

Axioms for health care resource allocation[☆]

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Abstract

This paper examines principles of health care resource allocation based on axioms for individual preferences and distributive justice. We establish axioms for representing individual preferences by quality-adjusted life years (QALYs), as well as axioms for existence of a social welfare function depending only on QALYs. A symmetric Cobb–Douglas social welfare function is characterized by an axiom stating that social welfare is anonymous with respect to the distribution of individual life years. Replacing this axiom with an axiom of non-age dependence, we obtain a characterization of a utilitarian social welfare function with certain weights. Further, we give axioms for a social welfare function being a weighted sum of power transformations of individual QALYs.

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1. Introduction

In health economic analyses, it is often assumed that an individual's preferences for own health can be represented by an index measuring quality-adjusted life years (QALYs).

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As pointed out by Wagstaff (1991) and others, a policy of maximizing the total number of QALYs in society may fail to capture distributive justice. How to adjust for this problem has not only evolved into a major research theme in health economics but it has also attracted attention from other social sciences, e.g. social philosophy and medical ethics. Fundamental to this problem is the question of whether it, at least in theory, is possible in a meaningful way to aggregate individual health indices (QALYs), in particular when individuals not necessarily have identical preferences.

The aim of this paper is to investigate one way of obtaining an explicit link between QALYs at the individual level and distributive justice at the societal level.

In recent literature, different ways of accounting for distributive justice via specific social welfare functions have been suggested. Williams (1997), for example, advocates a concrete interpretation of the *fair innings* principle which involves the estimation of a social welfare function defined on the distribution of expected QALYs which is a member of the family of CES functions (constant elasticity of substitution). Dolan (1998) suggests a Cobb–Douglas social welfare function on individual QALYs. Because social welfare is then strictly concave in individual QALYs, this reflects a particular concern for people with bad health. However, in a comment to Dolan's article, Johannesson (1999) argues that the approach suggested by Dolan has no theoretical foundation and should not be used as a basis for aggregating QALYs. Johannesson suggests the following alternative framework: each individual has a utility function which depends on the entire distribution of individual QALYs and the social welfare function is then defined as the sum of individual utilities. The idea is that since individual utilities already capture altruism or concerns for distributive justice, there is no need to take these things into account again via the curvature of the social welfare function.¹ In Nord (1993, 1999) and Nord et al. (1999) variants of the *cost-value* method are suggested and analyzed. The societal value of a health gain is defined as the product of certain weight functions (the utility gain, a weight determined by the severity of the initial condition, and a weight determined by the potential for health) and the method for health care resource allocation implied is radically different from those suggested by Williams, Dolan or Johannesson.²

It is difficult to evaluate the superiority of one functional form over the other. For instance, different specifications of individual utility and social welfare may represent the same underlying preference relations and concerns for justice in resource allocation; and it may not be clear for some approaches whether key properties such as the Pareto principle are satisfied. Also, the characteristics of these approaches depend crucially on how the individual health utilities (QALYs) have been constructed from observable data (see below). Thus, instead of comparing *functions*, with properties that are difficult to compare, in this paper we compare underlying *axioms* which in certain combinations lead to different allocation methods.

In this respect our approach complements that of Bleichrodt (1997) who investigates a situation with choice under uncertainty and aggregation of individual health utilities (interpreted as QALYs). Bleichrodt examines axioms related to decisions under uncertainty, whereas our focus primarily is axioms related to comparisons of deterministic outcomes.

In the above-mentioned papers, *health states* and *life years* do not enter the models explicitly, but abstract one-dimensional individual health utility indices are the primitives

¹ See also Dolan's reply (1999).

² See Østerdal (2003) for a discussion of the cost-value method.

of the model. However, when we speak of justice in health and health care, this is usually in terms of individual entitlement to life years and health states, not abstract health utilities. There are many possible ways of constructing an individual one-dimensional health index from relevant underlying data. For instance, a health index can be ‘timeless’ or depend on life years, and there are various ways of estimating parameters (rating-scales, standard gambles, time trade-offs, etc.). In other words, it is difficult to evaluate the reasonableness of methods based on the maximization of some function of health indices when it is not explicitly described how these health indices are derived from underlying data.

We characterize allocation methods by axioms which have fairly simple and intuitive interpretations in this particular context of health care resource allocation. These axioms are formulated as principles of distributive justice in particular ‘simple’ dilemmas, for example, involving comparisons of certain gains in life years for two different people at the same health state. Once having established a principle of justice in these cases, social welfare evaluations of arbitrary distributions of (changes in) life years and health states in society turn out as a consequence of the underlying axioms.

The plan of the paper is as follows. In Section 2, the necessary definitions are provided and the health care resource allocation problem that underlies these definitions is outlined. In Section 3, we derive an axiomatic characterization of QALY measures as individual utility functions in a deterministic framework. In Section 4, we give sufficient conditions for the existence of a continuous social welfare function being a function of individual QALYs only. Based on the representation of individual preferences, we derive axiomatic characterizations of two specific solutions to the health care resource allocation problem. The symmetric Cobb–Douglas solution is characterized by an anonymity axiom in terms of the distribution of individual life years for any fixed distribution of health states; an alternative characterization by means of equal social preferences for relative gains in life years for individuals at some health state is also obtained (Section 4.1). By replacing this axiom with an axiom of non-age dependence in terms of social preferences for gains in life years for individuals at a particular health state, we obtain a characterization of a utilitarian solution with certain weights (Section 4.2). Further, we identify two characterizations of the class of social welfare functions which can be written as a weighted sum of power transformations of individual QALYs (Section 4.3). The latter results are obtained as applications of a theorem by Bergson (Burk, 1936). Section 5 considers two extensions of the model, involving uncertainty and non-chronic health states, respectively, and explains how the same axioms for distributive justice (restricted to comparisons of cases with deterministic chronic health states) can be used to obtain similar characterizations of welfare functions (Sections 5.1 and 5.2). Finally, we briefly discuss status quo biases (Section 5.3), and provide some concluding remarks (Section 5.4).

2. Health care resource allocation

Let A be a set of conceivable *health states* of an individual. We think of a health state as a list of factors describing observable aspects of individual well-being. We do not make any assumptions on the mathematical structure of A , but in two cases we impose a certain richness condition (Theorems 3 and 8). We assume that an individual is always in the same

health state throughout life. The *health profile* of individual i is a pair $(a_i, t_i) \in A \times \mathbb{R}_+$ containing health state a_i and prospective lifetime t_i . We can interpret t_i as an absolute number of life years relative to an absolute zero interpreted as unborn or stillborn, or as gains relative to some status quo (see also remarks on this in Section 5). Let \succsim_i be a preference relation (a complete and transitive relation) on $A \times \mathbb{R}_+$ representing individual i 's preferences for own health. In Section 3, we give conditions for a representation q_i of the form

$$q_i(a_i, t_i) = f_i(a_i)t_i, \quad (1)$$

where f_i is a positive real-valued function on A .

Let $N = \{1, \dots, n\}$ be a list of individuals in society, $n \geq 3$, and let $a = (a_1, \dots, a_n)$ and $t = (t_1, \dots, t_n)$. $t > 0$ means that t is a vector with positive entries. Then $(a, t) = ((a_1, t_1), \dots, (a_n, t_n))$ is a *health distribution* and $q(a, t) = (q_1(a_1, t_1), \dots, q_n(a_n, t_n))$ is a *QALY distribution*. A *social welfare ordering* is a complete and transitive relation \succsim on $(A \times \mathbb{R}_+)^N$, the set of conceivable health distributions in society.

For any given stock of health care resources, we can imagine that it is possible to obtain different health distributions in society depending on the particular allocation of resources. The most preferred distribution(s) must then be identified by the social planner. This is *the health care resource allocation problem*. From the social welfare ordering \succsim we obtain the most preferred distribution(s) for any collection of feasible distributions.³ Thus \succsim induces a method for allocating health care resources.

Let U be a real-valued *social welfare function* on $(A \times \mathbb{R}_+)^N$. We say that U represents \succsim if $U(a, t) \geq U(a', t') \Leftrightarrow (a, t) \succsim (a', t')$. Given individual QALY indices q_i , a real-valued function W on \mathbb{R}_+^N represents \succsim if $W(q(a, t)) \geq W(q(a', t')) \Leftrightarrow (a, t) \succsim (a', t')$, i.e. W is a (Bergson–Samuelson) social welfare function which only depends on the distribution of individual QALYs.

3. Individual QALYs

QALY-based individual utility functions have been characterized in the literature by regularity conditions on preferences over health lotteries, see Pliskin et al. (1980), Bleichrodt et al. (1997) and Miyamoto et al. (1998). In this section, we derive a simple characterization of QALY-based individual preferences in a situation of choice under certainty.

