Part 1 - Gradient of the sigmoid
Let $g(z) = \frac{1}{1+e^{-z}}$. Show that $\frac{\delta g(z)}{\delta z} = g(z)(1 - g(z))$.

Solution
By the quotient rule for differentiation:

$$\begin{aligned} g'(z) &= \frac{d}{dz}\left(\frac{1}{1+e^{-z}}\right) \\ &= \frac{0 \cdot (1+e^{-z}) - 1 \cdot (-e^{-z})}{(1+e^{-z})^2} \\ &= \frac{e^{-z}}{(1+e^{-z})^2} \end{aligned}$$

With further simplification:

$$\begin{aligned} g'(z) &= \frac{e^{-z}}{(1+e^{-z})^2} \\ &= \frac{e^{-z}}{(1+e^{-z})(1+e^{-z})} \\ &= \frac{1}{1+e^{-z}} \cdot \frac{e^{-z}}{1+e^{-z}} \\ &= \frac{1}{1+e^{-z}} \cdot \left(1 - 1 + \frac{e^{-z}}{1+e^{-z}}\right) \\ &= \frac{1}{1+e^{-z}} \cdot \left(1 - \frac{1+e^{-z}}{1+e^{-z}} + \frac{e^{-z}}{1+e^{-z}}\right) \\ &= \frac{1}{1+e^{-z}} \cdot \left(1 - \frac{1}{1+e^{-z}}\right) \\ &= g(z) \cdot (1 - g(z)) \quad \text{since } g(z) = \frac{1}{1+e^{-z}} \end{aligned}$$