Linear Programming

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Lecture 5 Duality and Sentivity Analysis

Outline:
1.Dual linear program
2.Duality theory
3.Sensitivity analysis
4.Dual simplex method

● Sensitivity analysis

Sensitivity is a post-optimality analysis of a linear program

$$(P)\quad \begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

in which, some components of $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ may change after obtaining an optimal solution $\boldsymbol{x}^*$ with an optimal basis $\boldsymbol{B}^*$ and an optimal objective value z^*.

● Question of interests:

Will $\boldsymbol{x}^*$ remain optimal? $\boldsymbol{B}^*$ remain optimal? or How will they change accordingly?

● Fundamental concepts:

No matter how the data $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ change, we need to make sure that

1. Feasibility($\boldsymbol{c}$ is not involved):

$$\text{bs } \boldsymbol{x} \text{ is feasible} \Longleftrightarrow \boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$$

2. Optimality($\boldsymbol{b}$ is not involved):

$$\text{bfs } \boldsymbol{x} \text{ is optimal} \Longleftrightarrow r_q \geq 0, \quad \forall q \in \tilde{N}$$

$$r_q = c_q - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A}_q$$

● Change in the cost vector $\boldsymbol{c}$

Scenario: $\boldsymbol{x}^*$ is an optimal solution and $\boldsymbol{A} = [\boldsymbol{B} \mid \boldsymbol{N}]$.

Given $\boldsymbol{c}' = \begin{bmatrix} \boldsymbol{c}_B' \\ \boldsymbol{c}_N' \end{bmatrix} \in R^n$ be a perturbation.

$$\boldsymbol{c} := \boldsymbol{c} + \alpha\boldsymbol{c}' = \begin{bmatrix} \boldsymbol{c}_B + \alpha\boldsymbol{c}_B' \\ \boldsymbol{c}_N + \alpha\boldsymbol{c}_N' \end{bmatrix} = \bar{\boldsymbol{c}}$$

$$
\text{(P')} \quad \begin{array}{ll} \min & (\boldsymbol{c} + \alpha \boldsymbol{c}')^T \boldsymbol{x} = \bar{\boldsymbol{c}}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}
$$

● Question of change in $\boldsymbol{c}$

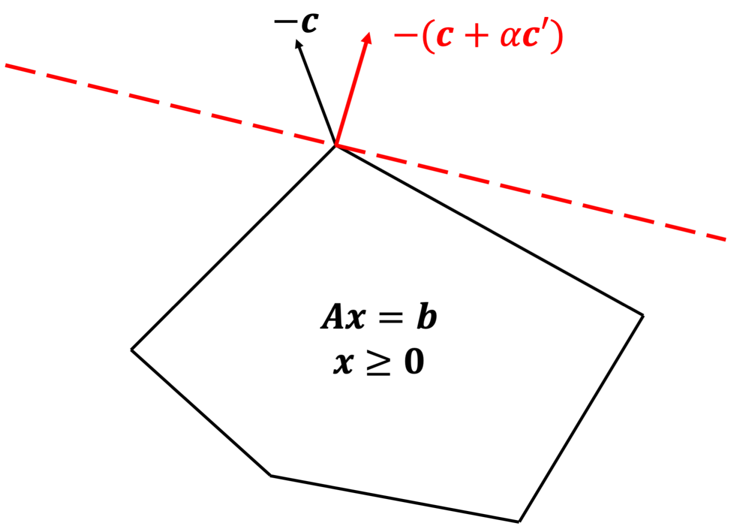
Q: Within which range of $[\underline{\alpha}, \overline{\alpha}]$, the current optimal solution $\boldsymbol{x}^*$ remains to be optimal?

A:

$$
\underline{\alpha} = \begin{cases} \max\{-\dfrac{r_q}{r_q'} \mid r_q' > 0,\ q \in \tilde{N}\} \\ -\infty, \quad \text{if } r_q' \leq 0,\ \forall q \in \tilde{N} \end{cases} \qquad \overline{\alpha} = \begin{cases} \min\{-\dfrac{r_q}{r_q'} \mid r_q' < 0,\ q \in \tilde{N}\} \\ +\infty, \quad \text{if } r_q' \geq 0,\ \forall q \in \tilde{N} \end{cases}
$$

(Reasons would be shown in the following text.)

