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Strategy-proof assignment on the full preference domain

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Abstract

We consider the problem of efficiently allocating several indivisible objects between agents who are to receive at most one object and whose preferences are private information. We examine this standard "assignment" problem from the perspective of mechanism design giving up the usual assumption of linear preferences and instead using a full preference domain (with indifferences permitted). We characterize two classes of mechanisms: (i) Bi-polar Serially Dictatorial Rules by Essential Single-Valuedness, Pareto Indifference, Strategy-Proofness and Non-Bossiness; and (ii) all selections from Bi-polar Serially Dictatorial Rules by Single-Valuedness, Efficiency, Strategy-Proofness and Weak Non-Bossiness. We compare the outcomes from the (Bi-polar) Serially Dictatorial Rules with the outcomes obtained using a market based approach, namely the "core" of the market. We show that all strongly efficient outcomes in the core can be generated using Serially Dictatorial Rules. Moreover, we argue that Serially Dictatorial Rules have an advantage over the market based approach in that they yield strongly efficient solutions for all preference profiles, making it possible to use randomization to restore equity. When preferences are private information, this type of *ex ante* equity cannot be implemented using the market based approach.

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1. Introduction

Consider the problem of allocating a number of indivisible objects (houses) to a group of (homeless) individuals such that each individual gets at most one house. The individuals do not pay for the houses; neither is any form of side payments between the individuals permitted. Clearly, this type of problem arises when assigning courses to professors, offices to employees, or public schools to students. [1] If agents' preferences over the houses are private information, how can one ensure that the houses are allocated efficiently? Two approaches to this problem, discussed below, have been explored in the literature.

The seminal analysis by Shapley and Scarf [12] uses the notion of Edgeworthian exchange to solve this problem. The houses are arbitrarily allocated among the individuals who are free to engage in multilateral negotiations and exchange by using their allocation either individually or as part of a coalition to block proposed allocations. The set of unblocked allocations is, of course, the core. In their classic paper, Shapley and Scarf [12] use Gale's "Top Trading Cycle" algorithm to establish the existence of the (weak) core of this "housing market." However, in general, the weak core of the housing market may not be strongly efficient in the sense that starting from an allocation in the weak core it may be possible to reallocate the houses to make some individuals better off without making anyone else worse off. A number of subsequent "early" papers have used this "exchange based" approach to investigate further the properties of cores of markets with indivisibilities (see Ma [8], Wako [15], Roth and Postlewaite [10]) showing that in the special case when preferences are restricted to be linear orderings, the strong core exists, is unique and efficient.

More recent works (Abdulkadiroğlu and Sönmez [1,2], Pápai [9], Svensson [13,14], Ehlers [5]) have viewed the assignment problem as a problem of mechanism design and have looked for strategy-proof mechanisms satisfying desirable properties which would provide an efficient and equitable solution to the problem. The allocation rules that this approach has yielded while efficient, have been shown to have a hierarchic structure. One such mechanism resembles queuing rules of a type used for distributing scarce indivisible and (possibly) heterogeneous objects like seats at a theater. Individuals queue up and the first person in the queue selects her best object. The second selects his best from the remaining objects. Each individual gets to select from the objects that remain after the individuals ahead of her in the queue have made their choices. In a recent paper, Svensson [14] shows that if preference orderings are linear, these queuing mechanisms are the only rules satisfying Strategy-Proofness, Neutrality and Non-Bossiness. A second important characterization result (also using linear orderings) can be found in Pápai [9]. She describes a class of rules, Hierarchical Exchange rules, and shows that this class is completely characterized by the properties of Group Strategy-Proofness, Efficiency and Reallocation-Proofness.

How do these two approaches compare? It can be argued that for the special domain of linear orderings the market-based approach using exchange and the mechanism design approach using hierarchical rules are, in fact, "equivalent". Pápai [9] shows that hierarchical rules can be thought of as imitating a market in which individuals exchange objects from hierarchically determined endowment sets in an iterative manner that closely resembles

[1] The introduction in Pápai [9] gives examples of other similar situations.

Gale's Top Trading Cycle procedure developed in the context of the housing market. [2] An examination of the queuing procedures in Svensson [14] reveals that the connection between the market approach and the mechanism design approach is even closer. The queuing mechanisms pick elements from the core of the housing market and any allocation in the core of the housing market can be viewed as being generated by some queuing mechanism (see Abdulkadiroğlu and Sönmez [1]). Non-stochastic applications of both approaches are inequitable. The inequity is explicit in the hierarchy of the queue with an implied disadvantage for those at the bottom of the hierarchy. [3] For the market-based approach the inequity is *implicit* in the initial distribution of houses, and is reflected in the queuing rule generating the core allocation of that market. One can of course introduce equity ex ante in both approaches by attaching equal probability to all possible queues on one hand and equal probability to all initial endowments of houses on the other. An insightful result in Abdulkadiroğlu and Sönmez [1] establishes the formal equivalence of these two procedures.

We have, so far, reported two types of results for the specialized domain of linear preference orderings: characterization results for queuing mechanisms and hierarchical exchange rules and results showing that these mechanisms are in a sense equivalent to market mechanisms. Do these types of results extend to the full domain of preferences in which indifference in individual preferences is permitted? There are both pragmatic and formal reasons for considering the full domain. One can think of many assignment problems in which indifference between objects is intuitively *plausible* where some individuals are indifferent between a pair of objects while others are not. (It would be perfectly reasonable, for instance, for an individual to be indifferent between two different seats in the front row of a theater, both of which are "off-center" by the "same amount", while for another individual who favors one ear over another to prefer the seat on one side.) There are also allocation problems where some number of the objects are "identical" (say books are being distributed, with multiple copies of some books being available). Here indifference in individual preferences is a *logical* implication and all rational individuals would be indifferent between the objects. Moreover, the full domain of preferences is considered to be the natural domain in economics and many of the papers using the market based approach allow for the full domain (see, for instance Shapley and Scarf [12]). To be able to compare the two approaches and to address problems where indifference arises naturally in the context of particular problems, clearly, there is a need for models with a full domain. From a technical point of view, the study of the wider domain is interesting not only because results true for a narrower domain may not extend to the wider domain, but more substantively, the richer domain may yield results which are *false* in the narrow domain. This makes the investigation of the wider domain interesting in its own right independently of the possibility of extending results obtained for a special domain. In our case, we will see instances of all these phenomena: new results which do not hold for the linear domain, results that do not extend to the full domain as well as results that do.

[2] Roth [16] proved that the rule implementing the Top Trading Cycle method is strategy-proof.

[3] While the earliest exchange based approaches were largely deterministic, some of the recent work on mechanism design has used probabilistic methods to make the solution more equitable (see Bogomolnaia and Moulin [3]).

With one exception, the literature devoted to the incentive-compatible mechanism design approach does not consider the full domain of individual preferences. In the general case, when individual preferences are weak orderings, we are left with the question as to which rules may be identified in terms of a set of desirable properties. For the market based approach, permitting indifference raises the possibility of the strong core being empty and the weak core being inefficient (see Roth and Postlewaite [10]). From this should we conclude that the mechanism design approach is less problematic? Is the mechanism design approach equivalent (as it is in the case of linear orderings), is it better or is it worse than the market-based approach?

In this paper, we will resolve two issues in the context of the full domain of individual preferences:

(a) The problem of providing a complete characterization of a class of allocation rules which are natural extensions of queuing rules.
(b) A comparison of these rules with allocation rules arising from the traditional exchange based approach to the housing market.

The only paper using the mechanism design approach that looks at the full preference domain is Svensson [13].[4] He gives an example of a class of rules, Serially Dictatorial Rules, that satisfy a number of desirable properties like Efficiency, Strategy-Proofness and Neutrality. But, since he provides no characterization of this class, he leaves open the question whether other rules with similar or more desirable properties exist. Our contribution should be viewed as fully resolving this issue. We will do this by completely characterizing (i) a class of rules we call Bi-polar Serially Dictatorial Rules using the assumptions of Strategy-Proofness, Non-Bossiness, Essential Single-Valuedness and Pareto Indifference and (ii) all selections from Bi-polar Serially Dictatorial Rules using the assumptions of Strategy-Proofness, Weak Non-Bossiness, Single-Valuedness and Efficiency.

Here (i) takes the welfarist view that the only basis for selecting amongst different allocations is the satisfaction that individuals derive from the allocations and thus if two allocations are "utility equivalent" either both or neither is selected. In the presence of a complete domain of preferences this forces the selection mechanism to be a correspondence rather than a function. On the other hand, (ii) assumes that the selection mechanism is a function rather than a correspondence and that a single allocation is selected. The argument that may be made in this case is that since individuals do not care which of two utility equivalent allocations are picked, no harm is done by breaking ties and selecting a single allocation.

The set of Bi-polar Serially Dictatorial Rules includes as a sub-class the class of rules (Serially Dictatorial Rules) proposed (but not characterized) by Svensson [13]. By adding the assumption of Neutrality to our results we can see that a characterization result of Svensson [14] for the linear preference domain does extend to the case of the full preference domain. However, since like Pápai's [9] characterization, our result does not use neutrality it is the comparison with Pápai's [9] result for the linear preference domain that is more interesting.

[4] Some recent papers allow for indifferences, but not for the full domain. For instance, Bogomolnaia and Moulin [4] use dichotomous preference domain and in Ergin [6] some objects may be identical but subjective indifferences are ruled out.

A comparison immediately shows that our characterization does *not* hold for the linear domain, while her result does not extend to the full domain! We will argue that the principal (characterization) result in Pápai [9] is quite sensitive to and can be significantly sharpened with the "full domain" assumption. Some *assumptions* of Pápai [9] can be derived as *conclusions* using the full preference domain and the large and complex family of rules described in Pápai [9], Hierarchical Exchange Rules, is replaced by our simpler class of allocation rules: Bi-polar Serially Dictatorial Rules. We will demonstrate that this "simplification" also has the unfortunate consequence of eliminating useful mechanisms of a type that have been effectively used by Abdulkadiroğlu and Sönmez [2] to solve the problem of house allocation with existing tenants for the strict preference domain. This raises a possibility that a solution to this important problem may not exist for the full preference domain.

In comparing Bi-polar Serially Dictatorial Rules with exchange-based market mechanisms we find that just as in the case of the linear domain of preferences the two approaches are similar in one respect: if we pursue strong efficiency, this inevitably leads to a hierarchic structure over the set of individuals. Specifically, we show that there is an isomorphism between the strong core of the housing market game, the set of possible outcomes that can arise from the Serially Dictatorial Rules and the set of all strongly efficient allocations of houses. Serially Dictatorial Rules, however, have an advantage in that for this method a strongly efficient solution always exists, making it possible to use randomization over all possible hierarchies of individuals to mitigate (ex ante) the inequity inherent in these hierarchic structures. Given our assumption that preferences are private information, since the strong core may not exist for some endowments, this type of randomization cannot be used over the endowments that give rise to allocations in the strong core as these allocations cannot be identified without first knowing the individuals' preferences. [5] This problem does not arise in the domain of linear preferences in which Abdulkadiroğlu and Sönmez [1] establish their equivalence result. Thus, in the full domain of preferences, their type of equivalence breaks down with the Serially Dictatorial Rule enjoying a distinct advantage over solutions to the assignment problem generated by exchange based market allocations.

2. Notation and definitions

We consider a situation where a non-empty finite set of indivisible alternatives $X = \{x_1, \ldots, x_m\}$ has to be allocated to the members of a finite society $N = \{1, \ldots, n\}$ of n agents in such a way that everybody receives at most one of these objects and no agent is denied an object as long as objects are available.

