The Local Volatility Model

Riccardo Longoni

riccardo.longoni@intesasanpaolo.com

Course Outline

1. Market Volatility Surface
   - Derivatives
   - Black model
   - Implied Volatility
   - Sticky Delta vs Sticky Strike
2. The Local Volatility Model
   - Stochastic Differential Equation
   - Kolmogorov equations
   - Dupire Equation
   - From Implied Volatility to Local Volatility
3. Local Volatility in practice
   - Parametrization of the Local Volatility Surface
   - Numerical Solution of Partial Differential Equations
   - Discretization Schemes
   - Calibration
4. Pricing
   - Monte Carlo Method
   - European vs Forward Starting Options
   - Hedging

Disclaimer

The opinions expressed here are solely those of the author and do not represent in any way those of his employers.

Course Outline

1. Market Volatility Surface
   - Derivatives
   - Black model
   - Implied Volatility
   - Sticky Delta vs Sticky Strike
2. The Local Volatility Model
3. Local Volatility in practice
4. Pricing

Derivative exchanges

- A **derivative** is a contract between two or more parties, whose price is dependent upon one or more underlying assets (e.g. stocks, FX rates, commodities, etc..).

- We can classify financial instruments according to their trading type:
  - an **exchange traded** (listed) product is a standardized financial instrument that is traded on an organized exchange
  - an over-the-counter (**OTC**) product is a customized financial instrument traded off an exchange

- **Derivative exchanges** are markets where people trade standardized derivative contracts that are defined by the exchange.

Derivative exchanges

An exchange traded derivative has a **market price**

E-mini S&P 500 Index Futures,De (ESZ17.CME)

CME - CME Delayed Price. Currency in USD

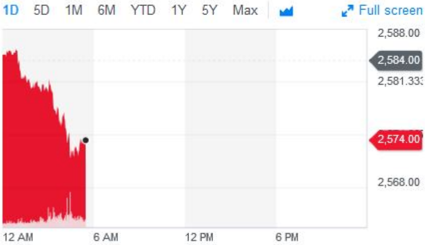
Add to watchlist

2,575.00 -9.25 (-0.35%)

As of 5:31AM EST. Market open.

Summary | Chart NEW | Futures

Pre. Settlement	N/A	Last Price	2,584.00
Settlement Date	2017-12-15	Day's Range	2,571.50 - 2,585.75
Open	2,582.00	Volume	182,573
Bid	2,574.50	Ask	2,574.75

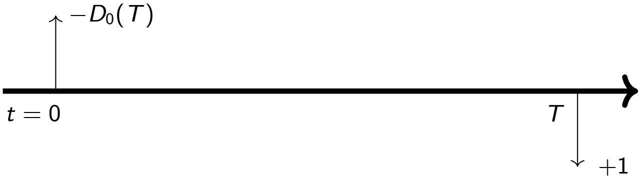


Derivative exchanges

- We are interested in the following contracts, that are actively traded on the exchanges
  - zero coupon bonds
  - forward contracts
  - plain vanilla options
- We will deal with simplified versions of the contracts that are actually traded on derivative exchanges
- We also assume that **exchange traded derivatives are risk-free** since the derivatives exchange itself acts as the counterparty for each transaction

Zero Coupon Bond

- A Zero Coupon (ZC) Bond is a debt security that is sold at a discount and does not pay any interest payments to the bondholder



- The **discount factor** $D_0(T)$ is the price at time $t = 0$ of a Zero Coupon Bond expiring in T
- We will see later that on the market one finds prices of Box Spreads, which are equivalent to ZC Bonds up to a simple transformation

Forward

- A forward contract is an agreement to buy or sell an asset at a certain future time T for a certain strike price K
- If $S(T)$ is the price of the asset at time T, then the payoff at T of the forward contract is
$$S(T) - K$$
- The **forward price** $F_0(T)$ at time $t = 0$ for a contract expiring in T is the value of K that makes a forward contract worth zero at $t = 0$
- Market quotes of $F_0(T)$ are usually available for several expiries T

Homework: If S is a stock that does not pay dividends, show that the only non-arbitrage value for a forward price at T is $F_0(T) = S(0)/D_0(T)$.
Hint: build a replication strategy of the forward contract

Plain Vanilla Options

- A **call option** (resp. **put option**) gives the right to buy (resp. sell) the underlying asset at time T at a fixed price K
- The payoff at T of an option is

$$\begin{cases} (S(T) - K)^+ & \text{Call} \\ (K - S(T))^+ & \text{Put} \end{cases}$$

- Let's denote by $C_0(K, T)$ (resp. $P_0(K, T)$) the price at $t = 0$ of a call (resp. put) contract
- Call and put options have a long history: they were traded since 18th century

Homework: Given $D_0(T)$ and $F_0(T)$, show that $C_0(K, T)$ must satisfy the no-arbitrage relations $D_0(T)\,(F_0(T) - K) \le C_0(K, T) \le D_0(T)\,F_0(T)$

Market vs model prices

- We haven't made any *model* assumption on prices yet: $D_0(T)$, $F_0(T)$, $C_0(K,T)$ are not computed but just observed on the exchanges
- One may ask if there's any explanation of observed prices, namely if there is an underlying model that justifies them
- We use the symbols $D_0(T)^{\text{model}}$, $F_0(T)^{\text{model}}$, $C_0(K,T)^{\text{model}}$ to denote prices which are computed from a model assumption

Black model

- Black and Scholes in 1973 made sense of a pricing formula for stock options by modeling a (arbitrage-free, complete) market with two assets depending on three parameters q, σ, r:
  - a "risk-free" bank account $B(t) = e^{rt}$
  - a stochastic process $S(t)$ evolving under the risk-neutral measure as

$$dS(t) = S(t)(r - q)dt + \sigma S(t)dW_t$$

- The *model* price at $t = 0$ of a derivative contract expiring in T with payoff $\Phi(S(T))$ is

$$\mathbb{E}\left[\frac{\Phi(S(T))}{e^{rT}}\right]$$

where $\mathbb{E}$ is the expectation under the risk-neutral measure

Black model

- Black and Scholes in 1973 made sense of a pricing formula for stock options by modeling a (arbitrage-free, complete) market with two assets depending on three parameters q, σ, r:
  - a "risk-free" bank account $B(t) = e^{rt}$
  - a stochastic process $S(t)$ evolving under the risk-neutral measure as

$$dS(t) = S(t)(r - q)dt + \sigma S(t)dW_t$$

- The *model* price at $t = 0$ of a derivative contract expiring in T with payoff $\Phi(S(T))$ is

$$\mathbb{E}\left[\frac{\Phi(S(T))}{e^{rT}}\right]$$

  where $\mathbb{E}$ is the expectation under the risk-neutral measure

Black model

In this framework, we can compute model prices

$$D_0(T)^{\text{model}} = \mathbb{E}\left[\frac{1}{e^{rT}}\right] = e^{-rT}$$

$$F_0(T)^{\text{model}} = \mathbb{E}[S(T)] = S(0)e^{(r-q)T}$$

$$\begin{aligned} C_0(K,T)^{\text{model}} &= \mathbb{E}\left[\frac{(S(T)-K)^+}{e^{rT}}\right] = \\ &= \underbrace{Bl\left(F_0(T)^{\text{model}}, K, T, \sigma, D_0(T)^{\text{model}}\right)}_{\text{Black's formula}} \end{aligned}$$

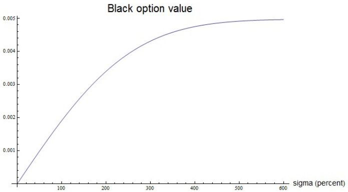
Black formula

- More explicitly, Black's formula is given by

$$Bl\,(F, K, T, \sigma, D) = D \cdot \left(F \cdot N(d_1) - K \cdot N(d_2) \right)$$

where $d_1 = \frac{\log(F/K)+\sigma^2 T/2}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$ and N is the cumulative normal distribution function

- Actually known to traders since 1965
- It's an invertible function of σ with $\lim_{\sigma\to 0} = D \cdot (F - K)$, $\lim_{\sigma\to+\infty} = D \cdot F$



Calibration of Black model

Suppose that for a certain future time T we have:

- market price $D_0(T)$ of a zero coupon bond expiring in T
- market price $F_0(T)$ of a forward contract expiring in T
- market price $C_0(K, T)$ of a call option with strike K expiring in T

On the other side, given the model parameters, we know how to compute model prices $D_0(T)^{\text{model}}$, $F_0(T)^{\text{model}}$, $C_0(K, T)^{\text{model}}$

Then we can "calibrate" Black model, namely find parameters r, q, σ such that **model prices** match **market prices**

Calibration of Black model

Let's find r, q, σ by matching model with market prices:

$$D_0(T) = D_0(T)^{\text{model}} = e^{-rT} \quad \Rightarrow \quad r = -\log(D_0(T))/T$$

$$F_0(T) = F_0(T)^{\text{model}} = S(0)e^{(r-q)T} \quad \Rightarrow \quad q = r - \frac{1}{T}\log\left(F_0(T)/S(0)\right)$$

$$C_0(K,T) = C_0(K,T)^{\text{model}} = Bl\left(F_0(T)^{\text{model}}, K, T, \sigma, D_0(T)^{\text{model}}\right)$$
$$\Rightarrow \quad \text{find } \sigma \text{ by inverting Black's formula}$$

A solution σ is called *implied (market) volatility*. Notice that a solution exists if and only if $C_0(K,T)$ satisfies no-arbitrage constraints (page 10)

Calibration of Black model

Let's find r, q, σ by matching model with market prices:

$$D_0(T) = D_0(T)^{\text{model}} = e^{-rT} \quad \Rightarrow \quad r = -\log(D_0(T))/T$$

$$F_0(T) = F_0(T)^{\text{model}} = S(0)e^{(r-q)T} \quad \Rightarrow \quad q = r - \frac{1}{T}\log\left(F_0(T)/S(0)\right)$$

$$C_0(K,T) = C_0(K,T)^{\text{model}} = Bl\left(F_0(T)^{\text{model}}, K, T, \sigma, D_0(T)^{\text{model}}\right)$$

$$\Rightarrow \quad \text{find } \sigma \text{ by inverting Black's formula}$$

A solution σ is called *implied (market) volatility*. Notice that a solution exists if and only if $C_0(K,T)$ satisfies no-arbitrage constraints (page 10)

Calibration of Black model

Let's find r, q, σ by matching model with market prices:

$$D_0(T) = D_0(T)^{\text{model}} = e^{-rT} \quad \Rightarrow \quad r = -\log(D_0(T))/T$$

$$F_0(T) = F_0(T)^{\text{model}} = S(0)e^{(r-q)T} \quad \Rightarrow \quad q = r - \frac{1}{T}\log\left(F_0(T)/S(0)\right)$$

$$C_0(K,T) = C_0(K,T)^{\text{model}} = Bl\left(F_0(T)^{\text{model}}, K, T, \sigma, D_0(T)^{\text{model}}\right)$$

$$\Rightarrow \quad \text{find } \sigma \text{ by inverting Black's formula}$$

A solution σ is called *implied (market) volatility*. Notice that a solution exists if and only if $C_0(K,T)$ satisfies no-arbitrage constraints (page 10)

Calibration of Black model

Let's find r, q, σ by matching model with market prices:

$$D_0(T) = D_0(T)^{\text{model}} = e^{-rT} \quad \Rightarrow \quad r = -\log(D_0(T))/T$$

$$F_0(T) = F_0(T)^{\text{model}} = S(0)e^{(r-q)T} \quad \Rightarrow \quad q = r - \frac{1}{T}\log\left(F_0(T)/S(0)\right)$$

$$C_0(K,T) = C_0(K,T)^{\text{model}} = Bl\left(F_0(T)^{\text{model}}, K, T, \sigma, D_0(T)^{\text{model}}\right)$$

$$\Rightarrow \quad \text{find } \sigma \text{ by inverting Black's formula}$$

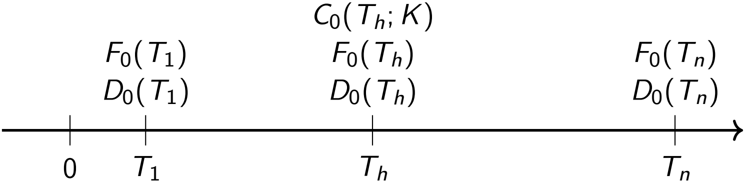
A solution σ is called *implied (market) volatility*. Notice that a solution exists if and only if $C_0(K,T)$ satisfies no-arbitrage constraints (page 10)

Homework

- Suppose $S(t)$ is a stochastic process following Black dynamics and define a new process $X(T) = \frac{S(T)}{F_0(T)^{\text{model}}}$
  - Using Itô formula, find the SDE for $X(t)$
  - Find the model forward price $\mathbb{E}[X(T)]$
  - Find the model price of a call contract on $X(T)$ in terms of model prices of call options on $S(T)$

Time-dependent Black model

Suppose we are given a set of **n market quotes** $\{D_0(T_i), F_0(T_i)\}_{i=1,\dots,n}$ for several expiries $\{T_1, \dots, T_n\}$ and a **single** option price $C_0(T_h, K)$ for $h \in \{1, \dots, n\}$



In general we cannot calibrate a Black model, namely there is no choice of r, q such that model prices $D_0(T_i)^{\text{model}}$, $F_0(T_i)^{\text{model}}$ are equal to market prices $D_0(T_i)$, $F_0(T_i)$ **for all i**

Time-dependent Black model

An easy extension of the Black-Scholes model

$$
\begin{aligned}
dS(t) &= S(t)(r(t) - q(t))dt + \sigma S(t)dW_t \\
dB(t) &= B(t)r(t)dt
\end{aligned}
$$

with time-dependent parameters $r(t), q(t)$, gives

$$D_0(T)^{\text{model}} = e^{-\int_0^T r(s)ds}$$

$$F_0(T)^{\text{model}} = S(0)e^{\int_0^T (r(s)-q(s))ds}$$

$$C_0(T,K)^{\text{model}} = Bl\left(F_0(T)^{\text{model}}, K, T, \sigma, D_0(T)^{\text{model}}\right)$$

Time-dependent Black model

- Let's match $D_0(T_i) = D_0(T_i)^{\text{model}}$ for all expiries $\{T_i\}_{i=1,\dots,n}$:

$$D_0(T_1) = e^{-\int_0^{T_1} r(s)ds} \Leftrightarrow \int_0^{T_1} r(s)ds = -\log\left(D_0(T_1)\right)$$

$$\frac{D_0(T_i)}{D_0(T_{i-1})} = e^{-\int_{T_{i-1}}^{T_i} r(s)ds} \Leftrightarrow \int_{T_{i-1}}^{T_i} r(s)ds = -\log\left(D_0(T_i)/D_0(T_{i-1})\right)$$

- We have information only on the integral of r between two expiries!
- So we make further assumptions on the shape of $r(t)$ (and the same for $q(t)$)

Time-dependent Black model

- There are several possible choices (see e.g. paper by Hagan-West [5]). The easiest is to assume that $r(t), q(t)$ are piecewise constant functions

$$r(t) = \sum_{i=1}^{n} \mathbf{1}_{(T_{i-1},T_i]}(t) \cdot r_i \qquad q(t) = \sum_{i=1}^{n} \mathbf{1}_{(T_{i-1},T_i]}(t) \cdot q_i \qquad (T_0 = 0)$$

- Then, r_i, q_i can be calibrated iteratively from $D_0(T_i), F_0(T_i)$

$$r_1 = \frac{1}{T_1} \int_0^{T_1} r(s)ds = -\frac{1}{T_1} \log\left(D_0(T_1)\right)$$

$$r_i = \frac{1}{(T_i - T_{i-1})} \int_{T_{i-1}}^{T_i} r(s)ds = -\frac{1}{(T_i - T_{i-1})} \log\left(D_0(T_i)/D_0(T_{i-1})\right)$$

