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# Representation and calculation of economic uncertainties: Intervals, fuzzy numbers, and probabilities☆

Hans Schjaer-Jacobsen

*Department of Production Engineering, Copenhagen Engineering College, 15 Lautrupvang, DK-2750 Ballerup, Denmark*

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**Abstract**

Management and decision making when certain information is available may be a matter of rationally choosing the optimal alternative by calculation of the utility function. When only uncertain information is available (which is most often the case) decision-making calls for more complex methods of representation and calculation and the basis for choosing the optimal alternative may become obscured by uncertainties of the utility function. In practice, several sources of uncertainties of the required information impede optimal decision making in the classical sense. In order to be able to better handle the economic uncertainties involved, different procedures have been suggested. This paper discusses the representation of economic uncertainties by *intervals*, *fuzzy numbers* and *probabilities*, including double, triple and quadruple estimates and the problems of applying the four basic arithmetical operations to uncertain economic numbers are discussed. When solving economic models for decision-making purposes calculation of uncertain functions will have to be carried out in addition to the basic arithmetical operations. This is a challenging numerical problem since improper methods of calculation may introduce additional uncertainties not present in the original economic problem. The paper will finally discuss the applicability and limitations of a few computational procedures based on available computer programs used for practical economic calculations with uncertain values. 

*Keywords:* Economic uncertainty; Interval; Fuzzy number; Probability

## 1. Introduction

> "As the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics". Zadeh [1].

Basically, this paper deals with a challenging problem facing managers in business and industry: Can it be made possible to establish by means of numerical calculations, a fairly useful and realistic picture of the economic consequences of strategic decisions despite the fact that little is known about the future?

Traditionally, the problem of calculating economic consequences of technological development in cases where information about the future is

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*E-mail address:* hsj@cph.ih.dk (H. Schjaer-Jacobsen).

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sufficient and *certain* has been extensively treated in the literature: For manufacturing technology a survey may be found in [2]. Following the classical paper by Knight [3], the concept of *certainty* presumes that a cognitively competent decision maker has certain knowledge and perfect foresight about all the relevant future states of the world and that the strategies (actions) of the decision maker does not influence these states. Under these circumstances a decision maker is able to precisely calculate the economic consequences of any strategy, he may pursue and thus choose the strategy that in an optimal sense fulfil his aspirations.

Many arguments may be found, however, that the preassumptions on the amount and quality of relevant information being available and the limited cognitive ability of decision makers make traditional decision theory and investment calculations obsolete or, at least, far from sufficient. Alternatively, efforts have been devoted to developing methods that were able to cope with *uncertainties* in strategic decision situations, in particular, involving calculations of the economic uncertainties. One of the early approaches used probabilities to represent *risk* [4]. This concept presumes that the probability of future events is objectively known (or is at least knowable) leaving the competent decision maker with the problem of calculating the expected value of the utility function taking into account the probabilities of events.

As far as the sources of *risk* and *uncertainty* is concerned, Kyläheiko [5] made an extensive study focussing on economic theory and methodology as well as strategic management of technology. Whereas many interpretations of *uncertainty* are found in economics, Kyläheiko [5, p. 37] stresses the need for a further subdivision.

However, in this paper we are mainly interested in the representation and calculation of economic *uncertainties*, i.e. an arithmetic of *uncertain* numbers applied to economic problems. The approach is to choose the representation of *uncertainty* according to the type and amount of information that can be made available in the process. Once the representation has been chosen the problem is to carry out subsequent calculations in a way that produces true results so that additional *uncertainty* is not introduced.

In this paper the next Section 2 will deal with the *interval* representation of uncertainty. This *fuzzy number* representation is found in Section 3 and finally, Section 4 deals with the *probabilistic* representation. This sequence of topics also reflects the increasing demand for information required in order to implement the representations in practical cases. Section 5 contains examples of economic applications and in Section 6 some conclusions are drawn.

## 2. The interval representation

Recently, it has been suggested to use *intervals* in order to represent *uncertainties* in connection with worst- and best-case (WBC) evaluation of economic consequences of technological development [6,7]. The interval approach was originally developed in 1962 by Moore [8,9] in order to be able to keep track of the lower and upper bounds to the exact result when carrying out numerical calculations on digital computers with a finite number of significant digits. Following Moore [9], we define an *interval number* as an ordered pair $[a, b]$ of real numbers with $a \leqslant b$. It may also be defined as an ordinary set of real numbers $x$ such that $a \leqslant x \leqslant b$, or

$$[a, b] = \{x \mid a \leqslant x \leqslant b\}. \tag{1}$$

In our application of intervals to represent economic uncertainties we may represent any uncertain economic parameter by its lower and upper limits in order to be able to keep track of the economic consequences of these uncertainties. For example, the demanded quantity of a certain product in future periods of time may be represented by uncertainty intervals of increasing width depending on the time distance of the future.

