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Bounded rationality: static versus dynamic approaches*

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Summary. Two kinds of theories of boundedly rational behavior are possible. Static theories focus on stationary behavior and do not include any explicit mechanism for temporal change. Dynamic theories, on the other hand, explicitly model the fine-grain adjustments made by the subjects in response to their recent experiences. The main contribution of this paper is to argue that the restrictions usually imposed on the distribution of choices in the static approach are generically not supported by a dynamic adjustment mechanism. The genericity here is understood both in the measure theoretic and in the topological sense.

Keywords and Phrases: Bounded rationality, Dynamic adjustment, Genericity.

JEL Classification Numbers: C0, D7.

1 Introduction

There is a growing empirical evidence that calls into question the utility maximization paradigm.[1] On the basis of this evidence, Conlisk (1996) convincingly argued for the incorporation of bounded rationality into economic models. Some early attempts in this direction were made by Alchian (1950), Simon (1957), and Nelson and Winter (1982) among others. But a universal model of boundedly rational behavior still does not exist. The existing models can be divided into two classes: *static* and *dynamic*.

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[1] For a description of systematic errors made by experimental subjects, see Arkes and Hammond (1986), Hogarth (1980), Kahneman, Slovic, and Tversky (1982), Nisbett and Ross (1980), and the survey papers by Payne, Bettman, and Johnson (1992) and by Pitz and Sachs (1984).

In *static* models choice is assumed to be probabilistic. It is typical in this type of models to impose some intuitive restrictions on the choice probabilities and study the probability distributions that satisfy these restrictions. They were introduced into economics by Luce (1959) and have already found their application. See, for example, McKelvey, Palfrey (1995, 1998), Chen, Friedman, Thisse (1997), Offerman, Schram, Sonnemans (1998), and Anderson, Goeree, and Holt (1998, 2001).

In *dynamic* models individuals are assumed to adjust their choices over time in directions that appear beneficial. The dynamic approach originated in the work of Bush and Mosteller (1955), was introduced in economics by Arrow and Hurwicz (1960), and is represented, for example, by papers of Foster and Young (1990), Fudenberg and Harris (1992), Kandori, Mailath, Rob (1993), Young (1993), Friedman and Yellin (1997), Anderson, Goeree, and Holt (1999), Friedman (2000), and Basov (2003b).

The distinctive feature of this type of models is an attempt to capture the fine-grain adjustments made by the individuals on the basis of their current experiences. On a very general level, such adjustments produce a stochastic process on the choice set. The probability distribution of choices of a static model can be naturally viewed as the steady state distribution of the stochastic process arising from a dynamic model. For a study of a broad class of dynamic adjustment processes, see Basov (2001).

This paper studies the connections between the properties of the static and the dynamic models. Many dynamic models assume that the process of choice adjustment leads to better choices on average. For the purposes of this paper, I will formalize this idea using the notion of *a locally improving* (LI) adjustment process, i.e. such a process that the vector of the expected adjustment points into a direction of the increase of the utility.

A question I address is: Does the steady state density of a generic locally improving process satisfy the usual axioms of the static approach? For this purpose I introduce two important concepts: *payoff monotonicity* (PM) and *Independence of Irrelevant Alternatives* (IIA). An adjustment process is PM if for any admissible choice set the density of the steady state distribution at x_1 is greater than the density of the steady state distribution at x_2 if and only if x_1 is preferred to x_2. An adjustment process satisfies IIA if the ratio of the steady state probability densities at any two feasible points do not depend on what other choices are available.

As we will see below, under some mild regularity assumptions, the steady state of each dynamic adjustment process is unique. This, together with the requirement that the restrictions on the steady state density should hold for *any* admissible choice set, implies that the payoff monotonicity and the IIA characterize the *process* rather then a particular distribution.

The first main result of the paper is the following:

(R1) *any PM process is LI.*

This result is rather intuitive. It claims that for the long-run choice probabilities to be increasing in the payoffs for any choice set the expected adjustment vector should point into a direction of the increase of the utility function. The second main result of the paper is less intuitive. To formulate it I consider a finite-dimensional subset of LI processes, PDS processes, i. e. processes for which the deterministic part of the generator is linked to the gradient of the utility by a symmetric positively definite linear transformation.

