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# On the statistical dynamics of economics

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**Abstract**

Statistical properties are explored for typical ergodic processes in economics. A paradox has been observed – although the long-run average of a variable may converge to a constant, its average growth rate in the long-run is always non-negative so as to induce an illusion of growth. The same characteristic exists even if a dynamic process is reversed. For the chaotic and ergodic processes involving a limited growth rate mechanism, it is demonstrated that chaos may be preferred to a equilibrium in the sense that the long-run average return may be higher than the equilibrium return.


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## 1. Motivations

While chaotic economic systems have been intensively studied over the last two decades, their statistical properties in the long-run, and especially their relations to ergodic dynamical systems, have not aroused much attention from economists.

A chaotic economic process is characterized by the sensitive dependence on initial conditions, observation and measurement accuracy, and computation errors that make long-term

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predictions of its trajectories become both meaningless and impossible. In many circumstances, economists have to resort to long-term statistical properties such as long-run averages, observed frequencies, correlation indices, and so on to understand the economic process. Long-term dynamical properties reveal the whole picture of a dynamical process with much richer information that is difficult to attain from an individual trajectory. Consequently, in analyzing chaotic economic processes, the long-run average is one crucial and indispensable measurement.

In this paper, statistical properties are explored for some typical ergodic processes widely applied in economic analysis. These properties not only help us to understand better the internal mechanism of an ergodic system, but also provide new insights about dynamical economic processes.

In Section 2, after briefly summarizing the characteristics of an ergodic process, we reveal some interesting statistical inequalities for a typical class of ergodic processes appearing frequently in the economic analysis. Section 3 then addresses the issue of illusion of average growth rate in a general ergodic process. The applications to nonlinear business cycle models are provided. Section 4 focuses on the chaotic and ergodic process involving caution mechanism. It is shown that *chaos may be preferred to an equilibrium* in the sense that the long-run average return may be higher than the equilibrium return. Finally, in Section 5, the paper is concluded.

## 2. Statistical properties of ergodic processes

Let $x_{t+1} = \theta(x_t)$ be a one-dimensional dynamical process defined on a domain $I = [a, b]$. Under some rather weak mathematical requirements such as period 3, the process will exhibit chaos, that is, perpetual and erratic fluctuations. Due to the sensitive dependence on the initial condition, it is meaningless, in some situations, to study an individual trajectory $\{x_t\}_{t=0}^{\infty}$. Instead, we are more concerned with the long-term characteristics, such as the long-run time-average given by

$$\langle x\rangle = \lim_{T\to\infty}(1/T)\sum_{t=0}^{T-1} x_t = \lim_{T\to\infty}(1/T)\sum_{t=0}^{T-1}\theta^t(x_0). \quad (1)$$

The time-average defined in (1), however, may not exist (Sigmund, 1992), and we can see that, even existing, its value still “sensitively” depends on the initial value $x_0$. Fortunately, such difficulties can be overcome if a chaotic process $\theta$ is an *ergodic* process as well.

Conceptually, a dynamical process defined on a domain $I$ is ergodic if it cannot be “decomposed” into some sub-processes, or in other words, its asymptotic behavior cannot be studied in a sub-space $A \subset I$.[1] Two chaotic but non-ergodic processes are illustrated in Fig. 1. In Fig. 1(a), the asymptotic dynamics can be studied under the restricted process defined on $A$, a sub-space of $I$. In Fig. 2(a), there exist two sub-spaces $A$ and $B$ that the

[1] The strict definition, however, involves measure theoretical concepts. Details can be seen in Lasota and Mackey (1985).

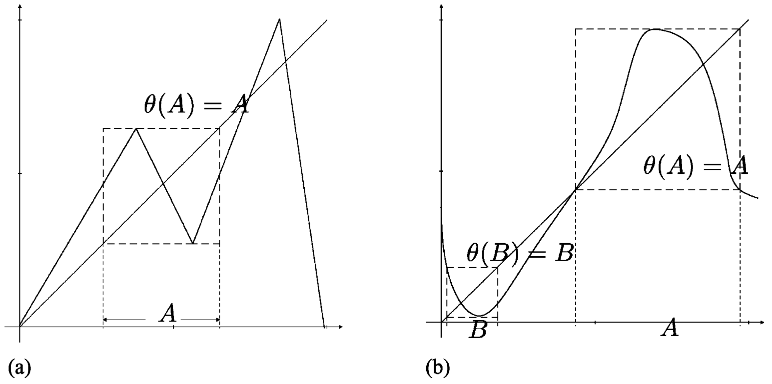



Fig. 1. Illustration of chaotic but non-ergodic processes.

trajectory $\{x_t\}_{t=0}^{\infty}$ may wander into, and hence, the long-run asymptotic behavior depends on the initial value $x_0$, as does the long-run time-average.

For an ergodic process, chaotic or not, the frequencies for trajectory $\{x_t\}_{t=0}^{\infty}$ to visit any subset of the interval $I$, the observed density, will be independent of the initial value and converge to a unique stable density function in the long-run, that is,

$$\lim_{N\to\infty}(1/N)\sum_{t=0}^{N-1}\chi_A(\theta^t(x_0)) \to \int_A \varphi(x)\,\mathrm{d}x, \quad \text{for any subset } A \text{ of } I,$$

where $\varphi(x)$ is commonly referred to as invariant density and possesses the same properties of a density function for a stochastic variable, that is, $\varphi(x) \geq 0$ for all $x$ and $\int_I \varphi(x)\,\mathrm{d}x = 1$.

While “chaosity” emphases the irregularity to distinguish it from the periodic behavior, “ergodicity” stresses uniqueness and observability, which includes stable periodic behavior. It is now clear to mathematicians that *a chaotic dynamical process is also an ergodic process if and only if the invariant density preserved by it is unique and absolutely continuous with respect to the Lebesgue measure*. Hence, through the invariant density, the two conceptually different classes of dynamical characteristics are linked together so that the long-term behavior of a chaotic dynamical process can be investigated by virtue of the fruitful results available in the field of stochastic processes.

Most important properties of an ergodic process $\theta$ are characterized by two well-known theorems: the Birkhoff–Von Neumann mean ergodic theorem and the central limit theorem, which can be summarized as follows:[2]

(i) Ergodicity property: for any continuous function of $x$, its space mean equals to the time-average. That is, the following identity holds for any $f$ that is essentially bounded

[2] See Lasota and Mackey (1985) or Day and Shafer (1987).

and almost all $x_0 \in I$:

$$\langle f(x)\rangle = \lim_{N\to\infty}(1/N)\sum_{t=0}^{N-1} f(\theta^t(x_0)) = \int_{\mathbb{X}} f(x)\varphi(x)\,\mathrm{d}x, \tag{2}$$

where $\varphi(x)$ is the invariant density preserved by $\theta$. Therefore, *the time-average of any statistical property is independent of trajectory* (initial point),

(ii) Stochasticity property: the time-average will obey the central limit theorem in the sense that the observed trajectory $\{x_t\}_{t=0}^{\infty}$ behaves like a stochastic process so that, for almost any initial value $x_0 \in X$, the sample mean defined by

$$x_k^{(m)} \hat{=} (1/m)\sum_{t=mk}^{(k+1)m-1} \theta^t(x_0) \quad \text{for}\, k = 0, 1, \ldots, K,$$

will converge in distribution to a normal distribution $N(\bar{x}, \sigma^2)$ when $m \to \infty$ and $K \to \infty$, and

(iii) Invariance: the time-average is invariant to the system's iterates, that is, if $\theta$ is an ergodic process, then for any $f \in C^1$ and positive integer $k$, the following identities hold true:

$$\langle f(\theta^k(x))\rangle = \langle f(x)\rangle = \langle f(\theta^{-k}(x))\rangle. \tag{3}$$

When $f \equiv x$, the identities (3) become

$$\langle \theta^{\pm k}(x)\rangle = \langle x\rangle \quad \text{for all positive integer}\, k. \tag{4}$$

With identity (4) in hand, we wonder how $\langle x\theta(x)\rangle$ behaves in general? Does there exist any qualitative relationship between $\langle x\theta(x)\rangle$ and $\langle x^2\rangle$? Actually, there is.

