Linear Programming

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Lecture 5 Duality and Sentivity Analysis

Outline:
1.Dual linear program
2.Duality theory
3.Sensitivity analysis
4.Dual simplex method

● Complementary slackness
Consider the symmetric pair LP

$$(P)\quad \begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad (D)\quad \begin{array}{ll} \max & \boldsymbol{b}^T\boldsymbol{w} \\ \text{s.t.} & \boldsymbol{A}^T\boldsymbol{w} \leq \boldsymbol{c} \\ & \boldsymbol{w} \geq \boldsymbol{0} \end{array}$$

Let $\boldsymbol{x}$ be primal feasible, $\boldsymbol{w}$ be dual feasible.
Define:
$\boldsymbol{s} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \geq \boldsymbol{0}$ $\qquad$ $\boldsymbol{s} \in R^m$: primal slackness
$\boldsymbol{r} = \boldsymbol{c} - \boldsymbol{A}^T\boldsymbol{w} \geq \boldsymbol{0}$ $\qquad$ $\boldsymbol{r} \in R^n$: dual slackness

Observations:
(i) If $\boldsymbol{r}^T\boldsymbol{x} = 0$ and $\boldsymbol{s}^T\boldsymbol{w} = 0$, then $(\boldsymbol{c}^T - \boldsymbol{w}^T\boldsymbol{A})\boldsymbol{x} = 0$ and $\boldsymbol{w}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) = 0$.

Hence
$$\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{w}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{w}^T\boldsymbol{b} = \boldsymbol{b}^T\boldsymbol{w}$$

Besides, we have already assume that $\boldsymbol{x}$ and $\boldsymbol{w}$ are feasible. According to the Weak Duality Theorem, $\boldsymbol{x}$ is primal optimal and $\boldsymbol{w}$ is dual optimal.

(ii) On the contrary side, for a feasible pair $(\boldsymbol{x}, \boldsymbol{w})$, $\boldsymbol{c}^T\boldsymbol{x} \geq \boldsymbol{w}^T\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{w}^T\boldsymbol{b}$.

If $\boldsymbol{x}$ is primal optimal and $\boldsymbol{w}$ is dual optimal, then according to the Strong Duality Theorem, $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{w}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{w}^T\boldsymbol{b}$.

Hence
$$(\boldsymbol{c}^T - \boldsymbol{w}^T\boldsymbol{A})\boldsymbol{x} = 0 \text{ and } \boldsymbol{w}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) = 0 \Longleftrightarrow \boldsymbol{r}^T\boldsymbol{x} = 0 \text{ and } \boldsymbol{s}^T\boldsymbol{w} = 0.$$

"complementary slackness" $\Longleftrightarrow$ "primal optimal, dual optimal"
(feasible)

● Complementary slackness theorem

Let (P) and (D) be a "symmetric pair", $\boldsymbol{x}$ is primal feasible, $\boldsymbol{w}$ is dual feasible. Then $\boldsymbol{x}, \boldsymbol{w}$ are a optimal solution pair if and only if

$$\begin{cases} r_j = 0 \text{ or } x_j = 0 & \forall j = 1, \cdots, n \\ s_i = 0 \text{ or } w_i = 0 & \forall i = 1, \cdots, m \end{cases}$$

● Complementary slackness for standard form LP

$$\text{(P)} \quad \begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad \text{(D)} \quad \begin{array}{ll} \max & \boldsymbol{b}^T\boldsymbol{w} \\ \text{s.t.} & \boldsymbol{A}^T\boldsymbol{w} \leq \boldsymbol{c} \\ & \boldsymbol{w} \in R^m \end{array}$$

(i) $\boldsymbol{x}$ is primal feasible, $\boldsymbol{w}$ is dual feasible.
(ii) The condition $\boldsymbol{s}^T\boldsymbol{w} = 0$ is always true($\boldsymbol{s} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} = \boldsymbol{0}$).
(iii) The complementary slackness condition reduces to $\boldsymbol{r}^T\boldsymbol{x} = 0$.

● The relationship between duality gap and complementary slackness

$\boldsymbol{x}$ is primal feasible, $\boldsymbol{w}$ is dual feasible.
For standard form,

$$\begin{aligned} \text{Duality Gap} &= \boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{w} \\ &= \boldsymbol{c}^T\boldsymbol{x} - (\boldsymbol{A}\boldsymbol{x})^T\boldsymbol{w} \\ &= \boldsymbol{x}^T\boldsymbol{c} - \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{w} \\ &= \boldsymbol{x}^T(\boldsymbol{c} - \boldsymbol{A}^T\boldsymbol{w}) \\ &= \boldsymbol{x}^T\boldsymbol{r} \\ &= \boldsymbol{r}^T\boldsymbol{x} \end{aligned}$$

For symmetric pair form,

$$\begin{aligned} \text{Duality Gap} &= \boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{w} \\ &= (\boldsymbol{w}^T\boldsymbol{A} + \boldsymbol{r}^T)\boldsymbol{x} - \boldsymbol{w}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{s}) \\ &= \boldsymbol{w}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{r}^T\boldsymbol{x} - \boldsymbol{w}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{w}^T\boldsymbol{s} \\ &= \boldsymbol{r}^T\boldsymbol{x} + \boldsymbol{w}^T\boldsymbol{s} \end{aligned}$$

● Kuhn-Tucker condition(assume that the optimal objective value is a finite number)

Theorem:
$\boldsymbol{x}$ is optimal for the problem

$$\text{(P)} \quad \begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

if and only if there exists $\boldsymbol{w}$ and $\boldsymbol{r}$ such that

(1) $\boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}$ (Primal feasibility)
(2) $\boldsymbol{A^T w} + \boldsymbol{r} = \boldsymbol{c},\ \boldsymbol{r} \geq \boldsymbol{0}$ (Dual feasibility)
(3) $\boldsymbol{r^T x} = 0$ (Complementary slackness)

Implication:
Solving a linear programming problem is equivalent to solving a system of linear inequalities and equalities.

$$\begin{cases} \boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0} \\ \boldsymbol{A^T w} \leq \boldsymbol{c} \\ \boldsymbol{c^T x} = \boldsymbol{b^T w} \end{cases}$$