Agents can have arbitrary ordinal weak preference orderings over the set X. Receiving no object ($\emptyset$) is ranked strictly below any element of X by all agents. Let $\mathcal{R}$ be the set of all such preferences over $X \cup \{\emptyset\}$. We denote preferences of agent i by R_i (R_i', R_i^1, etc.). We also write it as $\succsim_{R_i}$ with its asymmetric component $\succ_{R_i}$ and its symmetric component $\sim_{R_i}$ or simply as $\succsim_i$ etc., when it refers to R_i with the preference profile subscript reserved for the cases of R_i', R_i^1, etc. or whenever there is a possibility of ambiguity. $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$

[5] We would not know which endowments to randomize over, since without knowing the individuals' preferences we would not know which endowments would yield an empty core.

denotes a preference profile of the society. We use the standard notation (R_{-i}, R'_i) for a profile which differs from R in its ith component only. (R_{-i}, R'_i) is obtained from R by changing the preferences of agent i from R_i to R'_i.

An *allocation* is a function $\alpha : N \to X \cup \{\emptyset\}$ such that for any $i \neq j$ we can only have $\alpha(i) = \alpha(j)$ if $\alpha(i) = \alpha(j) = \emptyset$, and that $|\{i : \alpha(i) = \emptyset\}| = \max\{0, n - m\}$. We thus require that each agent receives at most one object, that each object goes to at most one agent, and that an agent ends up with no object only if there are no unallocated objects. Let A be the set of all allocations. Given a preference profile R, we say that two allocations α and β are *R-utility equivalent* if $\alpha(i) \sim_{R_i} \beta(i)$ for all $i \in N$. We denote by $\widetilde{\alpha}^R$ the class of all allocations which are R-utility equivalent to α.

An *assignment rule* is a non-empty correspondence $f : \mathcal{R}^N \rightarrow\!\!\!\rightarrow A$. Given two assignment rules f and f', we call f *a selection from* f' if $f(R)$ is single-valued and $f(R) \subseteq f'(R)$ for all $R \in \mathcal{R}^N$.

3. Properties of assignment rules

To accommodate the full domain of preferences, a mechanism allocating the objects needs to be able to specify an allocation for the case when all individuals are indifferent between all objects. From the individuals' perspective, in this case, all possible assignments (such that each individual receives an object) are equivalent to each other. Consequently, if one adopts the welfarist norm, that selection of allocations should only be based on individual utilities, and since in this case, all allocations are utility equivalent, all should be deemed to be equally good. Thus, any "assignment rule" is necessarily a correspondence from individual preference profiles to the set of allocations and such a correspondence should not distinguish between "utility equivalent" allocations.

Pareto indifference (PI): An assignment rule $f : \mathcal{R}^N \rightarrow\!\!\!\rightarrow A$ is *Pareto Indifferent*, if for all $R \in \mathcal{R}^N$ and all $\alpha \in f(R)$, we have that $\widetilde{\alpha}^R \subseteq f(R)$.

Pareto Indifference, however, does not require that all allocations selected as a solution to the assignment problem should be utility equivalent. Such a restriction would ensure that all allocations proposed by an allocation rule are equivalent from the individuals' perspective (i.e. for any preference profile all selected allocations are utility equivalent) and it thus becomes irrelevant which allocation is finally implemented.

Essential Single-Valuedness (ESV): An assignment rule $f : \mathcal{R}^N \rightarrow\!\!\!\rightarrow A$ is *essentially single-valued*, if for all $R \in \mathcal{R}^N$ and all $\alpha, \beta \in f(R)$, we have that $\alpha(i) \sim_{R_i} \beta(i)$ for all $i \in N$.

Most of the strategy-proof mechanism design literature on the allocation of indivisible goods have considered only single-valued rules (functions) defined over the domain of strict orderings.

Single-Valuedness (SV): An assignment rule $f : \mathcal{R}^N \rightarrow\!\!\!\rightarrow A$ is *Single-Valued*, if for all $R \in \mathcal{R}^N$, $|f(R)| = 1$.[6]

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[6] Here $|f(R)|$ denotes the cardinality of set $f(R)$.

Any single-valued assignment rule is essentially single-valued. On the domain of strict orderings it also satisfies PI. Moreover, on this restricted preference domain, our allocation rules, if they satisfy ESV, would generate such functions. Thus, arguably, if one wishes to retain the welfarist norm, ESV of the allocation rule is the appropriate analogue of single-valuedness under the assumption of full preference domain. Notice that the assumption of SV is inconsistent with PI and hence with welfarism, but is consistent with real world procedures where unique selections from equivalent alternatives are often made using tie breaking information based on non-utility information.[7] Thus, non-welfarist allocation rules satisfying SV and not PI are also clearly worth investigating.

It has been well known since the mid-1970s that multivalence of the social choice function (i.e. the social choice "function" being a correspondence) poses special problems (see for instance Gärdenfors [7]). In this context, even formally capturing the intuition underlying the concept of strategy-proofness may be problematic. In our model this problem will not arise. We will argue below that the assumption ESV brings with it the added advantage of simplifying considerably the formalization of the concept of strategy-proofness of our allocation rules.

For the game in which each individual reports her preference ordering and *f* selects a *set* of allocations, what is the appropriate notion of strategy-proofness? Consider the following intuition underlying strategy-proofness: "No matter what messages the other individuals send about their preference, no individual has an incentive to manipulate the system by sending a false signal". This intuition would be easy to interpret formally in the standard way had *f* been a function or if individuals had preferences defined over the *power set* of alternatives. Here, neither being the case we define below two concepts, "strong strategy-proofness" and "weak strategy-proofness," *both* of which would correspond to the standard definition of strategy-proofness for single-valued allocation rules. The two definitions are based on different intuition as to how, given their preferences over the objects, individuals may make their strategic decisions when presented with alternative *sets* of objects.

Strong Strategy-Proofness (SSP): An assignment rule $f : \mathcal{R}^N \twoheadrightarrow A$ is *strongly strategy-proof*, if for any R and R' such that $R' = (R_{-i}, R'_i)$ for some $i \in N$, we have that $\alpha(i) \succsim_{R_i} \beta(i)$ for all $\alpha \in f(R)$ and $\beta \in f(R')$.

We say that an agent i *can* manipulate at R via R', where $R' = (R_{-i}, R'_i)$, *only if* there exist $\beta \in f(R')$ and $\alpha \in f(R)$ such that $\beta(i) \succ_{R_i} \alpha(i)$.

The underlying notion of SSP is that no individual will ever seek to manipulate the rule unless by doing so she has the *possibility* of acquiring an outcome that is better for her than at least one of the outcomes she could get at the initial profile. This is clearly an appealing *sufficient* condition for *non-manipulability*. Now, consider the following definition which provides an appealing sufficient condition for *manipulability* (i.e. a good *necessary* condition for *non-manipulability*).

Weak Strategy-Proofness (WSP): An assignment rule $f : \mathcal{R}^N \twoheadrightarrow A$ is *weakly strategy-proof* if, for any R and R' such that $R' = (R_{-i}, R'_i)$ for some $i \in N$, we have either

[7] For instance, in voting ties are broken in favor of the status quo.

$\alpha(i) \succsim_{R_i} \beta(i)$ for all $\alpha \in f(R)$ and $\beta \in f(R')$, or there exist $\beta \in f(R')$ and $\alpha \in f(R)$ such that $\alpha(i) \succ_{R_i} \beta(i)$.

We say that an agent i *will* manipulate at R via $R' = (R_{-i}, R'_i)$ whenever $\beta(i) \succsim_{R_i} \alpha(i)$ for all $\alpha \in f(R)$ and $\beta \in f(R')$, and there exist $\beta \in f(R')$ and $\alpha \in f(R)$ such that $\beta(i) \succ_{R_i} \alpha(i)$.

Fortunately, under ESV we do not have to choose between an attractive necessary condition and an equally attractive sufficient condition for strategy-proofness (respectively, manipulability).

Proposition 1. *Let f be an Essentially Single-Valued assignment rule. Then, f satisfies SSP if and only if it satisfies WSP.*

Proof. SSP implies WSP follows immediately from the definitions. To see the converse, assume to the contrary that WSP is satisfied and SSP is not.

Violation of SSP implies that there exist R, $R' = (R_{-i}, R'_i)$ and $\beta \in f(R')$ such that $\beta(i) \succ_{R_i} \alpha(i)$ for some $\alpha \in f(R)$. Thus, by ESV, $\beta(i) \succ_{R_i} \gamma(i)$ for all $\gamma \in f(R)$. Now, consider $R'' = (R_{-i}, R''_i)$, where R''_i is such that $\beta(i)$ is the unique top choice of agent i, while all other objects she can get at $f(R')$ are indifferent to each other and strictly better for her than all remaining alternatives. By ESV, i either receives under R'' object $\beta(i)$ at all chosen allocations or i does not receive object $\beta(i)$ under R'' at any chosen allocation. If she does not get $\beta(i)$ at R'' then, given that all allocations from $f(R'')$ are indifferent for her and thus worse then $\beta(i)$, she *will* manipulate at R'' via R'. If agent i gets $\beta(i)$ at R'', then she *will* manipulate at R via R''. Hence, f violates WSP, a contradiction. □

From now on we say that a rule satisfies Strategy-Proofness (SP) whenever it satisfies both SSP and WSP.

In the presence of indifference in individual preferences we have to distinguish between different notions of Pareto efficiency. When we refer to the efficiency or inefficiency of an allocation rule we will be referring to "strong" rather than "weak" efficiency.

Efficiency (EFF): An assignment rule $f : \mathcal{R}^N \twoheadrightarrow A$ is *Efficient*, if for any $R \in \mathcal{R}^N$ and any $\alpha \in f(R)$, there is no $\beta \in A$ such that for all $i \in N$, $\beta(i) \succsim_{R_i} \alpha(i)$ and for some $j \in N$, $\beta(j) \succ_{R_j} \alpha(j)$.

The final conditions that we introduce are two versions of non-bossiness first introduced by Sonnenschein and Satterthwaite [11]. Some form of this assumption has been used in the vast majority of the papers applying the mechanism design approach to the allocation of private goods. An individual is "bossy" if she can alter the outcomes for others without affecting her own outcome. Again, with assignment rules being correspondences rather than functions some ambiguity may arise as to what an individual's "outcome" is. Bearing this in mind, we propose the following conservative (i.e. weak) versions of "non-bossiness":

Non-Bossiness (NB): An assignment rule $f : \mathcal{R}^N \twoheadrightarrow A$ is *Non-Bossy*, if for any R and R' such that $R' = (R_{-i}, R'_i)$ for some $i \in N$, the following is true: If $\alpha \in f(R)$ and for all $\beta \in f(R')$, $\alpha(i) = \beta(i)$, then $\alpha \in f(R')$.

Weak Non-Bossiness (WNB): An assignment rule $f : \mathcal{R}^N \twoheadrightarrow A$ is *Weakly Non-Bossy*, if for any R and R' such that $R' = (R_{-i}, R'_i)$ for some $i \in N$, the following is true: If

$\alpha \in f(R)$ and for all $\beta \in f(R')$, $\alpha(i) = \beta(i)$, then there exists $\beta \in f(R')$ such that β is R-utility equivalent to α.

