Master Seminar – p-adic Galois Representations
Talk 4: Étale φ-modules, part 2.

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Last time we proved $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{F}_p}(G_E) \rightleftarrows \Phi^{\text{ét}}_{E}$.

We start by generalizing the proof to $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E) \rightleftarrows \Phi^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$.

We complete the proof of

Proposition

1. *for any $V \in Rep^{cont}_{\mathbb{Z}_p}(G_E)$, $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ is an $\mathcal{O}_{\breve{\mathcal{E}}}$-linear, G_E-equivariant isomorphism of φ-modules over $\mathcal{O}_{\breve{\mathcal{E}}}$*
2. *the $\mathcal{O}_{\mathcal{E}}$-module $\mathbb{D}(V)$ is finitely generated*
3. *if $0 \to V' \xrightarrow{g} V \xrightarrow{h} V'' \to 0$ is a s.e.s. in $Rep^{cont}_{\mathbb{Z}_p}(G_E)$ (i.e. a s.e.s. of $\mathbb{Z}_p$-modules with g, h continuous G_E-equivariant $\mathbb{Z}_p$-linear maps), then*

$$0 \to \mathbb{D}(V') \xrightarrow{id \otimes g} \mathbb{D}(V) \xrightarrow{id \otimes h} \mathbb{D}(V'') \to 0$$

   is a well-defined exact sequence of $\mathcal{O}_{\mathcal{E}}$-modules

- case $pV = 0$: done
- torsion case ($p^n V = 0$ for some n)
- general case: passage to the limit

Proof for the torsion case

Suppose $p^{n+1}V = 0$

$$V' := p^n V, \quad V'' := V/p^n V \implies pV' = p^n V'' = 0$$

$0 \to V' \to V \to V'' \to 0$ exact, hence

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V' \arrow[r, "\Psi_1"] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \arrow[r, "\Psi_2"] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V'' \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{D}(V') \arrow[r] \arrow[u, hook] & \mathbb{D}(V) \arrow[r] \arrow[u, hook] & \mathbb{D}(V'') \arrow[u, hook] &
\end{tikzcd}$$

- the top row is exact (flatness of $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} (-)$)
- the bottom row (applied $(-)^{G_E}$ to the top row) is exact (direct check)

Claim

$\mathbb{D}(V) \to \mathbb{D}(V'')$ is surjective.

$\mathbb{D}(V) \to \mathbb{D}(V'')$ is surjective:

- for $z \in \mathbb{D}(V'') \subset \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V''$, let $y \in \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ with $y \mapsto z$

$$0 \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V' \xrightarrow{\Psi_1} \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \xrightarrow{\Psi_2} \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V'' \to 0$$
$$y \mapsto z$$

- z is a G_E-invariant, hence
  $\delta(\sigma) := \sigma y - y \in \ker \Psi_2 = \operatorname{im} \Psi_1 \cong \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V' \cong E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V'$
- $p^{n+1} V = 0$ implies $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \cong \frac{\mathcal{O}_{\breve{\mathcal{E}}}}{(p^{n+1})} \otimes_{\mathbb{Z}_p} V \cong \frac{\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}}{(p^{n+1})} \otimes_{\mathbb{Z}_p} V \cong \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} V$
- let E'/E finite Galois extension such that $\mathrm{Gal}(E^{\mathrm{sep}}/E')$ fixes y

$$G_E \twoheadrightarrow \mathrm{Gal}(E'/E) =: G \xrightarrow{\delta} E' \otimes_{\mathbb{F}_p} V' \hookrightarrow E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V'$$
$$\sigma \mapsto \sigma y - y$$

- $pV' = 0$ $(n = 1)$ $\implies$ $E' \otimes_{\mathbb{F}_p} V' \cong E' \otimes_E \mathbb{D}(V')$, action via $\sigma \otimes \mathrm{id}_{\mathbb{D}(V')}$
- **Normal basis theorem**: *there is an E-basis of E' of the form $(\sigma(x))_{\sigma \in G}$.*
  Then there is an E-linear G-equivariant isomorphism

$$E[G] \to E' \qquad \sum_{\sigma \in G} \lambda_\sigma \cdot \sigma \mapsto \sum_{\sigma \in G} \lambda_\sigma \cdot \sigma(x)$$

- it follows that we have

$$E' \otimes_{\mathbb{F}_p} V' \cong E[G] \otimes_E \mathbb{D}(V') \cong (\bigoplus_\sigma E \cdot \sigma) \otimes_E \mathbb{D}(V) \cong \bigoplus_{\sigma \in G} \mathbb{D}(V') \cdot \sigma$$