If the algebraic operations addition, subtraction, multiplication, and division is denoted by the symbol $\#$, we can define operations on two intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ based on the set-theoretic formulation:

$$I_1 \# I_2 = \{x \# y \mid a_1 \leqslant x \leqslant b_1, a_2 \leqslant y \leqslant b_2\}. \tag{2}$$

Instead of the set-theoretic definition in (2) we may give an alternative definition in terms of endpoints of the resulting intervals by

$$I_1 + I_2 = [a_1 + a_2, b_1 + b_2], \tag{3a}$$

$$I_1 - I_2 = [a_1 - b_2, b_1 - a_2], \tag{3b}$$

$$I_1 * I_2 = [\min(a_1a_2, a_1b_2, b_1a_2, b_1b_2), \max(a_1a_2, a_1b_2, b_1a_2, b_1b_2)], \tag{3c}$$

$$I_1/I_2 = [a_1, b_1]*[1/\mathrm{b}_2, 1/\mathrm{a}_2], \quad \text{if } 0 \notin [a_2, b_2]. \tag{3d}$$

It should be noted, however, that in some cases the straightforward application of interval arithmetic leads to too pessimistic (i.e. too wide) intervals. A simple example of this is the calculation of the expression $I*(1-I)$ where $I$ is an interval, $I = [0, 1]$. Application of the above-mentioned formulas gives the result $[0, 1]$ which obviously is a too wide interval. According to the fundamental definition of arithmetic operations on intervals based upon set-theory (2) [9], the narrowest possible resulting interval should be $[0, \frac{1}{4}]$. In this paper the terms "true" or "correct" is used to indicate the narrowest possible bounds that can be calculated for an uncertain number. By using global optimization methods, see e.g. Hansen [10], correct results may be obtained to an accuracy specified by the user. This feature has been implemented in a recent spread sheet program Interval Solver by Hyvönen and De Pascale [11,12].

## 3. The fuzzy representation

Since the introduction by Zadeh [13], *fuzzy sets* and *fuzzy numbers* have found a wide range of applications within the areas of engineering, management, and finance. A *fuzzy set* is a class of objects with a continuum of grades of membership defined by a *membership function* ranging from 0 to 1. The *fuzzy set* concept provides a convenient way of keeping precisely track with imprecise, vague, and uncertain informative statements such as "the class of all large investments", "costs will be considerably reduced in the coming period", and "the turn over will be a little larger next year".

Following Zadeh [13] a *fuzzy set* $A$ in $X$ where $X$ is a space of points (objects) with a generic element of $X$ denoted by $x$, i.e. $X = \{x\}$, is characterized by a *membership function* $f_A(x)$ which associates with each point in $X$ a real number in the interval [0, 1]. The value of the *membership function* $f_A(x)$ at $x$ represents the "grade of membership" of $x$ in $A$. Thus the closer the value of $f_A(x)$ to unity, the higher the grade of membership of $x$ in $A$. Note, that when $A$ is an ordinary set, i.e. *non-fuzzy*, the membership function can take only two values 0 and 1.

In other words, a *fuzzy set* is a set of ordered pairs $(x, f_A(x))$,

$$A = \{(x, f_A(x)) \mid x \in X\}. \tag{4}$$

It may also be useful to define the ordinary (*non-fuzzy*) set $A_\alpha$ as the $\alpha$-cut of $A$:

$$A_\alpha = \{x \in X \mid f_A(x) \geqslant \alpha, 0 \leqslant \alpha \leqslant 1\}. \tag{5}$$

In this paper we are mainly interested in the concept of *fuzzy numbers* as a means of representing *uncertain* or *fuzzy* information. A *fuzzy number* is more precisely described [14,15], as any *fuzzy subset* of the real line $R$ whose membership function is

$$\text{A continuous mapping from } R \text{ to the interval } [0, 1], \tag{6a}$$

$$\text{Constant on } ]-\infty, a]: \quad f(x) = 0 \ \forall x \in ]-\infty, a], \tag{6b}$$

$$\text{Strictly increasing on } [a, c], \tag{6c}$$

$$\text{Constant on } [c, d]: \quad f(x) = 1 \ \forall x \in [c, d], \tag{6d}$$

$$\text{Strictly decreasing on } [d, b], \tag{6e}$$

$$\text{Constant on } [b, \infty[: \quad f(x) = 0 \ \forall x \in [b, \infty[, \tag{6f}$$

where

$$a, c, d, \text{ and } b \text{ are real numbers, such that } a \leqslant c \leqslant d \leqslant b. \tag{6g}$$

Note, that if $a = c = d = b$ we have an ordinary (crisp) *non-fuzzy number*.