We make the assumption that if an individual obtains zero lifetime, health state does not matter. This property has been referred to as the *zero-condition* (Bleichrodt et al., 1997; Miyamoto et al., 1998).

ZERO : $(a_i, 0) \sim_i (a'_i, 0)$ for all $a_i, a'_i \in A$.

We will also make use of the following *lifetime continuity* axiom for individual preferences for life years, which says that for fixed health states, preferences are smooth in lifetime.

³ If the set of feasible distributions is finite then there is a non-empty set of maximal elements (optimal distributions) with respect to the social welfare ordering \succsim .

CONT: Let $a_i, a'_i \in A$, $t'_i, t_i^{(k)} \geq 0$ for all $k = 1, 2, 3, \dots$, and let $t_i^{(k)}$ be a converging sequence, $t_i = \lim_{k \rightarrow \infty} t_i^{(k)}$. If $(a_i, t_i^{(k)}) \succsim_i (a'_i, t'_i)$ for all k , then $(a_i, t_i) \succsim_i (a'_i, t'_i)$. If $(a'_i, t'_i) \succsim_i (a_i, t_i^{(k)})$ for all k , then $(a'_i, t'_i) \succsim_i (a_i, t_i)$.

Moreover, it is assumed that for any health state there exists a positive lifetime preferred to zero lifetime. We refer to this condition as *positivity*.

POS: Let $a_i \in A$. Then there exists $\bar{t}_i > 0$ such that $(a_i, \bar{t}_i) \succ_i (a_i, 0)$.

Assuming positivity is not vacuous, since individuals may consider some health states worse than being dead regardless of prospective lifetime (see also Section 5).

As we will see in Theorem 1 below, a representation of individual preferences by (1) hinges crucially on *lifetime scale independence*.⁴ This condition states that the relation between two health profiles (containing health states and life years) does not reverse if both lifetimes are multiplied with the same positive constant.

LSI: If $(a_i, t_i) \succsim_i (a'_i, t'_i)$ then $(a_i, ct_i) \succsim_i (a'_i, ct'_i)$ for $c, t_i, t'_i > 0$.

Consider, for example, a person who prefers 70 life years in a wheelchair to 60 years in perfect health. Then LSI tells that the person also prefers 35 years in a wheelchair to 30 years in perfect health and vice versa.

Pliskin et al. derive a characterization of individual QALY-based utility using a *constant proportional trade-off* axiom similar to LSI.⁵ Here, we shall demonstrate that ZERO, CONT, POS and LSI enable a time trade-off based foundation for QALYs as individual utilities without imposing axioms of expected utility, risk neutrality in life years or separability of preferences for health states and life years.

As a first step towards a characterization of QALY-based individual utility function, we demonstrate that ZERO, POS, CONT and LSI imply the following two useful properties. *Monotonicity* (MONO) is a strong variant of POS.

MONO: Let $a_i \in A$ and $t_i > t'_i \geq 0$. Then $(a_i, t_i) \succ_i (a_i, t'_i)$.

Another important property is that of *comparability* (COMP). This means that any health state $a_i \in A$ can be compared to any other health state in the following sense.

COMP: Let $a_i \in A$. Then for all $a'_i \in A$ there exists $t_i > 0$ such that $(a_i, t_i) \sim_i (a'_i, 1)$.

We may now obtain that monotonicity and comparability follows from the preceding axioms. This observation bears similarities to a recent result by Maccheroni (2001, Proposition 5). All proofs can be found in [Appendix A](#).

Lemma 1. *Let individual preferences \succsim_i satisfy ZERO, POS, CONT and LSI. Then MONO and COMP are satisfied.*

⁴ See Moulin (1988) for a treatment of *scale independence*. Scale independence has also been referred to as *homotheticity* (e.g. Maccheroni, 2001).

⁵ Under expected utility, where the expectation of a utility function $u_i(a_i, t_i)$ governs preferences over health lotteries, *constant proportional trade-off* holds if $u_i(a_i, t_i) = u_i(a'_i, t'_i)$ implies that $u_i(a_i, ct_i) = u_i(a'_i, ct'_i)$ for all $c \geq 0$ (Pliskin et al., 1980).

Let $a^* \in A$ be an arbitrary health state which is fixed in the following. The role of a^* is to be a benchmark that the quality of all other health states is measured against. It may be helpful to think of a^* as a state of ‘perfect health’ but it is not required that a^* is better than any other health state (and it is not assumed that such health state exists). We shall use the same reference health state for all individuals because it simplifies exposition and there is no loss of generality. Define

$$h_i(a_i) = \{t_i \in \mathbb{R}_{++} | (a^*, t_i) \sim_i (a_i, 1)\}. \quad (2)$$

As a corollary of Lemma 1 we observe that with our four axioms, h_i is a real-valued function.

Corollary 1. *Let individual preferences \succsim_i satisfy ZERO, CONT, POS and LSI. Then $h_i(a_i)$ is a singleton for all $a_i \in A$.*

In the following, the health state index h_i satisfying (2) is referred to as the *time trade-off index* for individual i . We may now formulate the following characterization of QALY-based individual utilities.⁶

Theorem 1. *Individual preference \succsim_i satisfies ZERO, CONT, POS and LSI if and only if there is a function $f_i : A \rightarrow \mathbb{R}_{++}$, which is a positive linear transformation of the time trade-off index h_i , such that*

$$(a_i, t_i) \succsim_i (a'_i, t'_i) \Leftrightarrow f_i(a_i)t_i \geq f_i(a'_i)t'_i. \quad (3)$$

Note that individuals obeying axioms ZERO, CONT, POS, and LSI do not necessarily think that living for a hundred years is ‘twice as good’ as living 50 years at the same health state. We do not require assessments of ‘preference intensity’. For example, assume that an individual’s preferences over health profiles are represented by the ‘cardinal’ utility function $f_i(a_i)\sqrt{t_i}$ (representing ‘preference intensity’). As ZERO, CONT, POS, and LSI are satisfied, it follows from Theorem 1 that $q_i(a_i, t_i) = h_i(a_i)t_i$ represents the (induced) preference relation \succsim_i , where h_i is the time trade-off index.

4. Aggregating QALYs

Having introduced assumptions on individual preferences for own health, we now go on to aggregation of QALYs.

We have imposed a rather restrictive structure on preferences for own health. On the other hand, we do not assume that individuals have symmetric preferences (with one

⁶ After this work was submitted and revised, I learned of an independent paper by Doctor and Miyamoto (2003) which in a similar model obtains a characterization of the linear QALY model using a set of conditions constituting a *constant proportional trade-off structure*.

exception—Theorem 8). The possibility that individuals have different views on the trade-off between lifetime and health states is accounted for, and health states may even be ranked differently by the individuals.

The social welfare ordering \succsim satisfies the *Pareto principle* if, when the health profiles in one health distribution is preferred to the health profiles in another distribution by all individuals, then the former distribution is also preferred according to the social welfare ordering. Formally,

PAR : $(a, t) \succ (a', t')$ if $(a_i, t_i) \succ_i (a'_i, t'_i)$ for all $i \in N$.

We have assumed that a small change in lifetime has a small effect on individual welfare (*lifetime continuity*), and therefore it seems natural that this property carries over to the societal level; that is, small changes in all lifetimes have a small effect on social welfare. We shall make use of a corresponding assumption of *lifetime continuity of social welfare*, which says that for fixed distributions of health states, the social welfare ordering is smooth in lifetimes.

CONT-S: Let $a, a' \in A^N$, $t', t^{(k)} \geq 0$ for all $k = 1, 2, 3, \dots$, and let $t^{(k)}$ be a converging sequence, $t = \lim_{k \rightarrow \infty} t^{(k)}$. If $(a, t^{(k)}) \succsim (a', t')$ for all k , then $(a, t) \succsim (a', t')$. If $(a', t') \succsim (a, t^{(k)})$ for all k , then $(a', t') \succsim (a, t)$.

We can now formulate a basic representation theorem, which we will make repeatedly use of in the following subsections.⁷

Theorem 2. *Let individual preferences satisfy ZERO, CONT, POS and LSI for all $i \in N$. If the social welfare ordering \succsim satisfies PAR and CONT-S then \succsim is represented by a continuous social welfare function depending only on the distribution of individual QALYs, i.e. there exists a continuous function $W : \mathbb{R}_+^N \rightarrow \mathbb{R}$ such that*

$$(a, t) \succsim (a', t') \Leftrightarrow W(q(a, t)) \geq W(q(a', t')),$$

where $q_i(a_i, t_i) = h_i(a_i)t_i$, and h_i is the time trade-off index (2) for all $i \in N$.

