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Keynesian conundrum: multiplicity and time consistent stabilization

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Abstract

This paper identifies a novel form of dynamic inconsistency of stabilization policy in increasing returns models that generate multiple equilibria. We present a two-period model with externalities and derive closed-form solutions for all endogenous variables in every perfect foresight equilibrium. We provide conditions under which the stabilization policy that maximizes time-zero consumer welfare is not time consistent. Furthermore, we characterize the time consistent stabilization policy. Our results cast doubts on the usefulness of government coordination of economic activity when the government lacks a commitment mechanism. Without commitment, a benevolent government can rule out multiplicity only by ensuring that a Pareto dominated equilibrium obtains.

Time inconsistent policy arises when a government maximizing and acting sequentially conducts a different policy from a government that chooses a plan and can commit not to revisit that plan later. In macroeconomics, time consistency has been studied extensively in the context of monetary policy and optimal taxation. Early research includes Barro and Gordon (1983), Chari et al. (1989), Fischer (1980) and Kydland and Prescott (1977). In this paper, we identify a new avenue for time inconsistency to arise: macroeconomic stabilization policy in the presence of multiple equilibria.

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Models that display multiple equilibria provide an interesting mechanism for understanding business cycles.[1] Multiplicity may also provide a role for government intervention. As Benhabib and Farmer (1994, p. 40) point out, "it may be important to explore the possibility that some classes of policy interventions may be associated with higher economic welfare." In a similar vein, Woodford (1991, p. 79) states that "it remains possible to distinguish policy regimes or institutional arrangements that allow for sunspot equilibria from those that do not, and the choice of policies or institutions of the latter sort to rule out one possible source of aggregate instability may itself be an appropriate object of public policy."

We consider an economy that, in absence of government intervention, has multiple equilibria. Then, we allow a government to coordinate household expectations to insure that a single equilibrium obtains. The government is benevolent and selects an equilibrium to maximize consumer welfare.[2] In this environment, the optimal policy may be time inconsistent.

The idea of government coordinating on an equilibrium by encouraging certain expectations has a long history in both economic theory as well as policy discussions. Here are three examples from academic research. Diamond and Fudenberg (1989, p. 607), in presenting a business cycle model with indeterminacy due to search, state that "because expectations can so strongly influence the economy, traders would prefer some way of coordinating on a 'good' equilibrium. It is not unheard of for governments to issue optimistic forecasts in the hope of inducing optimistic private forecasts; such forecasts could be self-fulfilling." Matsuyama (1991, p. 620), in the context of growth model with indeterminacy, states that "if coordination failure of agents' expectations is the cause of the problem, the state's role in initiating and sustaining the development process should be limited to promoting the optimism and the entrepreneurial spirits in the private sector or to preaching 'the Economics of Euphoria'." Evans et al. (1998) consider a model with indeterminacy due to strategic complementarities in investment. They state that "announcements about monetary policy are one example of a variable that could coordinate the expectations that different firms form" (p. 508).

In the US, financial market participants expectations seem to be highly correlated and responsive to forecasts and statements made by Federal Reserve Chairman Alan Greenspan. Of course, distinguishing 'expectation coordination' effects from the 'fundamental' economic effects of monetary policy is a difficult exercise. The October 19, 1992, Bush–Clinton–Perot presidential debate provides another example of a government official encouraging optimism, or at least discouraging pessimism, to avoid moving to a bad equilibrium outcome—a bank run. In response to a question concerning the quality of US banks' balance sheets, President George Bush Sr. answered, "I don't believe it would be appropriate for a president to suggest that the banking system is not sound." Pessimistic statements by prominent officials could unfortunately provide a signal that households use to switch to a bad equilibrium.

[1] Recent work on multiple, self-fulfilling rational expectations equilibria includes Azariadis and Guesnerie (1986), Chatterjee et al. (1993), Farmer and Woodford (1997), Kiyotaki (1988), Shleifer (1986) and Woodford (1988).

[2] One may think of this policy as a signal that households use to coordinate expectations.

To our knowledge, this issue has received no formal analysis. A first reaction might be that if the government can coordinate expectations on a particular equilibrium, then the answer is trivial: choose the expectations path that maximizes welfare. In a dynamic model, matters are not that simple. Although a strict Pareto ranking of allocations across different equilibria may obtain for the household utility function evaluated at any particular point in time, this ranking can change if the welfare function is evaluated at a different point in time. In other words, there is a time consistency problem. We show that this can occur in a workhorse indeterminacy model: the neoclassical growth model with external returns to capital and labor, studied extensively by Benhabib and Farmer (1994).[3]

In this model, externalities in production lead to increasing social returns to labor. If external effects are sufficiently strong, there is an upward sloping aggregate labor demand curve. This, together with capital externalities, can lead to multiple equilibria. Because of multiplicity, there is a potential for government coordination efforts, as suggested by Fudenberg and Tirole and Matsuyama. Government manipulation of household expectations about the future influences current savings and labor supply decisions; therefore, the government's problem is dynamic.

Let us sketch out the model and the intuition for the main result. Following a tradition in the study of time consistent policy, e.g. Persson and Tabellini (1990), the model has two periods. There is a labor supply decision in both periods and a capital accumulation decision between periods. The optimal policy under commitment has the government select an equilibrium with high initial period and low final period labor supply. The government chooses an equilibrium with high labor supply in the initial period because labor externalities lead to an undersupply of labor in equilibrium. Why does the government announce a low final period labor supply? If households believe the announcement, they respond by increasing initial period labor supply and savings.[4] With externalities, the initial period factor quantities become closer to their optimum.

Now imagine the government enters the final period and is allowed to reoptimize. It would like to select an equilibrium with high final period labor since the labor externality implies that final period labor is too low. Since the benefits of high initial period savings and labor cannot be reversed, the government has no incentive to stick to the previously announced low final period labor. Forward-looking households realize that a government acting sequentially cannot credibly commit to a policy intervention that is welfare-maximizing among the entire class of policy interventions.

[3] Baxter and King (1991) conduct an early study of production externalities in the real business cycle model. Variants of the model have been used extensively to understand the mechanism for understanding the role of sunspots in explaining business cycles. Farmer and Guo (1995) compare the impulse response functions from a discrete-time version of the model with a vector autoregression on US data. Schmitt-Grohé (2000) studies the ability of a two-sector increasing returns model with sunspot shocks to match empirical business cycle regularities. Basu and Fernald (1995, 1997), Burnside et al. (1995) and Burnside (1996) provide evidence against large increasing returns and large external effects in production. More recent theoretical work shows how indeterminacy may arise with more modest increasing returns. For example, Bennett and Farmer (2000) introduce non-separability between consumption and leisure in the utility function which reduces the degree of increasing returns necessary for indeterminacy.

[4] The result that initial period labor and savings increases in response to a decrease in future labor input depends upon the details of preferences and technology, as we explain below.

Our results do not hinge on the assumption that the government can choose an equilibrium. A number of researchers (such as (Cooper, 1999)), have demonstrated how tax policy can be used to rule out Pareto dominated equilibria. To generalize our main result where the government selects expectations, we adopt this tax approach. The same time consistency problem arises if a government, that cannot issue debt or collect lump-sum taxes, uses a state-contingent wage tax to maximize household welfare and render the equilibrium unique.[5] Our results also do not hinge on the particular assumptions used to develop our theorems. Through computations, we show that time inconsistency may arise in a more general model.

In the next section, we present a two-period competitive model with external returns and characterize the perfect foresight equilibria. In Section 2, we characterize the optimal stabilization policy with and without commitment and find conditions where the two differ. Section 3 considers several extensions. Section 4 concludes.

1. Two-period model with external effects

In this section, we present a two-period equilibrium model with increasing returns, production externalities and capital accumulation. We find the entire set of perfect foresight equilibria.