In the general case the full algebraic operations on *fuzzy numbers* require complex computations, e.g. using the extension principle by Dubois and Prade [14,15]. However, the general definition of a *fuzzy number* (6) may be simplified by assuming the membership function to be linear on the intervals $[a, c]$ and $[d, b]$. Thus, in addition to the simplest *fuzzy number*, namely the *interval* (2), we generate two more *fuzzy numbers*:

The first one is the *triangular fuzzy number* [16], that can be defined in the following way using real numbers $a \leqslant c \leqslant b$:

$$f(x) = (x - a)/(c - a), \quad a \leqslant x \leqslant c, \tag{7a}$$

$$= (b - x)/(b - c), \quad c \leqslant x \leqslant b, \tag{7b}$$

$$= 0, \quad \text{otherwise.} \tag{7c}$$

The second one is the *trapezoidal fuzzy number* [17], that may be given the following definition, $a \leqslant c \leqslant d \leqslant b$:

$$f(x) = (x - a)/(c - a), \quad a \leqslant x \leqslant c, \tag{8a}$$

$$= 1, \quad c \leqslant x \leqslant d, \tag{8b}$$

$$= (b - x)/(b - d), \quad d \leqslant x \leqslant b, \tag{8c}$$

$$= 0, \quad \text{otherwise.} \tag{8d}$$

Note that if $c = d$ in (8) we have a *triangular fuzzy number*.

"Full" algebraic operations on *triangular fuzzy numbers* may take the following form introducing the left $L(\alpha)$ and right $R(\alpha)$ representation of a *fuzzy number* $F$, refer to the $\alpha$-cut (5):

$$F = [L(\alpha), R(\alpha)], \tag{9a}$$

where

$$L(\alpha) = a + (c - a)\alpha \text{ and } \quad R(\alpha) = b + (c - b)\alpha,$$

$$\alpha \in [0,1]. \tag{9b}$$

As an example, consider the addition of two *triangular fuzzy numbers* $F_1$ and $F_2$ [16], where

$$F_1 = [1 + \alpha, 4 - 2\alpha] \quad \text{and} \quad F_2 = [2 + 3\alpha, 7 - 2\alpha]. \tag{10}$$

For the sum we get by using (3a): $F_1 + F_2 = [3 + 4\alpha, 11 - 4\alpha], \alpha \in [0, 1]$. As another example calculate the product of $F_1$ and $F_2$ by using (3c):

$$F_1 * F_2 = [3\alpha^2 + 5\alpha + 2, 4\alpha^2 - 22\alpha + 28], \quad \alpha \in [0,1]. \tag{11}$$

Notice, that whereas the *membership functions* of $F_1$ and $F_2$ and the sum as well are piecewise linear the *membership function* of the product (11) is piecewise quadratic. In the general case *membership functions* of arbitrary complexity may result from algebraic operations and make the practical calculations impossible. One way of overcoming this difficulty is to limit the calculations to a finite number of values of $\alpha$. As an example of this consider the *triple estimate* defined by $\alpha$-cuts corresponding to the values $\alpha = 0$ and 1, refer to (5).

Thus we have from (11) the product $F_1 * F_2 = [2, 10, 28]$, written on the *triple estimate* form $[a, c, b]$ because

$$a = L(0) = 3\alpha^2 + 5\alpha + 2\,|_{\alpha=0} = 2, \tag{12a}$$

$$c = L(1) = R(1) = 3\alpha^2 + 5\alpha + 2\,|_{\alpha=1}$$

$$= 4\alpha^2 - 22\alpha + 28\,|_{\alpha=1} = 10 \tag{12b}$$

and

$$b = R(0) = 4\alpha^2 - 22\alpha + 28\,|_{\alpha=0} = 28. \tag{12c}$$

Based on the above and also as a generalization of Kaufmann and Gupta [18] we may now define arithmetic operations on *triple estimate triangular fuzzy numbers* $F_1 = [a_1, c_1, b_1]$ and $F_2 = [a_2, c_2, b_2]$ in the following way:

$$F_1 + F_2 = [a_1 + a_2, c_1 + c_2, b_1 + b_2], \tag{13a}$$

$$F_1 - F_2 = [a_1 - b_2, c_1 - c_2, b_1 - a_2], \tag{13b}$$

$$F_1 * F_2 = [\min(a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2), c_1 c_2, \max(a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2)], \tag{13c}$$

$$F_1 / F_2 = [a_1, c_1, b_1] * [1/b_2, 1/c_2, 1/a_2] \quad \text{if } 0 \notin [a_2, c_2, b_2]. \tag{13d}$$