By Theorem 2, we observe that under mild assumptions for the social welfare orderings, we may restrict attention to continuous social welfare functions depending only on the level of individual QALYs.

4.1. Dolan's solution: a case of age dependence

Up to this point, we have only introduced relatively mild assumptions for the social welfare ordering: the Pareto principle and lifetime continuity. In what follows, we will demonstrate that the Cobb–Douglas solution proposed by Dolan may be derived from the following axiom, which turns out to be very powerful in combination with the QALY-based representation of the individual preferences. The axiom says that, for any fixed distribution

⁷ This result has also a parallel in recent work by Kaplow and Shavell (2001). Note that in this paper it is not assumed from the outset that social welfare is represented by a social welfare function; but a social welfare function is derived from underlying assumptions.

of health states, the social welfare ordering is anonymous with respect to the distribution of lifetimes. We refer to this condition as *lifetime anonymity*.

LTA : $(a, t) \sim (a, t_\sigma)$ for all $(a, t) \in (A \times \mathbb{R}_+)^N$, and all permutations $\sigma : N \rightarrow N$.

This axiom has a simple interpretation in the context of justice in distribution of health care resources. LTA states that each individual has a right to life years independently of health state: disabled individuals are not less or more entitled to life years than individuals in good health.

We shall make use of the following richness condition on A with respect to individual preferences, denoted *full health domain*, which assures that there is a continuum of health states values between 0 (not included) and 1.

FHD : For any $0 < t_i \leq 1$ there is $a_i \in A$ for which $(a_i, 1) \sim_i (a_*, t_i)$.

We are now able to characterize Dolan's solution to the health care resource allocation problem.

Theorem 3. *Let individual preferences satisfy ZERO, CONT, POS, LSI, and FHD for all $i \in N$. Then PAR, CONT-S, and LTA are satisfied if and only if the social welfare ordering \succsim is represented by a social welfare function of the form*

$$U(a, t) = \prod_{i \in N} q_i(a_i, t_i),$$

where $q_i(a_i, t_i) = h_i(a_i)t_i$, and h_i is the time trade-off index (2) for all $i \in N$.

Consider, for example, two individuals both facing 20 prospective life years in all. Assume that individual 1 is at a health state a_1 with no severe discomfort, but individual 2 is at a health state a_2 which is only slightly better than death from his own point of view. Assume that individual 1 attributes the health state value $h_1(a_1) = 1$ to health state a_1 , whereas the health state value for individual 2 is $h_2(a_2) = 0.1$. Now, assume that the authority responsible for provision of health care services is able to prolong the lifetime with, say, 10 years for one of the individuals (remaining at unchanged health state), but due to capacity constraints not for both. According to Dolan's solution, it is socially equally desirable to provide health care to obtain extra life years to either individual. Social preferences for the distribution of life years do not depend on the individuals' actual health states.

Consider the following axiom of *relative lifetime comparisons*.

RLC: There exists $a_- \in A$ such that for all $[(a_-, t_i), (a_-, t_j), (a_k, t_k)_{k \in N \setminus \{i, j\}}]$, $t_i, t_j > 0$, and $c > 0$.

$[(a_-, ct_i), (a_-, t_j), (a_k, t_k)_{k \in N \setminus \{i, j\}}] \sim [(a_-, t_i), (a_-, ct_j), (a_k, t_k)_{k \in N \setminus \{i, j\}}]$.

Axiom RLC asserts that there is a health state a_- such that for any health distribution with a pair of individuals (i and j) at this health state and life years different from zero, a change in life years for individual i is equally good as a change in life years for individual j in the same proportion.

Under our previous assumptions RLC may alternatively replace FHD and LTA in Theorem 3.

Theorem 4. In Theorem 3 we may replace FHD and LTA with RLC.

As demonstrated, Dolan's solution can be derived from a particular form of distributive justice with respect to health. With a multiplicative social welfare function, persons with short lifetimes are given high priority. A difficulty here is that if prospective lifetime is very close to zero, this bias becomes absurd.

We have established two ways of characterizing the symmetric Cobb–Douglas social welfare function from axioms related to justice in terms of individual rights to years of life. Technically, these observations are related to results on scale independent social welfare orderings (d'Aspremont and Gevers, 1977; Kaneko, 1984; Moulin, 1988).⁸ Kaneko (1984) showed that a social welfare ordering on \mathbb{R}_+^N satisfying Pareto, anonymity and scale independence is represented by the symmetric Cobb–Douglas function (also called the Nash social welfare function). We notice that the CONT-S axiom cannot be dispensed with in our case, because in addition to the axioms of individual preferences it is possible to find a social welfare ordering satisfying PAR and RLC that is not represented by a Cobb–Douglas social welfare function.⁹ An indirect way of proving Theorems 3 and 4 would be to show that anonymity and scale independence is implied, but it seems, however, to be slightly more troublesome.

4.2. Utilitarianism: a case of non-age dependence

Dolan's solution admits a very strong aversion to inequality in lifetimes within a population. For example, using Dolan's solution it is socially more desirable to prolong the life of a 1-year-old child with 1 year than prolonging the life of ten 15 years old persons with one year each. This is the case regardless of the specific health states of the persons. Dolan's solution can perhaps be considered a variant of the 'fair innings' reasoning (see Harris, 1985; Williams, 1997), reflecting the viewpoint that we should aim to give a 'fair' number of (quality-adjusted) life years to each person, the *average* number of (quality-adjusted) life years, and less effort should be carried out to improve or extend the life of persons at age beyond the average. Dolan's solution and the fair innings principle imply discrimination against the elderly and to some people this would not be reasonable.

It may appear sensible to permit age-discrimination if the number of remaining life years is smaller for an old individual than for a young individual facing the same medical treatment for the same chronic disease. However, one may hold the view that an individual is not less worthy of treatment on the *sole* ground that the individual has already lived for sufficiently many years. For a health state a_- , we formulate *non-age dependence* as follows.

NAD: Given $a_- \in A$, for all $[(a_-, t_i), (a_-, t_j), (a_k, t_k)_{k \in N \setminus \{i, j\}}]$ and all $c > 0$.

$[(a_-, t_i + c), (a_-, t_j), (a_k, t_k)_{k \in N \setminus \{i, j\}}] \sim [(a_-, t_i), (a_-, t_j + c), (a_k, t_k)_{k \in N \setminus \{i, j\}}]$.

⁸ A social welfare ordering \succsim_Q is scale independent if $(q_1, \dots, q_n) \succsim_Q (q'_1, \dots, q'_n) \Leftrightarrow (c_1 q_1, \dots, c_n q_n) \succsim_Q (c_1 q'_1, \dots, c_n q'_n)$ for all $q, q' \geq 0$ and $c_1, \dots, c_n \geq 0$.

⁹ For example, $(a, t) \succ (a', t')$ if $\prod_{i \in N} h_i(a_i) t_i > \prod_{i \in N} h_i(a'_i) t'_i$ or $\prod_{i \in N} h_i(a_i) t_i = \prod_{i \in N} h_i(a'_i) t'_i$ and $\sum_{i \in N} h_i(a_i) \geq h_i(a'_i)$.

In words, for any health distribution with a pair of individuals (i and j) at health state a_- , a gain in life years for individual i is equally good as a gain in life years for individual j for the same number of years.

With this version of non-discrimination against any age-group we have the following characterization.

Theorem 5. *Let individual preferences satisfy ZERO, CONT, POS and LSI for all $i \in N$, and let the social welfare ordering satisfy PAR and CONT-S. Then NAD w.r.t. a_- is satisfied if and only if the social welfare ordering \succsim is represented by a social welfare function of the form*

$$U(a, t) = \sum_{i \in N} h_i^{-1}(a_-) q_i(a_i, t_i),$$

where $q_i(a_i, t_i) = h_i(a_i) t_i$, and h_i is the time trade-off index (2) for all $i \in N$.

As an important special case we observe that if NAD holds for the reference health state a_* (i.e. $a_- = a_*$) then $h_i^{-1}(a_-) = 1$ for all i , and, without imposing symmetry assumptions, we obtain unweighted QALY-utilitarianism, $U(a, t) = \sum_{i \in N} q_i(a_i, t_i)$.

4.3. Other aggregation methods

If neither LTA nor NAD necessarily is acceptable, we may want to look for alternative solutions to the health care resource allocation problem.