**Proposition 1.** *For any ergodic economic process $x_{t+1} = \theta(x_t)$, there exists the following inequality:*

$$\langle x^m(\theta(x))^m\rangle < \langle x^{2m}\rangle, \tag{5}$$

*where m is a positive constant. If $\theta(x) \neq 0$ for all x in the domain, then the inequality will hold when m takes a negative value.*

**Proof.** The proof is straightforward. If $\theta(x)$ is an ergodic process, its time-average can be defined. It follows from the inequality $(x^m - (\theta(x))^m)^2 \geq 0$ that

$$\langle (x^m - (\theta(x))^m)^2\rangle > 0, \tag{6}$$

where strict inequality holds because $\theta(x_t) \neq x_t$ in general. Inequality (6) in turn implies that

$$\langle x^{2m}\rangle + \langle (\theta(x))^{2m}\rangle > 2\langle x^m(\theta(x))^m\rangle, \tag{7}$$

which yields the inequality (5) because of $\langle x^{2m}\rangle = \langle (\theta(x))^{2m}\rangle$. $\square$

**Remark 2.** It needs to be emphasized that, although inequality (5) seems trivial both in form and in its proof, it does not hold for a chaotic process in general, since the existence of time-average of the latter may not be guaranteed.

For a broad class of one-dimensional economic processes with one-dividing point as illustrated in Fig. 2, we have the following stronger and richer results.

**Theorem 3.** *Let $\theta$ be a one-dimensional ergodic dynamical process defined on a domain $I = [a, b]$. If there exists a unique dividing point $\bar{x}$ in the interior of the interval I such that,*

$$\theta(x) \gtrless x \quad \textit{for} \quad x \lessgtr \bar{x}, \tag{8}$$

*then for any functions $F \in C^1$ and $G \in C^1$ that are monotonic and finite in I, the following inequalities hold:*

$$\langle F(x)G(\theta(x))\rangle \lessgtr \langle F(x)G(x)\rangle, \ \textit{if} \ F'\,G' \gtrless 0. \tag{9}$$

**Proof.** See Appendix A. □

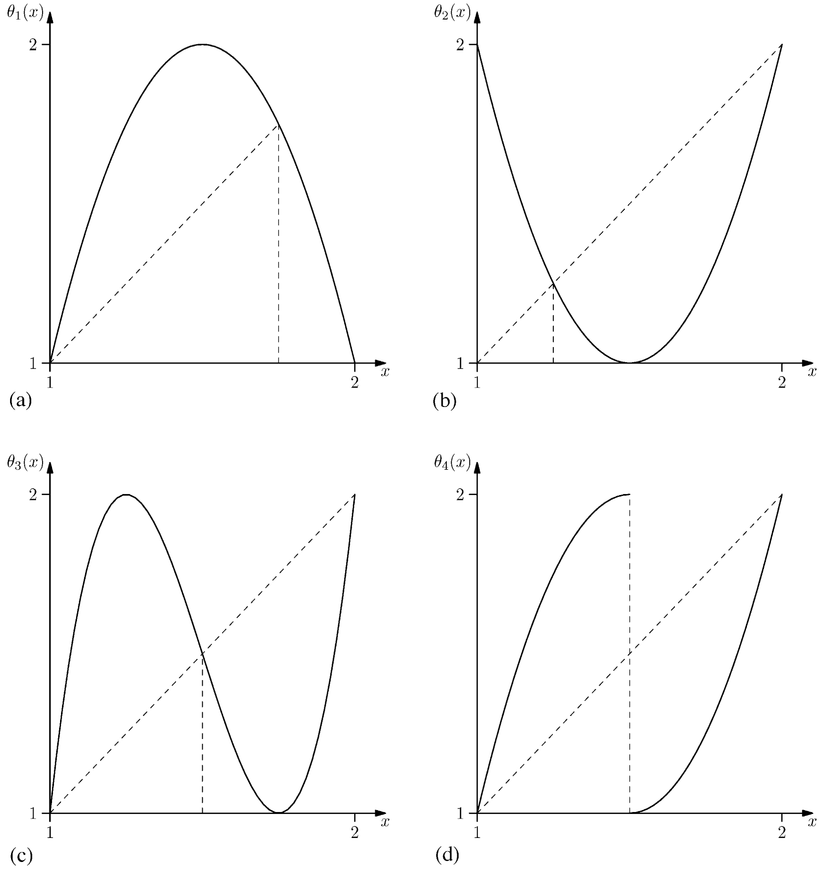



Fig. 2. Illustration of ergodic processes with one-dividing point.

The following proposition can be concluded directly from Theorem 3.

**Corollary 4.** *For any one-dimensional ergodic process $\theta$ that satisfies the condition specified in Theorem 3, the following inequalities hold for any positive integers m and n.*

$$\langle x^m(\theta(x))^n\rangle < \langle x^{m+n}\rangle, \tag{10}$$

$$\langle 1/x^m(\theta(x))^n\rangle < \langle 1/x^{m+n}\rangle, \tag{11}$$

$$if\ \theta(x) \neq 0 \quad for\ all\ x \in I\ and\ 0 \notin I. \tag{12}$$

$$\langle(\theta(x))^m/x^n\rangle > \langle x^{m-n}\rangle, \quad if\ 0 \notin I, \tag{13}$$

$$\langle x^m/(\theta(x))^n\rangle > \langle x^{m-n}\rangle, \quad if\ \theta(x) \neq 0\ for\ all\ x \in I. \tag{14}$$

Inequality (7) then follows directly for $m = n$ in (10).

Although the results stated in our theorem are restricted to "one-turn" type of dynamical processes, they actually cover almost all one-humped continuous processes so far discussed in the economic literature.

Due to the fact that if a property holds for an ergodic process $\theta$, it also holds for its iterates $\theta^k$, $k > 1$. The latter, however, is multiple-humped, so it would be expected that the inequality (9) holds for quite a broader class of dynamical processes.

When $m = n = 1$, (13) and (14) reveal that

$$\langle\theta(x)/x\rangle > 1 \quad \text{and} \quad \langle x/\theta(x)\rangle > 1, \quad \text{if}\ \theta(x) \neq 0\ \text{for all}\ x \in I,$$

which are equivalent to

$$\langle(\theta(x) - x)/x\rangle > 0, \tag{15}$$

and

$$\langle(x - \theta(x))/\theta(x)\rangle > 0, \quad \text{if}\ \theta(x) \neq 0\ \text{for all}\ x \in I. \tag{16}$$

While (15) implies the average growth rate is positive, (16) suggests the reverse growth rate is also positive. Computer simulations for such model are provided in Huang and Day (2001). Section 3 will then generalize (15) and (16) to general ergodic processes and multi-dimensional systems.

## 3. Paradox of average growth rate

In economics, the growth rate, the inflation rate, and their averages are among the most important economic indicators. In reality, they are frequently taken as major economic measurements for macroeconomic performance of an economy. In this article, we reveal a paradoxical but interesting property about the long-run average of the growth rate, which according to our knowledge has never been observed before. It is shown in theory and also by computer simulations that for any dynamical economic process, whether converging to a limited cycle or becoming chaotic asymptotically, the average growth rate for any endogenous variable will be consistently higher than zero. Moreover, this characteristic is preserved even if the economic process is reversed.

