Linear Programming

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Additional content 1: Dantzig-Wolfe Decomposition

<u>Outline:</u>

● Introduction

In this file, we would give some examples so show how Dantzig-Wolfe Decomposition works and discuss some further details.

● Example 5.1

Consider the linear program

$$
\begin{array}{lrrrrcl}
\text{Minimize} & -2x_1 - 3x_2 - 5x_3 - 4x_4 \\
\text{subject to} \\
& x_1+ & x_2+ & 2x_3 & & \leq & 4 \\
& & x_2+ & x_3+ & x_4 & \leq & 3 \\
& 2x_1+ & x_2 & & & \leq & 4 \\
& x_1+ & x_2 & & & \leq & 2 \\
& & & x_3+ & x_4 & \leq & 2 \\
& & & 3x_3+ & 2x_4 & \leq & 5 \\
& \multicolumn{6}{l}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0}
\end{array}
$$

The first two constraints

$$
\begin{array}{rrrrcl}
x_1+ & x_2+ & 2x_3 & & \leq & 4 \\
& x_2+ & x_3+ & x_4 & \leq & 3
\end{array}
$$

are the linking constraints and can be represented as $L_1\boldsymbol{x}^1 + L_2\boldsymbol{x}^2 \leq \boldsymbol{b}^0$ where

$$L_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ L_2 = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \boldsymbol{b}^0 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

and

$$\boldsymbol{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ \boldsymbol{x}^2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

Without these constraints, the linear program would consist of the two independent linear programming subproblems where the first independent subproblem is

$$\begin{array}{ll} \text{Minimize} & -2x_1 - 3x_2 \\ \text{subject to} & 2x_1 + x_2 \le 4 \\ & \quad x_1 + x_2 \le 2 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{array}$$

where

$$\boldsymbol{c}^1 = \begin{bmatrix} -2 \\ -3 \end{bmatrix},\ A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},\ \boldsymbol{b}^1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

and the second independent subproblem is

$$\begin{array}{ll} \text{Minimize} & -5x_3 - 4x_4 \\ \text{subject to} & \quad x_3 + \ x_4 \le 2 \\ & 3x_3 + 2x_4 \le 5 \\ & x_3 \ge 0,\ x_4 \ge 0 \end{array}$$

where

$$\boldsymbol{c}^2 = \begin{bmatrix} -5 \\ -4 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix},\ \boldsymbol{b}^1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

● Example 5.2

We illustrate the Dantzig-Wolfe Decomposition on the linear program in Example 5.1, which happens to be bounded, with the subproblem feasible sets that are also bounded, and so extreme directions will not exist.

Note: (information attained by Matlab)
The optimal solution for the original LP is $\boldsymbol{x}^* = (1,1,1,1)^T$ and $z^* = -14$.
The optimal solution for the first subproblem is $(\boldsymbol{x}^1)^* = (x_1^*, x_2^*)^T = (0,2)^T$ and $z_1^* = -6$.
The optimal solution for the second subproblem is $(\boldsymbol{x}^2)^* = (x_3^*, x_4^*)^T = (1,1)^T$ and $z_2^* = -9$.

The master problem has the form (after adding slack variables $\boldsymbol{s} = (s_1, s_2)^T$):

$$\text{Minimize} \quad \sum_{i=1}^{N_1} \lambda_i^1 f_i^1 + \sum_{i=1}^{N_2} \lambda_i^2 f_i^2$$

subject to

$$\begin{array}{llll}
\sum_{i=1}^{N_1} \lambda_i^1 L_1 \boldsymbol{v}_i^1 + & \sum_{i=1}^{N_2} \lambda_i^2 L_2 \boldsymbol{v}_i^2 + \boldsymbol{s} & = \boldsymbol{b}^0 & \\
\sum_{i=1}^{N_1} \lambda_i^1 & & = 1 & \\
 & \sum_{i=1}^{N_2} \lambda_i^2 & = 1 & \\
\lambda_i^1 \geq 0 & & & i = 1, \cdots, N_1 \\
 & \lambda_i^2 \geq 0 & & i = 1, \cdots, N_2 \\
\boldsymbol{s} \geq \boldsymbol{0} & & &
\end{array}$$

Recall that it is not necessary to know the quantities N_1 and N_2 in advance.