A social welfare ordering satisfies *separability* if social preferences for distributions of health (health states and life years) within a group $S \subset N$, do not depend on the health of individuals in the complement $N \setminus S$

$$\begin{aligned} \text{SEP} : [(a_i, t_i)_{i \in S}, (a_i, t_i)_{i \in N \setminus S}] \succsim [(a'_i, t'_i)_{i \in S}, (a_i, t_i)_{i \in N \setminus S}] &\Leftrightarrow \\ [(a_i, t_i)_{i \in S}, (a'_i, t'_i)_{i \in N \setminus S}] \succsim [(a'_i, t'_i)_{i \in S}, (a'_i, t'_i)_{i \in N \setminus S}], &\text{ for all } S \subseteq N. \end{aligned}$$

By Theorem 2, we may restrict attention to continuous social welfare functions depending on individual QALYs only. We can therefore immediately formulate the following variant of a theorem by [Debreu \(1960\)](#). Theorem 6 states that the social welfare ordering may be represented by a real-valued function which is additively separable in individual QALYs.

Theorem 6. *Let individual preferences satisfy ZERO, CONT, POS and LSI for all $i \in N$, and let the social welfare ordering satisfy PAR, CONT-S and SEP. Then there exist functions $v_i : \mathbb{R}_+ \rightarrow \mathbb{R}$, such that the social welfare ordering \succsim is represented by a social welfare function of the form*

$$U(a, t) = \sum_{i \in N} v_i(q_i(a_i, t_i)),$$

where $q_i(a_i, t_i) = h_i(a_i) t_i$, and h_i is the time trade-off index (2) for all $i \in N$.

Recall the LSI condition for individual preferences for own health which states that the ranking of a pair of health profiles does not reverse if both lifetimes are multiplied with

the same positive constant. This condition has a parallel for the social welfare ordering. A social welfare ordering satisfies *common lifetime scale independence* if the ranking of a pair of health distributions does not reverse when all lifetimes are multiplied with the same positive constant. Formally,

CLSI : If $(a, t) \succsim (a', t')$ then $(a, ct) \succsim (a', ct')$ for $c, t, t' > 0$.

If individual preferences are unaffected by proportional changes in lifetimes (LSI), it may seem natural that the same is the case for a social welfare ordering over health distributions (CLSI). As an application of a result which can be traced back to Bergson (Burk, 1936), we obtain that the preceding axioms impose quite narrow restrictions on the functional form of the social welfare function.

Theorem 7. *Let individual preferences satisfy ZERO, CONT, POS and LSI for all $i \in N$, and let the social welfare ordering satisfy PAR and CONT-S. Then SEP and CLSI are satisfied if and only if the social welfare ordering \succsim is represented by a social welfare function of the form*

$$U(a, t) = \sum_{i \in N} \alpha_i (q_i(a_i, t_i))^p,$$

where $q_i(a_i, t_i) = h_i(a_i)t_i$, h_i is the time trade-off index (2), α_i non-negative for all i and positive for some i , $p > 0$.

This class of welfare functions is a subfamily of the Bergson family of functions which, formulated in our context, is the class of functions where

$$U(a, t) = \text{sgn}(p) \sum_{i \in N} \alpha_i (q_i(a_i, t_i))^p, \quad p \neq 0.$$

In a health economic context, the Bergson family was discussed by Wagstaff (1991) and Williams (1997, footnote 22) (using other, but equivalent, representations). Here, we have shown that by extending the lifetime scale independence condition characterizing individual preferences to a common lifetime scale independence condition for social welfare orderings, given the basic QALY representation of individual preferences (1) we obtain a characterization of this particular class of social welfare functions with the restriction $p > 0$.

Since we have no restrictions on the complexity of the factors describing a health state, comparing individuals at different health states may be cognitively very difficult. On the contrary, as long as the health states are the same, lifetimes are non-negative real numbers relatively easy to compare. For practical purposes and empirical tests, a major simplification is obtained if comparisons between individuals, in terms of trade-offs for lifetimes, are required only for individuals at identical health states. The following axiom denoted *common health state independence* states that if two individuals have the same health state, then welfare comparisons of pairs of positive lifetimes for these two individuals (keeping the health profile of any other individual fixed) do not depend on the particular health state.

CHI : If $t_i, t_j, t'_i, t'_j > 0$ and $[(a_-, t_i), (a_-, t_j), (a_k, t_k)_{k \in N \setminus \{i, j\}}] \succsim [(a_-, t'_i), (a_-, t'_j), (a_k, t_k)_{k \in N \setminus \{i, j\}}]$ for some $a_- \in A$ then $[(a'_-, t_i), (a'_-, t_j), (a_k, t_k)_{k \in N \setminus \{i, j\}}] \succsim [(a'_-, t'_i), (a'_-, t'_j), (a_k, t_k)_{k \in N \setminus \{i, j\}}]$ for all $a'_- \in A$.

If axiom FHD is included in the list of axioms characterizing individual preferences for own health, we may replace CLSI with CHI. We shall, however, also need a *symmetry* axiom stating that the individuals have identical preferences.

SYM : $(a_i, t_i) \succsim_i (a'_i, t'_i) \Leftrightarrow (a_i, t_i) \succsim_j (a'_i, t'_i)$ for all $i, j \in N, (a_i, t_i), (a'_i, t'_i) \in A \times \mathbb{R}_+$.

Theorem 8. *If in addition SYM and FHD holds, then in Theorem 7 we may replace CLSI by CHI.*

The utilitarian solution is the special case $p=1$ and $\alpha_i = h_i^{-1}(a_-)$ for all i . Note that Dolan's solution is not a member of the Bergson family and does not satisfy SEP. If we restrict the domain of the social welfare ordering to *positive* lifetimes, in Theorems 7 and 8

Table 1

Axiom	Interpretation
Individual preferences \succsim_i	
Zero-condition (ZERO)	If lifetime is zero then the health state is irrelevant
Lifetime continuity (CONT)	Preferences are smooth in lifetime
Positivity (POS)	For any health state there is some positive lifetime preferred to zero lifetime
Lifetime scale independence (LSI)	The relation between two health profiles is invariant to proportional changes in lifetimes
Symmetry (SYM)	All individuals have identical preferences
Social preferences \succsim	
Pareto principle (PAR)	If the health profiles are better for all individuals in one health distribution relative to another, then the former is socially preferred
Lifetime continuity of social welfare (CONT-S)	Social welfare ordering is smooth in lifetimes
Lifetime anonymity (LTA)	Social welfare ordering is anonymous in the distribution of life years
Full health domain (FHD)	There is a continuum of health states values between 0 and 1
Relative lifetime comparisons (RLC)	There is some health state, such that for pairs of individuals (i and j) with positive lifetime at this health state, a change in lifetime for individual i is equally good as change in lifetime for individual j in the same proportion
Non-age dependence w.r.t. a_- (NAD)	For any pair of individuals (i and j) at health state a_- , c extra years to i is equally good as c extra years to individual j
Separability (SEP)	Social preferences for health distributions within a group $S \subset N$ are independent of the health of the other individuals in $N \setminus S$
Common lifetime scale independence (CLSI)	The relation between two health distributions is invariant to proportional changes in lifetimes
Common health state independence (CHI)	If two individuals have the same health state, then welfare comparisons of pairs of positive lifetimes for these two individuals do not depend on the particular health state

Table 2

Individual preferences \succsim_i	Social preferences \succsim	Representation $U(a, t)$	Theorem
–	–	$W(q_1(a_1, t_1), \dots, q_n(a_n, t_n))$	2
FHD	LTA	$\prod_{i \in N} q_i(a_i, t_i)$	3
–	RLC	$\prod_{i \in N} q_i(a_i, t_i)$	4
–	NAD w.r.t. a_-	$\sum_{i \in N} h_i^{-1}(a_-) q_i(a_i, t_i)$	5
–	NAD w.r.t. a^*	$\sum_{i \in N} q_i(a_i, t_i)$	5
–	SEP	$\sum_{i \in N} v_i(q_i(a_i, t_i))$	6 (Debreu)
–	SEP, CLSI	$\sum_{i \in N} \alpha_i (q_i(a_i, t_i))^p, \quad p > 0$	7
FHD, SYM	SEP, CHI	$\sum_{i \in N} \alpha_i (q_i(a_i, t_i))^p, \quad p > 0$	8

we would in addition to the Bergson family of functions also admit Cobb–Douglas social welfare functions

$$U(a, t) = \prod_{i \in N} (q_i(a_i, t_i))^{\alpha_i},$$

for which Dolan's solution is the special case $\alpha_1 = \dots = \alpha_n = 1$. This class of welfare functions (that is, the Bergson family plus the Cobb–Douglas family) has been studied by Samuelson (1965), Atkinson (1970), Blackorby and Donaldson (1982), Roberts (1980) and others, and in the specific context of QALYs discussed by Dolan (1998).

4.4. Overview of results

Table 1 contains a lists of all axioms giving the intuitive meanings.¹⁰ Table 2 contains an overview of results in Section 4, where it throughout is assumed that individual preferences satisfy ZERO, CONT, POS and LSI and social preferences satisfy PAR and CONT-S.