- Finally σ is implied from $C_0(T_h, K)$ by inverting Black's formula

Time-dependent Black model

Further extension:

- suppose we are given a set of market quotes

$$\{D_0(T_i), F_0(T_i), C(T_i, K)\}_{i=1,\dots,n}$$

  (notice there is only one call option for each expiry)
- consider an extendend Black model

$$\begin{aligned} dS(t) &= S(t)(r(t) - q(t))dt + \sigma(t)S(t)dW_t \\ dB(t) &= B(t)r(t)dt \end{aligned}$$

  with time-dependent parameters $r(t), q(t), \sigma(t)$
- it is not difficult to derive an extension of Black formula and to show that this model can be calibrated to the above market data, e.g., by using piecewise-constant parameters

Foreign Exchange Rates

- Black model, which originally was used to explain prices of stock options, can be applied to options on foreign exchange (FX) rates
- An exchange rate $X(t)$ between a domestic and a foreign currency is the amount of domestic currency needed to buy 1 unit of foreign currency at time t
- $EUR/USD\,(t)$ is the amount of USD needed to buy 1 EUR at time t
  - USD is called the "domestic currency"
  - EUR is called the "foreign currency"
  - $EUR/USD\,(t)$ is an asset quoted in USD

Foreign Exchange Rates

- Domestic/foreign discount factors $D_0(T), D_0^{\text{for}}(T)$ are the prices of (risk-free) ZC Bonds in domestic/foreign currency
- Forward $F_0(T)$ is the fair value of an FX forward contract, i.e., the fair exchange rate at T, as determined in $t = 0$

- No-arbitrage condition:

$$\begin{array}{ccc} 1\ \text{FOR} & \xrightarrow{FX\ at\ t=0} & X(0)\ \text{DOM} \\ \Big\downarrow {\scriptstyle ZCB^{\text{for}}} & & \Big\downarrow {\scriptstyle ZCB^{\text{dom}}} \\ 1/D_0^{\text{for}}(T) & \xrightarrow{FX\ at\ T} & F_0(T)/D_0^{\text{for}}(T) = X(0)/D_0(T) \end{array}$$

$$\Rightarrow \qquad F_0(T) = X(0)\frac{D_0^{\text{for}}(T)}{D_0(T)}$$

Foreign Exchange Rates

- Domestic/foreign discount factors $D_0(T)$, $D_0^{\text{for}}(T)$ are the prices of (risk-free) ZC Bonds in domestic/foreign currency
- Forward $F_0(T)$ is the fair value of an FX forward contract, i.e., the fair exchange rate at T, as determined in $t = 0$

- No-arbitrage condition:

$$
\begin{array}{ccc}
1\ \text{FOR} & \xrightarrow{\quad FX\ at\ t=0 \quad} & X(0)\ \text{DOM} \\
\Big\downarrow{\scriptstyle ZCB^{\text{for}}} & & \Big\downarrow{\scriptstyle ZCB^{\text{dom}}} \\
1/D_0^{\text{for}}(T) & \xrightarrow{FX\ at\ T} & F_0(T)/D_0^{\text{for}}(T) = X(0)/D_0(T)
\end{array}
$$

$$
\Rightarrow \qquad F_0(T) = X(0)\frac{D_0^{\text{for}}(T)}{D_0(T)}
$$

Black model for FX rates

- Black dynamics is applied to an FX rate $X(t)$ as follows

$$
\begin{aligned}
dX(t) &= X(t)(r(t) - r_f(t))dt + \sigma X(t)dW_t \\
dB(t) &= B(t)r(t)dt \quad \text{(domestic bank account)} \\
dB_f(t) &= B_f(t)r_f(t)dt \quad \text{(foreign bank account)}
\end{aligned}
$$

- We have similar pricing formulas

$$
\begin{aligned}
D_0(T)^{\text{model}} &= e^{-\int_0^T r(s)ds} \\
F_0(T)^{\text{model}} &= X(0)\frac{e^{-\int_0^T r_f(s)ds}}{e^{-\int_0^T r(s)ds}} \\
C_t(K,T)^{\text{model}} &= Bl(F_0(T)^{\text{model}}, K, T, \sigma, D_0(T)^{\text{model}})
\end{aligned}
$$

with prices computed under the domestic risk-neutral measure

Black model for FX rates

- Just as in the equity case, we can calibrate a time-dependent FX Black model to:
  - a set of FX forwards $\{F_0(T_i)\}_{i=1\dots,n}$
  - a set of discount factors $\{D_0(T_i)\}_{i=1\dots,n}$ or $\{D_0^{\text{for}}(T_i)\}_{i=1\dots,n}$
  - a quote $C_0(T_h, K)$ of a plain vanilla option
- Namely we can find $r(t), r_f(t), \sigma$ such that:

$$
\begin{aligned}
D_0(T_i)^{\text{model}} &= D_0(T_i) = \frac{X(0)}{F_0(T_i)} D_0^{\text{for}}(T_i) \\
F_0(T_i)^{\text{model}} &= F_0(T_i) \\
C_0(T_h, K)^{\text{model}} &= C_0(T_h, K)
\end{aligned}
$$

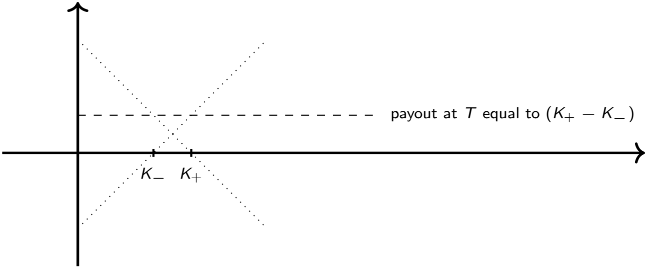
Market quotes

- So, let's have a closer look at options prices quoted on derivative exchanges
- Different derivative exchanges have different conventions
- We will first consider equity options and then FX options

Equity Market quotes (Box Spread)

- Zero Coupon Bonds are not quoted directly
- Rather, we have quotations for Box Spread strategies (with expiry T):

  - buy a Call with strike K_-, sell a Call with strike K_+
  - buy a Put with strike K_+, sell a Put with strike K_-



- Price of a Box Spread is a multiple of the discount factor at T:

$$D_0(T) \cdot (K_+ - K_-)$$

Equity Market quotes (Box Spread)

Example of quotations of Box Spread strategies:

Ticker	Struct	Legs	Ratio	Bid	Offer
E-CORP US	BOX	Dec26	1000 6000	4605	4680
E-CORP US	BOX	Dec25	1000 6000	4707	4767
E-CORP US	BOX	Dec24	1000 6000	4792	4846
E-CORP US	BOX	Dec23	1000 6000	4875	4913
E-CORP US	BOX	Dec22	1000 6000	4940	4967
E-CORP US	BOX	Dec21	1000 6000	4988	5009
E-CORP US	BOX	Dec20	1000 6000	5019	5035
E-CORP US	BOX	Dec19	1000 6000	5029	5039
E-CORP US	BOX	Dec18	1000 6000	5016	5021

Figure: Prices of Box Spreads on "E-CORP US"

Homework: find the discount factors at all available expiries

Equity Market quotes (Call-Put Parity)

- Forward contracts are not quoted directly
- Suppose we can find quotes for Call/Put options with the same strike/maturity
- We can set up the following strategy:
  - buy a Call with strike K and expiry T
  - buy a Put with strike K and expiry T
- Payoff of this strategy is

$$(S(T)-K)^{+}-(K-S(T))^{+}=S(T)-K$$

  and its (market) price is

$$C_0(T,K)-P_0(T,K)=D_0(T)\,(F_0(T)-K)$$

- Hence $F_0(T)$ can be deduced from $C_0(T,K), P_0(T,K), D_0(T)$

Equity Market quotes (Vanilla Options)

Derivative exchanges provide quotations for a large number of call/put options:

Term	15 Jun 2018				21 Sep 2018				18 Jan 2019			
Put/Call	Call		Put		Call		Put		Call		Put	
Strike	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
162.50												
165.00	1.88	2.04	21.95	22.25	2.92	3.10	23.65	23.95	4.15	4.40	25.50	25.75
167.50												
170.00	1.21	1.33	26.15	26.85	2.05	2.19	27.65	28.25	3.20	3.40	29.25	29.75
172.50												
175.00	0.770	0.870	29.35	32.85	1.43	1.56	31.90	32.60	2.38	2.57	33.15	34.35
180.00	0.490	0.570	33.95	37.45	0.990	1.10	35.60	39.45	1.79	1.95	37.50	38.85
185.00	0.310	0.390	38.70	42.30	0.680	0.780	40.05	43.95	1.33	1.49	42.05	43.40
190.00	0.200	0.280	43.40	47.10	0.470	0.640	44.70	49.00	0.980	1.18	45.50	50.00
195.00	0.140	0.210	48.45	52.05	0.330	0.430	48.80	53.50	0.740	0.890	50.00	54.50
200.00	0.090	0.170	53.35	56.95	0.240	0.330	54.00	58.50	0.560	0.690	56.10	59.00
205.00												
210.00	0.040	0.110	63.05	66.90	0.120	0.210	63.50	68.00	0.350	0.450	63.55	68.50
215.00												
220.00	0.020	0.080	72.95	76.65	0.050	0.130	73.40	78.00	0.200	0.320	73.55	78.50

Figure: Prices of listed options on "E-CORP US"

Equity Market quotes

- If we look at quoted prices we notice that:
  - options are priced over a predetermined grid of strikes and expiries

$$\left\{ C_0(T_i, K_{i,j}) \right\}_{i=1\ldots,n \quad j=1,\ldots,m}$$

  - discount factors $\{D_0(T_i)\}_{i=1\ldots,n}$ can be deduced from box spreads
  - forward prices $\{F_0(T_i)\}_{i=1\ldots,n}$ can be deduced from call-put parity

- Can we calibrate a Black model?
  - calibrate $\{r_i\}$ so that $D_0(T_i)^{\text{model}} = D_0(T_i)$ for $i = 1\ldots,n$
  - calibrate $\{q_i\}$ so that $F_0(T_i)^{\text{model}} = F_0(T_i)$ for $i = 1\ldots,n$
  - but are all options priced with the same volatility σ? [NO!]

Equity Market quotes

- If we look at quoted prices we notice that:
    - options are priced over a predetermined grid of strikes and expiries

$$\left\{ C_0(T_i, K_{i,j}) \right\}_{i=1\dots,n \quad j=1,\dots,m}$$

    - discount factors $\{D_0(T_i)\}_{i=1\dots,n}$ can be deduced from box spreads
    - forward prices $\{F_0(T_i)\}_{i=1\dots,n}$ can be deduced from call-put parity

- Can we calibrate a Black model?
    - calibrate $\{r_i\}$ so that $D_0(T_i)^{\text{model}} = D_0(T_i)$ for $i = 1\dots,n$
    - calibrate $\{q_i\}$ so that $F_0(T_i)^{\text{model}} = F_0(T_i)$ for $i = 1\dots,n$
    - but are all options priced with the same volatility σ? [NO!]

Implied Volatility

- In general, Black's model cannot be calibrated to market prices, i.e., cannot reproduce **exactly** all market prices
- However one can still use Black formula to price quoted options at the cost of having a **different volatility** for each strike/expiry:

$$C_0(T_i, K_{ij}) = Bl(F_0(T_i), K, T_i, \sigma_{mkt}(T_i, K_{ij}), D_0(T_i))$$

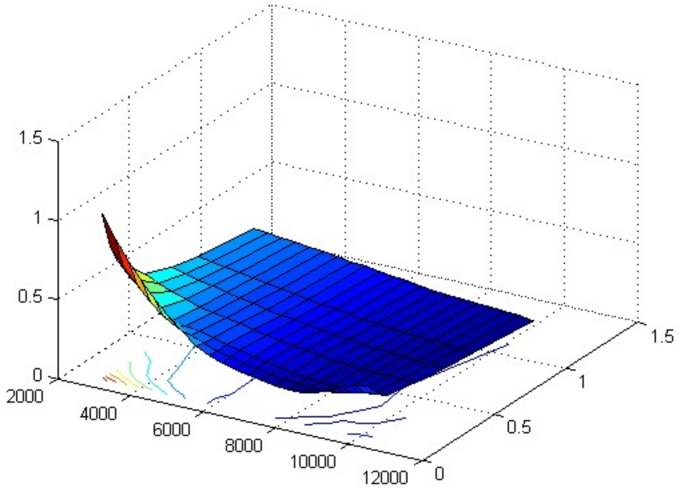
- The matrix $\{\sigma_{mkt}(T_i, K_{ij})\}_{i,j}$ is the so-called *market volatility surface*
- The function

$$K \mapsto \sigma_{mkt}(T, K)$$

  obtained by interpolating market volatilities for a fixed T is called *smile* function

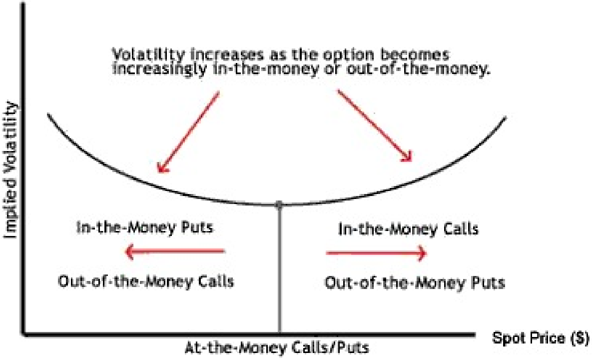
Smile

A typical shape of a "market volatility surface":



Smile

The smile function $K \mapsto \sigma_{mkt}(T, K)$ is a slice of the "market volatility surface" and usually looks like this:



Equity Volatility Surface

	Call						
	Ticker	Strike	Bid	Ask	Ultimo	IVM	Volm
	20-Oct-17 (8g); VolC 100; R 1.20; IFwd 147.51						
1)	10/20/17 C144	144.00	4.95y	5.15y	4.85y	31.11	4
2)	10/20/17 C145	145.00	4.25y	4.35y	4.27y	30.24	470
3)	10/20/17 C146	146.00	3.60y	3.75y	3.61y	30.25	30
4)	10/20/17 C147	147.00	3.00y	3.15y	3.01y	29.58	50
5)	10/20/17 C148	148.00	2.50y	2.59y	2.48y	29.26	211
6)	10/20/17 C149	149.00	2.03y	2.08y	2.05y	28.92	120
7)	10/20/17 C150	150.00	1.57y	1.64y	1.61y	27.97	269
8)	10/20/17 C152.5	152.50	.80y	.91y	.81y	27.53	61
9)	10/20/17 C155	155.00	.37y	.47y	.41y	27.33	1857
	17-Nov-17 (36g); VolC 100; IDiv 1.34 USD; R 1.24; IFwd 146.32						
10)	11/17/17 C130	130.00	15.70y	18.35y	16.10y		
11)	11/17/17 C135	135.00	11.45y	13.45y	14.07y		
12)	11/17/17 C140	140.00	8.45y	9.40y	8.45y	22.95	19
13)	11/17/17 C145	145.00	4.70y	4.80y	4.65y	18.82	78
14)	11/17/17 C150	150.00	1.98y	2.10y	2.06y	17.59	277
15)	11/17/17 C155	155.00	.65y	.74y	.70y	17.08	159
16)	11/17/17 C160	160.00	.19y	.23y	.19y	17.45	38
17)	11/17/17 C165	165.00	.05y	.08y	.12y	17.99	
18)	11/17/17 C170	170.00		.05y	.03y	20.97	7

