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Rearrangement inequalities in non-convex insurance models

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Abstract

This paper provides an existence theorem for a class of infinite-dimensional non-convex problems arising in symmetric and asymmetric information models. Sufficient conditions for monotonicity of solutions are also given. The proofs are very simple and rely on rearrangement techniques and the concept of supermodularity. Several applications to insurance are given.

JEL classification: C61; D81; D82; G22

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1. Introduction

In many infinite-dimensional optimization problems that appear in economics, concavity of the criterion and convexity of the feasible set are required to prove existence of a solution. Furthermore, as was already mentioned by Milgrom and Shannon (1994), analyses of qualitative properties of solutions are based on first order conditions and the implicit function theorem. Hence many non-convex infinite-dimensional problems cannot be dealt with asymmetric information, non-convex costs, non-quasi-concave criteria. This is all the

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more striking since the finite-dimensional versions can usually, easily, be coped with. The purpose of this paper is to provide an existence theorem for a class of infinite-dimensional non-convex problems and sufficient conditions for monotonicity of optimal solutions. The proofs are elementary and based on rearrangements and on the concept of supermodularity. Our main motivation is insurance and more particularly principal agent problems with adverse selection and moral hazard. Rearrangement tools also turn out to provide a solution to other problems in insurance (non-convex costs, non-expected utility criteria, background risk, etc.) and unify a number of results available in the literature. They may be also be used in other economic fields (incentive theory, see Carlier and Dana (in press); decision theory Carlier and Dana (2003)).

It is well known that, among pairs of random variables with given marginals, comonotone pairs, maximize correlation. This result is closely related to the concept of rearrangement and to rearrangement inequalities. These inequalities introduced by Hardy et al. (1988) have been extensively used in many areas of mathematics (graph theory, matrix theory, numerical analysis, geometry), statistics (rank order tests), calculus of variations (Cardaliaguet and Tahraoui, 2000) and probability. Marshall and Olkin (1979) provide a survey of applications up to the end of the seventies. Rearrangement inequalities are still used and give powerful results in various mathematical subjects. In contrast, rearrangement inequalities have barely been used in economics. A goal of this paper is to illustrate the power of these techniques in convex and non-convex problems arising in insurance.

Comonotone pairs of random variables with given marginals, also maximize a large class of measures of dependence that generalize correlation, based on supermodularity properties. Indeed, Hardy–Littlewood's inequalities have a supermodular extension. Supermodularity techniques have been used in a variety of economic subjects for the past 20 years, in particular to cope with lack of convexity. They were first used by Spence and Mirrlees (Spence, 1974; Mirrlees, 1976) in their signaling and taxation models. Besides incentive theory, they have been used, for example, for monotone comparative static techniques (Milgrom and Shannon, 1994), for sensitivity analysis of optimal growth problems (Amir et al., 1991; Amir, 1996), in supermodular games (Topkis, 1979, 1978; Vives, 1990; Milgrom and Roberts, 1990; Sobel, 1988) and for stochastic orders (see Gollier (2001) and Müller and Stoyan (2002)).

As already mentioned, the main tool of the paper is a supermodular version of Hardy–Littlewood's inequalities, originally proven by Lorentz (1953), in the case of Lebesgue's measure. It has then been extended by Cambanis et al. (1976). In order to emphasize the tight connection with incentive theory, we first provide, in the special case where they are the most often used, a proof by incentive theoretical arguments. The supermodular Hardy–Littlewood's inequalities are then used to prove an existence theorem for a class of infinite-dimensional non-convex problems with equality and inequality constraints. We further give sufficient conditions for monotonicity of optimal solutions.

In order to show that our results may be applied to a large variety of convex or non-convex problems arising in insurance, we consider three problems: a problem in decision theory, a problem with non-convex costs and finally a problem with adverse selection. Companion papers (Carlier and Dana, in press; Dana and Scarsini, 2005) deal with the case of moral hazard and background risk.

The first model we consider is a model where agents have strictly "Schur concave" (or strictly second order stochastic dominance preserving) utilities. Let us recall that most

"Schur concave" utilities are not concave. We prove that optimal contracts exist and prove that, as in von Neumann Morgenstern models, agents' wealth are comonotone. To our knowledge, the existence result is new. The comonotonicity property is well known (see Landsberger and Meilijson (1999)), however our proof is elementary and directly follows from rearrangement techniques.

We next deal with costs. The hypothesis that costs are convex, standard in microeconomics, has been quite criticized in insurance because of the presence of administrative and audit costs (see Gollier (1987), Hubermann et al. (1983) and Picard (2000a,b)). Gollier (1987) and Picard (2000a,b) consider a cost function with a jump at zero and affine elsewhere, while Hubermann et al. (1983) use concave cost functions. As concave costs may be dealt with very easily, we focus on the case of Gollier's and Picard's cost. We show that optimal contracts exist and that any solution has to be a disappearing deductible. Though these results are not new (see Picard (2000a), Gollier (1987) and Carlier and Dana (2003)), the proof is again a direct consequence of rearrangement inequalities.

We lastly turn to a principal agent model. We study a deterministic version of an adverse selection model considered by Landsberger and Meilijson (1999). We prove existence of optimal contracts and describe some of their qualitative properties.

The paper is organized as follows. In Section 2, we recall the concept of rearrangement of a Borel function on $[0, 1]$ with respect to a non-atomic probability measure. We then state a supermodular version of Hardy–Littlewood's inequality. We prove it, in a special case, by incentive theoretical arguments. In Section 3, we prove an existence theorem for non-convex problems based on rearrangements and provide sufficient conditions for monotonicity of optimal solutions. We then turn to applications. A section is devoted to the problem of efficient insurance contracts. The case where insurer and insured have symmetric information and are von Neumann Morgenstern maximizers is first studied, as a benchmark, by rearrangement techniques. Properties of efficient insurance contracts that remain true in non-convex settings are emphasized. We next generalize the model to non-expected utility maximizers with "Schur concave" utilities. We then study the non-convex problem of efficient insurance contracts when the cost structure includes a fixed cost per claim. The last section is devoted to a deterministic version of Landsberger and Meilijson's adverse selection model (Landsberger and Meilijson, 1999). We prove existence of optimal contracts and give some of their qualitative properties.

2. Rearrangements: a brief review

In this section, we recall the concept of rearrangement of a function with respect to any non-atomic probability measure and state some of its properties.

Given x, a Borel function defined on $[0, 1]$, in the rearrangement literature, one usually considers the rearrangement of x with respect to Lebesgue measure. It is also called the "generalized inverse of the distribution function of x" in probability theory (x is then considered as a random variable on the space $([0, 1], \mathcal{B}, \lambda)$ with λ the Lebesgue measure and $\mathcal{B}$ the Borel σ-algebra of $[0, 1]$). It is defined as follows: let F_x be the distribution function of x: $F_x(t) = \lambda\{s \mid x(s) \leq t\}$. The rearrangement of x (with respect to λ) or generalized inverse of F_x is defined by:

$$F_x^{-1}(t) = \inf\{z \in \mathbb{R} : F_x(z) > t\} \quad \text{for all } t \in]0, 1].$$

One easily verifies that F_x^{-1} is nondecreasing and that it is distributed like x.