Note the similarities between (3) and (13). This implies that arithmetic operations on *triple estimate triangular fuzzy numbers* can be performed as a combination of conventional arithmetic operations yielding the central values (corresponding to $\alpha = 1$) and *interval* arithmetic operations yielding the upper and lower bounds (corresponding to $\alpha = 0$).

In the general case when calculating uncertain functions with *triple estimate* arguments care must be taken in order to produce true lower and upper limits. As an example consider the expression $F*(1-F)$, where $F$ is a *triple estimate* $[0, \frac{1}{2}, 1]$. The true value of this expression turns out to be $[0, \frac{1}{4}, \frac{1}{4}]$, which is not obtained by straight forward application of (13) because the variable $F$ appears twice in the expression. The remedy is to use a recent spread sheet program Interval Solver by Hyvönen and De Pascale [11,12] to calculate the true lower and upper bounds.

## 4. The representation by probabilities

Probability theory and statistics today represent a well-tested mathematical theory with clear axioms and has reached an advanced stage of development. Since in our application the concept of repetitive experiments has little meaning we shall concentrate on the "subjectivists" interpretation of probabilities as a measure of the feeling of uncertainty. Accordingly, the numerical value of a probability is interpreted as being proportional to the sum of money a rational individual would be willing to pay should a proposition he asserts prove false. The measure of uncertainty so defined can be shown to obey the axioms of probability theory. Although criticism has been raised towards probability theory as being a too normative framework to take all the aspects of uncertain judgements into account [19], new developments and applications indicate that there is more to be said.

Let two uncertain economic parameters be represented by the random variables $X$ and $Y$ with marginal probability density functions (marginal pdf's) $f_X(x)$ and $f_Y(y)$ and marginal cumulative distribution functions (marginal cdf's) $F_X(x)$ and $F_Y(y)$. (As for the basics of probability theory, refer to e.g. Leon-Garcia [20]). The joint cumulative distribution function (joint cdf) $F_{X,Y}(x,y)$ is then defined as the probability of the product-form event $\{X \leqslant x\} \cap \{Y \leqslant y\}$:

$$F_{X,Y}(x,y) = P(X \leqslant x, Y \leqslant y). \tag{14}$$

The joint cdf is also defined in terms of the joint pdf $f_{X,Y}(x,y)$ by integrating over the semi-infinite rectangle defined by $-\infty$ and $(x,y)$.

As an example of a basic algebraic operation, consider addition where the random variable $Z$ denotes the sum of $X$ and $Y$, $Z = X + Y$. The cdf of $Z$ is then found by integrating the joint pdf of $X$ and $Y$ over the region of the plane corresponding to the event $\{Z \leqslant z = x + y\}$ giving rise to a *superposition* integral. If $X$ and $Y$ are independent random variables the pdf of $Z$, $f_Z(z)$ is given by the *convolution* integral of the pdf's of $X$ and $Y$. In case of subtraction, multiplication, and division we may proceed in a similar manner.

In order to handle the calculations numerically, the discussion now falls into two parts:

1. The variables $X$ and $Y$ are *independent*. In this case, the calculations are carried out by the convolution integral and the result is a random variable $Z$ represented by its pdf or cdf as an expression of the uncertainty connected with the economic variable $Z$. Furthermore, the conventional expected value and standard deviation may be used to characterize the uncertainty of $Z$.
2. The variables $X$ and $Y$ are potentially *dependent*, but nothing is known about their dependency, i.e. the joint pdf and cdf are unknown. In this case not only the statistical uncertainty represented by the marginal pdf's of the original variables $X$ and $Y$ will contribute to the uncertainty of $Z$ but additional uncertainty will be created because of the dependency having to be ignored in the numerical calculations. Recent work by Williamson and Downs [21] and Berleant and Goodman-Strauss [22] has demonstrated that the uncertainty introduced this way may be bound within limits by using intervals.