	Put						
	Ticker	Strike	Bid	Ask	Ultimo	IVM	Volm
	20-Oct-17 (8g); VolC 100; R 1.20; IFwd 147.51						
54)	10/20/17 P144	144.00	1.30y	1.40y	1.38y	30.66	18
55)	10/20/17 P145	145.00	1.62y	1.70y	1.64y	30.51	1874
56)	10/20/17 P146	146.00	1.98y	2.05y	1.99y	30.14	38
57)	10/20/17 P147	147.00	2.36y	2.47y	2.49y	29.68	93
58)	10/20/17 P148	148.00	2.83y	2.95y	2.89y	29.36	706
59)	10/20/17 P149	149.00	3.35y	3.45y	3.40y	28.96	40
60)	10/20/17 P150	150.00	3.90y	4.05y	4.00y	28.15	40
61)	10/20/17 P152.5	152.50	5.50y	5.80y	5.70y	27.08	6
62)	10/20/17 P155	155.00	7.65y	7.85y	7.75y	27.21	23
	17-Nov-17 (36g); VolC 100; IDiv 1.34 USD; R 1.24; IFwd 146.3						
63)	11/17/17 P130	130.00	.30y	.36y	.36y	25.01	51
64)	11/17/17 P135	135.00	.61y	.68y	.69y	22.43	48
65)	11/17/17 P140	140.00	1.24y	1.38y	1.36y	20.12	61
66)	11/17/17 P145	145.00	2.72y	2.83y	2.81y	18.52	45
67)	11/17/17 P150	150.00	5.40y	5.50y	5.54y	17.83	49
68)	11/17/17 P155	155.00	8.60y	9.50y	8.20y	16.10	
69)	11/17/17 P160	160.00	12.00y	15.60y	14.86y	18.65	
70)	11/17/17 P165	165.00	16.35y	20.80y		16.87	
71)	11/17/17 P170	170.00	21.55y	26.00y		27.95	

Figure: Implied volatilities of "E-CORP US"

Remarks on Implied Volatility

- Option quotations can be given equivalently in terms of prices or of implied volatilities provided discount factors and forwards are given
- Until 1987 all prices could be reconstructed using Black's formula with the same volatility σ, but after the "Black Monday" stock market crash, Black-Scholes model did not predict option prices anymore
- Black model cannot explain option prices but rather gives a "quotation" mechanism

FX Market quotes

FX options have different conventions than equity options:

- options expiries are rolling (e.g. 1 week, 1 month, 1 year) instead of fixed (third Friday of the month)
- (domestic) discount factors are derived from **"IR" market**
- forward prices are derived from the **"FX swap" market**
- traders quote implied volatilities $\{\sigma(T,\Delta)\}$ of options with expiry T whose **Black's delta** is equal to a predetermined value Δ. This is the naming convention for quoted options:

$$\begin{aligned}
\text{"10D Put" option} \quad &\mapsto \quad \Delta = 90\% \\
\text{"25D Put" option} \quad &\mapsto \quad \Delta = 75\% \\
\text{"ATM" option} \quad &\mapsto \quad \Delta = 50\% \\
\text{"25D Call" option} \quad &\mapsto \quad \Delta = 25\% \\
\text{"10D Call" option} \quad &\mapsto \quad \Delta = 10\%
\end{aligned}$$

FX Market quotes (Discount Factors)

Term	Market Rate	Shift	Shifted Rate	Zero Rate	Discount
6 MO	0.56000	+0.00	0.56000	0.56000	0.997177
CKFR0A	0.74300	+0.00	0.74300	0.60050	0.996481
CKFR0B	0.74000	+0.00	0.74000	0.64091	0.995662
CKFR0CI	0.80000	+0.00	0.80000	0.66928	0.994954
CKFR0DJ	0.86500	+0.00	0.86500	0.70154	0.994114
CKFR0EK	0.90000	+0.00	0.90000	0.73131	0.993222
CKFR0F1	0.95000	+0.00	0.95000	0.75684	0.992384
CKFR011	1.15300	+0.00	1.15300	0.88957	0.986633
2 YR	0.98500	+0.00	0.98500	0.98607	0.980299
3 YR	1.13019	+0.00	1.13019	1.13265	0.966292
4 YR	1.23405	+0.00	1.23405	1.23799	0.951228
5 YR	1.31500	+0.00	1.31500	1.32056	0.935623
6 YR	1.37952	+0.00	1.37952	1.38668	0.919602
7 YR	1.43272	+0.00	1.43272	1.44146	0.903343
8 YR	1.48252	+0.00	1.48252	1.49315	0.886656
9 YR	1.52763	+0.00	1.52763	1.54029	0.869737
10 YR	1.57000	+0.00	1.57000	1.58492	0.852552
12 YR	1.64238	+0.00	1.64238	1.66189	0.818181

Figure: *CZK* discount factors

FX Market quotes (Forwards)

T	Date	Bid pt	Ask pt	Bid fwd	Ask fwd
ON	10/11/17	0.613	0.643	1.1804712	1.1805762
TN	10/12/17	0.625	0.645	1.1805355	1.1806375
SP	10/12/17	1.1806	1.1807	1.1806	1.1807
SN	10/13/17	0.610	0.640	1.1806610	1.1807640
1W	10/19/17	4.28	4.38	1.181028	1.181138
2W	10/26/17	8.55	8.67	1.181455	1.181567
3W	11/02/17	12.93	13.07	1.181893	1.182007
1M	11/13/17	19.76	19.90	1.182576	1.182690
2M	12/12/17	37.70	37.90	1.184370	1.184490
3M	01/12/18	63.06	63.31	1.186906	1.187031
4M	02/12/18	84.25	84.65	1.189025	1.189165
5M	03/12/18	103.35	103.85	1.190935	1.191085
6M	04/12/18	126.95	127.55	1.193295	1.193455
9M	07/12/18	195.04	196.17	1.200104	1.200317
1Y	10/12/18	266.41	267.87	1.207241	1.207487
15M	01/14/19	341.78	344.47	1.214778	1.215147
18M	04/12/19	412.14	417.36	1.221814	1.222436
2Y	10/15/19	561.93	566.93	1.236793	1.237393
3Y	10/13/20	847.50	857.50	1.265350	1.266450

Figure: EUR/USD forward prices

FX Market quotes (Plain Vanilla)

	ATM	25D Call EUR	25D Put EUR	10D Call EUR	10D Put EUR
Scd	Bid / Ask	Bid / Ask	Bid / Ask	Bid / Ask	Bid / Ask
1D	6.495 / 8.995	6.511 / 9.674	6.115 / 9.280	5.543 / 11.467	4.821 / 10.789
1W	6.170 / 6.990	6.445 / 7.475	5.995 / 7.025	6.482 / 8.358	5.681 / 7.559
2W	6.070 / 6.595	6.389 / 7.049	5.931 / 6.591	6.567 / 7.763	5.801 / 6.999
3W	7.170 / 7.560	7.521 / 8.011	7.064 / 7.554	7.769 / 8.656	6.968 / 7.857
1M	7.095 / 7.390	7.463 / 7.834	6.991 / 7.362	7.776 / 8.447	6.973 / 7.644
2M	6.760 / 6.985	7.136 / 7.419	6.701 / 6.984	7.533 / 8.045	6.775 / 7.287
3M	6.850 / 7.065	7.255 / 7.525	6.820 / 7.090	7.719 / 8.209	6.976 / 7.466
4M	6.990 / 7.200	7.392 / 7.656	6.999 / 7.263	7.878 / 8.357	7.213 / 7.692
5M	7.075 / 7.281	7.480 / 7.740	7.117 / 7.376	8.032 / 8.503	7.400 / 7.871
6M	7.140 / 7.345	7.556 / 7.814	7.206 / 7.464	8.147 / 8.615	7.545 / 8.013
9M	7.605 / 7.805	7.949 / 8.201	7.809 / 8.061	8.592 / 9.048	8.322 / 8.778
1Y	7.680 / 7.885	8.045 / 8.302	7.923 / 8.180	8.773 / 9.239	8.566 / 9.032
18M	7.920 / 8.175	8.242 / 8.563	8.187 / 8.508	8.896 / 9.476	8.814 / 9.394
2Y	8.065 / 8.365	8.342 / 8.718	8.342 / 8.718	8.946 / 9.629	8.941 / 9.624
3Y	8.420 / 8.920	8.619 / 9.246	8.724 / 9.351	9.111 / 10.249	9.321 / 10.459
4Y	8.815 / 9.325	8.985 / 9.625	9.130 / 9.770	9.428 / 10.589	9.776 / 10.937
5Y	9.105 / 9.615	9.239 / 9.879	9.436 / 10.076	9.683 / 10.844	10.061 / 11.222

Figure: EUR/USD implied volatility surface

FX Market quotes

- Concretely, for a given expiry T and delta Δ we have the following market data:
  - market value of discount factor $D_0(T)$
  - market value of forward $F_0(T)$
  - market implied volatility $\sigma(T, \Delta)$ of an option with expiry T and delta equal to Δ
- In order to have the price of the quoted option, we first have to compute its strike K, which is implicitly defined by

$$N(d_1(K)) = \Delta$$

where $d_1(K) = \left(\log \frac{F_0(T)}{K} + \sigma(T, \Delta)T/2\right) / \sigma(T, \Delta)\sqrt{T}$

FX Market quotes

- In general the equation $N(d_1(K)) = \Delta$ is solved numerically for K

$$K_\Delta = K(\Delta, F_0(T), \sigma(T, \Delta))$$

- Once K_Δ found, the price of the quoted option is given by

$$C_0(T, K_\Delta) = Bl(F_0(T), K_\Delta, T, \sigma(T, \Delta), D_0(T))$$

- Notice that for every expiry T the ATM case ($\Delta = 50\%$) is

$$K_{ATM} := K_{0.5} \simeq F_0(T)$$

Homework: find the exact value for K_{ATM}

FX Market quotes

- Things are actually a bit more complicated, for example, depending on the FX rate, the quotation may not be given for a value of the undiscounted delta $N(d_1)$, but rather for the discounted Delta $D_0(T)\,N(d_1)$.
- For a complete discussion on FX options see the book by Iain J. Clark, *Foreign Exchange Option Pricing: A Practitioner's Guide*, Wiley Finance (2010)

Sticky Delta vs Sticky Strike

What happens to equity/FX option prices during the day?

- Stock and FX spot prices change continuously when the relevant market is open

$$S(0) \to S(0+\delta t) = S(0) + \delta S$$

- The forward price changes accordingly

$$F_0(T) \to F_{0+\delta t}(T) = F_0(T) + \delta F$$

- Implied volatilities do not change very often; for small δt we observe

$$\begin{aligned} \sigma_0(T,K) &\to \sigma_{0+\delta t}(T,K) = \sigma_0(T,K) \qquad &\text{(equity mkt)} \\ \sigma_0(T,\Delta) &\to \sigma_{0+\delta t}(T,\Delta) = \sigma_0(T,\Delta) \qquad &\text{(FX mkt)} \end{aligned}$$

- Finally, assume for simplicity that $D_{0+\delta t}(T) = D_0(T)$

Sticky Strike

In the equity market, options are quoted for a predetermined set of strikes $\{K_i\}_i$ for each expiry T. So, when the spot price moves:

- price of quoted options has same volatility (but different forward)

$$C_{0+\delta t}(T, K_i) = Bl(F_{0+\delta t}(T), K_i, T, \sigma(T, K_i), D_0(T))$$

- ATM-spot option is priced with a different volatility

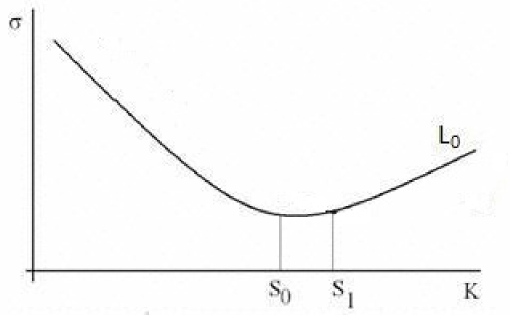
$$C_{0+\delta t}(T, S(0+\delta t)) = Bl(F_{0+\delta t}(T), S(0+\delta t), T, \sigma(T, S(0+\delta t)), D_0(T))$$

Sensitivity to the spot (delta) is given by

$$\frac{dC_0(T,K)}{dS(0)} = \frac{\partial Bl(F_0(T),\ldots)}{\partial F_0(T)} \frac{\partial F_0(T)}{\partial S(0)} = D_0(T) N(d_1) \frac{\partial F_0(T)}{\partial S(0)} = \Delta_{BS}$$

Sticky Strike

Here is a graphical representation of what happens to the smile when the spot moves



Sticky Delta

- If the FX spot moves then

$$X(0) \to X(0+\delta t)$$
$$F_0(T) \to F_{0+\delta t}(T)$$
$$\sigma(T,\Delta) \to \sigma(T,\Delta)$$

  but the strike K_Δ of a quoted option changes too!

$$K(\Delta, F_0(T), \sigma(T,\Delta)) \to K(\Delta, F_{0+\delta t}(T), \sigma(T,\Delta))$$

- For example the ATM option ($\Delta = 50\%$) has a new strike

$$K_{ATM} \simeq F_{0+\delta t}(T)$$

  but is priced with the same volatility

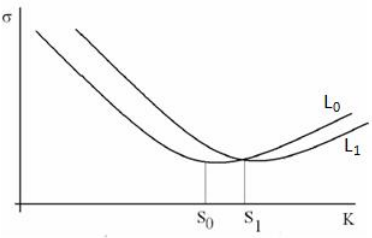
$$C_{0+\delta t}(T, K_{ATM}) \simeq Bl(F_{0+\delta t}(T), F_{0+\delta t}(T), T, \sigma(T, 50\%), D_0(T))$$

Sticky Delta

- More generally, the Δ-option, though corresponding to different strike, is priced with the same implied volatility

$$C_{0+\delta t}(T, K_\Delta) = Bl(F_{0+\delta t}(T), K(\Delta, F_{0+\delta t}(T)), T, \sigma(T, \Delta), D_0(T))$$

- In other words, when the FX spot moves, the implied volatility surface "translates" along the direction of the spot



Sticky Delta

- If on the contrary we fix a strike K, the option will be priced with a different volatility

$$C_0(T,K) = Bl(F_0(T), K, T, \sigma(T,K), D_0(T))$$
$$\Downarrow$$
$$C_{0+\delta t}(T,K) = Bl(F_{0+\delta t}(T), K, T, \sigma(T,K) + \delta\sigma, D_0(T))$$

- Sensitivity to the spot (delta)

$$\frac{dC_0(T,K)}{dS(0)} = \Delta_{BS} + \underbrace{\frac{\partial Bl(\ldots,\sigma(T,K),\ldots)}{\partial \sigma(T,K)} \cdot \frac{\partial \sigma(T,K)}{\partial S(0)}}_{\text{extra term due to delta-stickiness}}$$

Sticky Delta vs Sticky Strike

Remarks:

- We have seen that quoting options in strikes or deltas reflects the *stickyness* of market, namely how market implied volatility surfaces evolve in time
- Given a model, one has to perform similar analysis on model prices and determine if they have a *sticky-delta* or *sticky-strike* behaviour
- What happens if we calibrate a *sticky-delta* model to a *sticky-strike* market?

