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Reliability Computation Using Logistic and Extreme-Value Distributions

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ABSTRACT

The estimation of the reliability, R, where R = P(Y < X), has been considered in the literature in both distribution-free and parametric cases. In particular, when X and Y are assumed to be independent identically distributed (iid) random variables, R has been extensively studied in the literature using various types of distribution functions. In this paper, utilizing the Maximum Likelihood Estimator (MLE), considering logistic and extreme-value type I distributions, estimation of the unknown parameters of these distributions are discussed and estimation of R is given. The effect of sample size and values of ratios of parameters to different estimates of R in different cases is also studied. The MSE analysis is carried on the assumed distributions.

KEYWORDS: Reliability; Logistic; Extreme-Value; Distributions; MLE

Mathematics Subject Classifications: 62D05; 62P10

Journal of Economic Literature (JEL): C13

1. INTRODUCTION AND BACKGROUND

The product reliability seems, finally, becoming the top priority for the third millennium and a technically sophisticated customer. Manufacturers and all other producing entities are sharpening their tools to satisfy such customer. Estimation of the reliability has become a concern for many quality control professionals and statisticians. Let Y represents the random value of a stress (or supply) that a device (or a component) will be subjected to in service, and X represents the strength (or demand) that varies from product to product in the population of devices. The device fails at the instant that the stress applied to it exceeds the strength and functions successfully whenever $X > Y$. Then, the reliability (or the measure of reliability) R is defined as $P(Y < X)$, i.e., the probability that a randomly selected device functions successfully.

Algebraic forms of R for different distributions have been studied in the literature. Among distributions considered are normal, exponential, gamma, Weibull, Pareto, and recently extreme value family by

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Nadarajah, 2003. In all cases, X and Y are assumed to be independent random variables. If $H_Y(y)$ and $f_X(x)$ are the *cdf* and *pdf* of Y and X, respectively, then it is well known that

$$R = P(Y < X) = \int_{-\infty}^{\infty} H_Y(z) f_X(z) dz . \tag{1}$$

The estimation of R is a very common concern in statistical literature in both distribution-free and parametric cases. Enis and Geisser, 1971, studied Bayesian approach to estimate R. Different distributions have been assumed for the random variables X and Y. Downton, 1973, and Church and Harris, 1970, have discussed the estimation of R in the normal and gamma cases, respectively. Studies of stress-strength model and its generalizations have been gathered in Kotz et al., 2003. Gupta and Lvin, 2005, studied the monotonicity of the failure rate and mean residual life function of the generalized log-normal distribution.

Out of 12 different forms of *cdf* Burr, 1942, introduced for modeling survival data, Type X and Type XII have received an extensive attention. Awad and Gharraf, 1986, used the Burr type X model to simulate comparison of three estimates for R: the minimum variance unbiased, the maximum likelihood, and Bayes estimators. They also studied the sensitivity of the Bayes estimator to the prior parameters. Ahmad et al., 1997, used the same assumptions except they further assumed that the scale parameters are known. Then, they used maximum likelihood, Bayes, and empirical techniques to deal with the estimation of R in such a case. They compared the three methods of estimation using the Monte-Carlo simulation. Additionally, they presented comparison among the three estimators and some characterization of the distribution. The first characteristic they studied was based on the recurrence relationship between two successively conditional moments of a certain function of the random variable, whereas the second was given by the conditional variance of the same function. Surles and Padgett, 1998, introduced the scaled Burr Type X distribution and name it as scaled Burr Type X or the generalized Rayleigh distribution. They considered the inference on R when X and Y are independently distributed Burr Type X random variables and discussed the existing and new results on estimation of R and, using *MLE,* introduced a significance test for R. They also presented Bayesian inference on R when the parameters are assumed to have independent gamma distribution. They, further, offered an algorithm for finding the highest posterior density interval for R, and, using different methods of estimation, calculated the value of R under the assumed distributions.

Recently, the effect of the sample size and the values of distribution parameters on the estimation of R have appeared in the literature. Kundu and Gupta, 2005, considered estimation of R when X and Y are assumed to be two independent generalized exponential distributions (GE) with different shape parameters but having the same scale parameters. (The GE is a sub-model of the exponential Weibull distribution. The two-parameter GE is an alternative to the two-parameter gamma, Weibull and log-normal distributions.) They obtained the maximum likelihood estimator of R, using the Monte Carlo simulation as their preferred method to compare different confidence intervals. Raqab and Kundu, 2005, considered the estimation of R, when X and Y are two independent scaled Burr Type X distributions with the same scale parameters. They used *MLE* and its asymptotic distribution to construct an asymptotic confidence interval of *R*. Using Monte Carlo simulation, they compared different methods used and corresponding confidence intervals. They also analyzed a data set for illustration purpose. Kundu and Gupta, 2006, through simulation, compared the different estimate of *R*. They also compared their results with the Bayes' estimates.

Howlader and Weiss, 1989a,b, considered the Bayes estimators of the reliability of the logistic distribution. Squared-error and log-odds error loss functions were used. Nadarajah, 2003, considered the class of extreme-value distributions (including the Pareto distribution) and derived the corresponding forms for the reliability *R* by utilizing special functions and assuming the two parameters families.

In this paper we will find an estimate for *R* and study the effect of the sample sizes and parameters' ratio on estimating *R*, when *X* and *Y* are assuming the logistic and extreme-value Type I distributions. In section 2, we will consider the logistic distribution as the underlying assumption for the stress *Y* and strength *X*, simultaneously. In section 3, we will consider extreme-value distribution as the underlying assumption. *MSE* computation for both distributions, when parameters are estimated, is presented in section 4. Conclusions and recommendations are given in section 5.

2. LOGISTIC PROBABILITY DISTRIBUTION FOR *Y* AND *X*

Generally *cdf* and *pdf* of a two-parameter logistic distribution are given, respectively, as:

$$F_X(x) = \frac{1}{1+e^{-\frac{x-\alpha}{\beta}}}, \quad f_X(x) = \frac{e^{-\frac{x-\alpha}{\beta}}}{\beta\left(1+e^{-\frac{x-\alpha}{\beta}}\right)^2}, \quad -\infty < x < \infty, \ -\infty < \alpha < \infty, \ \beta > 0,$$

where α is the location parameter, β is the scale parameter.

We consider the unknown-one-parameter logistic distributions for both *X* and *Y*, i.e., we assume that the location parameter α is 0. Hence, *cdf* and *pdf* are, respectively, as follow:

$$cdf\text{: } H_Y(y) = \frac{1}{1+e^{-\frac{y}{\beta_1}}}, \qquad pdf\text{: } h_Y(y) = \frac{e^{-\frac{y}{\beta_1}}}{\beta_1\left(1+e^{-\frac{y}{\beta_1}}\right)^2}, \quad -\infty < y < \infty, \quad \beta_1 > 0. \tag{2}$$

$$cdf\text{: } F_X(x) = \frac{1}{1+e^{-\frac{x}{\beta_2}}}, \quad pdf\text{: } f_X(x) = \frac{e^{-\frac{x}{\beta_2}}}{\beta_2\left(1+e^{-\frac{x}{\beta_2}}\right)^2}, \ -\infty < x < \infty, \quad \beta_2 > 0. \tag{3}$$

Theorem 2.1.