(R2) *a generic PDS process is neither PM nor IIA. Moreover, if the utility function is additively separable a generic PDS process that satisfies IIA is not PM.*

These findings suggest that the usual restrictions on the probability density in the static approach are too strong since they are not supported by a generic dynamic adjustment process. Therefore, an explicit modelling of the dynamic adjustment process is important when describing boundedly rational behavior.

This paper is organized as follows. Section 2 describes a broad class of stochastic adjustment processes. In Section 3 I define the main concepts of the paper. Section 4 describes an example of a locally improving process for which the steady state is not payoff monotone. Section 5 contains the main results. I give the proofs in Section 6. Section 7 concludes.

2 A model of individual behavior

Let us assume that an individual repeatedly faces with a problem of choosing an alternative from an open, bounded set $\Omega \subset R^n$ with a smooth boundary. She adjusts her choices gradually in response to her recent experiences. The adjustment rule produces a stochastic process on the choice set.

Assume that the adjustment process is Markov and define

$$\mu(x) = \lim_{\tau \to 0} \frac{1}{\tau} E(x(t+\tau) - x(t)\,|x(t) = x) \tag{1}$$

$$\Gamma(x) = \lim_{\tau \to 0} \frac{1}{\tau} Var(x(t+\tau) - x(t)\,|x(t) = x). \tag{2}$$

Vector μ captures the deterministic trend in the adjustment rule, which can be interpreted as an attempt to increase the individual's utility, while matrix Γ is the covariance matrix of the experimentation errors.

Assume that limits (1) and (2) exist. Then, under a mild additional regularity assumption, the Markov process is completely characterized by vector μ and matrix Γ (see, for example, Kanan, 1979). I sometimes refer to it as process (μ, Γ). Assume further that matrix Γ is non-degenerate. From an economic perspective, it means that experimentation has a full range. Then, if the initial distribution of choices can be characterized by a density function, it can also be characterized by a density function at any $t > 0$ and the evolution of the density is determined by the following system:

$$\frac{\partial f}{\partial t} + div(\mu(x) f) = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2(\Gamma_{ij}(x) f)}{\partial x_i \partial x_j}, \tag{3}$$

$$\sum_{i=1}^{n}\left(\frac{1}{2}\sum_{j=1}^{n}\frac{\partial(\Gamma_{ij}(x)f)}{\partial x_j}-\mu_i(x)f\right)n_i(x)=0 \text{ on } \partial\Omega, \tag{4}$$

where $n(x)$ is the unit vector normal to the boundary of the choice set $\partial\Omega$ (Ito, 1992). In the rest of the paper I will assume that matrix Γ does not depend on x. The assumption is made for the sake of simplicity only and does not seriously affect the results.[2] The preferences of the individual are given by a twice continuously differentiable utility function $U(\cdot)$.

3 The definitions of some classes of adjustment processes

In this section I am going to define the main classes of Markov adjustment processes studied in the paper.

Definition 1 *A Markov adjustment process is called locally improving (LI) if*

$$\langle\mu(x),\nabla U(x)\rangle\geq 0 \textit{ for } \forall x\in\Omega. \tag{5}$$

Here and throughout the paper $\langle\cdot,\cdot\rangle$ denotes the inner product of two vectors. In words, a process is LI if the vector of the expected adjustment of the choice, μ, points into a direction of an increase of the utility. The space of all LI processes is a functional space of an infinite dimension. Let us define two finite-dimensional subsets of LI.

Definition 2 *Markov adjustment process* (μ,Γ) *is called PD (PDS) if*

$$\mu(x)=B\nabla U,$$

where B is a (symmetric) positively definite matrix with constant coefficients.

Note that $PDS\subset PD\subset LI$. The dimension of *PDS is* $n(n+1)$, while that of *PD* is $n^2+n(n+1)/2$. Hence, both of them can be embedded into space R^k with an appropriate k endowed with the Lebesgue measure and be considered as measure spaces.