5. Discussion

We have so far considered the simplest possible relevant model where the health profile of an individual is summarized in two pieces of information, health state and life years. We discuss in turn extensions of the model to incorporate uncertainty and to incorporate non-chronic health states, respectively.

5.1. Uncertainty

It was argued by Broome (1993) that uncertainty is a complication rather than an essential part of the problem of valuing lives, and it ought not to be introduced into the analysis

¹⁰ Note that the symmetry axiom SYM is somewhat different from the other axioms related to individual preferences, because it relates to the identity of the individual preference relations, not to a property of a single preference relation.

earlier than needed. In this spirit, we have considered a formulation of the health care resource allocation problem which contains no explicit element of risk, and in which we obtain characterizations of allocation methods separated from assumptions made on the individuals' and social planner's risk attitudes.

Axioms for choice between deterministic prospects can be useful for decision under uncertainty (see e.g. Wakker, 1996). In a more general model where each individual faces a lottery over health profiles we will, however, need some assumptions on risk attitudes.

Consider a probabilistic model where each individual has a preference relation \succsim_i^L on finite lotteries over health profiles. Under the axioms of expected utility (see e.g. Mas-Colell et al., 1995) there is a function

$$u_i(a_i, t_i) \quad (4)$$

such that for individual i the preference relation \succsim_i^L over health lotteries is governed by the expectation $E(u_i(a_i, t_i))$. It follows from Bleichrodt et al. (1997) that under expected utility, risk neutrality in life years, the zero-condition ($u_i(a_i, 0) = u_i(a'_i, 0)$ for all a_i, a'_i) and positivity ($u_i(a_i, t_i) > u_i(a_i, 0)$ for all a_i and some t_i) then

$$u_i = h_i(a_i)t_i, \quad (5)$$

where h_i is the time trade-off index ($h_i(a_i) = \{t_i \in \mathbb{R}_{++} | u_i(a_i, t_i) = u_i(a_i, 1)\}$).

In this context, the natural extension of the social welfare ordering is now an ordering \succsim^L on distributions of individual health lotteries.

Let $(a_i, t_i, p_i) = [(a_{i1}, t_{i1}, p_{i1}), \dots, (a_{im_i}, t_{im_i}, p_{im_i})]$, where $p_{i1} + \dots + p_{im_i} = 1$, denote a health lottery where individual i obtains (a_{i1}, t_{i1}) with probability p_{i1} , etc. Likewise, let $(a, t, p) = [(a_{i1}, t_{i1}, p_{i1}), \dots, (a_{im_i}, t_{im_i}, p_{im_i})]_{i=1}^n$ denote a distribution of individual health lotteries.

In this probabilistic context, let (a_i, t_i) denote the degenerate health lottery $(a_i, t_i, 1)$ (i.e. (a_i, t_i) is obtained with unit probability), and let the health distribution $(a, t) = ((a_1, t_1), \dots, (a_n, t_n))$ denote the distribution of degenerate health lotteries.

The natural generalizations of the Pareto principle and lifetime continuity of social welfare are then:

$\text{PAR}^L : (a, t, p) \succ^L (a', t', p')$ if $(a_i, t_i, p_i) \succ_i^L (a'_i, t'_i, p'_i)$ for all $i \in N$.

CONT-S^L : Let $t^{(k)} = [t_{i1}^{(k)}, \dots, t_{im_i}^{(k)}]_{i=1}^n$ be a converging sequence, $t = \lim_{k \rightarrow \infty} t^{(k)}$. If $(a, t^{(k)}, p) \succsim^L (a', t', p')$ for all k , then $(a, t) \succsim^L (a', t')$. If $(a', t', p') \succsim^L (a, t^{(k)}, p)$ for all k , then $(a', t', p') \succsim^L (a, t, p)$.

We now have the following variant of Theorem 2.

Theorem 9. Assume that each individual has preferences over health lotteries governed by the expectation $E(h_i(a_i)t_i)$, where $h_i(a_i)$ is the time trade-off index. Then if the social welfare ordering \succsim^L satisfies PAR^L and CONT-S^L , it is represented by a continuous social welfare function depending only on the distribution of individual expected QALYs, i.e. there exists a continuous function $W : \mathbb{R}_+^N \rightarrow \mathbb{R}$ such that

$$(a, t, p) \succsim^L (a', t', p') \Leftrightarrow W(E(q(a, t))) \geq W(E(q(a', t'))),$$

where $E(q(a, t)) = (E(q_1(a_1, t_1)), \dots, E(q_n(a_n, t_n)))$, $q_i(a_i, t_i) = h_i(a_i)t_i$, and h_i is the time trade-off index for all $i \in N$.

The proof is close to the proof of Theorem 2 and omitted.

By Theorem 9, we can find a social welfare function where the distribution of expected linear QALYs is the argument. Since an arbitrary expected QALY value $E \geq 0$ can be obtained by the degenerate health lottery (a^*, E) (recall that $E(q_i(a^*, E)) = h_i(a^*)E = E$) for every individual i , the social welfare function W is entirely determined from the restriction of \succsim^L to distributions of degenerate health lotteries.

Theorems 3–8 therefore apply to the extended probabilistic model with the following changes:

- Replace individual preferences \succsim_i with individual preferences over health lotteries \succsim_i^L and assume that these are governed by the expectation $E(h_i(a_i)t_i)$, where $h_i(a_i)$ is the time trade-off index. For Theorems 3 and 8, let FHD apply to the restriction of \succsim_i^L to degenerate health lotteries, and for Theorem 8 let SYM apply to these restrictions as well.
- Replace the social welfare ordering \succsim with the social welfare ordering \succsim^L on distributions of health lotteries. Replace PAR and CONT-S with PAR^L and CONT-S^L , and let the other axioms defined for \succsim (LTA, RLC, NAD, SEP, CLSI, CHI) apply to the restriction of \succsim^L to distributions of degenerate health lotteries.

5.2. Non-chronic health

We have so far assumed that an individual is always at the same health state throughout life. As noted by Pliskin et al. (1980) we can interpret these health states as some sort of averages over the entire life span. But restricting attention to chronic health states also blurs some important aspects of QALY measurement at the individual level (Hansen and Østerdal, 2004).

As mentioned in Section 2, we can alternatively interpret life years in the model as *gains* and *losses* relative to some status quo possibly different from the natural zero (i.e. zero life years to all individuals). This might well be a relevant interpretation, since the status quo health of a population could be unknown to the social planner. If such gains and losses in life years are relatively small compared to the whole life span, assuming chronic health is less restrictive. In addition, we are then able to deal with health states worse than death since we then obtain a natural interpretation of negative values for the time trade-off indices h_i (and thereby may allow for negative individual QALYs).¹¹ In the case of a utilitarian social welfare criterion negative QALYs can be used. It remains an open question how to modify Dolan's method and the class of methods characterized in Section 4.3 to deal in a reasonable way with health states that are worse than being dead.

In a model where health varies over time, it is useful to introduce a health state a_0 called 'death' since a health profile can then be described by a function l_i from the non-negative

¹¹ Note, however, that we cannot obtain a representation similar to Theorem 1 (allowing for negative time trade-off scores) simply by omitting the POS axiom. The time trade-off function h_i may then not be well defined.

reals to A such that for some t_i (the age at death for individual i) we have $l_i(s) \neq a_0$ for all $s < t_i$ and $l(s) = a_0$ for all $s \geq t_i$.

Now consider an individual preference relation \succsim_i^N defined on some class of life profiles L which contains the chronic health profiles. We then say that \succsim_i^N is represented by a linear QALY model if

$$q_i(l_i) = \int h_i(l_i(s)) ds, \quad (6)$$

where h_i is the time trade-off index, and

$$l_i \succsim_i^N l'_i \Leftrightarrow q_i(l_i) \geq q_i(l'_i),$$

for all $l_i, l'_i \in L$.

Let \succsim^N be a social welfare ordering on distributions of life profiles drawn from L , and let $l = (l_1, \dots, l_n)$ and $l' = (l'_1, \dots, l'_n)$ denote distributions of life profiles in society.

The natural generalizations of the Pareto principle and lifetime continuity of social welfare are then:

$\text{PAR}^N : l \succ^N l'$ if $l_i \succ_i^N l'_i$, for all $i \in N$.

CONT-S^N : For each i , let l_i be a health profile with t_i life years, let $l_i[\varepsilon]$ be a health profile with $t_i + \varepsilon$ life years which agrees with l_i on the interval $[0, \min\{t_i, t_i + \varepsilon\}]$, let $l[\varepsilon] = (l_1[\varepsilon], \dots, l_n[\varepsilon])$, and let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots\}$ be a sequence where $|\varepsilon_k| < k^{-1}$. If $l[\varepsilon_k] \succsim^N l'$ for all k , then $l \succsim^N l'$. If $l' \succsim^N l[\varepsilon_k]$ for all k , then $l' \succsim^N l$.