- for $\tau \in G$, write

$$\delta(\tau) = \sum_{\sigma \in G} \delta(\tau)(\sigma) \cdot \sigma \in \bigoplus_\sigma \mathbb{D}(V') \cdot \sigma$$

  with $\delta(\tau)(\sigma) \in \mathbb{D}(V')$ for all $\sigma \in G$

$$x := \sum_{\sigma \in G} \delta(\sigma^{-1})(1) \cdot \sigma \in \bigoplus_\sigma \mathbb{D}(V') \cdot \sigma \cong E' \otimes_{\mathbb{F}_p} V'$$

- check $\delta(\tau) = \tau x - x$ for all $\tau \in G$
- $x \in E' \otimes_{\mathbb{F}_p} V' \hookrightarrow \ker \Psi_2$ hence $x + y \in \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ also maps to z
- moreover

$$\tau(x+y) - (x+y) = \tau y - y - (\tau x - x) = \delta(\tau) - (\tau x - x) = 0$$

  for all $\tau \in G$, so $x + y \in \mathbb{D}(V)$

$\square$ (claim)

We now have the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V' \arrow[r, "\Psi_1"] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \arrow[r, "\Psi_2"] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V'' \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{D}(V') \arrow[r] \arrow[u, hook] & \mathbb{D}(V) \arrow[r] \arrow[u, hook] & \mathbb{D}(V'') \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

Applying $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} (-)$ to the bottom row we get the following, with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V' \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V'' \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V') \arrow[r] \arrow[u] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) \arrow[r] \arrow[u] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V'') \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

By the induction hypothesis:

- the outer vertical arrows are bijections; hence, also the middle arrow
- $\mathbb{D}(V')$ and $\mathbb{D}(V'')$ are finitely generated over $\mathcal{O}_{\mathcal{E}}$; hence, also $\mathbb{D}(V)$

$\square$ (1),(2) torsion case

Consider an arbitrary exact $0 \to V' \to V \to V'' \to 0$ of $\mathbb{Z}_p$-torsion modules. The same diagram now gives

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V' \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V'' \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V') \arrow[r] \arrow[u, "\cong"'] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) \arrow[r] \arrow[u, "\cong"'] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V'') \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd}$$

- as before, we already know $0 \to \mathbb{D}(V') \to \mathbb{D}(V) \to \mathbb{D}(V'')$ is exact
- top row exact (flatness) $\implies$ bottom row exact (vertical isomorphisms)
- by faithful flatness $\mathbb{D}(V) \to \mathbb{D}(V'')$ is surjective

$\square$ (3) torsion case

General case: passage to the limit

Proposition

Let R be Noetherian, $\mathfrak{m}$ an ideal such that R is $\mathfrak{m}$-adically separated and complete; then any finitely generated R-module is $\mathfrak{m}$-adically separated and complete.

Let $V \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E)$ be arbitrary.

- $V \cong \varprojlim_m V/p^m V$ is p-adically separated and complete
- $V/p^m V \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E)$ is $\mathbb{Z}_p$-torsion

V is finitely generated over $\mathbb{Z}_p$, hence, by right-exactness of $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} (-)$, $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ finitely generated over $\mathcal{O}_{\breve{\mathcal{E}}}$. Then

$$\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \cong \varprojlim_m (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V)/p^m(\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V) \cong \varprojlim_m \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V/p^m V$$

- the first isomorphism follows from the proposition
- the second from applying $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} (-)$ to $V \xrightarrow{p^m} V \to V/p^m V \to 0$

- G_E acts componentwise on $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \cong \varprojlim_m \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V/p^m V \implies \mathbb{D}(V) \cong \varprojlim_m \mathbb{D}(V/p^m V)$
- $V/p^m V \xrightarrow{p^n} V/p^m V \to V/p^n V \to 0$ exact in $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(G_E)$ for $m \geq n \geq 1$
- apply $\mathbb{D}$: the sequence is exact because we are in the $\mathbb{Z}_p$-torsion case
- now apply $\varprojlim_{m \geq n}$:

$$\begin{tikzcd}
\mathbb{D}(V) \arrow[d, phantom, "\cong" rotate=90] & & \mathbb{D}(V) \arrow[d, phantom, "\cong" rotate=90] & & & \\
\varprojlim_{m\geq n} \mathbb{D}(V/p^m V) \arrow[r, "p^n"] \arrow[d, hook] & & \varprojlim_{m\geq n} \mathbb{D}(V/p^m V) \arrow[r, "\Psi"] \arrow[d, hook] & \mathbb{D}(V/p^n V) \arrow[r] \arrow[d, "\mathrm{diag}"] & 0 \\
\prod_{m\geq n} \mathbb{D}(V/p^m V) \arrow[r, "p^n"] & & \prod_{m\geq n} \mathbb{D}(V/p^m V) \arrow[r, "\Psi'"] & \prod_{m\geq n} \mathbb{D}(V/p^n V) \arrow[r] & 0
\end{tikzcd}$$

