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A New Matrix Theorem: Interpretation in Terms of Internal Trade Structure and Implications for Dynamic Systems

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Economic systems often are described in matrix form as $x = Mx$. We present a new theorem for systems of this type where M is square, nonnegative and indecomposable. The theorem discloses the existence of additional economic relations that have not been discussed in the literature up to now, and gives further insight in the economic processes described by these systems. As examples of the relevance of the theorem we focus on static and dynamic closed Input-Output (I-O) models. We show that the theorem is directly relevant for I-O models formulated in terms of difference or differential equations. In the special case of the dynamic Leontief model the system's behavior is shown to depend on the properties of matrix $M = A + C$ where A and C are the matrices of intermediate and capital coefficients, respectively. In this case, C is small relative to A and a perturbation result can be employed which leads directly to a statement on the system's eigenvalues. This immediately suggests a solution to the well-known problem of the instability of the dynamic Leontief model.

Keywords: leverage matrix, stability analysis, input-output analysis, dynamic input-output models.

JEL Classification: C62, C67, D57.

1 Introduction

We present a new theorem, basically a new matrix decomposition, which has an interpretation in terms of the economy's circular flows of goods and money. As we shall see, it is relevant for many economic models. The theorem in itself can be expected to have applications beyond the type of models we discuss.

Traditional equilibrium approaches which incorporate an I-O framework, such as the I-O model or the CGE model, are based on the equality between receipts from sales and outlays for inputs. These relations are well known, and need no further discussion. However, as we shall show in the next section, there exists a *second* type of equilibrium relations which also express the above-mentioned equality. These additional relations, which have not been discussed in the literature up to now, will give us more insight in the described economic processes and may, in particular, help to explain their often surprising dynamic behavior.

In the well-known static closed model we have two equations describing the output (real) and the price (nominal) side:

$$\begin{aligned} x &= Mx, \\ p &= pM, \end{aligned} \tag{1.1}$$

where M is the matrix of intermediate inputs. We observe that M has Perron-Frobenius eigenvalue equal to unity. The model tells us that sector receipts (measured over the rows) must equal outlays (measured over the columns). So for sector i, the price of the (sold) commodity bundle $\sum_j m_{ij}x_j$ must equal its expenses, i.e., the price of the bundle $\sum_i m_{ij}x_j$. This is precisely what is said by the set of Eqs. (1.1). However, the I-O system's equilibrium must satisfy a second type of relations.

This second type of relations is based on a property of eigenvectors systems like (1.1). More formally we have:

Theorem 1: Let M be a (n, n) nonnegative indecomposable matrix with Perron-Frobenius eigenvalue $\mu = 1$ and right-hand Perron-Frobenius eigenvector $x = [x_i]$. Let $m_{.i}$ and $m_{i.}$ (i = 1,…, n) stand for the i-th column and row of M, respectively. Then with $L_i \equiv m_{.i}m_{i.}$ we have the following two alternative forms:

$$x = Mx = (x_1)m_{.1}m_{1.} + \ldots + (x_n)m_{.n}m_{n.} = \sum_{i=1}^{n}(x_i)L_i, \tag{1.2}$$

$$x = Mx = m_{.1}m_{1.}x + \ldots + m_{.n}m_{n.}x = \left(\sum_{i=1}^{n} L_i\right)x. \tag{1.3}$$

The proof is elementary. We also have directly:

Theorem 2: Let p be the Perron-Frobenius left-hand eigenvector of matrix M in Theorem 1. Then:

$$p = p\left(\sum_{i=1}^{n} L_i\right). \tag{1.4}$$

Below we will use these and similar properties for matrices where L is the sum of two or more nonnegative and indecomposable matrices. First we will discuss the implications of the matrix L for "standard" input-output analysis. Subsequently, we will discuss the consequences for dynamic systems.

2 A New Decomposition of I-O-based Models: The Leverage Matrix and I-O Analyses

In this section, we will show what the implications are of the above-mentioned theorems for static I-O analyses. We will first explain the implications in general terms and subsequently give a specific numerical example.

Suppose we have a system of equations like (1.1). The first industry *buys* at the unit level the bundle $m_{.1}$, the first column of matrix M, to produce one unit of its own good. The demand for sector 1 goods equals the bundle $m_{1.}$ times total demand. Thus, given total demand we can determine the demand for goods in the economy via the demand structure of sector 1 goods (the row) and the intermediate demand structure of sector 1 (the column). This represents the "demand strength" of sector 1 as it gives the demand effect on the economy via sector 1. We shall call this the leverage of sector 1. Clearly, we may also determine the leverage of the other sectors. Adding up the leverage of all sectors gives the economy's leverage. In this way the concept of an economy's leverage can be used to describe the behavior of the economy in terms of *internal trade* relations.