We now have this variant of Theorem 2.

Theorem 10. Assume that each individual has preferences over health profiles governed by the utility function $\int h_i(l_i(s)) ds$. Then if the social welfare ordering \succsim^N satisfies PAR^N and CONT-S^N , it is represented by a continuous social welfare function depending only on the distribution of individual QALYs, i.e. there exists a continuous function $W : \mathbb{R}_+^N \rightarrow \mathbb{R}$ such that

$$l \succsim^N l' \Leftrightarrow W(q_1(l_1), \dots, q_n(l_n)) \geq W(q_1(l'_1), \dots, q_n(l'_n)),$$

where $q_i(l_i) = \int h_i(l_i(s)) ds$, and h_i is the time trade-off index for all $i \in N$.

The proof is again close to the proof of Theorem 2 and omitted.

Since an arbitrary QALY value $E \geq 0$ can be obtained by a health profile where the individual is always at the same health state throughout life ($l_i(s) = a^*$ for $s < E$ and $l_i(s) = a_0$ otherwise) for every individual i , the social welfare function W is entirely determined from the restriction of \succsim^N to distributions of health profiles with chronic health states.

More precisely, Theorems 3–8 apply to the case of non-chronic health profiles with the following changes:¹²

¹² See Hasman and Østerdal (2004) for an examination of a principle of (possibly age-dependent) equal value of continued life in a model with non-chronic health states.

- Replace individual preferences \succsim_i with individual preferences over non-chronic health profiles \succsim^N and assume that these are governed by a linear QALY model (6). For Theorems 3 and 8, let FHD apply to the restriction of \succsim_i^N to chronic health profiles, and for Theorem 8 let SYM apply to this restriction as well.
- Replace the social welfare ordering \succsim with the social welfare ordering \succsim^N on distributions of (possibly non-chronic) health profiles. Replace PAR and CONT-S with PAR^N and CONT-S^N , and let the other axioms defined for \succsim (LTA, RLC, NAD, SEP, CLSI, CHI) apply to the restriction of \succsim^N to distributions of chronic health profiles.

We leave the combining of uncertainty and non-chronic health to the reader.

5.3. *Status quo bias*

In recent years, there has been a growing interest in modelling situations where individual or social preferences depend on a reference point or a ‘status quo’ (Dolan and Robinson, 2001; Munro and Sugden, 2002; see also Section 3.3 in Dolan, 2000). We have not attempted to model explicitly the effect of status quo. In this paper we can think of the zero-distribution with zero life years to all individuals as the status quo, or we may simply assume that status quo is irrelevant, because the whole problem is how we want society to allocate resources when this has to be decided before we know the health profiles of named individuals. Alternatively, we could allow for status quo effects via asymmetry in individual or social preferences. However, further research on status quo dependent social choice in general and in the context of health care resource allocation is needed.

5.4. *Concluding remarks*

This paper has investigated possible theoretic foundations for QALY-based approaches to health care resource allocation. The degree to which the various axioms are reasonable is a question that we can give no definitive answer to; it is a value judgement and depends on the context.

In practice, a questionnaire study can be used in which respondents’ preferences for own health profiles are elicited from time trade-off scores and attitudes toward axioms for distributive justice (LTA, RLC, NAD, CLSI, CHI, etc.) evaluated by means of, for example, pairwise comparisons of axioms or ranking exercises. From this approach, we can avoid asking respondents to weigh abstract QALY gains between different individuals. For many respondents, it is presumably difficult conceptually to distinguish between individual utility functions (their own and others) and a social welfare function on QALY distributions, and the combined effect of the curvatures of these functions. Using an axiomatic approach, we may obtain one way of getting around such difficulties.

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Appendix A. Proofs

Proof of Lemma 1 (*POS, CONT, LSI* \Rightarrow *MONO*). Assume that there exists $a_i \in A$ such that $(a_i, t'_i) \succsim_i (a_i, t_i)$ for some $t_i > t'_i \geq 0$. First, suppose that $t'_i > 0$. By POS there is $\bar{t}_i > 0$ such that $(a_i, \bar{t}_i) \succ_i (a_i, 0)$. By LSI, $(a_i, \bar{t}_i(t'_i/t_i)) \succsim_i (a_i, \bar{t}_i)$. By LSI and transitivity, $(a_i, \bar{t}_i(t'_i/t_i)^k) \succsim_i (a_i, \bar{t}_i)$ for all $k=1, 2, 3, \dots$. By CONT $(a_i, 0) \succsim_i (a_i, \bar{t}_i)$ contradicting POS. Next, suppose that $t'_i = 0$, i.e. $(a_i, 0) \succsim_i (a_i, t_i)$ for some $t_i > 0$ but otherwise $(a_i, t'_i) \succ_i (a_i, t''_i)$ for all $t'_i > t''_i > 0$. Since $(a_i, t_i) \succ_i (a_i, t_i/2)$ by transitivity we have $(a_i, 0) \succ_i (a_i, t_i/2)$. However, since $(a_i, t_i/2) \succ_i (a_i, t_i/k)$ for $k=3, 4, \dots$ by CONT $(a_i, t_i/2) \succsim_i (a_i, 0)$, a contradiction.

(*ZERO, CONT, LSI, MONO* \Rightarrow *COMP*). Let a_i be an arbitrary health state. Now assume that there is some $a'_i \in A$, $a_i \neq a'_i$ such that there exists no $t_i > 0$ for which $(a_i, t_i) \sim_i (a'_i, 1)$. By CONT and MONO, then either (i) $(a_i, t_i) \succ_i (a'_i, 1)$ for all $t_i > 0$ or (ii) $(a'_i, 1) \succ_i (a_i, t_i)$ for all $t_i > 0$.

Case i. By CONT then $(a_i, 0) \succsim_i (a'_i, 1)$ and by ZERO and transitivity $(a'_i, 0) \succsim_i (a'_i, 1)$ contradicting MONO.

Case ii. By LSI $(a'_i, 1/t_i) \succ_i (a_i, 1)$ for all $t_i > 0$ and by CONT $(a'_i, 0) \succsim_i (a_i, 1)$. By ZERO and transitivity $(a_i, 0) \succsim_i (a_i, 1)$ contradicting MONO. \square

Proof of Corollary 1. Let $a_i \in A$. By Lemma 1 (COMP), the set $h_i(a_i)$ is non-empty, and by MONO it is a singleton. \square

Proof of Theorem 1. *Sufficiency* (\Rightarrow). By Corollary 1, h_i is a real-valued function. Moreover, by Lemma 1 (COMP) we observe that $h_i(a_i) > 0$ for all $a_i \in A$.

Since $(a_i, 1) \sim_i (a_*, h_i(a_i))$, by we have $(a_i, t_i) \sim_i (a_*, t_i h_i(a_i))$, and similarly $(a'_i, t'_i) \sim_i (a_*, t'_i h_i(a'_i))$, $t_i, t'_i \geq 0$. By transitivity and Lemma 1 (MONO) $(a_i, t_i) \succsim_i (a'_i, t'_i) \Leftrightarrow (a_*, h_i(a_i)t_i) \succsim_i (a_*, h_i(a'_i)t'_i) \Leftrightarrow h_i(a_i)t_i \geq h_i(a'_i)t'_i$. Thus (3) holds if $f_i = h_i$.

Now, for some f_i assume that there is no $c > 0$ such that $f_i = ch_i$. Clearly, if $f_i = ch_i$ for $c < 0$ then (3) does not hold. Thus, for some $a_i, a'_i \in A$, $f_i(a_i)/f_i(a'_i) \neq h_i(a_i)/h_i(a'_i)$. Let $t_i > 0$ and let t'_i be defined such that $(a_i, t_i) \sim_i (a'_i, t'_i)$. Then $h_i(a_i)t_i = h_i(a'_i)t'_i$ but $f_i(a_i)t_i \neq f_i(a'_i)t'_i$ contradicting (3).