### 3.1. AGR bias in business cycles

Generally, for a one-dimensional discrete economic process $x_{t+1} = \theta(x_t)$, where $x_t \geq \epsilon \gg 0$, we define the growth rate as

$$g_t \hat{=} (x_{t+1} - x_t)/x_t = \theta(x_t)/x_t - 1. \tag{17}$$

We define the long-run average of $x_t$ as $\langle x_t \rangle \hat{=} \lim_{T\to\infty}(1/T)\sum_{t=1}^{T} x_t$, and similarly define the average growth rate (AGR for short) as

$$\langle g_t \rangle \hat{=} \lim_{T\to\infty}(1/T)\sum_{t=1}^{T} g_t. \tag{18}$$

The growth rate defined by (17) (and hence AGR defined by (18)) is invariant to the scaling but variant to shifting. To see this, letting $X_t = \alpha x_t$ and $Y_t = x_t + b$, we obtain

$$g_t^X = (X_{t+1} - X_t)/X_t = (x_{t+1} - x_t)/x_t = g_t^x,$$
$$g_t^Y = (Y_{t+1} - Y_t)/Y_t = (x_{t+1} - x_t)/(x_t + b) \lesseqgtr g_t^x, \quad \text{if } b \gtreqless 0.$$

Given an economic process $x_{t+1} = \theta(x_t)$, whether the long-run $\langle x_t \rangle$ exists or not generally depends on the internal structure and asymptotic characteristics in the long-run. We start with the following.

Case I: Monotonic process

When the trajectories of a process $\theta$ are asymptotically monotonic in the long-run, that is, there exists a time $t^*$ such that $\theta(x_t) \gtrless x_t$ for all $t > t^*$, we will have $g_t \gtrless 0$ for all $t > t^*$.

If the process is monotonically increasing at a constant or relatively stable rate, that is, $g_t = \rho + \epsilon_t$, with $\rho > 0$ and $\epsilon_t$ white noise, the long-run average $\langle x_t \rangle$ does not exist, while the average growth rate exists and equals the positive constant $\rho$, that is, $\langle g_t \rangle = \rho$.

If the process is monotonically decreasing at a constant or relatively stable rate, that is, $g_t = -\rho + \epsilon_t$, with $0 < \rho < 1$ and $\epsilon_t$ white noise, then the long-run average $\langle x_t \rangle$ will approach zero (if $0 < \rho \ll 1$),[3] while the average growth rate will be equal to a negative constant: $\langle g_t \rangle = -\rho$.

Case II: Periodic process

While the trajectories of the process $\theta$ are periodic in the long-run, that is, there exists a time $t^*$ and a positive integer $k$ such that $\theta(x_{t+k}) = x_t$, for all $t > t^*$, the long-run average $\langle x_t \rangle$ will approach a positive constant when $T$ is large enough. On the other hand, the growth rate $g_t$ will fluctuate cyclically between positive and negative values. When $T$ is large enough, the long-run average growth rate can be defined and evaluated. Intuitively, since $x_t$ fluctuates between certain periodic states, we may expect that AGR in the long-run either approaches zero or preserves a certain sign that depends on the values of particular periodic orbit.

The next theorem, however, shows exactly the contrary.

[3] This is due to our assumption that $x_t \geq \epsilon \gg 0$. If we abandon this assumption, then $\langle x_t \rangle$ does not exist either. For convenience, we shall use notation $R^{++}$ to reflect such positiveness.

**Theorem 5.** *Let $\{\overline{x}_i\}_{i=1}^m$ (with $\overline{x}_i \in R_+$ for all $i$) be any m-periodic orbit exhibited in a dynamical system of $x_{t+1} = \theta(x_t)$ so that $\overline{x}_{i+1} = \theta(\overline{x}_i)$ for all $i = 1, 2, \ldots, m-1$ and $\overline{x}_1 = \theta(\overline{x}_{m+1})$. The average growth rate is always greater than zero, that is,*

$$\langle g \rangle_m \hat{=} (1/m) \sum_{i=1}^{m} (\overline{x}_{i+1} - \overline{x}_i)/\overline{x}_i > 0. \tag{19}$$

**Proof.** See Huang (2004). □

Therefore, in a periodic economic process (business cycle), although after certain periods, the process always returns to its original state so as to appear with "zero" growth rates for all endogenous variables involved, the actually calculated AGRs always exhibit certain "positiveness biases".[4] In reality, due to the apparent regularity in its trajectories, it seems unlikely for an economic agent to be "cheated" by such AGR bias in a periodic process if the cycle length (periods) is short and identifiable. There do, however, exist in practice situations in which a cycle length is too long to be observable.[5] Also under some circumstances, an economic variable will grow at a regular rate for certain periods, giving an illusion of constant economic growth, then suddenly fall back to its initial value.

Theorem 5 provides an essential foundation for our understanding of AGR under such non-periodic or aperiodic processes.

Mathematically, "chaos" has been repeatedly defined in various contexts but rather inconsistent senses. Loosely speaking, a discrete dynamical process $x_{t+1} = \theta(x_t)$ is chaotic if the trajectories from almost all initial values will fluctuate irregularly and persistently in a certain economically plausible range $X$. However, for a positive ergodic process $\theta$, the existence of long-run averages $\langle x_t \rangle$ and the average growth rate $\langle g_t \rangle$ can be justified from the ergodicity property (2) by letting $f(x) = x$ and $f(x) = \theta(x)/x - 1$, respectively.[6]

The stochasticity property, however, indicates that the fixed-period averages such as monthly averages or weekly averages for daily processes, daily averages for hourly processes, and so on, essentially fluctuate "randomly" around a mean value. Similar random phenomena can be observed for the fixed-periods average of growth rates defined by

$$g_k^{(m)} \hat{=} (1/m) \sum_{t=mk}^{m(k+1)-1} g_t.$$

[4] For the convenience of reference, we shall refer it as "AGR bias" hereafter.

[5] For instance, it is found in the literature that for some dynamical processes, stable cycles exist for $10^5$ periods.

[6] By positiveness of $\theta$, we mean $x_t \geq \epsilon > 0$ for all $t$, which is essential for the boundedness of the function $\theta(x)/x - 1$.

It can be seen that what the mean value of the normal distribution that $g_k^{(m)}$ converges to is identical to the long-run average $\langle g_t \rangle$ defined by (18), since we have

$$\langle g_k^{(m)} \rangle \hat{=} \lim_{T\to\infty} (1/T) \sum_{k=1}^{T} g_k^{(m)} = \lim_{T\to\infty} (1/T)(1/m) \sum_{k=1}^{T} \sum_{t=mk}^{m(k+1)-1} g_t = \langle g_t \rangle,$$

as implied by the ergodicity property.

The same conclusion applies to another most often used index – the moving averages such as the M-days averages widely utilized in financial analysis and time-series analysis, which is defined by

$$\bar{g}_t^{(m)} \hat{=} (1/m) \sum_{i=1}^{m} g_{t+i}.$$

That is, $\langle \bar{g}_t^{(m)} \rangle = \langle g_t \rangle$, and $\bar{g}_t^{(m)}$ converges to a normal distribution $N(\langle g_t \rangle, \sigma^2)$ when $m$ is sufficiently large.

Therefore, exploring the nature of $\langle g_t \rangle$ helps in getting further insight about the overall dynamical natures of an ergodic process. A result analogous to Theorem 5 is thus similarly established.

**Theorem 6.** *For any ergodic dynamical system of $x_{t+1} = \theta(x_t)$, if there exists a $\epsilon \gg 0$ such that $x_t \geq \epsilon$ for all t, then the average growth rate is always greater than zero. That is,*

$$\langle g_t \rangle \hat{=} \lim_{T\to\infty} (1/T) \sum_{t=1}^{T} \frac{x_{t+1} - x_t}{x_t} > 0.$$

**Proof.** The proof is a simple extension of the proof of Theorem 5:

$$\begin{aligned}
\langle g_t \rangle &= \lim_{T\to\infty} (1/T) \sum_{t=1}^{T} (x_{t+1} - x_t)/x_t = \lim_{T\to\infty} (1/T) \sum_{t=1}^{T} x_{t+1}/x_t - 1 \\
&= \lim_{T\to\infty} (1/T) \left[ \left( x_1/x_T + \sum_{t=1}^{T-1} x_{t+1}/x_t \right) + (x_{T+1}/x_T - x_1/x_T) \right] - 1 \\
&= \lim_{T\to\infty} (1/T)(x_1/x_T + \sum_{t=1}^{T-1} x_{t+1}/x_t) + \lim_{T\to\infty} (1/T)(x_{T+1} - x_1)/x_T - 1.
\end{aligned}$$

From Theorem 5, the first limit term is greater than unity since we have actually constructed a $T$-periodic orbit. By the chaosity and ergodicity, the second limit term obviously approaches zero due to the boundedness of $(x_{T+1} - x_1)/x_T$, which results from our assumptions that $x_t \geq \epsilon \gg 0$ for all $t$.