Assuming *X* and *Y* independent random variables with logistic probability distribution, we have

$$R(\beta_1,\beta_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{n+m}}{1+\frac{m}{n+1}\frac{\beta_2}{\beta_1}}. \tag{4}$$

Proof

Based on the assumption of the Theorem 2.1, it is easy to see that

$$R(\beta_1,\beta_2) = P(Y < X) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} \frac{e^{-\frac{y}{\beta_1}}}{\beta_1\left(1+e^{-\frac{y}{\beta_1}}\right)^2} \frac{e^{-\frac{x}{\beta_2}}}{\beta_2\left(1+e^{-\frac{x}{\beta_2}}\right)^2} dydx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{x}{\beta_2}}}{\beta_2\left(1+e^{-\frac{x}{\beta_2}}\right)^2} \frac{1}{\left(1+e^{-\frac{y}{\beta_1}}\right)\Bigg|_{-\infty}^{x}} dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{x}{\beta_2}}}{\beta_2\left(1+e^{-\frac{x}{\beta_2}}\right)^2} \frac{1}{\left(1+e^{-\frac{x}{\beta_1}}\right)} dx \ . \tag{5}$$

Let $z = e^{-\frac{x}{\beta_2}}$. Then, $dz = -\frac{1}{\beta_2} e^{-\frac{x}{\beta_2}} dx$, and $dx = -\beta_2 e^{\frac{x}{\beta_2}} dz = -\beta_2 \frac{1}{z} dz$. When $x = -\infty$, $z = \infty$; when $x = \infty$, $z = 0$. Then, from (5) we will have:

$$R(\beta_1,\beta_2) = -\int_0^{\infty} \frac{z}{\beta_2(1+z)^2} \frac{1}{\left(1+z^{\frac{\beta_2}{\beta_1}}\right)} (-\beta_2 z^{-1} dz)$$

$$= \int_0^\infty \frac{1}{(1+z)^2} \frac{1}{\left(1+z^{\frac{\beta_2}{\beta_1}}\right)} dz \ .$$

(6)

Now,

$$\frac{1}{(1+z)^2} = -\frac{d}{dz}\frac{1}{1+z} = -\frac{d}{dz}\sum_{n=0}^{\infty}(-1)^n z^n = \frac{d}{dz}\sum_{n=0}^{\infty}(-1)^{n+1} z^n$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{d}{dz}z^n = \sum_{n=1}^{\infty}(-1)^{n+1} nz^{n-1} = \sum_{n=1}^{\infty}(-1)^{n+1} nz^{n-1} = \sum_{n=0}^{\infty}(-1)^n (n+1)z^n \ .$$

Using the ratio test, we will have

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)z^{n+1}}{(-1)^n(n+1)z^n}\right| = |z| < 1.$$

(7)

Since for finite values of *z* we have $z = e^{-\frac{x}{\beta_2}} > 0$, it follows from (7) that $0 < z < 1$. Thus, the series $\sum_{n=0}^{\infty}(-1)^n(n+1)z^n$ converges for $0 < z < 1$.

Also let $u = z^{\frac{\beta_2}{\beta_1}}$. Then,

$$\frac{1}{\left(1+z^{\frac{\beta_2}{\beta_1}}\right)} = \frac{1}{(1+u)} = \sum_{n=0}^{\infty}(-1)^n u^n = \sum_{n=0}^{\infty}(-1)^n\left(z^{\frac{\beta_2}{\beta_1}}\right)^n = \sum_{n=0}^{\infty}(-1)^n z^{\frac{\beta_2}{\beta_1}n}.$$

However,

$$\left|z^{\frac{\beta_2}{\beta_1}}\right| < 1 \Rightarrow |z|^{\frac{\beta_2}{\beta_1}} < 1 \Rightarrow \frac{\beta_2}{\beta_1}\ln|z| < 0 \Rightarrow \ln|z| < 0 \Rightarrow 0 < z < 1.$$

Therefore, for 0 < *z* <1 we have

$$\frac{1}{\left(1+e^{-\frac{x}{\beta_2}}\right)^2}\frac{1}{\left(1+e^{-\frac{x}{\beta_1}}\right)} = \sum_{n=0}^{\infty}(-1)^n(n+1)z^n\sum_{m=0}^{\infty}(-1)^m z^{\frac{\beta_2}{\beta_1}m}$$

$$= \sum_{m=0}^{\infty}(-1)^m z^{\frac{\beta_2}{\beta_1}m} - 2z\sum_{m=0}^{\infty}(-1)^m z^{\frac{\beta_2}{\beta_1}m} + 3z^2\sum_{m=0}^{\infty}(-1)^m z^{\frac{\beta_2}{\beta_1}m} + \cdots$$

$$= \sum_{m=0}^{\infty} (-1)^m z^{\frac{\beta_2}{\beta_1}m} - 2\sum_{m=0}^{\infty} (-1)^m z^{1+\frac{\beta_2}{\beta_1}m} + 3\sum_{m=0}^{\infty} (-1)^m z^{2+\frac{\beta_2}{\beta_1}m} + \cdots$$

Now let $\theta = \dfrac{\beta_2}{\beta_1}$. Then, from (5) we will have:

$$R(\theta) = \int_0^1 \left[\sum_{m=0}^{\infty} (-1)^m z^{\theta m} - 2\sum_{m=0}^{\infty} (-1)^m z^{1+\theta m} + 3\sum_{m=0}^{\infty} (-1)^m z^{2+\theta m} + \cdots \right] dz$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m z^{1+\theta m}}{1+m\theta}\Bigg|_0^1 - 2\sum_{m=0}^{\infty} \frac{(-1)^m z^{2+\theta m}}{2+m\theta}\Bigg|_0^1 + 3\sum_{m=0}^{\infty} \frac{(-1)^m z^{3+\theta m}}{3+m\theta}\Bigg|_0^1 + \cdots$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{1+m\theta} - 2\sum_{m=0}^{\infty} \frac{(-1)^m}{2+m\theta} + 3\sum_{m=0}^{\infty} \frac{(-1)^m}{3+m\theta} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1) \sum_{m=0}^{\infty} \frac{(-1)^m}{(n+1)+m\theta} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+m}}{1+\dfrac{m}{n+1}\theta}.$$

Hence,

$$R(\beta_1, \beta_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+m}}{1+\dfrac{m}{n+1}\dfrac{\beta_2}{\beta_1}}, \tag{8}$$

and this proves Theorem 2.1.