Note that the above definitions are crafted in such a way that there is no ambiguity about the fact that between $f(R)$ and $f(R')$ individual i's "utility level" evaluated using R_i does not change and that neither does that of any other individual. If the utility of the deviating agent does not change, then (i) the first condition (NB) requires that the initially chosen allocation is also chosen at the new profile and (ii) the second condition (WNB) requires that the initially chosen allocation is utility equivalent to some allocation which is chosen after the deviation. Obviously, NB implies WNB. In the presence of PI the two conditions are equivalent. Any non-bossy condition which has been proposed in the mechanism design literature for single-valued rules would imply one of our weak versions of non-bossiness. [8]

4. Results

An assignment rule that has featured prominently in the literature on allocation rules (see Abdulkadiroğlu and Sönmez [1,2], Svensson [14]) for societies with *strict* individual preferences is based on a hierarchic structure or queue over the set of individuals. This ranking of individuals (in the queue) is used to determine the allocation of objects as follows: the first individual in the queue picks his best alternative and leaves with that alternative, from what remains the second individual in the queue picks, from what remains the third agent in the queue picks and so on. The first issue that needs to be addressed is what kind of mechanism on the full domain would be equivalent to the kind of "serial dictatorship" represented by a queuing rule? In the presence of indifference a straightforward application of a queue can lead to inefficiencies. Say, that the first person in the queue is indifferent between two of his best alternatives and leaves with one of the alternatives that happens to be person number two's only best alternative. The allocation that arises is clearly inefficient. The only paper (other than the present paper) that takes the approach of strategy-proof mechanism design *and* allows for full domain of individual preferences including the possibility of individuals being indifferent between distinct objects is Svensson [13]. He proposes a natural extension of the queuing rule observing that these rules are efficient, strategy-proof and "weakly" fair. However, he does not provide a full characterization of this class of rules in terms of these desirable properties. Are there other such rules with the same properties or properties that are even more desirable?

We present below and characterize a family of rules that we call Bi-polar Serially Dictatorial Rules. We start by describing the sub-class of these rules proposed in Svensson [13] (we call them Serially Dictatorial Rules), and then use this definition to describe the whole family of Bi-polar Serially Dictatorial Rules.

[8] Some definitions replace the requirement "$\alpha(i) = \beta(i)$" by "$\alpha(i) \sim_{R_i} \beta(i)$". This replacement would yield a stronger condition than NB (WNB) which is incompatible with ESV and EFF. To see this consider two agents and two objects, x_1 and x_2. Suppose that both agents strictly prefer x_1 to x_2. By ESV, the rule allocates x_1 to 1 and x_2 to 2 or x_1 to 2 and x_2 to 1. Let 1 receive x_1. If 1 is indifferent between both objects and 2 strictly prefers x_1 to x_2, then by EFF, the rule allocates x_2 to 1 and x_1 to 2. This is a violation of NB (WNB) when the requirement "$\alpha(i) = \beta(i)$" is replaced by "$\alpha(i) \sim_{R_i} \beta(i)$".

Serially Dictatorial Rule with respect to a ranking $i_1, i_2, i_3 \ldots, i_n$. An assignment rule $p : \mathcal{R}^N \twoheadrightarrow A$ is called the *Serially Dictatorial Rule* with respect to a given ranking $i_1, i_2, i_3 \ldots, i_n$ of agents (where $i_1, \ldots, i_n$ is a permutation of $1, \ldots, n$), if for any preference profile R we have that $p(R) = S_n$, where S_n is defined as follows. Let $S_0 = A$ (the set of all possible allocations),

$$
\begin{aligned}
S_1 &= \left\{\alpha \in S_0 : \alpha(i_1) \in \max_{\succsim_{i_1}} \{\beta(i_1) | \beta \in S_0\}\right\}\\
S_2 &= \left\{\alpha \in S_1 : \alpha(i_2) \in \max_{\succsim_{i_2}} \{\beta(i_2) | \beta \in S_1\}\right\}\\
&\vdots\\
S_k &= \left\{\alpha \in S_{k-1} : \alpha(i_k) \in \max_{\succsim_{i_k}} \{\beta(i_k) | \beta \in S_{k-1}\}\right\}\\
&\vdots\\
S_n &= \left\{\alpha \in S_{n-1} : \alpha(i_n) \in \max_{\succsim_{i_n}} \{\beta(i_n) | \beta \in S_{n-1}\}\right\}.
\end{aligned}
$$

A Bi-polar Serially Dictatorial Rule with respect to a ranking $\{i_1, i_2\}, i_3 \ldots, i_n$ of agents and a given partition $\{X_1, X_2\}$ of the set of alternatives ($X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = X$) coincides with the Serially Dictatorial Rule with respect to the ranking $i_1, i_2, i_3 \ldots, i_n$, unless $\max_{\succsim_{i_1}} \{X\} = \max_{\succsim_{i_2}} \{X\} = \{x\} \in X_2$. In this last case it coincides with the Serially Dictatorial Rule with respect to the ranking $i_2, i_1, i_3 \ldots, i_n$.

The above rule gives individuals a priority in choice of alternatives according to the specified ordering, with the exception that first two agents, i_1 and i_2, receive different priorities depending on their preferences: i_1 has the right of the first choice over X_1, and i_2 over X_2. We illustrate a Bi-polar Serially Dictatorial in the following example.

Example 1. Let $N = \{1, 2, 3, 4\}$ and $X = \{a, b, c, d\}$. Let p be the Bi-polar Serially Dictatorial Rule with respect to the ranking $\{1, 2\}, 3, 4$ and the partition $X_1 = \{b, c\}$ and $X_2 = \{a, d\}$. Let R be the profile such that

R_1	R_2	R_3	R_4
abc	a	a	a
d	bc	bd	cd
	d	c	b

Then (since $\max_{\succsim_1} \{X\} = \{a, b, c\} \neq \max_{\succsim_2} \{X\} = \{a\}$) our rule coincides with the Serially Dictatorial Rule with respect to the ranking 1, 2, 3, 4 and $S_1 = \{\alpha \in A : \alpha(1) \neq d\}$, $S_2 = \{(b, a, c, d), (b, a, d, c), (c, a, b, d), (c, a, d, b)\}$, $S_3 = \{(b, a, d, c), (c, a, b, d), (c, a, d, b)\}$, and $S_4 = \{(b, a, d, c), (c, a, b, d)\}$. Therefore, $p(R) = \{(b, a, d, c), (c, a, b, d)\}$.

If we would replace R_1 by R_4 in profile R, say $R' = (R_4, R_2, R_3, R_4)$, then (since now $\max_{\succsim_1} \{X\} = \max_{\succsim_2} \{X\} = \{a\} \in X_2$) our rule coincides with the Serially Dictatorial Rule with respect to the ranking 2, 1, 3, 4 and $S'_1 = \{\alpha \in A : \alpha(2) = a\}$, $S'_2 = \{(c, a, b, d), (c, a, d, b), (d, a, b, c), (d, a, c, b)\}$, $S'_3 = \{(c, a, b, d), (c, a, d, b),$

$(d, a, b, c)\}$, and $S_4' = \{(c, a, b, d), (d, a, b, c)\}$. Therefore, $p(R') = \{(c, a, b, d), (d, a, b, c)\}$.

If $X_2 = \emptyset$, then all individuals including i_1 and i_2 are ordered and we get Serially Dictatorial Rules, the class of mechanisms defined by Svensson [13]. For Serially Dictatorial Rules the individual ranked first gets one of her best alternatives, the second individual gets his best *subject to* leaving the first individual one of her best alternatives, the third individual gets one of her best choices available, given that 1st and 2nd individuals get their (constrained) "bests", etc. For the Bi-polar case, when $X_2 \neq \emptyset$, the rule is similar except that individual 1 gets her best alternative unless her unique top choice coincides with the unique top choice of the second individual and lies in X_2, in which case she gets one of her second bests. A similar condition attaches to assignment of the best alternative to individual 2, the co-leader of the hierarchy. The allocation process for individuals 3, ..., n is identical to that of the Serially Dictatorial Rule. Notice that these rules do not ensure that any individual gets any specific alternative but rather that they receive one from a utility equivalent group of alternatives. While the choices of individuals who are ranked at the bottom of the hierarchy may determine what alternative a higher ranking individual gets, this choice cannot affect the utility level of the higher ranking individual.

It is easy to see that S_n is a class of R-utility equivalent allocations and to verify that any Bi-polar Serially Dictatorial Rule satisfies ESV, PI, SP and NB. Our principal objective is not to find a rule that satisfies these conditions, we already know from Svensson [13] that such rules exist. However, results in the case of linear preference orders by Pápai [9] show a large and rich class of rules satisfying these and other desirable conditions, rules that do *not* belong to the family of Bi-polar Serially Dictatorial Rules. So the main question we need to resolve is whether (like in the case of linear preference orderings) there are other rules outside the class of Bi-polar Serially Dictatorial Rules that satisfy ESV, PI, SP and NB.

Theorem 1. *An assignment rule f satisfies Essential Single-Valuedness, Pareto Indifference, Strategy-Proofness and Non-Bossiness, if and only if it is a Bi-polar Serially Dictatorial Rule.*

The proofs of Theorem 1 and all other results are relegated to the appendix. The following examples establish the independence of the axioms in Theorem 1.

Example 2. *Pareto Indifferent, Weakly Strategy-Proof and Non-Bossy rule*: For all profiles choose the set of all strongly Pareto efficient allocations. [9]

Example 3. *Essentially Single-Valued, Strategy-Proof and Non-Bossy rule*: Fix a Bi-polar Serially Dictatorial Rule *f* and an ordering σ over the set of all allocations A. Let g be the selection from *f* choosing for each profile R the allocation in $f(R)$ which is ranked highest according to σ among the allocations in $f(R)$.

[9] Strictly speaking any rule which violates ESV also violates SSP. However, we could reformulate SSP in such a way such that there exist rules satisfying SSP but violating ESV. For example, we could add to the definition of SSP that i cannot manipulate at R via R' if $\{\alpha(i) \mid \alpha \in f(R)\} = \{\beta(i) \mid \beta \in f(R')\}$. Then the rule choosing all allocations for each profile satisfies PI, SSP and NB.

Example 4. *Essentially Single-Valued, Pareto Indifferent and Non-Bossy rule*: Fix an object $x_0 \in X$ and for each profile R define $S_0(R)$ to be the set of all allocations at which agent n receives an object which she weakly prefers to x_0 under her preference relation. Then select the set of utility equivalent allocations corresponding to the set S_n of allocations obtained by defining a sequence of sets $S_1, S_2, \dots, S_n$ where S_1 is the set of all allocations in which individual 1 receives the highest utility level in the set $S_0(R)$ and $S_2, \dots, S_n$ are obtained by applying the Serially Dictatorial Rule corresponding to the ordering $2, 3, 4, \dots, n$ of individuals. [10]

Example 5. *Essentially Single-Valued, Pareto Indifferent and Strategy-Proof rule*: Use the Serially Dictatorial Rule corresponding to the ordering $1, 2, 3, 4, \dots, n$ of agents, whenever the preferences of agent 1 are strict (no indifferences). Use the Serially Dictatorial Rule corresponding to the ordering $1, 3, 2, 4, \dots, n$, otherwise.