Course Outline

1. Market Volatility Surface
2. The Local Volatility Model
   - Stochastic Differential Equation
   - Kolmogorov equations
   - Dupire Equation
   - From Implied Volatility to Local Volatility
3. Local Volatility in practice
4. Pricing

The Local Volatility Model

- Black-Scholes model cannot reproduce market data; it can be just used as a quoting mechanism
- Can we find a SDE that explains all market quotes? and what about the sticky delta / sticky strike feature?
- Dupire and Derman-Kani proposed

$$dS(t) = S(t)(r(t) - q(t))dt + \varsigma(t, S(t))\, S(t)dW_t$$

where the **local volatility function** $\varsigma(t, s)$ is a deterministic function

Existence and uniqueness

- Suppose there exists a constant K such that

$$\begin{aligned} |(r(t)-q(t))| &\le K \\ |x\varsigma(t,x)-y\varsigma(t,y)| &\le K\,|x-y| \\ |x(r(t)-q(t))|+|x\varsigma(t,x)| &\le K(1+|x|) \end{aligned}$$

- Then there exists a unique (adapted, t-continuous) solution $S(t)$ to the Local Volatility SDE with starting point $S(0)=S_0$

We don't need full generality, so we make the following stronger assumptions on $\varsigma(t,x)$:

- $\exists \varsigma_{\max}$ such that $0 \le \varsigma(t,x) \le \varsigma_{\max}$, $\forall t \in \mathbb{R}_+, \forall t \in \mathbb{R}_+$
- $\exists C > 0$ such that $\left| x \frac{\partial \varsigma(t,x)}{\partial x} \right| \le C$, $\forall x \in \mathbb{R}_+, \forall t \in \mathbb{R}_+$

Model prices

Similarly to the Black case, one can show no-arbitrage/completeness. Therefore model price of a derivative with payoff $\Phi(S(T))$ is

$$\mathbb{E}\left[\frac{\Phi(S(T))}{e^{\int_0^T r(s)ds}}\right]$$

In particular

$$\begin{aligned} D_0(T)^{\mathrm{LV}} &= e^{-\int_0^T r(s)ds} \\ F_0(T)^{\mathrm{LV}} &= S(t)e^{\int_0^T (r(s)-q(s))ds} \\ C_0(T,K)^{\mathrm{LV}} &= \text{No closed formula available!} \end{aligned}$$

Model calibration

- Model calibration means setting **model parameters** $\mathbf{r}(\mathbf{t}), \mathbf{q}(\mathbf{t}), \varsigma(\mathbf{t}, \mathbf{s})$ so that model prices coincide with market prices

- We can choose $\mathbf{r}(\mathbf{t}), \mathbf{q}(\mathbf{t})$ so that model discount factors and forwards perfectly reproduce the market prices

$$D_0(T_i)^{\mathrm{LV}} = D_0(T_i), \qquad F_0(T_i)^{\mathrm{LV}} = F_0(T_i)$$

  for all market expiries $\{T_i\}_{i=1,\ldots,n}$ (use same formulas of page 21)

- In order to calibrate $\varsigma(\mathbf{t}, \mathbf{s})$ it is crucial to find an algorithm that computes $C_0(T, K)^{LV}$ for any T, K

The Local Volatility Model

- In order to simplify formulas we set

$$X(t) := \frac{S(t)}{F_0(t)^{\mathrm{LV}}}$$

- The SDE for X is

$$dX(t) = \eta(t, X(t))\, X(t) dW_t$$

where $\eta(t, X(t)) := \varsigma(t, S(t))$, with initial condition $X(0) = 1$

- If we set $k = K/F_0(T)^{\mathrm{LV}}$, then we have:

$$C_0(T, K)^{\mathrm{LV}} = F_0(T)^{\mathrm{LV}} D_0(T)^{\mathrm{LV}} c_0(T, k)^{\mathrm{LV}}$$

where

$$c_0(T, k)^{\mathrm{LV}} := \mathbb{E}\left[(X(T) - k)^+\right]$$

Kolmogorov equations

We recall few general results on SDE and PDE.

Suppose $Y(t)$ solves the following SDE

$$dY(t) = \mu(t, Y(t))dt + \zeta(t, Y(t))dW(t)$$

and denote by $p(y, T; x, t)$ the transition probability density of $Y(t)$, namely the probability of being at y at time T given that it started at x at time t.

Thus, given a function $\Phi(\cdot)$, the expectation at t of $\Phi(Y(T))$ is given by

$$\mathbb{E}_{Y(t)=x}\left[\Phi\left(Y(T)\right)\right] = \int_{\mathbb{R}} \Phi(y)p(y, T; x, t)dy$$

Kolmogorov equations

Let's apply Ito lemma to a function $v(t, Y(t))$

$$dv(t, Y(t)) = \zeta(t, Y(t))v_y(t, Y(t))dW(t) +$$
$$+ \left(v_t(t, Y(t)) + \mu(t, Y(t))v_y(t, Y(t)) + \frac{1}{2}\zeta(t, Y(t))^2 v_{yy}(t, Y(t)) \right) dt$$

or, in integral form

$$v(T, Y(T)) - v(t, Y(t)) = \int_t^T \zeta(s, Y(s))v_y(s, Y(s))dW(s) +$$
$$+ \int_t^T \left(v_t(s, Y(s)) + \mu(s, Y(s))v_y(s, Y(s)) + \frac{1}{2}\zeta(s, Y(s))^2 v_{yy}(s, Y(s)) \right) ds$$

Kolmogorov equations

By taking expectation at t on both sides we get

$$\mathbb{E}_{Y(t)=x}\left[v(T, Y(T))\right] - v(t, Y(t)) =$$
$$= \int_{\mathbb{R}} \int_t^T \left(v_t(s,y) + \mu(s,y) v_y(s,y) + \frac{1}{2}\zeta(s,y)^2 v_{yy}(s,y) \right) p(y,s;\, x,t) ds\, dy$$

Further, we can introduce the operator $\mathcal{L}v = \mu v_y + \frac{1}{2}\zeta^2 v_{yy}$, so that the equation becomes

$$\mathbb{E}_{Y(t)=x}\left[v(T, Y(T))\right] - v(t, Y(t)) =$$
$$= \int_{\mathbb{R}} \int_t^T \left(v_t + \mathcal{L}v \right)(s,y) p(y,s;\, x,t) ds\, dy \quad (1)$$

Kolmogorov equations

Consider now a payoff function $\Phi(\cdot)$, and set

$$u(t,x) := E_{Y(t)=x}\left[\Phi(Y(T))\right]$$

Eq. (1) holds for any v. In particular for $v = u$ one deduces that $u(t,x)$ solves the following PDE, called the **Kolmogorov Backward Equation**

$$\begin{cases} (u_t + \mathcal{L}u)(t,x) = 0 & \forall t < T \\ u(T,x) = \Phi(x) \end{cases}$$

Similarly one can show that the discounted version

$$\bar{u}(t,x) := E_{Y(t)=x}\left[e^{-\int_t^t r(s,Y(s))ds}\Phi(Y(T))\right]$$

for some $r(t, Y(t))$, solves instead

$$\bar{u}_t(t,x) + \mu(t,x)\bar{u}_x(t,x) + \frac{1}{2}\zeta(t,x)^2\ \bar{u}_{xx}(t,x) - r(t,x)\bar{u}(t,x) = 0 \quad \forall t < T$$

Kolmogorov equations

With similar arguments, we can derive a PDE for the probability density p. In fact, if we integrate by parts eq. (1) we get

$$\mathbb{E}_{Y(t)=x}\left[v(T, Y(T))\right] - v(t, Y(t)) = \\ = \int_{\mathbb{R}} \int_t^T v(s, y) \left(-p_t + \mathcal{L}^* p\right)(y, s;\, x, t) ds\, dy$$

where $\mathcal{L}^* p = -\frac{\partial}{\partial y}(\mu p) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\zeta^2 p)$. Again, this equation holds for any choice of v, but notice that the l.h.s. only depends on v at times t and T, while the r.h.s. depends on v at every time. Therefore one can conclude that $(-p_t + \mathcal{L}^* p)$ must vanish.

Kolmogorov equations

Assume that at time $t = 0$ the process starts at y_0 and, in order to simplify notations, set $p(t, y) := p(y, t;\, y_0, 0)$.

We have thus shown that $p(t, y)$ solves the so-called **Forward Kolmogorov Equation** or *Fokker-Plank Equation*

$$\begin{cases} \left(-p_t + \mathcal{L}^* p\right)(t, y) = 0 & \forall t > 0 \\ p(0, y) = \delta(y - y_0) & \end{cases}$$

where

$$\left(\mathcal{L}^* p\right)(t, y) = -\frac{\partial}{\partial y}\Big(\mu(t, y) p(t, y)\Big) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\Big(\zeta(t, y)^2 p(t, y)\Big)$$

The Local Volatility Model

Now we can apply these result to our Local Volatility process

$$dX(t) = \underbrace{\eta(t, X(t))\, X(t)}_{\zeta(t, X(t))} dW_t$$

- The distribution of $X(T)$, namely the transition probability $P_T(B) = \mathbb{Q}(X(T) \in B | X(0) = 1)$, has **transition density** $p(T, x)$

$$P_T(B) = \int_B p(T, x) dx$$

- The time evolution of $p(t, x)$ is given by the **forward Kolmogorov equation**

$$\frac{\partial p(t, x)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[\eta(t, x)^2 x^2 p(t, x) \right]$$

with initial condition $p(0, x) = \delta(x - 1)$

Dupire Equation

- The relation between $c_0(T,k)^{LV}$ and $p(T,x)$ is known in general for all models that have transition densities:

$$c_0(T,k)^{LV} = \mathbb{E}[(X_T - k)^+] = \int_0^\infty (x-k)^+ p(T,x)dx = \\ = \int_k^\infty (x-k)p(T,x)dx$$

and, by differentiating twice w.r.t. the strike

$$\frac{\partial^2}{\partial^2 k} c_0(T,k)^{LV} = p(T,k)$$

- *This result is very important: it tells that the set of vanilla option prices contains all information regarding the transition probabilities from present day and present spot to whatever future time and future spot*

Dupire Equation

If we differentiate w.r.t. T and apply Kolmogorov equation, we have

$$\frac{\partial}{\partial T} c_0(T,k)^{LV} = \int_k^\infty (x-k)\,\frac{\partial}{\partial T} p(T,x)dx =$$

$$= \int_k^\infty (x-k)\,\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\eta(T,x)^2 x^2 p(T,x)\right] dx =$$

$$= \left[(x-k)\,\frac{1}{2}\frac{\partial}{\partial x}\left[\eta(T,x)^2 x^2 p(T,x)\right]\right]_{x=k}^{x=\infty} - \int_k^\infty \frac{1}{2}\frac{\partial}{\partial x}\left[\eta(T,x)^2 x^2 p(T,x)\right] =$$

$$= -\frac{1}{2}\left[\eta(T,x)^2 x^2 p(T,x)\right]_{x=k}^{x=\infty} = \frac{1}{2}\eta(T,k)^2 k^2 p(T,k) =$$

$$= \frac{1}{2}\eta(T,k)^2 k^2 \frac{\partial^2}{\partial^2 k}\, c_0(T,k)^{LV}$$

Dupire Equation

- We have proven that the model price function $c_0(T,K)^{LV}$ for call options on $X(t)$ satisfies **Dupire equation**

$$\frac{\partial}{\partial T} c_0(T,k)^{LV} = \frac{1}{2}\eta(T,k)^2 k^2 \frac{\partial^2}{\partial^2 k} c_0(T,k)^{LV}$$

with initial condition $c_0(0,k)^{LV} = (1-k)^+$

- As for the model prices of call options on $S(t)$

$$C_0(T,K)^{LV} = D_0(T)^{LV} F_0(T)^{LV} c_0(T,k)^{LV} \qquad k = K/F_0(T)^{LV}$$

- Notice that solving this (forward) differential equation yields all model prices $C_0(T,K)^{LV}$ for all T and K!

Dupire Equation

- We have proven that the model price function $c_0(T, K)^{LV}$ for call options on $X(t)$ satisfies **Dupire equation**

$$\frac{\partial}{\partial T} c_0(T, k)^{LV} = \frac{1}{2}\eta(T, k)^2 k^2 \frac{\partial^2}{\partial^2 k} c_0(T, k)^{LV}$$

with initial condition $c_0(0, k)^{LV} = (1 - k)^+$

- As for the model prices of call options on $S(t)$

$$C_0(T, K)^{LV} = D_0(T)^{LV} F_0(T)^{LV} c_0(T, k)^{LV} \qquad k = K / F_0(T)^{LV}$$

- Notice that solving this (forward) differential equation yields all model prices $C_0(T, K)^{LV}$ for all T and K!

Dupire Equation

- We have proven that the model price function $c_0(T,K)^{LV}$ for call options on $X(t)$ satisfies **Dupire equation**

$$\frac{\partial}{\partial T}c_0(T,k)^{LV} = \frac{1}{2}\eta(T,k)^2 k^2 \frac{\partial^2}{\partial^2 k}c_0(T,k)^{LV}$$

with initial condition $c_0(0,k)^{LV} = (1-k)^+$

- As for the model prices of call options on $S(t)$

$$C_0(T,K)^{LV} = D_0(T)^{LV} F_0(T)^{LV} c_0(T,k)^{LV} \qquad k = K/F_0(T)^{LV}$$

- Notice that solving this (forward) differential equation yields all model prices $C_0(T,K)^{LV}$ for all T and K!

From option prices to local volatility function

Suppose we have a continuous set of market prices $D_0(T)$, $F_0(T)$, $C_0(T,K)$, for any $T>0$ and $K>0$, and define the corresponding market prices $c_0(T,k)$ for the normalized asset $X(t)=\frac{S(t)}{F_0(t)}$ by the relation

$$C_0(T,K)=D_0(T)F_0(T)c_0(T,k) \qquad k:=K/F_0(T)$$

If $c_0(T,k)$ is a differentiable function of T and k, then we can **calibrate the model** by setting

$$\eta(T,k)^2=\frac{\frac{\partial}{\partial T}c_0(T,k)}{\frac{1}{2}k^2\frac{\partial^2}{\partial^2 k}c_0(T,k)} \tag{2}$$

From option prices to local volatility function

Suppose we have a continuous set of market prices $D_0(T)$, $F_0(T)$, $C_0(T, K)$, for any $T > 0$ and $K > 0$, and define the corresponding market prices $c_0(T, k)$ for the normalized asset $X(t) = \frac{S(t)}{F_0(t)}$ by the relation

$$C_0(T, K) = D_0(T)F_0(T)c_0(T, k) \qquad k := K/F_0(T)$$

If $c_0(T, k)$ is a differentiable function of T and k, then we can **calibrate the model** by setting

$$\eta(T, k)^2 = \frac{\frac{\partial}{\partial T}\, c_0(T, k)}{\frac{1}{2} k^2 \frac{\partial^2}{\partial^2 k}\, c_0(T, k)} \tag{2}$$

From option prices to local volatility function

- Match $D_0(T)$, $F_0(T)$ by choosing $r(t), q(t)$ as on page 56)
- By choosing $\eta(T, k)$ as in equation (2) then by Dupire equation we have $c_0(T, k)^{LV} = c_0(T, k)$ and hence

$$C_0(T, K)^{LV} = C_0(T, K)$$

- Market prices $c_0(T, k)$ must form a differentiable function. So if we start from real market data (which are a finite set!), we need an interpolator to build $c_0(T, k)$
- Actually not every $c_0(T, k)$ gives rise to a "valid" LV model; $c_0(T, k)$ must satisfy additional "no-arbitrage constraints"

No-arbitrage conditions on option prices

No-arbitrage theorem: a smooth price function $c_0(T,k)$ is arbitrage-free if and only if it satisfies the following constraints:

- bounds for the call price are

$$(1-k)^+ \leq c_0(T,k) \leq 1$$

- in order to avoid strike arbitrage we must have

$$-1 \leq \frac{\partial}{\partial k} c_0(T,k) \leq 0 \qquad \frac{\partial^2}{\partial^2 k} c_0(T,k) > 0$$

- in order to avoid calendar arbitrage we must have

$$\frac{\partial}{\partial T} c_0(T,k) \geq 0$$

From implied volatility to local volatility

- How can we write $c_0(T, k)$ in terms of market volatilities?
- We know that option prices for $S(T)$ can be equivalently expressed in terms of an implied volatility function $\sigma_{mkt}(T, K)$

$$C_0(T, K) = Bl(F_0(T), K, T, \sigma_{mkt}(T, K), D_0(T))$$

- The corresponding market price for the normalized asset $X(t) = \frac{S(t)}{F_0(t)}$ is given by the relation

$$C_0(T, K) = D_0(T)F_0(T)c_0(T, k)$$

From implied volatility to local volatility

- We would like to write market prices $c_0(T,k)$ in terms of $\sigma_{mkt}(T,K)$, namely the market implied volatilities of $S(T)$
- We can write

$$\begin{aligned} C_0(T,K) &= Bl(F_0(T),K,T,\sigma_{mkt}(T,K),D_0(T)) = \\ &= D_0(T)\,\left(F_0(T)N(d_1) - KN(d_2)\right) = \\ &= D_0(T)F_0(T)\,\left(N(d_1) - K/F_0(T)N(d_2)\right) = \\ &= D_0(T)F_0(T)\;Bl(1,K/F_0(T),T,\sigma_{mkt}(T,K),1) \end{aligned}$$

- In other words, we have

$$c_0(T,k) = Bl(1,k,T,\Sigma_{mkt}(T,k),1)$$

where $k = \frac{K}{F_0(T)}$ and $\Sigma_{mkt}(T,k) = \sigma_{mkt}(T,K)$

From implied volatility to local volatility

- Can we write the equation $\eta(T,k)^2 = \frac{\frac{\partial}{\partial T} c_0(T,k)}{\frac{1}{2}k^2 \frac{\partial^2}{\partial^2 k} c_0(T,k)}$ in terms of $\Sigma_{mkt}(T,k)$ instead of $c_0(T,k)$?