Theorem 2.2. ***Computation of the MLE of R in case of the Logistic Distribution***

The maximum likelihood estimate (*MLE*) for *R* in case of logistic distribution for *X* and *Y* is

$$\widehat{R}\left(\widehat{\beta}_1, \widehat{\beta}_2\right) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+l}}{1+\dfrac{l}{k+1}\dfrac{\widehat{\beta}_2}{\widehat{\beta}_1}}.$$

Proof

To compute the *MLE* of *R* when *X* and *Y* are independent logistic random variables, we will obtain the *MLE* of each β_1 and β_2 (parameters of the one-parameter logistic distributions). Thus, suppose $Y_1, \cdots, Y_M$ and $X_1, \cdots, X_N$ are random samples from one-parameter-logistic distributions with parameters β_1 and β_2, respectively. We let $y_1, \cdots, y_m$ and $x_1, \cdots, n_n$ denote values of these random samples, respectively. Hence, the joint density function for each sample is the product of their marginal density functions, i.e.,

$$h_Y\left(y_1,\cdots,y_M;\beta_1\right)=\frac{e^{-\frac{y_1}{\beta_1}}}{\beta_1\left(1+e^{-\frac{y_1}{\beta_1}}\right)^2}\cdots\frac{e^{-\frac{y_M}{\beta_1}}}{\beta_1\left(1+e^{-\frac{y_M}{\beta_1}}\right)^2}=\frac{e^{-\frac{1}{\beta_1}\sum_{m=1}^{M}y_m}}{\beta_1^M\prod_{m=1}^{M}\left(1+e^{-\frac{y_m}{\beta_1}}\right)^2}. \tag{9}$$

$$f_X\left(x_1,\cdots,x_N;\beta_2\right)=\frac{e^{-\frac{x_1}{\beta_2}}}{\beta_2\left(1+e^{-\frac{x_1}{\beta_2}}\right)^2}\cdots\frac{e^{-\frac{x_N}{\beta_2}}}{\beta_2\left(1+e^{-\frac{x_N}{\beta_2}}\right)^2}=\frac{e^{-\frac{1}{\beta_2}\sum_{n=1}^{N}x_i}}{\beta_2^N\prod_{n=1}^{N}\left(1+e^{-\frac{x_n}{\beta_2}}\right)^2}. \tag{10}$$

Since the random variables *Y* and *X* are independent, their joint distribution is the product of their density functions. On the other hand the *MLE*'s of their parameters can be obtained by taking derivative of the natural logarithm of each with respect to their respective parameter being equal to zero. Therefore, from (9) and (10) we will have the followings:

$$\begin{aligned}\ln h_Y\left(y_1,\cdots,y_M;\beta_1\right)&=\ln\left(e^{-\frac{1}{\beta_1}\sum_{m=1}^{M}y_m}\right)-\ln\left[\beta_1^M\prod_{m=1}^{M}\left(1+e^{-\frac{y_m}{\beta_1}}\right)^2\right]\\&=-\frac{1}{\beta_1}\sum_{m=1}^{M}y_m-M\ln\beta_1-2\sum_{m=1}^{M}\ln\left(1+e^{-\frac{y_m}{\beta_1}}\right).\end{aligned} \tag{11}$$

Similarly,

$$\ln f_X\left(x_1,\cdots,x_N;\beta_2\right)=-\frac{1}{\beta_2}\sum_{n=1}^{N}x_n-N\ln\beta_2-2\sum_{n=1}^{N}\ln\left(1+e^{-\frac{x_n}{\beta_2}}\right). \tag{12}$$

Hence,

$$\frac{\partial\ln h_Y}{\partial\beta_1}=\frac{1}{\beta_1^2}\sum_{m=1}^{M}y_m-\frac{M}{\beta_1}+\frac{2}{\beta_1^2}\sum_{m=1}^{M}\frac{y_m e^{-\frac{y_m}{\beta_1}}}{1+e^{-\frac{y_m}{\beta_1}}}=0 \tag{13}$$

and

$$\frac{\partial\ln f_X}{\partial\beta_2}=\frac{1}{\beta_2^2}\sum_{n=1}^{N}x_n-\frac{N}{\beta_2}+\frac{2}{\beta_2^2}\sum_{n=1}^{N}\frac{x_n e^{-\frac{x_n}{\beta_2}}}{1+e^{-\frac{x_n}{\beta_2}}}=0. \tag{14}$$

Now, we focus on the right hand side of (13), simply:

$$\frac{1}{\beta_1^2}\sum_{m=1}^{M}y_m-\frac{M}{\beta_1}+\frac{2}{\beta_1^2}\sum_{m=1}^{M}\frac{y_m e^{-\frac{y_m}{\beta_1}}}{1+e^{-\frac{y_m}{\beta_1}}}=0. \tag{15}$$

Multiplying both side of (15) by $\frac{\beta_1^2}{M}$, we obtain:

$$\beta_1 = \overline{y} + \frac{2}{M}\sum_{m=1}^{M}\frac{y_m e^{-\frac{y_m}{\beta_1}}}{1+e^{-\frac{y_m}{\beta_1}}}$$

or

$$\beta_1 = \overline{y} + \frac{2}{M}\sum_{m=1}^{M}\frac{y_m}{1+e^{\frac{y_m}{\beta_1}}}, \tag{16}$$

where $\overline{y} = \frac{1}{M}\sum_{m=1}^{M} y_m$.

Equation (16) is a nonlinear equation in β_1. To solve it, we use iterative approximation. Hence, β_1 can be found as the solution of the equation

$$\beta_1 = f(\beta_1) , \tag{17}$$

where

$$f(\beta_1) = \overline{y} + \frac{2}{M}\sum_{m=1}^{M}\frac{y_m e^{-\frac{y_m}{\beta_1}}}{1+e^{-\frac{y_m}{\beta_1}}}.$$

Then,

$$f\left(\beta_{1(k)}\right) = \beta_{1(k+1)}, \tag{18}$$

where $\beta_{1(k)}$ means the k^{th} iteration of $\widehat{\beta}_1$. The iterating process may stop as $\left|\beta_{1(k+1)} - \beta_{1(k)}\right| < \varepsilon_1$, happens, where ε_1 is a small positive number. Hence,

$$\widehat{\beta}_1 = \overline{y} + \frac{2}{M}\sum_{m=1}^{M}\frac{y_m}{1+e^{\frac{y_m}{\widehat{\beta}_1}}} . \tag{19}$$

Similarly, we will have $g\left(\beta_{2(l)}\right) = \beta_{2(l+1)}$, where $\beta_{2(l)}$ means the l^{th} iteration of $\widehat{\beta}_2$. Thus,

$$\widehat{\beta}_2 = \overline{x} + \frac{2}{N}\sum_{n=1}^{N}\frac{x_n}{1+e^{\frac{x_n}{\widehat{\beta}_2}}} , \tag{20}$$

where $\overline{x} = \frac{1}{N}\sum_{n=1}^{N} x_n$.