Note: when $\boldsymbol{c}' = (0,\cdots,0,1,0,\cdots,0)^T$, we have the regular sensitivity analysis on each cost coefficient.



● Analysis

Compare

$$
\text{(P)} \quad \begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad \text{(P')} \quad \begin{array}{ll} \min & (\boldsymbol{c} + \alpha \boldsymbol{c}')^T \boldsymbol{x} = \bar{\boldsymbol{c}}^T \boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}
$$

(1) (P) and (P') have the same feasible domain, hence $\boldsymbol{x}^*$ is feasible to (P') for any α.

(2) $\boldsymbol{x}^*$ remains optimal to (P') if

$$
\bar{\boldsymbol{r}}_N^T = \bar{\boldsymbol{c}}_N^T - \bar{\boldsymbol{c}}_B^T \boldsymbol{B}^{-1} \boldsymbol{N} \geq \boldsymbol{0}
$$

namely

$$
(\boldsymbol{c}_N + \alpha \boldsymbol{c}_N')^T - (\boldsymbol{c}_B + \alpha \boldsymbol{c}_B')^T \boldsymbol{B}^{-1} \boldsymbol{N} \geq \boldsymbol{0}
$$

$$(\boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N}) + \alpha[(\boldsymbol{c}_N')^T - (\boldsymbol{c}_B')^T \boldsymbol{B}^{-1} \boldsymbol{N}] \geq \boldsymbol{0}$$

$$\alpha (\boldsymbol{r}_N')^T \geq -\boldsymbol{r}_N^T$$

(3)Case 1: for $r_q' > 0$, $q \in \tilde{N}$,

$$\alpha \geq -\frac{r_q}{r_q'} \quad \text{is required,}$$

thus

$$\underline{\alpha} = \max\{-\frac{r_q}{r_q'} \mid r_q' > 0,\ q \in \tilde{N}\}$$

Otherwise

$$\underline{\alpha} = -\infty, \text{ if } r_q' \leq 0,\ \forall q \in \tilde{N}.$$

Case 2: for $r_q' < 0,\ \ q \in \tilde{N}$

$$\alpha \leq -\frac{r_q}{r_q'} \quad \text{is required,}$$

thus

$$\overline{\alpha} = \max\{-\frac{r_q}{r_q'} \mid r_q' < 0,\ q \in \tilde{N}\}$$

Otherwise

$$\overline{\alpha} = +\infty, \text{ if } r_q' \geq 0,\ \ \forall q \in \tilde{N}.$$

(4) For $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $\boldsymbol{x}^*$ remains optimal.

$$\begin{aligned} z^*(\alpha) &= [\boldsymbol{c}_B^T + \alpha(\boldsymbol{c}_B')^T]\boldsymbol{B}^{-1}\boldsymbol{b} \\ &= \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} + \alpha(\boldsymbol{c}_B')^T \boldsymbol{B}^{-1}\boldsymbol{b} \\ &= z^* + k\alpha \end{aligned}$$

Thus $z^*(\alpha)$ is linear in α.