Without going further into details it should be mentioned here that whereas the theory of probabilities allows to take into account interdependencies between variables (if sufficient information is available) the fuzzy set approach does not. The concept of possibility emerged from the notion of fuzzy sets (representing imprecision) and is able to quantify uncertainty only through the fuzzy set membership functions.

Obviously, the calculations involved in practical applications of the probability representation may become extremely complex, particularly in the case of dependent and potentially dependent variables. In addition to *interval* and *fuzzy number* calculations a recent program package Risc Calc by Ferson et al. [23] also implements calculations by *probabilities*, including *probability bounds*.

## 5. Examples of application

*The first example* considered is the problem of calculating the present value PVA of an ordinary annuity of \$1 over $n$ periods at the interest rate $i$,

$$\mathrm{PVA} = (1 - (1 + i)^{-n})/i. \tag{15}$$

In the case of the interest rate being subject to uncertainty, the present value will also be uncertain. PVA has been calculated for $n = 5$ and different uncertainty representations of the interest rate. From Table 1 it is seen (the first three rows) that using *interval* and *fuzzy number* representation of the interest rate produces different results with Interval Solver and Risk Calc, the latter giving too wide ranges. In the last row the normal distribution with mean value 5% and standard deviation 0.8% has been chosen because it corresponds to the interest rate $i$ being bound approximately inside the interval [3,7]. In the last row the resulting PVA calculated by Risc Calc may be interpreted this way: The probability of PVA being less then 1.910 is practically equal to zero and being less than 9.833 practically equal to unity whereas the numbers in square brackets represent bounds on the mean of the distribution. The reason for Risc Calc producing too pessimistic uncertainties is that the uncertain interest rate $i$ is appearing more than once (actually twice) in the expression for PVA.

The results in the right column of Table 1 have been obtained by straight forward application of Risc Calc. However, it is possible to obtain narrower bounds by doing some additional programming and ongoing research is aimed at making calculations with repeated parameters more easily.

*The other example* is an investment calculation of a new product and production development project using *triple estimates* or *triangular fuzzy numbers* with $\alpha$-cuts corresponding to $\alpha = 0$ and 1, to represent the uncertain economic values. The results of the calculations are shown in Table 2. It is seen that at the actual discount rate the net present value of the investment is most possibly equal to \$2.094 mil. However, the lower bound (worst case) is $-6.239$ mil. and the upper bound (best case) is 9.656 mil. Under the actual assumptions and the model chosen for calculation of working capital these bounds are true bounds.

The investment calculation and its uncertainties is easily understood by people without any specific knowledge of calculation with uncertainties because the center value of all *triple estimates* represents the conventional investment calculation with scalar (crisp) numbers whereas the extreme values forms the true upper and lower bounds taking the judgements of uncertainty into account. Consequently, this way of representing uncertainties forms an excellent vehicle of communication between financial and other experts in the company and the management: Could the most possible present value of the investment be improved? May some of the uncertainties be reduced by seeking further information or changing some aspects of the project? Is the risk of economic project failure

Table 1
Present value of annuity of \$1 with uncertain rate of interest

| Interest rate $i$ (%) | PVA by Interval Solver | PVA by Risk Calc |
|---|---|---|
| [3, 7] | [4.100, 4.580] | [1.963, 9.567] |
| [3, 5, 7] | [4.100, 4.329, 4.580] | [1.963, 4.329, 9.567] |
| [3, 4, 6, 7] | [4.100, 4.212, 4.452, 4.580] | [1.963, 2.968, 6.319, 9.567] |
| Normal (5, 0.8) | — | (1.910 [4.284, 4.582] 9.833) |

Table 2
Investment example using Interval Solver with *triple estimates triangular fuzzy numbers*. Shaded numbers are calculated, non-shaded are input variables

