Submission for Apoorv Walia and Marcos Lagos

2.2 $Beta(\theta/a,b) \propto \theta^{a-1}(1-\theta)^{b-1}$

$$f(\theta) = L(\theta)\, Beta(\theta \mid a,b)$$

$$= \theta^{\sum_{i=1}^{m} x_i}(1-\theta)^{m-\sum_{i=1}^{m} x_i}\,\theta^{a-1}(1-\theta)^{b-1}$$

$$= \theta^{\sum_{i=1}^{m} x_i + a - 1}(1-\theta)^{m-\sum_{i=1}^{m} x_i + b - 1}$$

$$\log(f(\theta)) = \left(\sum_{i=1}^{m} x_i + a - 1\right)\log(\theta) + \left(m - \sum_{i=1}^{m} x_i + b - 1\right)\log(1-\theta)$$

$$\frac{\partial \log(f(\theta))}{\partial\theta} = \frac{\sum_{i=1}^{m} x_i + a - 1}{\theta} - \frac{m - \sum_{i=1}^{m} x_i + b - 1}{1-\theta}$$

Setting the derivative equal to 0, we get

$$(1-\theta)\left(\sum_{i=1}^{m} x_i + a - 1\right) - \theta\left(m - \sum_{i=1}^{m} x_i + b - 1\right) = 0$$

$$= \sum_{i=1}^{m} x_i + a - 1 - \cancel{\theta\sum_{i=1}^{m}} x_i - \theta a + \theta - \theta m + \cancel{\theta\sum_{i=1}^{m}} x_i - \theta b + \theta = 0$$

$$= \sum_{i=1}^{m} x_i + a - 1 - \theta(m+a+b-2) = 0$$

$$\boxed{\hat{\theta} = \frac{\sum_{i=1}^{m} x_i + a - 1}{m+a+b-2}}$$

under the conditions of uniform prior $\therefore a = 1, b = 1$, we get

$\hat{\theta} = \frac{\sum_{i=1}^{m} x_i}{m}$, which is the same as MLE