2.1. Definitions and basic properties

Definition 1. A Borel measure μ on $[0, 1]$ is nonatomic if and only if $\mu(\{t\}) = 0$ for all $t \in [0, 1]$.

From now on, we assume that we are given a nonatomic Borel probability measure μ on $[0, 1]$.

Definition-Property 1. Two Borel functions on $[0, 1]$, x and y are equimeasurable with respect to μ if and only if they fulfill one of the following equivalent conditions:

1. $\mu(x^{-1}(B)) = \mu(y^{-1}(B))$ for every Borel subset B of $\mathbb{R}$.
2. For every bounded continuous function f:

$$\int_0^1 f(x(t))\,\mathrm{d}\mu(t) = \int_0^1 f(y(t))\,\mathrm{d}\mu(t).$$

3. For every random variable Z with probability distribution μ, $x(Z)$ and $y(Z)$ have the same probability distribution.

In the sequel, the fact that two Borel functions x and y are equimeasurable with respect to μ will simply be denoted by $x \sim y$. Given x, there exists a unique (up to μ a.e. equivalence) *nondecreasing* function which is equimeasurable to x. This function is called the nondecreasing rearrangement of x with respect to μ.

Proposition 1 (1). *Let x be some real-valued Borel function on* $[0, 1]$ *and* μ *be a nonatomic Borel probability measure on* $[0, 1]$. *Then there exists a unique* (*up to* μ*-a.e. equivalence*) *nondecreasing function which is equimeasurable to x. This function denoted* $\tilde{x}$ *is given by the explicit formula*:

$$\tilde{x}(t) := \inf\{u \in \mathbb{R} : v(u) > t\},$$

where

$$v(u) := \inf\{\alpha \in [0, 1] : \mu([\alpha, 1]) \leq \mu(\{s : x(s) \geq u\})\} = \inf\{\alpha \in [0, 1] : \mu([\alpha, 1]) = \mu(\{s : x(s) \geq u\})\}.$$

The proof can be found in Appendix A. From now on, $\tilde{x}$ will denote the nondecreasing rearrangement of x (with respect to μ). We shall also use the nonincreasing rearrangement of x which equals: $-(\widetilde{-x})$.

Remark. The necessity of the nonatomicity condition on μ in Proposition 1 is quite clear, as the following example shows: assume that $\mu = (1/3)\delta_0 + (2/3)\delta_1$ and let x be the characteristic function of $[0, 1/2]$. Obviously, there exists no $\tilde{x}$ such that $\tilde{x}$ is nondecreasing, $\mu(\{\tilde{x} = 0\}) = \mu(\{x = 0\}) = 2/3$ and $\mu(\{\tilde{x} = 1\}) = \mu(\{x = 1\}) = 1/3$.

We end this paragraph by Ryff's decomposition theorem: roughly speaking any function x can be written as the composition of its nondecreasing rearrangement $\tilde{x}$ and a *permutation*. What we mean by permutation relies on the concept of measure preserving maps.

Definition 2. A Borel map $s : [0, 1] \to [0, 1]$ is measure-preserving with respect to μ if $\mu(s^{-1}(B)) = \mu(B)$ for every Borel subset B of $[0, 1]$.

In other words, s is measure-preserving with respect to μ if its nondecreasing rearrangement with respect to μ is the identity map. Equivalently:

$$\int_0^1 f(t)\,\mathrm{d}\mu(t) = \int_0^1 f(s(t))\,\mathrm{d}\mu(t)$$

for every bounded continuous function f.

The following result was proven by Ryff (1970), for the sake of simplicity, we assume, as in Ryff (1970), that μ is the Lebesgue measure on $[0, 1]$:

Proposition 2. *Let μ be the Lebesgue measure on* $[0, 1]$. *Then for every real-valued Borel function x defined on* $[0, 1]$, *there exists s, a measure-preserving map with respect to μ such that $x = \tilde{x} \circ s$.*

Ryff's decomposition in fact extends to the case of more general measures and even to a multidimensional framework but we omit it here for the sake of simplicity.

2.2. Rearrangement inequalities

In this paragraph, we shall see that integral expressions of the form:

$$\int_0^1 L(x(t), y(t))\,\mathrm{d}\mu(t)$$

increase when one replaces the arbitrary functions $x(\cdot)$ and $y(\cdot)$ by their nondecreasing rearrangements $\tilde{x}(\cdot)$ and $\tilde{y}(\cdot)$ when the integrand L satisfies a *supermodularity* condition. By definition, a function L: $\mathbb{R}^2 \to \mathbb{R}$ is called *supermodular* if for all $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$ such that $x_2 \geq x_1$ and $y_2 \geq y_1$:

$$L(x_2, y_2) + L(x_1, y_1) \geq L(x_1, y_2) + L(x_2, y_1). \tag{1}$$

The function L is called *strictly supermodular* if for all $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$ such that $x_2 > x_1$ and $y_2 > y_1$:

$$L(x_2, y_2) + L(x_1, y_1) > L(x_1, y_2) + L(x_2, y_1). \tag{2}$$

Theorem 1. *Let μ be a nonatomic probability measure on $[0, 1]$, x and y be in $L^\infty([0, 1], \mathcal{B}, \mu)$ and L be supermodular. We then have*:

$$\int_0^1 L(\tilde{x}(t), \tilde{y}(t))\,\mathrm{d}\mu(t) \geq \int_0^1 L(x(t), y(t))\,\mathrm{d}\mu(t).$$

Moreover if L is continuous and strictly supermodular, then the inequality is strict unless x and y are comonotone, that is fulfill:

$$(x(t) - x(t'))(y(t) - y(t')) \geq 0 \mu \otimes \mu\text{-}a.e.\,(t, t').$$

In particular, if L is continuous and strictly supermodular, then

$$\int_0^1 L(t, \tilde{x}(t))\,\mathrm{d}\mu(t) > \int_0^1 L(t, x(t))\,\mathrm{d}\mu(t) \quad unless\, x = \tilde{x}\mu\text{-}a.e.$$

Note that the inequality given above can be written in probabilistic terms as:

$$E[L(\tilde{x}(Z), \tilde{y}(Z))] \geq E[L(x(Z), y(Z))]$$

for every random variable Z with probability law μ.

It seems that the first proof of Theorem 1 was given by Lorentz (1953). The first assertion follows from Cambanis et al. (1976). Indeed, in Cambanis et al. (1976), it is proven that if $(\Omega, \mathcal{A}, P)$ and $(\Omega', \mathcal{A}', P')$ are two probability spaces and X and Y (respectively, X' and Y') are pairs in $L^\infty(\Omega, \mathcal{A}, P)$ (respectively, in $L^\infty(\Omega', \mathcal{A}', P')$), then if $X \sim X'$, $Y \sim Y'$ and X' and Y' are comonotone (that is satisfy $(X'(\omega) - X'(\omega'))(Y'(\omega) - Y'(\omega')) \geq 0 P' \otimes P'$-a.e.), then

$$E(L(X', Y')) \geq E(L(X, Y))$$

whenever $L(\cdot, \cdot)$ is supermodular. The result may then be applied to the pairs $(x(Z), y(Z))$ and $(\tilde{x}(Z), \tilde{y}(Z))$ since by construction $\tilde{x}(Z) \sim x(Z)$ and $\tilde{y}(Z) \sim y(Z)$ and the pair $(\tilde{x}(Z), \tilde{y}(Z))$ is comonotone. Finally, we refer the reader to Cardaliaguet and Tahraoui (2000) for a generalization of Theorem 1.

