Problem 5

Let $A \in \mathbb{R}^{n \times n}$ be symmetric matrix. We say that A is positive semi-definite if $\forall x \in \mathbb{R}^n$, $x^T A x \geq 0$. Show that if A is positive semi-definite, then all eigenvalues of A are non-negative.

Solution

Consider an eigenvector v of A so that $Av = \lambda v$:

$$Av = \lambda v$$
$$v^T Av = \lambda v^T v$$

Since A is positive semi-definite, we know that $x^T Ax \geq 0$. In addition, $v^T v$ is the squared norm of vector v so $v^T v \geq 0$.

$$x^T Ax \geq 0$$

$$\lambda v^T v \geq 0$$

If λ were negative then $\lambda v^T v$ would also be negative since $v^T v$ is non-negative, but we have already established that $v^T v \geq 0$. Therefore, we can conclude that all eigenvalues of A are non-negative.

Problem 6 continued on next page. . .