Therefore, we have $\langle g_t \rangle > 0$, which completes the proof. □

**Remark 7.** Just as for the case of periodic processes, although the trajectory generated from a chaotic process will look like the one from a stochastic process, its average growth rate is consistently "positively biased".

In our proofs of Theorems 5 and 6, except for periodicity and ergodicity, no explicit internal structure for the process $\theta$ is involved. Therefore, the results can be generalized to any multi-dimensional or higher-dimensional ergodic process. Mathematically, a stable periodic process (that is, the process in which almost all trajectories will become periodic) is a special type of ergodic process (with support limited on periodic points, see Lasota and Mackey, 1994), Theorem 6 implies Theorem 5. On the other hand, ergodicity is actually a requirement stronger than necessary. For instance, the conclusions of Theorem 6 also apply to the two non-ergodic processes illustrated in Fig. 1. For an $n$-dimensional ergodic process, since AGR for each and every variable is defined and calculated with respect to each individual dimension only, as long as the trajectory of each variable is not monotonic, it can be treated as $n$ different chaotic processes, and thus the conclusion about AGR bias applies. Hence, we are able to generalize Theorem 6 to the following without proof.

**Theorem 8.** *For any n-dimensional ergodic process defined by*

$$\mathbf{X}_{t+1} = \mathbf{F}(\mathbf{X}_t, \mathbf{X}_{t-1}, \ldots, \mathbf{X}_{t-k}), \tag{20}$$

*where* $k > 1$, $\mathbf{X}_t = (x_{1t}, x_{2t}, \ldots, x_{nt})$, *and* $F = (f_1, f_2, \ldots, f_n)$, *with* $f_i$ *being well-defined functions on a domain inside* $R^n_+$, *we always have*

$$\langle g_{it} \rangle = \lim_{T\to\infty} (1/T) \sum_{t=1}^{T} (x_{it+1} - x_{it})/x_{it} > 0 \quad \textit{for } i = 1, 2, \ldots, n. \tag{21}$$

*Therefore, whatever variables are involved in a dynamical process, as long as their trajectories fluctuate perpetually in* $R^n_+$ *(periodically or aperiodically), their average growth rates are always positive.*

The implication is interesting yet against economic intuition. For an economic process involving price and income, the ergodicity would suggest that both the inflation rate and the growth rate are positive on average, which seems in some sense in contrast with the covariance relationship of the Phillips curve. Moreover, it also means that for any ergodic economic process involving employment (labor), both the unemployment rate and the employment rate will "grow" with a positive average rate.

More paradoxically, if we define a reverse growth rate as

$$\tilde{g}_{it} = (x_{it} - x_{it+1})/x_{it+1} = x_{it}/\theta(x_{it}) - 1, \tag{22}$$

then, for any dynamical process fulfilling the conditions specified in Theorem 8, the following inequality always holds true:

$$g_{it}\tilde{g}_{it} < 0 \quad \text{for all } t. \tag{23}$$

However, the inequality does not apply to the long-run average. Formally, we have the following.

**Corollary 9.** *For the n-dimensional ergodic process defined by (20), we always have*

$$\langle \tilde{g}_{it} \rangle = \lim_{T\to\infty} (1/T) \sum_{t=1}^{T} (x_{it} - x_{it+1})/x_{it+1} > 0 \quad \textit{for } i = 1, 2, \ldots, n. \tag{24}$$

**Proof.** The proof follows the same principle as in Theorem 6, except noticing that the ergodicity implies that the existence of $\langle \tilde{g}_{it} \rangle$ (let $f = x_i/f_i(X_{t+1}) - 1$ in (2)). □

Therefore, for an ergodic process given by (20), we have

$$\langle g_{it} \rangle \langle \tilde{g}_{it} \rangle > 0, \tag{25}$$

which is seemingly contradictory to the inequality (23), a characteristic preserved by every and each individual iteration.

It needs to be mentioned that, the reverse growth rate defined by (22) is nothing but the growth rate for a reverse dynamical process implicitly defined by

$$\mathbf{X}_{t+k} = \mathbf{F}(\mathbf{X}_{t+k-1}, \mathbf{X}_{t+k-2}, \ldots, \mathbf{X}_t),$$

so we are in a position to say that the AGR bias property concurrently holds for the reverse dynamical process.

The significance of statistical inequalities provided in this paper are yet to be explored. Numerical simulations can be found in Huang (2004). A direct but interesting application in Walras' tatonnement-type process is provided in Huang (2001).

### *3.2. Implications for models of business cycles*

The implications and applications of inequalities (21) and (24) are not limited to AGR bias only. They can be directly applied to draw additional statistical information from an ergodic dynamical system. We shall show that these statistical inequalities provide additional measures to check the validities of different discrete models and to test nonlinearity or plausible specifications from empirical time series.

#### *3.2.1. Gabisch multiplier-accelerator model*

The original multiplier-accelerator model (Samuelson, 1939) used three relations:

(1) Net investments were proportionate to the delayed income difference: $I_t = v(Y_{t-1} - Y_{t-2})$.

    This was due to the assumed simple production structure with a capital stock always proportionate to output (i.e., real income). Net investments, by definition, the change of capital stock, thus became proportionate to the change of income.

(2) Consumption was assumed to be proportionate to income, a fraction $(1 - s)$ being spent, and there was usually a delay, so that $C_t = (1 - s)Y_{t-1}$.

(3) Finally, income was generated by consumption and investment, specifically, $Y_t = C_t + I_t$.

Since the emergence of chaos theory at the end of the 1970s, modifications through introducing nonlinearity have been repeatedly attempted. Gabisch (1984) modified Samuleson's model to

$$Y_t = C_t + I_t, \tag{26}$$

$$I_t = \gamma(Y_t - Y_{t-1}), \quad v > 1, \tag{27}$$

$$C_t = c_0(Y_{t-1})^{\alpha}, \quad \alpha \geq 1,\ 0 < c_0 < 1. \tag{28}$$

Gabisch's modification also suggests a general occurrence of chaotic dynamics in the sense of Li and Yorke. Moreover, within a certain range of parameter values, the trajectories of the dynamics are highly sensitive to a variation of the parameters or initial values so that the long-term predictions are useless.[7]

Let $\bar{Y}$, $\bar{C}$ and $\bar{I}$ indicate the equilibrium output, consumption and investment, respectively. It follows that

$$\bar{I} = 0, \quad \bar{Y} = c_0^{1/(1-\alpha)} \quad \text{and} \quad \bar{C} = c_0^{(2-\alpha)/(1-\alpha)}. \tag{29}$$

Let $\langle\cdot\rangle$ denote the long-run average (time-mean). Taking the time-average on both sides of each of Eqs. (26)–(28) leads to

$$\langle Y_t\rangle = \langle C_t\rangle + \langle I_t\rangle,$$
$$\langle I_t\rangle = \gamma(\langle Y_t\rangle - \langle Y_{t-1}\rangle),$$
$$\langle C_t\rangle = c_0\langle (Y_{t-1})^{\alpha}\rangle.$$

The ergodicity then would imply $\langle Y_t\rangle = \langle Y_{t-1}\rangle$, and hence $\langle I_t\rangle = 0$, which is consistent with the equilibrium, and

$$\langle I_t\rangle/\langle Y_t\rangle = 0 \quad \text{and} \quad \langle C_t\rangle/\langle Y_t\rangle = 1. \tag{30}$$

Divide both sides of (26) and (27) by $Y_t$, respectively, and rearrange

$$1 = C_t/Y_t + I_t/Y_t, -(1/k)I_t/Y_t = (Y_{t-1} - Y_t)/Y_t.$$

It follows from Theorem 8 that

$$\langle I_t/Y_t\rangle < 0 \quad \text{and} \quad \langle C_t/Y_t\rangle > 1. \tag{31}$$

Combining (30) and (31) yields

$$\langle I_t/Y_t\rangle < \langle I_t\rangle/\langle Y_t\rangle, \tag{32}$$

and

$$\langle C_t/Y_t\rangle > \langle C_t\rangle/\langle Y_t\rangle, \tag{33}$$

[7] The Gabisch model is rather ad hoc, and we do not intend to provide any justification from economic fundamentals. Instead, we examine what would happen to the long-term statistical properties of relevant macroeconomic variables if the system indeed behaves ergodically.

which can be viewed as the necessary conditions for the ergodicity of the multiplier-accelerator model.