When $L \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, supermodularity of L is equivalent to:

$$\frac{\partial^2 L}{\partial x \partial y} \geq 0 \tag{3}$$

and a sufficient condition for strict supermodularity of L is the Spence–Mirrlees type condition:

$$\frac{\partial^2 L}{\partial x \partial y} > 0. \tag{4}$$

There are many examples of functions L that satisfy (1). Let us mention $L(x, y) = f(x)g(y)$ with both f and g nondecreasing or both f and g nonincreasing, $L(x, y) = \min(x, y)$,

$L(x, y) = -(x-y)_+$, $L(x, y) = f(x+y)$ with f convex, $L(x, y) = U(x-y)$ with U concave. The *concave* case $U(x-y)$ will show of particular interest in applications. If $L(x, y) = xy$, we get the classical Hardy–Littlewood's inequality:

$$\int_0^1 \tilde{x}(t)\tilde{y}(t)\,\mathrm{d}\mu(t) \geq \int_0^1 x(t)y(t)\,\mathrm{d}\mu(t),$$

which also means that the covariance of $\tilde{x}(Z)$ and $\tilde{y}(Z)$ is greater than that of $x(Z)$ and $y(Z)$ for every random variable Z with probability law μ.

We have defined the concept of rearrangement of a Borel function with respect to a probability measure μ on $[0, 1]$. The definition and properties may be extended to non-atomic measures on $[0, \bar{x}]$, for any $\bar{x}$.

2.3. Rearrangements and adverse selection models

In this section, we aim to emphasize the tight connections between rearrangement techniques and unidimensional adverse selection models à la Mussa and Rosen (1978). In this paragraph we shall prove Theorem 1 in the special case where x is the identity function and the distribution function of μ is the Lebesgue measure by means of incentive-theoretical arguments.

Let us consider a standard (quasi-linear) averse selection model in which the unobservable type is scalar and the agent's utility function is:

$$U(t, x, p) := L(t, x) - p,$$

where $t \in [0, 1]$ is the agent's unobservable characteristic or type, $x \in \mathbb{R}_+$ is a consumption and $p \in \mathbb{R}_+$ is a price. Assume for simplicity, that types are uniformly distributed in the population i.e. according to the Lebesgue measure on $[0, 1]$ and that L is twice continuously differentiable and fulfills Spence–Mirrlees condition (4).

The purpose of an uninformed principal is to set an incentive compatible contract that is a pair of measurable functions

$$t \in [0, 1] \mapsto (x(t), p(t))$$

such that:

$$L(t, x(t)) - p(t) \geq L(t, x(t')) - p(t') \quad \text{for all}\,(t, t') \in [0, 1]^2. \tag{5}$$

$x(\cdot)$ is the *allocation* or *physical* part of the contract $(x(\cdot), p(\cdot))$ and $p(\cdot)$ its monetary part. An allocation function $t \mapsto x(t)$ is *implementable* if there exists $p(\cdot)$ such that the pair $(x(\cdot), p(\cdot))$ is incentive compatible.

Proposition 3. *Let x be a measurable allocation function*: $[0, 1] \to \mathbb{R}$ *such that $t \mapsto L(t, x(t))$ is Lebesgue integrable, then the following assertions are equivalent*:

1. *x is implementable*.
2. *x solves*:

$$\sup\left\{\int_0^1 L(t, y(t))\,\mathrm{d}t : y \sim x\right\}.$$

3. *x is nondecreasing*.

Proof. The equivalence between (1) and (3) is well-known (see for instance Rochet (1987)). To prove that (3) $\Rightarrow$ (2), let us assume that x is nondecreasing. (3) being equivalent to (1), x is implementable. Let $p(\cdot)$ be such that $(x(\cdot), p(\cdot))$ is incentive compatible and let $y \sim x$. Then $\tilde{y} = \tilde{x} = x$. From Ryff's decomposition, there exists a Lebesgue measure-preserving map $s : [0, 1] \to [0, 1]$ such that $y = x \circ s$. Incentive-compatibility yields for all t:

$$L(t, x(t)) - p(t) \geq L(t, x(s(t))) - p(s(t)) = L(t, y(t)) - p(s(t)).$$

Integrating this inequality over $[0, 1]$ and using the fact that s is measure-preserving, we obtain assertion (2):

$$\int_0^1 L(t, x(t))\,\mathrm{d}t \geq \int_0^1 L(t, y(t))\,\mathrm{d}t.$$

Let us finally prove that (2) $\Rightarrow$ (3). Assume that x satisfies (2). Let t and t' be two Lebesgue points[1] of $s \mapsto L(s, x(s))$ with $0 < t < t' < 1$ and let $\varepsilon > 0$ be such that $t + \varepsilon < t' - \varepsilon$. Let us prove that $x(t) \leq x(t')$. Let $x_\varepsilon = x$ on $(0, 1) \setminus (t - \varepsilon, t + \varepsilon) \cup (t' - \varepsilon, t' + \varepsilon)$, $x_\varepsilon(s) = x(s + t' - t)$ for $s \in (t - \varepsilon, t + \varepsilon)$ and $x_\varepsilon(s) = x(s + t - t')$ for $s \in (t' - \varepsilon, t' + \varepsilon)$. Then, $x_\varepsilon \sim x$, hence from (2), we get:

$$\frac{1}{2\varepsilon}\left(\int_0^1 L(t, x(t)) - L(t, x_\varepsilon(t))\,\mathrm{d}t\right) \geq 0.$$

Passing to the limit, we obtain that:

$$L(t', x(t')) - L(t, x(t')) \geq L(t', x(t)) - L(t, x(t)),$$

which, with the strict supermodularity of L, yields $x(t') \geq x(t)$. This proves that for Lebesgue a.e. $(t, t') \in (0, 1)^2$ with $t' \geq t$ one has $x(t') \geq x(t)$ so that x coincides a.e. with a nondecreasing function. □

[1] Let us recall a fine property of integrable functions, we refer to Rudin (1987) for details. If f is a Lebesgue integrable function on $(0, 1)$, then a.e. $t \in (0, 1)$ is a *Lebesgue point* of f which, by definition, means:

$$f(t) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} f.$$

The equivalence between (1) and (2) can be interpreted as follows: x is implementable if and only if there is no other allocation profile with the same distribution that induces a higher surplus. Finally, let us note that a straightforward consequence of the previous proposition is the rearrangement inequality:

$$\int_0^1 L(t, \tilde{x}(t))\,\mathrm{d}t \geq \int_0^1 L(t, x(t))\,\mathrm{d}t.$$

3. Existence via monotonicity

In this section, we prove the existence of nondecreasing solutions for a class of optimization problems by using Theorem 1. We mention that a similar method was used by Cardaliaguet and Tahraoui (2000), for a broad class of calculus of variations problems.