1.1. Household problem

The economy consists of a large number of identical households who live for two periods. Households derive utility from consumption and not working according to

$$u(c_0) - v(n_0) + h\big[u(c_1) - v(n_1)\big] \tag{1}$$

where c_t and n_t denote period t consumption and labor supply and $1 \geqslant h > 0$. Each household is endowed with an initial quantity of capital k_0 and a share in the representative firm of the economy.

In addition to choosing consumption and leisure, agents may also hold capital k_1 between periods. Each household maximizes (1) by choice of $\{c_0, c_1, k_1, n_0, n_1\}$, subject to

$$k_{t+1} = r_t k_t + (1-d)k_t + w_t n_t - c_t \tag{2}$$

for $t = 0, 1$, where $c_t, n_t \geqslant 0$, $k_0 > 0$, $k_2 = 0$. The capital depreciation rate d lies on the open unit interval. Also, it is convenient to define the capital retention rate as $\delta = 1 - d$. Finally, households have rational expectations regarding and take as given w_t and r_t, the period t wage and real interest rate.

Optimization by households at an interior requires

$$\frac{v'(n_t)}{u'(c_t)} = w_t$$

[5] It is important to maintain the realistic assumption that the government cannot issue lump-sum taxes. With lump-sum taxes to finance factor price subsidies, the Pareto optimum can be recovered and no time consistency problem arises.

for $t = 0, 1$,

$$\frac{u'(c_0)}{hu'(c_1)} = r_1 + 1 - d,$$

and that the budget constraint (2) is satisfied with equality for $t = 0, 1$.

This completes our description of the household problem and the associated necessary conditions for optimization. Next, consider the firm's problem.

1.2. Firm problem

The economy consists of a large number of identical firms, each of which produces output according to a production function with externalities.[6] We assume that if the average economy-wide level of labor is positive, the firm-level production function is

$$y_t = (k_t)^a (n_t)^b \left(\bar{k}_t\right)^{\alpha - a} (\bar{n}_t)^{\beta - b} \tag{3}$$

where $\bar{k}_t$ and $\bar{n}_t$ represent the average economy-wide levels of capital and labor. If $\bar{n}_t = 0$, firm-level production is given by $y_t = (k_t)^\alpha (n_t)^\beta$. This technical assumption rules out zero labor supply equilibria.

With many firms, these external effects are exogenous from the perspective of a single firm. Throughout the paper, we consider non-negative externalities: $\alpha \geqslant a$, $\beta \geqslant b$. Finally, each firm privately faces constant returns-to-scale, $a + b = 1$, and $a, b > 0$. Constant returns ensures zero profits.

We study symmetric equilibria where $k_t = \bar{k}_t$ and $n_t = \bar{n}_t$. In a symmetric equilibrium, the aggregate production function is

$$y_t = (k_t)^\alpha (n_t)^\beta \tag{4}$$

where $\alpha, \beta > 0$ and $\alpha + \beta > 1$.

In each period, a firm chooses capital and labor to maximize profits taking factor prices as given. Profit maximization implies

$$r_t k_t = a y_t, \tag{5}$$

$$w_t n_t = b y_t, \tag{6}$$

for $t = 0, 1$. This completes our description of the firm problem and the associated necessary conditions for profit maximization.

1.3. Equilibrium

We summarize the necessary conditions that the endogenous variables $\{n_0, n_1, c_0, c_1, k_1\}$ must satisfy in any interior perfect foresight equilibrium (PFE) by substituting out factor prices using (5) and (6). These conditions are:

[6] Ireland (1997) uses monopolistic competition and sticky prices to study time consistency of monetary policy. In our model, positive production externalities can be thought of as a public good. Time inconsistency of government plans arises in models where distortionary taxes are used to finance public good provision, as in Chari et al. (1989). Thus, this paper demonstrates that time inconsistency arises even in the absence of sticky prices or distortionary taxes.

$$v'(n_t) = bu'(c_t)(k_t)^{\alpha}(n_t)^{\beta-1} \quad \text{for } t = 0, 1, \tag{7}$$

$$u'(c_0) = hu'(c_1)\big(a(k_1)^{\alpha-1}(n_1)^{\beta} + 1 - d\big), \tag{8}$$

as well as the resource constraint

$$k_{t+1} = (k_t)^{\alpha}(n_t)^{\beta} + (1-d)k_t - c_t \quad \text{for } t = 0, 1, \tag{9}$$

where $k_0 > 0$, $k_2 = 0$.

Next, we make functional form assumptions in order to construct closed-form expressions for the entire set of PFE. Assume $\alpha = h = 1$, $\beta = 2$, $u(c) = \log(c)$ and $v(n) = n$.[7] In this case, our necessary conditions become

$$c_t = bk_t n_t \quad \text{for } t = 0, 1, \tag{10}$$

$$c_1 = c_0\big(a(n_1)^2 + 1 - d\big), \tag{11}$$

$$c_1 = k_1(n_1)^2 + (1-d)k_1, \tag{12}$$

$$c_0 + k_1 = k_0(n_0)^2 + (1-d)k_0. \tag{13}$$

We will shortly study the optimal policy when the government is able to select among equilibrium resource allocations. The government maximizes (1) subject to the resource constraints (12) and (13), as well as the constraint that the labor supply choice is consistent with an equilibrium (10), and that the capital allocation is consistent with an equilibrium (11). This Ramsey problem is non-standard because the government selects from a set of equilibria rather than tax rates.

External returns to labor are sufficiently strong to imply increasing social returns to labor and lead to multiple final period equilibrium values. A forward-looking government may increase initial period savings and labor by coordinating on a equilibrium with low final period labor income.

Below is the present value budget constraint:

$$c_0 + \frac{c_1}{r_1 + \delta} = \left[(r_0 + 1 - d)k_0 + w_0 n_0 + \frac{w_1 n_1}{r_1 + \delta}\right].$$

Using (11) to substitute out c_1, the savings of an individual household is given by

$$s \equiv y_0 - c_0 = \frac{1}{2}\left[(r_0 + 1 - d)k_0 + w_0 n_0 - \frac{w_1 n_1}{r_1 + \delta}\right].$$

Since there are multiple future equilibria, the coordinating government can treat the future average inputs as controls.[8] Recall $\bar{k}_1, \bar{n}_1$ denote economy-wide averages. Use the firm first-order conditions to substitute out final period factor prices:

$$s = y_0 - c_0 = \frac{1}{2}\left[(r_0 + 1 - d)k_0 + w_0 n_0 - \frac{b\bar{k}_1\bar{n}_1}{a(\bar{n}_1)^2 + \delta}\bar{n}_1\right].$$

[7] These assumptions are useful in deriving analytic results. They are not necessary for the optimal stabilization policy to be time inconsistent. In Section 3.1, we show that time inconsistency occurs with more modest increasing returns and $h < 1$.

[8] Note $\bar{n}_1, \bar{k}_1$ must satisfy optimality and market clearing, as well.

Conjecture that $n_1 = \bar{n}_1$. This will require us to go back and check that n_1 is optimal for the household. The above equation becomes:

$$s = \frac{1}{2}\left[(r_0 + 1 - d)k_0 + w_0 n_0 - z(\bar{n}_1)\bar{k}_1\right] \tag{14}$$

where $z(n) \equiv b[a + (1-d)/(n^2)]^{-1}$ is the discounted value of future labor income per unit of the capital stock. For $n > 0$, $z, z' > 0$.

Examining (14), the government can increase savings by reducing $\bar{n}_1$, which increases private savings holding fixed $\bar{k}_1$ and initial period quantities and prices (w_0, n_0, r_0). Reducing permanent income induces greater savings, which moves capital closer to its Pareto optimal level. This is the first beneficial effect of reducing $\bar{n}_1$.

Once prices adjust, initial period labor will respond as well. Increasing savings lowers initial consumption and drives up the marginal utility of consumption. At the original initial wage, households increase labor supply. With increasing returns to labor, the real wage and initial labor both rise. Because of the labor externality, the general equilibrium effect of announcing low future labor can bring initial period labor closer to the optimal level.

