# Problem 2

Prove that the sum of two independent Poisson random variables is also a Poisson random variable.

**Solution**

Consider two independent random variables s.t. $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$. Let $Z = X + Y$.
Let $\Omega_X = \Omega_Y = \Omega_Z = \{1, 2, ...\}$. The convolution formula for discrete distributions is (for $n \in \Omega_Z$ and $i \leq n$):

$$p_Z(n) = \sum_{i=0}^{n} p_X(i)p_Y(n-i)$$

In addition, the binomial theorem is as follows:

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^{n-k} b^k$$

From the following, we can conclude that $Z \sim Poi(\lambda_1 + \lambda_2)$:

$$\begin{aligned}
p_Z(n) &= \sum_{i=0}^{n} p_X(i)p_Y(n-i) \\
&= \sum_{i=0}^{n} e^{-\lambda_1} \frac{\lambda_1^i}{i!} \cdot e^{-\lambda_2} \frac{\lambda_2^{n-i}}{(n-i)!} \\
&= e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{n} \frac{\lambda_1^i \lambda_2^{n-i}}{i!(n-i)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \lambda_1^i \lambda_2^{n-i} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n
\end{aligned}$$