Let us first recall a compactness property of the set of nondecreasing functions due to Helly (see for instance Natanson (1955)).

Lemma 1. *Let (x_n) be a uniformly bounded sequence of nondecreasing functions defined on* $[0, 1]$. *Then there exists a nondecreasing function x defined on* $[0, 1]$ *and a subsequence, again denoted* (x_n), *which converges pointwise to x on* $[0, 1]$.

Let μ be a nonatomic Borel probability measure on $[0, 1]$, we consider the following optimization problem:

$$\sup_{(x,P)\in\mathcal{C}} J(x, P) := \int_0^1 L\left(t, x(t), \int_0^1 \gamma(x(s))\,\mathrm{d}\mu(s), P\right) \mathrm{d}\mu(t), \tag{6}$$

where $\mathcal{C}$ is a constrained set of the form $\mathcal{C} := \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3$ with:

$$\mathcal{C}_1 := \{(x, P) : P \in K, x(\cdot)\text{Borel:} x_0 \leq x \leq x_1 \mu\text{-a.e.}\},$$

$$\mathcal{C}_2 := \left\{(x, P) : P \in K, x(\cdot)\text{Borel:} \int_0^1 f_i(x(t), P)\,\mathrm{d}\mu(t) = 0, i = 1, \ldots, k_1\right\},$$

$$\mathcal{C}_3 := \left\{(x, P) : P \in K, x(\cdot)\text{Borel:} \int_0^1 g_j(t, x(t), P)\,\mathrm{d}\mu(t) \geq 0, j = 1, \ldots, k_2\right\}$$

and we make the following assumptions:

- K is a compact subset of $\mathbb{R}^k$, L is continuous on $[0, 1] \times \mathbb{R}^2 \times K$ and for every $(z, P) \in \mathbb{R} \times K$, $(t, x) \mapsto L(t, x, z, P)$ is supermodular.
- Either $\gamma(\cdot)$ is continuous or $\gamma(\cdot)$ is l.s.c. and bounded from below and L satisfies the additional monotonicity condition: for all $(t, x, z, z', P) \in [0, 1] \times \mathbb{R}^3 \times K$, $z \leq z' \Rightarrow L(t, x, z, P) \geq L(t, x, z', P)$.
- x_0 and x_1 are nondecreasing and bounded on $[0, 1]$ with $x_0 \leq x_1$.
- f_i is continuous on $\mathbb{R} \times K$, for $i = 1, \ldots, k_1$.

- g_j is continuous on $[0,1]\times\mathbb{R}\times K$, for $j=1,\dots,k_2$, and for all $P\in K$, $(t,x)\mapsto g_j(t,x,P)$ is supermodular.
- $\mathcal{C}\neq\emptyset$.

Let us first make a few comments on the assumptions. We assume that the criterion is of the form $E(L(Z,x(Z),E(\gamma(x(Z))),P))$ with L supermodular with respect to its first two arguments. We have in mind insurance problems where typically the criterion depends on the risk Z, the contract $x(Z)$, the premium P. When the insurer is risk neutral, then the premium equals the expected cost of the contract $E(\gamma(x(Z)))$ and the criterion becomes $E(L(Z,x(Z),E(\gamma(x(Z)))))$. It should be noticed that no convexity assumption is made on the cost function γ; we only require it to be l.s.c. In particular the cost function may be concave (see Hubermann et al. (1983)) or may have a discontinuity at zero (see Gollier (1987)). As far as the constraints are concerned, the functions x_0 and x_1 model state by state constraints. Typically $x_0=0$ and $x_1=\mathrm{Id}$. $\mathcal{C}_2$ and $\mathcal{C}_3$ are motivated by asymmetric information problems (see the section below on Landsberger and Meilijson's (1999)model). For further use, let us mention that if $L(t,x,z)=U(x-t-z)$ with U concave, then L satisfies the first assumption listed above.

The next result expresses that $\mathcal{C}$ is stable under rearrangements.

Lemma 2. *For $i=1,2,3$, if $(x,P)\in\mathcal{C}_i$, then $(\tilde{x},P)\in\mathcal{C}_i$.*

Proof. For notational purpose we may without loss of generality omit the dependence in P in this proof. To prove that $\mathcal{C}_1$ is stable under rearrangements, let us remark that if $(x,y)\in(L^\infty([0,1],\mathcal{B},\mu))^2$ are such that $x\le y\mu$-a.e., then $\tilde{x}\le\tilde{y}\mu$-a.e. This may be checked directly from the formulas defining $\tilde{x}$ and $\tilde{y}$ but let us give a proof using Theorem 1. Indeed, let $L(x,y):=-(x-y)_+$ for every $(x,y)\in\mathbb{R}^2$. If $x\le y$ μ-a.e., as L satisfies (1), we have:

$$0=\int_0^1(x(t)-y(t))_+\mathrm{d}\mu(t)\ge\int_0^1(\tilde{x}(t)-\tilde{y}(t))_+\mathrm{d}\mu(t),$$

which proves that $\tilde{x}\le\tilde{y}\mu$-a.e. $\mathcal{C}_2$ is stable under rearrangements by equimeasurability. Lastly a direct consequence of Theorem 1 is that integral constraints of the form

$$\int_0^1 g(t,x(t))\,\mathrm{d}\mu(t)\ge 0$$

are also stable under rearrangement, provided that the function g is supermodular. Hence $\mathcal{C}_3$ is stable under rearrangements. □

Let us now turn to our main existence result:

Theorem 2. *Under the assumptions listed above, problem (6) has a solution. Furthermore, if (x,P) is a solution of (6), then $(\tilde{x},P)$ is also a solution of (6). If, in addition, L is strictly supermodular with respect to its first two arguments, then any solution (x,P) of (6) satisfies $x=\tilde{x}\mu$-a.e..*

The proof can be found in Appendix A.

Remark. In moral hazard problem, the measure μ depends on endogenous effort parameters. The previous existence theorem may easily be extended provided μ has a density continuous with respect to these parameters (see Carlier and Dana (in press)).

4. Pareto efficient insurance contracts

4.1. The benchmark model

We first recall the model of efficient insurance contracts when insurer and insured have symmetric information and are von Neumann Morgenstern maximizers. Our aim is to establish properties of efficient insurance contracts that will remain true in non-convex settings, by using tools from the previous sections.

Insurance buyers face a loss X where X is a random variable with support $[0, \bar{x}]$ and nonatomic probability law μ. The insurance market provides insurance contracts for this loss. A contract is characterized by a premium P and an indemnity schedule $I : [0, \bar{x}] \to \mathbb{R}$ which satisfies $0 \leq I(x) \leq x\mu$ a.e. on $[0, \bar{x}]$. When the insured buys the contract, he is endowed with the random wealth $W(X) := w_0 - P - X + I(X)$ where w_0 is the insured's initial wealth. For the sake of simplicity, from now on, we normalize $w_0 = 0$.