Step 0: Start with as basic variables $s_1 = b_1^0 = 4$, $s_2 = b_2^0 = 3$, $\lambda_1^1 = 1$ and $\lambda_1^2 = 1$, so $\boldsymbol{x}_B = (s_1, s_2, \lambda_1^1, \lambda_1^2)^T$ with initial extreme points $\boldsymbol{v}_1^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_1^2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Then, the restricted master problem is

$$\text{Minimize} \quad \lambda_1^1 f_1^1 + \lambda_1^2 f_1^2$$

subject to

$$\begin{array}{lll}
\lambda_1^1 L_1 \boldsymbol{v}_1^1 + & \lambda_1^2 L_2 \boldsymbol{v}_1^2 + \boldsymbol{s} & = \boldsymbol{b}^0 \\
\lambda_1^1 & & = 1 \\
 & \lambda_1^2 & = 1 \\
\lambda_1^1 \geq 0 & & \\
 & \lambda_1^2 \geq 0 & \\
\boldsymbol{s} \geq \boldsymbol{0} & &
\end{array}$$

and since the initial extreme point for each subproblem is the zero vector we get

$$
\begin{array}{lllllll}
\text{Minimize} & 0 & & & & & \\
\text{subject to} & & & & & & \\
& & & & s_1 & & = 3 \\
& & & & & s_2 & = 4 \\
& \lambda_1^1 & & & & & = 1 \\
& & & \lambda_1^2 & & & = 1 \\
& \lambda_1^1 \geq 0 & & & & & \\
& & & \lambda_1^2 \geq 0 & & & \\
& \boldsymbol{s} \geq \mathbf{0} & & & & &
\end{array}
$$

The basis matrix $\boldsymbol{B} = \boldsymbol{I}$, i.e., the 4 by 4 identity matrix.

Iteration 1:

Step 1:

$$\boldsymbol{f}_B = \begin{bmatrix} c_{s_1} \\ c_{s_2} \\ f_1^1 \\ f_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (\boldsymbol{c}^1)^T \boldsymbol{v}_1^1 \\ (\boldsymbol{c}^2)^T \boldsymbol{v}_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$, then solving $\boldsymbol{B}^T \boldsymbol{\pi} = \boldsymbol{f}_B$ gives

$$\boldsymbol{\pi} = \begin{bmatrix} \boldsymbol{\pi}^1 \\ \pi_1^2 \\ \pi_2^2 \end{bmatrix} = \begin{bmatrix} \pi_1^1 \\ \pi_2^1 \\ \pi_1^2 \\ \pi_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Step 2:

Now the objective function of (SP_1) is

$$\sigma_1 = \left((\boldsymbol{c}^1)^T - (\boldsymbol{\pi}^1)^T L_1\right) \boldsymbol{x}^1 = (\boldsymbol{c}^1)^T \boldsymbol{x}^1 = -2x_1 - 3x_2$$

so (SP_1) is

$$
\begin{array}{ll}
\text{Minimize} & -2x_1 - 3x_2 \\
\text{subject to} & 2x_1 + x_2 \leq 4 \\
& \quad x_1 + x_2 \leq 2 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{array}
$$

The optimal solution is $\boldsymbol{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ with objective function value $\sigma_1^* = -6$ and so $r_*^1 = \sigma_1^* - \pi_1^2 = -6 - 0 = -6$. Let $\boldsymbol{v}_2^1 = \boldsymbol{x}^1$.

Now the objective function of (SP_2) is

$$\sigma_2 = \left((\boldsymbol{c}^2)^T - (\boldsymbol{\pi}^1)^T L_2\right) \boldsymbol{x}^2 = (\boldsymbol{c}^2)^T \boldsymbol{x}^2 = -5x_3 - 4x_4$$

So (SP_2) is

$$\begin{array}{ll} \text{Minimize} & -5x_3 - 4x_4 \\ \text{subject to} & x_3 + x_4 \leq 2 \\ & 3x_3 + 2x_4 \leq 5 \\ & x_3 \geq 0,\ x_4 \geq 0 \end{array}$$

The optimal solution is $\boldsymbol{x}^2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with objective function value $\sigma_2^* = -9$ and so $r_*^2 = \sigma_2^* - \pi_2^2 = -9 - 0 = -9$. Let $\boldsymbol{v}_2^2 = \boldsymbol{x}^2$.

Step 3:

$r_{\min} = r_*^2 = -9$ and so $\bar{\boldsymbol{a}} = \begin{bmatrix} \boldsymbol{q}_2^2 \\ \boldsymbol{e}_2 \end{bmatrix} = \begin{bmatrix} L_2 \boldsymbol{v}_2^2 \\ \boldsymbol{e}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$.

