Linear Programming

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Lecture 3 Geometry of LP

Outline:
1.Terminologies
2.Background Knowledge
3.Graphic Method
4.Fundamental Theorem of LP

● Terminologies

Baseline model:

$$(\text{LP}) \quad \begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

● Feasible domain

$$P = \{\boldsymbol{x} \in R^n \,|\, A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$$

● Feasible solution

$\boldsymbol{x}$ is a feasible solution if $\boldsymbol{x} \in P$.

● Consistency

When $P \neq \phi$, LP is *consistent*.

● Bounded feasible domain

P is bounded if $\exists M > 0$ such that $\|x\| \leq M$, $\forall \boldsymbol{x} \in P$. In this case, we say "LP has bounded feasible domain."

● Bounded LP

LP is bounded if $\exists M \in R$ such that $\boldsymbol{c}^T\boldsymbol{x} \geq M$, $\forall \boldsymbol{x} \in P$.

Bounded feasible domain ⇒ Bounded LP. (√)
(Explanation: the objective function is a linear function which is continuous, thus the objective value must be bounded!)

Bounded LP. ⇒ Bounded feasible domain(×, not necessarily correct!)
Counter-example:

$$\begin{array}{ll} \min & \mathbf{0}^T\boldsymbol{x} \ (\equiv \mathbf{0}) \\ \text{s.t.} & \mathbf{0}\boldsymbol{x} = \mathbf{0} \\ & \boldsymbol{x} \geq \mathbf{0} \end{array}$$

Its feasible domain is R^n_+, which is a cone and not bounded!

- Optimal solution

$\boldsymbol{x}^*$ is an optimal solution if $\boldsymbol{x}^* \in P$ and $\boldsymbol{c}^T\boldsymbol{x}^* = \min_{x \in P} \boldsymbol{c}^T\boldsymbol{x}$.

- Optimal solution set

$$P^* = \{\boldsymbol{x}^* \,|\, \boldsymbol{x}^* \text{ is optimal}\}$$

- We say $\boldsymbol{x}^*$ solves LP, if $\boldsymbol{x}^* \in P$.

- Background knowledge

Each equality constraint in the standard form LP is a "hyperplane" in the solution space.

- Definition:

For a vector $\boldsymbol{a} \in R^n$, $\boldsymbol{a} \neq \mathbf{0}$, and a scalar $\beta \in R$, define

$$H = \{\boldsymbol{x} \in R^n \,|\, \boldsymbol{a}^T\boldsymbol{x} = \beta\}.$$

$H_U = \{\boldsymbol{x} \in R^n \,|\, \boldsymbol{a}^T\boldsymbol{x} \geq \beta\}$
$H_U^i = H_U - H = \{\boldsymbol{x} \in R^n \,|\, \boldsymbol{a}^T\boldsymbol{x} > \beta\}$
$H_L = \{\boldsymbol{x} \in R^n \,|\, \boldsymbol{a}^T\boldsymbol{x} \leq \beta\}$ closed half space
$H_L^i = \{\boldsymbol{x} \in R^n \,|\, \boldsymbol{a}^T\boldsymbol{x} < \beta\}$ open half space

- Properties of hyperplanes

Property 1: The normal vector $\boldsymbol{a}$ is orthogonal to all vectors in the hyperplane H.

Property 2: The normal vector is directed toward the upper half space.

