Linear Regression Assumptions

DATA SCIENCE BOOTCAMP

First, some notes about Covariance

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- The sign (+/-) shows the tendency of the linear relationship between X and Y

- The magnitude is harder to interpret.

Covariance (math)

Let's say X and Y are two random variables, where:

$E(X) = \mu_X$ and $E(Y) = \mu_Y$

The <u>covariance</u> between X and Y is:

$$
\begin{aligned}
Cov(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\
&= E(XY) - \mu_X \mu_Y \\
&= \sigma_{XY}
\end{aligned}
$$

Covariance, normalized

- Pearson's correlation coefficient

$$\rho_{X,Y} = \frac{\mathrm{cov}(X, Y)}{\sigma_X \sigma_Y}$$

- Here, magnitude shows strength of the linear relationship.
- $-1 \leq \rho_{X,Y} \leq 1$

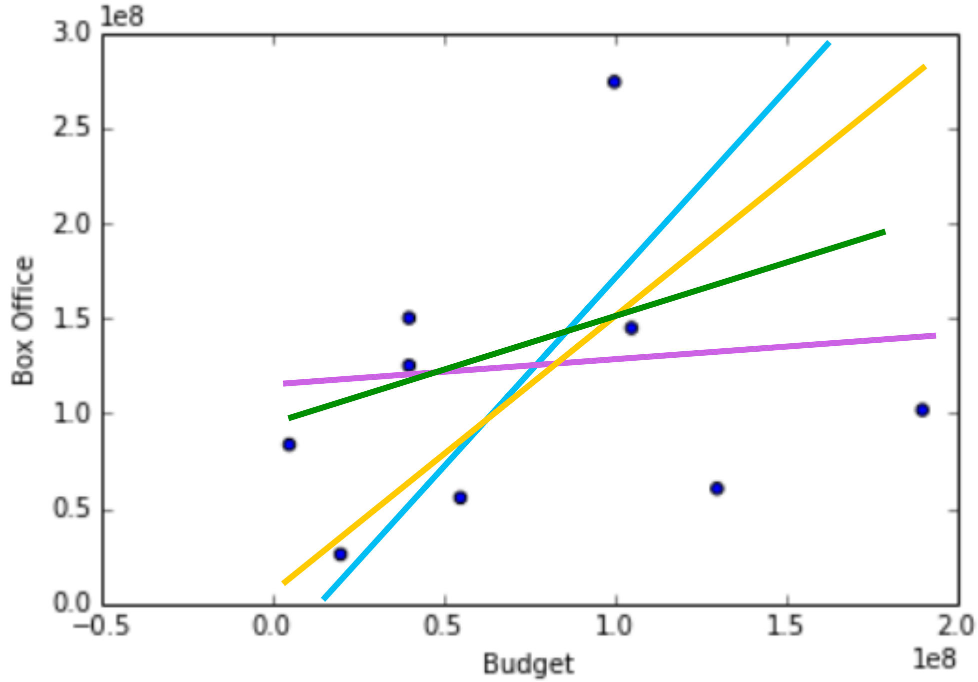
Covariance (more math facts)

- $\text{Cov}(X, X) = \text{Var}(X)$
- $\text{Cov}(X, Y) = \text{Cov}(Y, X)$
- $\text{Cov}(X, aY) = a\text{Cov}(X, Y)$; a is any constant number
    - $\text{Var}(aX) = \text{Cov}(aX, aX)$
      $= a^2\text{Var}(X)$

Covariance

- If random variables X and Y are independent,
  Then $Cov(X, Y) = 0$

- BUT if $Cov(X, Y) = 0$, it *does not necessarily* mean that X and Y are independent!



$$y_\beta(x) = \beta_0 + \beta_1 x + \varepsilon$$

$\beta_0 = 80million$	$\beta_0 = 0$	$\beta_0 = 120million$	$\beta_0 = 30million$
$\beta_1 = 0.5$	$\beta_1 = 1.5$	$\beta_1 = 0.1$	$\beta_1 = 2$

Classical Assumptions of Ordinary Least Squares

1. Linear in parameters
2. Identifiability / No exact pairwise collinearity
   / No exact multicollinearity
3. Either: the covariates (X_i's) are fixed, OR,
   if X_i's are random variables, then X_i's are independent of ε
   i.e.: $Cov(X_1, \varepsilon) = Cov(X_2, \varepsilon) = \ldots = Cov(X_p, \varepsilon) = 0$
4. Number of observations > number of β parameters
5. Sufficient variation in the values of the X variables
6. Errors ε are normally distributed
7. Mean of the errors ε is 0
   i.e.: $E(\varepsilon) = 0$
8. Homoskedasticity. $Var(\varepsilon_i) = x^2$ for all i observations
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If those assumptions highlighted in blue are true, then the OLS estimators $\hat{\beta}_0, \hat{\beta}_0, \ldots, \hat{\beta}_p$ are the

Best:

Linear

Unbiased:

Estimators

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Linear

Unbiased: $E(\hat{\beta}_i) = \beta_i =$ for i from 1, 2, …, p

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If an estimate is *not efficient* (but still unbiased), you're still generally OK if you use enough data, i.e.: your estimate will be asymptotically correct.

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    - Here, the underlying models are nonlinear
- (bad): $Y = \beta_0 + e^{\beta_1} X^{\beta_2} + \varepsilon$

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Test:

- Ask yourself: is Y numerical? Are you sure Y is not a rank?
- Try partial regressions and plots: $Y \sim X_i$, see if there's a linear relationship
- If all your standard errors are really big, might suspect nonlinearity
- (Assumption #8) residuals vs fitted plot: nonlinear

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Remedies:

- Give up (try a nonlinear model)

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<u>Test:</u>

- Check 1 versus 1 scatterplots of suspect pairs of X_i's
- High R^2 and significant F-statistic but mostly insignificant t-statistics

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  - Still minimum possible variances

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- BUT:
  - Large variances (large standard errors)
    - In perfect collinearity, standard errors would be infinite
  - Wide confidence intervals

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Consequences (continued):

- t-tests tend to fail to reject the null (statistically insignificant covariates)
    - Thus you would be incorrectly concluding that covariates aren't related to Y, when in actuality, they are.
- Tiny changes in data → large differences in $\hat{\beta}$ and $\hat{Y}$

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Remedies:

- Feature selection; then see if standard errors get smaller
  - Regularize (Ridge/Lasso) – this gets rid of some of the overlapping
- Be careful: sometimes it's better to have near-collinearity than a loss in signal.

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    - % discount levels (10%, 20%, 25%, etc.) are fixed.

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Test:

- Mostly you can assume the former. Else: don't worry about it.

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Consequences:
- Model may be mis-specified (see Assumption #10)

Remedies:
- Mostly you can assume X is fixed.
- Specify the model as best you can.
- Most importantly: be aware, but don't worry too much.

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Examples:

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  - Number of data points > (p + 1)
- (bad) Fitting all possible X_i's, their interactions (1000 covariates), but only having 100 movies in your dataset

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Test:

- Count.

Consequences:

- Overfit

Remedies:

- Feature selection / Regularization
- Get more data

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  - Then you try to predict revenues of films with $50,000 budget

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Test:

- Look. (Be smart).

Consequences:

- Wrong about anything outside of your covariate (X_i) range.

Remedies:

- Don't.

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