Of course, the government choice of equilibrium is trivial if the equilibrium is unique. We must establish that there exist multiple PFE. Substituting out c_1 in (12) using (10) at $t = 1$,

$$k_1(n_1)^2 - bk_1n_1 + (1-d)k_1 = 0.$$

Since k_1 must be positive if consumption is positive at $t = 1$, we may divide both sides by k_1. We are left with a quadratic equation in n_1:

$$(n_1)^2 - bn_1 + 1 - d = 0. \tag{15}$$

Applying the quadratic formula to (15), there are two possible values of labor supply at $t = 1$:

$$n_1^h = \frac{1}{2}\left(b + \sqrt{b^2 - 4(1-d)}\right), \tag{16}$$

$$n_1^l = \frac{1}{2}\left(b - \sqrt{b^2 - 4(1-d)}\right). \tag{17}$$

In order for n_1^h, n_1^l to be real valued, we assume that $b^2 > 4(1-d)$. Note that n_1 does not depend on k_1. A larger capital stock k_1 induces a positive income effect in the final period, because undepreciated capital is consumed, which tends to reduce n_1. On the other hand, a larger capital stock raises the marginal product of labor, which tends to increase n_1. These two effects exactly cancel with log utility in consumption and constant social returns to capital. This simplifies our analysis because the government choice of labor supply entering the final period will be independent of initial period actions.

At $t = 1$, labor supply will either be high or low and less than b, which we state as a lemma.

Lemma 1. $0 < n_1^l < b/2 < n_1^h < b < 1$.

Proof. All proofs are in Appendix A. □

Lemma 1 will be useful in characterizing the initial period labor supply. In particular, if households expect that w_1 will be low because aggregate labor supply is low at time one, then they will increase time-zero labor supply because of a wealth effect. We also verify that households prefer positive to zero labor at equilibrium prices in all equilibria.

Substituting (10) and (13) into (11) gives us a quadratic equation in n_0:

$$n_1(n_0)^2 - \left[bn_1 + 1 - d + a(n_1)^2\right]n_0 + (1-d)n_1 = 0.$$

For a given value of n_1, two distinct values for n_0 solve this quadratic equation. Since there are two potential values for n_1, there are a total of four possible values of labor supply at $t = 0$, which we denote n_0^{ij} for $i = h, l$ and $j = h, l$. Letting $\theta^j = bn_1^j + 1 - d + a(n_1^j)^2$, the four period-zero equilibrium values of labor supply are:

$$n_0^{hj} = \frac{1}{2n_1^j}\left(\theta^j + \sqrt{\left(\theta^j\right)^2 - 4(1-d)\left(n_1^j\right)^2}\right), \tag{18}$$

$$n_0^{lj} = \frac{1}{2n_1^j}\left(\theta^j - \sqrt{\left(\theta^j\right)^2 - 4(1-d)\left(n_1^j\right)^2}\right), \tag{19}$$

for $j = h, l$. Summarizing, there are four equilibria, each of which is described by a pair (n_0^{ij}, n_1^j) for $i = h, l$ and $j = h, l$.[9]

By examining (18) and (19), it is straightforward to show that $n_0^{hl} > n_0^{hh}$. This provides a partial explanation of why choosing a low future labor equilibrium maximizes time-zero household welfare. Expectation of n_1^l instead of n_1^h leads to greater labor supply in the initial period. According to (15), final period labor supply does not depend on k_1. This occurs because there were offsetting income and substitution effects on final period labor supply.

One might expect that these offsetting income and substitution effects would imply that n_0 and k_1 do not depend on n_1. Why is this not the case? First, announcing that labor supply will be lower in the future period reduces income in the future period. This income effect tends to increase n_0 and k_1. This effect was emphasized in the introduction. On the other hand, announcing a low future labor supply also reduces the real interest rate since the future marginal product of capital will be lower if future labor supply is low. This effect tends to reduce n_0 and k_1. However, this effect is smaller in magnitude than the opposing income effect because there are capital externalities. Equation (11) demonstrates that only part of the returns to capital, in the form of higher labor productivity, are internalized; therefore, only part of this real interest rate effect is internal to the firm.

Consumption in each equilibrium may be calculated using the above necessary conditions:

$$c_0^{ij} = bn_0^{ij}k_0, \tag{20}$$

$$c_1^{ij} = bn_1^j\left[\left(n_0^{ij}\right)^2 - bn_0^{ij} + 1 - d\right]k_0. \tag{21}$$

This completes our characterization of the model's perfect foresight equilibria. In the next section, we present the optimal stabilization policy with and without commitment.

[9] Time-zero labor is positive in every equilibrium by straightforward examination of (18) and (19).

2. Optimal stabilization policy

Equilibrium multiplicity may provide a justification for government intervention. We study the policy of a benevolent government that selects from among the set of perfect foresight equilibria with and without commitment. The timing of decisions in both cases is presented in Fig. 1. Also, since initial capital does not effect equilibrium labor supply, let $k_0 = 1$ for the remainder of this section without loss of generality.

2.1. Without commitment

Without commitment, solve the government problem by working backwards. First, consider the optimal policy in the final period.

2.1.1. Final period

Entering the final period, k_1 is given by previous actions of households, firms and the government. With k_1 given, the government chooses a value of n_1 to maximize consumer welfare at $t = 1$ that is consistent with equilibrium. Let z_1 denote the household's utility entering the final period with capital k_1 if the government behaves optimally:

$$z_1(k_1) = \max_{n_1 \in \{n_1^h, n_1^l\}} \left\{ \log(c_1) - n_1 \right\} \tag{22}$$

where $c_1 = bk_1n_1$. This restriction on c_1 is the incentive compatibility constraint (10) at $t = 1$.

If n_1 were a continuous choice variable, the value function would be maximized at labor equal to one. This is independent of the value of k_1 for reasons described in Section 1.

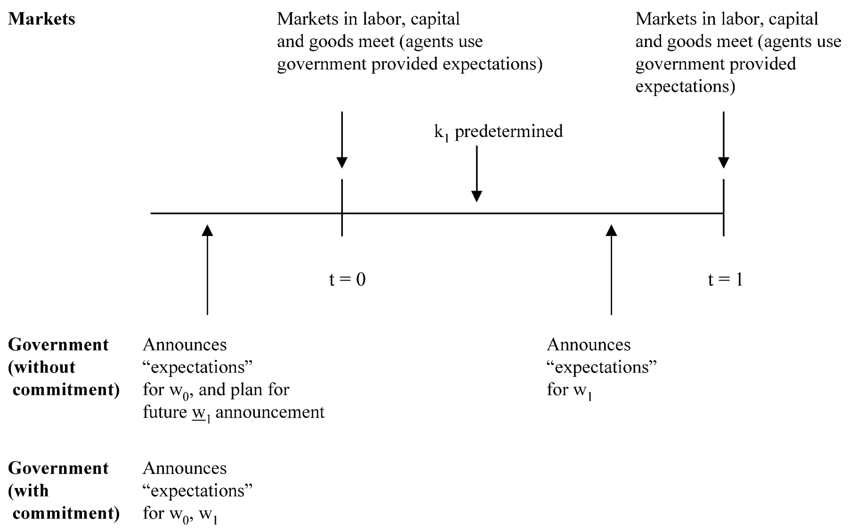

Fig. 1. Two alternative timing structures for government stabilization policy. *Note*: At each node, government chooses policies in order to maximize existing (or remaining) household utility. Without commitment, time consistency requires that $\underline{w}_1 = w_1$.

In addition, the objective is concave and increasing in n_1 for $n_1 \leqslant 1$. Since both choices for labor $\{n_1^h, n_1^l\}$ are less than one by Lemma 1, the optimizing government will choose the larger value n_1^h.