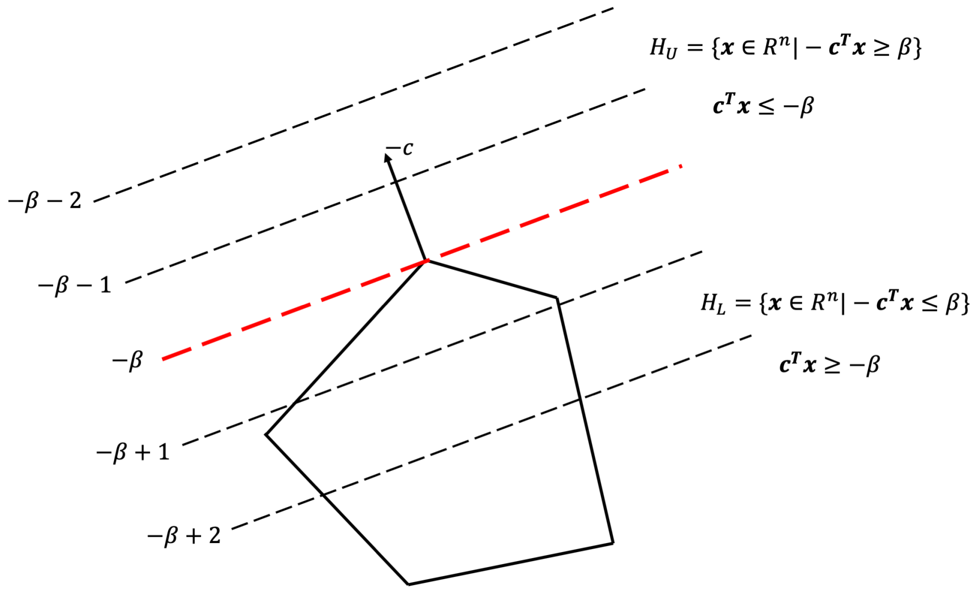
Properties of feasible solution set.
Definition: A polyhedral set or polyhedron is a set formed by the intersection of a finite number of closed half spaces. If it is nonempty and bounded, it is a polytope.

Property 3: The feasible domain of a standard LP, namely

$$P = \{\boldsymbol{x} \in R^n \,|\, \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \mathbf{0}\}$$

is a polyhedral set.

Property 4: If $P \neq \phi$ and $\beta \in R$ such that $P \subset H_L := \{\boldsymbol{x} \in R^n \,|\, -\boldsymbol{c}^T\boldsymbol{x} \leq \beta\}$, then $\min_{x \in P} \boldsymbol{c}^T\boldsymbol{x} \geq -\beta$.
(A little abstract, the following figure shows the Property 4 more directly. It also indicates the mechanism of graphic method for solving LP!)



● Definition of convex, affine and cone combination/set

Let $\boldsymbol{x^1}, \boldsymbol{x^2}, \cdots, \boldsymbol{x^p} \in R^n$, $\lambda_1, \lambda_2, \cdots, \lambda_p \in R$. And $\boldsymbol{x} = \sum_{i=1}^{p} \lambda_i \boldsymbol{x^i}$. We say $\boldsymbol{x}$ is a linear combination of $\{\boldsymbol{x^1}, \boldsymbol{x^2}, \cdots, \boldsymbol{x^p}\}$.

$$\begin{cases} \sum_{i=1}^{p} \lambda_i = 1 & \text{affine combination} \\ \sum_{i=1}^{p} \lambda_i = 1, \lambda_i \geq 0, \ \forall i & \text{convex combination} \\ \lambda_i \geq 0, \ \forall i & \text{conic combination} \end{cases}$$

Set S is an affine set $\Longleftrightarrow$ $\forall \boldsymbol{x_1}, \boldsymbol{x_2} \in S, \ \forall \lambda_1, \lambda_2 \in R, \ \lambda_1 + \lambda_2 = 1,$
we have $\lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2 \in S$.

Set S is a convex set $\Longleftrightarrow$ $\forall \boldsymbol{x_1}, \boldsymbol{x_2} \in S, \lambda_1, \lambda_2 \in [0,1], \lambda_1 + \lambda_2 = 1,$
we have $\lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2 \in S$.

Set S is a cone $\Longleftrightarrow$ $\forall \boldsymbol{x} \in S, \forall \lambda \geq 0$, we have $\lambda \boldsymbol{x} \in S$.