The insured is assumed to have von Neumann Morgenstern preferences over random wealth: $\int_0^{\bar{x}} U(W(x))\,\mathrm{d}\mu(x)$ where $U : \mathbb{R} \to \mathbb{R}$ is strictly concave, and strictly increasing. His indirect utility over contracts is:

$$u(P, I) := \int_0^{\bar{x}} U(W(x))\,\mathrm{d}\mu(x) = \int_0^{\bar{x}} U(-P - x + I(x))\,\mathrm{d}\mu(x).$$

By selling the contract the insurer gets P and promises to pay $I(x)$ if loss x occurs. His profit $\pi(P, I)$ is assumed here to be of the following form

$$\pi(P, I) := P - \int_0^{\bar{x}} c(I(x))\,\mathrm{d}\mu(x),$$

where the cost function $c : \mathbb{R}_+ \to \mathbb{R}$ satisfies: $c(0) = 0$ and c is convex, continuous and increasing on $[0, +\infty)$.

Our aim is to study Pareto efficient contracts. For the sake of completeness, we recall some definitions.

Definition 3.

(1) A contract (P, I) is Pareto efficient iff there exists no contract (P', I') such that $u(P', I') \geq u(P, I)$, $\pi(P', I') \geq \pi(P, I)$ with at least one strict equality.
(2) Two contracts (P, I) and (P', I') are utility equivalent iff $u(P, I) = u(P', I')$ and $\pi(P', I')$.
(3) The contract (P, I) dominates (resp. strictly dominates) the contract (P', I') if $u(P, I) \geq u(P', I')$ and $\pi(P, I) \geq \pi(P', I')$ (resp. with at least one strict inequality).

Let Id: $\mathbb{R}_+ \to \mathbb{R}_+$ be defined by $\mathrm{Id}(t) = t$ for all $t \geq 0$. As usual, we may parameterize Pareto efficient contracts by the profit level λ of the insurer. A contract (P, I) is Pareto efficient iff there exists $\lambda \in \mathbb{R}$ such that it is a solution of[2]

$$\mathcal{P}(\lambda) \ \sup_{P,I}\{u(P, I)\,\text{s.t.}\,\pi(P, I) \geq \lambda, 0 \leq I \leq \mathrm{Id}\}.$$

The following result is very well-known. We however provide a new proof as a prototype of proof that we shall use in more complicated models.

Proposition 4. *For every profit level λ, $\mathcal{P}(\lambda)$ has a unique solution (P^*, I^*). Moreover I^* and $\mathrm{Id} - I^*$ are nondecreasing, hence I^* is 1-Lipschitz.*

Proof. Since $\pi(\cdot, I)$ is increasing and $u(\cdot, I)$ is decreasing, one may replace in $\mathcal{P}(\lambda)$ the constraint $\pi(P, I) \geq \lambda$ by $\pi(P, I) = \lambda$ or, equivalently by $P = \lambda + E(c(I))$. Let $L(t, x, z) := U(x - t - z - \lambda)$. Then $\mathcal{P}(\lambda)$ is equivalent to:

$$\sup_I \left\{ \int_0^{\bar{x}} L(t, I(t), E(c(I)))\,\mathrm{d}\mu(t) \text{s.t.}\, 0 \leq I \leq \mathrm{Id} \right\}.$$

Since $(t, x) \to L(t, x, z)$ is supermodular as a concave function of $x - t$, Theorem 2 applies: there exists a nondecreasing solution I^* of $\mathcal{P}(\lambda)$. Its uniqueness follows from the strict concavity of U and the convexity of c. The associated premium P^* verifies $P^* = E(c(I^*))$.

To prove the last assertion of the proposition, let $Z^* := -I^* + \mathrm{Id}$, and $\widetilde{Z^*}$ be the nondecreasing rearrangement of Z^*. Then Z^* clearly is the optimal solution of the problem:

$$\sup_Z \left\{ E(U(-Z - P^*))\,\text{s.t.}\, \int_0^{\bar{x}} c(t - Z(t))\,\mathrm{d}\mu \leq P^* - \lambda, 0 \leq Z \leq \mathrm{Id} \right\}. \quad (7)$$

Applying Theorem 1 to the function $g(t, z) = c(t - z)$ (with c convex), we have

$$\int_0^{\bar{x}} c(t - \widetilde{Z^*}(t))\,\mathrm{d}\mu \leq \int_0^{\bar{x}} c(t - Z^*(t))\,\mathrm{d}\mu \leq P^* - \lambda$$

and $0 \leq \widetilde{Z^*} \leq \mathrm{Id}$, therefore $\widetilde{Z^*}$ is also a solution of (7). Since the solution is unique, $\widetilde{Z^*} = Z^*$ which proves the desired result. □

Remark.

1. We have assumed here that a contract is a function of the loss. We could have assumed more generally that the loss is a random variable on a probability space $(\Omega, \mathcal{A}, P)$ and that a contract is a function $I : \Omega \to \mathbb{R}_+$. It may be shown (see for example Carlier and Dana (2003)) that Pareto efficient contracts exist by a topological argument and that they are comonotone, hence nondecreasing 1-Lipschitz function of X.
2. A much finer characterization of efficient contract may be obtained in this model by using first order conditions and the additively separable structure. The optimal contract

[2] The equivalence follows from the strict monotonicity of π and u with respect to P.

is a "generalized deductible". It satisfies $I(x) = 0$ for $x \in [0, a]$ and the functions I and $\mathrm{Id} - I$ are nondecreasing on $x \geq a$. Our point is to emphasize the properties of contracts that are robusts to perturbations of the model.

3. More generally, we could have considered a risk-averse insurer who bears costs with indirect utility function

$$\pi(P, I) := \int_0^{\bar{x}} V(P - c(I(x)))\,\mathrm{d}\mu(x)$$

with $V : \mathbb{R} \to \mathbb{R}$ strictly increasing, strictly concave.

4.2. *The case of non-expected utilities*

The model is as in the previous section except that we do not assume agents to be expected utility maximizers. The risk has a nonatomic probabilty law μ with compact support $[0, \bar{x}] \subset \mathbb{R}_+$. In what follows, L^∞ denotes $L^\infty(\mathbb{R}, \mathcal{B}, \mu)$.

Let $(Z, Y) \in (L^\infty)^2$. Let us recall that $Z \succeq_2 Y$ (respectively, $X \succ_2 Y$) if $E[U(Z)] \geq E[U(Y)]$ for every concave increasing function $U : \mathbb{R} \to \mathbb{R}$ (respectively, the inequality is strict for any strictly concave increasing function $U : \mathbb{R} \to \mathbb{R}$).

Let $V : L^\infty \to \mathbb{R}$ and $W : \mathbb{R}_+ \times L^\infty \to \mathbb{R}$ be the insured and insurer's utility. In this new setting, a contract (P, I) is a Pareto-efficient contract if and only there exists $\lambda \in \mathbb{R}$ such that (P, I) is a solution of $\mathcal{Q}(\lambda)$[3]

$$\mathcal{Q}(\lambda) \quad \sup_{P,I}\{V(-X + I(X) - P)\,\text{s.t.}\, 0 \leq I \leq \mathrm{Id},\, W(P, -I(X)) \geq \lambda,\, P \geq 0\}.$$

We assume the following on V and $W(P, Z)$:

(U1).