- One can compute $\frac{\partial^2}{\partial^2 k} c_0(T,k)$ and $\frac{\partial}{\partial T} c_0(T,k)$ by taking derivatives of Black's formula and plug into Dupire's equation

- The result of these tedious computations is

$$\eta^2(T,k) = \frac{\Sigma^2 + 2\Sigma T \frac{\partial \Sigma}{\partial T}}{\left(1 - \frac{ky}{\Sigma}\frac{\partial \Sigma}{\partial k}\right)^2 + k\Sigma T\left(\frac{\partial \Sigma}{\partial k} - \frac{1}{4}k\Sigma T\left(\frac{\partial \Sigma}{\partial k}\right)^2 + k\left(\frac{\partial^2 \Sigma}{\partial k^2}\right)\right)}$$

where $\Sigma = \Sigma_{mkt}(T,k)$, $y = \log(k)$

From implied volatility to local volatility

- We found a formula that allows a perfect calibration of the local volatility model to any **smooth** implied volatility surface
- In practice market implied volatilities are a **discrete** set of quotes $\{\sigma_{mkt}(T_i, K_{i,j})\}_{i,j}$, not a surface!
- In order to use the previous formula one has to interpolate $c_0(T, k)$ or $\sigma_{mkt}(T, K)$ on market quotes
- Problems:
    - $\eta(t, k)$ is very sensitive to the choice of the interpolation
    - calibration is instable, small change on market quotes produces big changes on $\eta(T, k)$

Course Outline

1. Market Volatility Surface
2. The Local Volatility Model
3. Local Volatility in practice
   - Parametrization of the Local Volatility Surface
   - Numerical Solution of Partial Differential Equations
   - Discretization Schemes
   - Calibration
4. Pricing

Parametrization of the Local Volatility Surface

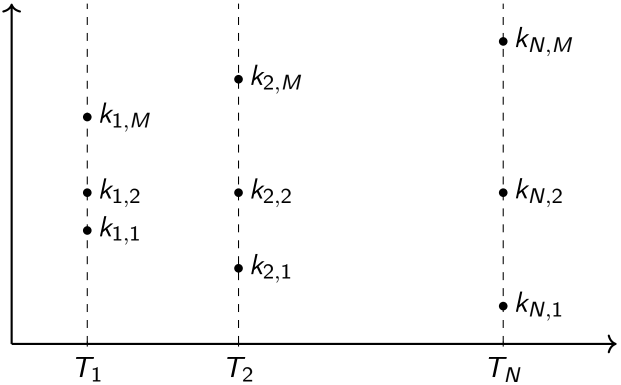
Here is the strategy to calibrate a local volatility model to a discrete set of market prices, without interpolating market data:

- find a parametrization of the local volatility function η in terms of a finite number of parameters $\{v_{i,j}\}$
- find a numerical algorithm that computes model prices as a function of η and hence of the parameters $\{v_{i,j}\}$
- using a best-fit procedure, choose the parameters $\{v_{i,j}\}$ so that model prices match a given set of market prices

Parametrization of the Local Volatility Surface

- We choose to parametrize the local volatility function on a **bi-dimensional grid** $\{k_{i,j}\}$ for $i = 1, \ldots, N$, $j = 1, \ldots, M$ in time $\{T_1, \ldots, T_N\}$ and in normalized strike price $\{k_{i,j}\}$
- For each of these points $k_{i,j}$ we denote the value of the local volatility function as $v_{i,j} = \eta(T_i, k_{i,j})$

Parametrization of the Local Volatility Surface



The LV function at these nodes is determined by $\{v_{i,j} = \eta(T_i, k_{i,j})\}_{i,j}$

Parametrization of the Local Volatility Surface

- Having set $v_{i,j}$ as the value of the LV function at the node $k_{i,j}$, we define the value for a generic point by choosing an interpolation and extrapolation scheme f, so that

$$\eta(T,k) = f\left(\{v_{i,j}\}\right)$$

- The easiest choice is **piecewise constant in time** and **linear in strike with flat extrapolation**, but other choices are possible, e.g., spline interpolation
- Once the interpolation/extrapolation and the nodes nodes $\{k_{i,j}\}$ are fixed, the function η is **completely determined by a finite number of parameters**, namely by the matrix $\{v_{i,j}\}$

Parametrization of the Local Volatility Surface

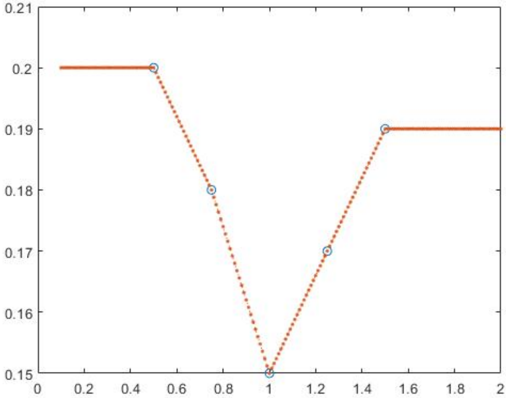


Figure: Function $\eta(T,\cdot)$ using linear interpolation and flat extrapolation

Parametrization of the Local Volatility Surface

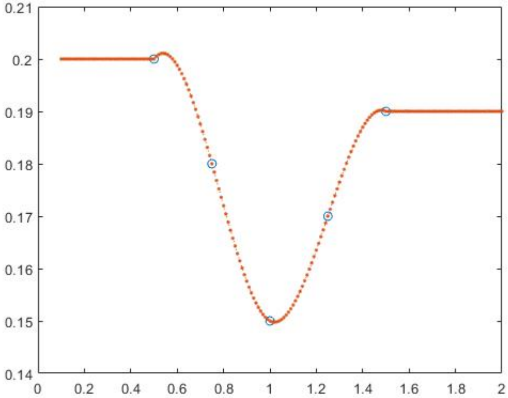


Figure: Function $\eta(T, \cdot)$ using spline interpolation and flat extrapolation

Parametrization of the Local Volatility Surface

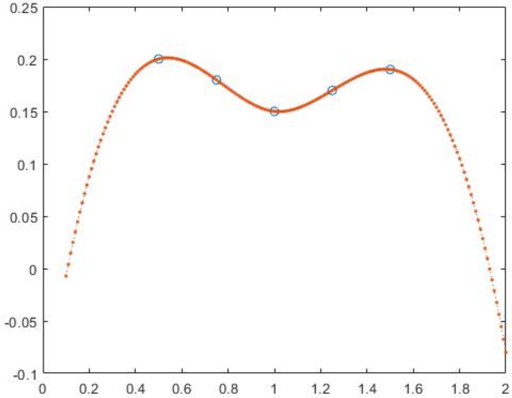
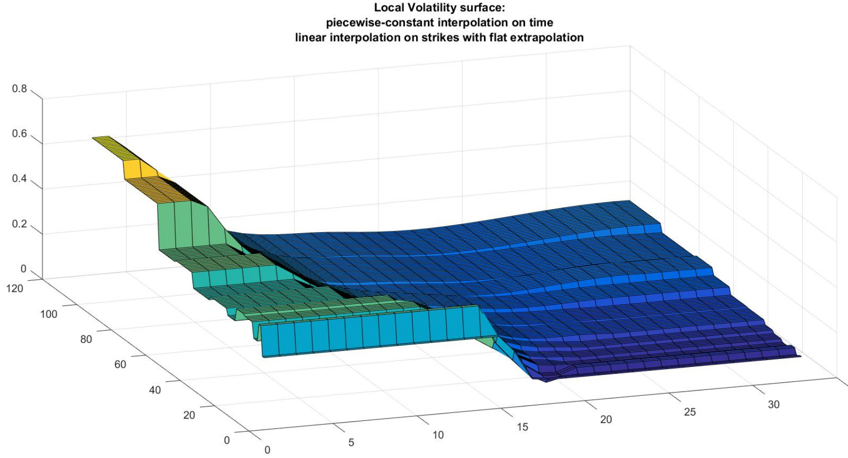


Figure: Function $\eta(T, \cdot)$ using Matlab spline interpolation

Parametrization of the Local Volatility Surface



Numerical Solution of Partial Differential Equations

- Now we need an algorithm to compute, at least approximatively, the model price of call options as a function of the model parameter η
- Given the parametrization of η as seen before, we will have model prices of call options as function of the matrix $\{v_{i,j}\}$
- The strategy is to solve **numerically** Dupire equation

$$\frac{\partial}{\partial T} c_0(T,k)^{LV} = \frac{1}{2}\eta(T,k)^2 k^2 \frac{\partial^2}{\partial^2 k} c_0(T,k)^{LV}$$

$$c_0(0,k)^{LV} = (1-k)^+$$

Truncation of the computational domain

- If we want to solve Dupire equation to find $c_0(T,k)^{LV}$, then the domain is $[0,T] \times \mathbb{R}_+$
- However, since we are going to find a solution using a numerical method, we need to **truncate the computational domain** to $[0,T] \times [0,k_{max}]$
- A call with $k \gg 1$ has price almost zero, so we enforce the following boundary condition on the artificial boundary $k = k_{max}$:

$$c_0(t,k_{max})^{LV} = 0 \quad \forall t \in [0,T]$$

- One can prove that the error introduced by truncation goes to zero as $k_{max} \to \infty$, and in any case k_{max} should be big enough so that the approximate solution does not depend on k_{max}

Change of variable

- A convenient change of variable is $h = log(k)$. Then Dupire's equation becomes

$$\left(\frac{\partial}{\partial T} + \frac{1}{2}\bar{\eta}(T,h)^2\left(\frac{\partial}{\partial h} - \frac{\partial^2}{\partial h^2}\right)\right)\bar{c}_0(T,h)^{LV} = 0$$

$$\bar{c}_0(0,h)^{LV} = (1-e^h)^+$$

where $\bar{\eta}(t,h) = \eta(t,e^h)$ and $\bar{c}_0(T,h)^{LV} = c_0(T,e^h)^{LV}$

- Working with log-strikes, we have to further restrict the domain to $[0,T]\times[h_{min},h_{max}]$ where $h_{max} = \log(k_{max})$, $h_{min} = -h_{max}$

- As for the boundary conditions, we assume

$$\bar{c}_0(t,h_{min})^{LV} = 1 - e^{h_{min}}, \quad \bar{c}_0(t,h_{max})^{LV} = 0 \qquad \forall t \in [0,T]$$

Finite Difference Operators

Consider a smooth function $u : I \subset \mathbb{R} \to \mathbb{R}$ and $x, x+h \in I$. In order to approximate $u'(x) = \frac{du(x)}{dx}$ we can use the following formulas

$$D_h^+ u(x) = \frac{u(x+h) - u(x)}{h} \qquad \text{forward f.d.}$$

$$D_h^- u(x) = \frac{u(x) - u(x-h)}{h} \qquad \text{backward f.d.}$$

$$D_h^c u(x) = \frac{u(x+h/2) - u(x-h/2)}{h} \qquad \text{centered f.d.}$$

while the second derivative $u^{(2)}(x) = \frac{d^2 u(x)}{dx^2}$ is approximated by

$$D_h^2 u(x) = D_h^- D_h^+ u(x) = D_h^c D_h^c u(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

Finite Difference Operators

Using Taylor's formula is easy to give an estimate of these approximations:

$$\left|D_h^{\pm}u(x) - u'(x)\right| \leq \frac{h}{2}||u^{(2)}||_{\infty}$$

$$\left|D_h^{c}u(x) - u'(x)\right| \leq \frac{h^2}{24}||u^{(3)}||_{\infty}$$

$$\left|D_h^{2}u(x) - u^{(2)}(x)\right| \leq \frac{h^2}{12}||u^{(4)}||_{\infty}$$

Notice that $D_h^{\pm}u(x)$ is a first order approximation of $\frac{du(x)}{dx}$, while $D_h^{c}u(x)$ and $D_h^{2}u(x)$ are second order approximations of $\frac{du(x)}{dx}$ and $\frac{d^2u(x)}{dx^2}$ respectively

Computational Grid

We divide the domain into sub-intervals:

- $[h_{min}, h_{max}]$ is divided into $L_h + 1$ sub-intervals of equal length $\Delta h = (h_{max} - h_{min}))/(L_h + 1)$
- $[0, T]$ in L_t sub-intervals of equal length $\Delta t = T/L_t$

and introduce the uniform grid

$$
\begin{aligned}
h_j &= h_{min} + j\Delta h & j &= 0, \ldots, L_h + 1 \\
t_i &= i\Delta t & i &= 0, \ldots, L_t
\end{aligned}
$$

Remark: we use a uniform grid for simplicity but one can also consider non-uniform grids

Computational Grid

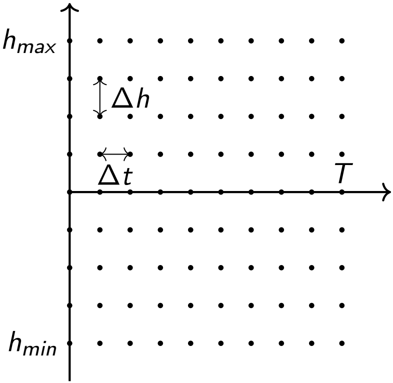


Figure: Uniform grid in $[h_{min}, h_{max}] \times [0, T]$

Forward Euler Method

At each point (t_i, h_j) of the grid we approximate derivatives:

$$\begin{aligned} \frac{\partial \bar{c}_0(t_i, h_j)}{\partial T} &\quad\rightarrow\quad D^+_{\Delta t}\bar{c}_0(t_i, h_j) \\ \frac{\partial \bar{c}_0(t_i, h_j)}{\partial h} &\quad\rightarrow\quad D^c_{\Delta h}\bar{c}_0(t_i, h_j) \\ \frac{\partial^2 \bar{c}_0(t_i, h_j)}{\partial h^2} &\quad\rightarrow\quad D^2_{\Delta h}\bar{c}_0(t_i, h_j) \end{aligned}$$

This is the so-called **Forward Euler Method** since time derivative is approximated with $D^+_{\Delta t}$.