Step 4:

$\boldsymbol{Bd} = -\bar{\boldsymbol{a}}$ and so $\boldsymbol{d} = -\begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$.

Step 5:

$\alpha = \min\{\frac{4}{2}, \frac{3}{2}, \frac{1}{1}\} = 1$ and so $\lambda_2^2 = \alpha = 1$.

Step 6:

$\boldsymbol{x}_B = \begin{bmatrix} s_1 \\ s_2 \\ \lambda_1^1 \\ \lambda_1^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} -2 \\ -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, so λ_1^2 leaves the basis and λ_2^2 enters the basis and the updated basic variable set is $\boldsymbol{x}_B = \begin{bmatrix} s_1 \\ s_2 \\ \lambda_1^1 \\ \lambda_2^2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Step 7: Column $\bar{\boldsymbol{a}}$ enters the basis and the column $(0,0,0,1)^T$ associated with λ_1^2 leaves the basis and so $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Go to Step 1.

Iteration 2:

Step 1:

$\boldsymbol{f}_B = \begin{bmatrix} c_{s_1} \\ c_{s_2} \\ f_1^1 \\ f_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (\boldsymbol{c}^1)^T \boldsymbol{v}_1^1 \\ (\boldsymbol{c}^2)^T \boldsymbol{v}_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -9 \end{bmatrix}$, then solving $\boldsymbol{B}^T \boldsymbol{\pi} = \boldsymbol{f}_B$ gives

$$\boldsymbol{\pi} = \begin{bmatrix} \boldsymbol{\pi}^1 \\ \pi_1^2 \\ \pi_2^2 \end{bmatrix} = \begin{bmatrix} \pi_1^1 \\ \pi_2^1 \\ \pi_1^2 \\ \pi_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -9 \end{bmatrix}$$

Step 2: Now the objective function of (SP_1) and (SP_2) remain the same as $\boldsymbol{\pi}^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so the subproblems remain the same. Since the optimal solution $\boldsymbol{v}_2^2$ for (SP_2) and its multipler $\lambda_2^2 = 1$ are already in the restricted master problem, thus $r_{\min} = r_*^1 = -6$, so we enter into the besis λ_2^1 of the restricted master problem along with $\boldsymbol{v}_2^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$(the optimal extreme point of (SP_1)).

Step 3:

$$\text{So } \bar{\boldsymbol{a}} = \begin{bmatrix} \boldsymbol{q}_2^1 \\ \boldsymbol{e}_1 \end{bmatrix} = \begin{bmatrix} L_1 \boldsymbol{v}_2^1 \\ \boldsymbol{e}_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Step 4:

$\boldsymbol{B}\boldsymbol{d} = -\bar{\boldsymbol{a}}$ and so $\boldsymbol{d} = -\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

Step 5:

$\alpha = \min\{\frac{2}{2}, \frac{1}{2}, \frac{1}{1}\} = 0.5$ and so $\lambda_2^1 = \alpha = 0.5$.

Step 6:

$$\boldsymbol{x}_B = \begin{bmatrix} s_1 \\ s_2 \\ \lambda_1^1 \\ \lambda_2^2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} + (0.5)\begin{bmatrix} -2 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0.5 \\ 1 \end{bmatrix}$$

, so s_2 leaves the basis and λ_2^1 enters the basis and the updated basic variable set is $\boldsymbol{x}_B = \begin{bmatrix} s_1 \\ \lambda_2^1 \\ \lambda_1^1 \\ \lambda_2^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \\ 1 \end{bmatrix}$

Step 7: Column $\bar{\boldsymbol{a}}$ enters the basis and the column $(0,1,0,0)^T$ associated with s_2 leaves the basis, and so

$$\boldsymbol{B} = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\bar{B}$ is the same as before, except λ_2^1 is in the position of the basis that s_2 previously occupied.

Go to Step 1.

Iteration 3:

Step 1:

$$\boldsymbol{f}_B = \begin{bmatrix} c_{s_1} \\ f_2^1 \\ f_1^1 \\ f_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ (\boldsymbol{c}^1)^T\boldsymbol{v}_2^1 \\ (\boldsymbol{c}^1)^T\boldsymbol{v}_1^1 \\ (\boldsymbol{c}^2)^T\boldsymbol{v}_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 0 \\ -9 \end{bmatrix}$$

, then solving $\boldsymbol{B}^T\boldsymbol{\pi} = \boldsymbol{f}_B$ gives

$$\boldsymbol{\pi} = \begin{bmatrix} \boldsymbol{\pi}^1 \\ \pi_1^2 \\ \pi_2^2 \end{bmatrix} = \begin{bmatrix} \pi_1^1 \\ \pi_2^1 \\ \pi_1^2 \\ \pi_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \\ -3 \end{bmatrix}.$$

Step 2:

Now the objective function of (SP_1) and (SP_2) change since $\boldsymbol{\pi}^1 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$.