● What's the geometric meaning of the feasible domain?

$$P = \{\boldsymbol{x} \in R^n | \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}\}$$

1. P is a polydedral set. (Not necessarily to be a polytope!)
2. P is a convex set.
3. P is the intersection of m hyperplanes and the cone of the first orthant.
4. "$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}$" means that the rhs vector $\boldsymbol{b}$ falls in the cone generated by the columns of constraint matrix $\boldsymbol{A}$. (A brilliant proposition!)

$$\boldsymbol{Ax} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$\boldsymbol{Ax} = \sum_{j=1}^{n} x_j \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

5. Actually, the set

$$A_c = \{\boldsymbol{y} \in R^m \,|\, \boldsymbol{y} = \boldsymbol{Ax},\ \boldsymbol{x} \in R^n,\ \boldsymbol{x} \geq 0\}$$

is a convex cone genearted by the columns of matrix A.

● Definition of interior and boundary points

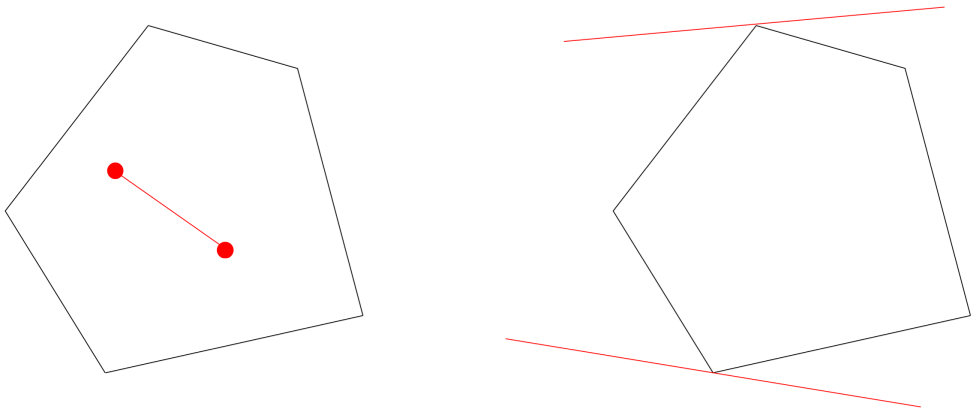
Given a set $S \subset R^n$, a point $\boldsymbol{x} \in S$ is an interior point of S, if $\exists \epsilon > 0$ such that the ball $B = \{\boldsymbol{y} \in R^n \,|\, \|\boldsymbol{x} - \boldsymbol{y}\| < \epsilon\} \subset S$. Otherwise, $\boldsymbol{x}$ is a boundary point of S.

We denote that
$\text{int}(S) = \{\boldsymbol{x}$ is an interior point of $S\}$.
$\text{bdry}(S) = \{\boldsymbol{x}$ is an boundary point of $S\}$.

● Separation Theorem of convex sets:

$S \subset R^n$ is convex, then $\forall \boldsymbol{x} \in \text{bdry}(S)$, $\exists$ a hyperplane H such that:
(i)$\boldsymbol{x} \in H$; (ii)either $S \subset H_L$ or $S \subset H_U$. (H is called "supporting hyperplane"!)

Two different ways to comprehend *convex set.*



From inner points' perspective　　　　From boundary points' perspective

● Definition of extreme points:

$\boldsymbol{x}$ is an extreme point of a convex set S, if $\boldsymbol{x}$ cannot be expressed as a convex combination of other points in S. Geometrically, the extreme point corresponds to a vertex.

● Obervations

$\boldsymbol{Ax} = \boldsymbol{b}$ has n variables in m linear equations. (m hyperplanes geometically)

When $n > m$ (generally), we only need to consider m variables in m equations for solving a system of linear equations.(Knowledge from LA, such as Cramer's Rule)

An extreme point of P is obtained by setting $n - m$ variables *to be zero* and solving the remaining m variables in m equations.

The *columns* of $\boldsymbol{A}$ corresponding to the non-zero(positive) variables better be linear independent.