The concept of a locally improving process is a dynamic concept. The next two concepts I define, *payoff monotonicity* and *independence of irrelevant alternatives*, can be thought of as static concepts, since they put restrictions on the steady state density function. However, if one demands that these restrictions should hold for any choice set, they will become properties of the process. To ensure the soundness of this procedure, we need the following result from the theory of stochastic processes.

Lemma 1 *Assume that matrix Γ is non-degenerate. There exists a unique twice continuously differentiable stationary solution of system (3)–(4) satisfying the normalization condition*

$$\int_{\Omega} f_{s,\Omega}(x)dx=1.$$

[2] This assumption is made to assure that the space of *PDS* processes (defined below) is finite-dimensional. All results will continue to hold if one assumes that Γ is finite-parametric, e. g. its matrix elements are polynomials of degree no larger than N, where N can be arbitrary large but fixed.

Moreover, it is positive everywhere on Ω and asymptotically stable.

For a proof, see Ito (1992). The result states that the steady state density is well-defined and is determined by the process rather than by the initial condition. This allows us to give the following definition.

Definition 3 *A Markov adjustment process is called payoff monotone (PM) if for any choice set $\Omega \subset R^n$ and any $x_1, x_2 \in \Omega$*

$$(f_{s,\Omega}(x_1) \geq f_{s,\Omega}(x_2)) \Leftrightarrow (U(x_1) \geq U(x_2)). \tag{6}$$

In words, a Markov adjustment process is PM if for a sufficiently small neighborhood of point x_1 the steady state probability that the individual's choice is within it is higher than the probability that it is within an equimeasurable neighborhood of point x_2 if and only if alternative x_1 is preferred to alternative x_2. Another important restriction often imposed in the static approach is IIA.

Definition 4 *A Markov adjustment process satisfies independence of irrelevant alternatives (IIA) if for any two choice sets Ω_1 and Ω_2 and any $x_1, x_2 \in \Omega_1 \cap \Omega_2$*

$$\frac{f_{s,\Omega_1}(x_1)}{f_{s,\Omega_1}(x_2)} = \frac{f_{s,\Omega_2}(x_1)}{f_{s,\Omega_2}(x_2)}. \tag{7}$$

In words, IIA states that the ratio of the steady state probability densities of two choices does not depend on what other choices are available. In the next section I study a simple example of an adjustment process.

4 An example

In this section I am going to consider a particular example of a locally improving process and show that its steady state density is not payoff monotone. Consider the following adjustment process:

$$dx = \nabla U(x)dt + \Lambda dW. \tag{8}$$

Here $U(\cdot)$ is a twice continuously differentiable utility function, Λ is a 2×2 diagonal matrix with non-zero elements on the main diagonal and $W = (W_1, W_2)$ is a vector of independent standard Wiener processes. Note that the probability density of choices generated by process (8) is governed by (3)-(4) with and μ and Γ given by

$$\mu(x) = \nabla U(x)$$
$$\Gamma = \Lambda^T \Lambda = \begin{matrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{matrix}.$$

The first term in (8) corresponds to the gradient dynamics and states that the individual adjust her choices in the direction of the maximal increase of her utility. The second term states that this adjustment is subject to a random error or experimentation.

Process (8) is locally improving. To find the steady state explicitly, assume that

$$U(x_1, x_2) = u_1(x_1) + u_2(x_2),$$

Then

$$f_{s,\Omega}(x) = C_\Omega \exp\left(\frac{u_1(x_1)}{\sigma_1^2} + \frac{u_2(x_2)}{\sigma_2^2}\right).$$

To see that the steady state is not payoff monotone, consider two choice vectors $x = (x_1,\, x_2)$ and $y = (y_1,\, y_2)$. Assume $u(x_1) = u(x_2) = 5$, $u(y_1) = 4$, $u(y_2) = 8$, $\sigma_1^2 = 1$, $\sigma_2^2 = 10$. Then $u(x) < u(y)$ but $f(x) > f(y)$. Hence, the steady state density is not payoff monotone. Note that such pair of choices can be found for *any* choice set Ω as long as $\sigma_1 \neq \sigma_2$. This result is stronger than the lack of payoff monotonicity by process (8) that states that the steady state density will fail to be payoff monotone for *some* choice set. On the other hand, if $\sigma_1 = \sigma_2$ the direct inspection of the formula for the steady state proves that the process is payoff monotone.