Intuitively, with positive external returns to labor, part of productive labor effort is not internalized by the firm. In the final period, every equilibrium value of labor is inefficiently low. From among the equilibrium values of labor, n_1^h and n_1^l, a higher labor is preferred. Having chosen the optimal final period policy, consider the optimal initial period policy.

2.1.2. Initial period

In the initial period, the government chooses labor n_0 to maximize

$$z_0 = \max_{n_0 \in \{n_0^{hh}, n_0^{lh}\}} \log(c_0) + \log(c_1) - n_0 - n_1^h, \tag{23}$$

subject to (20) and (21) which define the law of motion for consumption. The time-zero government takes the final period labor as given at n_1^h. Substituting (20) and (21) into (23), we have

$$z_0 = \max_{n_0 \in \{n_0^{hh}, n_0^{lh}\}} U\big(n_0, n_1^h\big)$$

where

$$U(n_0, n_1) = \log(bn_0) + \log(bn_1) + \log\big[(n_0)^2 - bn_0 + 1 - d\big] - n_0 - n_1.$$

Next, we must find the optimal n_0. The next three lemmas will be useful.

Lemma 2. *For $n_0 \in [b, 2b)$, $U(n_0, n_1)$ is increasing in n_0.*

The intuition for Lemma 2 is straightforward. Since labor is undersupply due to positive labor externalities in both periods, every equilibrium involves too little labor supply relative to the Pareto optimum. The only caveat to this statement is that for very low levels of n_0, U may be decreasing in n_0. When n_0 is very low, labor's marginal product is low and the marginal disutility of working is one. This does not occur in the relevant region $n_0 \in [b, 2b)$. The next lemma provides three inequalities concerning initial period labor in the alternative equilibria.

Lemma 3. (i) $n_0^{lh} < b/2$; (ii) $n_0^{hh} > b$; (iii) $n_0^{hh} < 2b$.

Lemma 4. $U(b, n_1) > U(n_0, n_1)$ *if* $0 < n_0 < b$.

Theorem 1. *The time consistent optimal government stabilization policy is* (n_0^{hh}, n_1^h).

We previously established that the benevolent final period government chooses the high labor equilibrium; therefore, the initial government must choose labor supply that is consistent with both equilibrium as well as high final period labor supply. The government may select either n_0^{lh} or n_0^{hh}. Together, Lemmas 2 and 4 show that in a relevant range, higher labor supply is strictly preferred. This occurs because positive labor externalities lead to an undersupply of labor in every equilibrium. Finally, Lemma 3 demonstrates that n_0^{lh} is less than n_0^{hh} and that both n_0^{lh} and n_0^{hh} are in the relevant range of labor values. Therefore, the welfare-maximizing time consistent equilibrium involves high labor in both periods.

In the next section, we show that the time-zero welfare-maximizing equilibrium may involve high initial and low future labor supply.

2.2. Under commitment

Consider the government problem under commitment. With commitment, the government selects (n_0, n_1) from the entire set of PFE to maximize $U(n_0, n_1)$ subject to the restriction $(n_0, n_1) \in \{(n_0^{ij}, n_1^j)\}_{i=h,l;\, j=h,l}$. First, we present some bounds on equilibrium values of labor supply.

Lemma 5. (i) $n_0^{ll} < n_0^{hl}$; (ii) $n_0^{lh} < n_0^{hh}$; (iii) $n_0^{hl} > b$; (iv) $n_0^{hl} < 2b$.

The following theorem provides conditions under which a benevolent government under commitment would choose the production bunching equilibrium. The main expression in Theorem 2 that must be checked is (24), which determines whether higher utility is achieved under (n_0^{hl}, n_1^l) rather than (n_0^{hh}, n_1^h). This expression is complicated.

Theorem 2. *The time-zero optimal government stabilization policy under commitment is (n_0^{hl}, n_1^l) if and only if*

$$\begin{aligned}\frac{k_1^{hl}\phi^l}{k_1^{hh}\phi^h} &> \exp\Big(b\sqrt{b^2-4(1-d)} - b^2 + 4(1-d)\Big)^{1/2}\\ &\quad\times\Big\{\exp\Big[b^2(4-b)\sqrt{b^2-4(1-d)}\Big]\Big\}^{1/4}\end{aligned} \tag{24}$$

where $\phi^j = \theta^j + \sqrt{(\theta^j)^2 - 4(1-d)(n_1^j)^2}$.

The proof of Theorem 2 consists of three pairwise time-zero utility comparisons of different equilibria. Let U_0^{ij} denote the time-zero household utility of equilibrium (n_0^{ij}, n_1^j). First, we show that $U_0^{hh} > U_0^{lh}$. Lemmas 2 and 4 provided conditions under which higher n_0 holding fixed n_1 results in higher household utility. These results apply since, by Lemma 5(ii), $n_0^{hh} > n_0^{lh}$. Second, as a part of the proof of Theorem 1, we establish $U_0^{hl} > U_0^{ll}$. Recall that there are returns to labor that are not internal to the firm. In every equilibrium, there is an undersupply of labor. Intuitively, an equilibrium with low labor in both periods is Pareto dominated by an equilibrium with high labor in at least one period.

The third and final required inequality is $U_0^{hl} > U_0^{hh}$. This inequality holds if and only if (24) is true. Because of external returns to labor, there is too little labor and output in every equilibrium. At first look, it may seem that the equilibrium (n_0^{hh}, n_1^h) generates the greatest total output. This is not necessarily true. Imagine a household at time zero has a change in expectations that wages at time one will be low instead of high. The reduction in permanent income induces the household to increase time-zero labor. In fact, it is possible to show that $n_0^{hl} > n_0^{hh}$. Because of increasing returns to labor, a higher n_0 increases the marginal product of labor and therefore the wage. Thus, the income effect of a falling future wage is amplified by the substitution effect. Larger initial period wage income increases initial savings. This is also Pareto improving because of capital externalities.

Not every parameter configuration generates a time consistency problem. In Section 3.2, we compute numerically which values of the underlying parameters (b, δ) imply $U_0^{hl} > U_0^{hh}$. If $U_0^{hl} < U_0^{hh}$ or, equivalently, (24) holds, then the highest time-zero utility equilibrium is (n_0^{hh}, n_1^h) and there is no time inconsistency issue.

In part, Theorem 2 holds because a benevolent stabilization authority recognizes the usefulness of concentrating production when there are externalities. It may not be obvious why high labor is preferred in the initial period over high labor in the final period. Why is $U_0^{hl} \neq U_0^{lh}$? When labor is high in the initial period, the capital stock is high between the initial and final periods. This has two effects. First, it allows the household to save output across periods. Second, a high capital stock in the final period raises the final period labor productivity. This second channel is not operative if production is bunched in the final period.

Let us take a sample parameter configuration. Consider the following selection of underlying parameter values: $b = 0.7$, $d = 0.9$.[10] In this case, the assumptions of Theorem 1 clearly hold and the time consistent strategy involves high labor supply in both periods. In addition, the assumptions of Theorem 2 hold and the optimal policy with commitment is (n_0^{hl}, n_1^l). Figure 2 lists the endogenous variables and time-zero household utility in every equilibrium.

The highest utility equilibrium involves high initial and low future labor. As explained in Theorem 1, this equilibrium is not time consistent because the final period government, if given the opportunity to deviate, will choose high labor equilibrium. It is important not to focus on total labor supply across both periods as a measure of economic activity. Note that total labor effort is greater in the high–high equilibrium, that is $n_0^{hh} + n_1^h > n_0^{hl} + n_1^l$. If the production function is convex in labor, then an unevenly distributed labor supply with a lower total may generate more output than an evenly distributed labor supply with a higher total. In addition, the capital stock is much larger in the production bunching equilibrium, which is welfare improving because of the external effects of capital.

