Master Seminar – p-adic Galois Representations
Talk 3: Étale φ-modules.

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The goal of talks 3 and 4

- Define the category $\Phi_R^{\text{ét}}$ of *étale φ-modules* over a ring R
- For a field E of characteristic p, $G_E := \mathrm{Gal}(E^{\mathrm{sep}}/E)$ prove category equivalences between $\mathbb{F}_p$-representations (resp. $\mathbb{Z}_p$-, $\mathbb{Q}_p$-) of G_E and the categories of étale φ-modules over specific rings

$$\mathrm{Rep}_{\mathbb{F}_p}^{\mathrm{cont}}(G_E) \rightleftarrows \Phi_E^{\text{ét}} \qquad \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cont}}(G_E) \rightleftarrows \Phi_{\mathcal{O}_{\mathcal{E}}}^{\text{ét}} \qquad \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cont}}(G_E) \rightleftarrows \Phi_{\mathcal{E}}^{\text{ét}}$$

$M \in R$-Mod, $f\colon M \to M$, $\varphi \in \mathrm{End}(R)$

Definition

- f is *φ-semilinear* if $f(m+m') = f(m) + f(m')$ and $f(rm) = \varphi(r)f(m)$
- A *φ-module* over R is (M, f) where f is φ-semilinear
- A morphism $(M, f) \to (N, g)$ is $F \in \mathrm{Hom}_R(M, N)$ such that $F \circ f = g \circ F$

Let $R \in R$-Mod via $r \cdot s := \varphi(r)s$

- $R\ {}_{\varphi}\!\otimes_R M := R \otimes_R M, \quad r * (r' \otimes m) := rr' \otimes m$
- $\psi\colon R \times M \to M, \quad (r, m) \mapsto rf(m) \quad$ is R-balanced: $\mathbb{Z}$-linear and

$$\psi(r \cdot s, m) = \psi(\varphi(r)s, m) = \varphi(r)sf(m) = sf(rm) = \psi(s, rm)$$

Then

$$\begin{tikzcd} R \times M \arrow[r] \arrow[dr, "\psi"] & R \otimes_R M \arrow[d, dashed, "f_R"] \\ & M \end{tikzcd}$$

group hom. $f_R\colon R\ {}_{\varphi}\!\otimes_R M \to M, \quad r \otimes m \mapsto rf(m)$

$$f_R \colon R\ {}_{\varphi}\!\otimes_R M \to M, \quad r \otimes m \mapsto rf(m)$$

is actually an R-Mod morphism:

$$f_R(r * (r' \otimes m)) = f_R(rr' \otimes m) = rr'f(m) = rf_R(r' \otimes m).$$

Definition

(M, f) is *étale* if M is finitely generated and f_R is bijective.
Denote this category by $\Phi_R^{\text{ét}}$.

We will need the following

Proposition

Let R be a d.v.r., $k = R/\mathfrak{m}$ its residue field.
Let $\varphi \colon R \to R$ be a ring morphism with $\varphi(\mathfrak{m})R = \mathfrak{m}$; then

1. *a φ-module (M, f) over R with M fin. gen. is étale iff f_R is surjective;*
2. *if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of φ-modules, then M is étale iff M' and M'' are.*

The objects involved

1. E is a complete d.v.f. of char. $p > 0$
   $E^{\mathrm{sep}} = \{x \in \overline{E} \mid x \text{ separable over } E\}$
2. $\mathcal{O}_{\mathcal{E}}$ is a Cohen ring of E ($=$ c.d.v.r. C with max. ideal pC and $C/pC = E$)
3. $\mathcal{E} := \mathrm{Quot}(\mathcal{O}_{\mathcal{E}})$ is a c.d.v.f. with residue field $\mathcal{O}_{\mathcal{E}}/p\mathcal{O}_{\mathcal{E}} \cong E$
4. $\mathcal{E}^{\mathrm{ur}} = \cup_{F/\mathcal{E} \text{ fin.ur.}} F$ is the maximal unramified extension of $\mathcal{E}$
   - $\mathcal{E}^{\mathrm{ur}}$ is a d.v.f. with $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} = \cup_{F/\mathcal{E} \text{ fin.ur.}} \mathcal{O}_F$ and res. field $\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} = E^{\mathrm{sep}}$
   - $\mathcal{E}^{\mathrm{ur}}/\mathcal{E}$ is Galois with a canonical isomorphism