Boundary conditions are:

$$\begin{aligned} \bar{c}_0(0, h_j) &= (1 - e^{h_j})^+ & j &= 0, \ldots, L_h + 1 \\ \bar{c}_0(t_i, h_{min}) &= 1 - e^{h_{min}} & i &= 0, \ldots, L_t \\ \bar{c}_0(t_i, h_{max}) &= 0 & i &= 0, \ldots, L_t \end{aligned}$$

Forward Euler Method

We denote by $c_{i,j}$ the (discretized) approximation of the exact solution $\bar{c}_0(t_i, h_j)$, so that the approximate Dupire equation becomes

$$\frac{c_{i+1,j} - c_{i,j}}{\Delta t} = \\ -\frac{1}{2}\bar{\eta}^2(t_{i+1}, h_j)\left(\frac{c_{i,j+1} - c_{i,j-1}}{2\Delta h} - \frac{c_{i,j+1} - 2c_{i,j} + c_{i,j-1}}{\Delta h^2}\right)$$

for the internal points $i = 0, \ldots, L_t - 1$ and $j = 1, \ldots, L_h$

with boundary conditions

$$\begin{aligned} c_{0,j} &= (1 - e^{h_j})^+ && j = 0, \ldots, L_h + 1 \quad \text{init condition} \\ c_{i,0} &= 1 - e^{h_0} && i = 0, \ldots, L_t \quad \text{left boundary} \\ c_{i,L_h+1} &= 0 && i = 0, \ldots, L_t \quad \text{right boundary} \end{aligned}$$

Forward Euler Method

Let us write the discretized equation in matrix form for each time index $i = 0, \ldots, L_t - 1$:

$$\frac{1}{\Delta t}\left(C_{i+1} - C_i\right) = -A_i C_i$$

where $C_i = (c_{i,0}, \ldots, c_{i,L_h+1})^t$ is a column vector, and A_i is a tri-diagonal $(L_h + 1) \times (L_h + 1)$ matrix

$$A_i = \begin{bmatrix} 0 & 0 & 0 & & & & \\ \alpha_{i,1} & \beta_{i,1} & \gamma_{i,1} & & & & \\ 0 & \alpha_{i,2} & \beta_{i,2} & \gamma_{i,2} & & & \\ & & \alpha_{i,3} & \ddots & \ddots & & \\ & & & \ddots & & & \\ & & & & \alpha_{i,L_h+1} & \beta_{i,L_h+1} & \gamma_{i,L_h+1} \\ & & & & 0 & 0 & 0 \end{bmatrix}$$

Forward Euler Method

$$
-A_i C_i = -\begin{bmatrix} 0 & 0 & 0 & & & \\ \alpha_{i,1} & \beta_{i,1} & \gamma_{i,1} & & & \\ 0 & \alpha_{i,2} & \beta_{i,2} & \gamma_{i,2} & & \\ & & \alpha_{i,3} & \ddots & \ddots & \\ & & & \ddots & & \end{bmatrix} \begin{bmatrix} c_{i,0} \\ \vdots \\ c_{i,1} \\ \vdots \\ c_{i,L_h+1} \end{bmatrix} =
$$

$$
= -\begin{bmatrix} 0 \\ \vdots \\ c_{i,j-1}\alpha_{i,j} + c_{i,j}\beta_{i,j} + c_{i,j+1}\gamma_{i,j} \\ \vdots \\ 0 \end{bmatrix}
$$

Forward Euler Method

It is easy to compute the coefficients of A_i

$$\alpha_{i,j} = \frac{1}{2}\eta_{i+1,j}\left(-\frac{1}{2\Delta h} - \frac{1}{\Delta h^2}\right)$$
$$\beta_{i,j} = \frac{1}{2}\eta_{i+1,j}\left(\frac{2}{\Delta h^2}\right)$$
$$\gamma_{i,j} = \frac{1}{2}\eta_{i+1,j}\left(\frac{1}{2\Delta h} - \frac{1}{\Delta h^2}\right)$$

where $\eta_{i+1,j} = \bar{\eta}(t_{i+1}, h_j)$

Forward Euler Method

- The solution C_{i+1} at time t_{i+1} is obtained from the solution C_i at time t_i by setting (**explicit method**)

$$C_{i+1} = (\mathbf{1} - \Delta t A_i)\, C_i$$

- Does the approximate solution C_i **converge** to the exact solution $C(t_i) = (\bar{c}_0(t_i, h_1), \ldots, \bar{c}_0(t_i, h_{L_h}))$? The question in other term is if there exists a constant such that

$$\max_{i=0,\ldots,L_t} ||C_i - C(t_i)||_\infty \le const\left(\Delta t + \Delta h^2\right)$$

- This is true but only if Δt **is small enough**, namely

$$\Delta t \le \frac{\Delta h^2}{||\eta||_\infty}$$

If this condition is not satisfied, C_i "explodes" and has nothing to do with the exact solution!

Stability of Forward Euler Method

In order to show convergence, we define

$$e_{i,j} = c_{i,j} - c_0(t_i, h_j)$$

and apply Dupire f.d. operators on both sides

$$\left(D^+_{\Delta_t} + \frac{1}{2}\eta_{i+1,j} \left(D^c_{2\Delta_h} - D^c_{\Delta_h^2} \right) \right) e_{i,j} = 0 + \epsilon_{i,j}$$

since $c_{i,j}$ satisfies discretized equation. Moreover $\epsilon_i = (\epsilon_{i,1}, \ldots, \epsilon_{i,L_h})^t$ satisfies $||\epsilon_i||_\infty \leq C_\epsilon(\Delta t + \Delta h^2)$ for some constant $C_\epsilon > 0$.

Define $E_i = (e_{i,1}, \ldots e_{i,L_h})$. We want to show that

$$\Delta t \leq \frac{\Delta h^2}{||\eta||_\infty} \quad \Rightarrow \quad ||E_{i+1}||_\infty \leq ||E_i||_\infty \text{ for } \Delta h \to 0$$

Stability of Forward Euler Method

The proof goes as follows:

- if $\Delta t \leq \frac{\Delta h^2}{||\eta||_\infty}$, then $1 - \Delta t \beta_{i,j} = 1 - \Delta t \frac{\eta_{i+1,j}}{\Delta h^2} \geq 0$
- since moreover $\alpha_{i_j} < 0$ and $\gamma_{i,j} < 0$ we have:

$$
\begin{aligned}
&|e_{i+1,j}| \leq \\
&\quad \leq |(1 - \Delta t \beta_{i,j})| \, |e_{i,j}| + |\alpha_{i,j}| \, |e_{i,j-1}| + |\gamma_{i,j}| \, |e_{i,j+1}| + |\epsilon_{i,j}| \leq \\
&\quad\quad \leq \underbrace{|1 - \Delta t \beta_{i,j} - \alpha_{i,j} - \gamma_{i,j}|}_{1} \, ||E_i||_\infty + ||\epsilon_i||_\infty
\end{aligned}
$$

- hence

$$||E_{i+1}||_\infty \leq ||E_i||_\infty + C_\epsilon(\Delta t + \Delta h^2)$$

This shows convergence of the approximate to the exact solution since by construction $E_0 = 0$ and hence $||E_{L_t}||_\infty \to 0$

Stability of Forward Euler Method

An analysis of the eigenvalues of A also shows that a small Δt is also a necessary condition. Suppose in fact that λ_{min} is the minimum eigenvalue of A and denote by E_{min} its eigenvector, so that

$$(1 - \Delta t A_i)E_{min} = (1 - \Delta t \lambda_{min})E_{min}$$

It is easy to see that $\lambda_{min} \sim 1/\Delta h^2$. If the ratio $\Delta t/\Delta h^2$ is not small enough, then $|1 - \Delta t \lambda_{min}| \geq const > 1$, and hence

$$||E_{L_t}||_\infty \geq |1 - \Delta t \lambda_{min}|^{L_t} ||E_{min}||_\infty$$

which diverges for $\Delta t \to 0$

Backward Euler Method

The backward Euler scheme overcomes the instability problem using a **backward** finite difference approximation of the time derivative

$$\frac{\partial \bar{c}_0(t_i, h_j)}{\partial T} \quad \rightarrow \quad D^-_{\Delta t}\bar{c}_0(t_i, h_j)$$

In the internal points $i = 0, \ldots, L_t - 1$ and $j = 1, \ldots, L_h$ we have

$$\frac{c_{i+1,j} - c_{i,j}}{\Delta t} = \\ -\frac{1}{2}\bar{\eta}^2(t_{i+1}, h_j)\left(\frac{c_{i+1,j+1} - c_{i+1,j-1}}{2\Delta h} - \frac{c_{i+1,j+1} - 2c_{i+1,j} + c_{i+1,j-1}}{\Delta h^2}\right)$$

while boundary conditions remain the same

Backward Euler Method

- In matrix form the Backward Euler Method gives rise to the **implicit** scheme

$$\frac{1}{\Delta t}\left(C_{i+1} - C_i\right) = -A_{i+1} C_{i+1}$$

which can be written as a linear system of equation

$$\left(\mathbf{1} + \Delta t A_{i+1}\right) C_{i+1} = C_i$$

- Notice that $\mathbf{1} + \Delta t A_i$ is a tri-diagonal matrix and hence can be easily inverted.
- The solution C_i obtained with the backward Euler method converges to the exact solution $\bar{c}_0(t_i, h)$ **without any restriction on $\boldsymbol{\Delta}$t**:

$$\max_{i=0,\dots,L_t} ||C_i - C(t_i)||_\infty \leq const\left(\Delta t + \Delta h^2\right)$$

Stability of Backward Euler Method

The backward method always give a bounded solution since, by analyzing the minimum eigenvalue λ_{min} of A, and exploiting the fact that η is bounded, one can easily prove that

$$||V||^2 \leq V^t(1 + \Delta t A_i)V$$

for any vector V. In particular

$$||E_{i+1}||^2 \leq (E_{i+1})^t(1 + \Delta t A_i)E_{i+1} = (E_{i+1})^t(E_i + \epsilon_{i+1})$$

and hence

$$||E_{i+1}|| \leq ||E_i|| + C_\epsilon(\Delta t + \Delta h^2)$$

which shows that the distance between approximate and exact solution converges to zero

Crank-Nicolson method

The Crank-Nicolson method is a linear combinations of the forward and backward method

$$C_{i+1} = C_i + \frac{1}{2}\Delta t\left(-A_{i+1}C_{i+1}\right) + \frac{1}{2}\Delta t\left(-A_i C_i\right)$$

or

$$\left(\mathbf{1} + \frac{1}{2}\Delta t A_{i+1}\right) C_{i+1} = \left(\mathbf{1} - \frac{1}{2}\Delta t A_i\right) C_i$$

It turns out that Crank-Nicolson method is a **2nd order scheme**, namely there exists a constant such that

$$\max_{i=0,\ldots,L_t} \|C_i - C(t_i)\|_\infty \leq const\left(\Delta t^2 + \Delta h^2\right)$$

without restrictions on Δt.

Numerical Solitution of Dupire Equation

- Summarizing, independently from the chosen numerical scheme, we find iteratively C_{i+1} from C_i starting from C_0 which is a known initial condition

- $C_{L_t} = (c_{L_t,1}, \ldots, c_{L_t,L_h})^t$ is the vector of approximated model prices:

$$c_{L_t,j} \simeq \bar{c}_0(T, h_j) = c_0(T, e^{h_j})^{LV}$$

- For a generic strike, $c_0(T,k)^{LV}$ is approximated by interpolating in $[c_{L_t,j}, c_{L_t,j+1})$ where j is such that $e^{h_j} \leq k < e^{h_{j+1}}$

- This approximation of $c_0(T,k)^{LV}$ is a function of the $N \times M$ model parameters $\{v_{i,j}\}$ which enter into the discretized equation through the coefficients of A

Calibration

- Let us assume that market implied volatilities are a **discrete** set of quotes $\{\sigma_{mkt}(T_i, K_{i,j})\}_{i=1,\dots,N, j=\dots,M}$ corresponding to a a **bi-dimensional strike grid** $\{K_{i,j}\}$
- We choose to define the local volatility matrix $\{v_{i,j}\}$ on the same (normalized) grid $\{k_{i,j} = K_{i,j}/F_0(T_i)\}$
- Given an initial guess for the LV parameters $\{v_{i,j} = v^0_{i,j}\}$, we can compute model prices $c_0(T_i, k_{i,j})^{LV} = c_0(T_i, k_{i,j}; \{v_{i,j}\})^{LV}$ at all market coordinates
- Do they match market prices? And what is a measure of the error between model and market prices?

Calibration

We can follow two different approaches:

- We define the (i,j)-**calibration error (in price)** as

$$\delta\pi_{i,j} = \left| c_0(T_i, k_{i,j}; \{v_{i,j}\})^{LV} - c_0(T_i, k_{i,j}) \right|$$

where $c_0(T_i, k_{i,j}) = Bl(1, k_{i,j}, T_i, \sigma_{mkt}(T_i, K_{i,j}), 1)$

- We define the (i,j)-**calibration error (in volatility)** as

$$\delta\sigma_{i,j} = |\sigma_{LV}(T_i, K_{i,j}) - \sigma_{mkt}(T_i, K_{i,j})|$$

where the model implied volatility $\sigma_{LV}(T_i, K_{i,j})$ is such that $c_0(T_i, k_{i,j}; \{v_{i,j}\})^{LV} = Bl(1, k_{i,j}, T_i, \sigma_{LV}(T_i, K_{i,j}), 1)$

Calibration

We can follow two different approaches:

- We define the (i,j)-**calibration error (in price)** as

$$\delta\pi_{i,j} = \left| c_0(T_i, k_{i,j}; \{v_{i,j}\})^{LV} - c_0(T_i, k_{i,j}) \right|$$

  where $c_0(T_i, k_{i,j}) = Bl(1, k_{i,j}, T_i, \sigma_{mkt}(T_i, K_{i,j}), 1)$

- We define the (i,j)-**calibration error (in volatility)** as

$$\delta\sigma_{i,j} = |\sigma_{LV}(T_i, K_{i,j}) - \sigma_{mkt}(T_i, K_{i,j})|$$

  where the model implied volatility $\sigma_{LV}(T_i, K_{i,j})$ is such that $c_0(T_i, k_{i,j}; \{v_{i,j}\})^{LV} = Bl(1, k_{i,j}, T_i, \sigma_{LV}(T_i, K_{i,j}), 1)$

Calibration on prices

Following the first approach, we define the **price calibration function**

$$
\begin{aligned}
f_\pi(\{v_{i,j}\}) = \sum_{i=1}^{N}\sum_{j=1}^{M} w_{i,j}\,(\delta\pi_{i,j})^2 = \\
= \sum_{i=1}^{N}\sum_{j=1}^{M} w_{i,j}\left(c_0(T_i, k_{i,j}; \{v_{i,j}\})^{LV} - c_0(T_i, k_{i,j})\right)^2
\end{aligned}
$$

Calibration means finding $\{v_{i,j}\}$ such that

$$f_\pi(\{v_{i,j}\}) \le \Theta$$

for a chosen *calibration threshold* Θ.

A possible choice for the weights is $w_{i,j} = Vega(T_i, k_{i,j})$.

Calibration on volatilities

Calibration on volatilities means looking for $\{v_{i,j}\}$ such that

$$f_\sigma(\{v_{i,j}\}) \leq \Theta$$

for a chosen *calibration threshold* Θ, where the **volatility calibration function** is given by

$$f_\sigma(\{v_{i,j}\}) = \sum_{i=1}^{N}\sum_{j=1}^{M}(\delta\sigma_{i,j})^2 = \sum_{i=1}^{N}\sum_{j=1}^{M}\Big(\sigma_{LV}(T_i, K_{i,j}) - \sigma_{mkt}(T_i, K_{i,j})\Big)^2$$

An alternative is to set

$$f_\sigma(\{v_{i,j}\}) = \max_{i,j}(\delta\sigma_{i,j}) = \max_{i,j}\Big|\sigma_{LV}(T_i, K_{i,j}) - \sigma_{mkt}(T_i, K_{i,j})\Big|$$

Calibration on volatilities

- There are several numerical algorithms that can find minimum of a convex function (up to a tolerance Θ), e.g., Levenberg–Marquardt algorithm. We can apply them to find the minimum of f_π or f_σ, thus solving the calibration problem.
- However we can hope to do better than standard minimization algorithms if we have a good hint on the "search direction".
- We will set up the calibration algorithm based on known results for the model implied volatility.