The correctness and significance of the inequalities we obtained are yet to be explored, but they do provide a basis for detecting “chaos” from time-series data.

Moreover, inequalities (21) and (24) are especially suitable for those processes that are discrete versions of certain established continuous economic processes, that is, the discrete processes resulting from replacing a derivative term such as $\mathrm{d}x(t)$ with the difference term $(x_{t+1} - x_t)$. This point can be made clearer through the following example.

*3.2.2. Kaldor’s model of the business cycle*

Kaldor’s nonlinear model of the business cycle (Kaldor, 1940) is often present (under various forms) in modern treatments of business cycle theory. Its mathematical representation was originally given in a continuous time framework:

$$\mathrm{d}Y/\mathrm{d}t = \alpha(I(Y, K) - S(Y, K)),$$
$$\mathrm{d}Y/\mathrm{d}t = I(Y, K) - \delta K,$$

where $Y$ = national income, $K$ = capital, $I(Y, K)$ is the investment function, $S(Y, K)$ the saving function, and $0 \leq \delta < 1$ the depreciation rate.

The following discrete version has been frequently adopted:

$$\Delta Y_{t+1} = Y_{t+1} - Y_t = \alpha(I_t - S_t), \tag{34}$$

$$\Delta K_{t+1} = K_{t+1} - K_t = I_t - \delta K_t \tag{35}$$

with $I_t = I(Y_t, K_t)$ and $S_t = S(Y_t, K_t)$.

Under classical assumptions proposed by Kaldor about the analytical properties of $I$ and $S$:

$$I_Y > S_Y > 0, \quad I_K < 0, \quad S_K > 0,$$

it can be verified that Hopf bifurcation occurs for certain parameter ranges and the system converges to a limited cycle. If the saving function is additionally assumed to be of Kaldorian S-shape type, strange attractors appear, that is, the system becomes chaotic (Gandolfo, 1996).

Dividing both sides of (34) by $Y_t$ and (35) by $K_t$ gives us

$$(Y_{t+1} - Y_t)/Y_t = \alpha(I(Y_t, K_t)/Y_t - S(Y_t, K_t)/Y_t), \tag{36}$$

$$(K_{t+1} - K_t)/K_t = I(Y_t, K_t)/K_t - \delta. \tag{37}$$

By Theorem 8, no matter whether the Kaldorian system becomes cyclic or chaotic, we always have $\langle (Y_{t+1} - Y_t)/Y_t \rangle > 0$ and $\langle (K_{t+1} - K_t)/K_t \rangle > 0$, or, equivalently,

$$\langle I_t/Y_t \rangle > \langle S_t/Y_t \rangle \quad \text{and} \quad \langle I_t/K_t \rangle > \delta. \tag{38}$$

Therefore, in addition to the usual conclusions that the average growth rates of income and capital are both greater than zero, we also arrive at the following observations:

(1) the investment ratio $I_t/Y_t$ is greater than the saving ratio $S_t/Y_t$ on average, and
(2) the proportion of investment on capital is greater than the depreciation rate $\delta$.

These two results are interesting, since, by taking the time-average operator $\langle\cdot\rangle$ on both sides of (34) and (35), respectively, and utilizing the facts that $\langle Y_{t+1}\rangle = \langle Y_t\rangle$ and $\langle K_{t+1}\rangle = \langle K_t\rangle$, we also obtain

$$\langle I_t\rangle = \langle S_t\rangle = \delta\langle K_t\rangle,$$

which in fact suggests that

$$\langle I_t\rangle/\langle Y_t\rangle = \langle S_t\rangle/\langle Y_t\rangle \quad \text{and} \quad \langle I_t\rangle/\langle K_t\rangle = \delta. \tag{39}$$

Combining (38) and (39) gives us a necessary condition for the system to be ergodic:

$$\langle I_t/Y_t\rangle > \langle I_t\rangle/\langle Y_t\rangle, \quad \text{and} \quad \langle I_t/K_t\rangle > \langle I_t\rangle/\langle K_t\rangle. \tag{40}$$

**Remark 10.** Noticing the contrast between (32) and (40), we see that our results do provide an additional means to test the specifications and assumptions of different models.

## 4. Equilibrium return versus long-run average return

Consider a one-dimensional discrete dynamical process defined by

$$x_{t+1} = \Theta(x_t) = \min\{(1+\beta)x_t, \theta(x_t)\}, \tag{41}$$

where $\theta(x_t)$ is either a monotonically decreasing continuous function, or a continuous single-humped map in the following sense:

(i) there exists $\hat{x}$ in $\mathfrak{R}$ such that $\theta(\hat{x}) > \hat{x}$ and $\theta'(x) \lesseqgtr 0$ if and only if $x \gtreqless \hat{x}$;
(ii) there exists a unique equilibrium steady state $\bar{x}$ in the interval $[\bar{x}, \theta(\bar{x})]$ so that $\theta(\bar{x}) = \bar{x}$;
(iii) there are no fixed points in the interval $[\theta^2(\bar{x}), \bar{x}]$.

A continuous return function $\pi : \mathfrak{R} \to \mathfrak{R}$ serves as a performance evaluation for the process so that a sequence return $\pi_t$ is obtained: $\pi_t = \pi(x_t)$.

The type of system defined by (41) has been widely applied in economic modeling and analysis (Day, 1978).

The purpose of this section is to compare the equilibrium return $\bar{\pi} \equiv \pi(\bar{x})$ with the long-run average return, defined by,

$$\langle\pi\rangle_\beta = \lim_{T\to\infty}(1/T)\sum_{t=1}^{T}\pi(x_t), \tag{42}$$

should such an average exist. Such comparison is of great importance if the dynamical system is chaotic.

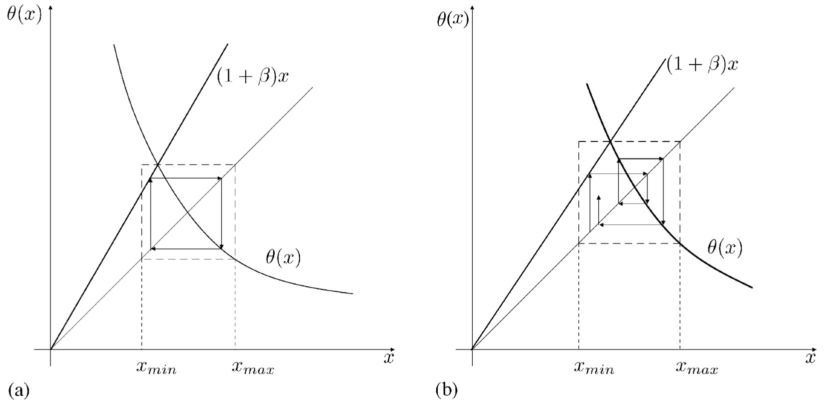



Fig. 3. Illustration of limited growth rate mechanism.

