Linear Programming

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Lecture 7 Robust linear optimization

Outline:
1.Motivation
2.Robust model
3.Solution methods

● Motivation:
Example: Pharmaceutic decision making (from Prof. Tom Luo)
-One active agent A for treating a disease.
-Two possible drugs: D_1, D_2
-Two possible raw material: R_1, R_2.

● Data Sheet:

	D_1	D_2
Selling price ($/1K packs)	6,200	6,900
Agent A (grams/1K packs)	0.5	0.6
Man power (hr/1K packs)	90	100
Equipment usage (hr/1K packs)	40	50
Operating cost ($/1K packs)	700	800

	Buying price ($/kg)	Content of A (g/kg)
R_1	100	0.01
R_2	199	0.02

Budget ($)	Man power (hr)
100,000	2,000
Equipment (hr)	Storage (kg)
800	1,000

- LP formulation

$$
\begin{aligned}
\text{Max} \quad & 6200D_1 + 6900D_2 - 100R_1 - 199R_2 - 700D_1 - 800D_2 \\
\text{s.t.} \quad & \\
& 0.01R_1 + 0.02R_2 - 0.5D_1 - 0.6D_2 \geq 0 && \text{(Balance of A)} \\
& R_1 + R_2 \leq 1000 && \text{(Storage)} \\
& 90D_1 + 100D_2 \geq 2000 && \text{(Man power)} \\
& 40D_1 + 50D_2 \leq 800 && \text{(Equipment)} \\
& 100R_1 + 199R_2 + 700D_1 + 800D_2 \leq 100000 && \text{(Budget)} \\
& R_1, R_2, D_1, D_2 \geq 0
\end{aligned}
$$

<u>Optimal Solution:</u>

$$z^* = 9205.79$$

$$\boldsymbol{x}^* : \begin{cases} R_1^* = 0, & R_2^* = 438.79 \\ D_1^* = 17.55, & D_2^* = 0 \end{cases}$$

<u>Another Feasible Solution:</u>

$$\bar{z} = 8294.5$$

$$\bar{\boldsymbol{x}} : \begin{cases} \bar{R}_1 = 877.73, & \bar{R}_2 = 0 \\ \bar{D}_1 = 17.467, & \bar{D}_2 = 0 \end{cases}$$

- Situation analysis:

Error/Uncertainty in Data

Content of Agent A (g/kg)

R_1	$0.01 \rightarrow \pm 0.5\,\% : [0.00995, 0.01005]$
R_2	$0.02 \rightarrow \pm 0.2\,\% : [0.0196, 0.0204]$

(1) $\boldsymbol{x}^*$ becomes infeasible.

(2) To keep the same plan

$$\begin{cases} R_1^* = 0, & R_2^* = 438.79 \\ & D_2^* = 0 \end{cases} \quad \text{remains the same,}$$

D_1^* has to be reduced to $D_1^* \times 0.98 = 17.201$.

(3) z^* is reduced from \$9205.79 to \$7286.29.

(4) This means the profit is reduced by 21%.

(5) $\bar{\boldsymbol{x}}$ remains feasible with a profit of \$8294.5. Thus this is a more “robust” solution!

● Robust LP model

$$\begin{aligned} \min \quad & z = \boldsymbol{c}^T \boldsymbol{x} + d \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

Data: $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c}^T & d \end{bmatrix}_{(m+1)\times(n+1)}$ sometimes are typically uncertain.

Dimension: (m, n) certain.