Now the objective function of (SP_1) is $\sigma_1 = \left((\boldsymbol{c}^1)^T - (\boldsymbol{\pi}^1)^T L_1\right)\boldsymbol{x}^1 = -2x_1$, so (SP_1) is

$$\begin{array}{ll} \text{Minimize} & -2x_1 \\ \text{subject to} & 2x_1 + x_2 \leq 4 \\ & \quad x_1 + x_2 \leq 2 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

The optimal solution is $\boldsymbol{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ with objective function value $\sigma_1^* = -4$ and so $r_*^1 = \sigma_1^* - \pi_1^2 = -4 - 0 = -4$. Let $\boldsymbol{v}_3^1 = \boldsymbol{x}^1$.

Now the objective function of (SP_2) is $\sigma_2 = \left((\boldsymbol{c}^2)^T - (\boldsymbol{\pi}^1)^T L_2\right) \boldsymbol{x}^2 = -2x_3 - x_4$ so (SP_2) is

$$\begin{array}{ll} \text{Minimize} & -2x_3 - x_4 \\ \text{subject to} & \ x_3 + \ x_4 \ \leq 2 \\ & 3x_3 + 2x_4 \leq 5 \\ & x_3 \geq 0,\ x_4 \geq 0 \end{array}$$

The optimal solution is $\boldsymbol{x}^2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ 0 \end{bmatrix}$ with objective value $\sigma_2^* = -\dfrac{10}{3}$ and so $r_*^2 = \sigma_2^* - \pi_2^2 = -\dfrac{10}{3} - (-3) = -\dfrac{1}{3}$.

Step 3:

$$\bar{\boldsymbol{a}} = \begin{bmatrix} \boldsymbol{q}_3^1 \\ \boldsymbol{e}_1 \end{bmatrix} = \begin{bmatrix} L_1 \boldsymbol{v}_3^1 \\ \boldsymbol{e}_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Step 4:

$\boldsymbol{B}\boldsymbol{d} = -\bar{\boldsymbol{a}}$ and so $\boldsymbol{d} = -\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Step 5:

$\alpha = \min\{\dfrac{1}{2}, \dfrac{0.5}{1}\} = 0.5$ and so $\lambda_3^1 = \alpha = 0.5$.

Step 6:

$$\boldsymbol{x}_B = \begin{bmatrix} s_1 \\ \lambda_2^1 \\ \lambda_1^1 \\ \lambda_2^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \\ 1 \end{bmatrix} + (0.5)\begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ 1 \end{bmatrix}$$, so λ_1^1 leaves the basis, and λ_3^1 enters the basis, and the updated basic variable set is $\boldsymbol{x}_B = \begin{bmatrix} s_1 \\ \lambda_2^1 \\ \lambda_3^1 \\ \lambda_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 1 \end{bmatrix}$.

Step 7: Column $\bar{\boldsymbol{a}}$ enters the basis and the column $(0,0,1,0)^T$ associated with λ_1^1 leaves the basis and so

$$\boldsymbol{B} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Go to Step 1.

Iteration 4:

Step 1:

$$\boldsymbol{f}_B = \begin{bmatrix} c_{s_1} \\ f_2^1 \\ f_3^1 \\ f_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ (\boldsymbol{c}^1)^T \boldsymbol{v}_2^1 \\ (\boldsymbol{c}^1)^T \boldsymbol{v}_3^1 \\ (\boldsymbol{c}^2)^T \boldsymbol{v}_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ -4 \\ -9 \end{bmatrix}$$, then solving $\boldsymbol{B}^T \boldsymbol{\pi} = \boldsymbol{f}_B$ gives

$$\boldsymbol{\pi} = \begin{bmatrix} \boldsymbol{\pi}^1 \\ \pi_1^2 \\ \pi_2^2 \end{bmatrix} = \begin{bmatrix} \pi_1^1 \\ \pi_2^1 \\ \pi_1^2 \\ \pi_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -4 \\ -7 \end{bmatrix}.$$