     $$\mathrm{Gal}(\mathcal{E}^{\mathrm{ur}}/\mathcal{E}) \xrightarrow{\sim} G_E := \mathrm{Gal}(E^{\mathrm{sep}}/E), \quad \sigma \mapsto \overline{\sigma}$$

     where $\overline{\sigma}(x + p\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}) := \sigma(x) + p\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$
5. $\breve{\mathcal{E}} := \widehat{\mathcal{E}^{\mathrm{ur}}}$ is the completion of $\mathcal{E}^{\mathrm{ur}}$
   - it is a c.d.v.f. with $\mathcal{O}_{\breve{\mathcal{E}}} = \varprojlim_n \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p^n\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ and res. field $\mathcal{O}_{\breve{\mathcal{E}}}/p\mathcal{O}_{\breve{\mathcal{E}}} = E^{\mathrm{sep}}$
6. $\varphi\colon \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}$ is a Frobenius lift, i.e. $\varphi(x) \equiv x^p \bmod p\mathcal{O}_{\mathcal{E}}$ for all $x \in \mathcal{O}_{\mathcal{E}}$
   - φ extends uniquely to a Frobenius lift $\varphi\colon \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \to \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$
   - φ extends further to a Frobenius lift $\varphi\colon \mathcal{O}_{\breve{\mathcal{E}}} \to \mathcal{O}_{\breve{\mathcal{E}}}$

Example

Suppose E is a c.d.v.f. of char. $p > 0$ with perfect res. field k. Then

1. $k((t)) \cong E$, $\sum_i a_i t^i \mapsto \sum_i r(a_i)\pi^i$ for a canonical $r\colon k \to \mathcal{O}_E$
2. if $W(k)$ is a Cohen ring for k, then a Cohen ring for $k((t))$ is

$$\left\{ \sum_{n\in\mathbb{Z}} c_n t^n \mid c_n \in W(k), \lim_{n\to-\infty} v(c_n) = \infty \right\} =: \mathcal{O}_{\mathcal{E}}$$

Definition (Ring of Witt vectors)

- $\Phi_n(X_0, \ldots, X_n) := \sum_{i=0}^n p^i X_i^{p^{n-i}} \in \mathbb{Z}[X_0, X_1, \ldots]$ n-th Witt polynomial
- map of sets $\Phi_k\colon k^{\mathbb{N}} \to k^{\mathbb{N}}$, $(a_n)_n \mapsto (\Phi_n(a_0, \ldots, a_n))_n$
- There is a ring $W(k) = (k^{\mathbb{N}}, \oplus, \odot)$ such that $\Phi_k\colon W(k) \to (k^{\mathbb{N}}, +, \cdot)$ becomes a ring homomorphism

   For k perfect, $W(k)$ is the unique (up to iso) absolutely unramified $(v(p) = 1)$ c.d.v.r. with res. field k ($=$ a Cohen ring)

3. a Frobenius lift is $\varphi\colon \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}$, $t \mapsto (1+t)^p - 1$

$$\begin{array}{ccccc} \mathcal{E} & \subset & \mathcal{E}^{\mathrm{ur}} & \subset & \breve{\mathcal{E}} \\ \mathcal{O}_{\mathcal{E}}/(p) = E & \subset & \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/(p) = E^{\mathrm{sep}} & = & \mathcal{O}_{\breve{\mathcal{E}}}/(p) = E^{\mathrm{sep}} \end{array}$$

Universal property of completion

Let $M, N \in R$-Mod have the topologies defined by descending sequences of submodules $(M_j)_j$, $(N_j)_j$ respectively. Assume N is Hausdorff and complete. Then, any R-linear continuous $\varphi\colon M \to N$ extends uniquely to an R-linear continuous $\widehat{\varphi}\colon \widehat{M} \to N$.

