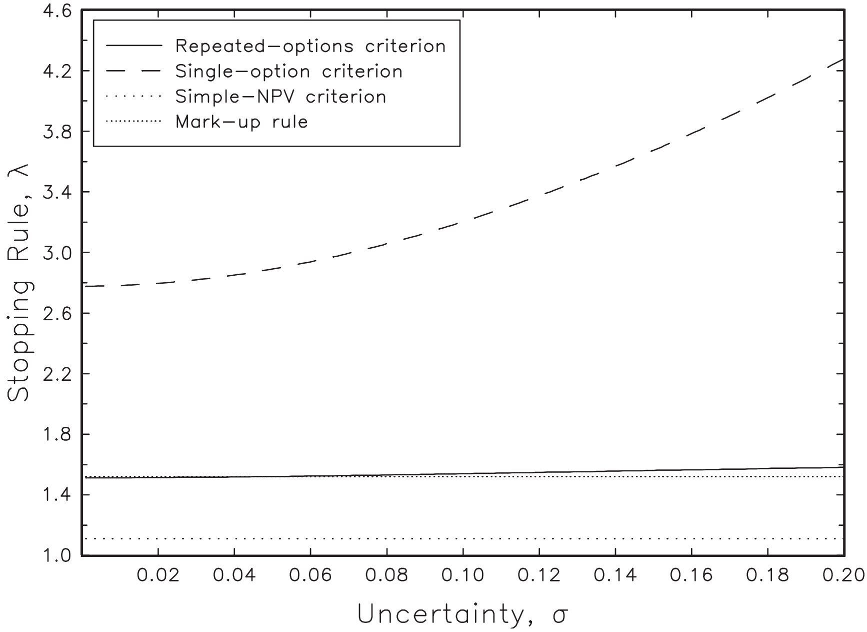

Fig. 2. Sensitivity of stopping rules with respect to the uncertainty, σ, when $\alpha = 0.03$, $r = 0.05$, and $c = 2$.

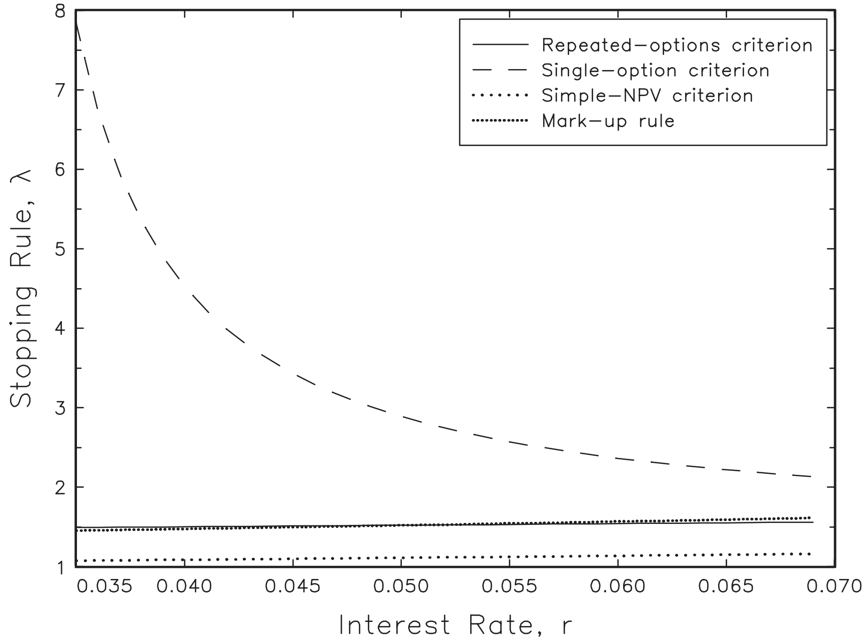

Fig. 3. Sensitivity of stopping rules with respect to the interest rate, r, when $\alpha = 0.03$, $\sigma = 0.05$, and $c = 2$.

resemble those observed using the simple NPV rule, which is insensitive to changes in α and σ and only increases slightly with r.

This suggest that in the case of repeated options, it might be much easier to find a simple and yet useful rule of thumb than in the case of a single option. One possible rule of thumb could be to use the simple NPV rule with a mark-up on the interest rate. The use of this type of rule is common practice, and is usually referred to as a "hurdle-rate criterion", see, e.g., Wambach (2000) and Dixit and Pindyck (1994).[10]

Figs. 1–3 include a hurdle-rate rule computed using a 12 percentage point mark-up on the interest rate. In the baseline case, this means that r is marked up from 5% to 17%.[11] It follows immediately from the figures that this investment rule comes rather close to the optimal repeated-options rule. At least, it clearly outperforms the decision rule derived under the simplifying assumption that there is only a single investment option. This is confirmed by Monte Carlo computations of the expected present value obtained using each of the three decision criteria – the repeated-options rule, the single-option rule, and the hurdle-rate rule – when investment options are actually repeated.[12]

For values of α between 0.01 and 0.04 while keeping other parameters constant, using a rule of thumb based on a 12 percentage points mark-up on the interest rate does not imply an expected loss of more than 1%. The expected loss from using the single-option rule, however, is up to 27% compared to the value obtained by using the exact repeated-options rule. Similarly, for values of σ between 0.01 and 0.2, a maximum expected loss of less than 0.1% is realised when using the rule of thumb. Again, the single-option rule might cause a loss of more than 20%. Finally, varying r between 0.04 and 0.07, the expected loss from using the rule of thumb varies between 0 and 0.1%, whereas the expected loss from using the single-option rule is up to 25%.