The iterating process in this case may stop as $\left|\beta_{2(l)} - \beta_{2(l+1)}\right| < \varepsilon_2$ happens, where ε_2 is a small positive number. Having found $\widehat{\beta}_1$ and $\widehat{\beta}_2$, we will have

$$\widehat{R}\left(\widehat{\beta}_1,\widehat{\beta}_2\right)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{k+l}}{1+\frac{l}{k+1}\frac{\widehat{\beta}_2}{\widehat{\beta}_1}}, \tag{21}$$

and this proves the Theorem 2.2.

The logistic distribution assumes that the scale parameter, β, is positive. However, if β approaches infinity, the *cdf* will be equal to 1/2, a degenerated distribution that is not desirable for us. Thus, we assume that both β_1 and β_2 are positive finite numbers.

Now we will consider infinite sample sizes for the *MLE* of β_1 and β_2, i.e., $M \to \infty$ and $N \to \infty$.

Theorem 2.3. ***Asymptotic Behavior***

(i) For $M = \infty$, as $m \to M$ if y_m approaches either zero or infinity, then

$$\widehat{\beta}_1 \to \overline{Y}. \tag{22}$$

(ii) For $N = \infty$, as $n \to N$ if x_n approaches either zero or infinity, then

$$\widehat{\beta}_2 \to \overline{X}. \tag{23}$$

(iii) For $M = \infty$ and $N = \infty$ we have

$$\widehat{R}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{k+l}}{1+\frac{l}{k+1}\frac{\overline{X}}{\overline{Y}}}. \tag{24}$$

Proof

(i) We will focus on $\widehat{\beta}_1$ in (19). Similar results will be true for $\widehat{\beta}_2$ in (20). As we need to choose a random sample, no matter how large *M* is, it will be a finite number. Thus, from (19), $\lim_{M\to\infty}\widehat{\beta}_1=\overline{Y}$ if $\sum_{m=1}^{\infty}\frac{y_m}{1+e^{\frac{y_m}{\beta_1}}}$ is finite, i.e., if it is a convergent series. To be so, we need to show that

$$\lim_{m\to\infty}\left(\frac{y_m}{1+e^{\frac{y_m}{\beta_1}}}\right)=0. \tag{25}$$

Now, $\widehat{\beta}_1$ is finite because β_1 is. On the other hand, no matter what the value of *m* is, the value of y_m, although finite, is not under our control since it is chosen randomly. Therefore, the limit in (25) is zero either if y_m is closed to zero and $1+e^{\frac{y_m}{\widehat{\beta}_1}}$ is not, or $1+e^{\frac{y_m}{\widehat{\beta}_1}}$ is extremely larger than y_m to make the ratio zero. If y_m is closed to zero, then $1+e^{\frac{y_m}{\widehat{\beta}_1}} > 0$ since $\widehat{\beta}_1 > 0$, and thus, the ratio is zero. On the other hand, if y_m is extremely large, then $1+e^{\frac{y_m}{\widehat{\beta}_1}}$ will approach infinity. Thus, the ratio will approach zero. This completes part (i) of Theorem 2.1.

To prove part (ii), we reason as we did for part (i). Part (iii) follows from parts (i) and (ii). However, we have to note that (24) is a double alternating series. Since denominator increases as *k* and *l*, terms are decreasing. Thus, the series is convergent and $\widehat{R}$ is finite.

3. EXTREME-VALUE PROBABILITY DISTRIBUTION FOR *Y* AND *X*

In this section we assume that both *X* and *Y* are independent and each has an unknown-one-parameter extreme-value type 1 (or double exponential or Gumbel-type) distributions. Without loss of any generality, we assume that the location parameter is zero for each *X* and *Y*. Hence, the *cdf* and *pdf* for *Y* and *X* are, respectively, as follows:

$$F_Y(y;\theta_1) = e^{-e^{-y/\theta_1}}, \quad f(y;\theta_1) = \frac{1}{\theta_1} e^{-y/\theta_1} e^{-e^{-y/\theta_1}}, -\infty < y < \infty, \quad \theta_1 > 0. \tag{26}$$

$$H_X(x;\theta_2) = e^{-e^{-x/\theta_2}}, \quad h(x;\theta_2) = \frac{1}{\theta_2} e^{-x/\theta_2} e^{-e^{-x/\theta_2}}, -\infty < x < \infty, \quad \theta_2 > 0. \tag{27}$$

Theorem 3.1.

If

$$k\frac{\theta_2}{\theta_1} < 1, \ k = 0,1,2,\cdots,$$

Then,

$$R(\theta_1,\theta_2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma\left(k\frac{\theta_2}{\theta_1}+1\right).$$

Proof

We assume that *X* and *Y* are independent random variables. Then, it is trivial that

$$R = P(Y < X) = \int_{-\infty}^{\infty}\int_{0}^{x}\frac{1}{\theta_1}e^{-\frac{y}{\theta_1}}e^{-e^{-\frac{y}{\theta_1}}}\frac{1}{\theta_2}e^{-\frac{x}{\theta_2}}e^{-e^{-\frac{x}{\theta_2}}}dydx\ . \tag{28}$$

To simplify (28), let $u = e^{-\frac{y}{\theta_1}}$. Then, $du = -\frac{1}{\theta_1}e^{-\frac{y}{\theta_1}}dy$, and $dy = -\theta_1 e^{\frac{y}{\theta_1}}du = -\theta_1\frac{1}{u}du$. Similarly, let $v = e^{-\frac{x}{\theta_2}}$. Then, $dv = -\frac{1}{\theta_2}e^{-\frac{x}{\theta_2}}dx$, and $dx = -\theta_2 e^{\frac{x}{\theta_2}}dv = -\theta_2\frac{1}{v}dv$.

Note that $v = e^{-\frac{x}{\theta_2}}$ implies that $\ln v = -\frac{x}{\theta_2}$ or $\frac{x}{\theta_2} = \ln\frac{1}{v}$.