● Theorem of extreme points(more specific verision of the definition of ep!)

A point $\boldsymbol{x} \in P = \{\boldsymbol{x} \in R^n \,|\, \boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ is an extreme point(ep) of P

$\Longleftrightarrow$

The columns of $\boldsymbol{A}$ corresponding to the positive components of $\boldsymbol{x}$ are linear independent.

Let $\boldsymbol{A}$ be an m by n matrix with $m \leq n$, we say $\boldsymbol{A}$ has full rank(full row rank) if $\boldsymbol{A}$ has m linear independent columns.

● Definiton of basic solution and basic feasible solution:

Arrange

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x_B} \\ \boldsymbol{x_N} \end{pmatrix},\ \boldsymbol{A} = (\boldsymbol{B} \,|\, \boldsymbol{N})$$

If we set $\boldsymbol{x_N} = \boldsymbol{0}$ and solve $\boldsymbol{x_B}$ for $\boldsymbol{Ax} = \boldsymbol{Bx_B} = \boldsymbol{b}$, then $\boldsymbol{x}$ is a *basic solution(bs)*.
Furthermore, if $\boldsymbol{x_B} \geq \boldsymbol{0}$, then $\boldsymbol{x}$ is a *basic feasible solution(bfs).*

Matrix $\boldsymbol{A}$ must have full rank, if not, then comes the following two cases:
case 1: some constraints are redundant. Remove them and everything is OK.
case 2: $P = \phi$.

● Some corollaries:

A point $\boldsymbol{x}$ in P is an extreme point of $P \Longleftrightarrow \boldsymbol{x}$ is a bfs correponding to some basis B.
The polydedral set P has only a finite number of extreme points.

● An strong property of extreme points(ep are core members to express other points)

When $P = \{\boldsymbol{x} \in R^n \,|\, \boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ is a nonempty *polytope(P is bounded!)*, then any point in P can be represented as a convex combination of the extreme points of P.
PS: this property is very good, but does it still work when P is not bounded?
Yes! See *Resolution Theorem* in the following text.

● Extremal direction for unboundedness

Definition: A vector $\boldsymbol{d}(\neq \boldsymbol{0}) \in R^n$ is an extremal direction of P, if

$$\{\boldsymbol{x} \in R^n \,|\, \boldsymbol{x} = \boldsymbol{x}^0 + \lambda \boldsymbol{d},\ \lambda \geq 0\} \subset P,\ \forall \boldsymbol{x}^0 \in P.$$

P is unbounded $\iff$ P has an extremal direction.
$\boldsymbol{d}(\neq \boldsymbol{0})$ is an extremal direction of $P \iff \boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}$ and $\boldsymbol{d} > \boldsymbol{0}$.
(A more useful way to judge and find extremal direction!)

● Resolution Theorem:

Let $V = \{\boldsymbol{v}^i \in R^n \,|\, i \in I\}$ be a set of all extreme points of P, I is a finite index set, then $\forall x \in P$, we have

$$\boldsymbol{x} = \sum_{i \in I} \lambda_i \boldsymbol{v}^i + \boldsymbol{d}$$

where

$$\sum_{i \in I} \lambda_i = 1,\ \lambda_i \geq 0,\ \forall i \in I \text{ (convex combination of eps plus vector } \boldsymbol{d})$$

and either $\boldsymbol{d} = \boldsymbol{0}$ or $\boldsymbol{d}$ is an extremal direction of P.

We can also write $\boldsymbol{x} = \sum_{i \in I} \lambda_i \boldsymbol{v}^i + s\boldsymbol{d}$, for some $s \geq 0$.

● Fundamental Theorem of LP:

For a standard LP, if its feasible domain P is nonempty, then the optimal objective value of $z = \boldsymbol{c}^T \boldsymbol{x}$ over P is either unbounded below, or it is attained at(at least) an extreme point of LP.

(Skip the proof.)