Theorem 2, which I will formulate in the next section, states that the situation described in the above example is generic, i. e. LI processes are usually not PM.

5 The main results

In this section I study the connections between LI, PM and IIA processes and formulate three Theorems that contain the main results of the paper.

Theorem 1 *Assume that*

1.

$$\forall x \in \Omega \; (\mu(x) = 0) \Leftrightarrow (\nabla U(x) = 0) \tag{9}$$

2. Set U_C of the critical points of the utility defined by

$$U_C = \{x \in \Omega : \nabla U(x) = 0\} \tag{10}$$

is finite. Then $PM \subset LI$.

The first assumption states that there is no deterministic adjustment at the critical points of the utility.[3] The second is a regularity assumption. It will always hold if the utility function is analytical and non-constant. Theorem 1 states that LI is necessary for the process to be PM. It is, however, not sufficient. Moreover, a typical LI process is not PM. To formalize this idea I will restrict attention to the finite-dimensional subclasses of LI, PD and PDS.

Theorem 2 *Let assumptions of Theorem 1 hold and also assume that the Hessian of the utility has full rank and* $n > 1$. *Then*

1. $PD \cap PM$ can be embedded in PD as a submanifold of a lower dimension

[3] It is satisfied for *PD* processes.

2. *$PDS \cap PM$ can be embedded in PDS as a submanifold of a lower dimension.*

Theorem 3 states typical LI process is not IIA. However, even if one restricts attention to LI processes that satisfy IIA this will not allow us to escape the conclusion of Theorem 2 for all utility functions.[4] Assume that the utility is additively separable in the components of x, then a typical process that is both PDS and IIA is not PM.

Theorem 3 *Under the assumptions of Theorem 2:*

1. *$PM \subset IIA$*
2. *If the utility function is additively separable, i. e.*

$$U(x) = \sum_{i=1}^{n} u_i(x_i) \tag{11}$$

$PDS \cap PM$ can be embedded in $PDS \cap IIA$ as a submanifold of a lower dimension

3. *If for any non-degenerate constant matrix C there exist i, k such that*

$$\frac{\partial^2 U}{\partial x_i' \partial x_k'} \neq 0, \tag{12}$$

where

$$x' = Cx, \tag{13}$$

then $PDS \cap IIA$ can be embedded in PDS as a submanifold of a lower dimension.

Assumptions (12)–(13) state that the utility is not additively separable and does not become additively separable under any linear transformation. Theorem 3 states that: either the set of IIA processes is small in PDS or a typical process, which is both LI and IIA is not PM.

The main message of these results is that the assumptions of the probability density of choices in the static approach are unlikely to hold. Hence, an explicit modelling of the dynamic adjustment process is needed. In doing so it may be useful to restrict attention to LI processes, or even to its finite-dimensional subclasses (for example, PD or PDS). The next section provides a proof of the main results, which proceeds through a sequence of lemmata that are of an independent interest.

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[4] Remember that we think of a utility function as fixed and parametrize the adjustment processes by coefficients of matrices B and Γ.

6 Proof of the main results

In this section I develop a sequence of lemmata that lead to the proof of Theorems 1–3. Throughout this section I always assume that the assumptions of Theorem 1 hold without stating them explicitly. The following two lemmata and the corollary constitute a proof of Theorem 1.