Step 2:

The objective functions of (SP_1) and (SP_2) change since $\boldsymbol{\pi}^1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

Now the objective function of (SP_1) is $\sigma_1 = \left((\boldsymbol{c}^1)^T - (\boldsymbol{\pi}^1)^T L_1\right) \boldsymbol{x}^1 = -2x_1 - 2x_2$, so (SP_1) is

$$\begin{aligned} \text{Minimize} \quad & -2x_1 - 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \le 4 \\ & x_1 + x_2 \le 2 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

The optimal solution is $\boldsymbol{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ with objective function value $\sigma_1^* = -4$, and so $r_*^1 = \sigma_1^* - \pi_1^2 = -4 - (-4) = 0$.

Now the objective function of (SP_2) is $\sigma_2 = \left((\boldsymbol{c}^2)^T - (\boldsymbol{\pi}^1)^T L_2\right) \boldsymbol{x}^2 = -4x_3 - 3x_4$, so (SP_2) is

$$\begin{array}{ll} \text{Minimize} & -4x_3 - 3x_4 \\ \text{subject to} & x_3 + x_4 \le 2 \\ & 3x_3 + 2x_4 \le 5 \\ & x_3 \ge 0,\ x_4 \ge 0 \end{array}$$

The optimal solution is $\boldsymbol{x}^2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with objective function value $\sigma_2^* = -7$, and so $r_*^2 = \sigma_2^* - \pi_2^2 = -7 - (-7) = 0$. Since $r_{\min} = 0$ we STOP, and the current basis $\boldsymbol{B}$ represents an optimal solution. That is, the optimal basic variables to the restricted master problem are

$$\boldsymbol{x}_B = \begin{bmatrix} s_1 \\ \lambda_2^1 \\ \lambda_3^1 \\ \lambda_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 1 \end{bmatrix}$$

and so the optimal solution in terms of the original variables can be recovered as

$$\boldsymbol{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_2^1 \boldsymbol{v}_2^1 + \lambda_3^1 \boldsymbol{v}_3^1 = 0.5 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + 0.5 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\boldsymbol{x}^2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \lambda_2^2 \boldsymbol{v}_2^2 = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The optimal objective function value is $z^* = -14$. That is, the optimal solution for the original problem is in this instance a convex combination of extreme points of the subproblems.

● Example 5.3 (Unbounded subproblem case)

Consider the following linear program:

$$\begin{array}{ll} \text{Maximize} & 2x_1 + 3x_2 + 2x_3 \\ \text{subject to} & x_1 + x_2 + x_3 \le 12 \\ & -x_1 + x_2 \le 2 \\ & -x_1 + 2x_2 \le 8 \\ & x_3 \le 1 \\ & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 \end{array}$$

The linear program can be seen to exhibit block angular structure with the following partitions:

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^1 \\ \boldsymbol{x}^2 \end{bmatrix} \text{ where } \boldsymbol{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \boldsymbol{x}^2 = [x_3]$$

$$\boldsymbol{c} = \begin{bmatrix} \boldsymbol{c}^1 \\ \boldsymbol{c}^2 \end{bmatrix} \text{ where } \boldsymbol{c}^1 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \text{ and } \boldsymbol{c}^2 = [c_3] = [-2]$$

$$\boldsymbol{b}^0 = [12], \boldsymbol{b}^1 = \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \text{ and } \boldsymbol{b}^2 = [1]$$

$$L_1 = [1 \quad 1] \text{ and } L_2 = [1]$$

$$A_1 = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} \text{ and } A_2 = [1]$$

Step 0: (Initialization)
Start with as basic variables $s_1 = b_1^0 = 12$, $\lambda_1^1 = 1$ and $\lambda_1^2 = 1$, so $\boldsymbol{x}_B = (s_1, \lambda_1^1, \lambda_1^2)^T$ with initial extreme points $\boldsymbol{v}_1^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_1^2 = [x_3] = [0]$.