- If $Z_n \to Z$ pointwise, then $\limsup V(Z_n) \leq V(Z)$.
- If $P_n \to P$ and $Z_n \to Z$ pointwise, then $\limsup W(P_n, Z_n) \leq W(P, Z)$.
- For every Z, $W(\cdot, Z)$ is continuous.

(U2).

- V is "strictly monotone": for all $Y \geq 0$ a.e. $Y \neq 0$ and all $Z \in L^\infty$, $V(Z + Y) > V(Z)$. Furthermore $V(-P) \to -\infty$ as $P \to \infty$.
- $W(\cdot, Z)$ is strictly increasing for every Z.

[3] The equivalence follows from the monotonicity assumptions (U2).

(U3).

- $Z \succeq_2 Y$ implies $V(Z) \geq V(Y)$ for any $(Z, Y) \in (L^\infty)^2$.
- $Z \succeq_2 Y$ implies $W(P, Z) \geq W(P, Y)$ for any $(Z, Y, P) \in (L^\infty)^2 \times \mathbb{R}_+$. Furthermore $Z \succ_2 Y$ implies $W(P, Z) > W(P, Y)$ for any $(Z, Y, P) \in (L^\infty)^2 \times \mathbb{R}_+$.

Assumption **U3** is often called "strong risk aversion" (resp "strictly strong aversion"). Examples and characterizations of strongly risk averse utilities may be found in Chew and Mao (1995).

Remark. It follows from (U3) that if X and Y have same distribution, then $V(X) = V(Y)$ and $W(P, X) = W(P, Y)$.

Theorem 3. Assume that V and W satisfy (U1), (U2) and (U3). Then, for every profit level λ, $\mathcal{Q}(\lambda)$ has an optimal solution (P^*, I^*) with I^* and $\mathrm{Id} - I^*$ nondecreasing. Hence I^* is 1-Lipschitz. If furthermore V satisfies "strong risk aversion", then any optimal contract I^*, is such that I^* and $\mathrm{Id} - I^*$ are nondecreasing.

Proof. We may assume without loss of generality that a contract is nondecreasing. Indeed let I be any feasible contract and let $\tilde{I}$ be its nondecreasing rearrangement with respect to μ. Let us show that $W(P, -I(X)) = W(P, -\tilde{I}(X))$ and that $V(-X + \tilde{I}(X) - P) \geq V(-X + I(X) - P)$. Since $W(P, \cdot)$ fulfills (U3), and since $I \sim \tilde{I}$, $W(P, -I(X)) = W(P, -\tilde{I}(X))$. We have $0 \leq \tilde{I} \leq \mathrm{Id}$. For any U, $E(U(-X + I(X) - P)) \geq E(U(-X + \tilde{I}(X) - P))$, since I and $\tilde{I}$ have same distribution w.r. to μ, hence from (U3) $V(-X + \tilde{I}(X) - P) \geq V(-X + I(X) - P)$. Thus $(P, \tilde{I})$ dominates (P, I) (strictly if V is "strong risk averse").

To show the existence of an optimal solution, let (P_n, I_n) be a maximizing sequence with I_n nondecreasing. From (U2), $V(-P) \to -\infty$ as $P \to \infty$. Hence the sequence P_n is bounded and by Helly's theorem, the sequence (P_n, I_n) has a limit point (P^*, I^*)(with $I_n \to I^*$ pointwise and I^* nondecreasing). Clearly $0 \leq I^* \leq \mathrm{Id}$. Since W fulfills (U1), $W(P^*, -I^*) \geq \limsup W(P_n, -I_n) \geq \lambda$, therefore I^* is feasible. Since V fulfills (U1) $V(-X + I^*(X) - P^*) \geq \limsup V(-X + I_n(X) - P_n)$, hence (P^*, I^*) is optimal.

To show that $\mathrm{Id} - I^*$ is nondecreasing, let $Z^* = -\mathrm{Id} + I^*$ and let $\tilde{Z}^*$ be its nonincreasing rearrangement w.r. to μ. We have $V(Z^*(X) - P^*) = V(\tilde{Z}^*(X) - P^*)$ since V is distribution invariant. If $Z^* \neq \tilde{Z}^*$, then by Theorem 1, for any U strictly concave, $E(U(-X - \tilde{Z}^*(X))) > E(U(-X - Z^*(X)))$, hence $W(P^*, -\mathrm{Id} - \tilde{Z}^*) > W(P^*, -\mathrm{Id} - Z^*)$ since W fulfills (U2). From (U1), $W(\cdot, Z)$ is continuous for every Z, thus $(P^* - \varepsilon, \tilde{Z}^* + \mathrm{Id})$ for some $\varepsilon > 0$ is feasible and dominates (P^*, I^*) contradicting its optimality. Hence $Z^* = \tilde{Z}^*$ as was to be proven. □

4.3. *Pareto efficient insurance contracts when the insurer's cost function is discontinuous*

We consider the problem of efficient insurance contracts when the cost structure includes a fixed cost per claim in a framework introduced by Gollier (1987). This problem appears naturally in existing models of deterministic auditing (see Picard (2000a,b) and Carlier and Dana (2003)).

The model is the same as in Section 4.1 except that the cost function $c : \mathbb{R}_+ \to \mathbb{R}$ has a discontinuity at zero. It satisfies: $c(0) = 0 < c(0^+)$ and c is convex and increasing on $(0, +\infty)$. The jump $c(0^+) > 0$ is interpreted as a fixed cost. An important example is the case of an audit cost. Under our assumptions, the cost function is only lower semi-continuous and is not convex on $\mathbb{R}_+$.

Since c is l.s.c., one may apply Theorem 2 to obtain the following proposition.

Proposition 5. *For every profit level* λ, $\mathcal{P}(\lambda)$ *has a solution* I^* *and every solution is nondecreasing. In particular* $\{I^* = 0\}$ *is an interval of the form* $[0, s^*]$.

We next prove that efficient contracts have classical monotonicity properties on the set of damages with positive indemnity.

Proposition 6. *If* I^* *is a solution of* $\mathcal{P}(\lambda)$, *then* $\mathrm{Id} - I^*$ *is nondecreasing on the set* $\{I^* > 0\}$. *Hence* I^* *is 1-Lipschitz on the set* $\{I^* > 0\}$.