	Low labor at $t = 1$	High labor at $t = 1$
Low labor at $t = 0$	$U_0 = -8.12$ $n_0 = 0.09$; $n_1 = 0.20$ $c_1 = 0.007$; $k_1 = 0.047$	$U_0 = -7.52$ $n_0 = 0.11$; $n_1 = 0.50$ $c_1 = 0.013$; $k_1 = 0.037$
High labor at $t = 0$	$U_0 = -3.96$ $n_0 = 1.17$; $n_1 = 0.20$ $c_1 = 0.092$; $k_1 = 0.658$	$U_0 = -4.11$ $n_0 = 0.94$; $n_1 = 0.50$ $c_1 = 0.116$; $k_1 = 0.330$

Note: In this example, $b = 0.7$ and $d = 0.9$. For every equilibrium, $c_0 = b(n_0)$.

Fig. 2. Time inconsistent stabilization policy—a numerical example.

[10] Our restriction $b^2 > 4(1 - d)$ imposes an empirically unrealistic high depreciation rate for admissible values of b. As described later in the paper, this restriction is not crucial for the existence of time inconsistency in the more general model.

3. Extensions

3.1. Adding realistic depreciation and modest increasing returns

Several assumptions on production and preferences were required to derive closed form solutions. These assumptions eased the task of proving Theorems 1 and 2, which established the potential for time inconsistency of the optimal stabilization policy. The skeptical reader may wonder whether these assumptions, specifically high capital depreciation and social returns to factors, are necessary for the result. This subsection addresses that question.

We show that time inconsistency may arise under much weaker (but still increasing) returns-to-scale and a realistic depreciation rate. We proceed in two steps:

(i) give two parameter sets which generate time inconsistency, one from the theorem's assumptions and another which we established by calculation,
(ii) provide a brief justification for the second set and an intuition for what features are necessary to generate the conundrum.

The first two columns of Fig. 3 give the name and notation of each variable. The third column, labeled 'Theorem,' reports the restrictions used in the previous section. For the theorems, we assume $\alpha = 1$ and $\beta = 2$. Under these assumptions, equilibrium labor in each period solves a quadratic equation and was independent of the capital stock. These are large relative to some estimates that find evidence of social increasing returns-to-scale. Next, the theorems require only standard assumptions on b, the private returns to labor; however, together with the depreciation rate d, we require $b^2 > 4(1 - d)$. Using a realistic private return to labor, such as 0.70, implies that the depreciation rate must be at least 0.8775. This is unrealistically high. For simplicity, we did not introduce discounting $h = 1$.

The final column provides a configuration which also generates multiplicity and time inconsistent policy. Here, one period lasts five years. Our choices of d and h imply a relatively standard annual capital retention rate and time discount factor of 0.88 and 0.974. Therefore, time inconsistency can arise with reasonable discounting and capital depreciation.

Variable	Notation	Theorem	Computation
Social returns to labor	β	2	1.20
Private returns to labor	b	$*$	0.70
Social returns to capital	α	1	0.75
Private returns to capital	a	$1 - b$	0.30
Capital depreciation rate	d	$*$	0.40
Discount rate	h	1	0.90
Initial capital stock	k	$*$	0.10
One-period utility from c	$u(c)$	$\log(c)$	$\log(c)$
One-period disutility from n	$v(n)$	n	n

Fig. 3. Time inconsistency under an alternative model parameterization.

Next, note that time inconsistency arises under much lower returns-to-scale than the theorems require. Total returns in the computation equals 1.95, as opposed to 3 in the theorems. Note that increasing social returns to labor is necessary for time inconsistency. Without this feature, each period's aggregate labor demand curve is downward sloping and the equilibrium allocation is unique.

Our assumptions on the private returns to capital and labor are in the range of existing empirical estimates. To generate time inconsistency, we do require a significant difference between the private and social returns to capital: $\alpha = 0.75$, $a = 0.30$. The commitment solution involves low future labor because it induced greater labor effort and savings in the initial period. Without this motive, there is no time consistency problem. By coordinating an equilibrium with low future income, the government induces greater initial labor effort due to the wealth effect; however, lower future labor input also lowers the future marginal product of capital. This reduces the incentive to save in the initial period. Without a capital externality, these effects offset exactly. The externality dampens the future marginal product of capital effect so that the wealth effect on initial period labor supply dominates.

3.2. Varying private returns and depreciation

Returning to the model with $\alpha = 1$, $\beta = 2$, we next trace out the parameters for which our conundrum arises. Figure 4 identifies the highest time-zero utility equilibrium for all configurations of b and $1-d$. Values of $1-d$ are on the vertical axis and values of b are on the horizontal axis. An '$\times$' denotes a $(b, 1-d)$ pair with highest utility occurring at (n_0^{hl}, n_1^l) and an '$\circ$' denotes a $(b, 1-d)$ pair with highest utility occurring at (n_0^{hh}, n_1^h). The upper left side of the graph is blank. In this range, $b^2 < 4(1-d)$ and there are no equilibria where households are at an interior solution. Also, there is no region where the other two equilibria—both of which involve low time-zero labor—maximize time-zero welfare. Over the entire region, Theorem 1 demonstrates that the optimal time consistent equilibrium is (n_0^{hh}, n_1^h).

From Fig. 4, we see that the region where (n_0^{hh}, n_1^h) maximizes time-zero utility under commitment has low values of b and $1-d$. First, consider the private returns to labor, b. As b falls, the private return to capital approaches its social return. As explained earlier, a capital externality is necessary for our result. For sufficiently small b, the incentive to commit to low future input is too weak to generate time inconsistency.

Second, consider the capital retention rate $\delta = 1-d$. Holding fixed b, for a sufficiently large δ the model has no final period equilibria. This is apparent from Fig. 4. On the other hand, for sufficiently small δ, (n_0^{hh}, n_1^h) is optimal and there is no time consistency problem. Recall from (14) that initial savings is decreasing in the discounted value of future wage income, where final period factor prices are expressed as functions of final period aggregates:

$$s = \frac{1}{2}\left\{(r_0 + \delta)k_0 + w_0 n_0 - \frac{b\bar{k}_1(\bar{n}_1)^2}{a(\bar{n}_1)^2 + \delta}\right\}.$$

Holding fixed $\bar{k}_1$ and initial period quantities and prices, a lower retention rate δ reduces the impact of final period labor input on household savings. A small effect on initial period savings will also imply a small effect on initial period labor. In this case, the government

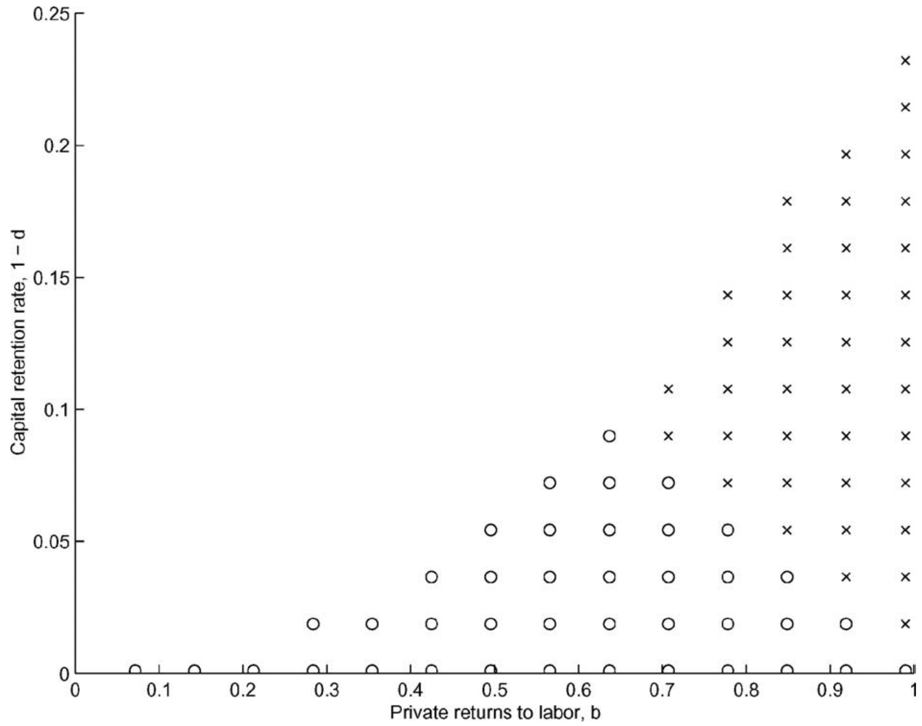

Fig. 4. Time consistency/inconsistency under alternative parameters.

has no incentive to announce a low future labor equilibrium; the benefits of slightly greater savings and initial period labor are less than the cost of reducing final period labor. In the extreme case where $d = 1$, the equilibrium is unique and $n_0 = b + (1 - b)b$, $n_1 = b$.