Then, the restricted master problem is

$$\begin{array}{ll} \text{Minimize} & \lambda_1^1 f_1^1 + \lambda_1^2 f_1^2 \\ \text{subject to} & \\ & \begin{array}{lllll} \lambda_1^1 L_1 \boldsymbol{v}_1^1 + & \lambda_1^2 L_2 \boldsymbol{v}_1^2 & + \boldsymbol{s} & = \boldsymbol{b}^0 \\ \lambda_1^1 & & & = 1 \\ & \lambda_1^2 & & = 1 \\ \multicolumn{4}{l}{\lambda_1^1 \geq 0,\ \lambda_1^2 \geq 0,\ \boldsymbol{s} \geq \boldsymbol{0}} \end{array} \end{array}$$

and since the initial extreme point for each subproblem is the zero vector we get

$$\begin{array}{ll} \text{Minimize} & 0 \\ \text{subject to} & \\ & \begin{array}{llll} & & s_1 & = 12 \\ \lambda_1^1 & & & = 1 \\ & \lambda_1^2 & & = 1 \\ \multicolumn{4}{l}{\lambda_1^1 \geq 0,\ \lambda_1^2 \geq 0,\ \boldsymbol{s} \geq \boldsymbol{0}} \end{array} \end{array}$$

The basis matrix $\boldsymbol{B} = \boldsymbol{I}$, i.e., the 3 by 3 identity matrix.

Iteration 1:

Step 1:

$$f_B = \begin{bmatrix} c_{s_1} \\ f_1^1 \\ f_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ (\boldsymbol{c}^1)^T \boldsymbol{v}_1^1 \\ (\boldsymbol{c}^2)^T \boldsymbol{v}_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$, then solving $\boldsymbol{B}^T \boldsymbol{\pi} = f_B$ gives

$$\boldsymbol{\pi} = \begin{bmatrix} \boldsymbol{\pi}^1 \\ \pi_1^2 \\ \pi_2^2 \end{bmatrix} = \begin{bmatrix} \pi_1^1 \\ \pi_1^2 \\ \pi_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Step 2:(Note that (SP_2) is arbitrarily solved first; in general, the ordering of solving subproblems does not affect the decomposition).

Now the objective function of (SP_2) is

$$\sigma_2 = \left((\boldsymbol{c}^2)^T - (\boldsymbol{\pi}^1)^T L_2\right) \boldsymbol{x}^2 = (\boldsymbol{c}^2)^T \boldsymbol{x}^2 = -2x_3$$

so (SP_2) is

$$\begin{array}{ll} \text{Minimize} & -2x_3 \\ \text{subject to} & x_3 \le 1 \\ & x_3 \ge 0 \end{array}$$

The optimal solution is $\boldsymbol{x}^2 = [x_3] = [1]$ with objective function value $\sigma_2^* = -2$ and so $r_*^2 = \sigma_2^* - \pi_2^2 = -2 - 0 = -2$. Let $\boldsymbol{v}_2^2 = \boldsymbol{x}^2$.

Now the objective function of (SP_1) is

$$\sigma_1 = \left((\boldsymbol{c}^1)^T - (\boldsymbol{\pi}^1)^T L_1\right) \boldsymbol{x}^1 = (\boldsymbol{c}^1)^T \boldsymbol{x}^1 = -2x_1 - 3x_2$$

So (SP_1) is

$$\begin{array}{ll} \text{Minimize} & -2x_1 - 3x_2 \\ \text{subject to} & -x_1 + x_2 \le 2 \\ & -x_1 + 2x_2 \le 8 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{array}$$

(SP_1) is unbounded below. Go to Step 3.

Step 3:

(SP_1) is unbounded, and so

$$\bar{\boldsymbol{a}} = \begin{bmatrix} \boldsymbol{q}_1^{-1} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} L_1 \boldsymbol{d}_1^1 \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} [1 \quad 1] \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

Step 4:

$\boldsymbol{Bd} = -\bar{\boldsymbol{a}}$ and so $\boldsymbol{d} = -\begin{bmatrix}3\\0\\0\end{bmatrix}$.

Step 5:

$\alpha = \min\{-\frac{12}{-3}\} = 4$ and so $\mu_1^1 = \alpha$.

Step 6:

$$\boldsymbol{x}_B = \begin{bmatrix}s_1\\\lambda_2^1\\\lambda_1^1\end{bmatrix} = \begin{bmatrix}12\\1\\1\end{bmatrix} + 4\begin{bmatrix}-3\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix},$$

so s_1 leaves the basis and μ_1^1 enters the basis and the updated basic variable set is

$$\boldsymbol{x}_B = \begin{bmatrix}\mu_1^1\\\lambda_1^1\\\lambda_1^2\end{bmatrix} = \begin{bmatrix}4\\1\\1\end{bmatrix}$$

Step 7: Column $\bar{\boldsymbol{a}}$ enters the basis and takes the place of column $(1,0,0)^T$ associated with s_1, which exits the basis and so

$$\boldsymbol{B} = \begin{bmatrix}3 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

Go to Step 1.