2.1 The Leverage Matrix: A Numerical Illustration

Let us consider a numerical example. Consider a closed I-O system characterized by the input coefficients matrix

$$M = \begin{pmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{pmatrix}.$$

The proportions of the corresponding real output vector are straightforwardly obtained. Output vectors x have fixed proportions with α indicating the scale of operation. Let us arbitrarily choose $\alpha = 500$. We then have:

$$x = \alpha\begin{pmatrix} 0.714 \\ 1 \end{pmatrix} = \begin{pmatrix} 357 \\ 500 \end{pmatrix},$$

where $\begin{pmatrix} 0.714 \\ 1 \end{pmatrix}$ is the Perron-Frobenius eigenvector. So we have for the output (real) side:

$$\begin{pmatrix} 357 \\ 500 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{pmatrix}\begin{pmatrix} 357 \\ 500 \end{pmatrix} \tag{2.1}$$

or, equivalently:

$$\begin{pmatrix} 357 \\ 500 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} 357 + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} 500. \tag{2.2}$$

This is very familiar, of course. Now, let us look at the relation between goods sold and goods bought from the perspective of an individual industry. We define the different rows and columns of the matrix M as follows:

$$m_{.1} = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}, \quad m_{1.} = (\,0.3 \quad 0.5\,),$$
$$m_{.2} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \quad m_{2.} = (\,0.7 \quad 0.5\,).$$

Clearly, the first industry has to buy the bundle $m_{.1}$ to be able to produce one unit of a sector 1 good. The demand for sector 1 goods equals the demand from all sectors in the economy times $m_{1.}$, the first row of matrix M. Thus the demand generated by sector 1, given the sector demand in the economy, may be described as follows:

$$\begin{pmatrix} d_{11} \\ d_{21} \end{pmatrix} = \left(\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}[\,0.3 \quad 0.5\,]\right)\begin{pmatrix} 357 \\ 500 \end{pmatrix}, \tag{2.3}$$

where d_{11} and d_{21} are the demand from sector 1 (resulting from the economy's demand for both sector goods) for sector 1 and sector 2 goods, respectively.

We define the leverage matrix L_1 such that it describes the leverage that sector 1 has via its demand structure on the economy:

$$L_1 \equiv m_{.1} m_{1.} = \begin{pmatrix} 0.09 & 0.15 \\ 0.21 & 0.35 \end{pmatrix}.$$

In the same way we find L_2, the leverage matrix for sector 2:

$$L_2 \equiv m_{.2} m_{2.} = \begin{pmatrix} 0.35 & 0.25 \\ 0.35 & 0.25 \end{pmatrix}.$$

Note that it is still the case that a sector can only buy goods in the same amount as it "earns" by selling its own produce. The leverage matrix L of the economy as a whole is defined as the sum over the leverage matrices of the different sectors and represents the economy's trade leverage over the sectors.

$$L \equiv L_1 + L_2 = \begin{pmatrix} 0.44 & 0.40 \\ 0.56 & 0.60 \end{pmatrix}.$$

2.2 The Leverage Matrix: Implications for I-O analyses

Now we should take into account that total supply equals total demand. This directly leads to the observation that the system (1.1) contains *a second set of equilibrium relations that also must be satisfied*. This is based on the above discussed decomposition of the I-O system. The decomposition is based on the trade relations between sectors and presents an internal structure entirely *different* from (1.1) or (2.1) which focused on production functions. This is expressed by the relation:

$$x = Lx. \tag{2.4}$$

With respect to our example we obtain the system:

$$\begin{pmatrix} 357 \\ 500 \end{pmatrix} = (L_1 + L_2) \begin{pmatrix} 357 \\ 500 \end{pmatrix} = \begin{pmatrix} 0.44 & 0.40 \\ 0.56 & 0.60 \end{pmatrix} \begin{pmatrix} 357 \\ 500 \end{pmatrix}. \tag{2.5}$$

The above suggests a reformulation in terms of what we may call "quasi input coefficient matrices", i.e., coefficient matrices based on the proportions in the industrial transactions and the trading strength or "leverage" of the industries. We see that we have obtained a different expression between the two sectors, obtained by the two leverage matrices. Prices, naturally, must satisfy (1.1) to (2.5). As we easily observe, unit prices will support the equilibrium. We should stress that (2.3) to (2.5) do not represent the technology, as does (1.1). It represents an alternative interpretation of (1.1) based on the trade relations between sectors, and gives an additional relation that *must* be satisfied. Moreover, it gives insight in the interrelation of different industries via their trade relations.

In this paper, as mentioned, we shall primarily confine ourselves to closed models. In this context, it will be interesting to explore cases where M is the sum of two or more sub-matrices. As we shall see, this will directly lead to applications in economic models such as the well-known case where a dynamic Leontief economy grows along its von Neumann ray. This involves a system matrix that is the sum of *two* matrices, namely the matrix of intermediate input coefficients and the matrix of capital coefficients.[1] On the basis of our proposed decomposition we can predict how the spectrum of eigenvalues of specific matrix products will look like. This will provide direct insight in the behavior of the dynamic versions of the basic model. First, again, we shall present a general result. Hereafter, in the next section, we shall discuss the dynamic model and the implications of our results.

Theorem 3: Let M in Theorem 1 be the sum of two (n, n) nonnegative indecomposable matrices A and B. We then have:

$$\begin{aligned} x &= Mx \\ &= (A+B)x \\ &= [(\alpha_1 A_1 + \ldots + \alpha_n A_n) + (\beta_1 B_1 + \ldots + \beta_n B_n)]x \\ &= \left[\sum\nolimits_i \alpha_i A_i + \sum\nolimits_i \beta_i B_i\right] x \end{aligned} \tag{2.6}$$

with, respectively, $a_{.i}$ and $a_{i.}$ the i-th column and row of A, $b_{.i}$ and $b_{i.}$ the i-th column and row of B, with $A_i \equiv a_{.i}a_{i.}$ and $B_i \equiv b_{.i}b_{i.}$, and with $\alpha_i > 0$ $(i = 1, \ldots, n)$ and $\beta_i > 0$ $(i = 1, \ldots, n)$ appropriate scalars.