Necessity (\Leftarrow). Conversely, if $f_i : A \rightarrow \mathbb{R}_{++}$ is a function satisfying (3) then it is easy to show that ZERO, CONT, POS and LSI hold. \square

Proof of Theorem 2. Let $q_i(a_i, t_i) = h_i(a_i)t_i$. By Theorem 1, $q_i(a_i, t_i) = q_i(a'_i, t'_i)$ if and only if $(a_i, t_i) \sim_i (a'_i, t'_i)$. Moreover, if $q_i(a_i, t_i) = q_i(a'_i, t'_i)$ for all $i \in N$ then $(a, t) \sim (a', t')$. To see this, assume otherwise for some $(a, t), (a', t') \in (A \times \mathbb{R}_+)^N$, where $(a_i, t_i) \sim_i (a'_i, t'_i)$ for all i that $(a, t) \succ (a', t')$. Let $t'(\varepsilon) = (t'_1 + \varepsilon, \dots, t'_n + \varepsilon)$. By Lemma 1 (MONO) and transitivity $(a'_i, t'_i + \varepsilon) \succ_i (a_i, t_i)$ for all $\varepsilon > 0$, $i \in N$. Then by PAR, $(a', t'(\varepsilon)) \succ (a, t)$ for all $\varepsilon > 0$, but $(a, t) \succ (a', t'(0))$ contradicting CONT-S.

Let \succsim_Q be a binary relation on $Q = \mathbb{R}_+^N$ such that $q \succsim_Q q'$ if there exists $(a, t), (a', t') \in (A \times \mathbb{R}_+)^N$, $q = q(a, t)$ and $q' = q(a', t')$ for which $(a, t) \succsim (a', t')$. Since $(a, t) \sim (a', t')$ if $q_i(a_i, t_i) = q_i(a'_i, t'_i)$ for all $i \in N$, \succsim_Q is complete and transitive representing \succsim in the sense that $q(a, t) \succsim_Q q(a', t')$ if and only if $(a, t) \succsim (a', t')$, for all $(a, t), (a', t') \in (A \times \mathbb{R}_+)^N$.

It remains to show that \succsim_Q is continuous. That is, for any $q \in \mathbb{R}_+^N$, and any converging sequence $q^{(k)}$, $\lim_{k \rightarrow \infty} q^{(k)} = q$, we require that (i) if $q^{(k)} \succsim_Q q'$ for all k then $q \succsim_Q q'$ and (ii) if $q' \succsim_Q q^{(k)}$ for all k then $q' \succsim_Q q$. We will verify that case (i) holds. Let (a', t') satisfy $q_i(a'_i, t'_i) = q'_i$ for all $i \in N$. By Lemma 1 (COMP) there exists $t^{(k)}$ such that $q^{(k)} = (q_1(a_*, t_1^{(k)}), \dots, q_n(a_*, t_n^{(k)}))$. Since $q^{(k)}$ converges, $\lim_{k \rightarrow \infty} t^{(k)} = t$ exists. By CONT-S, $((a_*, t_1), \dots, (a_*, t_n)) \succsim ((a'_1, t'_1), \dots, (a'_n, t'_n))$ hence $q \succsim_Q q'$.

As case (ii) follows by a similar argument, we may conclude that \succsim_Q is continuous. Thus \succsim_Q can be represented by a continuous function $W : \mathbb{R}_+^N \rightarrow \mathbb{R}$ (cf. Proposition 3.C.1 in Mas-Colell et al., 1995). \square

Proof of Theorem 3. *Sufficiency* (\Rightarrow). By Theorem 2, the social welfare ordering \succsim may be represented by a continuous social welfare function $W(q_1, \dots, q_n)$, where $q_i = h_i(a_i)t_i$. By PAR $W(q_1, \dots, q_n) > W(q'_1, \dots, q'_n)$ if $q_i > q'_i$ for all $i \in N$, i.e. W is monotonic. By LTA, $W(h_1(a_1)t_1, \dots, h_n(a_n)t_n) = W(h_1(a_1)t_{\sigma(1)}, \dots, h_n(a_n)t_{\sigma(n)})$ for all (a, t) and σ .

Consider a distribution $q = (q_1, \dots, q_n) > 0$. Let $p = \{i, j\}$ be an unordered pair $i, j \in N$, $i \neq j$, and let $K = 1 + 2 + \dots + (n - 1)$ be the number of unordered pairs from N . Then let $\{p_1, \dots, p_K\}$ be a list of unordered pairs where each pair appears exactly once.

For $0 \leq k \leq K$ define $q(k)$ recursively as follows: $q(0) \equiv (q_1, \dots, q_n)$. Given $q(k)$ and $p_{k+i} = (i', j')$, $i' < j'$, define $q(k+1)$ as

$$\left(q_1(k), \dots, q_{i'-1}(k), \frac{q_{i'}(k)}{\sqrt[n]{q_{i'}}}, \sqrt[n]{q_{j'}} q_{i'+1}(k), \dots, q_{j'-1}(k), \frac{q_{j'}(k)}{\sqrt[n]{q_{j'}}}, \sqrt[n]{q_{i'}} q_{j'+1}(k), \dots, q_n(k) \right).$$

Note that

$$q_i(K) = \frac{q_i}{(\sqrt[n]{q_i})^{n-1}} (\sqrt[n]{q_1}) \cdots (\sqrt[n]{q_{i-1}}) (\sqrt[n]{q_{i+1}}) \cdots (\sqrt[n]{q_n}) = \sqrt[n]{q_1} \sqrt[n]{q_2} \cdots \sqrt[n]{q_n},$$

for all $i \in N$.

We claim that $W(q(k)) = W(q(k+1))$ for all $0 \leq k \leq K-1$. For this, let $p_{k+1} = (i', j')$ and for $c > 0$ define

$$\bar{h}_{i'} = \frac{q_{i'}(k)}{c \sqrt[n]{q_{i'}}}, \quad t_{i'} = c \sqrt[n]{q_{i'}},$$

and

$$\bar{h}_{j'} = \frac{q_{j'}(k)}{c \sqrt[n]{q_{j'}}}, \quad t_{j'} = c \sqrt[n]{q_{j'}}.$$

For c sufficiently large $\bar{h}_{i'}, \bar{h}_{j'} \in]0, 1]$, and by FHD there exist health states $a_{i'}, a_{j'}$ such that $h_{i'}(a_{i'}) = \bar{h}_{i'}$ and $h_{j'}(a_{j'}) = \bar{h}_{j'}$. Hence by LTA $W(q(k)) = W(q(k+1))$, and because this holds for all k we conclude that $W(q) = W(q(K))$.

Since

$$W(q_1, \dots, q_n) = W(\sqrt[n]{q_1} \sqrt[n]{q_2} \cdots \sqrt[n]{q_n}, \dots, \sqrt[n]{q_1} \sqrt[n]{q_2} \cdots \sqrt[n]{q_n}),$$

for any $q > 0$, by continuity we find that $W(q_1, \dots, q_n) = 0$ if $q_i = 0$ for some i . By monotonicity we then have $W(q) = F(q_1 q_2 \cdots q_n)$, where F is a strictly increasing function on \mathbb{R}_+ .

Necessity (\Leftarrow). It is easily verified that the axioms are implied by the social welfare function $U(a, t) = \prod_{i \in N} h_i(a_i) t_i$. \square

Proof of Theorem 4. Let individual preferences satisfy ZERO, CONT, POS and LSI for all $i \in N$; moreover assume that the social welfare ordering satisfies PAR and CONT-S. We claim that RLC is satisfied if and only if the social welfare function is $U(a, t) = \prod_{i \in N} h_i(a_i) t_i$.

Sufficiency (\Rightarrow). Since

$$[(a_-, ct_i), (a_-, t_j), (a_k, t_k)_{k \in N \setminus \{i, j\}}] \sim [(a_-, t_i), (a_-, ct_j), (a_k, t_k)_{k \in N \setminus \{i, j\}}],$$

for all $t_i, t_j, c > 0$, we have by Theorems 1 and 2 that

$$\begin{aligned} & W(ch_i(a_-)t_i, h_j(a_-)t_j, (h_k(a_k)t_k)_{k \in N \setminus \{i, j\}}) \\ &= W(h_i(a_-)t_i, ch_j(a_-)t_j, (h_k(a_k)t_k)_{k \in N \setminus \{i, j\}}), \end{aligned}$$

where W represents the social welfare ordering \succsim . That is, for any $(q_1, \dots, q_n) > 0, c > 0$

$$W(cq_i, q_j, q_k)_{k \in N \setminus \{i, j\}} = W(q_i, cq_j, q_k)_{k \in N \setminus \{i, j\}},$$

which in turn implies

$$W(q_1, \dots, q_n) = W\left(\prod_{k \in N} q_k, 1, \dots, 1\right)$$

Since W is continuous we find that $W(q_1, \dots, q_n) = 0$ if $q_i = 0$ for some i . Thus by monotonicity

$$W = F\left(\prod_{k \in N} q_k\right),$$

for some strictly increasing function F on \mathbb{R}_+ .