Consequence

$G_E \cong \mathrm{Gal}(\mathcal{E}^{\mathrm{ur}}/\mathcal{E})$ acts on $\mathcal{E}^{\mathrm{ur}}$. G_E acts on $\mathcal{O}_{\breve{\mathcal{E}}}$:

$$G_E \times \mathcal{O}_{\breve{\mathcal{E}}} \to \mathcal{O}_{\breve{\mathcal{E}}}, \quad (\sigma, \alpha) \mapsto \widehat{\sigma}(\alpha)$$

where $\sigma\colon \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}} \to \mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}$ extends uniquely to $\widehat{\sigma}\colon \mathcal{O}_{\breve{\mathcal{E}}} \to \mathcal{O}_{\breve{\mathcal{E}}}$
By uniqueness: $\widehat{\sigma\tau} = \widehat{\sigma}\widehat{\tau}$, $\widehat{\mathrm{id}} = \mathrm{id}$, $\widehat{\sigma^{-1}} = \widehat{\sigma}^{-1}$
G_E acts on $\breve{\mathcal{E}}$: write $x = \alpha/\beta$ with $\alpha, \beta \in \mathcal{O}_{\breve{\mathcal{E}}}$

$$G_E \times \breve{\mathcal{E}} \to \breve{\mathcal{E}}, \quad (\sigma, x) \mapsto \widehat{\sigma}(\alpha)/\widehat{\sigma}(\beta) =: \sigma(x)$$

We want to prove category equivalences

$$\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{F}_p}(G_E) \underset{\mathbb{V}}{\overset{\mathbb{D}}{\rightleftarrows}} \Phi^{\text{ét}}_E \qquad \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E) \underset{\mathbb{V}}{\overset{\mathbb{D}}{\rightleftarrows}} \Phi^{\text{ét}}_{\mathcal{O}_\mathcal{E}} \qquad \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(G_E) \underset{\mathbb{V}}{\overset{\mathbb{D}}{\rightleftarrows}} \Phi^{\text{ét}}_{\mathcal{E}}$$

Equivalence of categories
$=$ there are isomorphisms of functors $\mathbb{V} \circ \mathbb{D} \cong \mathrm{id}$, $\mathbb{D} \circ \mathbb{V} \cong \mathrm{id}$
$=$ there are canonical isomorph.'s $\mathbb{V}(\mathbb{D}(V)) \cong V$, $\mathbb{D}(\mathbb{V}(M)) \cong M$ for all V, M.

We are going to

- define explicitly the functors $\mathbb{D}$ and $\mathbb{V}$;
  show that they are quasi-inverse to each other
- **today:** state the theorems for $\mathbb{Z}_p$, prove the result for $\mathbb{F}_p$ as a special case
- **next week:** prove the result for $\mathbb{Z}_p$ and for $\mathbb{Q}_p$

Proposition

1. $\mathcal{O}_{\breve{\mathcal{E}}}^{G_E} = \{x \in \mathcal{O}_{\breve{\mathcal{E}}} \mid \sigma(x) = x \ \ \forall \sigma \in G_E\} = \mathcal{O}_{\mathcal{E}}, \quad \breve{\mathcal{E}}^{G_E} = \mathcal{E}$
2. $\mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1} = \{x \in \mathcal{O}_{\breve{\mathcal{E}}} \mid \varphi(x) = x\} \cong \mathbb{Z}_p, \quad \breve{\mathcal{E}}^{\varphi=1} \cong \mathbb{Q}_p$

Proof of 1.

$\mathcal{O}_{\mathcal{E}} \subset \mathcal{O}_{\breve{\mathcal{E}}}^{G_E}$ is clear. Conversely, let $x \in \mathcal{O}_{\breve{\mathcal{E}}}^{G_E}$;

$$x + p\mathcal{O}_{\breve{\mathcal{E}}} \in (\mathcal{O}_{\breve{\mathcal{E}}}/p\mathcal{O}_{\breve{\mathcal{E}}})^{G_E} \cong (\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}}/p\mathcal{O}_{\mathcal{E}^{\mathrm{ur}}})^{G_E} = (E^{\mathrm{sep}})^{G_E} = E \cong \mathcal{O}_{\mathcal{E}}/p\mathcal{O}_{\mathcal{E}}$$

