Linear Programming

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Lecture 5 Duality and Sentivity Analysis

Outline:
1.Dual linear program
2.Duality theory
3.Sensitivity analysis
4.Dual simplex method

● Economic interpretation of duality
Is there any special meaning of the dual variables?
What is a dual problem tring to do?
What's the role of the complementary slackness in decision making?

● Dual variables
Consider a nondegenerate linear program:

$$\begin{array}{llll} & \min & \boldsymbol{c}^T\boldsymbol{x} & \text{(minimize total cost)} \\ \text{(P)} & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} & \text{(satisfy demands)} \\ & & \boldsymbol{x} \geq \boldsymbol{0} & \text{(different services)} \end{array}$$

Assume that $\boldsymbol{x}^*$ is a nondegenerate optimal bfs

$$\boldsymbol{x}^* = \begin{pmatrix} \boldsymbol{x}_B^* \\ \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{pmatrix}$$

$$z^* = \boldsymbol{c}^T\boldsymbol{x}^* = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b}$$

Since $\boldsymbol{x}_B^* = \boldsymbol{B}^{-1}\boldsymbol{b} > \boldsymbol{0}$, we have $\boldsymbol{B}^{-1}(\boldsymbol{b} + \Delta\boldsymbol{b}) > \boldsymbol{0}$ when $\Delta\boldsymbol{b}$ is small enough!

Thus $\bar{\boldsymbol{x}}^* = \begin{pmatrix} \bar{\boldsymbol{x}}^* \\ \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}^{-1}(\boldsymbol{b} + \Delta\boldsymbol{b}) \\ \boldsymbol{0} \end{pmatrix}$ is an optima bfs to

$$\begin{array}{lll} & \min & \boldsymbol{c}^T\boldsymbol{x} \\ (\overline{\text{P}}) & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \Delta\boldsymbol{b} \\ & & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

with $\bar{z}^* = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}(\boldsymbol{b} + \Delta\boldsymbol{b})$ (Why? No change in r_q!)

● Dual variable for shadow price

Moreover, we have

$$\begin{aligned}\Delta z &= \bar{z}^* - z^* \\ &= \boldsymbol{c}_B^T \boldsymbol{B}^{-1}(\boldsymbol{b} + \Delta \boldsymbol{b}) - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} \\ &= \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\Delta \boldsymbol{b} \\ &= \boldsymbol{w}^T \boldsymbol{b}\end{aligned}$$

where

$\boldsymbol{w}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$ is the simplex multiplier for (P) at optimum!

Hence w_i is the "marginal price" of the ith demand.

Note: dual variable w_i indicates the minimum unit price that one has to charge for additional demand i. It is also called *shadow price* or *equilibrium price*.

● Dual LP problem

Consider the following production scenario:

n products to be produced: x_j = amount of product j, $j = 1,\cdots,n$

m resources in hand: b_i = amount of resource i, $i = 1,\cdots,m$

market selling price for each product is known: $c_1, c_2, \cdots, c_n$

technology matrix is given by $[a_{ij}]$: each product j consumes a_{ij} units of resources i, $i = 1,\cdots,m, j = 1,\cdots,n$.

● A manufacturer's view

Maximize total sales $\sum_{j=1}^{n} c_j x_j$

Resource limitation

$a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n \le b_i, \quad i = 1,\cdots,m.$

Production requirement $x_j \ge 0, \quad j = 1,2,\cdots,n.$

$$\begin{array}{lll} & \max & \boldsymbol{c}^T\boldsymbol{x} \\ (\text{P}) & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ & & \boldsymbol{x} \ge \boldsymbol{0}\end{array}$$

● A wholesaler's view(supplier)

m resources to purchase from a supplier:

w_i = unit price to purchase resource i, $i = 1,2,\cdots,m$.

Free information market:

Supplier knows your selling price c_j for product x_j and he/she wants to get the most out of you, *i.e.*

$$a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{mj}w_m \geq c_j \quad j = 1, \cdots, n$$

Therefore, we have the dual

$$\begin{array}{llll} & \min & \boldsymbol{b}^T\boldsymbol{w} & \text{(minimize total spending)} \\ \text{(D)} & \text{s.t.} & \boldsymbol{A}^T\boldsymbol{w} \geq \boldsymbol{c} & \text{(price accepted by the supplier)} \\ & & \boldsymbol{w} \geq \boldsymbol{0} & \end{array}$$

Summary:

$$\begin{array}{lll} & \max & \boldsymbol{c}^T\boldsymbol{x} \\ \text{(P)} & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad \begin{array}{lll} & \min & \boldsymbol{b}^T\boldsymbol{w} \\ \text{(D)} & \text{s.t.} & \boldsymbol{A}^T\boldsymbol{w} \geq \boldsymbol{c} \\ & & \boldsymbol{w} \geq \boldsymbol{0} \end{array}$$

Q: how to transform (D) to (P) using the definition of dual LP?