Proof. We may not apply directly the argument of Proposition 2, since the cost function of the insurer is not convex. A finer argument is needed. Let $Z^* = I^* - \mathrm{Id}$ and $\tilde{Z}^*$ be such that $\tilde{Z}^* 1_{]s^*,\bar{x}]}$ equals the nonincreasing rearrangement on $(]s^*, \bar{x}], \mathcal{B}, \mu|_{]s^*,\bar{x}]})$ of $Z^* 1_{]s^*,\bar{x}]}$ and $\tilde{Z}^* 1_{[0,s^*]} = Z^* 1_{[0,s^*]}$. We have (by equimeasurability):

$$\int_0^{\bar{x}} U(Z^* - P)\,\mathrm{d}\mu = \int_0^{s^*} U(Z^* - P)\,\mathrm{d}\mu + \int_{s^*}^{\bar{x}} U(Z^* - P)\,\mathrm{d}\mu.$$

Hence $\int_0^{\bar{x}} U(Z^*(x) - P)\,\mathrm{d}\mu(x) = \int_0^{\bar{x}} U(\tilde{Z}^*(x) - P)\,\mathrm{d}\mu(x)$. Similarly,

$$\begin{aligned}\int_0^{\bar{x}} c(Z^*(x) + x - P)\,\mathrm{d}\mu &= \int_0^{s^*} c(Z^*(x) + x - P)\,\mathrm{d}\mu + \int_{s^*}^{\bar{x}} c(Z^*(x) + x - P)\,\mathrm{d}\mu \\ &\geq \int_0^{s^*} c(Z^*(x) + x - P)\,\mathrm{d}\mu + \int_{s^*}^{\bar{x}} c(\tilde{Z}^*(x) + x - P)\,\mathrm{d}\mu.\end{aligned}$$

The last inequality is obtained by applying Theorem 1 to the nonincreasing rearrangement of Z^* on $]s^*, \bar{x}]$. Furthermore the constraints $-\mathrm{Id} \leq \tilde{Z}^* \leq 0$ still hold on $]s^*, \bar{x}]$ since $-\mathrm{Id}$ is decreasing. Since U is strictly concave, $Z^* = \tilde{Z}^*$ on $]s^*, \bar{x}]$ which proves the desired result. Lastly since I^* and $\mathrm{Id} - I^*$ are nondecreasing on the set $\{I^* > 0\}$, they are comonotone. Hence I^* is 1-Lipschitz on the set $\{I^* > 0\}$. $\square$

Remark.

1. In the particular case where the cost function is affine on $(0, +\infty)$, one may easily show that any optimal contract is of the form $I(x) = 1_{[s^*,\bar{x}]}(x)(x - D)$ with $s^* > D$.
2. Utilities preserving second order stochastic dominance may also be considered in this example. One may show that any solution is nondecreasing and for any solution I^*, $\mathrm{Id} - I^*$ is nondecreasing on the set $\{I^* > 0\}$.

3. In order to obtain existence and the monotonicity of any solution, any lower-semi continuous cost function may be used. In particular, optimal solutions exist and are nondecreasing in the case of concave continuous costs.

5. A deterministic version of Landsberger and Meilijson adverse selection model

We consider now a deterministic version of an adverse selection model introduced by Landsberger and Meilijson (1999).

There are two types $l = H, L$ of insurance buyers who face a loss $X_l, l = H, L$ where X_l is a random variable with support $[0, \overline{x}]$ and probability law μ_l. It is assumed that μ_H is absolutely continuous with respect to μ_L with density R. We assume that R fulfills the monotone likelihood ratio property:

(H1). R is nondecreasing

The insurance market provides menus of insurance contracts. A menu of contracts is a pair $((P_l, I_l), l = H, L)$ where P_l is the premium paid by agent of type l and $I_l : [0, \overline{x}] \to \mathbb{R}$ is the indemnity schedule. When the insured of type l buys the contract (P_l, I_l), he is endowed with the random wealth $W_l(X_l) := w_{0l} - P_l - X_l + I_l(X_l)$. For the sake of simplicity, from now on, we normalize $w_{0l} = 0$. The insured of type l is assumed to have von Neumann Morgenstern preferences over random wealth: $\int_0^{\overline{x}} U_l(W_l(x))\,\mathrm{d}\mu_l(x)$ where $U_l : \mathbb{R} \to \mathbb{R}$ is strictly concave, strictly increasing and C^1. His indirect utility over contracts is: $v_l(P_l, t_l) := \int_0^{\overline{x}} U_l(W_l(x))\,\mathrm{d}\mu_l(x)$.

Definition 4. A menu $((P_l, I_l), l = H, L)$ is feasible if

$$0 \leq I_l(t) \leq t \text{ a.e. on } [0, \overline{x}], \quad l = H, L.$$

$$v_H(P_H, I_H) \geq E(U_H(-X_H)) \quad \text{I.R.H.}$$

$$v_L(P_L, I_L) \geq E(U_L(-X_L)) \quad \text{I.R.L.}$$

$$v_H(P_H, I_H) \geq v_H(P_L, I_L) \quad \text{I.C.H.}$$

$$v_L(P_L, I_L) \geq v_L(P_H, I_H) \quad \text{I.C.L.}$$

We denote by F the set of feasible menus. Constraint I.R.H (respectively, I.R.L) is the individual rationality constraint for an H (respectively, L) type agent. Constraint I.C.H (respectively, I.C.L) is the incentive compatibility constraint for an H (respectively, L) type agent.

Let C_l^0 denote the certainty equivalent of the no-insurance wealth of type l: $C_l^0 = U_l^{-1}E(U_l(-X_l))$. We assume:

(H2). $C_L^0 > C_H^0$.

We assume that the insurer is a risk neutral monopolist who maximizes profit. There is a proportion λ of agents with high risk. The insurer therefore solves the following problem:

$$\max_F \Pi(P_H, P_L, I_H, I_L) = \lambda(P_H - E(I_H(X_H))) + (1-\lambda)(P_L - E(I_L(X_L))).$$

Definition 5. A menu $((P_l, I_l), l = H, L) \in F$ is (weakly) dominated if there exists another menu $((\tilde{P}_l, \tilde{I}_l), l = H, L) \in F$, such that $v_l(\tilde{P}_l, \tilde{I}_l) \geq v_l(P_l, I_l), l = H, L$ and $\Pi(\tilde{P}_H, \tilde{P}_L, \tilde{I}_H, \tilde{I}_L) > \Pi(P_H, P_L, I_H, I_L)$ (resp. $\Pi(\tilde{P}_H, \tilde{P}_L, \tilde{I}_H, \tilde{I}_L) \geq \Pi(P_H, P_L, I_H, I_L)$).

We first state a lemma which may be found in Landsberger and Meilijson.

Lemma 3. *Assume* **(H2)**. *Any feasible menu* $((P_l, I_l), l = H, L)$ *such that insured with high risk do not get full insurance is dominated by a feasible menu where insured with high risk get full insurance*.

We may therefore reduce the insurer's problem to the following:

$$\max_{P_H, P_L, I_L} \lambda P_H + (1-\lambda)(P_L - E(I_L(X_L))) \text{ s.t.,}$$

$$0 \leq I_L(t) \leq t \text{ a.e. on } [0, \bar{x}], \tag{8}$$

$$-P_H \geq C_H^0 \quad \text{I.R.H,} \tag{9}$$

$$v_L(P_L, I_L) \geq E(U_L(-X_L)) \quad \text{I.R.L,} \tag{10}$$

$$v_H(P_H, \mathrm{Id}) = U_H(-P_H) \geq v_H(P_L, I_L) \quad \text{I.C.H,} \tag{11}$$

$$v_L(P_L, I_L) \geq U_L(-P_H) \quad \text{I.C.L.} \tag{12}$$

Let $Z_L = \mathrm{Id} - I_L$. As $v_L(P_L, I_L) = \int_0^{\bar{x}} U_L(-P_L - Z_L(x))\, \mathrm{d}\mu_l(x)$, $v_H(P_L, I_L) = \int_0^{\bar{x}} U_H(-P_L - Z_L(x))R(x)\, \mathrm{d}\mu_l(x)$ and $E(I_L(X_L)) = \int_0^{\bar{x}} (-Z_L(x) + x)\, \mathrm{d}\mu_L(x)$, we may rewrite the above problem as

$$\max_{P_H, P_L, Z_L} \lambda P_H + (1-\lambda)(P_L + E(Z_L(X_L))) \text{ s.t.,}$$

$$0 \leq Z_L \leq \mathrm{Id} \;\; - P_H \geq C_H^0, \;\; \int_0^{\bar{x}} U_L(-P_L - Z_L(x))\, \mathrm{d}\mu_l(x) \geq E(U_L(-X_L)),$$

$$U_H(-P_H) \geq \int_0^{\bar{x}} U_H(-P_L - Z_L(x))R(x)\, \mathrm{d}\mu_l(x),$$

$$\int_0^{\bar{x}} U_L(-P_L - Z_L(x))\, \mathrm{d}\mu_l(x) \geq U_L(-P_H).$$

We thus have the following proposition.