Also, when $x = -\infty$, $v = \infty$; when $x = \infty$, $v = 0$; and when $y = -\infty$, $u = \infty$; when $y = x$, $u = e^{-\frac{x}{\theta_1}} = e^{-\frac{\theta_2}{\theta_2}\frac{x}{\theta_1}} = e^{-\frac{\theta_2}{\theta_1}\frac{x}{\theta_2}} = e^{-\alpha\frac{x}{\theta_2}} = e^{-\alpha\ln\frac{1}{v}} = e^{\alpha\ln v} = e^{\ln v^{\alpha}} = v^{\alpha}$, where $\alpha = \frac{\theta_2}{\theta_1} > 0$. Thus,

$$\begin{aligned} R = P(Y < X) &= -\int_{0}^{\infty}(-1)\int_{v^{\alpha}}^{\infty}\frac{1}{\theta_1}ue^{-u}\frac{1}{\theta_2}ve^{-v}\left(-\theta_1\frac{1}{u}du\right)\left(-\theta_2\frac{1}{v}dv\right) \\ &= \int_{0}^{\infty}\int_{v^{\alpha}}^{\infty}e^{-u}e^{-v}dudv = \int_{0}^{\infty}-e^{-u}\Big)_{v^{\alpha}}^{\infty}e^{-v}dv = \int_{0}^{\infty}\left(-\left(0-e^{-v^{\alpha}}\right)\right)e^{-v}dv \end{aligned}$$

or

$$R(\alpha) = \int_{0}^{\infty}e^{-v^{\alpha}}e^{-v}dv\ . \tag{29}$$

Applying the series expansion of the exponential function, i.e., $e^{-u} = \sum_{k=0}^{\infty}\frac{(-1)^k u^k}{k!}$, on the first part of the integrand in (29), we obtain

$$\begin{aligned} R(\alpha) &= \int_{0}^{\infty}e^{-v}\left(\sum_{k=1}^{\infty}\frac{(-1)^k(v^{\alpha})^i}{k!}\right)dv = \int_{0}^{\infty}e^{-v}\left(\sum_{k=1}^{\infty}\frac{(-1)^k v^{k\alpha}}{k!}\right)dv \\ &= \int_{0}^{\infty}e^{-v}\left(1-\frac{v^{\alpha}}{1!}+\frac{v^{2\alpha}}{2!}-\frac{v^{3\alpha}}{3!}+\frac{v^{4\alpha}}{4!}-\cdots+\frac{(-1)^k v^{i\alpha}}{k!}-\cdots\right)dv \\ &= \int_{0}^{\infty}\left(\frac{e^{-v}}{0!}-\frac{v^{\alpha}e^{-v}}{1!}+\frac{v^{2\alpha}}{2!}e^{-v}-\frac{v^{3\alpha}}{3!}e^{-v}+\frac{v^{4\alpha}}{4!}e^{-v}-\cdots+\frac{(-1)^k v^{i\alpha}}{k!}e^{-v}-\cdots\right)dv \\ &= \int_{0}^{\infty}e^{-v}dv-\int_{0}^{\infty}v^{\alpha}e^{-v}dv+\frac{1}{2}\int_{0}^{\infty}v^{2\alpha}e^{-v}dv-\frac{1}{3!}\int_{0}^{\infty}v^{3\alpha}e^{-v}dv+\cdots-\frac{1}{k!}\int_{0}^{\infty}v^{k\alpha}e^{-v}dv+\cdots. \end{aligned}$$

(30)

Now, for $k = 0,1,2,\cdots$, let $k\alpha = \tau - 1$, i.e., $\tau = k\alpha + 1 = k\frac{\theta_2}{\theta_1}+1$. $\alpha = \frac{\theta_2}{\theta_1} > 0$ implies that $\tau > 1$ and

$$\int_0^\infty v^{k\alpha} e^{-v} dv = \int_0^\infty v^{\tau-1} e^{-v} dv = \Gamma(\tau) = \Gamma(k\alpha+1) = \Gamma\left(k\frac{\theta_2}{\theta_1}+1\right),\ k = 0,1,2,\cdots. \tag{31}$$

Hence, using (31), (30) can be rewritten as

$$R(\theta_1,\theta_2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma\left(k\frac{\theta_2}{\theta_1}+1\right), \tag{32}$$

as stated in the statement of the Theorem 3.1.

Now, since $0 \le R(\theta_1,\theta_2) \le 1$, the infinite series in (32) must converge. From the Euler formula for the gamma function (see Whittaker & Watson, 1965, p. 237) we have

$$\Gamma(z) = \frac{1}{z} \Pi_{n=1}^{\infty} \frac{(n+1)^z}{n^{z-1}(n+z)}, \quad |z| < 1. \tag{33}$$

Thus, letting

$$z = k\frac{\theta_2}{\theta_1} \tag{34}$$

in (32) and based on the difference equation satisfied by the gamma function, namely,

$$\Gamma(z+1) = z\Gamma(z), \tag{35}$$

(see Whittaker & Watson, 1965, p. 237), we have

$$R(\theta_1,\theta_2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[\Pi_{n=1}^{\infty} \frac{(n+1)^{k\frac{\theta_2}{\theta_1}}}{n^{k\frac{\theta_2}{\theta_1}-1}\left(n+k\frac{\theta_2}{\theta_1}\right)} \right]. \tag{36}$$

This is an alternative form for *R* stated in the Theorem 3.1.

To complete the proof of Theorem 3.1, note that condition in (33) and notation defined in (34) imply that

$$k\frac{\theta_2}{\theta_1} < 1\,,\; k = 0,1,2,\cdots. \tag{37}$$

Thus, (37) and (32) complete the proof.

Theorem 3.2. ***Computation of the MLE of R in case of Extreme –Value Distribution***

For $k\frac{\hat{\theta}_2}{\hat{\theta}_1} < 1$, $k = 0,1,2,\cdots$, we have:

$$\hat{R}\left(\hat{\theta}_1,\hat{\theta}_2\right) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma\left(k\frac{\hat{\theta}_2}{\hat{\theta}_1}+1\right).$$

Proof

To compute the *MLE* of *R*, we will find the *MLE* of both θ_1 and θ_2. Based on a random sample for each of *Y* and *X* of sizes *M* and *N*, respectively, we will have the following *MLE* estimators for θ_1 and θ_2, respectively,

$$\hat{\theta}_1 = \overline{Y} - \frac{\sum_{m=1}^{M} y_m e^{-y_m/\hat{\theta}_1}}{\sum_{m=1}^{M} e^{-y_m/\hat{\theta}_1}} \text{ and } \hat{\theta}_2 = \overline{X} - \frac{\sum_{n=1}^{N} x_n e^{-x_n/\hat{\theta}_2}}{\sum_{n=1}^{N} e^{-x_n/\hat{\theta}_2}}. \tag{38}$$

(See Johnson, Kotz, and Balakrishnan, 1995, p. 28) Since the likelihood equations (38) do not admit explicit solution, the estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ can be found numerically using methods such as iterative.

It is clear that both θ_1 and θ_2 being positive imply that both $\hat{\theta}_1$ and $\hat{\theta}_2$ have to be positive and, therefore, from (38) we must have $\frac{\sum_{m=1}^{M} y_m e^{-y_m/\hat{\theta}_1}}{\sum_{m=1}^{M} e^{-y_m/\hat{\theta}_1}} < \overline{Y}$ and $\frac{\sum_{n=1}^{N} x_n e^{-x_n/\hat{\theta}_2}}{\sum_{n=1}^{N} e^{-x_n/\hat{\theta}_2}} < \overline{X}$.

