Linear Programming

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Lecture 4 Simplex Method

<u>Outline:</u>
<u>1.Simplex method</u>
<u>2.Phase one method</u>
<u>3.Big M method</u>

● Key steps of Simplex Method

Step 1: Find a bfs $\boldsymbol{x}$ with $\boldsymbol{A} = [\boldsymbol{B} \,|\, \boldsymbol{N}]$.

Step 2: Check for n.b.v.'s

$$r_q = \boldsymbol{c}^T \boldsymbol{d}_q = c_q - \boldsymbol{c}_q \boldsymbol{B}^{-1} \boldsymbol{A}_q$$

If $r_q \geq 0$, $\forall$ nonbasic x_q, then the current bfs is optimal.
Otherwise, pick one(Brand's Rule to prevent cycling) $r_q < 0$. Go to next step.

Step 3: If $\boldsymbol{d}_q \geq 0$, then LP is unbounded below.
Otherwise, find

$$\alpha = \min_{i:\text{basic}} \{\frac{x_i}{-d_{qi}} \,|\, d_{qi} < 0\}$$

Then $\boldsymbol{x} := \boldsymbol{x} + \alpha \boldsymbol{d}_q$ is a new bfs.
Update $\boldsymbol{B}$ and $\boldsymbol{N}$. Go to Step 2.

● More details

Q: How to judge when LP has infinitely many optimal solutions? Of course in this case the optimal objective value is a finite number.
A: When we find one optimal solution, consider the reduced costs for the n.b.v.'s. If $r_q = 0$ for some n.b.v. x_q, then LP has infinitely many optimal solutions.

● How to start the simplex method?

Two-phase method(Phase I problem) or Big-M method.

● Two-phase method

Step 1: Make the right hand side vector nonnegative:

$$
\text{(LP)} \quad
\begin{array}{ll}
\min & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}(\geq \boldsymbol{0}) \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{array}
$$

Step 2: Add m artificial variables for Phase I problem:

$$
\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}
\qquad
\text{(PhI)} \quad
\begin{array}{ll}
\min & \sum_{i=1}^{m} u_i \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{I}\boldsymbol{u} = \boldsymbol{b}(\geq \boldsymbol{0}) \\
& \boldsymbol{x}, \boldsymbol{u} \geq \boldsymbol{0}
\end{array}
$$

● Observations for Phase I problem

1. $\boldsymbol{u} = \boldsymbol{b}, \boldsymbol{x} = \boldsymbol{0}$ is a bfs of (PhI).
2. (PhI) is bounded below by 0.
3. (LP) is feasible $\Longleftrightarrow z^*_{\text{PhI}} = 0$.
4. Underdegeneracy, if $z^*_{\text{PhI}} = 0$, then the optimal solution of (PhI) is a bfs of (LP).

How about degenerate cases?

5. If $z^*_{\text{PhI}} = 0$ at an optimal bfs which is degenerate with at least one artificial variable u_j in the basis.

Suppose that $u_i = 0$ is the kth basic variable in the current basis, then:

(1) if $\boldsymbol{e}_k^T \boldsymbol{B}^{-1} \boldsymbol{A}_q \neq 0$ for a nonbasic variable x_q, then u_i can be replaced by x_q to form a starting basis.

(2) if $\boldsymbol{e}_k^T \boldsymbol{B}^{-1} \boldsymbol{A}_q = 0$, $\forall$n.b.v. x_q, then the kth row of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is redundant. We remove it and start again.

(A little hard to comprehend? Please see the following explanation)

Reduced cost: $r_q = c_q - \boldsymbol{c}_{\boldsymbol{B}}^T \boldsymbol{B}^{-1} \boldsymbol{A}_q$.

$\boldsymbol{B} = [\boldsymbol{A}_{q1} | \boldsymbol{A}_{q2} | \cdots | \boldsymbol{A}_{qk} | \cdots | \boldsymbol{A}_{qm}]$, $\boldsymbol{c}_{\boldsymbol{B}}^T = [c_{q1}, c_{q2}, \cdots, c_{qk}, \cdots, c_{qm}]$,

$\boldsymbol{c}_{\boldsymbol{B}}^T = [0,0,\cdots,1,\cdots,0]$.

For n.b.v. x_q, $c_q = 0$, $\boldsymbol{c}_{\boldsymbol{B}}^T = \boldsymbol{e}_k^T$. Thus $r_q = -\boldsymbol{e}_k^T \boldsymbol{B}^{-1} \boldsymbol{A}_q$.

● Big-M method (no further discussion)

$$
\text{(PhI)} \quad
\begin{array}{ll}
\min & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} M u_i \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{I}\boldsymbol{u} = \boldsymbol{b}(\geq \boldsymbol{0}) \\
& \boldsymbol{x}, \boldsymbol{u} \geq \boldsymbol{0}
\end{array}
$$