Linear Programming

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Lecture 4 Simplex Method

Outline:
1.Simplex method
2.Phase one method
3.Big M method

● Basic idea of the simplex method(geometrically)
-Phase I:
Step 1: starting
Find an initial extreme point or declare P is null.

-Phase II:
Step 2: checking optimality
If the current ep is optimal, STOP!

Step 3: pivoting
Move to a better ep.
Return to Step 2.

Q: when we move from one *bfs* to another *bfs*, do we really move from one *ep* to another *ep*?
A: It depends on whether the LP is nondegenerate.

● Relationship between extreme points and basic feasible solutions
If an ep is determined by a bfs with exactly m positive basic variables and $n-m$ zero non-basic variables, then the correspondence is one-to-one. The bfs is nondegenerate.

Only when there exists at least one basic variable becoming 0, then the ep may correspond to more than one bfs. The bfs is degenerate.

An LP is nondegenerate if every bfs is nondegenerate.

● Some properties
Property 1: If a bfs $\boldsymbol{x}$ is nondegenerate, then $\boldsymbol{x}$ is uniquely determined by n hyperplanes. (why not m? See matrix $\boldsymbol{M}$)

Let:

$$\boldsymbol{A} = (\boldsymbol{B} \mid \boldsymbol{N}) \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{N} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}$$

Then $\boldsymbol{M}$ is nonsingular and

$$\boldsymbol{Mx} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{N} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$$

$$\boldsymbol{x} = \boldsymbol{M}^{-1} \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1} & -\boldsymbol{B}^{-1}\boldsymbol{N} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$$

$\boldsymbol{M}$(or $\boldsymbol{M}^{-1}$) is the fundamental matrix of LP.

Property 2: If a bfs $\boldsymbol{x}$ is degenerate, then $\boldsymbol{x}$ is over-determined by more than n hyperplanes.

Other than the n hyperplanes of $\begin{bmatrix} \boldsymbol{B} & \boldsymbol{N} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$, there exists at least one basic variable such that $x_i = 0$, which is another hyperplane.

Property 3: For a degenerate bfs $\boldsymbol{x}$ with $p(< m)$ positive components, we may have up to

$$\binom{n-p}{n-m} = C_{n-p}^{n-m} = \frac{(n-p)!}{(n-m)!(m-p)!}$$

different bfs corresponding to the same ep.

- Simplex method under nondegeneracy(the correpondence between ep and bfs is one-to-one)

Basic idea:
-Moving from one bfs(ep) to another bfs(ep) with a simple pivoting scheme.
-Instead of considering all bfs(ep) at the same time, just consider some neighboring bfs(ep).

- Definition of neighboring bfs

Two basic feasible solutions are adjacent if they have $m-1$ basic variables(not their values) in common.

- Analysis of pivoting

-Under nondegeneracy, every basic feasible solution(extreme point) has exactly $n - m$ adjacent neighbors.
-For a bfs, each adjacent bfs can be reached by increasing one nonbasic variable from zero to positive and decreasing one basic variable from positive to zero.

-Pivoting:
One nonbasic variable enters the basis and one basic variable leaves the basis.

$\boldsymbol{x}^1 = \boldsymbol{x}^0 + \lambda \boldsymbol{d}_q$, $\lambda > 0$: step length, $\boldsymbol{d}_q$: edge direction.

● Analysis of edge directions
Consider the fundamental matrix of LP:

$$\boldsymbol{M}^{-1} = \begin{bmatrix} \boldsymbol{B}^{-1} & -\boldsymbol{B}^{-1}\boldsymbol{N} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}$$

$$\begin{bmatrix} -\boldsymbol{B}^{-1}\boldsymbol{N} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{B}\boldsymbol{A}_{q1} & -\boldsymbol{B}\boldsymbol{A}_{q2} & \cdots & -\boldsymbol{B}\boldsymbol{A}_{q(n-m)} \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 & \cdots & \boldsymbol{e}_{(n-m)} \end{bmatrix}$$

We have the conclusion(skip the proof):

$$\boldsymbol{d}_q = \begin{pmatrix} -\boldsymbol{B}^{-1}\boldsymbol{A}_q \\ \boldsymbol{e}_q \end{pmatrix}$$

Q: Is an edge direction $\boldsymbol{d}_q$ always a feasible direction?
A: Yes when LP is nondegenerate.