Proof: By assumption we have $x = Mx = (A+B)x$. We define

$$g \equiv Ax. \tag{2.7}$$

1 To be precise, the matrix of capital coefficients multiplied by a specific factor, see Sect. 4.

Let g_i represent the i-th element of the column vector g. We then have:

$$g_i = a_{i.}x. \tag{2.8}$$

We also have straightforwardly:

$$g = x_1 a_{.1} + \cdots + x_n a_{.n}, \tag{2.9}$$

where x_i is the i-th element of x. As a next step, we define appropriate scalars $\alpha_i > 0$ implicitly via:

$$x_i = \alpha_i g_i. \tag{2.10}$$

Substituting (2.10) in (2.9), we have:

$$g = \alpha_1 g_1 a_{.1} + \cdots + \alpha_n g_n a_{.n}. \tag{2.11}$$

Using (2.8), we now obtain:

$$\begin{aligned} g &= \alpha_1 a_{.1} a_{1.} x + \cdots + \alpha_n a_{.n} a_{n.} x \\ &= (\alpha_1 A_1 + \cdots + \alpha_n A_n) x \\ &= \sum\nolimits_i (\alpha_i A_i) x. \end{aligned} \tag{2.12}$$

Similarly, we write

$$h \equiv Bx \tag{2.13}$$

which via a similar procedure gives

$$\begin{aligned} h &= (\beta_1 B_1 + \cdots + \beta_n B_n)\, x \\ &= \sum\nolimits_i (\beta_i B_i) x. \end{aligned} \tag{2.14}$$

Combining terms, we have:

$$\begin{aligned} x &= g + h \\ &= (\alpha_1 A_1 + \cdots + \alpha_n A_n) x + (\beta_1 B_1 + \cdots + \beta_n B_n) x \\ &= \left(\sum\nolimits_i \alpha_i A_i + \sum\nolimits_i \beta_i B_i \right) x. \end{aligned} \tag{2.15}$$

We observe that if either A or B is the zero matrix, we have $\beta_i = 1$ or $\alpha_i = 1$, $(i = 1, \ldots, n)$, respectively. Note that we may also write:

$$x = \left(A + \sum_i \beta_i B_i\right) x \tag{2.16}$$

or

$$x = \left(\sum_i \alpha_i A_i + B\right) x. \tag{2.17}$$

The price equation can be decomposed in an analogous way. However, we should note that the usual duality results *do not* apply straightforwardly. We have:

Theorem 4: Let M be as in Theorem 3. We then have:

$$\begin{aligned} p &= pM \\ &= p(A + B) \\ &= p[(\gamma_1 A_1 + \cdots + \gamma_n A_n) + (\delta_1 B_1 + \cdots + \delta_n B_n)] \\ &= p\left[\sum_i \gamma_i A_i + \sum_i \delta_i B_i\right] \end{aligned} \tag{2.18}$$

with appropriate scalars $\gamma_i > 0$ $(i = 1, \ldots, n)$ and $\delta_i > 0$ $(i = 1, \ldots, n)$.

Note that we also may write:

$$p = p\left[A + \sum_i \delta_i B_i\right] \tag{2.19}$$

or

$$p = p\left[\sum_i \gamma_i A_i + B\right]. \tag{2.20}$$

Now let us return to the economic implications of the decomposition described above. A common decomposition in I-O analysis would be to divide the matrix M into a final demand part and an intermediate demand part. We can easily apply this using our decomposition methodology. In this way we can analyse the effects of changes in the system through the final demand and the intermediate demand parts of the model. We will return to this economic interpretation of the decomposition in the next section where we discuss the dynamic behavior of the dynamic Leontief model.

3 Implications for Dynamic I-O Models

After having discussed the implications of the decomposition for the static I-O model we turn here to the implications for Leontief's dynamic interpretation of the I-O model. More specifically, we turn to the original dynamic Leontief model (DLM) and, using our theorem, we will explain the inherent instability that has characterized these types of models.

Ever since its presentation by Leontief in 1953, the dynamic Leontief model (DLM) has been plagued by problems that have extensively been discussed in the literature. However, after three decades the discussion basically ends without having reached a satisfactory conclusion. The discussion as it evolved was mainly focused on proposing alternative forms that circumvented the problems inherent to the DLM. The core problem, in overview, was not properly identified, nor solved. This is a serious flaw in the literature because this core problem is also present in proposed alternative versions. Based on the theorem presented in the introduction, we shall argue that the problem is due to a not well-understood property of (functions of) a fundamental system matrix. In numerical work this property determines the outcome (basically: stability or instability).[2]

The key concept is relative stability. It has been introduced to describe the behavior of arbitrary growth paths with respect to a specific growth path, which then serves as a reference path. The DLM has such a reference path in the so-called balanced or proportional growth path, in which the outputs of all industries or sectors grow at the same rate. (It is not difficult to show that the DLM in its various specifications possesses such a growth path, see Brody, 1970).

Suppose the economy is described by the difference equation $x_{t+1} = Mx_t$ with M the system matrix and x_t output during period t. Suppose x_t^* is on a balanced growth path. Suppose also that $\hat{x}_t$ is on some other growth path starting from an initial position $\hat{x}_0 \geq 0$. The balanced growth path is relatively stable if

$$\lim_{t\to\infty} \frac{\hat{x}_{it}}{x_{it}^*} = \sigma \tag{3.1}$$

2 Definitions will be given below.

exists, with $0 < \sigma < \infty$ and where the suffix *it* refers to the output of industry *i* during period *t*.[3] Let us now turn to the DLM.

