Linear Programming

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Lecture 5 Duality and Sentivity Analysis

Outline:
1.Dual linear program
2.Duality theory
3.Sensitivity analysis
4.Dual simplex method

● Definition of dual linear program

For a standard from LP

$$\begin{array}{lll} & \min & \boldsymbol{c}^T\boldsymbol{x} \\ (\text{LP}) & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

its dual linear program is

$$\begin{array}{lll} & \max & \boldsymbol{b}^T\boldsymbol{w} \\ (\text{D}) & \text{s.t.} & \boldsymbol{A}^T\boldsymbol{w} \leq \boldsymbol{c} \\ & & \boldsymbol{w} \in R^m \end{array}$$

● Dual of LP in other form: symmetric pair

$$\begin{array}{lll} & \min & \boldsymbol{c}^T\boldsymbol{x} \\ (\text{P}) & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad \begin{array}{lll} & \max & \boldsymbol{b}^T\boldsymbol{w} \\ (\text{D}) & \text{s.t.} & \boldsymbol{A}^T\boldsymbol{w} \leq \boldsymbol{c} \\ & & \boldsymbol{w} \geq \boldsymbol{0} \end{array}$$

Why? This is left as an exercise to the reader.
Hint: transfrom (P) to standard form and use the definition to solve the problem.

● Many interesting questions(No need to answer now. After learning weak and strong duality theorem, you should have the ability to answer them. This is also left as an exercise for you.)

-Feasibility
Can problems (P) and (D) be both feasible?
One is feasible, while the other is infeasible?
Both are infeasible?
-Basic solutions
Is there any relation between the basic solutions of (P) and that of (D)? bfs? optimal solutions?
-Optimality
Can problems (P) and (D) both have a unique optimal solution?

Both have infinitely many?
One unique, the other infinitely many?

- Some examples

-Both (P) and (D) are infeasible:

$$
\text{(P)} \quad \begin{array}{ll} \min & x_1 - 2x_2 \\ \text{s.t.} & -x_1 + x_2 \geq 1 \\ & x_1 - x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{array} \qquad \text{(D)} \quad \begin{array}{ll} \max & w_1 + 2w_2 \\ \text{s.t.} & -w_1 + w_2 \leq 1 \\ & w_1 - w_2 \leq -2 \\ & w_1, w_2 \geq 0 \end{array}
$$

-Both (P) and (D) have infinitely many optimal solutions:

$$
\text{(P)} \quad \begin{array}{ll} \min & x_1 - x_2 \\ \text{s.t.} & -x_1 + x_2 \geq 1 \\ & x_1 - x_2 \geq -1 \\ & x_1, x_2 \geq 0 \end{array} \qquad \text{(D)} \quad \begin{array}{ll} \max & w_1 - w_2 \\ \text{s.t.} & -w_1 + w_2 \leq 1 \\ & w_1 - w_2 \leq -1 \\ & w_1, w_2 \geq 0 \end{array}
$$

- Lemma: Dual of the dual = Primal.

Proof is left as an exercise. Hints: (i)taking $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$; (ii)adding slacks $\boldsymbol{s}$.

- Weak dualiy theorem

If $\boldsymbol{x}$ is a primal feasible solution to (P) and $\boldsymbol{w}$ is a dual feasible solution to (D), then

$$\boldsymbol{c}^T\boldsymbol{x} \geq \boldsymbol{b}^T\boldsymbol{w}.$$

Proof:

$\boldsymbol{x}$ is primal feasible, thus $\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. $\boldsymbol{w}$ is dual feasible, thus $\boldsymbol{A}^T\boldsymbol{w} \leq \boldsymbol{c}$.
Then:

$$\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{c} \geq \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{w} = (\boldsymbol{A}\boldsymbol{x})^T\boldsymbol{w} = \boldsymbol{b}^T\boldsymbol{w}, \;\; i.e. \; \boldsymbol{c}^T\boldsymbol{x} \geq \boldsymbol{b}^T\boldsymbol{w}.$$

Corollaries:

1. If $\boldsymbol{x}$ is a primal feasible solution to (P) and $\boldsymbol{w}$ is a dual feasible solution to (D), and $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{w}$, then $\boldsymbol{x}$ is primal optimal, and $\boldsymbol{w}$ is dual optimal.
2. If the primal is unbounded below, then the dual is infeasible.
3. If the dual is unbounded above, then the primal is infeasible.

- Strong duality theorem

(a)If either the primal or the dual has a finite optimum, then so does the other and $\min \boldsymbol{c}^T\boldsymbol{x} = \max \boldsymbol{b}^T\boldsymbol{w}$.
(b)If either problem has an unbounded objective, then the other has no feasible solution.

