Linear Programming

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Lecture 5 Duality and Sentivity Analysis

Outline:
1.Dual linear program
2.Duality theory
3.Sensitivity analysis
4.Dual simplex method

● Dual simplex method
What's the dual simplex method?
~~**It is a simplex based algorithm that works on the dual problem directly. In other words, it hops from one vertex to another vertex along some edge directions in the dual space.**~~
It keeps dual feasibility and complementary slackness, but seeks for primal feasibility. Actually, the dual simplex only solves the primal problem just like the simplex method that we have introduced.

● When to use the dual simplex method?
The dual simplex method is suitable when it may be difficult to obtain an initial primal basic feasible solution, but a dual feasible solution is readily available. However, its more substantial value may be seen in its use in sensitivity analysis.

● Motivation
Applying the (revised)simplex method to solve the dual problem:

$$\begin{array}{lll} & \min \quad \boldsymbol{b}^T\boldsymbol{w} & (m \text{ variables}) \\ \text{(D)} & \text{s.t.} \quad \boldsymbol{A}^T\boldsymbol{x} \le \boldsymbol{c} & (n \text{ constraints}) \end{array}$$

Converting to standard form:

$$\begin{array}{lll} (-)\min & -\boldsymbol{b}^T\boldsymbol{u} + \boldsymbol{b}^T\boldsymbol{v} + \boldsymbol{0}^T\boldsymbol{s} & (2m+n \text{ variables}) \\ \text{(D')} \quad \text{s.t.} & \boldsymbol{A}^T\boldsymbol{u} - \boldsymbol{A}^T\boldsymbol{v} + \boldsymbol{I}\boldsymbol{s} = \boldsymbol{c} & (n \text{ constraints}) \\ & \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{s} \ge \boldsymbol{0} & \end{array}$$

(i) Dimensionality becomes larger**(thus it is not very wise to solve the dual problem directly!)**. **Thus the dual simplex method works directly on the**

primal basis B and not on the dual basis. We do not concern the dual optimal solution but the primal optimal solution!

(ii) Keep (dual) feasibility. Maintain complementary slackness. Seek (primal) feasibility.

● Corresponding basic solutions

At a primal basic solution $\boldsymbol{x}$ with basis $\boldsymbol{B}$, we define a dual basic solution

$$\boldsymbol{w}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$$

● Basic ideas of dual simplex method

-Starting with a dual basic feasible solution.

(1)Start with a basis

$$\boldsymbol{A} = [\boldsymbol{B} \,|\, \boldsymbol{N}]$$

such that $\boldsymbol{w}^T := \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$ is dual feasible, *i.e.* $\boldsymbol{A}^T \boldsymbol{w} \le \boldsymbol{c}$. **(But how? We would show in the end of this file.)**

(2)Further define

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{pmatrix} = \begin{pmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{pmatrix}$$

then

$$\boldsymbol{A}\boldsymbol{x} = [\boldsymbol{B} \,|\, \boldsymbol{N}] \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix} = [\boldsymbol{B} \,|\, \boldsymbol{N}] \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{b}$$

and

$$\begin{aligned} \boldsymbol{r}^T \boldsymbol{x} &= (\boldsymbol{c}^T - \boldsymbol{w}^T \boldsymbol{A})\boldsymbol{x} \\ &= \boldsymbol{c}^T \boldsymbol{x} - \boldsymbol{w}^T \boldsymbol{A} \boldsymbol{x} \\ &= \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} \\ &= 0 \end{aligned}$$

namely complementary slackness condition holds.

-Checking optimality

(3)Since the dual feasibility and complementary slackness conditions are satisfied in this setting. If $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} \ge \boldsymbol{0}$, then we have primal feasibility and $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix}$ is primal optimal, $\boldsymbol{w}^T := \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$ is dual optimal.