Thus, from (32) and (38) we will have

$$\hat{R}\left(\hat{\theta}_1,\hat{\theta}_2\right) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma\left(k\frac{\hat{\theta}_2}{\hat{\theta}_1}+1\right). \tag{39}$$

Similar condition to (37) holds for (39) as well. However, care should be taken since $\hat{\theta}_1$ and $\hat{\theta}_2$ are calculated rather than being chosen. Thus, if "infinity" is taken as a finite number, say 100, then, it suffices to choose θ_1 and θ_2 such that $\frac{\theta_2}{\theta_1} < \frac{1}{100}$ for (36) to make sense and (39) to give desirable value. This completes proof of Theorem 3.2.

Note that the bias and *MSE* of the estimation may be computed using (36) and (39).

4. NUMERICAL EXAMPLE

We choose 7 different values of each β_1 and β_2 in pairs, i.e., (10, 01), (03, 1.5), (3.3, 03), (05, 05), (06, 07), (07, 14), (08, 20) such that the ratio β_2 / β_1 are 0.10, 0.50, 0.9091, 1.0, 1.1667, 2.0, and 2.5. Computationally, we have chosen $\infty \equiv 1000,$ and 2000 in infinite series. To solve the functional equations for β_1 and β_2 by iteration method, we use 1000 iterations. For the iteration processes, we, begin with the initial values of β_1 and β_2 as the means of the logistic random samples found in Step 6 below, denoted by $\bar{Y}$ and $\bar{X}$.

To find *MLE*'s for β_1 and β_2, we solve the functional equations using iteration method. To assure the difference of consecutive terms approaching zero, we have chosen 1000 iterations that cause the difference less than $\varepsilon = 10^{-6}$.

We choose two random sample vectors, *Y*, *X* of small sizes (*m*, *n*), i.e., $Y = (Y_1, Y_2, \cdots Y_m)$ and $X = (X_1, X_2, \cdots X_n)$, such that the ratio of *m*/*n* takes the values: 0.1, 0.5, 0.75, 0.9, 1, 1,1, 1.11, 1.33, 1.5, 1.9, 2, 2.5 and 3.

To choose random samples from logistic distribution we take the following steps (first for *Y* and similarly for *X*):

Step 1. Choose a random sample of size, say *M*, from the uniform distribution (between 0 and 1), say $\{p_1, p_2, \cdots p_M\}$.

Step 2. Set each point of the random sample found in Step 1 equal to the logistic distribution. If *p* is such a point, then write

$$p = \frac{1}{1 + e^{-\frac{y}{\beta_1}}}. \tag{40}$$

Step 3. Solve (40) for *y*, i.e., $p + pe^{-\frac{y}{\beta_1}} = 1$, $pe^{-\frac{y}{\beta_1}} = 1 - p$, $e^{-\frac{y}{\beta_1}} = \frac{1-p}{p}$, $-\frac{y}{\beta_1} = \ln\left(\frac{1-p}{p}\right)$,

$y = -\beta_1 \ln\left(\frac{1-p}{p}\right)$, and, thus

$$y = \beta_1 \ln\left(\frac{p}{1-p}\right). \tag{41}$$

Step 4. The set of *y*'s found in Step 3 form a random sample of size *M* from the logistic distribution with parameter β_1. However, some of the sample points may be negative and are not desirable.

Step 5. Since β_1 (and β_2) must be positive, choose the maximum value from the set in Step 4.

Step 6. Repeat Step 1 through Step 4, say *L* times, to generate a logistic random sample of size *L*.

Step 7. Similarly, choose a random sample of size *N* to estimate β_2. Let *q* be the logistic point in Step 2 and find *x*, say, from (41), i.e.,

$$x = \beta_2 \ln\left(\frac{q}{1-q}\right). \tag{42}$$

x's found according to (42) will produce a logistic sample of size *N*. We follow Step 5 and Step 6 to have a logistic random sample of the same size, *L*, for estimating β_2.

The results of the computations are given in the tables below based on the ratio of *M*/*N* for both logistic and extreme-value distributions.

		Given *M* = 5, *N* = 50 *M*/*N* = 0.10			Given *M* = 10, *N* = 20 *M*/*N* = 0.50		
(β_1, β_2)	β_2 / β_1	$\hat{R}(\hat{\beta}_1, \hat{\beta}_2)$	Bias	*MSE* 1.0e-004*	$\hat{R}(\hat{\beta}_1, \hat{\beta}_2)$	Bias	*MSE* 1.0e-005*
(10,01)	0.1000	0.7471	0.0001	0.0277	0.7475	0.0004	0.0554
(03,1.5)	0.5000	0.7437	- 0.0004	0.0093	0.7446	0.0005	0.1681
(3.3,03)	0.9091	0.7504	0.0008	0.0768	0.7481	- 0.0015	0.6907
(05,05)	1.0000	0.7497	- 0.0009	0.0417	0.7504	- 0.0003	0.4575
(06,07)	1.1667	0.7532	0.0008	0.1281	0.7527	0.0004	0.1979
(07,14)	2.0000	0.7561	- 0.0013	0.0297	0.7568	- 0.0005	0.1538
(08,20)	2.5000	0.7579	- 0.0007	0.0113	0.7583	- 0.0003	0.1294

Table 1-LA: *Logistic, M/N* < 1.0 Table 1-LB: *Logistic, M/N* < 1.0

		Given M = 5, *N* = 50, *M*/*N* = 0.10			*Given M* = 10, *N* = 20, *M*/*N* = 0.50		
(θ_1,θ_2)	θ_2/θ_1	$\hat{R}(\hat{\theta}_1,\hat{\theta}_2)$	Bias	*MSE* 1.0e-005*	$\hat{R}(\hat{\theta}_1,\hat{\theta}_2)$	Bias	*MSE* 1.0e-005*
(100,01)	0.0100	0.3703	- 0.2633	0.0791	0.3700	- 0.2637	0.0303
(210,02)	0.0095	0.3696	- 0.2640	0.1007	0.3703	- 0.2633	0.1756
(330,03)	0.0091	0.3699	- 0.2636	0.0464	0.3695	- 0.2640	0.0443
(450,04)	0.0089	0.3700	- 0.2634	0.0623	0.3702	- 0.2633	0.1329
(650,05)	0.0077	0.3701	- 0.2632	0.0255	0.3696	- 0.2636	0.0399
(850,06)	0.0071	0.3695	- 0.2637	0.0982	0.3692	- 0.2640	0.0138
(1000,7)	0.0070	0.3694	- 0.2638	0.0114	0.3695	- 0.2637	0.0249