3.3. *Stabilization policy as a state-contingent tax policy*

Despite several examples of models where this kind of stabilization policy is discussed, notably Diamond and Fudenberg (1989), Evans et al. (1998) and Matsuyama (1991), the reader may question the policy's realism or workability. Instead, several authors have studied tax policies that rule out multiplicity, such as Christiano and Harrison (1999) and Guo and Lansing (1998). In this section, we describe how our stabilization policy can be reinterpreted as a certain state-contingent labor tax policy. We do not employ lump-sum taxes. Lump-sum taxes would allow the government to achieve the unconstrained Pareto optimum and eliminate the time consistency problem.

Assume the government selects a tax on labor $\tau_t = \tau_t(n_t)$ for $t = 0, 1$ as a function of individual labor supply in order to maximize household welfare. In addition, assume that the government cannot borrow or lend. The government may issue lump-sum rebates but cannot collect lump-sum taxes. These assumptions together imply $\tau_t \geqslant 0$ in order that the government budget constraint is satisfied. Let the government choose a set of functions (τ_0, τ_1) to maximize household welfare subject to the constraint that the equilibrium implied by the choice of $\tau_t(n_t)$ for $t = 0, 1$ is unique.

First, consider the optimal time consistent policy. Entering the final period, the government would not impose a tax on labor supply in equilibrium since every equilibrium n_1 involves too little labor supply in the final period.[11] Although the government will not tax labor in equilibrium, it would prefer n_1^h obtain. It can accomplish this by setting $\tau_1(n_1^l)$ very high. This makes the low labor input inconsistent with consumer optimization. Without commitment, the tax policy guarantees final period labor equals n_1^h.

Working backwards, the initial period government maximizes time-zero utility subject to the constraint that equilibrium taxes will be zero at $t = 1$ and that $n_1 = n_1^h$. By the same argument as above, the government will not tax initial period labor in equilibrium and it prefers n_0^{hh} to n_0^{lh}. It can accomplish this by setting $\tau_0(n_0^{lh})$ very high and $\tau_0(n_0^{hh}) = 0$. Therefore, the time consistent allocation where the government coordinates households expectations is identical to the time consistent allocation where the government selects a state-contingent tax policy.

Next, consider the solution to the government problem with commitment. The allocation (n_0^{hl}, n_1^l) is feasible under this policy. This equilibrium obtains if the government sets $\tau_0(n_0^{hl}) = \tau_1(n_1^l) = 0$ and τ_t very high at all other values of labor supply. This demonstrates that there is a time consistency problem when government chooses a tax policy. There exists a feasible allocation with commitment (n_0^{hl}, n_1^l) which dominates the optimal allocation without commitment (n_0^{hh}, n_1^h).

In fact, although (n_0^{hl}, n_1^l) is feasible under commitment, it may not be optimal. Under commitment, the government may want to have a positive final period labor tax in equilibrium since a value of n_1 below n_1^l may induce greater initial period savings and labor input that increases welfare. Thus, using tax policy instead of equilibrium selection as the policy instrument may magnify the time consistency problem by increasing welfare under commitment and leaving welfare unchanged without commitment.

We could have considered alternative tax instruments to those presented above; however, several of these would either

(a) trivially resolve the time consistency problem by giving the government access to lump-sum taxes, or
(b) introduce new time consistency problems.

First, if the government can tax the initial capital stock, this is equivalent to a lump-sum tax. An effective lump-sum tax can be used to finance subsidies to undersupplied factors and allow the government to achieve the first-best allocation. Alternatively, the government could have taxed capital in the second period or issue debt in the initial period. Either of these fiscal instruments *would introduce well-understood time consistency problems of their own*.

[11] Again, the government would like to subsidize labor in equilibrium because of the externality. This is ruled out by our assumption that the government may not issue lump-sum taxes or borrow.

3.4. A longer horizon

For simplicity, we studied a two-period model. It is straightforward to demonstrate that the time consistency problem would extend to a model with arbitrarily longer, but finite number of periods, where the preferences and technology are modified appropriately. First of all, the optimal time consistent solution involves n_0^{hh}, n_1^h in the final two periods of a finite horizon model. Since these were computed by backward induction and do not depend upon the capital stock, these values are independent of the number of preceding periods or the level of the capital stock carried into the second to last period.

If there is no time consistency problem, then the Ramsey and the optimal time consistent solution coincide. Furthermore, there would be no incentive for any future governments to deviate from the initial government's announced policy strategy. However, entering the penultimate period, as long as (24) holds, that government would prefer to depart from the 'high labor from now on' strategy and announce low labor in the final period. This preference of the second to last government does not vary with the quantity of capital, the only potentially relevant state variable. We are careful not to claim that our results extend to an infinite horizon. With an infinite horizon, it may be possible to support the Ramsey solution with reputation.

4. Conclusion

In this paper, we have identified an interesting form of time inconsistency in a standard model of endogenous fluctuations. In a dynamic model with external returns to capital and labor, a stabilization policy that selects the equilibrium with highest time-zero utility may not be time consistent. If the government can revisit the stabilization plan at a future date, it may wish to change its plan. Our expectation coordination policy was reinterpreted as a tax policy.

Our result casts doubt on the usefulness of stabilization policy to remove fluctuations in a class of 'animal spirit' models. The government may stabilize the economy by selecting an equilibrium; however, without a commitment mechanism, the stabilization policy guarantees that the highest time-zero utility equilibrium cannot be reached and a Pareto dominated equilibrium is chosen instead.

This potentially leaves the government with a difficult conundrum: if it chooses to stabilize an economy by selecting an equilibrium, it will ensure that a Pareto dominated equilibrium occurs with probability one. If the government is able to "walk away" from conducting a stabilization policy, then there may be some chance that households and firms will coordinate on the highest time-zero utility equilibrium. Stated another way, imagine that the government believes that households have sufficiently optimistic expectations such that there is a strong chance a high utility equilibrium obtains without government intervention. Then, the government may prefer to conduct no stabilization policy if it can feasibly do so.

There are deeper implications of the connection between time inconsistency and indeterminacy that go beyond the particular production externality model considered here. Other frictions, such as search and credit market frictions, generate indeterminacy by admitting multiple paths for an external effect. When the external effect implies too little economic activity early on, a government, attempting to coordinate expectations, may coordinate on low future economic activity. This 'pessimistic' forecast about the future may encourage current economic activity. If the time distribution of these external effects favors the initial household utility at the expense of household utility evaluated in the future, then benevolent future governments will want to revisit the policy.

Our result is likely to extend to other models with longer and even infinite horizon; however, it is not true that indeterminacy will always lead to time inconsistent optimal stabilization policy. For the time-zero government to prefer coordinating on an equilibrium with low future activity, the welfare gain from increased activity initially must outweigh the discounted welfare loss from the future recession. Even in the two-period model, there are parameter values where coordinating on low future activity does not increase time-zero utility. In this case, the optimal policy will be time consistent.