- there is $x_1 \in \mathcal{O}_{\mathcal{E}}$ with $x - x_1 \in p\mathcal{O}_{\breve{\mathcal{E}}}$, say $x - x_1 = px_1'$
- $x \in \mathcal{O}_{\breve{\mathcal{E}}}^{G_E}$ and $x_1 \in \mathcal{O}_{\mathcal{E}} \subset \mathcal{O}_{\breve{\mathcal{E}}}^{G_E} \implies px_1' \in \mathcal{O}_{\breve{\mathcal{E}}}^{G_E}$, so $x_1' \in \mathcal{O}_{\breve{\mathcal{E}}}^{G_E}$:

$$\sigma(px_1') = px_1' \iff p\sigma(x_1') = px_1' \iff p(\sigma x_1' - x_1') = 0 \iff \sigma x_1' = x_1'$$

- Inductively, find $x_i \in \mathcal{O}_{\mathcal{E}}$, $x_i' \in \mathcal{O}_{\breve{\mathcal{E}}}$ with $x = \sum_{i=1}^n p^{i-1}x_i + p^n x_n'$
  $\sum_{i=1}^n p^{i-1}x_i$ converges in $\mathcal{O}_{\mathcal{E}}$ (by completeness) to x, hence $x \in \mathcal{O}_{\mathcal{E}}$

$\breve{\mathcal{E}}^{G_E} = \mathcal{E}$ follows from $\breve{\mathcal{E}} = \{xp^n \mid x \in \mathcal{O}_{\breve{\mathcal{E}}}, n \in \mathbb{Z}\}$ and $p \in \mathcal{O}_{\mathcal{E}} = \mathcal{O}_{\breve{\mathcal{E}}}^{G_E}$. $\square$

Proposition

1. $\mathcal{O}_{\breve{\mathcal{E}}}^{G_E} = \{x \in \mathcal{O}_{\breve{\mathcal{E}}} \mid \sigma(x) = x \ \ \forall \sigma \in G_E\} = \mathcal{O}_{\mathcal{E}}, \quad \breve{\mathcal{E}}^{G_E} = \mathcal{E}$
2. $\mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1} = \{x \in \mathcal{O}_{\breve{\mathcal{E}}} \mid \varphi(x) = x\} \cong \mathbb{Z}_p, \quad \breve{\mathcal{E}}^{\varphi=1} \cong \mathbb{Q}_p$

Proof of 2.

- $\mathcal{O}_{\breve{\mathcal{E}}}$ separated and complete, φ cont. $\implies$ $\mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1}$ separated and complete; hence $\mathbb{Z} \to \mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1}$ extends to $\mathbb{Z}_p \to \mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1}$
- $\mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1}$ domain $\implies$ $\ker(\mathbb{Z}_p \to \mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1})$ is prime, i.e. (0) or (p).
  It is (0) because $p\mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1} \neq 0$. Hence $\mathbb{Z}_p \to \mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1}$ is injective.
- surjectivity: $x \in \mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1}$, work mod p: $\overline{x} = \overline{\varphi(x)} = \overline{x}^p \implies \overline{x} \in \mathbb{F}_p$
  hence $\mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1}/(p) \subset \mathbb{F}_p$ and by approximation as before we get $\mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1} \subset \mathbb{Z}_p$

For the second point

- $\ker(\varphi\colon \mathcal{O}_{\breve{\mathcal{E}}} \to \mathcal{O}_{\breve{\mathcal{E}}})$ is prime, char $\mathcal{O}_{\mathcal{E}} \neq p$, so φ is injective and therefore extends to $\varphi\colon \breve{\mathcal{E}} \to \breve{\mathcal{E}}$ with $\varphi(p) = p$
- $\breve{\mathcal{E}} = \{xp^n \mid x \in \mathcal{O}_{\breve{\mathcal{E}}}, n \in \mathbb{Z}\}$, hence $\breve{\mathcal{E}}^{\varphi=1} = \{xp^n \mid x \in \mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1}, n \in \mathbb{Z}\} = \mathbb{Q}_p$