- the bottom row is exact because each factor makes an exact sequence; for the top row we have:

$$\ker \Psi = \ker \Psi' \cap \mathbb{D}(V) = \mathrm{im}\left(p^n \colon (y_m)_m \mapsto (p^n y_m)_m\right) \cap \mathbb{D}(V) =$$

$$\left\{(p^n y_m)_m \mid \varphi_m^{m'}(p^n y_{m'}) = p^n y_m \text{ for all } m' \geq m \geq n\right\} = \varprojlim_{m\geq n} p^n \mathbb{D}(V/p^m V)$$

$$\begin{tikzcd}
\mathbb{D}(V) \arrow[d, phantom, "\cong" rotate=90] & \mathbb{D}(V) \arrow[d, phantom, "\cong" rotate=90] & & \\
\varprojlim_{m\geq n} \mathbb{D}(V/p^m V) \arrow[r, "p^n"] \arrow[d, hook] & \varprojlim_{m\geq n} \mathbb{D}(V/p^m V) \arrow[r, "\Psi"] \arrow[d, hook] & \mathbb{D}(V/p^n V) \arrow[r] \arrow[d, "\text{diag}"] & 0 \\
\prod_{m\geq n} \mathbb{D}(V/p^m V) \arrow[r, "p^n"] & \prod_{m\geq n} \mathbb{D}(V/p^m V) \arrow[r, "\Psi'"] & \prod_{m\geq n} \mathbb{D}(V/p^n V) \arrow[r] & 0
\end{tikzcd}$$

To prove exactness at the first step it remains to prove

$$\varprojlim_{m\geq n} p^n\mathbb{D}(V/p^m V) = p^n \varprojlim_{m\geq n} \mathbb{D}(V/p^m V) =: \operatorname{im} p^n$$

- Remembering that $\mathbb{D}(V) \cong \varprojlim_{m\geq n} \mathbb{D}(V/p^m V)$, we want to prove: $\varprojlim_{m\geq n} p^n\mathbb{D}(V/p^m V) = p^n\mathbb{D}(V)$
- equivalently, prove surjectivity of

$$\mathbb{D}(V) \cong \varprojlim_{m\geq n} \mathbb{D}(V/p^m V) \to \varprojlim_{m\geq n} p^n\mathbb{D}(V/p^m V)$$
$$(x_m)_m \mapsto (p^n x_m)_m$$

The Mittag-Leffler Condition

An inverse system $(A_i, \varphi_{ji})_{i \leq j \in \mathbb{N}}$ is *Mittag-Leffler* if, for every fixed i, the sequence $\varphi_{ji}(A_j) \subset A_i$ $(j \geq i)$ becomes stationary, i.e. there exists $j(i)$ such that

$$\varphi_{ki}(A_k) = \varphi_{j(i),i}(A_{j(i)}) \quad \text{for all } k \geq j(i)$$

Proposition

Consider an exact sequence $0 \to A_i \to B_i \to C_i \to 0$ *of inverse systems of abelian groups, indexed over* $\mathbb{N}$.
If $(A_i, \varphi_{ji})_{i \leq j \in \mathbb{N}}$ *is Mittag-Leffler, then* $0 \to \varprojlim_i A_i \to \varprojlim_i B_i \to \varprojlim_i C_i \to 0$ *is exact; all we have to check is surjectivity of*

$$\varprojlim_i B_i \to \varprojlim_i C_i$$

$V_m := V/p^m V$, $V_m[p^n] := \ker(V_m \xrightarrow{p^n} V_m)$. Then

- $0 \to V_m[p^n] \to V_m \xrightarrow{p^n} p^n V_m \to 0$ exact in $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E)$, torsion modules
- applying $\mathbb{D}$, the resulting sequence is exact

$$0 \to \mathbb{D}(V_m[p^n]) \to \mathbb{D}(V_m) \to \mathbb{D}(p^n V_m) = p^n \mathbb{D}(V_m) \to 0$$

- we check the M-L condition on the first term, which will imply the exactness of

$$0 \to \varprojlim \mathbb{D}(V_m[p^n]) \to \varprojlim \mathbb{D}(V_m) \to \varprojlim p^n \mathbb{D}(V_m) \to 0$$

If $m' \geq m+n$, $v + p^{m'} V \in V_{m'}[p^n]$ then

$$p^n v \in p^{m'} V \subset p^{n+m} V \implies p^n v = p^{n+m} w \, (\exists w \in V) \implies v - p^m w \in V[p^n]$$

Therefore

- $\mathrm{im}(V_{m'}[p^n] \to V_m[p^n]) = \mathrm{im}(V[p^n] \to V_m[p^n])$ independently of m', i.e. $(V[p^n] + V_m)/V_m$
- applying $\mathbb{D}$, the same is true for $M_m := \mathbb{D}\left(\frac{V[p^n]+V_m}{V_m}\right)$.