Calibration on volatilities

The "most likely path approximation" given an approximation of model implied volatility in terms of the local volatility function

$$\sigma_{LV}^2(T_i, K_{i,j}) \simeq \frac{1}{T_i}\mathbb{E}\left[\int_0^{T_i} \eta^2(t, X_t)dt \,\middle|\, X_{T_i} = k_{i,j}\right]$$

Therefore we can guess that

$$\sigma_{LV}^2(T_i, K_{i,j})\,T_i - \sigma_{LV}^2(T_{i-1}, K_{i,j})\,T_{i-1} \sim v_{i,j}^2\,(T_i - T_{i-1})$$

Calibration on volatilities

- Let us define the **forward implied volatility** between T_i and T_{i+1} as

$$\sigma_{LV}^{fwd}(T_{i-1}, T_i, K_{i,j}) = \sqrt{\frac{\sigma_{LV}^2(T_i, K_{i,j})T_i - \sigma_{LV}^2(T_{i-1}, K_{i,j})T_{i-1}}{T_i - T_{i-1}}}$$

  Similarly define $\sigma_{LV}^{fwd}(T_i, T_{i+1}, K_{i,j})$ in terms of market volatilities σ_{mkt}^{fwd}
- Then the most likely path approximation becomes

$$\sigma_{LV}^{fwd}(T_{i-1}, T_i, K_{i,j}) \sim v_{i,j}$$

- If we multiply a local volatility parameter $v_{i,j}$ by a constant, then the resulting model (forward) implied volatility at $(T_i, k_{i,j})$ will be approximately the old implied volatility multiplied by the same constant

Calibration on volatilities

Then we can setup a fixed-point algorithm (see [7]):

1. Suppose we have LV parameters $\{v_{i,j}^u\}$ and compute the model volatilities $\sigma_{LV}(T_i, K_{i,j})^u$ associated to model prices
2. If $f_\sigma(\{v_{i,j}^u\}) \leq \Theta$ then stop the algorithm, otherwise continue to the next step
3. Set new LV parameters

$$v_{i,j}^{u+1} = v_{i,j}^u \; \frac{\sigma_{mkt}^{fwd}(T_{i-1}, T_i, K_{i,j})}{\sigma_{LV}^{fwd}(T_{i-1}, T_i, K_{i,j})}$$

and go to step 1

The algorithm starts from an initial choice of LV parameters, for example $\{v_{i,j}^0 = \sigma_{mkt}(T_i, K_{i,j})\}$

Calibration on volatilities

- Fixed-point procedure is more stable especially for short maturities, and moreover converges faster
- What if the calibration fails, namely if the threshold is not reached within the maximum number of iterations? This might be a symptom of arbitrages present in the input data

Course Outline

1. Market Volatility Surface
2. The Local Volatility Model
3. Local Volatility in practice
4. Pricing
    - Monte Carlo Method
    - European vs Forward Starting Options
    - Hedging

Monte Carlo

- Since we can calibrate the Local Volatility model to market prices $D_0(T)$, $F_0(T)$, $C_0(T,K)$, we are ready to **price OTC options**!

- The model price of a derivative with payoff $\Phi(S(T))$ is

$$D_0(T)\,\mathbb{E}\left[\Phi(S(T))\right]$$

- Unfortunately there is no closed formula for $S(T)$ under LV dynamics, and computing the expectation is difficult, so the strategy will be:

  - approximate $S(T)$ by recursively applying a discretized version of the model SDE

  - compute the expected value using Central Limit Theorem

Monte Carlo

- First, we divide the interval $[0, T]$ into M equi-spaced subintervals

- Then determine $S(t_{i+1})$ from $S(t_i)$ for $i = 0, \dots M-1$ by the discretized SDE

$$S(t_{i+1}) \simeq S(t_i) + S(t_i)\,(r(t_i) - q(t_i))\,\Delta t_i + S(t_i)\varsigma(t_i, S(t_i))W_i$$

where $\Delta t_i = t_{i+1} - t_i$ and W_i is a normal random variable with mean 0 and variance Δt_i

Monte Carlo

In order to avoid negative values for the approximated $S(t)$, one can discretize the SDE of the logarithm of the normalized process

$$Y(t) = \log X(t) = \log \left(\frac{S(t)}{F_0(t)} \right)$$

By applying Ito's lemma we have

$$d \log X(t) = -\frac{1}{2}\bar{\eta}^2 \left(t, \log X(t)\right) dt + \bar{\eta} \left(t, \log X(t)\right) dW_t$$

Then we approximate

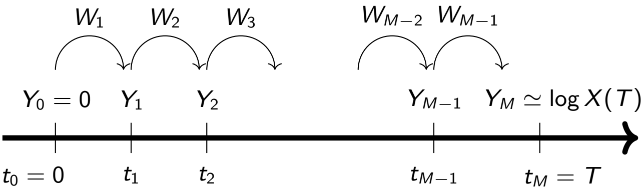
$$\log X(t_{i+1}) \simeq \log X(t_i) - \frac{1}{2}\bar{\eta}^2 \left(t, \log X(t_i)\right) \Delta t_i + \bar{\eta} \left(t, \log X(t_i)\right) W_i$$

Monte Carlo

In other terms, if we define the random variables $Y_0, \ldots, Y_M$ as $Y_0 = 0$ and iteratively

$$Y_{i+1} = Y_i - \frac{1}{2}\bar{\eta}^2(t, Y_i)\,\Delta t_i + \bar{\eta}(t, Y_i)\,W_i$$

then $Y_M = Y_M(W_1, \ldots, W_M)$ approximates $\log X(T)$



Monte Carlo

- Once we approximate $\log X(T)$ with Y_M, we also obtain

$$S(T) = e^{\log X(T)} F_0(T) \simeq e^{Y_M} F_0(T)$$

- We remark that the approximant of $S(T)$ is a function of the browninan motions by writing

$$e^{Y_M} F_0(T) = \mathcal{S}(W_1, \ldots, W_M)$$

- If Φ is a given payoff function, then $\Phi(S(T))$ is approximated by a random variable $\Phi(\mathcal{S}(W_1, \ldots, W_M))$ which is a function of M normal random variables

- Thus the model price of the derivative with payoff Φ is approximated by $\mathbb{E}\left[\Phi(\mathcal{S}(W_1, \ldots, W_M))\right]$

Monte Carlo

- Recall Central Limit Theorem: let $\Phi_1, \ldots, \Phi_N$ a sequence of i.i.d. random variables with mean μ and variance v, then for $N \to \infty$ the random variable $\frac{1}{N}\sum_{j=1}^{N} \Phi_j$ converges in distribution to a normal $\mathcal{N}(\mu, v/N)$ with mean μ and variance v/N

- We will apply this theorem to Φ_j having the same distribution of the payoff random variable $\Phi(S(T))$:

$$\frac{1}{N}\sum_{j=1}^{N} \Phi_j \to \mathcal{N}(\mu, v/N) \qquad \text{for } N \to \infty$$

where

$$\mu = \mathbb{E}[\Phi(S(T))] \qquad v = Var[\Phi(S(T))]$$

Monte Carlo

Central Limit Theorem tells that

$$\mathcal{P}\left[\frac{\frac{1}{N}\sum_{j=1}^{N}\Phi_j-\mu}{\sqrt{v/N}}<x\right]\simeq CumulNormDistrib(x)$$

or

$$\mathcal{P}\left[\frac{\left|\frac{1}{N}\sum_{j=1}^{N}\Phi_j-\mu\right|}{\sqrt{v/N}}<z_{(1+\alpha)/2}\right]=\alpha$$

where z_α is the quantile of the standard normal distribution. For example

$$\mathcal{P}\left[\left|\frac{1}{N}\sum_{j=1}^{N}\Phi_j-\mu\right|<\frac{\sqrt{v}}{\sqrt{N}}\right]\simeq 68\%$$

$$\mathcal{P}\left[\left|\frac{1}{N}\sum_{j=1}^{N}\Phi_j-\mu\right|<2\frac{\sqrt{v}}{\sqrt{N}}\right]\simeq 95\%$$

Monte Carlo

- Suppose we evaluate $\sum_{j=1}^{N} \Phi_j$ at a point ω of the probability space:

$$\hat{\mu}_N := \frac{1}{N} \sum_{j=1}^{N} \Phi_j(\omega)$$

  Then we have a confidence interval for μ of level α:

$$\left[\hat{\mu}_N - z_{(1+\alpha)/2} \frac{\sqrt{v}}{\sqrt{N}},\ \hat{\mu}_N + z_{(1+\alpha)/2} \frac{\sqrt{v}}{\sqrt{N}}\right]$$

- We can estimate v with the sample variance

$$\hat{v}_N := \frac{1}{N-1} \sum_{j=1}^{N} \left(\Phi_j(\omega) - \hat{\mu}_N\right)^2$$

- Loosely speaking, μ is equal to $\hat{\mu}_N$ with an error $\pm\sqrt{\hat{v}_N}/\sqrt{N}$

Monte Carlo

- Now let's choose independent random variables $\{\Phi_j\}$ with the same distribution as $\Phi(\mathcal{S}(W_1, \dots, W_M))$ by setting

$$\Phi_j = \Phi(\mathcal{S}(W_1^j, \dots, W_M^j))$$

where $\{W_i^j\}_j$ are independent normal random variables with mean zero and standard deviation $\sqrt{\Delta t_i}$

- Then the random variable

$$\frac{1}{N} \sum_{j=0}^{N} \Phi\left(\mathcal{S}(W_1^j, \dots, W_M^j)\right)$$

tends to a normal with mean $E[\Phi(\mathcal{S}(W_1, \dots, W_M))] \simeq \mathbb{E}[\Phi(S(T))]$

Monte Carlo

- Once chosen M and N, the numbers $W_i^j(\omega)$ are generated according to their (known) distribution
- We compute the sample average

$$\hat{\mu}_N = \frac{1}{N} \sum_{j=1}^{N} \Phi\left(\mathcal{S}(W_1^j(\omega), \dots, W_M^j(\omega))\right)$$

and the sample variance

$$\hat{v}_N = \frac{1}{N-1} \sum_{j=1}^{N} \left(\Phi\left(\mathcal{S}(W_1^j(\omega), \dots, W_M^j(\omega))\right) - \hat{\mu}_N\right)^2$$

- Then $\mathbb{E}[\Phi(S(T))]$ is equal to $\hat{\mu}_N$ with an error $\pm\sqrt{\hat{v}_N}/\sqrt{N}$

Monte Carlo

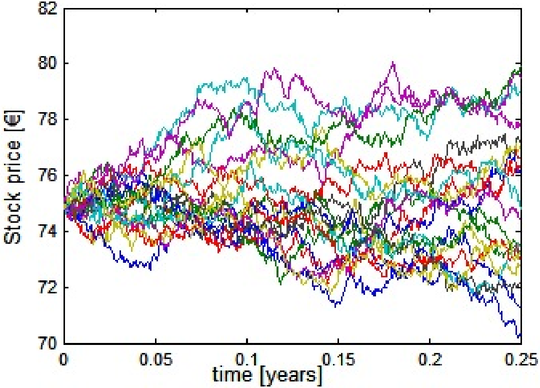


Figure: Example of Monte Carlo simulation of $S(T)$

Monte Carlo

Remark: Monte Carlo method can be applied to Black model as well.

For Black's model, solution to the SDE is known in closed form:

$$S(T) = S(0)\exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right)$$

so that the estimator for $\mathbb{E}[\Phi(S(T))]$ is

$$\frac{1}{N}\sum_{j=0}^{N}\Phi\left(S(0)\exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma W^j\right)\right)$$

where W^j are i.i.d. random variables with mean zero and standard deviation $\sqrt{T}$

European vs Forward Starting Options

- Monte Carlo method is a very powerful tool to price a wide variety of derivatives
- In particular it can be extended to price derivatives depending on the value of S at multiple times $T_1 < \ldots < T_N$ before the final expiry T_N
- We will elaborate on prices under Local Valatility dynamics of two categories of derivatives:
  - European derivatives with a single expiry T
  - Forward starting options with start date T_1 and expiring in $T_2 > T_1$

European derivatives

- Market prices of call options $C_0(T, K)$ completely determine the transition density $p_S(t, s)$ of $S(T)$

- But transition densities completely determine prices of European derivatives. In fact, the price of a derivative expiring in T with payoff $\Phi(S(T))$ can be written as

$$D_0(T)\mathbb{E}\left[\Phi(T)\right] = D_0(T)\int_{\mathbb{R}} \Phi(s)\, p_S(T, s) ds$$

- Any model calibrated to option prices, automatically produces prices of European derivatives coherent with the market

European derivatives

Examples of derivatives whose prices are usually computed using the Local Volatility model:

- Digital options with strike K:

$$\Phi(T) = \mathbf{1}_{S(T)>K}$$

- Asian options with strike K and asian fixing dates $\{T_1, \ldots, T_n\}$

$$\Phi(T) = \left(\frac{1}{n} \sum_{i=1}^{n} S(T_i) - K \right)^+$$

(not strictly "European" but with similar properties)

Forward starting options

- A forward starting option with start date T_1 and expiry $T_2 > T_1$ has payoff

$$\Phi(T_1, T_2) = (S(T_2) - \varkappa S(T_1))^+ , \quad \varkappa > 0, \quad T_2 > T_1 > 0$$

- In general its model price at time t is given by

$$C_t(T_1, T_2, \varkappa)^{model} = D_t(T_2)\,\mathbb{E}\left[\Phi(T_1, T_2)\middle|\, \mathcal{F}_t\right]$$

- Today's price can be computed using a Monte Carlo simulation, but it is worth expressing it in another form

Forward starting options

By the law of iterated expectations, today's price is given by

$$C_0(T_1, T_2, \varkappa)^{model} = D_0(T_2)\mathbb{E}\left[\mathbb{E}\left[\Phi(T_1, T_2)\middle|\mathcal{F}_{T_1}\right]\right]$$

Let's define $p_S(T_2, y; T_1, x)$ to be the **forward transition density** of the forward transition probabilities $\mathbb{Q}(S(T_2) \in B_2 | S(T_1) \in B_1)$ from a future date T_1 to a farther future date T_2. Then

$$\mathbb{E}\left[\Phi(T_1, T_2)\middle|\mathcal{F}_{T_1}\right] = \int_{\mathbb{R}} (s_2 - \varkappa S(T_1))^+ \, p_S(T_2, s_2; T_1, S(T_1)) ds_2$$

and hence

$$\begin{aligned} C_0(T_1, T_2, \varkappa)^{model} &= \\ &= D_0(T_2) \int_{\mathbb{R}} \int_{\mathbb{R}} (s_2 - \varkappa s_1)^+ \, p_S(T_2, s_2; T_1, s_1) p_S(T_1, s_1) ds_2 ds_1 \end{aligned}$$

Remarks on forward densities

- Therefore, independently from the pricing model:
  - price of a European derivative with expiry T depends only on spot transition density $p_S(T, s)$
  - price of a forward starting options also depends on the forward transition density $p_S(T_2, s_2; T_1, s_1)$

- A calibrated model has spot transition densities which are coherent with market quotes of plain vanilla options. However **each model will generate its own forward densities**

- Computing model prices of forward starting options is a way to understand how the model generates forward densities