Proposition 7. *Assume* **(H1)** *and* **(H2)**. *There exists an optimal feasible menu. At any optimal menu, agents with high risk are offered full insurance. Furthermore, at any optimal* *menu, there exists an optimal menu where insured have same utilities and the insurer same profit such that the wealth of the insured with low risk is nonincreasing*.

Proof. Since $P_l \leq -C_l^0, l = H, L$, premiums are bounded. As the function $(x, y) \to U_H(-P_L - y)R(x)$ is supermodular, we may apply Theorem 2, hence there exists an optimal feasible menu with $\mathrm{Id} - I_L^*$ nondecreasing. Agents with high risk are offered full insurance and the insured with low risk has nonincreasing wealth. The second assertion follows from the fact that any feasible menu $(P_H, \mathrm{Id}, P_L, I_L)$ is weakly dominated by a feasible menu $(P_H, \mathrm{Id}, P_L, \tilde{I}_L)$ where $\mathrm{Id} - \tilde{I}_L$ is the nondecreasing rearrangement of $\mathrm{Id} - I_L$. $\square$

Remark. Landsberger and Meilijson further show that at any optimal menu, the high risk incentive constraint (I.C.H) is binding while the individual rationality constraint for low risk (I.R.L) is binding.

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Appendix A

A.1. Proof of proposition 1

It may be checked that $\tilde{x}$ given in the proposition is right-continuous nondecreasing, that v is left-continuous nondecreasing and one has:

$$u \leq \tilde{x}(t) \Leftrightarrow v(u) \leq t. \tag{13}$$

Let us prove that $\tilde{x} \sim x$, to that end let us note that, using (13) for $u \in \mathbb{R}$, we have:

$$\mu(\{\tilde{x}(\cdot) \geq u\}) = \mu([v(u), 1])$$

but by definition of v and nonatomicity of μ, we have

$$\mu([v(u), 1]) = \mu(\{x(\cdot) \geq u\})$$

since this holds for all u we have $x \sim \tilde{x}$.

It remains to prove that $\tilde{x}$ defined above is the unique nondecreasing function distributed as x up to μ-a.e. equivalence. Assume that f is nondecreasing and $f \sim x$ hence $f \sim \tilde{x}$. Since f is nondecreasing, for every $u \in \mathbb{R}$ $\{f(\cdot) \geq u\}$ is either empty or an interval of the form $[g(u), 1]$ or $(g(u), 1]$ where g is nondecreasing. Since $f \sim \tilde{x}$, μ is nonatomic and by (13), for all $u \in \mathbb{R}$, we get:

$$\mu(\{f(\cdot) \geq u\}) = \mu(\{\tilde{x}(\cdot) \geq u\}) = \mu([v(u), 1]) = \mu([g(u), 1]). \tag{14}$$

If $\tilde{x}(t) < f(t)$, let $q \in \mathbb{Q} \cap (\tilde{x}(t), f(t)]$. We then have $g(q) \leq t < v(q)$. In particular $t \in [g(q), v(q)]$ and by (14), $\mu([g(q), v(q)]) = 0$, since $q \in \mathbb{Q}$ is arbitrary in the previous reasoning, we may conclude that $\{\tilde{x} < f\}$ is μ-negligible. Similarly, one obtains that $\{\tilde{x} > f\}$ is μ-negligible. This proves the claim of uniqueness.

A.2. Proof of Theorem 2

Let us first remark that our assumptions imply that the value of program (6) is finite. Note also that, as $\int_0^1 \gamma(x)\,\mathrm{d}\mu = \int_0^1 \gamma(\tilde{x})\,\mathrm{d}\mu$, it follows from Theorem 1 that $J(\tilde{x}, P) \geq J(x, P)$.

Let (x_n, P_n) be a maximizing sequence of (6). From Lemma 2, $(\tilde{x}_n, P_n)$ fulfills the constraints and $J(\tilde{x}_n, P_n) \geq J(x_n, P_n)$. Hence, replacing x_n by $\tilde{x}_n$, we still have a maximizing sequence. We may thus assume that each x_n is nondecreasing. As elements of $\mathcal{C}$ are uniformly bounded by $x_0(0_+)$ and $x_1(1_-)$, we may assume by Lemma 1 that x_n converges pointwise to some nondecreasing function x^*, by compactness it may also be assumed that P_n converges to some $P^* \in K$. Obviously, $(x^*, P^*) \in \mathcal{C}_1$ and by Lebesgue's dominated convergence theorem, one also has $(x^*, P^*) \in \mathcal{C}_2 \cap \mathcal{C}_3$.

It remains to prove that (x^*, P^*) is a solution of (6). If γ is continuous, it follows from Lebesgue's dominated convergence theorem that

$$\lim \int_0^1 \gamma(x_n)\,\mathrm{d}\mu = \int_0^1 \gamma(x^*)\,\mathrm{d}\mu.$$

Applying Lebesgue's dominated convergence theorem one more time, we obtain that $J(x_n, P_n)$ converges to $J(x^*, P^*)$ which implies that (x^*, P^*) is a solution of (6).

If γ is only l.s.c. and L is nonincreasing with respect to its third argument, we first get:

$$\gamma(x^*) \leq \liminf \gamma(x_n).$$

Fatou's Lemma then yields:

$$\int_0^1 \gamma(x^*)\,\mathrm{d}\mu \leq \liminf \int_0^1 \gamma(x_n)\,\mathrm{d}\mu.$$

As L is nonincreasing with respect to its third argument, this implies for every $t \in [0, 1]$,

$$\limsup L\left(t, x_n(t), \int_0^1 \gamma(x_n)\,\mathrm{d}\mu, P_n\right) \leq L\left(t, x^*(t), \int_0^1 \gamma(x^*)\,\mathrm{d}\mu, P^*\right).$$

Applying Fatou's lemma again, we get

$$\limsup J(x_n, P_n) \leq J(x^*, P^*),$$

which proves that (x^*, P^*) is a solution of (6).

To prove the last assertion, let us note that if L is strictly supermodular with respect to (t, x), then by Theorem 1, $J(\tilde{x}, P) > J(x, P)$ unless $x = \tilde{x}\,\mu$-a.e.

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