Problem 8

A function f is convex on a given set S iff for $\lambda \in [0, 1]$ and for all $x, y \in S$, the following holds:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Moreover, a univariate function $f(x)$ is convex on a set S iff its second derivative $f''(x)$ is non-negative everywhere in the set. Prove the following assertions:

1. $f(x) = x^3$ is convex for $x \geq 0$

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2. $f(x_1, x_2) = max(x_1, x_2)$ is convex on $\mathbb{R}$
3. If univeriate functions f and g are convex on S, then $f + g$ is convex on S
4. If univariate functions f and g are convex and non-negative on S, and have their minimum within S at the same point, then fg is convex on S

Solution

1. $f(x)$ is a univariate function. Let us take the second derivative:

$$\begin{aligned} f(x) &= x^3 \\ f'(x) &= 2x^2 \\ f''(x) &= 6x \end{aligned}$$

Given that $6x \geq 0$ for $x \geq 0$, then we can conclude that $f(x) = x^3$ is convex for $x \geq 0$.
2. Consider $\lambda f(x) + (1 - \lambda)f(y)$:

$$\begin{aligned} \lambda f(x) + (1 - \lambda)f(y) &= \lambda \max(x, y) + (1 - \lambda) \max(x, y) \quad \text{[Using the definition of } f(x_1, x_2)] \\ &= \max(\lambda x, \lambda y) + \max((1 - \lambda)x, (1 - \lambda)y) \\ &= \begin{cases} \lambda x & \text{if } \lambda x \geq \lambda y \\ \lambda y & \text{if } \lambda x < \lambda y \end{cases} + \begin{cases} (1 - \lambda)x & \text{if } (1 - \lambda)x \geq (1 - \lambda)y \\ (1 - \lambda)y & \text{if } (1 - \lambda)x < (1 - \lambda)y \end{cases} \end{aligned}$$

Now, we have two cases to consider. If $\lambda x \geq \lambda y$ and $(1 - \lambda)x \geq (1 - \lambda)y$, then:

$$\begin{aligned} \lambda f(x) + (1 - \lambda)f(y) &= \lambda x + (1 - \lambda)x \quad \text{[Both } \lambda x \geq \lambda y \text{ and } (1 - \lambda)x \geq (1 - \lambda)y] \\ &= x \end{aligned}$$

If either $\lambda x < \lambda y$ or $(1 - \lambda)x < (1 - \lambda)y$, then:

$$\begin{aligned} \lambda f(x) + (1 - \lambda)f(y) &= \lambda y + (1 - \lambda)y \quad \text{[Either } \lambda x < \lambda y \text{ or } (1 - \lambda)x < (1 - \lambda)y] \\ &= y \end{aligned}$$

We can conclude for $\lambda \in [0, 1]$ and for all $x, y \in \mathbb{R}$, the inequality holds and the function $f(x_1, x_2) = \max(x_1, x_2)$ is convex on $\mathbb{R}$:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

3. Given that f, g are univariate functions convex on S, then we also know that $f'', g'' \geq 0$ for all $x \in S$. Consider $h(x) = f(x) + g(x)$ for all $x \in S$. It follows that:

$$h''(x) = f''(x) + g''(x)$$

Since both $f'', g'' \geq 0$ by definition, then $h'' \geq 0$ or in words h'' is non-negative $\forall x \in S$. We can then conclude that $f + g$ is convex on set S.
4. Given that functions f, g are convex and non-negative on S, then we know that f, g is also non-negative on S. This follows from the fact that f, g minimums are ≥ 0. This means that f, g at its lowest possible convex point is 0. We can then conclude that f, g is convex.