Fresnel integrals

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The Fresnel integrals are defined as

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \quad \text{and} \quad C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt \tag{1}$$

which are transcendental functions[1]. They are related to Fresnel diffraction in classical optics. In order to investigate the properties of these integrals, they will in the following be calculated numerically using a numerical integration routine in C# and compared to tabulated values given in https://www.dwc.knaw.nl/DL/publications/PU00011466.pdf. The tabulated values in X are calculated using the power series expansion of 1 which are given by

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)!(4n+3)} \tag{2}$$

and

$$C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!(4n+1)}. \tag{3}$$

The result of the numerical integration is presented in figure 1. Analytically, equation 1 may be shown to converge to 1/2 in the limit $x \to \infty$. To test convergence, the numerical integration was made for higher values of x as depicted in figure 2. Issues of the numerical integration routine are clearly seen in the intervals $x \in [18, 21]$, $x \in [22, 30]$ and $x \in [37, 47]$. Otherwise, the integrals seem to converge for all other values of x to 1/2. Since the test of convergence is not made for $x =$ machine infinity, it is not very indicative.

[1] A transcendental function is an analytic function which does not satisfy a polynomial equation (as opposed to algebraic functions)

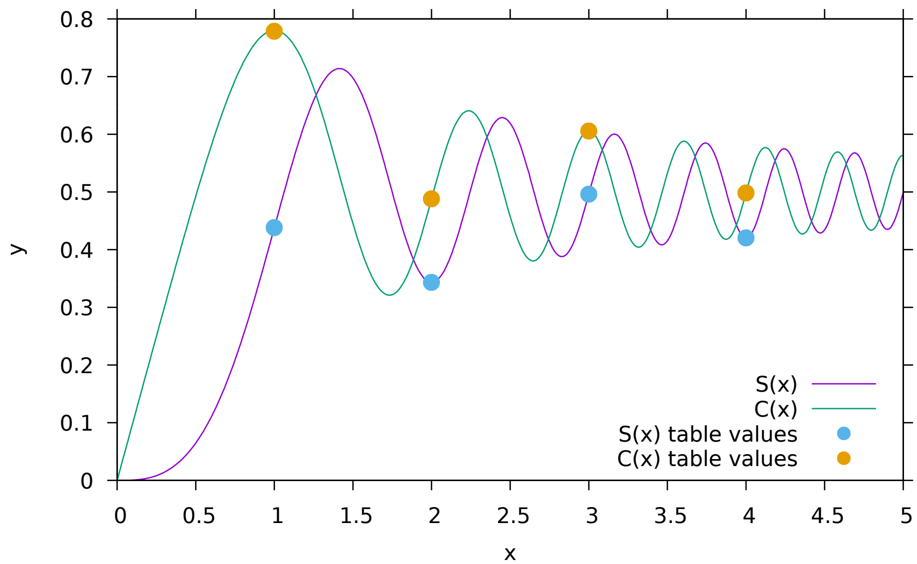

Figure 1: Numerical integration of the Fresnel integrals given by equation 1 along with the tabulated values given by REFERENCE.

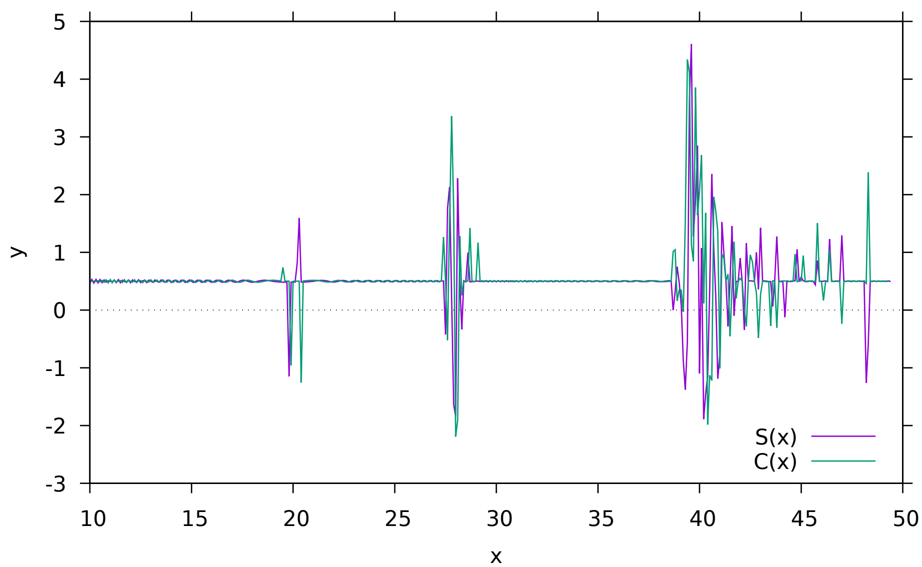

Figure 2: Large values of x in numerical integration of equation 1 to test convergence toward 1/2.