Source of Uncertainty:
-some entries may be missing
-measurement error
-prediction error
-quality statistics
-…

● Simple form robust LP

Define:

$$\begin{aligned} [\boldsymbol{A}_0 - \Delta\boldsymbol{A}, \boldsymbol{A}_0 + \Delta\boldsymbol{A}] &= \mathfrak{A} \\ [\boldsymbol{b}_0 - \Delta\boldsymbol{b}, \boldsymbol{b}_0 + \Delta\boldsymbol{b}] &= \mathfrak{B} \\ [\boldsymbol{c}_0 - \Delta\boldsymbol{c}, \boldsymbol{c}_0 + \Delta\boldsymbol{c}] &= \mathfrak{C} \\ [d_0 - \Delta d, d_0 + \Delta d] &= \mathfrak{D} \end{aligned}$$

Consider:

$$(\text{RLP}) \begin{cases} \min & \{\boldsymbol{c}^T\boldsymbol{x} + d \,|\, \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0}\} \\ \text{s.t.} & \boldsymbol{A} \in \mathfrak{A} \\ & \boldsymbol{b} \in \mathfrak{B} \\ & \boldsymbol{c} \in \mathfrak{C} \\ & d \in \mathfrak{D} \end{cases}$$

Question: what does this (RLP) mean mathmatically?

● Assumptions in decision making

(A1): $\boldsymbol{x}$ must be determined "here and now".
(A2): Decision maker is fully responsible for all consequences of data uncertainty.
$\Rightarrow$ $\boldsymbol{x}$ must be "robust feasible", i.e.,

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \quad & \forall A \in \mathfrak{A} \text{ and } \boldsymbol{B} \in \mathfrak{B} \\ \boldsymbol{x} \ge \boldsymbol{0} \quad & \end{aligned}$$

(A3): **Conservative** decision is adopted.

$$\Rightarrow \min_{\substack{\boldsymbol{c} \in \mathfrak{C} \\ d \in \mathfrak{D}}} \max\{\boldsymbol{c}^T\boldsymbol{x} + d \,|\, \boldsymbol{x} \text{ is robust feasible}\}$$

● Mathematics involved

(1)

$$
\begin{aligned}
\boldsymbol{A} \in \mathfrak{A} &\iff \boldsymbol{A} = \boldsymbol{A}_0 + t_{\mathfrak{a}} \Delta \boldsymbol{A}, \quad t_{\mathfrak{a}} \in [-1,1] \\
\boldsymbol{b} \in \mathfrak{B} &\iff \boldsymbol{b} = \boldsymbol{b}_0 + t_{\mathfrak{b}} \Delta \boldsymbol{b}, \quad t_{\mathfrak{b}} \in [-1,1] \\
\boldsymbol{c} \in \mathfrak{C} &\iff \boldsymbol{c} = \boldsymbol{c}_0 + t_{\mathfrak{c}} \Delta \boldsymbol{c}, \quad t_{\mathfrak{c}} \in [-1,1] \\
d \in \mathfrak{d} &\iff d = d_0 + t_{\mathfrak{d}} \Delta d, \quad t_{\mathfrak{d}} \in [-1,1]
\end{aligned}
$$

(2)

$\boldsymbol{x}$ is robust feasible

$$
\iff \begin{cases} \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \forall \boldsymbol{A} \in \mathfrak{A} \text{ and } \boldsymbol{b} \in \mathfrak{B} \\ \boldsymbol{x} \geq \boldsymbol{0} \end{cases}
$$

$$
\iff \begin{cases} (\boldsymbol{A}_0 + t_{\mathfrak{a}} \Delta \boldsymbol{A})\boldsymbol{x} \leq \boldsymbol{b}_0 + t_{\mathfrak{b}} \Delta \boldsymbol{b}, \quad \forall t_{\mathfrak{a}} \in [-1,1] \text{ and } t_{\mathfrak{b}} \in [-1,1] \\ \boldsymbol{x} \geq \boldsymbol{0} \end{cases}
$$