Iteration 2:

Step 1:

$$\boldsymbol{f}_B = \begin{bmatrix}f_1^{-1}\\f_1^1\\f_1^2\end{bmatrix} = \begin{bmatrix}(\boldsymbol{c}^1)^T\boldsymbol{d}_1^1\\(\boldsymbol{c}^1)^T\boldsymbol{v}_1^1\\(\boldsymbol{c}^2)^T\boldsymbol{v}_1^2\end{bmatrix} = \begin{bmatrix}[-2 \quad -3]\begin{bmatrix}2\\1\end{bmatrix}\\0\\0\end{bmatrix} = \begin{bmatrix}-7\\0\\0\end{bmatrix}$$ then solving $\boldsymbol{B}^T\boldsymbol{\pi} = \boldsymbol{f}_B$

gives

$$\boldsymbol{\pi} = \begin{bmatrix}\boldsymbol{\pi}^1\\\pi_1^2\\\pi_2^2\end{bmatrix} = \begin{bmatrix}\pi_1^1\\\pi_1^2\\\pi_2^2\end{bmatrix} = \begin{bmatrix}-2.33333333\\0\\0\end{bmatrix}.$$

Step 2:

Now the objective function of (SP_1) is

$$\sigma_1 = \left((\boldsymbol{c}^1)^T - (\boldsymbol{\pi}^1)^T L_1\right)\boldsymbol{x}^1 = 0.33333333x_1 - 0.66666667x_2$$

and so (SP_1) is

$$\begin{array}{ll} \text{Minimize} & 0.33333333x_1 - 0.66666667x_2 \\ \text{subject to} & -x_1 + \ x_2 \le 2 \\ & -x_1 + 2x_2 \le 8 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{array}$$

The optimal solution is $\boldsymbol{x}^1 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ with objective function value $\sigma_1^* = -2\frac{2}{3}$, and so $r_*^1 = \sigma_1^* - \pi_1^2 = -2\frac{2}{3} - 0 = -2\frac{2}{3}$. Let $\boldsymbol{v}_2^1 = \boldsymbol{x}^1$.

Now the objective function of (SP_2) is

$$\sigma_2 = \left((\boldsymbol{c}^2)^T - (\boldsymbol{\pi}^1)^T L_2\right)\boldsymbol{x}^2 = 0.33333333x_3$$

and so (SP_2) is

$$\begin{array}{ll} \text{Minimize} & 0.33333333x_3 \\ \text{subject to} & x_3 \le 1 \\ & x_3 \ge 0 \end{array}$$

The optimal solution is $\boldsymbol{x}^2 = [x_3] = [0]$ with objective function value $\sigma_2^* = 0$, and so $r_*^2 = \sigma_2^* - \pi_2^2 = 0 - 0 = 0$. Let $\boldsymbol{v}_3^2 = \boldsymbol{x}^2$.

Step 3:

$r_{\min} = r_*^1 = -2\frac{2}{3}$, and so

$$\bar{\boldsymbol{a}} = \begin{bmatrix} \boldsymbol{q}_2^1 \\ \boldsymbol{e}_1 \end{bmatrix} = \begin{bmatrix} L_1 \boldsymbol{v}_2^1 \\ \boldsymbol{e}_1 \end{bmatrix} = \begin{bmatrix} [1 \quad 1]\begin{bmatrix} 4 \\ 6 \end{bmatrix} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 0 \end{bmatrix}$$

Step 4:

$\boldsymbol{Bd} = -\bar{\boldsymbol{a}}$ and so $\boldsymbol{d} = \begin{bmatrix} -3.33333333 \\ -1 \\ 0 \end{bmatrix}$.

Step 5:

$\alpha = \min\{-\frac{4}{-3.33333333}, -\frac{1}{-1}\} = 1$ and so $\lambda_2^1 = \alpha$.

Step 6:

$$\boldsymbol{x}_B = \begin{bmatrix} \mu_1^1 \\ \lambda_1^1 \\ \lambda_1^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} -3.33333333 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.66666667 \\ 0 \\ 1 \end{bmatrix}$$

so λ_1^1 leaves the basis and λ_2^1 enters the basis and the updated basic variable set is

$$\boldsymbol{x}_B = \begin{bmatrix} \mu_1^1 \\ \lambda_2^1 \\ \lambda_1^2 \end{bmatrix} = \begin{bmatrix} 0.66666667 \\ 1 \\ 1 \end{bmatrix}$$

Step 7:

Column $\bar{\boldsymbol{a}}$ enters the basis and the column $(0,1,0)^T$ associated with λ_1^1 leaves the basis, and so

$$\boldsymbol{B} = \begin{bmatrix} 3 & 10 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Go to Step 1.