Lemma 2 *A Markov process is PM if and only if for any Ω there exists a strictly increasing continuously differentiable function $g_\Omega : U(\Omega) \to R_+/\{0\}$ such that*

$$f_{s,\Omega}(x) = g_\Omega(U(x)). \tag{14}$$

Proof. Consider a rational continuous preference relation $\succeq$ defined by

$$(x \succeq y) \Leftrightarrow (U(x) \geq U(y)). \tag{15}$$

The payoff monotonicity implies that $f_{s,\Omega}(\cdot)$ is a utility function that represents preferences relation $\succeq$, which is also represented by $U(\cdot)$. Hence, there exists continuous strictly increasing function $g_\Omega : U(\Omega) \to R$ such that

$$f_{s,\Omega}(x) = g_\Omega(U(x)). \tag{16}$$

According to Lemma 1, $f_{s,\Omega}(\cdot)$ is positive on Ω, hence $g_\Omega(\cdot) > 0$.

To prove that $g_\Omega(\cdot)$ is differentiable let us consider $\overline{U}$, $\overline{U} + \delta U \in U(\Omega)$ and $\delta U \neq 0$. Then $\exists x, x + \delta x \in \Omega$ such that

$$U(x) = \overline{U},\ U(x + \delta x) = \overline{U} + \delta U. \tag{17}$$

Note that, since set U_C is finite, it is always possible to select x in such a way that $\nabla U(x) \neq 0$, which in turn allows to select δx such that $\langle \delta x, \nabla U(x) \rangle \neq 0$. Continuity of $U(\cdot)$ implies that

$$(\delta U \to 0) \Leftrightarrow (\delta x \to 0). \tag{18}$$

Let $\delta x = a \left\| \delta x \right\|$, where a is a unit vector pointing in the direction δx. Then

$$\lim_{\delta U \to 0} \frac{g_\Omega(\overline{U} + \delta U) - g_\Omega(\overline{U})}{\delta U} = \frac{\langle \nabla f_{s,\Omega}, a \rangle}{\langle \nabla U, a \rangle}. \tag{19}$$

Equation (19) asserts that the limit on the left hand side exists, hence the function $g_\Omega(\cdot)$ is differentiable. Moreover, the derivative is continuous. Since the left hand side of (19) does not depend on a, the right hand side does not depend on a either. Therefore, $\nabla f_{s,\Omega}$ is proportional to ∇U with the coefficient of proportionality $g'_\Omega(U(x))$. □

Next, I am going to show that payoff monotonicity implies some connection between the deterministic part of the adjustment process, μ, and its stochastic part, Γ.

Lemma 3 *Markov process* (μ, Γ) *is PM if and only if there exists a continuous function* $c : U(\Omega) \rightarrow R_+/\{0\}$ *such that*

$$\mu = c(U)\Gamma\nabla U. \tag{20}$$

Moreover, if the Hessian of U has full rank $c(\cdot)$ *is differentiable.*

Proof. Suppose (20) holds. Define

$$\xi(z) = 2\int_0^z c(y)dy. \tag{21}$$

Then it is easy to check that

$$f_{s,\Omega}(x) = \frac{\exp(\xi(U(x)))}{\int_\Omega \exp(\xi(U(y)))dy} \tag{22}$$

is a normalized stationary solution of (3)-(4). According to Lemma 1, it is the unique normalized stationary solution. According to (21), $\xi'(\cdot) = 2c(\cdot) > 0$. Hence,

$$(f_{s,\Omega}(x_1) \geq f_{s,\Omega}(x_2)) \Leftrightarrow (U(x_1) \geq U(x_2)) \tag{23}$$

so the adjustment process is PM.

Now, suppose that the adjustment process is PM. Then, according to Lemma 2, there exists a continuously differentiable strictly increasing function $g_\Omega : R \rightarrow R_+/\{0\}$ such that

$$f_{s,\Omega}(x) = g_\Omega(U(x)). \tag{24}$$

Define vector κ by:

$$\kappa = g_\Omega(U(x))\mu(x) - \frac{g'_\Omega(U(x))}{2}\Gamma\nabla U(x). \tag{25}$$