The model comes in various specifications, which mainly differ according to the specification of the time lag. Certain specifications are invariably unstable, generating growth paths that do not make economic sense. However, other specifications appear to perform quite well. Particularly troublesome is the development of the price-output relation. The well-known dual-instability problem (Solow, 1959; Jorgenson, 1960; and others) illustrates this: if outputs "behave normally", the model seems to tell us that prices can go anywhere, and vice versa. More recently, some new model specifications were proposed, such as the Duchin and Szyld (1985) model that seemed to perform well. Nevertheless, again price implications were not readily available, and it was never satisfactorily explained what went wrong with the originally proposed models. In our view, it is necessary to return to these older forms for a fundamental cause of the inherent instability of the DLM. We have to know what was not well understood in the older versions of the DLM to understand the behaviour of alternative modern specifications.[4] This insight will give us a better understanding of the DLM and the mechanisms of growth and development.

We shall discuss the model in its forward specification

$$(I - A)x_t = B[x_{t+1} - x_t], \tag{3.2}$$

where A represents the indecomposable matrix of intermediate input coefficients, and B the nonsingular capital matrix. x_t stands for the quantity of each good being produced during period t (Leontief, 1953). We shall limit ourselves to the so-called closed model. The term "closed" in this context means that households' consumption has been endogenized.[5] Equation (3.2) represents the so-called forward-looking specification; we

3 Following convention, we say that the economy is stable if the balanced growth path is relatively stable and unstable if the balanced growth path is relatively unstable.

4 In Appendix B, we will show how the core problem of the DLM model also plays a role in more modern specifications such as the model proposed in Duchin and Szyld (1985), and in Leontief and Duchin (1986, Appendix A).

5 The use of the "open" model would only complicate matters without adding anything to our argument. An additional advantage of using the closed version of the model is the availability of several well-known empirical studies. In fact, the data that we use in this study have been compiled with this version in mind.

are "looking ahead". In view of the general applicability of our results, we might as well opt for the "backward-looking" specification.

$$(I - A)x_t = B[x_t - x_{t-1}]. \tag{3.3}$$

As we shall see, results will be irrespective of the choice of the forward or backward looking formulation. The reason we have opted for (3.1) is that this version is the original one proposed by Leontief.

The model asks for two types of coefficient matrices, a matrix of intermediate input coefficients A and a capital matrix B. The elements a_{ij} of the matrix A stand for the amount of good i used per unit of good j as intermediate input. The elements b_{ij} stand for the amount of capital of good i required per unit of output of good j. We assume that each good can be used either for intermediate uses or as a capital good. That is, there are no separate production processes to produce capital goods. A certain quantity of a good i becomes a capital good when it is set aside for further use as capital.

Solving (3.2) for x_{t+1} we obtain:

$$x_{t+1} = \left[I + B^{-1}(I - A)\right]x_t. \tag{3.4}$$

It is not difficult to show that this system has a balanced growth solution x_t^* (see Brody, 1970). The proportions of x_t^* are those of the Perron-Frobenius eigenvector $\underline{x} > \mathit{0}$ of matrix $(I - A)^{-1}B > 0$.[6] Let λ_1 be the corresponding Perron-Frobenius or dominant eigenvalue of $(I - A)^{-1}B$. In that case $\underline{x}$ also is an eigenvector of matrix $[I + B^{-1}(I - A)]$, with corresponding eigenvalue $(1 + 1/\lambda_1)$. That is, x_t^* also is a balanced growth solution of (3.3). However, in general, matrix $[I + B^{-1}(I - A)]$ will not be nonnegative. Therefore, $(1 + 1/\lambda_1)$ cannot be quaranteed to be the dominant eigenvalue. Relative stability is obtained if and only if

$$1 + 1/\lambda_1 > |1 + 1/\lambda_i| \quad i = 2, \ldots, n. \tag{3.5}$$

Using this formula, Leontief found that empirical versions for the United States rapidly produced a growth path that was not economically interpretable, predicting negative values within a few periods. Mathematically, it meant that (3.2) was unstable in the sense that the economy's outputs did not converge to the so-called balanced growth path. The same phenomenon was observed by many authors such as Tsukui (1968), Tokoyama and Murakami (1972), and Meyer and Schumann (1977). Brody (1970) devoted part of

6 Note that indecomposability of A implies $(I - A)^{-1} > 0$. With the nonsingularity of B we have $(I - A)^{-1}B > 0$.

his book to the problem. Contrarily, the "backward-looking" model (3.3) invariably was found to be stable in empirical work. It converged rapidly to the balanced growth path. Also the dynamic inverse (Leontief, 1970) appeared to be stable. However, it was found that accompanying price models were "stable" (i.e., producing economically sensible outcomes) if the physical model was unstable, and vice versa. For example, the equation

$$p_{t+1}[B + (I - A)] = (1 + r)p_t B \tag{3.6}$$

generates stable values if the real model (3.2) is unstable, a result known as the "dual instability property" (Solow, 1959, or Jorgenson 1960).[7] Following tradition, to study the behavior of the real model (3.2), we have to calculate the eigenvalues of matrix $[I + B^{-1}(I - A)]$. As we know, the eigenvector corresponding to the largest eigenvalue of that matrix will determine long run behavior. Because many economies had grown relatively smoothly, it was thought that the model would reproduce that. However, that was not the case. The eigenvalue corresponding to the von Neumann path usually was the *smallest* of all.