Remark 1. Bi-polar Serially Dictatorial Rules always select strongly Pareto efficient allocations. To see this, consider a Bi-polar Serially Dictatorial Rule with respect to a ranking $\{i_1, i_2\}, i_3, \dots, i_n$ of agents and an allocation $\alpha \in S_n$. Suppose that α can be Pareto improved and let i_t be the first agent in the ranking, who becomes strictly better off under this improvement. Note that agents i_1 and i_2 will get their best choices under α, unless they both have the same unique top object. In this last case, one of them will get her best alternative, and another one cannot be improved without hurting her. Thus, it must be $t \geqslant 3$. Since $i_1, \dots, i_{t-1}$ stay at the same "utility" levels, it means that agent i_t was not getting her maximal utility level given utility of agents before her, which contradicts the definition of a Bi-polar Serially Dictatorial Rule. Now, since Bi-polar Serially Dictatorial Rule always selects strongly Pareto efficient allocations it follows that in our context, Essential Single-Valuedness, Pareto Indifference, Strategy-Proofness and Non-Bossiness together imply strong Pareto efficiency. On the other hand, the rules discussed in Examples 2, 3, 4, 5 are strongly Pareto efficient. Thus, neither three of our properties together with Efficiency would imply the fourth one.

Remark 2. On the traditional domain used in the mechanism design literature (profiles of strict individual preferences and an allocation rule that is a function), Svensson [14] shows that the class of Serially Dictatorial Rules is characterized by Strategy-Proofness, Non-Bossiness and Neutrality (symmetry of the rule with respect to alternatives). Since the special domain of linear preferences together with the assumption that the allocation rule is a function rather than a correspondence already implies ESV, clearly, adding Neutrality to the hypothesis of our result extends Svensson's [14] result to the full domain of preferences. However, one has to be careful in interpreting this corollary: Our result does not imply

[10] To see why this rule can be manipulated consider a society of three individuals $\{1, 2, 3\}$ and three objects $\{x, y, z\}$ where allocations are restricted such that individual 3 receives an alternative at least as good as z and the serial dictatorship is given by 1, 2, 3 in that order. Consider the profile where 1 ranks z uniquely at the top, 2 ranks y uniquely at the top and 3 ranks y uniquely at the top with x and z indifferent. The allocation under the rule will assign z to 1, y to 2 and x to 3. Now if 3 expresses the preference which ranks y (strictly) over z and z (strictly) over x, the allocation rule will award z to 1, x to 2 and y to 3. Thus, 3 would have manipulated the rule at the initial profile. Clearly, this type of argument can be made for any $n \geqslant 3$.

Svensson's Theorem because a result which holds for a wide domain of preferences need not to hold if one confined oneself to a narrower domain. [11]

Observe that a selection from a Bi-Polar Serially Dictatorial Rule inherits Efficiency, SP and WNB (but not necessarily NB [12]). Therefore, a natural problem is to search for a characterization of Single-Valued rules. Any SV rule violates PI. Moreover, by Remark 1, Efficiency cannot be derived from the remaining properties in Theorem 1. The following theorem gives us a characterization of Single-Valued Rules satisfying EFF, SP and WNB.

Theorem 2. *An assignment rule f satisfies Single-Valuedness, Efficiency, Strategy-Proofness and Weak Non-Bossiness, if and only if it is a selection from a Bi-polar Serially Dictatorial Rule.*

Since NB implies WNB, the "only if-part" of Theorem 2 remains true if we replace WNB by NB. The following examples establish the independence of the axioms in Theorem 2. [13]

Example 6. *Efficient, Strategy-Proof and Weakly Non-Bossy rule*: Any Bi-polar Serially Dictatorial Rule.

Example 7. *Single-Valued, Strategy-Proof and Weakly Non-Bossy rule*: Any assignment rule which chooses the same allocation for all profiles.

Example 8. *Single-Valued, Efficient and Weakly Non-Bossy rule*: Any selection from the rule described in Example 4.

Example 9. *Single-Valued, Efficient and Strategy-Proof rule*: Any selection from the rule described in Example 5.

Remark 3. Group Strategy-Proofness requires that no group of agents can gain by misrepresentation. On the linear domain of preferences, Group Strategy-Proofness is equivalent to SP and WNB (or NB) for Single-Valued Rules (Pápai [9]). This equivalence breaks down on the full preference domain. As Ehlers [5] has shown, Efficiency and Group Strategy-Proofness are only compatible on the domain where indifferences are only allowed at the

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[11] Consider how impossibility results of voting theory can fail on the narrower domain of single-peaked preferences.

[12] To see this, consider three agents and three objects. Suppose that all agents are indifferent between all objects and let the selection f allocate x_i to agent i. Now suppose that x_1 becomes agent 1's most preferred object under her new preference. Let the selection f allocate x_1 to 1, x_3 to 2 and x_2 to 3. This is a violation of NB.

[13] In Theorem 2 it is not possible to replace EFF (which is "strong" efficiency) by "weak" efficiency. Any rule defined on the linear domain which satisfies SV, EFF, SP and WNB can be expanded to the full domain in a way such that SV, "weak" efficiency, SP and WNB are satisfied: (i) fix an exogenous order of the objects, say x_1, $x_2, \ldots, x_m$; (ii) associate with any profile $R \in \mathcal{R}^N$ a profile of linear orders by preserving each individual i's strict preference $\succ_{R_i}$ and breaking the ties of $\sim_{R_i}$ according to the order $x_1, x_2, \ldots, x_m$; and (iii) the expanded rule selects for each $R \in \mathcal{R}^N$ the allocation that the original rule (defined on the linear domain) chooses for the profile of linear orders associated with R.

bottom of a preference. No other indifferences can be permitted in the domain of a Single-Valued Rule if we insist on these two properties. Theorem 2 yields a characterization using SV, EFF, SP and WNB for rules defined on the full preference domain.

Remark 4. Since we do not use neutrality, our results are best viewed as the full domain analogue of Pápai's [9] linear domain characterization result. On the domain of strict orderings, dropping neutrality from Svensson's [14] axioms allows for a large and "messy" set of rules that do not belong to the Bi-polar Serially Dictatorial family. Pápai [9], using a linear domain of preferences and an allocation rule which like Svensson's is a function, successfully characterizes this important subset of the set of strategy-proof (possibly non-neutral) allocation rules. She uses Group Strategy-Proofness, an assumption that in her framework is equivalent to SP and WNB, together with EFF and Reallocation-Proofness to completely describe the family of Hierarchical Exchange Mechanisms. [14] Theorem 1 and 2 may be viewed as a solution to Pápai's problem with a full domain. [15] This has two implications. Firstly, Pápai's characterization tells us that our results do not hold if we restrict the domain of preferences to be strict. Secondly, our results show that unlike Svensson's [14] characterization, Pápai's characterization does not extend to the full domain. Allowing for the complete domain sharpens Pápai's type of characterization result by letting us *deduce* (instead of assume) in Theorem 1 Efficiency and in both Theorems 1 and 2 Reallocation-Proofness. It also allows us to whittle down the class of permissible rules to just Bi-polar Serially Dictatorial Rules and the selections from them.

Remark 5. Our characterizations also have an important implication for the house allocation problem with existing tenants analyzed by Abdulkadiroğlu and Sönmez [2]. In the domain of linear preferences, they show that by appropriately describing an "inheritance hierarchy" some members from the family of Hierarchical Exchange Mechanisms (thus rules that satisfy ESV, EFF, SP, NB and WNB) can be useful in solving the house allocation problem with existing tenants in a way that is efficient and respects the property rights of existing tenants in the sense of not making them worse off than at the *status quo*. In sharp contrast to Abdulkadiroğlu and Sönmez's [2] result, in our case with a full preference domain, since the Bi-polar Serially Dictatorial Rules do not respect "property rights," our characterization results imply that if there are more than two existing tenants, there are no ESV mechanisms satisfying EFF, SP and (Weak) NB which are capable of solving this assignment problem in a manner that leaves existing tenants no worse off than with their initial endowment.

Next, we will examine the relationship between the solutions to the assignment problem using (Bi-polar) Serially Dictatorial Rules and the ones using Edgeworthian exchange.

[14] Hierarchical Exchange Mechanisms are obtained through an iterative application of top trading cycle procedure, using hierarchical endowments. Each object is owned by an agent (some agents can own several objects, while others can own nothing), and for each object an inheritance rule specifies the order in which agents can inherit it. We start from applying top trading cycle for initial endowments. On the next step, agents who left the market with their top choices leave behind the objects they own, and any such object goes to the first agent in the inheritance order for it, who is still on the market. The top trading cycle procedure is then repeated, and so on.

[15] Our assumption of ESV is implied by the structure of her model.

Suppose we started with an arbitrary allocation of houses among the agents and allowed them to form groups to arrive at a negotiated solution. Any proposed solution that a (sub)group of individuals could improve upon using their own endowments of houses would be "blocked". The set of unblocked allocations, the core of the housing market, would be the only acceptable solution to the housing allocation problem. With individual indifference being possible, one needs to distinguish between the *weak core* (with the associated strongly blocking coalitions characterized by individuals all of whom are strictly better off) and the *strong core* (with the associated weakly blocking coalitions which may consist of some individuals who are strictly better off while others in the group are no worse off). While the weak core is non-empty for all possible initial allocations (see Shapley and Scarf [12]), elements in the weak core may be inefficient in that one may be able to reallocate the objects to improve the "utility" of some individuals without making any one worse off. While it is easy to check that the strong core may sometimes be empty, [16] allocations in the strong core will always be efficient if there are at least as many agents as objects and therefore this concept is normatively more compelling from the efficiency perspective. If there are more objects than agents, then a commonly most preferred object might not be the endowment of any agent, and thus would remain unallocated under the strong core. Then the strong core violates efficiency.

Given an allocation $\alpha \in A$ and $S \subseteq N$, let $\alpha(S) = \cup_{i \in S}\{\alpha(i)\}$ denote the set of objects allocated to the coalition S by α. An *endowment profile* is an allocation $\varepsilon \in A$. Given an endowment profile ε and a profile $R \in \mathcal{R}^N$, we have $Core(\varepsilon, R) = \{\alpha \in A \mid$ there is no $S \subseteq N$ and $\beta \in A$ such that (i) $\beta(S) = \varepsilon(S)$, (ii) for all $i \in S$, $\beta(i) \succsim_{R_i} \alpha(i)$, and (iii) for some $j \in S$, $\beta(j) \succ_{R_j} \alpha(j)\}$. Given a permutation $\sigma : N \to N$, let p^σ denote the Serially Dictatorial Rule based on the order σ. Let Σ denote the set of all permutations of N. For any preference profile R, let $\mathcal{A}_{Core}(R)$ stand for the set of all allocations which belong to the strong core for some endowment profile, i.e. $\mathcal{A}_{Core}(R) = \cup_{\varepsilon \in A} Core(\varepsilon, R)$; let $\mathcal{A}_{\text{Pareto}}(R)$ be the set of all strongly Pareto efficient allocations; and let $\mathcal{A}_{\text{Serially Dictatorial}}(R)$ be the set of all allocations which can be obtained using some Serially Dictatorial Rule, i.e. $\mathcal{A}_{\text{Serially Dictatorial}}(R) = \cup_{\sigma \in \Sigma} p^\sigma(R)$.

Our next theorem brings out the relationship between the strong core, strong Pareto efficiency *and* the Serially Dictatorial Rule.

Theorem 3. *Let* $|N| \geqslant |X|$. *Then for all* $R \in \mathcal{R}^N$,

(i) $\mathcal{A}_{Core}(R) \subseteq \mathcal{A}_{\text{Serially Dictatorial}}(R)$,
(ii) $\mathcal{A}_{\text{Serially Dictatorial}}(R) \subseteq \mathcal{A}_{\text{Pareto}}(R)$,
(iii) $\mathcal{A}_{\text{Pareto}}(R) \subseteq \mathcal{A}_{Core}(R)$.