Then (3)-(4) implies that it satisfies

$$div\kappa = 0 \text{ on } \Omega, \tag{26}$$

$$\langle \kappa, n \rangle = 0 \text{ on } \partial\Omega. \tag{27}$$

Moreover, the definition a PM process implies that (26)-(27) should hold for any Ω. Hence, $\kappa = 0$. Now (25) implies that

$$\frac{g'_\Omega(U(x))}{2g_\Omega(U(x))} = \frac{\langle \mu(x), \nabla U(x) \rangle}{\langle \nabla U(x), \Gamma\nabla U(x) \rangle} \tag{28}$$

provided that $\nabla U(x) \neq 0$. The right hand side of (28) does not depend on Ω, hence the left hand side should also not depend on Ω. Introduce

$$c(z) = \frac{g'_\Omega(z)}{2g_\Omega(z)}. \tag{29}$$

Since set U_C is finite, for any $z \in U(\Omega)$ the exists $x \in \Omega$ such that $U(x) = z$ and $\nabla U(x) \neq 0$. Hence, $c(\cdot)$ is defined on $U(\Omega)$. According to Lemma 2, $c(\cdot) \geq 0$ and according to (9) and (28) $c(\cdot) \neq 0$. Hence, $c(\cdot) > 0$. Finally, putting $\kappa = 0$ in (25) and using the definition of $c(\cdot)$ we get

$$\mu(x) = c(U(x))\Gamma\nabla U(x). \tag{30}$$

Proof of differentiability of $c(\cdot)$ is similar to the proof of differentiability of $g_\Omega(\cdot)$ in Lemma 2 and is omitted. □

Corollary 1 $PM \subset LI$.

Proof. According to Lemma 3, we can write

$$\mu(x) = c(U(x))\Gamma\nabla U(x). \tag{31}$$

for some positive real valued function $c(\cdot)$. Therefore,

$$\langle \mu(x), \nabla U(x)\rangle = c(U(x))\langle \nabla U(x), \Gamma\nabla U(x)\rangle \geq 0. \tag{32}$$

Hence, the process is locally improving. □

This completes the proof of Theorem 1. Lemmata 4 and 5 below constitute a proof of Theorem 2.

Lemma 4 *Let $x_0 \in \Omega$ and*

$$\Theta = \{x \in \Omega : U(x) = U(x_0)\} \tag{33}$$

be the indifference surface passing through x_0 and the Hessian of utility be non-degenerate. Then there exist n different points $x_1, .., x_n \in \Theta$ such that vectors $\nabla U(x_1), ..., \nabla U(x_n)$ are linearly independent.

Proof. First observe that if n vectors $b_1, .., b_n$ ($b_i \in R^n$) are linearly independent then $\exists \delta > 0$ such that for any $\varepsilon_1, ..., \varepsilon_n$ ($\varepsilon_i \in R^n$, $\|\varepsilon_i\| < \delta$) vectors $b_i + \varepsilon_i$ are also linearly independent (this follows from the fact that the determinant of the matrix formed by n vectors in R^n continuously depends on its columns). Since the Hessian of U has full rank, the indifference surface is not a hyperplane, therefore for $\forall \eta > 0$ there exist n different points $x_1, .., x_n \in \Theta$ such that vectors $(x_i - x_0)$ are linearly independent and $\|x_i - x_0\| < \eta$. Using the full rank assumption again, one concludes that vectors v_i defined by

$$v_i = D^2U(x_0) \cdot (x_i - x_0) + \nabla U(x_0)$$

are linearly independent. But

$$\nabla U(x_i) = v_i + o(\eta).$$

The observation made in the beginning of the proof implies that one can select η small enough small enough to ensure that $\nabla U(x_i)$ are linearly independent. □

Lemma 5 *Assume $n > 1$ and the Hessian of utility has full rank. Then $PD \cap PM$ can be embedded in PM as a submanifold of a lower dimension. In particular, this implies that the Lebesgue measure of the PM processes in class PD is zero and that the set of the PM processes is nowhere dense in PD.*