$\square$

$$V \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E) \implies \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \begin{cases} \in \mathcal{O}_{\breve{\mathcal{E}}}\text{-Mod via } \alpha \cdot (\beta \otimes v) := \alpha\beta \otimes v \\ \text{is a } \varphi\text{-module via } \varphi \otimes \mathrm{id}_V \ (\varphi\text{-semilinear}) \\ G_E \text{ acts via } \sigma \otimes \sigma : \sigma(r \otimes v) = \sigma(r) \otimes \sigma(v) \end{cases}$$

$$\mathbb{D}(V) := (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V)^{G_E} = \{m \in \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \mid \sigma(m) = m \ \ \forall \sigma \in G_E\}$$

$(\mathbb{D}(V), f := (\varphi \otimes \mathrm{id}_V)|_{\mathbb{D}(V)})$ is a φ-module over $\mathcal{O}_{\mathcal{E}}$

- $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \in \mathcal{O}_{\breve{\mathcal{E}}}\text{-Mod} \subset \mathcal{O}_{\mathcal{E}}\text{-Mod}$; $\mathbb{D}(V)$ is an $\mathcal{O}_{\mathcal{E}}$-submodule of $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$:

$$\sigma(\alpha \cdot (\beta \otimes v)) = \sigma(\alpha\beta \otimes v) = \sigma(\alpha\beta) \otimes \sigma(v) = \alpha\sigma(\beta) \otimes \sigma(v) = \alpha \cdot \sigma(\beta \otimes v)$$

- $\sigma^{-1} \circ \varphi \circ \sigma \colon \mathcal{O}_{\breve{\mathcal{E}}} \to \mathcal{O}_{\breve{\mathcal{E}}}$ equals φ on $\mathcal{O}_{\mathcal{E}}$ since $G_E \cong \mathrm{Gal}(\mathcal{E}^{\mathrm{ur}}/\mathcal{E})$
  By uniqueness $\sigma^{-1} \circ \varphi \circ \sigma = \varphi$, i.e. $\varphi \circ \sigma = \sigma \circ \varphi$ on $\mathcal{O}_{\breve{\mathcal{E}}}$ (hence on $\breve{\mathcal{E}}$)
- $\varphi \otimes \mathrm{id}_V$ restricts to $f \colon \mathbb{D}(V) \to \mathbb{D}(V)$: for $m = \sum_i \beta_i \otimes v_i \in \mathbb{D}(V)$

$$\begin{aligned} \sigma \cdot (\varphi \otimes \mathrm{id}_V)(m) &= \sigma(\textstyle\sum_i \varphi(\beta_i) \otimes v_i) = \sum_i \sigma\varphi(\beta_i) \otimes \sigma v_i = \\ &= \textstyle\sum_i \varphi\sigma(\beta_i) \otimes \sigma v_i = (\varphi \otimes \mathrm{id}_V)(\sum_i \sigma\beta_i \otimes \sigma v_i) = (\varphi \otimes \mathrm{id}_V)(m) \end{aligned}$$

$$
\begin{array}{ll}
(\mathbb{D}(V),\ f := (\varphi \otimes \mathrm{id}_V)|_{\mathbb{D}(V)}) & \text{is a } \varphi\text{-module over } \mathcal{O}_{\mathcal{E}} \\
(\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V),\ \varphi \otimes f) & \text{is a } \varphi\text{-module over } \mathcal{O}_{\breve{\mathcal{E}}} \text{ and } G_E \text{ acts via } \sigma \otimes \mathrm{id}_{\mathbb{D}(V)}
\end{array}
$$

There is a natural $\mathcal{O}_{\breve{\mathcal{E}}}$-linear map $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$:

$$
\mathcal{O}_{\breve{\mathcal{E}}} \times (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V, \quad (\alpha, \beta \otimes v) \mapsto \alpha\beta \otimes v
$$

restricts to $\mathcal{O}_{\breve{\mathcal{E}}} \times \mathbb{D}(V) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$, $\mathcal{O}_{\mathcal{E}}$-balanced:

$$
(\alpha\alpha', \beta \otimes v) \mapsto \alpha\alpha'\beta \otimes v \leftarrow\!\shortmid (\alpha, \alpha'\beta \otimes v) \quad \text{for } \alpha' \in \mathcal{O}_{\mathcal{E}}
$$

hence induces a group hom.