[16] Consider the case in which $N = \{1, 2, 3\}$, $X = \{x_1, x_2, x_3\}$, the initial endowment ε is given by $\varepsilon(i) = x_i$ for $i = 1, 2, 3$ and the preferences are given by $x_1 \sim_1 x_2 \sim_1 x_3$ and $x_1 \succ_i x_2 \succ_i x_3$ for $i = 2, 3$. The coalition $\{1, 2\}$ would weakly block any allocation in which 2 does not get x_1 and $\{1, 3\}$ would weakly block any allocation in which 3 does not get x_1. Thus, the strong core will be empty. The initial allocation belongs to the weak core but it violates strong Pareto efficiency.

Our result shows that the three sets are equal on the full preference domain. Notice that even though the strong core can be empty for some endowment profiles, $\mathcal{A}_{Core}(R)$ is never empty.

Remark 6. Given that the set of Serially Dictatorial Rules is a subset of the set of Bi-polar Serially Dictatorial Rules, for any profile R the set $\mathcal{A}_{\text{Serially Dictatorial}}(R)$ is a subset of the set of allocations which can be obtained using some Bi-polar Serially Dictatorial Rule, $\mathcal{A}_{\text{Bi-Serially Dictatorial}}(R)$. But from Remark 1 we have that $\mathcal{A}_{\text{Bi-Serially Dictatorial}}(R) \subseteq \mathcal{A}_{\text{Pareto}}(R)$. Thus, Theorem 3 implies $\mathcal{A}_{\text{Bi-Serially Dictatorial}}(R) = \mathcal{A}_{\text{Serially Dictatorial}}(R)$ and hence (i) and (ii) in Theorem 3 remain true if we changed $\mathcal{A}_{\text{Serially Dictatorial}}(R)$ to $\mathcal{A}_{\text{Bi-Serially Dictatorial}}(R)$.

The equivalence of $\mathcal{A}_{\text{Pareto}}$ and $\mathcal{A}_{Core}$ is in the spirit of the two fundamental theorems of welfare economics. In the first fundamental theorem, if the perfectly competitive equilibrium exists it will be efficient, here, *if the strong core exists*, the core allocations will be efficient. However, there is no guarantee that starting with an arbitrary allocation, a strong core will exist. In the second theorem of welfare economics, efficient allocations may be viewed as having been generated as a competitive equilibrium for some initial endowment of resources, here, efficient allocations may be viewed as having been generated as a strong core for some initial endowment of resources. This shows that (Bi-polar) Serially Dictatorial Rules have some advantages over the strong core as far as attaining efficient allocations is concerned. Firstly, just like for the strong core, efficient allocations can be viewed as outcomes supported by Serially Dictatorial Rules. But, if one starts with a Serially Dictatorial Rule, it will *always* lead to an efficient outcome. Secondly, unlike the strong core where the inequity of the system inherent in the endowment allocation is implicit (after all it implies a power structure which is equivalent to that of some Serially Dictatorial Rule!), for a Serially Dictatorial Rule the inequity is explicit as is a possible solution to this equity problem: a lottery to determine the ranking of individuals. Observe that if one looks at the restricted domain of linear individual orderings, as shown by Abdulkadiroğlu and Sönmez [1], the random priority rule with equal probability attached to each ranking of individuals is *equivalent* to core allocations with equal probability attached to each endowment profile. One method has no advantage over the other. Here, the equivalence *breaks down* because the strong core may be empty for some endowment allocations and hence attaching equal probabilities to all possible allocation endowments will not be equivalent to attaching equal probabilities to all possible Serially Dictatorial Rules. Attaching equal probabilities to all endowment allocations would entail a positive probability of the strong core being empty with unpredictable and possibly inefficient outcomes. [17] What would happen if equal probabilities are attached to endowment allocations with non-empty strong cores? Given the informational assumptions underlying our model, preferences being private information, this is not an important question since it is not possible to say a priori which endowment allocations will result in non-empty strong cores without prior information about preferences.

[17] This would be true if with an empty strong core, the allocation ended up in the weak core (which is always non-empty).

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Appendix

In the appendix we prove Theorem 1, Theorem 2 and Theorem 3.

Appendix A. A general result

Theorems 1 and 2 are easily derived from the following general result.

We say that two Essentially Single-Valued assignment rules f and f', are *utility equivalent* if, for all $R \in \mathcal{R}^N$, any $\alpha \in f(R)$ and any $\alpha' \in f'(R)$ are R-utility equivalent.

Theorem 4. *An assignment rule f satisfies Essential Single-Valuedness, Efficiency, Strategy-Proofness and Weak Non-Bossiness, if and only if it is utility equivalent to a Bi-polar Serially Dictatorial Rule.*

First we prove Theorem 4 and then show how Theorems 1 and 2 are derived from Theorem 4. [18]

A.1. Proof of Theorem 4

It is easy to check that any assignment rule, which is utility equivalent to a Bi-polar Serially Dictatorial Rule, satisfies the axioms of Theorem 4. We assume now that a rule f satisfies the premises of Theorem 4, and show that it must be utility equivalent to a Bi-polar Serially Dictatorial Rule.

First we prove a monotonicity lemma.

Lemma 1. *Let $R \in \mathcal{R}^N$, $\alpha \in f(R)$ and $i \in N$ be such that $\alpha(i) \neq \emptyset$. If $R'_i \in \mathcal{R}$ is such that (i) for all $x \in X\backslash\{\alpha(i)\}$, $\alpha(i) \succsim_{R_i} x$ implies $\alpha(i) \succ_{R'_i} x$ and (ii) there is no $y \in X\backslash\{\alpha(i)\}$ such that $y \sim_{R'_i} \alpha(i)$, then for all $\beta \in f(R'_i, R_{-i})$, $\beta(i) = \alpha(i)$ and $\beta(j) \sim_{R_j} \alpha(j)$ for all $j \in N\backslash\{i\}$.*

Proof. By ESV, for all $\gamma \in f(R)$, $\gamma(i) \sim_{R_i} \alpha(i)$. Thus, by (i) for R'_i, for all $\gamma \in f(R)$, $\gamma(i) \neq \alpha(i)$ implies $\alpha(i) \succ_{R'_i} \gamma(i)$. Hence, by SP, for all $\beta \in f(R'_i, R_{-i})$, $\beta(i) \succsim_{R'_i} \alpha(i)$ (otherwise i *can* manipulate at (R'_i, R_{-i}) via R). If for some $\beta \in f(R'_i, R_{-i})$, $\beta(i) \succ_{R'_i} \alpha(i)$, then by (i) for R'_i, $\beta(i) \succ_{R_i} \alpha(i)$ and i *can* manipulate at R via (R'_i, R_{-i}). Thus, by

[18] The independence of the axioms in Theorem 4 follows from Examples 2, 7–9.

(ii) for R'_i, ESV and SP, for all $\beta \in f(R'_i, R_{-i})$, $\beta(i) = \alpha(i)$. Hence, by ESV and WNB, $\beta(j) \sim_{R_j} \alpha(j)$ for all $j \in N \backslash \{i\}$. $\square$

Next, we show that the set of agents who receive the empty object is the same for all profiles.

Let $N_\emptyset(R)$ denote the set of agents who receive the empty object under R, i.e. $i \in N_\emptyset(R) \Leftrightarrow$ for some $\alpha \in f(R)$, $\alpha(i) = \emptyset$. (By ESV, we could replace "for some" by "for all".) Let $\overline{N}_\emptyset(R) = N \backslash N_\emptyset(R)$. Let $R_i^x \in \mathcal{R}$ denote the preference where agent i ranks object x as the unique best object and is indifferent between all other objects, i.e. for all $y, z \in X \backslash \{x\}$, $x \succ_{R_i^x} y \sim_{R_i^x} z$.

Lemma 2. *For all* $R, \bar{R} \in \mathcal{R}^N$, $N_\emptyset(R) = N_\emptyset(\bar{R})$.

Proof. Let R^{I} be the profile of "full indifference", i.e. for all $i \in N$ and all $x, y \in X$, $x \sim_{R_i^{\mathrm{I}}} y$. Let $R \in \mathcal{R}^N$. It suffices to show that $N_\emptyset(R^{\mathrm{I}}) = N_\emptyset(R)$. We will create a profile $R' \in \mathcal{R}^N$ and show that (a) $N_\emptyset(R) = N_\emptyset(R')$ and (b) $N_\emptyset(R^{\mathrm{I}}) = N_\emptyset(R')$.

(a) Let $\alpha \in f(R)$, $i \in \overline{N}_\emptyset(R)$ and $R'_i = R_i^{\alpha(i)}$. Since $\alpha(i)$ is the unique best object for i under R'_i, Lemma 1 implies that for all $\beta \in f(R'_i, R_{-i})$, $\beta(i) = \alpha(i)$, and $N_\emptyset(R'_i, R_{-i}) = N_\emptyset(R)$.

Repeated application of the same argument yields a profile $(R'_{\overline{N}_\emptyset(R)}, R_{N_\emptyset(R)})$ [19] such that (i) $N_\emptyset(R'_{\overline{N}_\emptyset(R)}, R_{N_\emptyset(R)}) = N_\emptyset(R)$, (ii) $f(R'_{\overline{N}_\emptyset(R)}, R_{N_\emptyset(R)}) = \{\delta\}$ for some $\delta \in A$, and (iii) for all $i \in \overline{N}_\emptyset(R)$, $R'_i = R_i^{\delta(i)}$. Let $R' = (R'_{\overline{N}_\emptyset(R)}, R^{\mathrm{I}}_{N_\emptyset(R)})$. Applying ESV, SP and WNB successively we obtain $f(R') = f(R'_{\overline{N}_\emptyset(R)}, R_{N_\emptyset(R)}) = \{\delta\}$.

(b) Now, start from R^{I}. Consider the (possibly trivial) partition of $\overline{N}_\emptyset(R')$ into the sets $\overline{N}_\emptyset(R^{\mathrm{I}}) \cap \overline{N}_\emptyset(R') = L_0$ and $N_\emptyset(R^{\mathrm{I}}) \cap \overline{N}_\emptyset(R') = L_1$. By construction of R', changing preferences of agents $i \in \overline{N}_\emptyset(R')$ from R_i^{I} to R'_i will lead to the profile R'.

We will first generate the profile $(R'_{L_0}, R^{\mathrm{I}}_{N \backslash L_0})$ by changing the preferences of individuals in L_0 one at a time to their preference in R' and argue that $N_\emptyset(R'_{L_0}, R^{\mathrm{I}}_{N \backslash L_0}) = N_\emptyset(R^{\mathrm{I}})$.

Let $|L_0| = n'$ and $L_0 = \{1, 2, \ldots, n'\}$.[20] Consider the sequence $R^1, R^2, \ldots, R^{n'}$ of profiles generated by changing the preferences of the individuals in L_0 one at a time to their preference in R'. We will argue that at every step in the sequence the set of deprived individuals remains unchanged, i.e. $N_\emptyset(R'_{L_0}, R^{\mathrm{I}}_{N \backslash L_0}) = N_\emptyset(R^{\mathrm{I}})$.