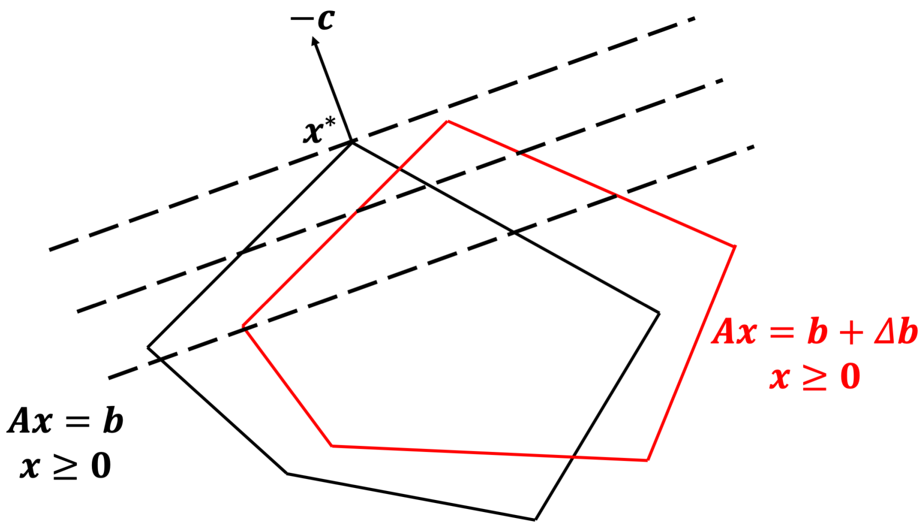
● Change in the r-h-s vector $\boldsymbol{b}$

Scenario:

Let $\boldsymbol{b}' \in R^m$ be a perturbation.

$$\text{(P')} \quad \begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \Delta \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Fact: $\boldsymbol{x}^*$ may be infeasible!



Q: within which range $[\underline{\alpha}, \overline{\alpha}]$, will $\boldsymbol{B}$ remain as an optimal basis?

Analysis:

(1) $\boldsymbol{B}$ is an optimal bais if

(i) $\boldsymbol{r}_N^T = \boldsymbol{c}_N^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{N} \geq \boldsymbol{0}$

(ii) $\boldsymbol{x}(\alpha) = \begin{pmatrix} \boldsymbol{B}^{-1}(\boldsymbol{b} + \alpha \boldsymbol{b}') \\ \boldsymbol{0} \end{pmatrix} \geq \boldsymbol{0}$

(2) (i) always holds, since $\boldsymbol{c}, \boldsymbol{B}, \boldsymbol{N}$ no change! But (ii) is not always true!

(3) We need $\boldsymbol{B}^{-1}(\boldsymbol{b} + \alpha \boldsymbol{b}') \geq \boldsymbol{0} \Longleftrightarrow \boldsymbol{B}^{-1}\boldsymbol{b} + \alpha \boldsymbol{B}^{-1}\boldsymbol{b}' \geq \boldsymbol{0}$.

We denote $\boldsymbol{B}^{-1}\boldsymbol{b}, \boldsymbol{B}^{-1}\boldsymbol{b}'$ by $\overline{\boldsymbol{b}}, \overline{\boldsymbol{b}}'$ separately.
Thus we get

$$\alpha \overline{\boldsymbol{b}}' \geq -\overline{\boldsymbol{b}}$$

Using the similar way to analysis, the conclusion is not difficult to attain:

$$\underline{\alpha} = \begin{cases} \max\{-\dfrac{\overline{b}_p}{\overline{b}'_p} \mid \overline{b}'_p > 0,\ p \in \tilde{B}\} \\ -\infty, \quad \text{if } \overline{b}'_p \leq 0,\ \forall p \in \tilde{B} \end{cases} \qquad \overline{\alpha} = \begin{cases} \min\{-\dfrac{\overline{b}_p}{\overline{b}'_p} \mid \overline{b}'_p < 0,\ p \in \tilde{B}\} \\ +\infty, \quad \text{if } \overline{b}'_p \geq 0,\ \forall p \in \tilde{B} \end{cases}$$

(4) When $\alpha \in [\underline{\alpha}, \overline{\alpha}]$,

$$
\begin{aligned}
x^*(\alpha) &= \begin{pmatrix} \boldsymbol{B}^{-1}(\boldsymbol{b} + \alpha \boldsymbol{b}') \\ \boldsymbol{0} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} + \alpha \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{pmatrix} + \alpha \begin{pmatrix} \boldsymbol{B}^{-1}\boldsymbol{b}' \\ \boldsymbol{0} \end{pmatrix} \\
&= \boldsymbol{x}^* + \alpha \begin{pmatrix} \boldsymbol{B}^{-1}\boldsymbol{b}' \\ \boldsymbol{0} \end{pmatrix}
\end{aligned}
$$

linear in α!