A:

$$\begin{array}{lll} & \min & \boldsymbol{b}^T\boldsymbol{w} \\ \text{(D)} & \text{s.t.} & \boldsymbol{A}^T\boldsymbol{w} \geq \boldsymbol{c} \\ & & \boldsymbol{w} \geq \boldsymbol{0} \end{array} \quad \Longleftrightarrow \quad \begin{array}{lll} & \min & \boldsymbol{b}^T\boldsymbol{w} + \boldsymbol{0}^T\boldsymbol{r} \\ \text{(D)} & \text{s.t.} & \boldsymbol{A}^T\boldsymbol{w} - \boldsymbol{I}\boldsymbol{r} \geq \boldsymbol{c} \\ & & \boldsymbol{w}, \boldsymbol{r} \geq \boldsymbol{0} \end{array}$$

$$\Updownarrow$$

$$\begin{array}{lll} & \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{(P)} & \text{s.t.} & \begin{bmatrix} \boldsymbol{A} \\ -\boldsymbol{I} \end{bmatrix} \boldsymbol{x} \leq \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} \end{array} \quad \Leftarrow \quad \begin{array}{lll} & \min & \begin{bmatrix} \boldsymbol{b}^T & \boldsymbol{0}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{r} \end{bmatrix} \\ \text{(D)} & \text{s.t.} & \begin{bmatrix} \boldsymbol{A}^T & -\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{r} \end{bmatrix} = \boldsymbol{c} \\ & & \boldsymbol{w}, \boldsymbol{r} \geq \boldsymbol{0} \end{array}$$

$$\Updownarrow$$

$$\begin{array}{lll} & \max & \boldsymbol{c}^T\boldsymbol{x} \\ \text{(P)} & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

● Complementary slackness condition

1. w_i^* is the maximum marginal price the manufacturer is willing to pay the supplier.

2. When resource i is not fully utilized, *i.e.*, $a_{i1}x_1^* + a_{i2}x_2^* + \cdots + a_{in}x_n^* < b_i$, the complementary slackness condition implies $w_i^* = 0$. This means the manufacturer is not willing to pay a penny for buying any additional amout!

3. When the supplier asks too much, *i,e.*, $a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{mj}w_m > c_j$, then $x_j = 0$. This means the manufacturer is not going to produce any product j.

● Additional explanation for complementary slackness

Consider a ***relaxed*** knapsack problem(the decision variables not must be integers, thus it is a relaxed problem):

$$\begin{aligned} \max \quad & 3x_1 + 4x_2 + 9x_3 + 2x_4 + 5x_5 \\ \text{s.t.} \quad & 4x_1 + 7x_2 + 10x_3 + 3x_4 + 7x_5 \le 20 \\ & x_j \ge 0, \quad j = 1,2,3,4,5. \end{aligned}$$

Its dual becomes

$$\begin{aligned} \min \quad & 20y \\ \text{s.t.} \quad & 4y \ge 3 \\ & 7y \ge 4 \\ & 10y \ge 9 \\ & 3y \ge 2 \\ & 7y \ge 5 \\ & y \ge 0 \end{aligned}$$

Obviously, $y^* = 0.9$.

Dual slack:

$4y^* - 3 = 0.6 \quad \to \quad x_1^* = 0$

$7y^* - 4 = 2.3 \quad \to \quad x_2^* = 0$

$10y^* - 9 = 0 \quad \to \quad x_3$

$3y^* - 2 = 0.7 \quad \to \quad x_4^* = 0$

$7y^* - 7 = 1.3 \quad \to \quad x_5^* = 0$

Dual variable: $y^* = 0.9 > 0$

Primal slack: $20 - (4x_1{}^* + 7x_2^* + 10x_3^* + 3x_4^* + 7x_5^*) = 0,\ x_3^* = 2,\ z^* = 18.$