Because all system matrices we shall discuss are functions of matrix $(I - A)^{-1}B$, we shall present the eigenvalues of that matrix. As we know, if λ is an eigenvalue of $(I - A)^{-1}B$, then $1 + 1/\lambda$ is an eigenvalue of matrix $[I + B^{-1}(I - A)]$. To illustrate, let us consider the eigenvalues calculated by Leontief (1953) for his 10-sector model of the USA:

Table 1. Eigenvalues of Matrix $(I - A)^{-1}B$ for the US Economy (Leontief, 1953)

Real part	Imag. part
8.333	
0.679	
0.278	
0.190	
0.149	
0.105	0.024
0.105	−0.024
0.076	
0.034	
−0.005	

7 r stands for the long run growth rate.

The largest eigenvalue here (λ_1 = 8.333) corresponds to the eigenvector which gives the proportions of the von Neumann path.[8] We immediately notice that this means that the *smallest* eigenvalue of matrix $[I + B^{-1}(I - A)]$, equal to 1 + 1/8.333, corresponds to the balanced growth path. Thus, the model will rapidly converge to the proportions of the eigenvector corresponding to the smallest eigenvalue of matrix $[I + B^{-1}(I - A)]$, which means that this path loses its economic interpretability. Tsukui called this property "complete instability", i.e., the property that

$$1 + 1/\lambda_1 < |1 + 1/\lambda_i| \quad i = 2, \ldots, n. \tag{3.7}$$

We see that in the case of model (3.2), precisely the opposite occurs; here the largest system eigenvalue corresponds to the eigenvector giving us the von Neumann ray. Consequently, that model rapidly converges to its von Neumann ray.[9]

3.1 Implications of the Decomposition

In this section, we shall argue that the dynamic properties are *not* related to the economic interpretation of equations like (3.1) or (3.2). The problem is caused by the fact that manipulation of the basic equations involves matrices that have a structure based on additional relations between rows and columns. As a consequence, specific forms like $(I - A)^{-1}B$ and $B^{-1}(I - A)$ have a particular structure. To prove the presence of this special structure, we shall present a new decomposition of I-O systems of the form $x = Mx$ where M is nonnegative and indecomposable. This theorem shows the existence of an additional relation between corresponding rows and columns of the matrices A and B. As we

8 See Szyld (1985) for conditions for the existence of a von Neumann path in the DLM.

9 For additional interpretations and applications we refer to Wurtele (1959), Lovell (1970), Szepesi and Székely (1972), Csepinzki and Rácz (1979), and Luenberger and Arbel (1977). Campisi and Nastasi (1992) recently presented an empirical analysis for an open version of the model.

shall see, this additional relation is a consequence of fundamental input-output logic.[10]

We shall start from the special equilibrium situation of the economy being on its von Neumann path. That is, outputs of each good are growing at the same, invariable rate $(1 + 1/\lambda_1)$, where again λ_1 is the Perron-Frobenius eigenvalue of matrix $(I - A)^{-1}B$. In this particular case, following Brody (1970), we may write:

$$(A + [1/\lambda_1]B)x = x \tag{3.8}$$

and

$$p(A + [1/\lambda_1]B) = p. \tag{3.9}$$

From (3.9), we see that output price proportions are given by the left-hand Perron-Frobenius eigenvector of matrix $(A + [1/\lambda_1]B)$. There is no particular numeraire, each good can serve as a numeraire. Below we shall assume that B has full rank. This is merely for convenience of presentation. Our basic result does not depend on this.[11]

Using the introduced decomposition method we may rewrite Eqs. (3.8) and (3.9) for an economy in a state of balanced growth equilibrium. From Theorem 3, we have straightforwardly:

$$x = \left(A + [1/\lambda_1]^2\{\alpha_1 B_1 + \cdots + \alpha_n B_n\}\right)x, \tag{3.10}$$

$$p = p\left(A + [1/\lambda_1]^2\{\beta_1 B_1 + \cdots + \beta_n B_n\}\right), \tag{3.11}$$

where again $B_i \equiv b_{.i}b_{i.}$ and where $\alpha_i > 0$ $(i = 1, \ldots, n)$ and $\beta_i > 0$ $(i = 1, \ldots, n)$ are appropriate scalars.

Here we shall consider the price system. However, the same results can be obtained, *mutatis mutandis*, for the output (real) system. Looking at

10 In an earlier paper (Steenge, 1990), it was suggested that on purely quantitative grounds certain restrictions on the coefficients in matrix A could be identified. On this basis, typical properties of the empirical models could be explained. However, no analytical proof or theoretical explanation was given. In this paper, we give an analytical proof based on new theorems which may have implications also outside this particular area.

11 For traditional theory where B is singular, see Luenberger and Arbil (1977), or Meyer (1980).

(3.11), we directly observe that we can make n different expressions by isolating one of the terms $[1/\lambda_1]^2\beta_i B_i, (i = 1, \ldots, n)$ and combining the remaining (n-1) terms with A to obtain a new matrix

$$A_i^* \equiv A + [1/\lambda_1]^2 \sum_{k=1,k\neq i}^{n} \beta_k B_k. \tag{3.12}$$

The price equation (3.11) now can be written:

$$p = p\left(A_i^* + [1/\lambda_1]^2\beta_k B_k\right),\ i \neq k. \tag{3.13}$$

We recall that we have n similar equations, each one giving an expression for the price equation in terms of a different matrix A_i^*, a different matrix B_k and a different β_k.