$$
\begin{tikzcd}
\mathcal{O}_{\breve{\mathcal{E}}} \times \mathbb{D}(V) \arrow[r] \arrow[dr] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) \arrow[d, dashed] \\
& \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta \otimes (\sum_i \alpha_i \otimes v_i) \arrow[d, mapsto] \\
\sum_i \beta\alpha_i \otimes v_i
\end{tikzcd}
$$

which is also $\mathcal{O}_{\breve{\mathcal{E}}}$-linear

Proposition

1. *for any* $V \in Rep^{cont}_{\mathbb{Z}_p}(G_E)$, $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ *is an* $\mathcal{O}_{\breve{\mathcal{E}}}$*-linear,* G_E*-equivariant isomorphism of* φ*-modules over* $\mathcal{O}_{\breve{\mathcal{E}}}$
2. *the* $\mathcal{O}_{\mathcal{E}}$*-module* $\mathbb{D}(V)$ *is finitely generated*
3. *if* $0 \to V' \xrightarrow{g} V \xrightarrow{h} V'' \to 0$ *is a s.e.s. in* $Rep^{cont}_{\mathbb{Z}_p}(G_E)$ *(i.e. a s.e.s. of* $\mathbb{Z}_p$*-modules with* g, h *continuous* G_E*-equivariant* $\mathbb{Z}_p$*-linear maps), then*

$$0 \to \mathbb{D}(V') \xrightarrow{id \otimes g} \mathbb{D}(V) \xrightarrow{id \otimes h} \mathbb{D}(V'') \to 0$$

is a well-defined exact sequence of $\mathcal{O}_{\mathcal{E}}$*-modules*

Corollary

The φ*-module* $(\mathbb{D}(V), f)$ *is étale, hence we have a functor*

$$\mathbb{D}\colon Rep^{cont}_{\mathbb{Z}_p}(G_E) \to \Phi^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}, \quad V \mapsto (\mathbb{D}(V), f)$$

Special case $pV = 0$ ($V = V/pV$ is a $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$-vector space)

for $\alpha \otimes v \in \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$

$$p \cdot (\alpha \otimes v) = p\alpha \otimes v = \alpha \otimes pv = 0,$$

hence the $\mathbb{Z}_p$-balanced map $\mathcal{O}_{\breve{\mathcal{E}}} \times V \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ factors through an $\mathbb{F}_p$-balanced map $E^{\text{sep}} \times V \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$:

$$\begin{tikzcd} \mathcal{O}_{\breve{\mathcal{E}}} \times V \arrow[r] \arrow[dr] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \arrow[d, dashed, shift right] \\ & E^{\text{sep}} \times V \arrow[u, shift right] \end{tikzcd} \qquad (E^{\text{sep}} \cong \mathcal{O}_{\breve{\mathcal{E}}}/p\mathcal{O}_{\breve{\mathcal{E}}})$$

Univ. prop. $\otimes \implies$ the induced map $E^{\text{sep}} \otimes_{\mathbb{F}_p} V \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ is bijective:

$$\begin{tikzcd} \mathcal{O}_{\breve{\mathcal{E}}} \times V \arrow[r] \arrow[dr] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \arrow[d, dashed, shift right] & E^{\text{sep}} \otimes_{\mathbb{F}_p} V \arrow[l, dashed] \\ & E^{\text{sep}} \times V \arrow[u, shift right] \arrow[ur] & \end{tikzcd}$$