### 4.1. State average versus equilibrium

As shown in Fig. 3, there exists a trapping set, $J_\beta = [x_{\min}, x_{\max}]$, such that all quantity trajectories will enter into $J$ after a finite period and will stay there forever, where[8]

$$x_{\max} = (1+\beta)\hat{x} \quad \text{and} \quad x_{\min} = \theta(x_{\max}),$$

so that the map (41) can be alternatively expressed as

$$x_{t+1} = \Theta(x_t) = \begin{cases} (1+\beta)x_t, & x_{\min} \le x_t \le \hat{x}, \\ \theta(x_t), & \hat{x} \le x_t \le x_{\max}, \end{cases} \tag{43}$$

where $\hat{x}$, the turning point, is uniquely determined by the identity: $(1+\beta)\hat{x} = \theta(\hat{x})$.

The unique nontrivial steady-state $\overline{x}$ is locally unstable if $\sigma = \theta'(\overline{x}) > 1$, where $\sigma$ so defined is commonly referred to as the multiplier of the steady state $\overline{x}$.

When $\sigma = 1$, the typical dynamics of (43) is aperiodic fluctuations. Occasionally, there can be periodic cycles. Theoretically, if we denote $J_\beta$ as the map constrained on $J$, then under some rather weak mathematical requirements such as nonpositive Schwarzian derivative for $x$ in $J_\beta$, the system will be chaotic and ergodic.

When $\Theta_\beta$ is chaotic, the long-run average return defined in (42) provides a good evaluation of performance.

For convenience, we denote $\pi_{(k)}$ as the average of the evaluation function for $k$ iterations, that is,

$$\pi^{(k)}(x_0) = (1/k)\sum_{t=0}^{k-1}\pi(x_t),$$

[8] From here on, $x_{\min}$ and $x_{\max}$ will be understood to be functions of $\beta$.

and the *long-run average* of $\pi_t$ starting with $x_0$ as

$$\langle\pi\rangle_\beta(x_0) = \lim_{k\to\infty} \pi_k(x_0)/k. \tag{44}$$

Then we have the following:

**Theorem 11.** *When a return function $\pi$ is downward-sloping (upward-sloping) around the unstable equilibrium $\bar{x}$, there exists a $x_m$, $x_m > \bar{x}$, such that the average return of two iterations $\{x, \Theta_\beta(x)\}$ is greater (less) than the equilibrium return if $x \in (\bar{x}, x_m)$. Moreover, the more unstable the equilibrium is, the greater the difference between the average return and the equilibrium return.*

**Proof.** See Appendix A. □

If $(x_1, x_2)$ is a pair periodic-2 points with $x_1 < \bar{x} < x_2$, then Theorem 11 indicates that

(i) if the return function is downward-sloping at the equilibrium, the average return of the periodic cycle will be greater (less) than the equilibrium return if $x_2$ ($x_1$) is close enough to the equilibrium $\bar{x}$.
(ii) if the return function is upward-sloping at the equilibrium, the average return of the periodic cycle will be greater (less) than the equilibrium return if $x_1$ ($x_2$) is close enough to the equilibrium $\bar{x}$.

With Theorem 11 in hand, we can proceed to prove the next theorem.

**Theorem 12.** *For an unstable discrete system specified by (43), there always exists a positive constant $\beta_m$ such that, when the parameter $\beta$ is set within the open interval $(0, \beta_m)$, then the long-run average return defined by (42) will be greater (less) than the equilibrium return for any initial $x \in J_\beta$ as long as the return function is downward-sloping (upward-sloping) at the equilibrium.*

**Proof.** See Appendix A. □

It should be pointed out at this moment, that the $\beta_m$ in Theorem 12 sometimes can be as large as the maximum possible value that makes the dynamical system (43) possible.

In economic applications, the return functions are either assumed quasi-concave, or quasi-convex. In these two cases, the shape of $\langle\pi\rangle_\beta - \bar{\pi}$, as a function of $\beta$, can be easily visualized. It will either increase (decrease) for all $\beta$, or increase (decrease) for $\beta \in (0, \beta_{\text{opt}})$ and then decrease (increase) for $\beta > \beta_{\text{opt}}$, where $\beta_{\text{opt}}$ is a positive parameter that depends on both $\pi$ and $\theta$.

When $\theta$ is ergodic, the long-run average defined by (42) should be independent of the initial value $x_0$, so we will denote it by $\langle\pi\rangle_\beta$. Furthermore, $\langle\pi\rangle_\beta$ can be analytically calculated from the density function $\phi_\beta(x)$ of $\Theta_\beta$, should the latter be explicitly given.[9] That is,

$$\langle\pi\rangle_\beta = \int_{x_{\min}}^{x_{\max}} \pi(x)\phi_\beta(x)\,\mathrm{d}x.$$

It can also be seen that the parameter $\beta$ itself can be utilized to serve as a choice variable to optimize long-run average return.

If we denote $\langle x\rangle_\beta$ as the long-run average of the state variable

$$\langle x\rangle_\beta = \lim_{k\to\infty}(1/k)\sum_{t=0}^{k-1} x_t, \tag{45}$$

which would equal the space average, that is, $\langle x\rangle_\beta = \int_{x_{\min}}^{x_{\max}} x\phi_\beta(x)\,\mathrm{d}x$, should $\Theta_\beta$ be ergodic.

Let $\pi(x) = x$ in Theorem 12, and we immediately get

**Corollary 13.** *For a chaotic discrete system defined by (43), there always exists a positive constant $\beta_x$ such that, when the parameter $\beta$ is set within the open interval $(0, \beta_x)$, the long-average of state defined by (45) will be less than the equilibrium state $\bar{x}$.*

*4.2. Economic growth in the very long-run*

Consider a classical growth model discussed in Day (1983).

In an egalitarian, agrarian economy, according to Malthusa, when the necessities of life were in abundance, population tended to grow at a maximal biological or natural rate, say $\beta$; when they were scarce, the net population birth rates were the maximum attainable under a culturally determined subsistence level, say $\alpha$. Accordingly, the population rate of growth in per capita terms is governed by the function

$$\Delta P/P = \min\{\beta, (w-\alpha)/\alpha\} \tag{46}$$

in which $\Delta P/P$ is the net birth rate and $w$ is the wage rate.

The aggregate output of the society is determined by a production function $Y = f(P)$, which is assumed to be continuous and single-humped.

Further, it is assumed that the output is distributed according to the average product so that

$$w = f(P)/P.$$

If we let $\Delta P = P_{t+1} - P_t$, the population growth dynamics of (46) become

$$P_{t+1} = \Theta_\beta(P_t) = \min\{(1+\beta)P_t, w_t P_t/\alpha\} = \min\{(1+\beta)P_t, f(P_t)/\alpha\}. \tag{47}$$

[9] In general, however, the invariant density of such two-piece map, if it exists, is not a classic function but a generalized function, or a finite (or infinite) superposition of classical functions and Dirac functions.

Since $f(P)$ is single-humped, there exists a unique equilibrium $\overline{P}$, at which $\Theta'_\beta(\overline{P}) < 0$. If the discrete system (47) is chaotic and ergodic, the following conclusions follow from Theorem 12 directly.

**Theorem 14.** *When the population dynamics specified by (47) are chaotic and ergodic, if the natural rate $\beta$ is small, then*

(i) *the long-run average of population $\langle P \rangle_\beta$ is less than the equilibrium population $\overline{P}$ and*
(ii) *the long-run averages of aggregate product $\langle Y \rangle_\beta$ is greater than the equilibrium product $\overline{Y}$.*

### 4.3. Cautious Cobweb dynamics

Cobweb dynamics has been widely applied to model the situation in which a perfectly competitive firm must make its output decision one period in advance of the actual sale, such as in agriculture, fishing, forestry, and construction, where the application of production inputs must precede the sale of the output by an appreciable length of time. Formally, consider a product produced by a firm at period $t$ with output $q_t$. The cost of this product is assumed to be $C(q_t)$, with $C'(\cdot) > 0$. The market inverse demand for the product is $D^{-1}(q_t)$, with $D^{-1'}(\cdot) < 0$. It is assumed that actual market price adjusts to demand so as to clear the market.