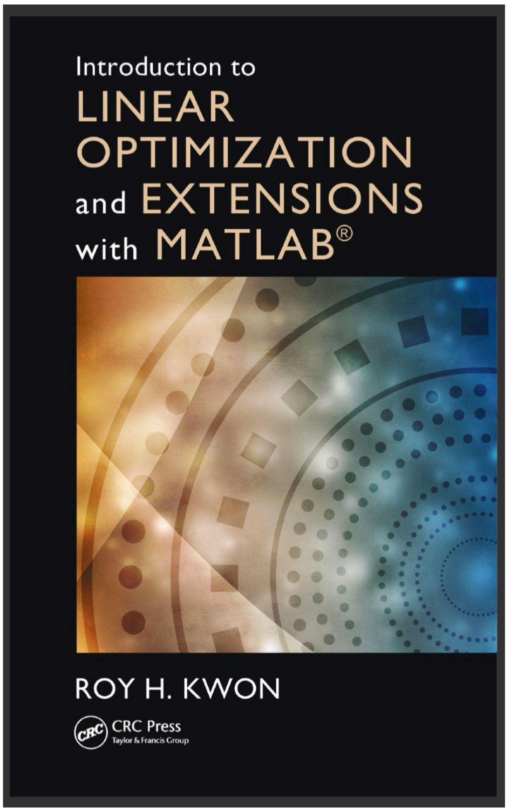
● Pivoting move:

If there exists $p \in \tilde{B}$ such that $x_p < 0$**(REMEMBER: x_p is not necessary to be pth basic variable!)**, then $\boldsymbol{A}_p$ should leave the basis, and $x_p := 0$ becomes nonbasic variable. We have to pivot-in an "appropriate" nonbasic variable x_q, for $q \in \tilde{N}$ **(REMEMBER: x_q is not necessary to be qth nonbasic variable!)**, namely enter the basis.

Of course, during this pivoting process, the dual feasibility and complementary slackness conditions should be maintained($\tilde{B}$ is the index set of basic variables in the primal problem).

Note that the complementary slackness conditions are always satisfied because of the way we define $\boldsymbol{w}$ and $\boldsymbol{x}$, hence we only have to concentrate on *dual feasibility*.

- The specific introduction of the dual simplex's principle



NOTE: the reference is the book shown by the figure.

In general, suppose that the leaving basic variable is the pth element of the current primal infeasible basis $\boldsymbol{x}_B$ denoted as $(\boldsymbol{x}_B)_p$ and $\bar{b}_p$ is the pth element of $\bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b}$. Then, since

$$\boldsymbol{x}_B + \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{B}^{-1}\boldsymbol{b} = \bar{\boldsymbol{b}},$$

the pth row is

$$(\boldsymbol{x}_B)_p + \sum_{l \in \tilde{N}} \tilde{a}_{p,l} x_l = \bar{b}_p < 0.$$

where $\boldsymbol{B}^{-1}\boldsymbol{N} = (\tilde{a}_{ij})_{m \times (n-m)}$.

A non-basic variable x_j where $j \in \tilde{N}$ must be selected to enter the basis such that $\tilde{a}_{p,j} < 0$**(if this is not easy to comprehend for you, please see the example in p154 of the book)**. In addition, the selection of the non-basic variable must ensure that dual feasibility is maintained (i.e., the reduced cost of the new primal solution must remain non-negative).

Suppose that a non-basic variable x_j with $\tilde{a}_{p,j} < 0$ will enter the primal basis. To shed light on the requirements for the entering non-basic variable x_j to maintain the non-negativity of the reduced costs, we start with the objective function at the current primal solution $\boldsymbol{x}$,

$$
\begin{aligned}
z &= \boldsymbol{c}^T \boldsymbol{x} \\
&= \boldsymbol{c}_B^T \boldsymbol{x}_B + \boldsymbol{c}_N^T \boldsymbol{x}_N \\
&= \boldsymbol{c}_B^T (\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N) + \boldsymbol{c}_N^T \boldsymbol{x}_N \\
&= \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} + (\boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{N})\boldsymbol{x}_N \\
&= z^* + \boldsymbol{r}_N^T \boldsymbol{x}_N \\
&= z^* + \sum_{l \in \tilde{N}} r_l x_l \\
&= z^* + r_j x_j + \sum_{\substack{l \in \tilde{N} \\ l \neq j}} r_l x_l
\end{aligned}
$$

where $z^* = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b}$ and $\boldsymbol{r}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{N}$, the vector of reduced costs for the current basis.