Proof:

(a)Note that the dual of the dual is the primal and the fact that "If $\boldsymbol{x}$ is primal feasible, $\boldsymbol{w}$ is dual feasible, and $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{w}$, then $\boldsymbol{x}$ is primal optimal and $\boldsymbol{w}$ is dual optimal".

We only need to show that "if the primal has a finite optimal bfs $\boldsymbol{x}$, then there exists a dual feasible solution such that $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{w}$".

Consider optimal bsf $\boldsymbol{x}$ with basis $\boldsymbol{B}$, we define: $\boldsymbol{w}^T = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}$.

Then

$$\begin{aligned}
\boldsymbol{c} - \boldsymbol{A}^T\boldsymbol{w} &= \begin{bmatrix} \boldsymbol{c}_B \\ \boldsymbol{c}_N \end{bmatrix} - \begin{bmatrix} \boldsymbol{B}^T \\ \boldsymbol{N}^T \end{bmatrix} \boldsymbol{w} \\
&= \begin{bmatrix} \boldsymbol{c}_B - \boldsymbol{B}^T(\boldsymbol{B}^{-1})^T\boldsymbol{c}_B \\ \boldsymbol{c}_N - \boldsymbol{N}^T(\boldsymbol{B}^{-1})^T\boldsymbol{c}_B \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{c}_B - \boldsymbol{B}^T(\boldsymbol{B}^{-1})^T\boldsymbol{c}_B \\ (\boldsymbol{c}_N^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{N})^T \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{r}_N \end{bmatrix}
\end{aligned}$$

$\boldsymbol{r}_N$ is a column vector!

$\boldsymbol{w}$ is dual feasible $\Longleftrightarrow \boldsymbol{A}^T\boldsymbol{w} \le \boldsymbol{c} \Longleftrightarrow \boldsymbol{c} - \boldsymbol{A}^T\boldsymbol{w} \ge \boldsymbol{0} \Longleftrightarrow \boldsymbol{r}_N \ge \boldsymbol{0}$
Since $\boldsymbol{x}$ is optimal, $\boldsymbol{r}_N \ge \boldsymbol{0}$ and $\boldsymbol{w}$ is dual feasible.

$$\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{c}_B^T\boldsymbol{x}_B = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{w}^T\boldsymbol{b} = \boldsymbol{b}^T\boldsymbol{w}$$

(b)It is a direct consequence of the Weak Duality Theorem.

- Some implications

(a)The simplex multiplier $\boldsymbol{w}$ corresponding to a primal optimal solution $\boldsymbol{x}$ is a dual optimal solution.
(b)At each iteration of the simplex method, the simplex multiplier $\boldsymbol{w}$ always satisfies that $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{w}$.(Why? $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{c}_B^T\boldsymbol{x}_B = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{w}^T\boldsymbol{b} = \boldsymbol{b}^T\boldsymbol{w}$) However, $\boldsymbol{w}$ is not dual feasible unless $\boldsymbol{r}_N \ge \boldsymbol{0}$.
(c)Revised Simplex Method: Keep primal feasibility and $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{w}$ but seeks for dual feasibility.

- Further implications of strong duality theorem

-Theorem of alternatives: Existence of solutions of systems of equalities and inequalities.

-Famous Farkas Lemma

The system (I) $\boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}$ has no solution

$$\Updownarrow$$

the system (II) $\boldsymbol{A}^T\boldsymbol{w} \leq \boldsymbol{0},\ \boldsymbol{b}^T\boldsymbol{w} > \boldsymbol{0}$ has solution.

(another form)

Two systems

$$\text{(I) } \boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}$$
$$\text{(II) } \boldsymbol{A}^T\boldsymbol{w} \leq \boldsymbol{0},\ \boldsymbol{b}^T\boldsymbol{w} > 0$$

Either (I) or (II) has a solution but NOT both.

Proof:

Consider LP and its dual problem:

$$\text{(P)} \quad \begin{array}{ll} \min & \boldsymbol{0}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{Ax} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad \text{(D)} \quad \begin{array}{ll} \max & \boldsymbol{b}^T\boldsymbol{w} \\ \text{s.t.} & \boldsymbol{A}^T\boldsymbol{w} \leq \boldsymbol{0} \end{array}$$

Since $\boldsymbol{w} = \boldsymbol{0}$ is dual feasible, namely (D) is impossible to be infeasible, we know:

$$\begin{array}{ccc} \text{(P) is infeasible.} & \Longleftrightarrow & \text{(D) is unbounded above.} \\ \Updownarrow & & \Updownarrow \\ \text{(I) has no solution.} & \Longleftrightarrow & \text{(II) has a solution.} \end{array}$$