Black forward volatility

Let's compute the Black model price of a forward starting option

$$
\begin{aligned}
C_0(T_1, T_2, \varkappa)^{Black} &= D_0(T_2)\mathbb{E}\left[(S(T_2) - \varkappa S(T_1))^+ \middle| \mathcal{F}_0\right] = \\
&= D_0(T_2)\mathbb{E}\left[S(T_1)\mathbb{E}\left[\left(\frac{S(T_2)}{S(T_1)} - \varkappa\right)^+ \middle| \mathcal{F}_{T_1}\right]\right]
\end{aligned}
$$

Now notice that at T_1 the process $S(T_2)/S(T_1)$ is log-normal with forward at T_2 given by $e^{\int_{T_1}^{T_2}(r(s)-q(s))ds} = F_0(T_2)/F_0(T_1)$ and hence

$$
\mathbb{E}\left[\left(\frac{S(T_2)}{S(T_1)} - \varkappa\right)^+ \middle| \mathcal{F}_{T_1}\right] = Bl\left(\frac{F_0(T_2)}{F_0(T_1)}, \varkappa, T_2 - T_1, \sigma, 1\right)
$$

Black forward volatility

Hence we can write

$$
\begin{aligned}
C_0(T_1, T_2, \varkappa)^{Black} &= D_0(T_2)\mathbb{E}\left[S(T_1)\mathbb{E}\left[\left(\frac{S(T_2)}{S(T_1)} - \varkappa\right)^+ \middle| \mathcal{F}_{T_1}\right]\right] = \\
&= D_0(T_2)\mathbb{E}\left[S(T_1)Bl\left(\frac{F_0(T_2)}{F_0(T_1)}, \varkappa, T_2 - T_1, \sigma, 1\right)\middle| \mathcal{F}_{T_1}\right] = \\
&= D_0(T_2)\underbrace{\mathbb{E}\left[S(T_1)\middle| \mathcal{F}_{T_1}\right]}_{F_0(T_1)} Bl\left(\frac{F_0(T_2)}{F_0(T_1)}, \varkappa, T_2 - T_1, \sigma, 1\right) = \\
&\qquad = Bl\left(F_0(T_2), \varkappa F_0(T_1), T_2 - T_1, \sigma, D_0(T_2)\right)
\end{aligned}
$$

Price is given by Black's formula with time-to-expiry $(T_2 - T_1)$

Implied forward volatility

- Question: can we express the price of a forward starting option in terms of some "forward implied volatility", just like the price of a call option is function of a "spot implied volatility"?
- The answer is positive since under the Black model we have an explicit formula

$$C_0(T_1, T_2, \varkappa)^{Black} = Bl\left(F_0(T_2), \varkappa F_0(T_1), T_2 - T_1, \sigma, D_0(T_2)\right)$$

- Given a (model or market) price $C_0(T_1, T_2, \varkappa)$ for a forward starting call option, the **forward implied volatility** is the value of Black volatility $\boldsymbol{\sigma}(\mathbf{T_1}, \mathbf{T_2}, \boldsymbol{\varkappa})$ that makes

$$C_0(T_1, T_2, \varkappa) = Bl\left(F_0(T_2), \varkappa F_0(T_1), T_2 - T_1, \boldsymbol{\sigma}(\mathbf{T_1}, \mathbf{T_2}, \boldsymbol{\varkappa}), D_0(T_2)\right)$$

Implied forward volatility

- Forward implied volatility $\sigma(T_1, T_2, \varkappa)$ is a generalization of the usual spot implied volatility $\sigma(T, K)$ and the two coincide when $T_1 = 0$ and $K = \varkappa S(0)$
- Forward implied volatility is not uniquely determined by the absence of arbitrage and the market prices of vanilla options; it is implied by a combination of market prices and model choice
- The map $\varkappa \mapsto \sigma(T_1, T_2, \varkappa)$ is called **forward volatility smile** at time T_1 for a maturity $(T_2 - T_1)$

Market forward volatility

Sometimes, for comparison purposes, the following model-free quantity, called **market forward volatility**, is considered

$$\sigma_{mkt}(T_1, T_2, K) := \sqrt{\frac{\sigma^2_{mkt}(T_2, K)T_2 - \sigma^2_{mkt}(T_1, K)T_1}{T_2 - T_1}}$$

where $\sigma_{mkt}(T_i, K)$ are market implied volatilities

This definition follows the observation that volatility σ and time-to-expiry T enter into Black's formula combined as "variance" $\sigma^2 T$. Therefore in a Black model

- A call with expiry T has variance $v(T) = \sigma^2 T$
- A forward starting call with start date T_1 and expiry T_2 has variance $v(T_1, T_2) = \sigma^2(T_2 - T_1) = v(T_2) - v(T_1)$

Forward volatility in the LV model

- What are the **forward implied volatilities** $\sigma(T_1, T_2, \varkappa)$ **implied by a Local Volatility model** calibrated to market prices of call options?

- To answer this question we must
  1. calibrate a Local Volatility model to market data
  2. find the model price of a forward starting option using Monte Carlo
  3. convert price into forward implied volatility

- A "reasonable" forward volatility smile should not be too different from the spot volatility smile. Is this the case?

Forward volatility in the LV model

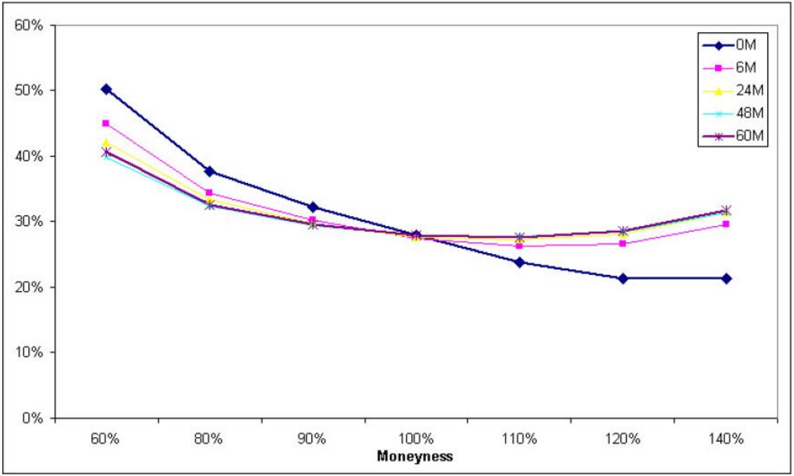


Figure: Example of three-month forward smile for several starting dates

Forward volatility in the LV model

- In the previous figure each line is the forward smile of three-month options, i.e., such that $(T_2 - T_1)$=3 months, for a given choice of T_1 from today to 60 months from today
- The forward smile flattens as T_1 increases, but this is not reasonable! Indeed market consensus is that **forward smile should be similar to spot smile**
- Forward smile flattening is a typical behavior of the LV model, and this means that OTC derivatives which strongly depend on forward transition densities will typically be **mispriced** under the LV model
- In the following slides we will sketch a solution to this problem

Stochastic Local Volatility

In order to get more control on forward volatility, let us consider a more general model

$$dY(t) = \ell(t, Y(t))\ \nu(t)\ Y(t)\ dW_t$$

where ℓ is a function and $\nu(t)$ is a stochastic process. Gyöngy Lemma states that prices of call options satisfy a Dupire's equation

$$\frac{\partial c_0(T,k)}{\partial T} = \frac{1}{2}\eta(T,k)^2 k^2 \frac{\partial^2 c_0(T,k)}{\partial^2 k} \qquad c_0(t,k) = (1-k)^2$$

where

$$\eta^2(T,k) = \ell^2(T,k)\mathbb{E}[\nu^2(T) \big| Y(T) = k]$$

The Local Volatility process $X(t)$ having Local Volatility function $\eta(t,k)$ is called the Markov projection of $Y(t)$

Stochastic Local Volatility

- Suppose we calibrate $X(t)$ to market quotes of vanilla options
- By setting
$$\ell^2(T,k) = \frac{\eta^2(T,k)}{\mathbb{E}[\nu^2(T)|\,Y(T)=k]}$$
we have that $Y(t)$ prices correctly the vanilla options too, since $X(t)$ and $Y(t)$ have the same spot transition densities (but different forward transition densities)
- Dynamics of $\nu(t)$ controls the forward transitions of $Y(t)$. It could be calibrated to market price of exotic derivatives (e.g. forward starting options, barrier options) which are sensitive to forward densities

Hedging

We now want to address the problem of hedging a derivative, and we need to recall few definitions about portfolio replicas

- Consider a market consisting of a risky asset $S(t)$ and a risk-free bank account $B(t) = e^{\int_0^t r(s)ds}$
- The value process $V^h(t)$ of a portfolio $h(t) = \big(h_0(t), h_1(t)\big)$ at t is

$$V^h(t) = h_0(t)\, B(t) + h_1(t)S(t)$$

- A portfolio is called self-financing if

$$dV^h(t) = h_0(t)\, dB(t) + h_1(t)\, dS(t)$$

Hedging

- Local Volatility is an arbitrage-free and complete model. Therefore, by general theorems, any European option on an asset $S(t)$ with expiry T and payoff $\Phi(S(T))$ can be replicated with a self financing portfolio
- In other words there exists a self-financing portfolio $h(t)$ such that

$$V^h(T) = \Phi(S(T))$$

- The *fair* price of the derivative at any time t is

$$\Pi(t) = V^h(t)$$

and can be computed as an expectation under the risk-neutral measure

$$\Pi(t) = \frac{1}{B(T)} \mathbb{E}\left[\Phi(S(T)) \middle| \mathcal{F}_t\right]$$

Hedging

- These general theorems actually tell us more than just price. They give us the explicit form of the portfolio strategy that replicates the derivative:
$$h_1(t) = \frac{\partial \Pi(t)}{\partial S(t)}$$
where we see $\Pi(t) = \Pi(t, S(t))$ as a function of the spot price $S(t)$ of the risky asset

- The quantity $\partial \Pi(t)/\partial S(t)$ is the so-called **delta** of the derivative

- The other part of the strategy $h_0(t)$ is completely determined by the self-financing condition

Hedging in practice

- Banks do not want to bear any risk in selling a derivative. So each time a derivative is sold, the trading desk has the task to set up a portfolio replica (also called or a *hedge*) that matches the value of the derivative.

- We have just seen that this portfolio must be rebalanced with new $h(t)$ in continuous time, but in practice it can be done only at discrete times

- We want to address how hedging works in practice when pricing with the Local Volatility model, and in particular how one can hedge against movements of the spot

Hedging in practice

Here is the list of steps that a trader should do in order to hedge a derivative

- At time t calibrate a Local Volatility model to market data and then compute the price $\Pi(t)$ (with a Monte Carlo simulation or any other pricing algorithm)
- Compute $h_1(t) = \partial \Pi(t, S(t))/\partial S(t)$ e.g. by a finite difference approximation
- Set up a (self-financing) portfolio by buying $h_1(t)$ shares of the risky asset and $h_0(t)$ risk-free bonds such that

$$\Pi(t) = V^h(t) = h_0(t)\, B(t) + h_1(t) S(t)$$

Hedging in practice

Suppose now that from t to time $t + \Delta t$ only the spot moves from $S(t)$ to $S(t + \Delta t)$, while all other market data (and in particular market implied volatilities) are fixed. Then the portfolio becomes

$$
\begin{aligned}
V^h(t + \Delta t) = h_0(t)B(t + \Delta t) + \frac{\partial \Pi(t)}{\partial S(t)} S(t + \Delta t) \simeq \\
\simeq V^h(t) + \frac{\partial \Pi(t)}{\partial S(t)} \cdot (S(t + \Delta t) - S(t))
\end{aligned}
$$

and we expect a first-order movement of $\Pi(t, S(t))$ as follows

$$
\Pi(t + \Delta t, S(t + \Delta t)) \simeq \Pi(t) + \frac{\partial \Pi(t)}{\partial S(t)} \cdot (S(t + \Delta t) - S(t))
$$

so that movement of the derivative is hedged by movement of portfolio

Hedging in practice

As for the approximation of the delta at t, we can compute

$$\frac{\partial \Pi(t)}{\partial S(t)} \simeq \frac{\Pi(t, S(t) + \delta S) - \Pi(t, S(t))}{\delta S}$$

However when computing $\Pi(t, S(t) + \delta S)$ we have two possibilities:

1. Use the same model parameters calibrated at t and just run the pricing algorithm with the new spot $S(t) + \delta S$
2. Perform a new calibration with the new market data at t and then do the pricing using the new parameters

Clearly the second approach is going to produce a better hedge, since $\Pi(t + \Delta t, S(t + \Delta t))$ (which is the quantity that will be guessed using $\partial \Pi(t)/\partial S(t)$) will be computed after a recalibration in order to be coherent with the new market prices

Hedging in practice

- A related question is *whether a calibration is really needed*, namely does a calibration to the new set of market data (with the shifted spot) produce different parameters than those calibrated to unperturbed data?
- It turns out that the answer depends on the type of underlying:
  - if $S(t)$ is an equity asset, namely *sticky-strike*, then calibration with a perturbed spot will yield different parameters
  - if $S(t)$ is an FX asset, namely *sticky-delta*, then calibration with a perturbed spot will yield the same parameters
- This means that the Local Volatility model has a *sticky-delta* behavior

Appendix: Implied volatility

- Fix a discount factor $D(T)$ and a forward $F(T)$ for an expiry T. Given an arbitrage-free price $C_0(T,K)$ of a plain vanilla option with expiry K and strike K, we define its **implied volatility** $\boldsymbol{\sigma}(\mathbf{T},\mathbf{K})$ by the equation

$$C_0(T,K) = Bl\left(F_0(T), K, T, \boldsymbol{\sigma}(\mathbf{T},\mathbf{K}), D_0(T)\right)$$

where Bl denotes Black's formula as defined on page 14

- Implied volatilities can be computed for **market prices**, as well as **model prices** e.g. of Black-Scholes or Local Volatility model

Appendix: Implied volatility

List of symbols used to denote implied volatilities:

Type	**Price**	**Implied volatility**	**Model param.**
Market	$C_0(T, K)$	$\sigma_{mkt}(T, K)$	–
LV model	$C_0^{LV}(T, K)$	$\sigma_{LV}(T, K)$	$\varsigma(\cdot, \cdot)$ or $\eta(\cdot, \cdot)$
BS model	$C_0^{BS}(T, K)$	σ	σ

Appendix: Implied volatility

- The set of implied volatilities $\{\sigma_{mkt}(T_i, K_{ij})\}$ for all quoted strikes $\{K_{ij}\}$ and maturities $\{T_i\}$, forms the **market (implied) volatility surface**. Sometimes the symbols $\sigma_{BS}(T, K)$ or $\sigma(T, K)$ are used in place of $\sigma_{mkt}(T, K)$

- The model parameter σ for the Black-Scholes model, called **Black volatility**, turns out to be the implied volatility for any price computed with this model

- The model parameter for the LV model, denoted by $\varsigma(\cdot,\cdot)$ or $\eta(\cdot,\cdot)$, is called **local volatility function** and should not be confused with the **model implied volatility surface** $\sigma_{LV}(\cdot,\cdot)$.

Bibliography

- Aronofsky, D. (1998) Pi. *Protozoa Pictures*
- Clark, I. (2010) Foreign Exchange Option Pricing: A Practitioner's Guide. *Wiley Finance*
- Dermann, E., Kani, I. (1994) Riding on a smile. *Risk Magazine* 7, 32-39.
- Dupire, B. (1994) Pricing with a smile. *Risk Magazine* 7, 18-20.
- Hagan, P., West, G. (2008) Methods for constructing a yield curve.
- Gatheral, J. (2006) The volatility surface, a practitioner's guide. *Wiley Finance*
- Reghai, A., Boya, G., Vong, G. (2012) Local Volatility: Smooth Calibration and Fast Usage. Working paper available on `SSRN.com`.
- Taleb, N. (2007) The Black Swan: The Impact of the Highly Improbable *Random House*