Not knowing the price $p_t$ that will actually prevail in the market, the competitive firm can only choose an output level $q_t$that makes its expected gross profit $\pi^{\mathrm{e}}(q_t) = p_t^{\mathrm{e}} q_t - C(q_t)$ as large as possible, where $p_t^{\mathrm{e}}$ is the expected price at period $t$. Without other constraints, the optimal output level is determined by the equality of the marginal cost, $\mathrm{MC}(q_t)$, to the expected price, that is, $p_t^{\mathrm{e}} = \mathrm{MC}(q_t)$, or,

$$q_t = \mathrm{MC}^{-1}(p_t^{\mathrm{e}}),$$

where $\mathrm{MC}^{-1}$ denotes the inverse marginal cost function.

The traditional Cobweb dynamics stems from the so-called "naive expectation" that the expected price is assumed to equal the actual price in the previous period, that is, $p_t^{\mathrm{e}} = p_{t-1}$, so that the output dynamics is formulated as

$$q_t = \theta(q_{t-1}) = \mathrm{MC}^{-1}(D^{-1}(q_{t-1})). \tag{48}$$

When both inverse demand and marginal cost functions are monotonic, $\theta(q)$ is a monotonic decreasing function of $q$. A unique equilibrium output $\overline{q}$ exists, which satisfies the identity: $\overline{q} = \theta(\overline{q})$, that is,

$$\mathrm{MC}(\overline{q}) = D^{-1}(\overline{q}).$$

Define the multiplier of the equilibrium $\overline{q}$ as $\sigma = |\theta'(\overline{q})| = -\mathrm{MC}^{-1}(\overline{p})/D'(\overline{p})$, where $\overline{p} = D^{-1}(\overline{q})$ is the equilibrium price. Then the necessary and sufficient condition for the equilibrium to be globally stable is $\sigma < 1$, or equivalently, *the demand elasticity must be higher than the supply elasticity*. Otherwise, if $\sigma > 1$, the equilibrium $\overline{q}$ is unstable.

In the real business world, it is commonly observed that a competitive firm will respond to fluctuating prices cautiously by limiting the growth rate of its output. This kind of strategy and the revised Cobweb model were first studied by Day (1978) and subsequently extended in Day (1994). The model assumes that the competitive firm will impose an upper bound, say, $\beta$, on the growth rate:

$$q_{t+1}/q_t \leq (1+\beta), \tag{49}$$

where $\beta \geq 0$. To see the implication of (49), we can define a flexibility bound by $q^u_{t+1} = (1+\beta)q_t$. Supposing that starting with period $t$, the constraint has governed the production process for $n$ periods, then $q^u_{t+n} = (1+\beta)^n q_t$. Therefore, along the exponential growth path of output (at the rate $\beta$), the firm becomes increasingly flexible.

The original Cobweb model with naive expectations (48) is thus modified to the following cautious Cobweb model:

$$q_{t+1} = \min\{(1+\beta)q_t, \theta(q_t)\}, \tag{50}$$

where $\theta$ is defined in (48).

Limiting the output growth rate is economically justified in the real world. It can be explained by capacity constraints, financial constraints and cautious response to price uncertainty by firms. In terms of dynamics, imposing an upper bound on the growth rate of output alone does not suffice to achieve an equilibrium in an originally unstable market. Instead, it only reduces the range of fluctuations so as to make the originally unbounded dynamics become a bounded one.

It is also important to notice that with the constraint on output growth, price is still determined by the inverse demand function. Hence the profit function denoted by $\pi$ is given by

$$\pi(q_t) = q_t D^{-1}(q_t) - C(q_t).$$

We notice that, $\pi'(q) = D^{-1}(q) + qD^{-1'}(q) - \mathrm{MC}(q)$. Since at the equilibrium $\bar{q}$, we have $D^{-1}(\bar{q}) = \mathrm{MC}(\bar{q})$, then

$$\pi'(\bar{q}) = \bar{q}D^{-1'}(\bar{q}) < 0,$$

that is, the profit function $\pi$ is downward-sloping at the equilibrium level. Then it follows from Theorem 12 that

**Theorem 15.** *When the cautious Cobweb dynamics specified by (50) are chaotic and ergodic, if the growth rate limit $\beta$ is small, then*

(i) *the long-run average of output $\langle q \rangle_\beta$ is less than the equilibrium output $\bar{q}$ and*
(ii) *the long-run averages of profit $\langle \pi \rangle_\beta$ is greater than the equilibrium profit $\bar{\pi} = \bar{q}D^{-1}(\bar{q}) - C(\bar{q})$, that is, chaos is preferred to the equilibrium.*

Detailed discussions and numerical simulations can be found in Huang (1995).

## 5. Concluding remarks

Some useful statistical inequalities of ergodic dynamical process are discussed and illustrated. It is demonstrated that economists can resort to long-term statistical properties to get critical information about the economic processes. In particular, the long-run average is one crucial and indispensable measurement in analyzing chaotic economic processes.

The paradox of average growth rate revealed in Section 3 is generic in the sense of being irrespective of the dynamic structure of a dynamic process. Economic interpretations, implications, and possible applications are yet to be explored. Nevertheless, it suggests that relying on the average growth rate alone may lead to systematically biased conclusions in a chaotic economy.

It can be hypothesized that the very "paradox" is a generic property exhibited in all positive processes that have no clear trend of growth or decay. It is the positivity of the processes that seems to be driving the results. This is because the growth rate of a positive process is unbounded above, but bounded below by $-100$. Therefore, our findings suggest that the long-run average growth rate for certain positive stochastic processes may also be biased upward. It would be worth future research to consider which aspects of the above results also hold for stochastic environments and to reexamine various econometric techniques for estimating average growth rates.

## Acknowledgement

I wish to express my sincere gratitude to Richard H. Day, for whom this conference is in honor, for his introduction to me of this interesting but "chaotic" field. I am grateful to an anonymous referee for his critical comments and helpful suggestions. A special debt is owed to my late colleague Grant A. Taylor who helped in improving the presentation and organization of the previous drafts. The usual disclaimer applies.

## Appendix A. Proofs of the main results

### *A.1. Proof of Theorem 3*

Let $\varphi(x)$ be the ergodic invariant density function of $\theta$. Due to the ergodicity the time-average should be equal to the space average

$$\langle F(x)G(\theta(x))\rangle = \int_a^b F(x)G(\theta(x))\varphi(x)\,\mathrm{d}x$$

$$\langle F(x)G(x)\rangle = \int_a^b F(x)G(x)\varphi(x)\,\mathrm{d}x,$$

$$\begin{aligned}\langle F(x)G(\theta(x))\rangle - \langle F(x)G(x)\rangle &= \int_a^{\bar{x}} F(x)[G(\theta(x)) - G(x)]\varphi(x)\,\mathrm{d}x \\ &\quad + \int_{\bar{x}}^{b} F(x)[G(\theta(x)) - G(x)]\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Referring to Fig. 2, the existence of a unique dividing point $\bar{x}$ guarantees that $\theta(x) \gtreqless x$ for $x \lesseqgtr \bar{x}$.