At this moment we recall from (3.9) that p is a left-hand eigenvector of matrix $(A + [1/\lambda_1]B)$.[12] We can therefore arbitrarily fix the size of p. Assuming this has been done, we have a completely determined price vector p. Given this pre-determined price vector p we introduce $\mu_k = pb_{.k}$ as a notational simplification. This gives:

$$p = pA_i^* + [1/\lambda_1]^2\beta_k\mu_k b_{k.},\ i \neq k. \tag{3.14}$$

We now can explain prices in terms of an equation with a specific Leontief inverse $\left[I - A_i^*\right]^{-1}$ and the row-vector $b_{k.}$ as a row of "quasi-primary factor coefficients". Defining $\gamma_k \equiv [1/\lambda_1]^2\beta_k\mu_k$ we have:

$$p = \gamma_k b_{k.}\left[I - A_i^*\right]^{-1},\ i \neq k. \tag{3.15}$$

So, p is proportional to vector $b_{k.}\left[I - A_i^*\right]^{-1}$.

From empirical studies we know that the second term on the right-hand side of Eq. (3.12) is a sum of matrices, which is small compared to A (see also Sect. 5). This implies that A_i^* is a perturbation of A, and that matrix $\left[I - A_i^*\right]^{-1}$ is a perturbation of the economy's Leontief inverse $[I - A]^{-1}$.[13] This means, considering Eq. (3.15), that p is a perturbation of the vector $b_{k.}[I - A]^{-1}$, which is the same as the k-th row of matrix

12 From immediately above we have $(A + [1/\lambda_1]B) = \left[A_i^* + (1/\lambda_1)^2 \beta_k B_k\right],\ i \neq k.$

13 See Appendix A for a more formal proof.

$B[I - A]^{-1}$. Clearly, we can perform this operation in n different ways, each time with a different perturbation of the Leontief inverse and a different row of quasi-primary factor coefficients. Each represents a different formulation for p, one for each $k = 1,\ldots, n$.[14] Thus, all of the formulations for the same price vector are (proportional to) perturbations of the corresponding row of $B[I - A]^{-1}$.

Thus, the above decomposition implies that all rows of $B[I - A]^{-1}$ are perturbations of each other; that is, they all "look alike". Consequently, we may expect matrix $B[I - A]^{-1}$ to be close to a matrix of rank one. Clearly, the left-hand eigenvector p of $B[I - A]^{-1}$ must reflect this.

Moreover, we have:

$$\begin{aligned} p &= pB[I - A]^{-1}, \\ pB &= pB[I - A]^{-1}B. \end{aligned} \tag{3.16}$$

Thus, following a line similar to the one above, we find that the rows of matrix $[I - A]^{-1}B$ are "close" to pB. If we are dealing with tables in *nominal values*, we have therefore that the rows are close to being proportional to the column sums of B.

Given that the matrix $B[I - A]^{-1}$ is close to a matrix of rank one, it is not surprising that the DLM system as described in Eq. (3.4) is characterized by inherent instability. In other words, the inherent instability of the DLM and the dual instability properties of the DLM, as described by Solow (1959), Jorgenson (1960), and others can in this way be very well explained and are to be expected.

4 Numerical Examples of the DLM

In Sect. 4, we have obtained a set of general expressions each of which represents a different way of looking at the price vector p. We realize that without additional background the equations can represent any system and do not necessarily represent an expression for a dynamic economic system. Clearly, we must look into the properties of matrices A and B if we wish to draw more definite conclusions.

14 There are many more if we decide to go into combinations; however, for our purpose that is not necessary, so we shall not further explore this direction.

There is a long-established tradition how these matrices are compiled in economic studies. There are a number of excellent descriptions, see, e.g., Tsukui (1969), Meyer and Schumann (1977), or Tsukui and Murakami (1979). Let us first take a look at matrix A. It consists of two parts, a traditional intermediate inputs part and a part that represents the endogenization of households' consumer demand. In the above mentioned studies the first part accounts for about half (or a little more) of the size of the elements of A, the household part accounting for the remaining part. Both combine to result in A having a sizeable dominant eigenvalue, usually around 0.9 or more.[15] In this respect we recall our discussion in Sect. 3 where we proposed to decompose the matrix M into a final demand and an intermediate demand part. The decomposition in case of the DLM involves the Investment matrix where Investment is only a small part of the total final demand.

The above analysis of the DLM now holds because A has indeed a large eigenvalue. The high eigenvalue of A implies that the elements of $[1/\lambda_1]B$ are relatively small compared to those of A. Therefore we may consider matrix $(A + [1/\lambda_1]B)$ as a perturbation of A.

4.1 The Spectrum of $B[I - A]^{-1}$

To show that the rows of the matrix $B[I - A]^{-1}$ indeed look alike we take a closer look at the spectrum of the matrix $B[I - A]^{-1}$ (our main indicator).[16] We will give three examples directly taken from the literature. These three examples are for completely different economies and time periods.

The first example is based on a miniaturized 3 x 3 model of the Austrian economy (chap. 4 in Fleissner et al., 1993), the second example is based on a study for Japan and is taken from Tsukui (1969), and the third example is the abovementioned study of Leontief (1953) for the United States.