Table 1-EA: *Extreme-Value, M/N* < 1.0 Table 1-EB: *Extreme-Value, M/N* < 1.0

		Given M = 15, *N* = 20, *M*/*N* = 0.75			*Given M* = 09, *N* = 10, *M*/*N* = 0.90		
(β_1,β_2)	β_2/β_1	$\hat{R}(\hat{\beta}_1,\hat{\beta}_2)$	Bias 1.0e-003*	*MSE* 1.0e-005*	$\hat{R}(\hat{\beta}_1,\hat{\beta}_2)$	Bias 1.0e-003*	*MSE* 1.0e-005*
(10,01)	0.1	0.7466	- 0.4558	0.0346	0.7472	0.1790	0.2224
(03,1.5)	0.5	0.7442	0.0490	0.0880	0.7439	- 0.2237	0.0802
(3.3,03)	0.9091	0.7498	0.2838	0.4911	0.7495	- 0.0031	0.3674
(05,05)	1.0	0.7513	0.6571	0.2924	0.7504	- 0.2459	0.3032
(06,07)	1.1667	0.7516	- 0.7823	0.2803	0.7523	- 0.0476	0.1478
(07,14)	2.0	0.7572	- 0.1470	0.1030	0.7571	- 0.2645	0.3440
(08,20)	2.5	0.7587	0.0968	0.0287	0.7580	- 0.6000	0.1463

Table 2-LA: *Logistic, M/N* < 1.0 Table 2-LB: *Logistic, M/N* < 1.0

		Given M = 15, *N* = 20, *M*/*N* = 0.75			*Given M* = 09, *N* = 10, *M*/*N* = 0.90		
(θ_1,θ_2)	θ_2/θ_1	$\hat{R}(\hat{\theta}_1,\hat{\theta}_2)$	Bias	*MSE* 1.0e-005*	$\hat{R}(\hat{\theta}_1,\hat{\theta}_2)$	Bias	*MSE* 1.0e-006*
(100,01)	0.0100	0.3700	- 0.2636	0.0214	0.3700	- 0.2636	0.6181
(210,02)	0.0095	0.3700	- 0.2636	0.0728	0.3699	- 0.2637	0.3951
(330,03)	0.0091	0.3698	- 0.2637	0.0276	0.3700	- 0.2635	0.4026
(450,04)	0.0089	0.3700	- 0.2635	0.1436	0.3697	- 0.2637	0.3487
(650,05)	0.0077	0.3691	- 0.2641	0.0242	0.3696	- 0.2636	0.4046
(850,06)	0.0071	0.3692	- 0.2640	0.0099	0.3694	- 0.2638	0.2906
(1000,7)	0.0070	0.3695	- 0.2637	0.0359	0.3693	- 0.2639	0.6049

Table 2-EA: *Extreme-Value, M/N* < 1.0 Table 2-EB: *Extreme-Value, M/N* < 1.0

		Given M = 10, *N* = 10, *M*/*N* = 1.0			*Given M* = 30, *N* = 30, *M*/*N* = 1.0		
(β_1,β_2)	β_2/β_1	$\hat{R}(\hat{\beta}_1,\hat{\beta}_2)$	Bias 1.0e-003*	*MSE* 1.0e-004*	$\hat{R}(\hat{\beta}_1,\hat{\beta}_2)$	Bias 1.0e-003*	*MSE* 1.0e-005*
(10,01)	0.1000	0.7473	0.2941	0.0073	0.7470	- 0.0862	0.0596
(03,1.5)	0.5000	0.7447	0.6135	0.0319	0.7443	0.1410	0.0477
(3.3,03)	0.9091	0.7503	0.7558	0.0620	0.7493	- 0.2520	0.1633
(05,05)	1.0000	0.7497	- 0.8780	0.1255	0.7511	0.4641	0.3147
(06,07)	1.1667	0.7518	- 0.5223	0.0385	0.7525	0.0907	0.1735
(07,14)	2.0000	0.7576	0.2833	0.0087	0.7575	0.1462	0.0818
(08,20)	2.5000	0.7583	- 0.2726	0.0136	0.7580	- 0.5291	0.0843

Table 3-LA: *Logistic, M/N* = 1.0 Table 3-LB: *Logistic, M/N* = 1.0

Given M = 10, N = 10 M/N = 1.0					Given M = 30, N = 30 M/N = 1.0		
(θ_1, θ_2)	θ_2 / θ_1	$\hat{R}(\hat{\theta}_1, \hat{\theta}_2)$	Bias	MSE 1.0e-005*	$\hat{R}(\hat{\theta}_1, \hat{\theta}_2)$	Bias	MSE 1.0e-006*
(100,01)	0.0100	0.3699	- 0.2638	0.0843	0.3697	- 0.2639	0.4026
(210,02)	0.0095	0.3699	- 0.2637	0.0542	0.3699	- 0.2637	0.7615
(330,03)	0.0091	0.3705	- 0.2630	0.2047	0.3698	- 0.2637	0.4983
(450,04)	0.0089	0.3702	- 0.2633	0.0775	0.3697	- 0.2637	0.4661
(650,05)	0.0077	0.3695	- 0.2638	0.0299	0.3698	- 0.2635	0.2500
(850,06)	0.0071	0.3699	- 0.2633	0.1463	0.3693	- 0.2639	0.0925
(1000,7)	0.0070	0.3693	- 0.2639	0.0354	0.3693	- 0.2639	0.4617

Table 3-EA: *Extreme-Value, M/N* = 1.0 Table 3-EB: *Extreme-Value, M/N* = 1.0

Given M = 10, N = 09 M/N = 1.11					Given M = 20, N = 15 M/N = 1.33		
(β_1, β_2)	β_2 / β_1	$\hat{R}(\hat{\beta}_1, \hat{\beta}_2)$	Bias	MSE 1.0e-004*	$\hat{R}(\hat{\beta}_1, \hat{\beta}_2)$	Bias 1.0e-003*	MSE 1.0e-005*
(10,01)	0.1	0.7474	0.0003	0.0193	0.7471	0.0270	0.1153
(03,1.5)	0.5	0.7441	- 0.0000	0.0109	0.7442	0.1018	0.0694
(3.3,03)	0.9091	0.7502	0.0006	0.0657	0.7491	- 0.4303	0.2002
(05,05)	1.0	0.7502	- 0.0004	0.1067	0.7506	- 0.0443	0.1751
(06,07)	1.1667	0.7520	- 0.0004	0.0710	0.7522	- 0.1433	0.3111
(07,14)	2.0	0.7563	- 0.0011	0.0244	0.7569	- 0.4218	0.1845
(08,20)	2.5	0.7587	0.0001	0.0078	0.7581	- 0.4450	0.0333