● Definition of reduced cost

$$\boldsymbol{x}(\lambda) = \boldsymbol{c}^T\boldsymbol{x} + \lambda \boldsymbol{c}^T\boldsymbol{d}_q$$

If $\boldsymbol{c}^T\boldsymbol{d}_q < 0$, then $\boldsymbol{x}(\lambda)$ is a better bfs.

$$\boldsymbol{c}^T\boldsymbol{d}_q = (\boldsymbol{c}_B^T\ ,\ \boldsymbol{c}_N^T)\begin{pmatrix} -\boldsymbol{B}^{-1}\boldsymbol{A}_q \\ \boldsymbol{e}_q \end{pmatrix} = -\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A}_q + c_q = c_q - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A}_q = r_q$$

r_q is "reduced cost", it looks like the cost coefficient of x_q substract something.

Definition: the quantity of

$$r_q = c_q - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A}_q$$

is called a reduced cost with respect to the variable x_q.

PS: although reduced cost is mainly used for non-basic variables, basic variables can also have reduced cost. Reduced cost of a basic variable is always 0. Want to know why? Please see the proof in the following text.

Proof (reduced cost of basic variable is always 0):
Note that:

$$\boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I}$$

and

$$\boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{B}^{-1}(\boldsymbol{A}_{q1}|\boldsymbol{A}_{q2}|\cdots|\boldsymbol{A}_{qm}) = (\boldsymbol{B}^{-1}\boldsymbol{A}_{q1}|\boldsymbol{B}^{-1}\boldsymbol{A}_{q2}|\cdots|\boldsymbol{B}^{-1}\boldsymbol{A}_{qm}) = (\boldsymbol{e}_1|\boldsymbol{e}_2|\cdots|\boldsymbol{e}_m)$$

Thus we can know, for a basic variable x_q:

$$r_q = c_q - \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{A}_q = c_q - \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{e}_q = c_q - c_q = 0$$

● Optimality condition

Given a bfs $\boldsymbol{x}^0 = \begin{pmatrix} -\boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{pmatrix}$ with basis $\boldsymbol{B}$, if $r_q \geq 0$, $\forall$n.b.v. x_q , then $\boldsymbol{x}^0$ is optimal.
(the converse statement is not always true when LP is degenerate)

● Analysis of step length

We have $\boldsymbol{x}(\lambda) = \boldsymbol{x} + \alpha\boldsymbol{d}_q$, $\alpha > 0$ with $r_q = c_q - \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{A}_q < 0$.
Remember that $\boldsymbol{A}\boldsymbol{d}_q = 0$(can be derived from $\boldsymbol{M}\boldsymbol{M}^{-1} = \boldsymbol{I}$), thus $\boldsymbol{A}\boldsymbol{x}(\alpha) = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Case 1: If $\boldsymbol{d}_q \geq \boldsymbol{0}$, then $\boldsymbol{x}(\alpha) \geq \boldsymbol{0}$, $\forall \alpha \geq 0$. Hence $\boldsymbol{x}(\alpha) \in P$, $\forall \alpha \geq 0$ and

$$\boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x}(\alpha) = \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} + \alpha\boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{d}_q \to -\infty, \text{ as } \alpha \to +\infty.$$

Case 2(minimum ratio test): If $\boldsymbol{d}_q$ has at least one component <0. To keep $\boldsymbol{x}(\alpha) \geq \boldsymbol{0}$, we have to choose

$$\alpha = \min_{i:\text{basic}} \{\frac{x_i}{-d_{qi}} | d_{qi} < 0\}$$