Iteration 3:

Step 1:

$$\boldsymbol{f}_B = \begin{bmatrix} f_2^{-1} \\ f_2^1 \\ f_1^2 \end{bmatrix} = \begin{bmatrix} (\boldsymbol{c}^1)^T \boldsymbol{d}_1^1 \\ (\boldsymbol{c}^1)^T \boldsymbol{v}_2^1 \\ (\boldsymbol{c}^2)^T \boldsymbol{v}_1^2 \end{bmatrix} = \begin{bmatrix} -7 \\ [-2 \quad -3] \begin{bmatrix} 4 \\ 6 \end{bmatrix} \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ -26 \\ 0 \end{bmatrix}$$

then solving $\boldsymbol{B}^T \boldsymbol{\pi} = \boldsymbol{f}_B$ gives

$$\boldsymbol{\pi} = \begin{bmatrix} \boldsymbol{\pi}^1 \\ \pi_1^2 \\ \pi_2^2 \end{bmatrix} = \begin{bmatrix} \pi_1^1 \\ \pi_1^2 \\ \pi_2^2 \end{bmatrix} = \begin{bmatrix} -2.33333333 \\ -2.66666667 \\ 0 \end{bmatrix}.$$

Step 2:

Now the objective function of (SP_1) is

$$\sigma_1 = \left((\boldsymbol{c}^1)^T - (\boldsymbol{\pi}^1)^T L_1\right) \boldsymbol{x}^1 = 0.33333333 x_1 - 0.66666667 x_2$$

and so (SP_1) is

$$\begin{aligned} \text{Minimize} \quad & 0.33333333x_1 - 0.66666667x_2 \\ \text{subject to} \quad & -x_1 + \ x_2 \leq 2 \\ & -x_1 + 2x_2 \leq 8 \\ & x_1 \geq 0, \ x_2 \geq 0 \end{aligned}$$

The optimal solution is $\boldsymbol{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ with objective function value $\sigma_1^* = -2.66666667$, and so $r_*^1 = \sigma_1^* - \pi_1^2 = -2.66666667 - (-2.66666667) = 0$. Let $\boldsymbol{v}_3^1 = \boldsymbol{x}^1$.

Now the objective function of (SP_2) is

$$\sigma_2 = \left((\boldsymbol{c}^2)^T - (\boldsymbol{\pi}^1)^T L_2\right) \boldsymbol{x}^2 = 0.33333333x_3$$

so (SP_2) is

$$\begin{aligned} \text{Minimize} \quad & 0.33333333x_3 \\ \text{subject to} \quad & x_3 \leq 1 \\ & x_3 \geq 0 \end{aligned}$$

The optimal solution is $\boldsymbol{x}^2 = [x_3] = [0]$ with objective function value $\sigma_2^* = 0$, and so $r_*^2 = \sigma_2^* - \pi_2^2 = 0 - 0 = 0$. Let $\boldsymbol{v}_4^2 = \boldsymbol{v}_3^2 = \boldsymbol{x}^2$.

Since $r_{\min} = 0$, STOP the optimal solution to the restricted master problem is

$$\boldsymbol{x}_B = \begin{bmatrix} \mu_1^1 \\ \lambda_2^1 \\ \lambda_1^2 \end{bmatrix} = \begin{bmatrix} 0.66666667 \\ 1 \\ 1 \end{bmatrix}$$

Thus, the optimal solution in the original variables are recovered as

$$\begin{aligned} \boldsymbol{x}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \lambda_2^1 \boldsymbol{v}_2^1 + \mu_1^1 \boldsymbol{d}_1^1 \\ &= 1 \begin{bmatrix} 4 \\ 6 \end{bmatrix} + 0.66666667 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 5.33333333 \\ 6.66666667 \end{bmatrix} \end{aligned}$$

and

$$\boldsymbol{x}^2 = [x_3] = \lambda_1^2 \boldsymbol{v}_1^2 = 1 \times 0 = 0$$

(Note: the exact optimal solution is $x_1 = 5\frac{1}{3}, x_2 = 6\frac{2}{3}, x_3 = 0$)