| ($1000) | YEAR 0 | YEAR 1 | YEAR 2 | YEAR 3 | YEAR 4 |
|---|---|---|---|---|---|
| Unit price | | [96, 100, 101] | [96, 100, 101] | [85, 90, 95] | [85, 90, 95] |
| Direct unit cost | | [58, 60, 62] | [53, 55, 58] | [46, 50, 54] | [46, 50, 54] |
| Unit margin | | [34, 40, 43] | [38, 45, 48] | [31, 40, 49] | [31, 40, 49] |
| Total sales (units) | | [45, 50, 52] | [130, 140, 145] | [190, 200, 210] | [140, 150, 160] |
| Total sales | | [4.320, 5.000, 5.252] | [12.480, 14.000, 14.645] | [16.150, 18.000, 19.950] | [11.900, 13.500, 15.200] |
| Direct cost | | [2.610, 3.000, 3.224] | [6.890, 7.700, 8.410] | [8.740, 10.000, 11.340] | [6.440, 7.500, 8.640] |
| Contribution margin | | [1.530, 2.000, 2.236] | [4.940, 6.300, 6.960] | [5.890, 8.000, 10.290] | [4.340, 6.000, 7.840] |
| Contribution margin (%) | | [34, 40, 45] | [38, 45, 50] | [33, 44, 58] | [33, 44, 58] |
| Marketing cost | [800, 1.000, 1.100] | [800, 1.000, 1.100] | [700, 800, 900] | [500, 600, 700] | [400, 500, 600] |
| RD&E cost | [2.900, 3.000, 3.300] | [1.400, 1.500, 1.800] | [200, 300, 500] | [50, 100, 200] | [50, 100, 200] |
| Operating income | [−4.400, −4.000, −3.700] | [−1.370, −500, 36] | [3.540, 5.200, 6.060] | [4.990, 7.300, 9.740] | [3.540, 5.400, 7.390] |
| Working capital | | [−806, −750, −652] | [−1.450, −1.175, −916] | [−1.113, −575, −82] | [1.651, 2.500, 3.369] |
| Investment | [−5.500, −5.000, −4.500] | [−2.500, −2.000, −1.500] | | | [0, 400, 500] |
| Net cash flow | [−9.900, −9.000, −8.200] | [−4.676, −3.250, −2.116] | [2.090, 4.025, 5.144] | [3.877, 6.725, 9.658] | [5.191, 8.300, 11.259] |
| Discount rate (% p.a.) | | [9, 10, 11] | [9, 10, 11] | [9, 10, 11] | [9, 10, 11] |
| Discounted cashflow | [−9.900, −9.000, −8.200] | [−4.290, −2.955, −1.907] | [1.696, 3.326, 4.329] | [2.835, 5.053, 7.457] | [3.420, 5.669, 7.976] |
| Net present value | [−6.239, 2.094, 9.656] | | | | |

compatible with the risk profile of the decision makers? What can be done during the project implementation to improve the situation?

## 6. Discussion and conclusion

When representing *risks* and *uncertainties* it can be made by "full" representation. In the case of *probabilistic* representation this implies the probability distribution of the actual economic parameters be known as (continous or discrete) functions of all possible values of the arguments and that this representation is carried all the way through the computation of a utility function. Likewize, in the case of a *fuzzy* representation all membership functions of the uncertain parameters will have to be specified and a utility function to be calculated for all values of $\alpha$, refer to (9). Obviously, this requires a vast amount of data and/or number of assumptions as far as the probability distributions or membership functions are concerned. Further, in a practical application, it requires the availability of an operational computer program that is able to handle algebraic operations on general probability distributions or general membership functions. Risc Calc by Ferson et al. [23] does this and also introduces the concept of *probability bounds* as an interesting way of making use of sparse and uncertain information.

Alternatively, as suggested in the paper, simplified representations may be chosen:

- A double estimate, the *interval* representation $[a, b]$.
- A *triple estimate* in the form $[a, c, b]$, being a simplification of the *triangular fuzzy number*.
- A *quadruple estimate* in the form $[a, c, d, b]$, being a simplification of the *trapezoidal fuzzy number*.
- Higher-order *fuzzy number estimates*, $[a, c, \ldots, d, b]$.

These representations are listed in a sequence of increasing requirements as far as the necessary number and quality of data and computational effort required is concerned. The simplest representation is the *interval* where no assumptions are made as to the "shape" of the *uncertainty* between lower and upper bounds. Faced with the question of which representation is the better one among those proposed the answer is: The best representation is the one that is able to handle all relevant information available. If only information on lower and upper bounds is available, the best representation is the *interval*. If additional information is available on most possible values, the best representation is the *triple estimate*. Do not use a representation that requires unavailable information.

It is suggested in this paper and supported by practical examples that the usage of *interval* and *triple estimate* calculations combines the advantage of easy communication with that of true bounds on the resulting uncertainties. Consequently, this approach is believed to be well suited for application to economic problems.

Whereas the focus in this paper has been on the representation and calculation aspects, further research is currently being conducted in order to be able to rank uncertain *interval* and *triple estimate* utility functions in relation to decision makers with known risk preferences. Thus, the classical theories of rational decision making may be developed to encompass uncertainties of various forms [24].

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