Hence $(\mathbb{D}(V_m[p^n]))_m$ satisfies the Mittag-Leffler condition

Exactness at the second step: $\varprojlim_{m\geq n} \mathbb{D}(V/p^m V) \xrightarrow{\Psi} \mathbb{D}(V/p^n V) \to 0$

- For $m \geq n$ we have surjections $V/p^{m+1}V \twoheadrightarrow V/p^m V \twoheadrightarrow V/p^n V$,
- $\mathbb{Z}_p$-torsion modules, hence we have surjections

$$\mathbb{D}(V/p^{m+1}V) \twoheadrightarrow \mathbb{D}(V/p^m V) \twoheadrightarrow \mathbb{D}(V/p^n V)$$

- given $d_0 \in \mathbb{D}(V/p^n V)$, inductively choose $d_i \in \mathbb{D}(V/p^{n+i}V)$ so that

$$d_{i+1} \mapsto d_i \quad \text{through} \quad \mathbb{D}(V/p^{n+i+1}) \to \mathbb{D}(V/p^{n+i}V),$$

  therefore we found an element of $\mathbb{D}(V)$ mapping to d_0:

$$\varprojlim_{m\geq n} \mathbb{D}(V/p^m V) \cong \mathbb{D}(V) \to \mathbb{D}(V/p^n V)$$

$$(d_i)_{i\geq 0} \mapsto d_0$$

$\square$ (exactness)

To recap, we proved the exactness of

$$\mathbb{D}(V) \xrightarrow{p^n} \mathbb{D}(V) \to \mathbb{D}(V/p^n V) \to 0,$$

whence $\mathbb{D}(V/p^n V) \cong \mathbb{D}(V)/p^n\mathbb{D}(V)$

Proof of (2) in the general case

- $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ p-adically separated $\implies$ the same for $\mathbb{D}(V) \subset \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$
- $\mathbb{D}(V/pV) \cong \mathbb{D}(V)/p\mathbb{D}(V)$ is fin. gen. over $\mathcal{O}_{\mathcal{E}}/p\mathcal{O}_{\mathcal{E}} \cong E$ (torsion case)
- let $d_1, \dots, d_r \in \mathbb{D}(V)$ be such that mod $p\mathbb{D}(V)$ they form an E-basis of $\mathbb{D}(V/pV)$; then $\mathbb{D}(V) = p\mathbb{D}(V) + \sum_i \mathcal{O}_{\mathcal{E}} d_i$
- given $d = d^{(0)} \in \mathbb{D}(V)$, find $\lambda_i^{(0)} \in \mathcal{O}_{\mathcal{E}}, d^{(1)} \in \mathbb{D}(V)$ with
  $d^{(0)} - \sum_i \lambda_i^{(0)} d_i = pd^{(1)} \in p\mathbb{D}(V)$. Similarly find $\lambda_i^{(1)}$, $d^{(2)}$ with
  $d^{(1)} - \sum_i \lambda_i^{(1)} d_i = pd^{(2)} \in p\mathbb{D}(V)$
  Hence $d^{(0)} - \sum_i (\lambda_i^{(0)} + p\lambda_i^{(1)}) = p^2 d^{(2)} \in p^2\mathbb{D}(V)$
- Inductively, find $\lambda_i^{(j)} \in \mathcal{O}_{\mathcal{E}}$ and $d^{(j+1)} \in \mathbb{D}(V)$ such that
$$d - \sum_{i=1}^r \left(\sum_{r=0}^j p^r \lambda_i^{(r)}\right) d_i = p^{j+1} d^{(j+1)} \in p^{j+1}\mathbb{D}(V)$$
- $\lambda_i := \sum_{r=0}^\infty p^r \lambda_i^{(r)}$ converges in $\mathcal{O}_{\mathcal{E}}$ with $d - \sum_i \lambda_i d_i \in \bigcap_{j \geq 0} p^j \mathbb{D}(V) = 0$;
  hence $\mathbb{D}(V) = \sum_{i=1}^r \mathcal{O}_{\mathcal{E}} d_i$, i.e. $\mathbb{D}(V)$ is finitely generated over $\mathcal{O}_{\mathcal{E}}$