Consider any step where $R^k = (R_1^{\delta(1)}, R_2^{\delta(2)}, \ldots, R_k^{\delta(k)}, R^{\mathrm{I}}_{N \backslash \{1,2,\ldots,k\}})$ and $R^{k+1} = (R_1^{\delta(1)}, R_2^{\delta(2)}, \ldots, R_k^{\delta(k)}, R_{k+1}^{\delta(k+1)}, R^{\mathrm{I}}_{N \backslash \{1,2,\ldots,k+1\}})$ such that in all previous steps the set of deprived individuals has not changed.

Let $\beta \in f(R^k)$. We know from $k+1 \in \overline{N}_\emptyset(R^{\mathrm{I}}) = \overline{N}_\emptyset(R^k)$, $\beta(k+1) \neq \emptyset$, and from EFF, that $\beta(k+1) \notin \{\delta(1), \delta(2), \ldots, \delta(k)\}$. Say, $\beta(k+1) = x$.

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[19] Given $R \in \mathcal{R}^N$ and $S \subseteq N$, R_S denotes the restriction of R to S.

[20] If $n' = 0$, then this step is complete.

First, notice that the step in which individual $(k+1)$'s preference is changing from R^{I}_k to R'_k (and thus altering the profile from R^k to R^{k+1}) can be factored into two parts: (i) A change from R^k to $\overline{R}^k = (R^x_{k+1}, R^k_{-(k+1)})$ and (ii) A change from $\overline{R}^k = (R^x_{k+1}, R^k_{-(k+1)})$ to $R^{k+1} = (R^{\delta(k+1)}_{k+1}, R^k_{-(k+1)})$. By SP and Lemma 1, we have $N_\emptyset(R^k) = N_\emptyset(\overline{R}^k)$ and for all $\mu \in f(\overline{R}^k)$, $\mu(k+1) = x$.

Suppose that $N_\emptyset(\overline{R}^k) \neq N_\emptyset(R^{k+1})$. By ESV and WNB, $\beta(k+1) = x \neq \delta(k+1)$. By SP, EFF and $\delta(k+1) \notin \{\delta(1), \ldots, \delta(k)\}$, we have for all $\alpha \in f(R^{k+1})$, $\alpha(k+1) = \delta(k+1)$. Let $R''_{k+1} \in \mathcal{R}$ be such that $\delta(k+1)$ and $\beta(k+1)$ are indifferent and all other objects are ranked below them, i.e. for all $y, z \in X \backslash \{\delta(k+1), \beta(k+1)\}$, $\delta(k+1) \sim_{R''_{k+1}} \beta(k+1) \succ_{R''_{k+1}} z \sim_{R''_{k+1}} y$. Let $\gamma \in f(R''_{k+1}, R^k_{-(k+1)})$. By $k+1 \notin N_\emptyset(\overline{R}^k)$, SP and EFF, we have $\gamma(k+1) \in \{\delta(k+1), \beta(k+1)\}$.

Case 1: $\gamma(k+1) = \delta(k+1)$.

Since for all $\alpha \in f(R^{k+1})$, $\alpha(k+1) = \delta(k+1)$, ESV and WNB imply $N_\emptyset(R''_{k+1}, R^k_{-(k+1)}) = N_\emptyset(R^{k+1})$. Let $j \in \overline{N}_\emptyset(R^{k+1}) \cap N_\emptyset(\overline{R}^k)$ and $R'' = (R^{\delta(k+1)}_j, R''_{k+1}, R^k_{-j,(k+1)})$. Since $\beta(k+1), \delta(k+1) \notin \{\delta(1), \ldots, \delta(k)\}$ and for all agents except j the objects $\beta(k+1)$ and $\delta(k+1)$ belong to the lowest indifference class of their preference, SP, EFF and $j \in \overline{N}_\emptyset(R^{k+1})$ imply for all $\mu \in f(R'')$, $\mu(j) = \delta(k+1)$. On the other hand, since $j \in N_\emptyset(\overline{R}^k)$, ESV, SP and WNB imply $N_\emptyset(R^{\delta(k+1)}_j, \overline{R}^k_{-j}) = N_\emptyset(\overline{R}^k)$. Thus, $k+1 \notin N_\emptyset(R^{\delta(k+1)}_j, \overline{R}^k_{-j})$ and by EFF, for all $v \in f(R^{\delta(k+1)}_j, \overline{R}^k_{-j})$, $v(k+1) = \beta(k+1)$. By SP and EFF, for all $\mu \in f(R'')$, $\mu(k+1) = \beta(k+1)$. Hence, by ESV and WNB,

$$N_\emptyset(R^{\delta(k+1)}_j, \overline{R}^k_{-j}) = N_\emptyset(R'').$$

Thus, by $N_\emptyset(\overline{R}^k) = N_\emptyset(R^{\delta(k+1)}_j, \overline{R}^k_{-j})$ and $j \notin N_\emptyset(R'')$, we obtain $j \notin N_\emptyset(\overline{R}^k)$, a contradiction.

Case 2: $\gamma(k+1) = \beta(k+1)$.

Since for all $\alpha \in f(\overline{R}^k)$, $\alpha(k+1) = \beta(k+1)$, ESV and WNB imply $N_\emptyset(R''_{k+1}, R^k_{-(k+1)}) = N_\emptyset(\overline{R}^k)$. Let $j \in \overline{N}_\emptyset(\overline{R}^k) \cap N_\emptyset(R^{k+1})$ and $R'' = (R^{\beta(k+1)}_j, R''_{k+1}, R^k_{-j,(k+1)})$. Using the same arguments as in Case 1 we obtain $j \notin N_\emptyset(R^{k+1})$, a contradiction.

Thus, we must have $N_\emptyset(R^{\mathrm{I}}) = N_\emptyset(R'_{L_0}, R^{\mathrm{I}}_{N\backslash L_0})$. Now, convert the profile $(R'_{L_0}, R^{\mathrm{I}}_{N\backslash L_0})$ to $R' = (R'_{L_0 \cup L_1}, R^{\mathrm{I}}_{N\backslash(L_0\cup L_1)})$ by changing one at a time the preferences of individuals in $L_1 \subseteq N_\emptyset(R^{\mathrm{I}}) = N_\emptyset(R'_{L_0}, R^{\mathrm{I}}_{N\backslash L_0})$. At each step, by ESV, SP and WNB these individuals will be assigned the empty alternative and the set of individuals being assigned the null alternative will remain unchanged. Thus, we get $N_\emptyset(R^{\mathrm{I}}) = N_\emptyset(R')$. $\square$

Lemma 2 tells us that under *f* the set of agents who do not receive an object is the same for all profiles. Fix preferences of these agents, and consider a restriction of *f* to the remaining individuals. Assume that any such restriction of *f* is utility equivalent to (some) Bi-polar Serially Dictatorial Rule. By ESV and WNB, $f(R)$ is utility equivalent to $f(R')$, whenever R' only differs from *R* by preferences of agents who never get an object. Hence, the restrictions of *f* to different preferences of agents in $N_\emptyset(R)$ are utility equivalent. It follows that all restrictions of *f*, and so the rule *f* itself, are utility equivalent to the same Bi-polar Serially Dictatorial Rule.

Hence, we can reduce our attention to the case where for all $R \in \mathcal{R}^N$, $N_\emptyset(R) = \emptyset$, $N \neq \emptyset$ and $|N| \leqslant |X|$. Thus, from now on we have for all $i \in N$, all $R \in \mathcal{R}^N$, and all $\alpha \in f(R)$, $\alpha(i) \neq \emptyset$.

Let $x \in X$ and let $R^x \in \mathcal{R}^N$ be the profile where everybody ranks x as the unique best object and is indifferent between all other objects (each agent i reports R_i^x). Let $\alpha \in f(R^x)$. By EFF, for some $j \in N$, $\alpha(j) = x$. Without loss of generality, let $j = 1$. We show that 1 has a "first priority" with respect to x, i.e. at each profile she (weakly) prefers her allotments to x.

Lemma 3. *For all $R \in \mathcal{R}^N$ and all $\alpha \in f(R)$, $\alpha(1) \succsim_{R_1} x$.*

Proof. Assume to the contrary that for some $\alpha \in f(R)$, $x \succ_{R_1} \alpha(1)$. Let $\gamma \in f(R_1^x, R_{-1})$. By SP, $\gamma(1) \neq x$. By EFF, for some $i \in N$, $\gamma(i) = x$. Without loss of generality, let $i = n$. By Lemma 1, for all $\beta \in f(R_1^x, R_n^x, R_{-1,n})$, $\beta(n) = x$. Using Lemma 1 repeatedly we obtain a preference profile $\tilde{R}$ such that for some $\delta \in A$ we have $\tilde{R} = (R_1^x, R_2^{\delta(2)}, R_3^{\delta(3)}, \ldots, R_n^{\delta(n)})$, $\delta \in f(\tilde{R})$ and $\delta(n) = x$.

Consider a sequence of n profiles $R^j = (\tilde{R}_{\{1,2,\ldots,j\}}, R^x_{N\setminus\{1,2,\ldots,j\}})$, $j = 1, 2, \ldots, n$. Clearly, $R^1 = R^x$ and (by the choice of individual n) $R^{n-1} = R^n = \tilde{R}$. By the way in which individual 1 has been defined, for all $\beta^1 \in f(R^1)$, $\beta^1(1) = x$.

Let $k \in \{2, \ldots, n-2\}$ and consider the change from R^k to R^{k+1}. Suppose that up to R^k individual 1 received x but at R^{k+1} she does not, i.e. for $\beta^k \in f(R^k)$, $\beta^k(1) = x$, and for $\beta^{k+1} \in f(R^{k+1})$, $\beta^{k+1}(1) \neq x$. By SP, EFF and $R_{k+1}^{k+1} = R_{k+1}^{\delta(k+1)}$, we have $\beta^{k+1}(k+1) = \delta(k+1)$.

Again notice that the change from R^k to R^{k+1} can be factored into two parts: (i) A change from R^k to $\overline{R}^k$, where $\overline{R}^k = (R_{k+1}^{\beta^k(k+1)}, R^k_{-(k+1)})$ and (ii) A change from $\overline{R}^k = (R_{k+1}^{\beta^k(k+1)}, R^k_{-(k+1)})$ to $R^{k+1} = (R_{k+1}^{\delta(k+1)}, R^k_{-(k+1)})$. By Lemma 1, in (i) we have for all $\mu \in f(\overline{R}^k)$, $\mu(k+1) = \beta^k(k+1)$ and $\mu(1) = x$.

Thus, by $\beta^{k+1}(1) \neq x$, there is a change in (ii). By WNB, $\beta^k(k+1) \neq \delta(k+1)$. Let $i \in N$ be such that $\beta^{k+1}(i) = x$. By EFF, $i \in \{k+2, k+3, \ldots, n\}$ and $\overline{R}_i^k = R_i^{k+1} = R_i^x$. Let $R'_{k+1} \in \mathcal{R}$ be such that $\delta(k+1)$ and $\beta^k(k+1)$ are indifferent and all other objects are ranked below, i.e. for all $y, z \in X\setminus\{\delta(k+1), x\}$, $\delta(k+1) \sim_{R'_{k+1}} \beta^k(k+1) \succ_{R'_{k+1}} z \sim_{R'_{k+1}} y$. Let $\gamma \in f(R'_{k+1}, R^k_{-(k+1)})$. By SP and EFF, $\gamma(k+1) \in \{\delta(k+1), \beta^k(k+1)\}$.

Case 1: $\gamma(k+1) = \delta(k+1)$.