We start with the case that both $F$ and $G$ are monotonic increasing functions of $x$, i.e. $F'(x) > 0$ and $G'(x) > 0$, for all $x$:

$$\begin{aligned}\int_a^{\bar{x}} F(x)[G(\theta(x)) - G(x)]\varphi(x)\,\mathrm{d}x &\le \int_a^{\bar{x}} F(\bar{x})[G(\theta(x)) - G(x)]\varphi(x)\,\mathrm{d}x, \\ \int_{\bar{x}}^{b} F(x)[G(\theta(x)) - G(x)]\varphi(x)\,\mathrm{d}x &\le \int_{\bar{x}}^{b} F(\bar{x})[G(\theta(x)) - G(x)]\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle F(x)G(\theta(x))\rangle - \langle F(x)G(x)\rangle &\le \int_a^{\bar{x}} F(\bar{x})[G(\theta(x)) - G(x)]\varphi(x)\,\mathrm{d}x \\ &\quad + \int_{\bar{x}}^{b} F(\bar{x})[G(\theta(x)) - G(x)]\varphi(x)\,\mathrm{d}x \\ &= F(\bar{x})\int_a^{b} (G(\theta(x)) - G(x))\varphi(x)\,\mathrm{d}x \\ &= F(\bar{x})(\langle G(\theta(x))\rangle - \langle G(x)\rangle) = 0.\end{aligned}$$

Here we utilize the fact that $\langle G(\theta)\rangle = \langle G(x)\rangle$.

The rest of the cases with different signs of derivatives of $F$ and $G$ follow the same principle.

### A.2. Proof of Theorem 11

The derivative of the average return of two iterations is

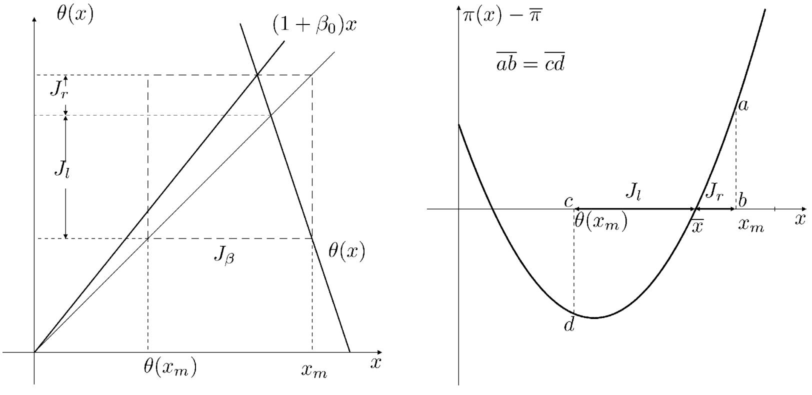
$$\pi'^{(2)}(x) = (\pi'(x) + \pi'(\Theta_\beta(x))\Theta'_\beta(x))/2,$$

which leads to $\pi'^{(2)}(\bar{x}) = (\pi'(\bar{x}) + \pi'(\theta(\bar{x}))\theta'(\bar{x}))/2 = -\pi'(\bar{x})(\sigma - 1)/2$, where $\sigma = |\theta(\bar{x})|$.

Since the equilibrium $\bar{x}$ is unstable ($\sigma > 1$), it follows that

$$\pi'^{(2)}(\bar{x}) \gtreqless 0 \text{ if and only } \pi'(\bar{x}) \lesseqgtr 0. \tag{51}$$

Notice that if $\pi^{(2)}(\bar{x}) = \pi(\bar{x})$, the inequality (51) implies that, for $x$ close enough to $\bar{x}$, the average return of a two-period iteration $\{x, \theta(x)\}$, with $\theta(x) > x$, will be greater (less) than the equilibrium return $\pi(\bar{x})$ if the return function is downward-sloping (upward-sloping) at the equilibrium $\bar{x}$.




(a) (b)

Fig. A.1. Illustration for Proof of Theorem 12.

If the return function is downward-sloping (upward-sloping) for all $x > \overline{x}$, there exists a $\tilde{x}$, $\tilde{x} > \overline{x}$, such that $\pi(\tilde{x}) < 0\ (> 0)$ and $\pi^{(2)}(\tilde{x}) = (\pi(\tilde{x}) + \pi(\theta(\tilde{x})))/2 = 0$. By the continuity of $\pi^{(2)}$ (from the continuity of $\pi$), we infer the existence of such $x_m$ that $\pi^{(2)}(x_m) = \pi^{(2)}(\overline{x}) = \pi(\overline{x})$.

Otherwise, $x_m$ may be as large as $x_{\max}$.

On the other hand, the facts that $\pi^{(2)}(\overline{x}) = \pi(\overline{x})$ and $|\pi'^{(2)}(\overline{x})| = (\sigma - 1)|\pi'(\overline{x})|/2$ indicate that the more unstable the equilibrium, the greater the difference between $\pi^{(2)}(x)$ and $\pi(\overline{x})$.

### *A.3. Proof of Theorem 12*

We provide the proof only for the case in which the return function is upward-sloping at the equilibrium.

From Theorem 11, there exists a $x_m$ such that the average return $\pi^{(2)}(x) < \pi(\overline{x})\ \forall x \in (\overline{x}, x_m)$. Referring to Fig. A.1, we choose $\beta_0 = x_m/\theta^{-1}(x_m) - 1$ so that $\hat{x} = \theta^{-1}(x_m)$, where $\hat{x}$ is the turning point.

Then the trajectory $\{x_t\}_{t=0}^{\infty}$ will be restricted in $J_0 = J_l \cup J_r = [\theta(x_m), \overline{x}] \cup [\overline{x}, x_m]$.

We note that whenever $x_t$ is greater than the equilibrium $\overline{x}$ (but less than $x_m$), $x_{t+1}$ is less than $\overline{x}$. In contrast, whenever $x_t$ is less than $\overline{x}$, $x_{t+1}$ can be either greater than $\overline{x}$ or less than $\overline{x}$. Hence, the trajectory $\{x_t\}_{t=0}^{\infty}$ either wanders into the left interval $J_l$ more frequently than into the right interval $J_r$ or visits them with the same frequency.

When $k$ is sufficiently large, for any $x_0 \in J_0$, we have

$$\begin{aligned}\pi^{(k)}(x_0) - \overline{\pi} &= (1/k)\sum_{t=0}^{k-1}(\pi(x_t) - \overline{\pi}) = (1/k)\sum_{t=0}^{k-1} d_t \\ &= (1/k)\left\{\Delta\pi_0 + \sum_{x_{t-1}\in J_l, x_t \in J_l}\Delta\pi_t + \sum_{x_{t-1}\in J_l, x_t\in J_r}\Delta\pi_t + \sum_{x_{t-1}\in J_r, x_t\in J_l}\Delta\pi_t\right\},\end{aligned}$$

where $\Delta\pi_t \doteq \pi(x_t) - \overline{\pi}$.

Notice that the number of terms in the second summation equals exactly the same as the number of terms in the third summation. Therefore, we can rewrite the last two terms into one so that

$$\pi^{(k)}(x_0) - \bar{\pi} = (1/k)\left\{\Delta\pi_0 + \sum_{x_{t-1}\in J_l, x_t\in J_l} \Delta\pi_t + 2\sum_{x_{t-1}\in J_l, x_t\in J_r} (\pi^{(2)}(x_t) - \bar{\pi})\right\}. \tag{52}$$

The construction of $\beta_0$ enforces $x_{\max} = x_m$, and

$$\pi^{(2)}(x_m) = [\pi(x_{\max}) + \pi(x_{\min})]/2 = \bar{\pi},$$

which implies that $\pi(x_{\min}) - \bar{\pi} = \bar{\pi} - \pi(x_{\max}) < 0$ (since $\pi' > 0$ around $\bar{x}$).

Then we must have $\pi(x) > \bar{\pi}$ for all $x \in J_l = [x_{min}, \bar{x}]$, and hence the first summation term in (52) is positive.

It then follows from Theorem 11 that the second summation term in (52) is negative as well. No matter whether $\Delta\pi_0$ is positive or negative, as long as $k$ is sufficiently large, we have $\pi^{(k)}(x_0) - \bar{\pi} < 0$.

Consequently, the existence of a $\beta_m$, $\beta_m \geq \beta_0$, such that $\lim_{k\to\infty} \pi^{(k)}(x_0) < \bar{\pi}$ for all $0 < \beta < \beta_m$ is guaranteed.

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