Proof of (1) in the general case

Consider the map $F\colon \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ of (1) for $V \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E)$.

$$(F \bmod p^n) : \frac{\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V)}{(p^n)} \to \frac{\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V}{(p^n)}$$

identifies with the map F_n below:

$$\frac{\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V)}{(p^n)} \cong \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \frac{\mathbb{D}(V)}{(p^n)} \cong \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(\tfrac{V}{p^n V}) \xrightarrow{F_n} \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \frac{V}{p^n V} \cong \frac{\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V}{(p^n)}$$

F_n are isomorphisms (torsion case); then so is F, as we now show:

- let $x \in \ker F$, then $(x \bmod p^n) \in \ker(F \bmod p^n) = 0$, hence $x \in \bigcap_{n \geq 1} p^n(\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V)) = 0$ because $\mathbb{D}(V)$ is fin. gen. over $\mathcal{O}_{\mathcal{E}}$
- $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V)$ fin. gen. over $\mathcal{O}_{\breve{\mathcal{E}}}$; by surjectivity of $F \bmod p$, choose $x_1, \ldots, x_r \in \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V)$ such that $F(x_1), \ldots, F(x_r) \bmod p$ generate $(\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V)/(p)$ over $\mathcal{O}_{\breve{\mathcal{E}}}/(p) = E^{\mathrm{sep}}$
- as before, this implies that $F(x_1), \ldots, F(x_r)$ generate $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ over $\mathcal{O}_{\breve{\mathcal{E}}}$, hence F is surjective

Proof of (3) in the general case

Same argument as before: an exact $0 \to V' \to V \to V'' \to 0$ gives

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V' \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V'' \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V') \arrow[r] \arrow[u, "\cong"'] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) \arrow[r] \arrow[u, "\cong"'] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V'') \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd}$$

- as before, we already know $0 \to \mathbb{D}(V') \to \mathbb{D}(V) \to \mathbb{D}(V'')$ is exact
- top row exact (flatness) $\implies$ bottom row exact (vertical isomorphisms)
- by faithful flatness $\mathbb{D}(V) \to \mathbb{D}(V'')$ is surjective

$\square$ general case

We complete the proof

Proposition

1. *for any* $M \in \Phi^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}$, $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$ *is an* $\mathcal{O}_{\breve{\mathcal{E}}}$*-linear,* G_E*-equivariant isomorphism of* φ*-modules over* $\mathcal{O}_{\breve{\mathcal{E}}}$
2. *the* $\mathbb{Z}_p$*-module* $\mathbb{V}(M)$ *is finitely generated*
3. *if* $0 \to M' \xrightarrow{g} M \xrightarrow{h} M'' \to 0$ *is exact in* $\Phi^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}$, *then*

$$0 \to \mathbb{V}(M') \xrightarrow{id \otimes g} \mathbb{V}(M) \xrightarrow{id \otimes h} \mathbb{V}(M'') \to 0$$

   is a well-defined exact sequence of $\mathcal{O}_{\mathcal{E}}$*-modules*

- case $pM = 0$: done
- torsion case ($p^n M = 0$ for some n)
- general case: passage to the limit

Proof for the torsion case

$(M, f) \in \Phi^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$ with $p^{n+1}M = 0$

$M' := p^n M, \quad M'' := M/p^n M$ with the induced φ-semilinear maps

- Recall: $0 \to M' \to M \to M'' \to 0$ exact, hence M', M'' are étale
- $0 \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M' \xrightarrow{\Phi} \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \xrightarrow{\Psi} \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M'' \to 0$ is exact (flatness)
- $0 \to \mathbb{V}(M') \to \mathbb{V}(M) \to \mathbb{V}(M'')$ is exact (direct check)

$\mathbb{V}(M) \to \mathbb{V}(M'')$ is surjective:

- let $z \in \mathbb{V}(M'')$, choose $y \in \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M : y \mapsto z$
- $(\varphi \otimes f \bmod p^n)(z) = z \implies y - (\varphi \otimes f)(y) \in \ker \Psi = \operatorname{im} \Phi = \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M'$
- $pM' = 0$ (case $n = 1$) $\implies \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M' \cong E^{\text{sep}} \otimes_E M' \cong E^{\text{sep}} \otimes_{\mathbb{F}_p} \mathbb{V}(M')$ iso of φ-modules
- E^{sep} separably closed
  $\implies (\mathrm{id}_{E^{\text{sep}}} - \varphi) \colon E^{\text{sep}} \to E^{\text{sep}}, \; x \mapsto x - x^p$ surjective
  $\implies (\mathrm{id}_{E^{\text{sep}}} - \varphi) \otimes \mathrm{id}_{\mathbb{V}(M')} \colon E^{\text{sep}} \otimes_{\mathbb{F}_p} \mathbb{V}(M') \to E^{\text{sep}} \otimes_{\mathbb{F}_p} \mathbb{V}(M')$ surjective
- this map is $\mathrm{id}_{E^{\text{sep}} \otimes_{\mathbb{F}_p} \mathbb{V}(M')} - \varphi \otimes \mathrm{id}_{\mathbb{V}(M')}$; so,
  $\mathrm{id}_{\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M} - \varphi \otimes f \colon \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M' \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M'$ surjective
- write $y - (\varphi \otimes f)(y) = x - (\varphi \otimes f)(x)$ with $x \in \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M'$:
  $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \ni y - x \mapsto z$ and $(\varphi \otimes f)(y - x) = y - x$, so $y - x \in \mathbb{V}(M)$

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M' \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M'' \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M') \arrow[r] \arrow[u, "\cong"'] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M) \arrow[r] \arrow[u] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M'') \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd}
$$

- exact rows (previous claim $+$ flatness)
- outer arrows bijective (induction hypothesis) $\implies$ middle arrow bijective

$\square$ (1) torsion case

- $0 \to \mathbb{V}(p^n M) \to \mathbb{V}(M) \to \mathbb{V}(M/p^n M) \to 0$ exact
- $\mathbb{V}(p^n M)$, $\mathbb{V}(M/p^n M)$ fin. gen. over $\mathbb{Z}_p$ (induction hypothesis) $\implies$ $\mathbb{V}(M)$ fin. gen.

$\square$ (2) torsion case

Let $0 \to M' \to M \to M'' \to 0$ be exact

- as recalled, $0 \to \mathbb{V}(M') \to \mathbb{V}(M) \to \mathbb{V}(M'')$ is exact

Moreover, we have

$$\begin{tikzcd}
\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M) \arrow[r, "\psi"] \arrow[d, "\cong"'] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M'') \arrow[d, "\cong"] \\
\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \arrow[r, "\varphi"'] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M''
\end{tikzcd}$$

- the vertical maps are bijections by (1)
- φ is surjective by right-exactness of $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} (-)$
- hence ψ is surjective
- $\mathbb{V}(M) \to \mathbb{V}(M'')$ surjective (faithful flatness)

$\square$ (3) torsion case

The general case

$M \in \Phi^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$

- M fin. gen. over $\mathcal{O}_{\mathcal{E}} \implies M \cong \varprojlim_n M/p^n M$
- $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$ fin. gen. over $\mathcal{O}_{\breve{\mathcal{E}}} \implies \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \cong \varprojlim_n (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)/(p^n)$
  $\cong \varprojlim_n \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \frac{M}{p^n M}$ (by right-exactness of $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} (-)$)
- $p^n M, M/p^n M$ φ-modules (induced structure) étale
- $\varphi \otimes f$ acts componentwise on $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \cong \varprojlim_n \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} (M/p^n M)$
  $\implies \mathbb{V}(M) \cong \varprojlim_n \mathbb{V}(M/p^n M)$
- $M \xrightarrow{p^n} M \to M/p^n M \to 0$ exact in $\Phi^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$
  $\implies \mathbb{V}(M) \xrightarrow{p^n} \mathbb{V}(M) \to \mathbb{V}(M/p^n M) \to 0$ exact (same proof as for $\mathbb{D}(V)$)
  $\implies \mathbb{V}(M)/p^n\mathbb{V}(M) \cong \mathbb{V}(M/p^n M)$
- $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$ p-adically separated $\implies \mathbb{V}(M)$ is too
- $\mathbb{V}(M)/p\mathbb{V}(M)$ fin. gen. over $\mathbb{Z}_p$ (torsion case)
  $\implies \mathbb{V}(M)$ fin. gen. over $\mathbb{Z}_p$ (same approximation argument as for $\mathbb{D}(V)$)

$\square$ (2) general case

Consider $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M) \xrightarrow{G} \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$

- $(G \mod p^n)$ are identified with $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M/p^nM) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M/p^nM$
- $(G \mod p^n)$ are bijective by the torsion case
- as before, this implies G is bijective