(3)

$$
\min_{\substack{\boldsymbol{c} \in \mathfrak{C} \\ d \in \mathfrak{D}}} \max\{\boldsymbol{c}^T \boldsymbol{x} + d \,|\, \boldsymbol{x} \text{ is robust feasible}\}
$$

$$
\iff \begin{array}{ll} \min & y \\ \text{s.t.} & (\boldsymbol{c}_0 + t_{\mathfrak{c}} \Delta c)^T \boldsymbol{x} + (d_0 + t_{\mathfrak{d}} \Delta d) \leq y \\ & \forall t_{\mathfrak{c}} \in [-1,1] \text{ and } t_{\mathfrak{d}} \in [-1,1] \\ & \boldsymbol{x} \text{ is robust feasible.} \end{array}
$$

● Semi-infinite LP

(RLP) becomes

$$
(\text{RLP})_{SI} \left\{ \begin{array}{lll} \min & y & \\ \text{s.t.} & (\boldsymbol{c}_0 + t_{\mathfrak{c}} \Delta c)^T \boldsymbol{x} + (d_0 + t_{\mathfrak{d}} \Delta d) \leq y & \forall t_{\mathfrak{c}} \in [-1,1] \text{ and } t_{\mathfrak{d}} \in [-1,1] \\ & (\boldsymbol{A}_0 + t_{\mathfrak{a}} \Delta \boldsymbol{A})\boldsymbol{x} \leq \boldsymbol{b}_0 + t_{\mathfrak{b}} \Delta \boldsymbol{b} & \forall t_{\mathfrak{a}} \in [-1,1] \text{ and } t_{\mathfrak{b}} \in [-1,1] \\ & \boldsymbol{x} \geq \boldsymbol{0} & \end{array} \right.
$$

● Properties of $(\text{RLP})_{SI}$

-It is a linear programming problem with $n+1$ variables and infinitely many constraints.

-It is a semi-infinite linear programming problem.

Question: How to solve $(\text{RLP})_{SI}$?

● Solution method 1

(1) Discretization method

-Pick k points form $[-1,1]^4$.

-Use these points to construct a regular linear program.

-As $k \to +\infty$, LP solutions approach a solution of $(\mathrm{RLP})_{SI}$.

Pros: Solved by LP.

Cons: An approximation solution.

● Solution method 2

(1) Cutting plane method

Consider a linear semi-infinite programming problem:

$$
\begin{aligned}
& & \text{Min} \quad & \sum_{j=1}^{n} c_j x_j \\
& (\text{LSIP}) & \text{s.t.} \quad & \sum_{j=1}^{n} f_j(t) x_j \le g(t), \quad t \in T \\
& & & x_j \ge 0, \quad j = 1,2,\cdots,n
\end{aligned}
$$

Step 1: Let $\epsilon > 0$ be sufficiently small, K and M be sufficiently large.
Choose any $t^1 \in T$ and set $k = 1$, $T_1 = \{t^1\}$, $z^0 = M$.

Step 2: Find an optimal solution $\boldsymbol{x}^k$ to

$$
\begin{aligned}
& & \text{Min} \quad & \sum_{j=1}^{n} c_j x_j \\
& (\text{LP}_k) & \text{s.t.} \quad & \sum_{j=1}^{n} f_j(t^i) x_j \le g(t^i), \quad i = 1,\cdots,k \\
& & & x_j \ge 0, \quad j = 1,2,\cdots,n
\end{aligned}
$$

Set $z^k = \boldsymbol{c}^T \boldsymbol{x}^k$.

Let $\Phi_{k+1}(t) \triangleq g(t) - \sum_{j=1}^{n} f_j(t) x_j^k, \ \forall t \in T$.

Fine a minimizer t^{k+1} of $\Phi_{k+1}(t)$ over T and calculate $\Phi_{k+1}(t^{k+1})$.

Step 3: If $\Phi_{k+1}(t^{k+1}) \ge 0$ or $|z^k - z^{k+1}| < \epsilon$ for $k > K$,
then STOP and output $\boldsymbol{x}^k$ as an optimal solution.
Otherwise, set
$T_{k+1} = T_k \cup \{t^{k+1}\}$ and $k := k + 1$.
Go to step 2.