Then by ESV and WNB, $\gamma(1) \neq x$ and $\gamma(i) = x$. Let $R'_1 \in \mathcal{R}$ be such that x is ranked first, $\delta(k+1)$ second and all other objects indifferent at the bottom, i.e. for all $y, z \in X\setminus\{\delta(k+1), x\}$, $x \succ_{R'_1} \delta(k+1) \succ_{R'_1} y \sim_{R'_1} z$. By SP, EFF, $\gamma(1) \neq x$ and $\delta(k+1) \neq x$, we have for all $\nu \in f(R'_1, R'_{k+1}, R^k_{-1,(k+1)})$, $\nu(1) = \delta(k+1)$. By EFF, for all $\nu \in f(R'_1, R'_{k+1}, R^k_{-1,(k+1)})$, $\nu(k+1) = \beta^k(k+1)$. On the other hand, by Lemma 1, for all $\alpha \in f(R'_1, \overline{R}^k_{-1})$, $\alpha(1) = x$ and $\alpha(k+1) = \beta^k(k+1)$. This contradicts WNB for agent 1 for the change from $(R'_1, \overline{R}^k_{-1})$ to $(R'_1, R'_{k+1}, R^k_{-1,(k+1)})$.

Case 2: $\gamma(k+1) = \beta^k(k+1)$.

Then by ESV and WNB, $\gamma(1) = x$ and $\gamma(i) \neq x$. Let $R'_i \in \mathcal{R}$ be such that x is ranked first, $\beta^k(k+1)$ second and all other objects indifferent at the bottom, i.e. for all $y, z \in X \setminus \{\beta^k(k+1), x\}$, $x \succ_{R'_i} \beta^k(k+1) \succ_{R'_i} y \sim_{R'_i} z$. By SP, EFF, $\gamma(i) \neq x$ and $\beta^k(k+1) \neq x$, we have for all $v \in f(R'_i, R'_{k+1}, R^k_{-i,(k+1)})$, $v(i) = \beta^k(k+1)$. By EFF, for all $v \in f(R'_i, R'_{k+1}, R^k_{-i,(k+1)})$, $v(k+1) = \delta(k+1)$. On the other hand, by Lemma 1, for all $\alpha \in f(R'_i, R^{k+1}_{-i})$, $\alpha(i) = x$ and $\alpha(k+1) = \delta(k+1)$. This contradicts WNB for agent i for the change from (R'_i, R^{k+1}_{-i}) to $(R'_i, R'_{k+1}, R^k_{-i,(k+1)})$.

Since $R^{n-1} = \tilde{R}$ we obtain that for all $\beta \in f(\tilde{R})$, $\beta(1) = x$. This contradicts ESV since $\delta \in f(\tilde{R})$ and $\delta(n) = x$. $\square$

We now define a hierarchy of individuals with respect to every object x by using the preferences in the profiles R^{I} (in which individuals are indifferent between all alternatives) and R^x (where x is the unique best and all other alternatives are indifferent to each other). The person who is assigned x under R^x gets the highest priority; if i has the highest priority with respect to x, the person who gets x under the profile $(R^{\mathrm{I}}_i, R^x_{N\setminus\{i\}})$ has the second highest priority and so on. Thus, for each object x we now inductively define an ordering $\sigma^x : N \to N$ as follows. Let $R^{\mathrm{I}} \in \mathcal{R}^N$ be such that for all $i \in N$ and all $y, z \in X$, $y \sim_{R^I_i} z$. Then $\sigma^x(1) = j \Leftrightarrow$ for all $\alpha \in f(R^x)$, $\alpha(j) = x$. For all $k < n$,

$$\sigma^x(k+1) = j \Leftrightarrow \text{ for all } \alpha \in f(R^{\mathrm{I}}_{\{\sigma^x(1),\ldots,\sigma^x(k)\}}, R^x_{N\setminus\{\sigma^x(1),\ldots,\sigma^x(k)\}}),\ \alpha(j) = x.$$

Note that by ESV and EFF of f, σ^x is well-defined.

Following the proof of Lemma 3 (and maintaining "universal" indifference for all the individuals in $\{\sigma^x(1), \ldots, \sigma^x(k)\}$ in all the profiles used in the proof) the following can easily be established:

Lemma 4. *For all $R \in \mathcal{R}^N$ such that $R_i = R^{\mathrm{I}}_i$ for all $i \in \{\sigma^x(1), \ldots, \sigma^x(k)\}$ and all $\alpha \in f(R)$, $\alpha(\sigma^x(k+1)) \succsim_{R_{\sigma^x(k+1)}} x$.*

Next, we use Lemma 4 to extend Lemma 3 in a slightly different direction. We show that for all profiles, if none of the agents belonging to $\{\sigma^x(1), \ldots, \sigma^x(k)\}$ *receives* x at some allocation chosen by the rule f, then agent $\sigma^x(k+1)$ weakly prefers all her allotments to x.

Lemma 5. *For all $R \in \mathcal{R}^N$, if for some $\alpha \in f(R)$, we have for all $i \in \{\sigma^x(1), \ldots, \sigma^x(k)\}$, $\alpha(i) \neq x$, then for all $\beta \in f(R)$, $\beta(\sigma^x(k+1)) \succsim_{R_{\sigma^x(k+1)}} x$.*

Proof. Without loss of generality, suppose that for all $l \in \{1, \ldots, k+1\}$, $\sigma^x(l) = l$. By ESV, it suffices to show that $\alpha(k+1) \succsim_{R_{k+1}} x$.

Assume to the contrary that $x \notin \{\alpha(1), \alpha(2), \ldots, \alpha(k)\}$ and $x \succ_{R_{k+1}} \alpha(k+1)$. Thus, by EFF, for some $i \in \{k+2, k+3, \ldots, n\}$, $\alpha(i) = x$. Without loss of generality, let $i = n$. By Lemma 1, for all $\beta \in f(R_n^x, R_{-n})$, $\beta(n) = x$ and $x \succ_{R_{k+1}} \beta(k+1)$. Using Lemma 1 repeatedly we obtain a preference profile $\tilde{R}'$ such that for some $\delta' \in A$ we have $\tilde{R}' = (R_1^{\delta'(1)}, R_2^{\delta'(2)}, \ldots, R_k^{\delta'(k)}, R_{k+1}, R_{k+2}^{\delta'(k+2)}, \ldots, R_n^{\delta'(n)})$, $\delta' \in f(\tilde{R}')$, $x \succ_{R_{k+1}} \delta'(k+1)$ and $\delta'(n) = x$. Let $\tilde{R} = (R_{k+1}^x, \tilde{R}'_{-(k+1)})$ and $\delta \in f(\tilde{R})$. By SP, $\delta(k+1) \neq x$. Thus, by EFF, for all $i \in N\backslash\{k+1\}$, $\delta(i) = \delta'(i)$ and $R_i^{\delta(i)} = R_i^{\delta'(i)}$ (in particular $\delta(n) = x$).

Let $R^0 = (R^I_{\{1,\ldots,k\}}, \tilde{R}_{N\backslash\{1,\ldots,k\}})$. By Lemma 4, for all $\beta^0 \in f(R^0)$, $\beta^0(k+1) = x$. Consider the sequence of k profiles $R^j = (\tilde{R}_{\{1,2,\ldots,j\}}, R^{\mathrm{I}}_{\{j+1,\ldots,k\}}, \tilde{R}_{N\backslash\{1,2,\ldots,k\}})$, $j = 1, 2, \ldots, k$.

Let $l \in \{0, 1, \ldots, k-1\}$ and consider the change from R^l to R^{l+1}. Suppose that up to R^l individual $k+1$ received x but at R^{l+1} she does not, i.e. for $\beta^l \in f(R^l)$, $\beta^l(k+1) = x$, and for $\beta^{l+1} \in f(R^{l+1})$, $\beta^{l+1}(k+1) \neq x$. By SP, EFF and $R_{l+1}^{l+1} = R_{l+1}^{\delta(l+1)}$, we have $\beta^{l+1}(l+1) = \delta(l+1)$.

Again the change from R^l to R^{l+1} can be factored into two parts: (i) A change from R^l to $\overline{R}^l$, where $\overline{R}^l = (R_{l+1}^{\beta^l(l+1)}, R^l_{-(l+1)})$ and (ii) A change from $\overline{R}^l = (R_{l+1}^{\beta^l(l+1)}, R^l_{-(l+1)})$ to $R^{l+1} = (R_{l+1}^{\delta(l+1)}, R^l_{-(l+1)})$. By Lemma 1, in (i) we have for all $\mu \in f(\overline{R}^l)$, $\mu(l+1) = \beta^l(l+1)$ and $\mu(k+1) = x$.

Thus, by $\beta^{l+1}(k+1) \neq x$, there is a change in (ii). Since all agents except $k+1$ and n rank x at the bottom of their preference under R^{l+1}, EFF implies $\beta^{l+1}(n) = x$. Now the same argument as in the proof of Lemma 3 establishes a contradiction. □

Next we show that if an agent has first priority with respect to some object, then this agent has either a first or a second priority with respect to any other object. In particular, this immediately implies that at most two agents have "first priority" on some object (i.e. $|\{\sigma^x(1) \mid x \in X\}| \leqslant 2$).

Lemma 6. *For all* $x_1, x_2 \in X$, $\sigma^{x_1}(1) \neq \sigma^{x_2}(1)$ *implies* $\sigma^{x_1}(2) = \sigma^{x_2}(1)$.

Proof. Without loss of generality, assume to the contrary that $\sigma^{x_1}(1) = 1$, $\sigma^{x_2}(1) = 2$ and $\sigma^{x_1}(2) = 3$. Let $R \in \mathcal{R}^N$ be such that for all $i \in N\backslash\{1, 2, 3\}$, $R_i = R_i^{\mathrm{I}}$, while $R_1, R_2 \in \mathcal{R}$ are such that for all $y \in X\backslash\{x_1, x_2\}$, $x_2 \succ_{R_1} x_1 \succ_{R_1} y$ and $x_1 \succ_{R_2} x_2 \succ_{R_2} y$, and $R_3 = R_3^{x_1}$. Let $\alpha \in f(R)$. By Lemma 3 and EFF, $\alpha(1) = x_2$ and $\alpha(2) = x_1$. But $\alpha(1) = x_2 \neq x_1$ and $\sigma^{x_1}(2) = 3$ together with Lemma 5 imply $\alpha(3) \succsim_{R_3} x_1$. But, since x_1 is individual 3's unique best element, this contradicts $\alpha(2) = x_1$. □

It follows from Lemma 6 that there are two mutually exclusive possibilities: either there is a unique individual who has the first "priority" for all objects ($|\{\sigma^x(1) \mid x \in X\}| = 1$), or there are *exactly* two individuals who always occupy first and second place in the "priority" for any object ($|\{\sigma^x(1) \mid x \in X\}| = 2$ and $\{\sigma^x(1) \mid x \in X\} = \{\sigma^x(2) \mid x \in X\}$). In the next lemma we show that in the first case a unique individual occupies each priority position, the individual being the same for every alternative and that in the second case that a unique individual occupies each priority position from the third priority onwards (the individual being the same for every alternative). This together with Lemma 5 and EFF will complete the proof of Theorem 4 by establishing that the assignment rule *f* is utility equivalent to a Bi-polar Serially Dictatorial Rule.

Lemma 7. *Let* $k_0 = |\{\sigma^x(1) \mid x \in X\}|$. *For all* $k \in \{k_0+1, \ldots, n\}$, $|\{\sigma^x(k) \mid x \in X\}| = 1$.