$\square$ (1) general case

Let $0 \to M' \to M \to M'' \to 0$ be exact

- as before, $0 \to \mathbb{V}(M') \to \mathbb{V}(M) \to \mathbb{V}(M'')$ is exact

$$\begin{tikzcd}
\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M) \arrow[r, "\psi"] \arrow[d, "\cong"'] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M'') \arrow[d, "\cong"] \\
\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \arrow[r, "\varphi"'] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M''
\end{tikzcd}$$

- the vertical maps are bijections by (1)
- φ is surjective by right-exactness of $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} (-)$
- hence ψ is surjective
- $\mathbb{V}(M) \to \mathbb{V}(M'')$ surjective (faithful flatness)

$\square$ (3) general case

The corollaries and the fact that $\mathbb{D}$, $\mathbb{V}$ are equivalences of categories were already proven for the $\mathbb{Z}_p$-case in Talk 3.

Hence, we have proven

$$\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E) \rightleftarrows \Phi^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$$

Definition

A φ-module (M, f) over $\mathcal{E}$ is said to have *slope 0* if

- $\dim_{\mathcal{E}} M$ is finite
- M has an f-stable $\mathcal{O}_{\mathcal{E}}$-lattice M_0 such that $(M_0, f) \in \Phi^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$

Recall

- An $\mathcal{O}_{\mathcal{E}}$-lattice is a finitely generated $\mathcal{O}_{\mathcal{E}}$-submodule of M which generates M over $\mathcal{E}$.
  Equivalently: it is a free $\mathcal{O}_{\mathcal{E}}$-module of rank $\dim_{\mathcal{E}} M$
- M_0 is f-stable if $f(M_0) \subseteq M_0$

Warning: φ-modules of slope 0 over $\mathcal{E}$ will be called *étale φ-modules over $\mathcal{E}$*, notation $\Phi^{\text{ét}}_{\mathcal{E}}$. However, they are **not** necessarily étale φ-modules in the sense of the previous definition!

Theorem

1. *for any* $V \in Rep^{cont}_{\mathbb{Q}_p}(G_E)$
   - $\mathbb{D}(V) := (\breve{\mathcal{E}} \otimes_{\mathbb{Q}_p} V)^{G_E} \in \Phi^{ét}_{\mathcal{E}}$ *via* $\varphi \otimes id_V$
   - $\breve{\mathcal{E}} \otimes_{\mathcal{E}} \mathbb{D}(V) \to \breve{\mathcal{E}} \otimes_{\mathbb{Q}_p} V$ *is a* G_E*-equivariant isomorphism of* φ*-mod. over* $\breve{\mathcal{E}}$
   - $\mathbb{D}\colon Rep^{cont}_{\mathbb{Q}_p}(G_E) \to \Phi^{ét}_{\mathcal{E}}$ *is exact*
2. *for any* $M \in \Phi^{ét}_{\mathcal{E}}$
   - $\mathbb{V}(M) := (\breve{\mathcal{E}} \otimes_{\mathcal{E}} M)^{\varphi=1} \in Rep^{cont}_{\mathbb{Q}_p}(G_E)$ *via* $\sigma \otimes id_M$
   - $\breve{\mathcal{E}} \otimes_{\mathbb{Q}_p} \mathbb{V}(M) \to \breve{\mathcal{E}} \otimes_{\mathcal{E}} M$ *is a* G_E*-equivariant isomorphism of* φ*-mod. over* $\breve{\mathcal{E}}$
   - $\mathbb{V}\colon \Phi^{ét}_{\mathcal{E}} \to Rep^{cont}_{\mathbb{Q}_p}(G_E)$ *is exact*
3. *the functors* $\mathbb{D}$ *and* $\mathbb{V}$ *are quasi-inverse to each other*

Proof of (1)

$V \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(G_E)$, $\Lambda \subset V$ G_E-stable $\mathbb{Z}_p$-lattice (see Talk 1)

- $\Lambda \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E)$, $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda \to V$ isomorphism in $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(G_E)$

$$\mathbb{D}(V) := (\breve{\mathcal{E}} \otimes_{\mathbb{Q}_p} V)^{G_E} \cong (\breve{\mathcal{E}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda)^{G_E} \cong (\breve{\mathcal{E}} \otimes_{\mathbb{Z}_p} \Lambda)^{G_E}$$

