Problem 3

Let A, B, C be events. Show that if $P(A|B, C) > P(A|B)$ then $P(A|B, C^c) < P(A|B)$. Here C^c denotes the complement of C. Assume that each event we are conditioning on has positive probability.

Solution

By the conditional probability formula:

$$P(A|B, C) = \frac{P(A \cap B \cap C)}{P(B \cap C)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

And by the complement rule:

$$P(C^c) = 1 - P(C)$$

From the following, we can conclude that $P(A|B, C) > P(A|B) \rightarrow P(A|B, C^c) < P(A|B)$:

$$P(A|B, C) > P(A|B)$$
$$\frac{P(A \cap B \cap C)}{P(B \cap C)} > \frac{P(A \cap B)}{P(B)}$$
$$P(A \cap B \cap C) > P(A \cap B)$$

Substituting $P(A \cap B \cap C^c) = P(A \cap B) - P(A \cap B \cap C)$:

$$P(A \cap B \cap C^c) < P(A \cap B)$$
$$\frac{P(A \cap B \cap C^c)}{P(B)} < \frac{P(A \cap B)}{P(B)}$$
$$\frac{P(A \cap B \cap C^c)}{P(B)} < P(A|B)$$
$$P(A|B, C^c) < P(A|B)$$