Proof. According to Lemma 3, for each PM process in class PD we can write

$$B\nabla U(x) = c(U(x))\Gamma\nabla U(x). \tag{34}$$

for some positive real valued function $c(\cdot)$. Fix $x_0 \in \Omega$. By Lemma 4, there exist $x_1, .., x_n \in I$ such that $\nabla U(x_1), ..., \nabla U(x_n)$ are linearly independent. Define

$$U(x_0) = U,\ U_{kj} = \frac{\partial U(x_k)}{\partial x_j} \tag{35}$$

$$y_j^i = b_{ij} - c(U)\gamma_{ij}, \tag{36}$$

where b_{ij} and γ_{ij} are matrix elements of matrices B and Γ respectively. Then for fixed but arbitrary i

$$\sum_{j=1}^{n} U_{kj} y_j^i = 0. \tag{37}$$

By linear independence of $\nabla U(x_1), ..., \nabla U(x_n)$, the unique solution of (37) is $y_j^i = 0$ for every i. Therefore, (36) implies

$$B = c(U)\Gamma. \tag{38}$$

Since both B and Γ are constant matrices, $c(U) = c > 0$ is also a constant. This means that the set of payoff monotone processes is given by $B = c\Gamma$, which is a smooth manifold of dimension $1+n(n+1)/2 < n^2+n(n+1)/2$, provided $n > 1$. (A point on the manifold can be uniquely determined by $n(n+1)/2$ elements of matrix Γ and c). □

This completes the proof of part 1 of Theorem 2. The proof of part 2 is almost the same and is omitted. Lemmata 6–9 below constitute a proof of Theorem 3.

Lemma 6 *A Markov adjustment process satisfies IIA if and only if the Jacobi matrix of the vector field $\Gamma^{-1}\mu(x)$, $D(\Gamma^{-1}\mu(x))$, is symmetric for $\forall x \in \Omega$.*

Proof. Let us introduce vector j by:

$$j(x) = -\nabla U(x) f(x) + \frac{1}{2}\Gamma\nabla f(x). \tag{39}$$

Then, in the steady state $j(x)$ should solve the following boundary problem

$$div(j(x)) = 0 \tag{40}$$

$$\langle j(x), n(x)\rangle = 0 \text{ on } x \in \partial\Omega. \tag{41}$$

Let $j^s(x)$ be the solution of (40)-(41). Then the distribution f is then determined by the following system of the first-order partial differential equations:

$$j^s(x) = -\mu(x) f(x) + \frac{1}{2}\Gamma\nabla f(x). \tag{42}$$

The IIA property implies that a change in Ω will result in multiplication of f, and hence of j, by a constant, that is $j_{new} = Cj_{old}$. This relation should hold at each point, which belongs to the intersection of the new and the old choice sets. Hence j_{new} should solve the same boundary problem, but on a different domain. The only vector j that would solve (40)-(41) for any domain is $j = 0$. Hence IIA, together with the definition of j, implies that the steady state density $f(x)$ solves the system

$$\mu(x)f(x) - \frac{1}{2}\Gamma\nabla f(x) = 0. \tag{43}$$

or

$$\frac{1}{2}\nabla \ln f(x) = \Gamma^{-1}\mu(x). \tag{44}$$

The Jacobi matrix of the left hand side of (44) is the Hessian matrix of $\ln f(x)$. Since, according to Lemma 1, $f(x)$ is positive and twice continuously differentiable this matrix is symmetric, so the Jacobi matrix of the right hand side should also be symmetric.

To prove the reverse, assume that the Jacobi matrix of $\Gamma^{-1}\mu(x)$ is symmetric and define $f(x)$ to be the solution of (44). According to the Frobenuous theorem, the solution exists and is unique up to a multiplicative constant. It is easy to see that $f(\cdot)$ such defined solves (3)-(4). The constant is chosen from the normalization condition. □

Lemma 7 completes the proof of part 1 of Theorem 3.

Lemma 7 *Assume the Hessian of the utility function is non-degenerate. Then* $PM \subset IIA$.

Proof. According to Lemma 3 payoff monotonicity implies that

$$\mu(x) = c(U(x))\Gamma\nabla U(x) \tag{45}$$

for some differentiable function $c : R \to R_+$. But then the matrix element $D(\Gamma^{-1}\mu)_{ij}$ is given by

$$\frac{\partial(\Gamma^{-1}\mu)_i}{\partial x_j} = c'(U(x))\frac{\partial U}{\partial x_i}\frac{\partial U}{\partial x_j} + c(U(x))\frac{\partial^2 U}{\partial x_i \partial x_j}. \tag{46}$$

Hence, the matrix is symmetric and the process satisfies IIA. □

Lemma 8 completes the proof of part 2 of Theorem 3.