- $\breve{\mathcal{E}} = \bigcup_{n \geq 0} p^{-n} \mathcal{O}_{\breve{\mathcal{E}}}$ and $\mathcal{E} = \bigcup_{n \geq 0} p^{-n} \mathcal{O}_{\mathcal{E}}$
- $\mathcal{O}_{\breve{\mathcal{E}}} \to p^{-n} \mathcal{O}_{\breve{\mathcal{E}}}$, $\alpha \mapsto p^{-n}\alpha$ ($\mathcal{O}_{\breve{\mathcal{E}}}$-linear G_E-equivariant isomorphisms)

$$\mathbb{D}(V) \cong \Big(\big(\bigcup_{n\geq 0} p^{-n}\mathcal{O}_{\breve{\mathcal{E}}}\big) \otimes_{\mathbb{Z}_p} \Lambda\Big)^{G_E} \cong \bigcup_{n\geq 0} \left(p^{-n}\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \Lambda\right)^{G_E} \cong$$
$$\bigcup_{n\geq 0} p^{-n} \left(\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \Lambda\right)^{G_E} \cong \bigcup_{n\geq 0} p^{-n}\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(\Lambda) = \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(\Lambda)$$

- Since $\mathbb{D}(\Lambda) \in \Phi^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$, $\mathbb{D}(V)$ has slope 0

The natural map $\breve{\mathcal{E}} \otimes_{\mathcal{E}} \mathbb{D}(V) \to \breve{\mathcal{E}} \otimes_{\mathbb{Q}_p} V$ is

$$\breve{\mathcal{E}} \otimes_{\mathcal{E}} \mathbb{D}(V) \cong \breve{\mathcal{E}} \otimes_{\mathcal{E}} \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(\Lambda) \cong \breve{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(\Lambda) \cong$$
$$\breve{\mathcal{E}} \otimes_{\mathcal{O}_{\breve{\mathcal{E}}}} \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(\Lambda) \cong \breve{\mathcal{E}} \otimes_{\mathcal{O}_{\breve{\mathcal{E}}}} \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \Lambda \cong \breve{\mathcal{E}} \otimes_{\mathbb{Z}_p} \Lambda \cong$$
$$\breve{\mathcal{E}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda \cong \breve{\mathcal{E}} \otimes_{\mathbb{Q}_p} V$$

Finally, let $0 \to V' \to V \to V'' \to 0$ be exact in $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(G_E)$,
$\Lambda \subset V$ G_E-stable $\mathbb{Z}_p$-lattice

- $\Lambda' := V' \cap \Lambda \subset V'$, $\quad \Lambda'' := (\Lambda + V')/V' \subset V''$ $\quad G_E$-stable lattices
- $0 \to \Lambda' \to \Lambda \to \Lambda'' \to 0$ exact in $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E) \implies$
  $0 \to \mathbb{D}(\Lambda') \to \mathbb{D}(\Lambda) \to \mathbb{D}(\Lambda'') \to 0$ exact in $\Phi^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$

$\implies$ $0 \to \mathbb{D}(V') \to \mathbb{D}(V) \to \mathbb{D}(V'') \to 0$ is exact, since $\mathbb{D}(V) \cong \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(\Lambda)$

Proof of (2)

(M, f) of slope 0, $\quad M_0$ f-stable $\mathcal{O}_{\mathcal{E}}$-lattice, $\quad (M_0, f) \in \Phi^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$
Analogously as before:

$$\begin{aligned}
\mathbb{V}(M) &= (\breve{\mathcal{E}} \otimes_{\mathcal{E}} M)^{\varphi=1} \cong (\breve{\mathcal{E}} \otimes_{\mathcal{E}} \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{E}}} M_0)^{\varphi=1} \cong (\breve{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}} M_0)^{\varphi=1} \cong \\
&\cong \Big(\bigcup_{n \geq 0} p^{-n} \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M_0 \Big)^{\varphi=1} \cong \bigcup_{n \geq 0} (p^{-n} \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M_0)^{\varphi=1} \cong \\
&\cong \bigcup_{n \geq 0} p^{-n} (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M_0)^{\varphi=1} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{V}(M_0)
\end{aligned}$$

- $\mathbb{V}(M_0) \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E)$, hence $\mathbb{V}(M) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{V}(M_0) \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(G_E)$ (action via $\mathrm{id}_{\mathbb{Q}_p} \otimes \sigma$)
- Rest of the proof as in (1).

Proof of (3)

The functors are quasi-inverse to each other:

$$\mathbb{D}(\mathbb{V}(M)) \overset{(2)}{\cong} \mathbb{D}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{V}(M_0)) \overset{(1)}{\cong} \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(\mathbb{V}(M_0)) \cong \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{E}}} M_0 \cong M$$

$$\mathbb{V}(\mathbb{D}(V)) \overset{(1)}{\cong} \mathbb{V}(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(\Lambda)) \overset{(2)}{\cong} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{V}(\mathbb{D}(\Lambda)) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda \cong V$$