Proof. Suppose that Lemma 7 is not true. Let $k \in \{k_0+1, \ldots, n\}$ be the minimal one ("the highest priority"), such that $|\{\sigma^x(k) \mid x \in X\}| \geqslant 2$. Without loss of generality, let $y, z \in X$, $\sigma^y(k) = 2$, and $\sigma^z(k) = 3$. Because of our choice of k, we have for all $k' \in \{k_0+1, \ldots, k-1\}$, $|\{\sigma^x(k') \mid x \in X\}| = 1$. Lemma 6 implies that for all $k' \in \{1, \ldots, k-1\}$, $2, 3 \notin \{\sigma^x(k') \mid x \in X\}$.[21] Let $1 \in \{\sigma^x(1) \mid x \in X\}$. Lemma 6 tells us that 1 should have first or second "priority" for any object, and, given that $k > k_0$, it follows $(\sigma^y)^{-1}(1) < k$ and $(\sigma^z)^{-1}(1) < k$.

Let $R \in \mathcal{R}^N$ be such that for all $i \in N \backslash \{1, 2, 3\}$, $R_i = R_i^{\mathrm{I}}$, and for all $x \in X \backslash \{y, z\}$,

$$y \sim_{R_1} z \succ_{R_1} x,\ z \succ_{R_2} y \succ_{R_2} x \text{ and } y \succ_{R_3} z \succ_{R_3} x.$$

Let $\alpha \in f(R)$. Since all agents, except 1, 2 and 3, are completely indifferent, EFF implies $y, z \in \{\alpha(1), \alpha(2), \alpha(3)\}$. Again, from Lemma 5, and since $(\sigma^y)^{-1}(1) < k$ and $(\sigma^z)^{-1}(1) < k$, we get that $\alpha(1) \in \{y, z\}$.

If $\alpha(1) = y$, then from Lemma 5, $\sigma^z(k) = 3$, and $2 \notin \{\sigma^z(k') \mid k' \in \{1, \ldots, k\}\}$, we have that $\alpha(3) = z$. Then an exchange of individual 1's and 3's objects results in a Pareto improvement, contradicting EFF. Similarly, if $\alpha(1) = z$, then $\alpha(2) = y$, which too would violate EFF, a contradiction. □

A.2. Proof of Theorem 1

It is easy to check that Bi-polar Serially Dictatorial Rules satisfy the axioms of Theorem 1. We show that any rule *f* satisfying ESV, PI, SP, and NB, must belong to the family of Bi-polar Serially Dictatorial Rules.

We check first that ESV, PI, SP and NB imply Efficiency.

Lemma 8. *f satisfies EFF.*

Proof. Let $R \in \mathcal{R}^N$ and $\alpha \in f(R)$. Suppose that α is inefficient for R. Thus, some $\beta \in A$ Pareto dominates α. In particular, $\beta \neq \alpha$. Let T denote the set of agents $i \in N$ such that $\beta(i) \succ_{R_i} \alpha(i)$. Note that $T \neq \emptyset$, for all $i \in T$, $\alpha(i) \neq \emptyset$, and for all $i \in N \backslash T$, $\alpha(i) \sim_{R_i} \beta(i)$.

Let $i \in T$ and $R_i' \in \mathcal{R}$ be such that for all $x \in X \backslash \{\beta(i), \alpha(i)\}$, $\alpha(i) \sim_{R_i'} \beta(i) \succ_{R_i'} x$. Let $R' = (R_i', R_{-i})$. By SP, for all $\gamma \in f(R')$, $\gamma(i) = \alpha(i)$, and by NB, $\alpha \in f(R')$. If $T = \{i\}$, then for all $j \in N$, $\alpha(j) \sim_{R_j'} \beta(j)$. Thus, by PI, $\beta \in f(R')$, which contradicts SP (*i can*

[21] Recall the two possible cases discussed immediately above the lemma. In the first case, for all k' smaller than k, $|\{\sigma^x(k') \mid x \in X\}| = 1$. In the second case, for all k' smaller than k but greater than 2, $|\{\sigma^x(k') \mid x \in X\}| = 1$ and $|\{\sigma^x(1) \mid x \in X\}| = 2$ and $\{\sigma^x(1) \mid x \in X\} = \{\sigma^x(2) \mid x \in X\}$ Thus, in either case neither 2 nor 3 could have had a higher priority for any object.

manipulate at R via R'). Thus, $T \neq \{i\}$. We then sequentially replace the preferences of the agents in $T\backslash\{i\}$ and apply the same arguments as above to deduce a contradiction. $\square$

By Lemma 8, *f* satisfies ESV, EFF, SP and NB. Furthermore, NB implies WNB. Thus, by Theorem 4, *f* is utility equivalent to a Bi-polar Serially Dictatorial Rule. From PI we obtain that *f* chooses for each profile the same set of allocations as the Bi-polar Serially Dictatorial Rule. Hence, *f* is a Bi-Polar Serially Dictatorial Rule, the desired conclusion of Theorem 1.

A.3. Proof of Theorem 2

It is easy to check that any selection from a Bi-polar Serially Dictatorial Rule satisfies SV, EFF, SP and WNB. In proving the converse, let *f* be an assignment rule satisfying these properties. Then *f* satisfies ESV. Thus, by Theorem 4, *f* is utility equivalent to a Bi-polar Serially Dictatorial Rule. Because any Bi-polar Serially Dictatorial Rule satisfies PI, it chooses for each profile the set of all allocations which are utility equivalent to each other. Hence, for each profile the allocation chosen by *f* must belong to this set and *f* must be a selection from the Bi-polar Serially Dictatorial Rule, the desired conclusion of Theorem 2.

Appedix B. Proof of Theorem 3

(i) Let α be any arbitrary allocation in the strong core. We need to find a ranking $I = i_1, \ldots, i_n$ of agents, such that α is an allocation chosen by the Serially Dictatorial Rule corresponding to this ranking of agents. We thus need to show that the object $\alpha(i_1)$ is one of the best objects for agent i_1. The object $\alpha(i_2)$ provides agent i_2 with the highest possible utility, given that agent i_1 receives an object at least as good as $\alpha(i_1)$, etc. For any *t*, the object $\alpha(i_t)$ must give agent i_t the highest possible utility, given that each of agents i_j, $1 \leq j < t$ receives an object at least as good as $\alpha(i_j)$.

Our construction is in the spirit of Gales "Top Trading Cycle Algorithm". We present below an algorithm which allows us to find an agent i_t, given that agents $i_1, \ldots, i_{t-1}$ are already chosen and $t - 1 < m$.[22,23]

Construct the following oriented graph. Let it have $n + m$ vertices, corresponding to n agents and m objects. Let each object point to the agent to whom it is allocated under α. Let $I_t = \{i_1, \ldots, i_{t-1}\}$. For all $i \in I_t$ let agent i point to all objects, utility equivalent to $\alpha(i)$. For all $j \notin I_t$, let agent j point to all objects, which maximize her satisfaction subject to the constraint that the agents in I_t each receive an object that is utility equivalent to that in allocation α.

It is important to note that *all* agents point only to the objects which are at least as good for them as objects they receive under allocation α. Indeed, the agents from I_t will point to the objects utility equivalent to those they get under α, while agents not from I_t will only point to best objects they can get, given that agents from I_t should receive something at least as good as what they get at α.

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[22] This algorithm also works when $t = 1$.

[23] If $t - 1 \geqslant m$, the rest of the ranking is irrelevant.

Now, start from an arbitrary agent $j_1 \notin I_t$ and construct a path $j_1, x_{j_1}, j_2, x_{j_2}, \ldots$, where each agent is followed by an object and each object is followed by an agent along the arrows in our graph, in the following way:

- For any object $x_{j_k} \neq \emptyset$, an agent j_{k+1} which follows it is the unique individual satisfying $\alpha(j_{k+1}) = x_{j_k}$.
- For any agent j_k, such that $j_k \notin I_t$ the object x_{j_k} which follows j_k is any arbitrary one of the objects to which this agent points.
- For the agent j_{k^*} such that and $j_{k^*} \in I_t$ and for all $l < k+1$, $j_l \notin I_t$,[24] choose any Serially Dictatorial Rule s based on a ranking of individual starting with $i_1, \ldots, i_{t-1}$ and choose $\sigma \in s$ such that $\sigma(j_{k^*-1}) = x_{j_{k^*-1}}$ and let the object following j_{k^*} be given by $\sigma(j_{k^*})$.
- For all $q \geq k^*$, for any agent $j_q \in I_t$, let next element in the sequence be given by $\sigma(j_q)$ where the allocation σ is the one selected in the last step.

Since the number of vertices of our graph is finite and because there is an arrow going out from each agent and each object in our path, there must be a vertex that is repeated in this path. Moreover, note that different objects *always* point at different agents (depending on the allocation α). Hence, the first repetition will be some *object*[25] $x_{j_p} = x_{j_q}$, $q < p$. We thus obtain the cycle $x_{j_q}, j_{q+1}, x_{j_{q+1}}, \ldots, j_p$ where all vertices are different, each one points to the following one, and the last one, j_p, points to the first one, x_{j_q}.

We will argue that there must be an agent j_k in this cycle, $q+1 \leq k \leq p$, such that $j_t \notin I_t$. Indeed, suppose that $j_{q+1}, \ldots, j_p \in I_t$. Then, by the construction of our path, there exist an agent j_r, $r \leq q$, such that $j_r \notin I_t$, $j_{r+1}, \ldots, j_p \in I_t$, and for some $\sigma \in s$, $\sigma(j_r) = x_{j_r}$ and we have that $x_{j_{r+1}} = \sigma(j_{r+1}), \ldots, x_{j_q} = \sigma(j_q), x_{j_{q+1}} = \sigma(j_{q+1}), \ldots, x_{j_p} = \sigma(j_p)$. Since $j_q \neq j_p$, it follows that $x_{j_q} = \sigma(j_q) \neq x_{j_p} = \sigma(j_p)$, which is a contradiction to our choice of x_{j_p} and x_{j_q}.

Now, consider the allocation where each agent j_l from our cycle is allocated x_{j_l} and agents j not from the cycle get $\alpha(j)$. Since an agent only points to an object if it is at least as good as one she gets at α, it follows that at this new allocation nobody is worse off then at α. Since α belongs to the strong core it will not be weakly blocked by the grand coalition. This implies that everybody must get an object at this new allocation that is indifferent to the object she would get at α.

Finally, let $i_t = j_k$, where j_k is an agent from our cycle such that $j_k \notin I_t$. A priority allocation σ corresponds to some ranking of agents starting from $i_1, \ldots, i_{t-1}, i_t$ and $\sigma(i_l) \sim_{i_l} \alpha(i_l)$ for all l, $1 \leq l \leq t$. Thus, we found an agent i_t, as desired.

(ii) See Remark 1.

(iii) Now, choose an arbitrary strongly Pareto efficient allocation α. It is easy to see that α will be in the strong core for the initial endowments where any agent i owns the object $\alpha(i)$. Indeed, assume to the contrary that a weakly blocking coalition exists. Any such (weakly) blocking coalition could just exchange their houses; such an exchange (since it

[24] j_{k^*} is the "first" element of I_t to occur in the path.

[25] Assume to the contrary that the first repeated vertex is an agent and not an object, then the object pointing to that agent must have appeared before: a contradiction.

leaves other agents with their initial allocations) would lead to a (weak) Pareto improvement, contradicting our assumption that the initial allocation was strongly Pareto efficient. □

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