More extreme deviations, where more than one parameter deviates from the baseline case, and thereby the optimality of the rule of thumb, such as $(r, \alpha, \sigma) = (0.1, 0.07, 0.2)$ or $(r, \alpha, \sigma) = (0.03, 0.01, 0.01)$, generate expected losses of 1.8% and 1.4%, respectively, from using the hurdle-rate rule, whereas the single-option rule implies losses of 15.5% and 3.8%, respectively.

Hence, a reasonable choice of a hurdle rate appears to be a simple and comforting way of handling repeated investment options. At least, it is much more comforting than using the more complicated single-option rule. Furthermore, the hurdle-rate rule is not very sensitive to the exact mark-up chosen. In fact, the elasticity of the hurdle-rate rule with respect to r is only about one half for $r = 0.17$ and $c = 2$. Hence, an "incorrect" mark-up of only 0.10 will imply a stopping rule of $\lambda = 1.42$ in the baseline case. This should be compared to the rule of $\lambda = 1.52$ derived using the "correct" mark-up of 0.12. This implies that some inaccuracy in choosing the

[10]Note that Wambach (2000) actually argues in favour of the use of hurdle rates in a single-option set-up. However, the present study shows that they are much more likely to be successful in a repeated-options set-up.

[11]This mark-up is, of course, not picked at random. It is (approximately) the mark-up that would make the simple NPV rule equal to the optimal repeated-options rule for our baseline parameter case, $c = 2$, $\alpha = 0.03$, $r = 0.05$, and $\sigma = 0.05$.

[12]The expected present value is equivalent to the value function associated with a given decision rule.

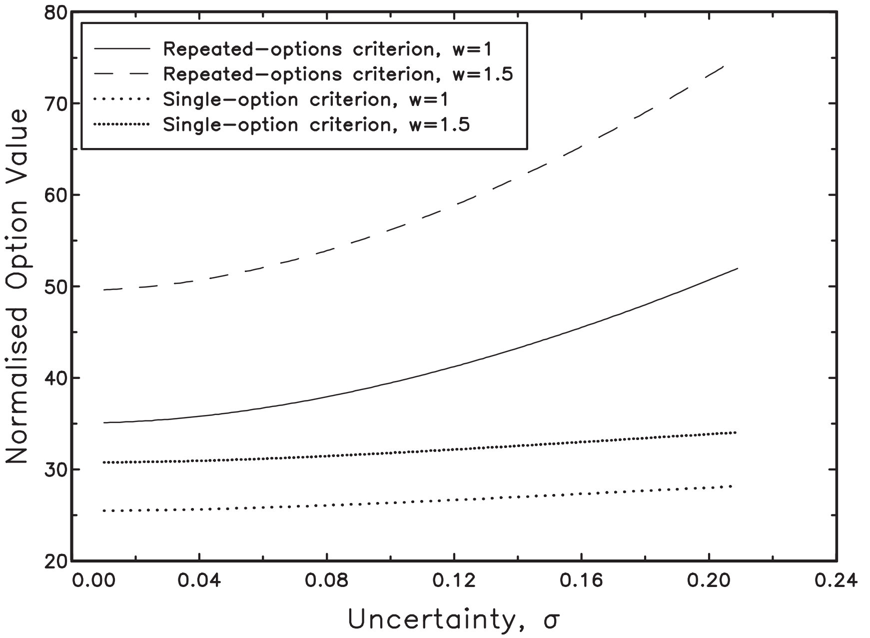

Fig. 4. Sensitivity of value functions with respect to the uncertainty, σ, when $\alpha = 0.03$, $r = 0.05$, and $c = 2$.

mark-up will not significantly decrease the usefulness of the hurdle-rate criterion. In short, the hurdle-rate rule seems to perform robustly both with respect to parameter changes and with respect to the choice of mark-up.

To show that the introduction of repeated options not simply eliminates the importance of uncertainty, Fig. 4 shows how the normalised value functions vary with changes in σ. Two values of normalised technology have been chosen, $w = 1$ and 1.5, which are both inside the continuation region given the chosen parameter values. The effect of uncertainty on the value functions is seen to be more pronounced for the repeated-options case. Although uncertainty does not affect the optimal stopping value, λ, by much, it definitely has positive accumulated effects on wealth in this case.

7. Conclusion

This paper analyses investment-under-uncertainty problems of a type closely related to those investigated by Dixit and Pindyck (1994), with the important extension that repeated investment options are allowed for. A model incorporating this feature is solved analytically. It is shown that when investment options are repeated: (i) the value of waiting is significantly smaller; and (ii) the implications of changes in underlying parameters are different and typically of a smaller magnitude. The latter implies that when investment options are repeated, finding a simple rule of thumb is much easier than in the case of a single option. To give an example of this, it is demonstrated how a hurdle-rate rule, based on the simple NPV criterion with a mark-up on the interest rate, behaves much like the optimal repeated-options investment rule. It definitely performs much better than the single-option criterion when investment options are in fact repeated. In addition, the hurdle-rate rule is not very sensitive to the exact choice of mark-up, which makes it even more attractive as a rule of thumb.

This might be comforting news for managers and decision makers who rarely have the time to undertake advanced option analyses and must deal with the effects of uncertainty and growth in a more ad hoc fashion. In particular, the above analysis implies that resorting to the single-option rule from the real-options literature might not be worth the effort – and may yield inferior solutions compared to a simple hurdle-rate criterion – when investment options are indeed repeated.

Acknowledgements

The authors wish to thank Finn Helles, Robert A. Jones, and Peter Skott for constructive suggestions, and Jens Iversen for research assistance. Financial support from the Danish Council for Development Research (Malchow-Møller) and the Danish Agricultural and Veterinary Research Council (Thorsen) is gratefully acknowledged.

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