Now suppose that the infeasible basic variable selected to leave is x_i and is the pth row of $\boldsymbol{x}_B + \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{B}^{-1}\boldsymbol{b} = \bar{\boldsymbol{b}}$, and so

$$
x_i + \tilde{a}_{p,j} x_j + \sum_{\substack{l \in \tilde{N} \\ l \neq j}} \tilde{a}_{p,l} x_l = \bar{b}_p < 0.
$$

Expressing the entering variable x_j in terms of the non-basic variables in this equation gives

$$
x_j = (\bar{b}_p - x_i - \sum_{\substack{l \in \tilde{N} \\ l \neq j}} \tilde{a}_{p,l} x_l)/\tilde{a}_{p,j},
$$

Now, substituting this expression for x_j in blue equation gives

$$
z = z^* + r_j \left[(\bar{b}_p - x_i - \sum_{\substack{l \in \bar{N} \\ l \neq j}} \tilde{a}_{p,l} x_l)/\tilde{a}_{p,j} \right] + \sum_{\substack{l \in \bar{N} \\ l \neq j}} r_l x_l
$$

$$
z = z^* + \frac{r_j}{\tilde{a}_{p,j}} \bar{b}_p - \frac{r_j}{\tilde{a}_{p,j}} x_i - \frac{r_j}{\tilde{a}_{p,j}} \sum_{\substack{l \in \bar{N} \\ l \neq j}} \tilde{a}_{p,l} x_l + \sum_{\substack{l \in \bar{N} \\ l \neq j}} r_l x_l
$$

$$
z = z^* + \bar{b}_p (r_j/\tilde{a}_{p,j}) - (r_j/\tilde{a}_{p,j}) x_i - \frac{r_j}{\tilde{a}_{p,j}} \sum_{\substack{l \in \bar{N} \\ l \neq j}} \tilde{a}_{p,l} x_l + \sum_{\substack{l \in \bar{N} \\ l \neq j}} r_l x_l
$$

$$z = z^* + \bar{b}_p(r_j/\tilde{a}_{p,j}) - (r_j/\tilde{a}_{p,j})x_i + \sum_{\substack{l \in \bar{N} \\ l \neq j}} [-\frac{r_j}{\tilde{a}_{p,j}}\tilde{a}_{p,l} + r_l]x_l$$

$$z = z^* + \bar{b}_p(r_j/\tilde{a}_{p,j}) - (r_j/\tilde{a}_{p,j})x_i + \sum_{\substack{l \in \bar{N} \\ l \neq j}} [r_l - (r_j/\tilde{a}_{p,j})\tilde{a}_{p,l}]x_l$$

Thus, we see that by entering x_j into the primal basis and exiting x_i that the **reduced cost of x_i is $-(r_j/\tilde{a}_{p,j})$ and the updated reduced cost for x_l for $l \in \tilde{N}$ and $l \neq j$ is $r_l := [r_l - \tilde{a}_{p,l}(r_j/\tilde{a}_{p,j})]$.**

Now we are in a position to address the issue of ensuring that the reduced costs of the non-basic variables are non-negative. First, observe that $-(r_j/\tilde{a}_{p,j})$ is non-negative, since r_j is non-negative, since it was the reduced cost of x_j prior to becoming a basic variable and $\tilde{a}_{p,j} < 0$.

Finally, we need the updated reduced costs $r_l - \tilde{a}_{p,l}(r_j/\tilde{a}_{p,j})$ to be non-negative and this will be ensured if we select the entering variable x_j such that $\min\{-\frac{r_j}{\tilde{a}_{p,j}} \mid \tilde{a}_{p,j} < 0\}$ is satisfied. The reason is shown:

$$r_l - \tilde{a}_{p,l}(r_j/\tilde{a}_{p,j}) \geq 0 \Longleftrightarrow r_l \geq \tilde{a}_{p,l}(r_j/\tilde{a}_{p,j}) \; (*)$$

Note that $r_j/\tilde{a}_{p,j} \leq 0$ and $r_l \geq 0$.

If $\tilde{a}_{p,l} = 0$, then $(*)$ can be satisfied.