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15 A dominant eigenvalue equal to unity implies that there is no investment; we then are back at the closed static Leontief model.

16 Notice that we would look at $[I - A]^{-1}B$ if we are dealing with tables in nominal values, and that it can be easily shown that the eigenvalues of $[I - A]^{-1}B$ are equal to the eigenvalues of $B[I - A]^{-1}$.

Table 2. Eigenvalues for Japan based on Tsukui (1969)

Matrix A			Matrix $(I - A)^{-1}B$	
Real part	Imag. Part	Modulus	Real part	Imag. part
0.919	–	0.919	7.788	–
0.456	–	0.456	0.198	–
0.391	–	0.391	−0.064	0.082
0.378	–	0.378	−0.064	−0.082
0.335	–	0.335	0.074	0.027
0.209	–	0.209	0.074	−0.027
0.158	0.053	0.167	0.045	–
0.158	−0.053	0.167	−0.033	–
0.163	–	0.163	0.028	–
0.057	–	0.057	0.018	–
0.042	–	0.042	0.003	0.003
0.007	–	0.007	0.003	−0.003

The eigenvalues of the matrix $[I - A]^{-1}B$ in the case of the Austrian example are equal to 16.042, 0.524 and −0.84, respectively. The eigenvalues for the USA and Japan are presented in Table 1 and Table 2, respectively.[17] Thus, we observe that in all presented empirical examples there is indeed one dominant eigenvalue.

5 Discussion

Based on a new theorem and an associated decomposition method we introduced the Leverage matrix in the context of I-O analyses. The original multiplier matrix in I-O analyses is based on supply and production characteristics. The proposed Leverage matrix gives an indication for the interdependency of industries based on the demand and trade structure of the economy.

We should recall here that the purpose of Sect. 3 has been to show that, in a state of equilibrium, matrices $(I - A)^{-1}B$ or $B(I - A)^{-1}$ can be closely approximated by a rank one matrix. This enables us to predict the behavior of specific forms of the DLM whenever these matrices (or functions thereof) appear in a particular model specification. Also the dual instability property now is straightforwardly understood. To illustrate, let us assume that system (3.2) is stable in the sense we have discussed. That

17 The corresponding matrices needed to determine the spectrum of $B[I - A]^{-1}$ are presented in the mentioned publications and/or can be obtained from the authors.

means that $(1+1/\lambda_1)$ is the *largest* eigenvalue of matrix $I+B^{-1}(I-A)$, and the system's growth will be determined by the corresponding (positive) eigenvector.[18] If we solve (3.6) for p_{t+1}, we find that the relevant system matrix is $[I+(I-A)B^{-1}]^{-1}$. Because the eigenvalues of matrix $(I-A)B^{-1}$ are equal to those of matrix $B^{-1}(I-A)$, we now find that $(1+1/\lambda_1)$ is the *smallest* eigenvalue of the system matrix $[I+(I-A)B^{-1}]^{-1}$. The price system, however, will develop according to the largest eigenvalue, and real and price systems will develop totally unconnected.

We may observe that Appendix A informs us that the results are symmetric regarding matrices A and B. We could just as well have worked towards propositions involving matrices $[I-B]^{-1}A$ or $A[I-B]^{-1}$. In our case, economic theory has suggested which matrices to investigate. Nevertheless, under certain conditions it may be worthwhile to also investigate such alternative forms. Furthermore, it may be useful to further investigate decompositions of the type we have proposed regarding matrix A. In the present context we shall not pursue this because additional benefits regarding the stability issue are not to be expected.

There have been several efforts to reformulate the original DLM along alternative basic ideas. The backward looking model (3.3) and the dynamic inverse (Leontief, 1970) are examples. The turnpike optimality models as presented, e.g., by Tsukui (1968), and Tsukui and Murakami (1979) provide another example. We should note that in these models (often after some manipulation) matrix $(I-A)^{-1}B$ also appears. The authors point out that the balanced growth path is reached in only a very limited number of rounds. Our results may provide an explanation, as a matrix close to rank one will direct the system essentially in only one direction.

Another strategy to obtain a well-performing dynamic model is to introduce constraints on certain variables to prevent economically nonsensible outcomes. This is one way to interpret the well-known Duchin-Szyld approach. This Duchin-Szyld model allows for different capital structures for each industry, but works with capacity and other limits on investment behaviour. The introduction of such nonlinearities may be seen as a first step towards a computable general equilibrium (CGE) approach, such as commonly used in empirical analyses since the work of Dervis et al. (1982) and which are mathematically more rigorously discussed in Ginsburgh and Keyzer (1997).

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18 Recall that λ_1 is the Perron-Frobenius eigenvalue of matrix $(I-A)^{-1}B$.

The introduction of constraints along the lines of Duchin-Szyld however, is not sufficient to obtain a full-fledged dynamic model. For instance, it is not clear how an accompanying price equation should be formulated. Prices are, of course, explicity modeled in a CGE approach. However, dynamic behavior described by the model may still be affected by the presence of matrices $(I-A)^{-1}B$ or $B(I-A)^{-1}$ at a certain stage. We should stress that our results apply to multi-sectoral models with *fixed* coefficients. It will be interesting to see to what extent the results are applicable to comparable models with *variable* coefficients. Interesting candidates for such an exercise are the Multi-Regional Variable Input-Output (MRVIO) and Dynamic Variable Input-Output (DVIO) models introduced by Liew and Liew (1985) and Liew (2000), respectively. Unlike conventional I-O models, these approaches stress specific neoclassical structural elements such as the cost- and price-sensitivity of the input coefficients. Partly their lineage dates back to the Hudson-Jorgenson (1974) KLEM line of CGE models, and partly to the various Leontief-type models. In the dynamic version, both intermediate and capital input coefficients include price terms making them variable instead of fixed. With Leontief these models share the fact that equilibrium outputs and prices are obtained as the solutions of *linear* equation systems. In this way, the transparency of the Leontief approach is maintained while capturing producers' and consumers' profit and utility maximizing behavior, respectively. Other properties include duality between production and price frontiers, and the interdependency of output and price equations. Because different methodologies are involved, exploring these hybrid types of models requires special attention. This falls outside the scope of the present contribution, however.