Lemma 8 *Assume that* $n > 1$, *the utility function is additively separable (i. e. has form (11)) and its Hessian has a full rank. Then* $PDS \cap PM$ *can be embedded in* $PDS \cap IIA$ *as a submanifold of a lower dimension.*

Proof. Following the same logic as in the proof of Lemma 5, it is easy to see that the set $PM \cap PDS$ has dimension $1 + n(n+1)/2$. On the other hand, Lemma 6 implies

$$a_{ji}u_i''(x_i) = a_{ij}u_j''(x_j) \tag{47}$$

for any (i, j) where a_{ik} is a matrix element of $A = \Gamma^{-1}B$. Equation (47) will be satisfied if $a_{ij} = 0$ for all $i \neq j$. It in turn will hold if

$$b_{ij} = c_j \Gamma_{ij}, \tag{48}$$

where $c = (c_1, ..., c_n)$ is an arbitrary constant vector. Hence, the dimension of $IIA \cap PDS$ is at least $n(n+1)/2 + n$. Lemma 7 implies that $PM \cap PDS \subset IIA \cap PDS$. Since $n(n+1)/2 + n > 1 + n(n+1)/2$ for $n > 1$, $PM \cap PDS$ can be embedded into $IIA \cap PDS$ as a submanifold of a lower dimension. □

Lemma 9 completes the proof of part 3 of Theorem 3.

Lemma 9 *Assume that* $n > 1$ *and for any non-degenerate constant matrix* C *there exist* i, k *such that*

$$\frac{\partial^2 U}{\partial x'_i \partial x'_k} \neq 0, \tag{49}$$

where

$$x' = Cx. \tag{50}$$

Then PDS $\cap$ *IIA can be embedded into PDS as a submanifold of lower dimension. In particular, this implies that the Lebesgue measure of the IIA processes in class PD is zero and that the set of the IIA processes is nowhere dense in PD.*

Proof. Since both matrices Γ^{-1} and B are positive definite, there exists a non-degenerate constant matrix C such that both $C^T\Gamma^{-1}C$ and C^TBC are diagonal, with all diagonal entries strictly positive (Gantmakher, 1989). Let us denote the i^{th} diagonal element of Γ^{-1} as $1/\sigma_i^2$ and the i^{th} diagonal element of B as b_i. Let $x' = Cx$. Then, according to Lemma 6, the process satisfies IIA if and only if

$$(\frac{b_i}{\sigma_i^2} - \frac{b_k}{\sigma_k^2})\frac{\partial^2 U}{\partial x'_i \partial x'_k} = 0. \tag{51}$$

Now (49) and (51) imply that there exists a pair of indices i, k such that

$$\frac{b_i}{\sigma_i^2} = \frac{b_k}{\sigma_k^2}. \tag{52}$$

This means that the the set of processes for which IIA holds is a smooth manifold with dimension at least by one smaller then $n(n+1)$ and therefore, the set of such processes has Lebesgue measure zero and is nowhere dense in *PDS*. □

To conclude, I have shown that for a sufficiently broad class of LI processes a generic process does not satisfy IIA and a generic process that is neither IIA nor PM. Note also that a PM process is always IIA, but an IIA process need not be LI.

7 Discussion and conclusions

The main contribution of this paper is to argue that the axioms of the static approach are not supported by a generic dynamic adjustment procedure. Therefore, when studying boundedly rational behavior, it would be desirable to start with explicit formulation of the learning process.

As a simple example when using a static concept may be misleading, Basov (2003a) a two player game and found a quantal response equilibrium that arises from a dynamic adjustment process. He showed that it is generically not monotone in payoffs. It shows that the basic modelling assumptions of static models are not robust to the dynamic modelling and calls for an explicit dynamic modelling, especially when the choice set is multidimensional.

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