To see this point more simply, consider the case when one of the multiple equilibria is Pareto efficient. Then, as long as households have time consistent preferences, no government would ever want to deviate from the equilibrium that is Pareto efficient. Of course, if the government has access to lump-sum taxes in order to finance corrective subsidies—or more simply, bypass the market altogether and directly allocate resources—both time inconsistency and indeterminacy will generally be avoided.

Within this class of models, however, when a government "issues optimistic forecasts" or promotes "the entrepreneurial spirits in the private sector," it is because of an externality-induced non-uniqueness that pronouncements could have a coordinating role. Current and future governments may then disagree over the preferred time path of the external effects.

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Appendix A. Proofs of theorems and lemmas

Proof of Lemma 1. This holds by (15), and because $1 - d > 0$. □

Proof of Lemma 2. Consider household utility as a function of n_0. Since utility is separable from n_1, it is sufficient to study

$$\widehat{U}(n_0) = \log(n_0) + \log\left(n_0^2 - bn_0 + \delta\right) - n_0.$$

We know that n_0 can only take on one of four values; however, it will be important to develop some facts about the function $\widehat{U}$. Because utility is unaffected up to a monotone transformation, we may equivalently study $V(n_0) = \exp(\widehat{U}(n_0))$. Note that

$$V(n_0) = \left(n_0^2 - bn_0 + \delta\right)\left[n_0 \exp(-n_0)\right].$$

Taking the first derivative,

$$V'(n_0) = \exp(-n_0)\left[-n_0^3 + (b+3)n_0^2 - (2b+\delta)n_0 + \delta\right].$$

Letting $f(n_0) = -n_0^3 + (b+3)n_0^2 - (2b+\delta)n_0 + \delta$, it is clear that $f(n_0) > 0$ implies $V'(n_0) > 0$.

Note that f is a cubic equation with a negative sign on the $(n_0)^3$ term. This implies f has only one increasing region. See Fig. A.1(a). Evaluating the function and its first derivative at $n_0 = b$, we have $f(b) = b^2 + \delta(1-b) > 0$ and $f'(b) = b(4-b) - \delta > 0$. Therefore, f is positive at $n_0 = b$ and it belongs to the increasing region of f. For any $x > b$, if $f'(x) > 0$ it must be the case that $f(n_0) > 0$ for all $n_0 \in (b, x)$. Simple algebra shows that $f'(2b) = 2(5-4b) - \delta > 0$. $\square$

Proof of Lemma 3. (i) Note from (19) that

$$n_0^{lh} = \frac{b}{2} + \frac{1}{2}\Delta\left(n_1^h\right) - \frac{1}{2}\sqrt{\left(b + \Delta\left(n_1^h\right)\right)^2 - 4\delta}$$

where $\Delta(n) = (1-b)n + \delta/n$. The conjectured inequality holds if

$$\Delta\left(n_1^h\right) < \sqrt{\left(b + \Delta\left(n_1^h\right)\right)^2 - 4\delta}.$$

Taking the square of both sides, this is equivalent to

$$\Delta\left(n_1^h\right)^2 < b^2 + 2b\Delta\left(n_1^h\right) + \Delta\left(n_1^h\right)^2 - 4\delta.$$

Simplifying,

$$4\delta < b^2 + 2b\Delta\left(n_1^h\right).$$

Since $b^2 > 4\delta$ by assumption and $\Delta > 0$, this inequality holds.

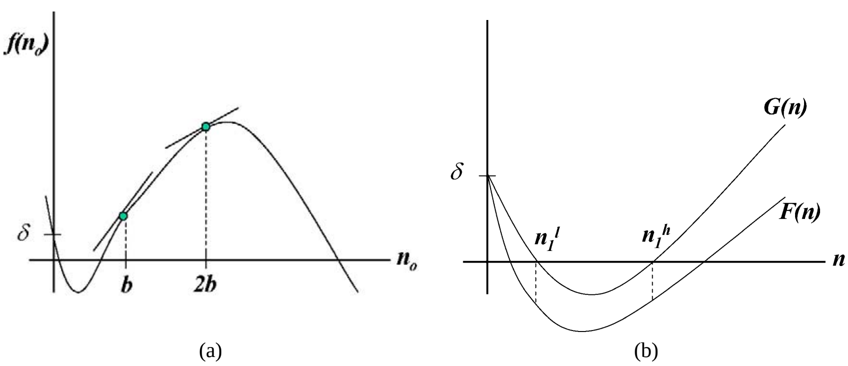

Fig. A.1. Properties of equilibrium labor allocations.

(ii) From (18) the definition of n_0^{hh}, the inequality is

$$\frac{b}{2}+\frac{1}{2}\Delta\left(n_1^h\right)+\frac{1}{2}\sqrt{\left(b+\Delta\left(n_1^h\right)\right)^2-4\delta}>b.$$

Simplifying

$$\sqrt{\left(b+\Delta\left(n_1^h\right)\right)^2-4\delta}>b-\Delta\left(n_1^h\right).$$

Squaring both sides and simplifying,

$$b\Delta\left(n_1^h\right)>\delta.$$

From the definition of $\Delta(n_1^h)$, this becomes

$$an_1^h+\frac{\delta}{n_1^h}>\frac{\delta}{b}.$$

We know that $0<n_1^h<b$ which guarantees that the inequality holds.

(iii) Proof by contradiction. Assume $n_0^{hh}\geqslant 2b$. From (18) and (19), note that $n_0^{hh}+n_0^{lh}=2y$ where $y=\frac{1}{2}[b+\delta/n_1^h+an_1^h]$. Then,

$$n_0^{lh}=b+\frac{\delta}{n_1^h}+an_1^h-n_0^{hh}. \tag{A.1}$$

We will show that $n_0^{hh}\geqslant 2b$ implies $n_0^{lh}<0$. From (A.1), it is clear that if $n_0^{hh}=2b$ implies $n_0^{lh}<0$, then $n_0^{hh}>2b$ will also imply $n_0^{lh}<0$. Letting $n_0^{hh}=2b$, we have

$$n_0^{lh}=-b+\frac{\delta}{n_1^h}+an_1^h.$$

Since we are interested in establishing the sign of n_0^{lh} under the contradiction, we can multiply the above expression by n_1^h. Then,

$$\operatorname{sgn}\left[n_0^{lh}\right]=\operatorname{sgn}\left[a\left(n_1^h\right)^2-bn_1^h+\delta\right]. \tag{A.2}$$

The quadratic term inside the brackets of the right-hand side of (A.2) is similar to the quadratic equation (15) that determines the value of n_1^h. In fact these two quadratic equations differ only in the coefficient multiplying the squared term. Let us define these two quadratic functions:

$$F(n)=an^2-bn+\delta,$$
$$G(n)=n^2-bn+\delta.$$

Figure A.1(b) plots F and G. First, F lies below G because $a<1$. Second, $G(n_1^h)=0$ since n_1^h is defined by (15). Since F lies beneath G, this implies that $F(n_1^h)<0$. From (A.5), however, n_0^{lh} is negative. Examining (19), it is clear n_0^{lh} is positive. We have reached the desired contradiction. Note that assuming $n_0^{hh}>2b$ will not reverse the sign of n_0^{lh}. □

Proof of Lemma 4. It is more convenient to study $V(n_0) = (n_0^2 - bn_0 + \delta)[n_0 \exp(-n_0)]$, a monotone transformation of the utility function where we drop terms involving n_1, since they are identical across the equilibria which we consider. Note,

$$V(b) = b\delta \exp(-b).$$

Therefore the inequality holds if

$$\delta b \exp(-b) > \left[n_0^2 - bn_0 + \delta\right]\left(n_0 \exp(-n_0)\right). \tag{A.3}$$

The function $n_0 \exp(-n_0)$ is monotonically increasing in n_0 for $n_0 < 1$. Since $b > n_0$, (A.3) certainly holds if

$$\delta > n_0^2 - bn_0 + \delta$$

holds. The above inequality holds since $b > n_0$. □

Proof of Theorem 1. From the final government's problem, $n_1 = n_1^h$. Since the time-zero government may only select from equilibrium choices of n_0 that are time consistent, the two possibilities are n_0^{lh}, n_0^{hh}. Combining Lemmas 2 and 4, we know that $U(n_0, n_1) > U(x, n_1)$ for all $n_0 \in [b, 2b)$ and $x < b$. From Lemma 3(i), $n_0^{lh} < b/2$. This implies $U(n_0, n_1^h) > U(n_0^{lh}, n_1^h)$ for $n_0 \in [b, 2b)$. Lemmas 3(ii) and 3(iii) guarantee that $n_0^{hh} \in [b, 2b)$ and the proof is complete. □

Proof of Lemma 5. (i) This inequality clearly holds by inspecting (18) and (19).