(5)When $\alpha \in [\underline{\alpha}, \overline{\alpha}]$,

$$
\begin{aligned}
z^*(\alpha) &= \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}} + \boldsymbol{x}^*(\alpha)_{\boldsymbol{B}} \\
&= \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}(\boldsymbol{x}_{\boldsymbol{B}}^{\boldsymbol{T}} + \alpha \boldsymbol{B}^{-1}\boldsymbol{b}') \\
&= z^* + k\alpha
\end{aligned}
$$

again, linear in α!

● Change in the constraint matrix

Since both feasibility and optimality are involved, a general analysis is difficult. We only consider simple cases such as adding a new variable, removing a variable, adding a new constraint.

● Adding a new variable

Why? A new product, service, or activity is introduced.

$$
\begin{array}{lll}
 & \min & \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} + c_{n+1}x_{n+1} \\
(\text{P}') & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}_{n+1}x_{n+1} = \boldsymbol{b} \\
 & & \boldsymbol{x} \geq \boldsymbol{0}, x_{n+1} \geq 0
\end{array}
$$

Analysis:

(1) $\begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{0} \end{bmatrix}$ is a bfs of (P') with $[\boldsymbol{B} \,|\, \boldsymbol{N}, \boldsymbol{A}_{n+1}]$.

(2) $\begin{bmatrix} \boldsymbol{x}^* \\ \boldsymbol{0} \end{bmatrix}$ is an optimal solution of (P') if $r_{n+1} = c_{n+1} - \boldsymbol{c}_{\boldsymbol{B}}^{\boldsymbol{T}}\boldsymbol{B}^{-1}\boldsymbol{A}_{n+1} \geq 0$.

(3) If $r_{n+1} < 0$, then x_{n+1} enters the basis and continue the revised simplex method to find an optimal solution of (P').

● Removing a new variable

Why? An activity is no longer available.

(a) if $x_k^* = 0$, then $\boldsymbol{x}^*$ remains optimal by deleting x_k^*.

(b) if $x_k^* > 0$, then x_k has to leave the basis. Can this be done?

Consider

$$\text{(Phase I)}\quad \begin{array}{ll} \min & x_k \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

(1) $\boldsymbol{x}^*$ is a current bfs to start the revised simplex method.

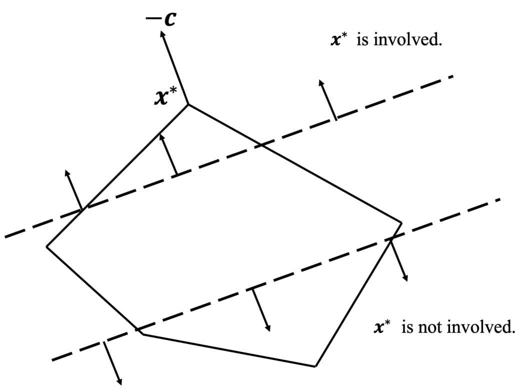
(2) If $z_{\mathrm{PhI}}^* = 0$, then we can start from there to solve the new problem.

If $z_{\mathrm{PhI}}^* > 0$, then removing x_k will cause infeasibility.

● Adding a new constraint

Why? A new restriction is enforces.

$$\text{(P')}\quad \begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{a}_{m+1}^T\boldsymbol{x} \leq b_{m+1} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$



Analysis:

(1) If $\boldsymbol{a}_{m+1}^T\boldsymbol{x}^* \leq b_{m+1}$, then $\boldsymbol{x}^*$ remains optimal!

(2) If not, $\boldsymbol{x}^*$ is not feasible and we have to find a new basis of demensionality $m+1$.