If $\tilde{a}_{p,l} > 0$, then $(*)$ can be satisfied.

If $\tilde{a}_{p,l} < 0$, then $r_l \geq \tilde{a}_{p,l}(r_j/\tilde{a}_{p,j}) \; (*) \Longleftrightarrow (r_l/\tilde{a}_{p,l}) \leq (r_j/\tilde{a}_{p,j})$ and $r_l/\tilde{a}_{p,l} \leq 0$.

So we should choose nonbasic variable x_j by $\min\{\frac{r_j}{\tilde{a}_{p,j}} \mid \tilde{a}_{p,j} < 0\}$.

In the case where there does not exist an $\tilde{a}_{p,j} < 0$, then $\tilde{a}_{p,j} \geq 0$ for all l in

$$x_i + \tilde{a}_{p,j}x_j + \sum_{\substack{l \in \tilde{N} \\ l \neq j}} \tilde{a}_{p,l}x_l = \bar{b}_p < 0$$

and we get

$$x_i = \bar{b}_p - \sum_{l \in \tilde{N}} \tilde{a}_{p,l}x_l.$$

So x_i will always be infeasible for any non-negative x_l with $l \in \tilde{N}$ since $\bar{b}_p < 0$.

Thus, the primal problem is infeasible.

After getting the basic variable to leave and the non-basic variable to enter the basis, we should further exploit the edge direction and step length for updating just like the simplex method we've learned. Actually, the step length $\alpha = \dfrac{\bar{b}_p}{\tilde{a}_{p,j}}$, and the edge direction is $\boldsymbol{d} = -\boldsymbol{B}^{-1}\boldsymbol{N}_j$.

● How to start the dual simplex method?
(How to find an initial solution that is dual feasibile?)

$$(\mathrm{P}) \quad \begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad\qquad (\mathrm{D}) \quad \begin{array}{ll} \max & \boldsymbol{b}^T\boldsymbol{w} \\ \text{s.t.} & \boldsymbol{A}^T\boldsymbol{w} \leq \boldsymbol{c} \end{array}$$

$\boldsymbol{A} = [\boldsymbol{B}\,|\,\boldsymbol{N}]$, $\boldsymbol{B}_{m\times m}$ is nonsingular, $\boldsymbol{w}^T = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}$.

$\boldsymbol{w}$ is dual feasible $\Longleftrightarrow \boldsymbol{A}^T\boldsymbol{w} \leq \boldsymbol{c}$.

Consider:

$$(\mathrm{P}') \quad \begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{e} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad\qquad (\mathrm{D}') \quad \begin{array}{ll} \max & \boldsymbol{e}^T\boldsymbol{B}^T\boldsymbol{w} \\ \text{s.t.} & \boldsymbol{A}^T\boldsymbol{w} \leq \boldsymbol{c} \end{array}$$

Observations:

(1) $\boldsymbol{x} = \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{0} \end{pmatrix}$ is a primal bfs of (P').

$$\text{Since } \boldsymbol{x} = \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{0} \end{pmatrix} \geq \boldsymbol{0} \text{ and } \boldsymbol{A}\boldsymbol{x} = (\boldsymbol{B}\,|\,\boldsymbol{N})\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{0} \end{pmatrix} = \boldsymbol{B}\boldsymbol{e} + \boldsymbol{N}\boldsymbol{0} = \boldsymbol{B}\boldsymbol{e}$$

(2) (D) and (D') have the same feasible domain.

(3) Apply the revised simplex method to (P'), either it stops at an optimal solution, or find (P') is unbounded below.

(4) It it stops at an optimal solution, then $(\boldsymbol{w}^*)^T = \boldsymbol{c}_{B^*}^T(\boldsymbol{B}^*)^{-1}$ is feasible to (D'). Hence $\boldsymbol{w}^*$ is feasible to (D).

(5) If (P') is unbounded below, then we can find an extremal direction $\boldsymbol{d}$, such that $\boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}$, $\boldsymbol{d} \neq \boldsymbol{0}$, and $\boldsymbol{c}^T\boldsymbol{d} < 0$. Hence (P) is also unbounded below and (D) must be infeasible.