(ii) This inequality clearly holds by inspecting (18) and (19).

(iii) From (18), the definition of n_0^{hl}, the inequality holds as long as

$$\frac{b}{2} + \frac{1}{2}\Delta\left(n_1^l\right) + \frac{1}{2}\sqrt{\left(b + \Delta\left(n_1^l\right)\right)^2 - 4\delta} > b.$$

Simplifying,

$$\sqrt{\left(b + \Delta\left(n_1^l\right)\right)^2 - 4\delta} > b - \Delta\left(n_1^l\right).$$

Squaring both sides and simplifying,

$$b\Delta\left(n_1^l\right) > \delta.$$

From the definition of $\Delta(n_1^l)$, this becomes

$$an_1^l + \frac{\delta}{n_1^l} > \frac{\delta}{b}.$$

We know that $0 < n_1^l < b$ which guarantees that the inequality holds.

(iv) Proof by contradiction. Assume $n_0^{hl} \geqslant 2b$. From (18) and (19), note that $n_0^{ll} + n_0^{hl} = 2y$ where $y = \frac{1}{2}[b + \delta/n_1^l + an_1^l]$. Then,

$$n_0^{ll} = b + \frac{\delta}{n_1^l} + an_1^l - n_0^{hl}. \tag{A.4}$$

We will show that $n_0^{hl} \geqslant 2b$ implies $n_0^{ll} < 0$. From (A.4), it is clear that if $n_0^{hl} = 2b$ implies $n_0^{ll} < 0$, then $n_0^{hl} > 2b$ will also imply $n_0^{ll} < 0$. Letting $n_0^{hl} = 2b$, we have

$$n_0^{ll} = -b + \frac{\delta}{n_1^l} + an_1^l.$$

Since we are interested in establishing the sign of n_0^{ll} under the contradiction, we can multiply the above expression by n_1^l. Then,

$$\operatorname{sgn}\bigl[n_0^{ll}\bigr] = \operatorname{sgn}\bigl[a\bigl(n_1^l\bigr)^2 - bn_1^l + \delta\bigr]. \tag{A.5}$$

The quadratic term inside the brackets of the right-hand side of (A.5) is similar to the quadratic equation (15) that determines the value of n_1^l. In fact these two quadratic equations differ only in the coefficient multiplying the squared term. Let us define these two quadratic functions:

$$F(n) = an^2 - bn + \delta,$$
$$G(n) = n^2 - bn + \delta.$$

Figure A.1(b) plots F and G. First, F lies below G because $a < 1$. Second, $G(n_1^l) = 0$ since n_1^l is defined by (15). Since F lies beneath G, this implies that $F(n_1^l) < 0$. From (A.5), however, n_0^{ll} is negative. Examining (19), n_0^{ll} is positive. We have reached the desired contradiction. Note that assuming $n_0^{hl} > 2b$ will not reverse the sign of n_0^{ll}. □

Proof of Theorem 2. Let U_0^{ij} define the time-zero household utility if time-zero labor is i and time-one labor is j. The proof consists of three parts: (i) $U_0^{hh} > U_0^{lh}$; (ii) $U_0^{hl} > U_0^{ll}$; (iii) $U_0^{hl} > U_0^{hh}$.

(i) Lemmas 2 and 4 compare time-zero utility as n_0 varies holding n_1 fixed. This makes Lemmas 2 and 4 appropriate for comparing allocations (n_0^{hh}, n_1^h) and (n_0^{lh}, n_1^h). Combining Lemmas 2 and 4, we know that $U(n_0, n_1^h) > U(x, n_1^h)$ for all $n_0 \in [b, 2b)$ and $x < b$. There are two cases. First, if $n_0^{lh} < b$, then $U_0^{hh} > U_0^{lh}$ by Lemmas 5(iii) and 5(iv), which establishes that $n_0^{hh} \in [b, 2b)$. Second, if $n_0^{lh} > b$, then $U_0^{hh} > U_0^{lh}$ by Lemmas 5(i), 5(iii) and 5(iv), because $U(n_0, n_1^h)$ is increasing in n_0 for all $n_0 \in [b, 2b)$.

(ii) This inequality is derived as part of the proof of Theorem 1.

(iii) $U_0^{hl} > U_0^{hh}$ is equivalent to

$$k_1^{hl}\bigl(n_0^{hl}\exp\bigl(-n_0^{hl}\bigr)\bigr)\bigl(n_1^l\exp\bigl(-n_1^l\bigr)\bigr) > k_1^{hh}\bigl(n_0^{hh}\exp\bigl(-n_0^{hh}\bigr)\bigr)\bigl(n_1^h\exp\bigl(-n_1^h\bigr)\bigr).$$

Let $n_0^{hj} = \phi^j/(2n_1^j)$ for $j = l, h$. The above inequality is equivalent to

$$\left(\frac{k_1^{hl}\phi^l}{k_1^{hh}\phi^h}\right)\exp\bigl(n_1^h - n_1^l\bigr) > \exp\bigl(n_0^{hl} - n_0^{hh}\bigr). \tag{A.6}$$

Next, it is straightforward to show that

$$n_1^h - n_1^l = \sqrt{b^2 - 4\delta},$$
$$n_0^{hl} - n_0^{hh} = \frac{1}{2}\Bigl[b\sqrt{b^2 - 4\delta} + \sqrt{\bigl(b + \Delta\bigl(n_1^l\bigr)\bigr)^2 - 4\delta} - \sqrt{\bigl(b + \Delta\bigl(n_1^h\bigr)\bigr)^2 - 4\delta}\Bigr]. \tag{A.7}$$

Using (A.7), we have an expression for the RHS of (A.6):

$$\exp\left(n_0^{hl} - n_0^{hh}\right) = \exp\left(b\sqrt{b^2 - 4\delta}\right)^{1/2} \left[\frac{\exp(\sqrt{(b + \Delta(n_1^l))^2 - 4\delta}\,)}{\exp(\sqrt{(b + \Delta(n_1^h))^2 - 4\delta}\,)}\right]^{1/2}.$$

Simplifying,

$$\exp\left(n_0^{hl} - n_0^{hh}\right) = \exp\left(b\sqrt{b^2 - 4\delta}\right)^{1/2} \left\{\exp\left[\left(\Delta^l\right)^2 - \left(\Delta^h\right)^2 + 2b\left(\Delta^l - \Delta^h\right)\right]\right\}^{1/4}.$$

This simplifies even further:

$$\exp\left(n_0^{hl} - n_0^{hh}\right) = \exp\left(b\sqrt{b^2 - 4\delta}\right)^{1/2} \left\{\exp\left[b^2(4 - b)\sqrt{b^2 - 4\delta}\right]\right\}^{1/4}.$$

The inequality (A.6) becomes

$$\left(\frac{k_1^{hl}\phi^l}{k_1^{hh}\phi^h}\right)\exp\left(b^2 - 4\delta\right)^{1/2} > \exp\left(b\sqrt{b^2 - 4\delta}\right)^{1/2} \left\{\exp\left[b^2(4 - b)\sqrt{b^2 - 4\delta}\right]\right\}^{1/4}.$$

This completes the proof. □

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