(3) Consider

$$\text{(P')}\quad \begin{array}{ll} \min & \boldsymbol{c}_B^T\boldsymbol{x}_B + \boldsymbol{c}_N^T\boldsymbol{x}_N + 0x_{n+1} \\ \text{s.t.} & \boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{N}\boldsymbol{x}_N + \boldsymbol{0}x_{n+1} = \boldsymbol{b} \\ & (\boldsymbol{a}_{m+1})_B^T\boldsymbol{x}_B + (\boldsymbol{a}_{m+1})_N^T\boldsymbol{x}_N + x_{n+1} = b_{m+1} \\ & \boldsymbol{x}_B, \boldsymbol{x}_N \geq \boldsymbol{0},\ x_{n+1} \geq 0 \end{array}$$

Then

$\overline{\boldsymbol{B}} = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{0} \\ (\boldsymbol{a}_{m+1})_B^T & 1 \end{pmatrix}$ is nonsigular $(m+1) \times (m+1)$ matrix, and

$\overline{\boldsymbol{B}}^{-1} = \begin{pmatrix} \boldsymbol{B}^{-1} & \boldsymbol{0} \\ -(\boldsymbol{a}_{m+1})_B^T B^{-1} & 1 \end{pmatrix}$, namely $\overline{\boldsymbol{B}}$ is a basis for (P').

(4) The reduced cost

$$
\begin{aligned}
r_q' &= c_q - \begin{bmatrix} \boldsymbol{c_B} \\ 0 \end{bmatrix}^T \overline{\boldsymbol{B}}^{-\mathbf{1}} \begin{bmatrix} \boldsymbol{A_q} \\ a_{m+1,q} \end{bmatrix} \\
&= c_q - [\boldsymbol{c_B} \mid 0] \begin{bmatrix} \boldsymbol{B}^{-1} & \mathbf{0} \\ -(\boldsymbol{a}_{m+1})_B^T B^{-1} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{A_q} \\ a_{m+1,q} \end{bmatrix} \\
&= c_q - [\boldsymbol{c_B^T B^{-1}} \mid 0] \begin{bmatrix} \boldsymbol{A_q} \\ a_{m+1,q} \end{bmatrix} \\
&= c_q - \boldsymbol{c_B^T B^{-1} A}_q \\
&= r_q \quad (\forall q \in \tilde{N})
\end{aligned}
$$

Since $\boldsymbol{B}$ is an optimal basis to (P), we know $r_q' = r_q \geq 0, \quad \forall q \in \tilde{N}$. Thus $\overline{\boldsymbol{B}}$ provides a dual feasible solution $\boldsymbol{w^T} = \boldsymbol{c_B^T B^{-1}}$ for (P').

(5) Define

$$
\overline{\boldsymbol{x}}_{\boldsymbol{B}} = \overline{\boldsymbol{B}}^{-\mathbf{1}} \begin{pmatrix} \boldsymbol{b} \\ b_{m+1} \end{pmatrix} \qquad \overline{\boldsymbol{x}}_{\boldsymbol{N}} = \mathbf{0}
$$

Then $\overline{\boldsymbol{x}} = \begin{pmatrix} \overline{\boldsymbol{x}}_B \\ \overline{\boldsymbol{x}}_N \end{pmatrix}$ is an optimal solution of (P') if $\overline{\boldsymbol{x}}_{\boldsymbol{B}} \geq \mathbf{0}$.

(6)If $\overline{\boldsymbol{x}}_{\boldsymbol{B}} = \overline{\boldsymbol{B}}^{-\mathbf{1}} \begin{pmatrix} \boldsymbol{b} \\ b_{m+1} \end{pmatrix} \ngeq \mathbf{0}$, then we can apply the dual simplex method with

$$
\begin{aligned}
\boldsymbol{w^T} &= \overline{\boldsymbol{c}}_{\boldsymbol{B}}^{\boldsymbol{T}} \overline{\boldsymbol{B}}^{-\mathbf{1}} \\
&= (\boldsymbol{c_B^T},0) \begin{bmatrix} \boldsymbol{B}^{-1} & \mathbf{0} \\ -(\boldsymbol{a}_{m+1})_B^T B^{-1} & 1 \end{bmatrix} \\
&= (\boldsymbol{c_B^T B^{-1}} \mid 0)
\end{aligned}
$$

to solve (P').