Necessity (\Leftarrow). It is easily verified that RLC is implied by the social welfare function $U(a, t) = \prod_{i \in N} h_i(a_i) t_i$. \square

Proof of Theorem 5. *Sufficiency* (\Rightarrow). By Theorems 1 and 2, the social welfare ordering may be represented by a continuous function $W(q_1, \dots, q_n)$, where $q_i = h_i(a_i) t_i$ and h_i satisfies (2). By PAR, $W(q_1, \dots, q_n) > W(q'_1, \dots, q'_n)$ if $q_i > q'_i$ for all $i \in N$, i.e. W is monotonic.

We shall demonstrate that $W(q_1, \dots, q_n) = W(q'_1, \dots, q'_n)$ if and only if $\sum_{i \in N} h_i^{-1}(a_-)q_i = \sum_{i \in N} h_i^{-1}(a_-)q'_i$. By Theorems 1 and 2

$$[(a_-, t_i + c), (a_-, t_j), (a_k, t_k)_{k \in N \setminus \{i, j\}}] \sim [(a_-, t_i), (a_-, t_j + c), (a_k, t_k)_{k \in N \setminus \{i, j\}}]$$

for all $i, j \in N$ and all $c > 0$ if and only if

$$\begin{aligned} & [(a_*, h_i(a_-)(t_i + c)), (a_*, h_j(a_-)t_j), (a_*, h_k(a_k)t_k)_{k \in N \setminus \{i, j\}}] \\ & \sim [(a_*, h_i(a_-)t_i), (a_*, h_j(a_-)(t_j + c)), (a_*, h_k(a_k)t_k)_{k \in N \setminus \{i, j\}}] \end{aligned}$$

for all $i, j \in N$ and all $c > 0$. By Theorem 2, we then observe that the social welfare function satisfies

$$W(q_i + ch_i(a_-), q_j, q_k)_{k \in N \setminus \{i, j\}} = W(q_i, q_j + ch_j(a_-), q_k)_{k \in N \setminus \{i, j\}}$$

for all $i, j \in N$, $c > 0$, implying that

$$W(q_i, q_j, q_k)_{k \in N \setminus \{i, j\}} = W(q_i + ch_i(a_-), q_j - ch_j(a_-), q_k)_{k \in N \setminus \{i, j\}}$$

for all $i, j \in N$, $c > 0$, $q_j - ch_j(a_-) \geq 0$.

Now, let (a, t) be given and let $q_i = h_i(a_i)t_i$ for all i . In addition, let \bar{c}_i satisfy $q_i - \bar{c}_i h_i(a_-) = 0$, i.e. $\bar{c}_i = q_i / h_i(a_-)$. Using NAD $n - 1$ times we have

$$\begin{aligned} W(q_1, \dots, q_n) &= W\left(q_1 + \sum_{i \in N \setminus \{1\}} h_1(a_-)\bar{c}_i, 0, \dots, 0\right) \\ &= W\left(q_1 + \sum_{i \in N \setminus \{1\}} h_1(a_-)\frac{q_i}{h_i(a_-)}, 0, \dots, 0\right) \\ &= W\left(h_1(a_-)\left(\sum_{i \in N} h_i^{-1}(a_-)q_i\right), 0, \dots, 0\right). \end{aligned}$$

Thus by monotonicity $W(q_1, \dots, q_n) = W(q'_1, \dots, q'_n)$ if and only if $\sum_{i \in N} h_i^{-1}(a_-)q_i = \sum_{i \in N} h_i^{-1}(a_-)q'_i$.

Necessity (\Leftarrow). It is easily verified that NAD is implied by the social welfare function $U(a, t) = \sum_{i \in N} h_i^{-1}(a_-)h_i(a_i)t_i$. \square

Proof of Theorem 7. *Sufficiency* (\Rightarrow). By Theorem 1, for any individual i , $q_i = h_i(a_i)t_i$ represents individual preferences for own health where h_i satisfies (2). By Theorem 2, there exists a continuous welfare function $W : \mathbb{R}_+^N \rightarrow \mathbb{R}$ representing the social welfare ordering. By Theorem 6, W is additive. As $h_i(a_i)ct_i = cq_i$, CLSI translates to $W(q_1, \dots, q_n) \geq W(q'_1, \dots, q'_n) \Leftrightarrow W(cq_1, \dots, cq_n) \geq W(cq'_1, \dots, cq'_n)$ for all $q, q' > 0$. By Bergson and Samuelson (Burk, 1936; Samuelson, 1965) a real-valued, continuous and additive function W^{++} on \mathbb{R}_{++}^N satisfying this property has one of the following functional forms:

1. $W^{++}(q_1, \dots, q_n) = \sum_{i \in N} \alpha_i q_i^p$, $p > 0$,
2. $W^{++}(q_1, \dots, q_n) = -\sum_{i \in N} \alpha_i q_i^p$, $p < 0$,
3. $W^{++}(q_1, \dots, q_n) = \sum_{i \in N} \alpha_i \log(q_i)$.

Hence there is a function W^{++} of the form 1, 2 or 3 such that $W = W^{++}$ on \mathbb{R}_{++}^N . Since there is no continuous extension of functions of the form 2 and 3 to \mathbb{R}_+^N we conclude that

$$W(q_1, \dots, q_n) = \sum_{i \in N} \alpha_i q_i^p, \quad p > 0,$$

where (by PAR) α_i is non-negative for all i and positive for some i .

Necessity (\Leftarrow). It is easily verified that the axioms are implied by each social welfare function. \square

Proof of Theorem 8. Let individual preferences satisfy ZERO, CONT, POS, LSI, FHD and SYM for all $i \in N$, and let the social welfare ordering satisfy PAR, CONT-S and SEP.

It is clear from Theorem 1 that $\succsim_i = \succsim_j$ if and only if $h_i = h_j$ (where h_i and h_j satisfy (2)). Therefore let $h = h_1 = \dots = h_n$.

By Theorem 6 there are real-valued functions v_1, \dots, v_n such that $W(q_1, \dots, q_n) = \sum_{i \in N} v_i(q_i)$ represents the social welfare ordering.

($CLSI \Rightarrow CHI$). Assume that CLSI holds. By Theorem 6 if $t, t' > 0$ and

$$\sum_{i \in N} v_i(h(a_i)t_i) \geq \sum_{i \in N} v_i(h(a'_i)t'_i)$$

then

$$\sum_{i \in N} v_i(h(a_i)ct_i) \geq \sum_{i \in N} v_i(h(a'_i)ct'_i),$$

for all $c > 0$. Thus, for any pair i and j , $i \neq j$, if

$$v_i(h(a_i)t_i) + v_j(h(a_j)t_j) \geq v_i(h(a_i)t'_i) + v_j(h(a_j)t'_j)$$

then

$$v_i(h(a_i)ct_i) + v_j(h(a_j)ct_j) \geq v_i(h(a_i)ct'_i) + v_j(h(a_j)ct'_j).$$

Therefore, if

$$v_i(h(a_-)t_i) + v_j(h(a_-)t_j) \geq v_i(h(a_-)t'_i) + v_j(h(a_-)t'_j)$$

then

$$v_i(h(a'_-)t_i) + v_j(h(a'_-)t_j) \geq v_i(h(a'_-)t'_i) + v_j(h(a'_-)t'_j),$$

for all $a'_- \in A$, since by COMP there is $c > 0$ such that $h(a'_-) = h(a_-)c$. Thus CHI holds.

($CHI \Rightarrow CLSI$). By CHI for any pair i and j , $i \neq j$, if $t_i, t_j, t'_i, t'_j > 0$ and

$$v_i(h(a_-)t_i) + v_j(h(a_-)t_j) \geq v_i(h(a_-)t'_i) + v_j(h(a_-)t'_j)$$

then

$$v_i(h(a'_-)t_i) + v_j(h(a'_-)t_j) \geq v_i(h(a'_-)t'_i) + v_j(h(a'_-)t'_j).$$

Let $q_i, q_j, q'_i, q'_j > 0$ and $c > 0$. By SYM and FHD there are health states a_- and a'_- such that $0 < h(a_-), h(a'_-) \leq 1$ and $h(a_-)c = h(a'_-)$. Now, let t_i, t_j and t'_i, t'_j be such that $h(a_-)t_k = q_k$ and $h(a_-)t'_k = q'_k$ for $k = i, j$. From CHI it then follows that

$$v_i(q_i) + v_j(q_j) \geq v_i(q'_i) + v_j(q'_j)$$

implies

$$v_i(q_i c) + v_j(q_j c) \geq v_i(q'_i c) + v_j(q'_j c).$$

From Bergson and Samuelson (Burk, 1936; Samuelson, 1965) and Theorem 7 we have $v_i(q_i) = \alpha_i q_i^p$, for some $p > 0$, $\alpha_i \geq 0$ for all i with $\alpha_i > 0$ for at least one i . Thus CLSI holds. \square

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