Table 4-LA: *Logistic, M/N* > 1.0 Table 4-LB: *Logistic, M/N* > 1.0

Given M = 10, N = 09 M/N = 1.11					Given M = 20, N = 15 M/N = 1.33		
(θ_1, θ_2)	θ_2 / θ_1	$\hat{R}(\hat{\theta}_1, \hat{\theta}_2)$	Bias	MSE 1.0e-005*	$\hat{R}(\hat{\theta}_1, \hat{\theta}_2)$	Bias	MSE 1.0e-005*
(100,01)	0.0100	0.3703	- 0.2634	0.1317	0.3702	- 0.2634	0.0829
(210,02)	0.0095	0.3700	- 0.2636	0.1295	0.3702	- 0.2633	0.0758
(330,03)	0.0091	0.3698	- 0.2637	0.0315	0.3698	- 0.2637	0.0310
(450,04)	0.0089	0.3699	- 0.2636	0.0245	0.3704	- 0.2630	0.1954
(650,05)	0.0077	0.3696	- 0.2637	0.0505	0.3697	- 0.2636	0.1032
(850,06)	0.0071	0.3693	- 0.2639	0.0086	0.3694	- 0.2638	0.0187
(1000,7)	0.0070	0.3693	- 0.2639	0.0322	0.3697	- 0.2635	0.0448

Table 4-EA: *Extreme-Value, M/N* > 1.0 Table 4-EB: *Extreme-Value, M/N* > 1.0

Given M = 20, N = 10 M/N = 2.0				
(β_1, β_2)	β_2 / β_1	$\hat{R}(\hat{\beta}_1, \hat{\beta}_2)$	Bias	MSE 1.0e-005*
(10,01)	0.1	0.7469	- 0.0001	0.0872
(03,1.5)	0.5	0.7441	0.0000	0.0449
(3.3,03)	0.9091	0.7503	0.0007	0.4179
(05,05)	1.0	0.7508	0.0002	0.3734
(06,07)	1.1667	0.7522	- 0.0002	0.4297
(07,14)	2.0	0.7562	- 0.0012	0.2922
(08,20)	2.5	0.7585	- 0.0000	0.0505

Table 5-L: *Logistic, M/N* > 1.0

Given M = 20, N = 10 M/N = 2.0				
(θ_1, θ_2)	θ_2 / θ_1	$\hat{R}(\hat{\theta}_1, \hat{\theta}_2)$	Bias	*MSE* 1.0e-005*
(100,01)	0.0100	0.3698	- 0.2638	0.0670
(210,02)	0.0095	0.3702	- 0.2634	0.0377
(330,03)	0.0091	0.3706	- 0.2629	0.1088
(450,04)	0.0089	0.3699	- 0.2635	0.0699
(650,05)	0.0077	0.3693	- 0.2640	0.0292
(850,06)	0.0071	0.3696	- 0.2636	0.0608
(1000,7)	0.0070	0.3693	- 0.2639	0.0236

Table 5-E: *Extreme-Value, M/N* > 1.0

Remark 4.1.

In case $\hat{\theta}_1$ and $\hat{\theta}_2$ are, respectively, large compared with y_m and x_n, the right hand sides of each expression in (36), respectively, takes the form

$$\hat{\theta}_1 = \bar{Y}\left(1 - \frac{S_y^2}{\bar{Y}\hat{\theta}_1}\right) \text{ and } \hat{\theta}_2 = \bar{X}\left(1 - \frac{S_x^2}{\bar{X}\hat{\theta}_2}\right), \tag{43}$$

where $\bar{Y}$ and $\bar{X}$ are the sample means and S_y^2 and S_x^2 are sample variances for Y and X, respectively (see Johnson and Kotz, 1970, p. 283).

Thus, for this case, from (36) and (39) we will have,

$$\hat{R}\left(\hat{\theta}_1, \hat{\theta}_2\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma\left(k \frac{\bar{X} - \frac{S_X^2}{\hat{\theta}_2}}{\bar{Y} - \frac{S_Y^2}{\hat{\theta}_1}} + 1 \right). \tag{44}$$

5. DISCUSSION

Industry is data rich and yet very often decisions are made based on a few, between 2 and 6, observations. It is the ratio between the distributions' scale parameters that mostly determines the value of R. The difference in values of the sample sizes for the stress and the strength, for a given ratio, does not affect the estimated value of R as long as the ratio of those sample sizes is kept fixed. Thus, this study shows that it is imperative to have the parameter of the stress much smaller than that of the strength despite that they share the same distribution.

It has been observed, through the computation, that when, in case of the logistic distribution, the ratio β_2 / β_1 (or θ_2 / θ_1 in case of extreme-value distribution) increases from 0.1 to 2.5, the estimated value of *R* does not fluctuate that much, barely 0.01. This small change has appeared and was not affected by the ratio of the sample sizes *M* and *N*, whether for small or large values of the sample sizes. Moreover, the same situation comes up again when *M* and *N* are, practically speaking, very large regardless of their ratio. The same argument goes on for the Bias and the *MSE* analysis for estimating *R*, as it is given in (6) and (32). This indicates that the *R* estimator is robust with regard to those variables, namely the ratio of the parameters or that of the sample sizes. The whole scenario applies on the extreme –value distribution as well.

Theoretically, the random variable under the logistic or extreme-value distribution can assume any value on the real. For estimating the positive parameters in each case, some restrictions on the sample values had to be applied. For simulation purposes the maximum in the generated sample was taken in both cases, and estimating the parameters was done on those new samples. Through simulation it has come clear that the ratio between the parameters has to taken very small. As mentioned above that ratio did not affect the calculation on *R*. In addition to that, in practice, the investigator needs to check on the data where does it fit. Checking is needed in order to decide which distribution will be suitable to use when the realization values are large. This is required since the numerical calculations had shown the *R* value is small for the extreme value when compared with that for the logistic.

Tables 1-LA, & 1-LB show the calculations for the logistic distribution while Tables 1-EA & 1-EB show the calculations for the extreme-value when the ratio of *M*/*N* <1. Tables 2-LA, & 2-LB show the calculations for the logistic distribution while Tables 2-EA & 2-EB show the calculations for the extreme-value when *M*/*N* = 1. Tables 3-LA, 3-LB 3-EA, 3-EB display the results when *M*/*N* > 1 for the logistic and extreme-value distributions respectively. Tables 4-A & 4-B tabulate the results in the asymptotic case, i.e., when *M* and *N* tend to infinity. For the extreme-value distribution, the results of the computations are given in Table 5.

It is not a good practical assumption for evaluating the reliability of a system, that as it is completely perceived, the larger the sample the better, Shayib, 2005. Although increasing the sample size will have a great effect on the reduction of variation, it has been shown through small samples that the ratio is more important that the sample size for calculation of reliability of a system. Attention should be paid to the values of the parameters that control that distribution for the stress and strength data.

It is completely clear that the insight knowledge for the underlined distribution and the ratio of its parameters will affect the calculated reliability of that item. The *MSE* analysis shows that the assumed estimated values of *R* under the considered distribution are very close to the actual values when the parameters assume their known values.

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