Whichever approach will eventually be adopted, it is extremely important to avoid directions that are not clear in the way output and price systems interact. Our equations depict the equilibrium situation. This suggests that in later work we may wish consider investigating the role of shifts in the α_i, the β_i, and the B_{i} matrices $(i = 1, \ldots, n)$.

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Appendix A

Sensitivity of A and the Leontief Inverse

We have a nonnegative indecomposable matrix A with $\sum_i a_{ij} < 1$ and a matrix A^* which is a perturbation of the matrix A, with a maximum deviation of d for every element of the matrix A. Thus, $(1-d)A < A^* < (1+d)A$ for all elements of A. Let us now introduce the matrix $C = (I-A)^{-1}$. We know that

$$C = I + A + A^2 + A^3 + A^4 + \dots \, . \tag{A.1}$$

Now we define the matrix $C^* = (I - A^*)^{-1}$. Given the lower and upper bounds for A^*, we have $(I-(1-d)A)^{-1} < C^* < (I-(1+d)A)^{-1}$. Using (A.1), we get for the lower and upper bound:

$$\begin{aligned} C^* &< I + (1+d)A + (1+d)^2A^2 + (1+d)^3A^3 + (1+d)^4A^4 + \dots \, , \\ C^* &> I + (1-d)A + (1-d)^2A^2 + (1-d)^3A^3 + (1-d)^4A^4 + \dots \, . \end{aligned} \tag{A.2}$$

Defining $\varepsilon \equiv dA + 2dA^2 + 3dA^3 + \dots$ and abstracting from very small terms involving d^2, d^3, d^4, et cetera, we may now write:

$$C - \varepsilon < C^* < C + \varepsilon. \tag{A.3}$$

Taking into account that $a_{ij} \geq 0, \sum_i a_{ij} \leq 1$, and $a_{ij} \leq 1$ it is easy to see that $\sum_{ij} a_{ij} > \sum_{ij} a_{ij}^2 > \sum_{ij} a_{ij}^3$ et cetera, where a_{ij}^2 and a_{ij}^3 are the elements of the matrices A^2 and A^3 if $\sum_i a_{ij} < 1$ for at least one column i. Since d is small (a perturbation) we may include that also the elements of the matrix ε will be small.

Appendix B

Other (Modern) Versions of the DLM

In Duchin and Szyld (1985), and in Leontief and Duchin (1986, Appendix B) a different approach is taken to address the instability of the DLM model. Their approach is based on introducing non-linearities to prevent negative or too high demand for capital goods and a reformulation of the standard model. However, also this reformulated model has the same problems as the original specification by Leontief. To show this we present a linear simplification of their model formulation. We abstract from a timelag between the production of capital goods and its use in the production process and formulate the model in its closed form. Moreover we abstract from any (nonlinear) limits on the different variables and assume that planned capacity equals realized capacity c_t. Now we have the following model:

$$\begin{aligned} [I-A]x_t &= Bo_t, \\ o_t &= c_t - c_{t-1}, \\ c_t &= 2x_{t-1} - x_{t-2}, \end{aligned} \tag{B.1}$$

where o_t is capacity expansion. After substitution of the equations and rewriting we get:

$$[I-A]x_t = B(2x_{t-1} - 3x_{t-2} + x_{t-3}). \tag{B.2}$$

Thus, the investment component is a weighted average of the extra capital needed to satisfy production increases over the last two periods. We simplify further by taking only the extra capital needed to satisfy production increases over the last period. Thus:

$$[I-A]x_t = B(x_{t-1} - x_{t-2}). \tag{B.3}$$

And,

$$\begin{aligned} x_t &= [I-A]^{-1}B(x_{t-1} - x_{t-2}), \\ &= \left([I-A]^{-1}B\right)^2 (x_{t-2} - 2x_{t-3} + x_{t-4}), \\ &= \left([I-A]^{-1}B\right)^3 (x_{t-3} - 3x_{t-4} + 3x_{t-5} - x_{t-6}), \\ &= \left([I-A]^{-1}B\right)^4 (x_{t-4} - 4x_{t-5} + 6x_{t-6} - 4x_{t-7} + x_{t-8}), \\ &= \left([I-A]^{-1}B\right)^5 (x_{t-5} - 5x_{t-6} + 10x_{t-7} - 10x_{t-8} + 5x_{t-9} - x_{t-10}), \\ &\text{et cetera.} \end{aligned}$$

From this, it is immediately clear that the dynamic behavior is still determined by the matrix $[I-A]^{-1}B$. Therefore, we argue that the basic problem inherent to the DLM is also inherent to this simplified model by Duchin and Szyld. Moreover, the dual-instability problem also applies here: if outputs "behave normally", prices are undetermined.

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