- V has the p-adic topology with $pV = 0 \implies V$ is discrete
- as in Talk 1, the G_E-action on V factors through finite $G := \mathrm{Gal}(E'/E) \cong G_E/G_{E'}$
- $(E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V)^{G_{E'}} \cong (E^{\mathrm{sep}} \oplus \cdots \oplus E^{\mathrm{sep}})^{G_{E'}} \cong E' \otimes_{\mathbb{F}_p} V$
- Taking G-invariants: $\mathbb{D}(V) \cong (E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V)^{G_E} = ((E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V)^{G_{E'}})^G \cong (E' \otimes_{\mathbb{F}_p} V)^G$

Hence it suffices to prove that $E' \otimes_E \mathbb{D}(V) \to E' \otimes_{\mathbb{F}_p} V$ is bijective, then apply $E^{\mathrm{sep}} \otimes_{E'} (-)$

Remark

A ring homomorphism $\varphi\colon R \to S$ is called *flat* if $S \otimes_R (-)$ is exact, i.e. if $0 \to M' \to M \to M'' \to 0$ exact in R-Mod implies $0 \to S \otimes_R M' \to S \otimes_R M \to S \otimes_R M'' \to 0$ exact in $\mathbb{Z}$-Mod

Proposition

$\varphi\colon R \to S$ is flat in each of the following cases:

1. S is free as an R-module via φ;
2. φ is injective, R is a p.i.d., S is an integral domain.

$E' \otimes_E \mathbb{D}(V) \to E' \otimes_{\mathbb{F}_p} V$ surjective:

- pick $(\lambda_1, \dots, \lambda_d)$ E-basis of E', $G = \{\sigma_1, \dots, \sigma_d\}$
- $\{\sigma_1, \dots, \sigma_d\}$ lin. independent set in the E'-vector space $\mathrm{End}_E(E')$
- the matrix $(\sigma_j(\lambda_i))_{ij} \in M_{d \times d}(E')$ is invertible
  (otherwise there are $a_1, \dots, a_d$ not all zero such that $\sum_j a_j \sigma_j(\lambda_i) = 0$ for all i, so $\sum_j a_j \sigma_j \in \mathrm{End}_E(E')$ is zero on a basis, hence $\sum_j a_j \sigma_j = 0$)
- for $w \in E' \otimes_{\mathbb{F}_p} V$, set $w_i := \sum_{j=1}^d \sigma_j(\lambda_i w) = \sum_{j=1}^d \sigma_j(\lambda_i)\sigma_j(w) \in \mathbb{D}(V)$; if $(a_{ij})_{ij}$ is the inverse matrix, we get $\sigma_i(w) = \sum_j a_{ij} w_j$
- since $\sigma_1 = \mathrm{id}$, w lies in the image

$E' \otimes_E \mathbb{D}(V) \to E' \otimes_{\mathbb{F}_p} V$ injective:

- $(v_i)_i$ E-basis of $\mathbb{D}(V)$ $\implies$ $(1 \otimes v_i)_i$ E'-basis of $E' \otimes_E \mathbb{D}(V)$
- claim: the $(v_i)_i$'s, seen inside $E' \otimes_{\mathbb{F}_p} V$, are lin. indep. over E':
  otherwise choose $\sum_{i=1}^r a_i v_i$ with minimal r and $a_1 = 1$

  by assumption $r \geq 2$ and not all a_i belong to E, so w.l.o.g. $a_2 \in E' \setminus E$;
  then, since $(E')^G = E$, there exists $\sigma \in G$ such that $\sigma(a_2) \neq a_2$

  then $0 = \sum_{i=2}^r (\sigma(a_i) - a_i) v_i$ is a dependency relation of length $\leq r - 1$

$E' \otimes_E \mathbb{D}(V) \to E' \otimes_{\mathbb{F}_p} V$ isomorphism
$\implies \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ isomorphism $\square(1)$

Consequence

$\dim_{\mathbb{F}_p}(V) = \dim_{E'}(E' \otimes_{\mathbb{F}_p} V) = \dim_{E'}(E' \otimes_E \mathbb{D}(V)) = \dim_E \mathbb{D}(V) < \infty$
$\implies \mathbb{D}(V)$ is a finitely generated $\mathcal{O}_{\mathcal{E}}$-module ($E$-module) $\square(2)$

Proof of 3

Let $0 \to V' \to V \to V'' \to 0$ be exact in $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{F}_p}(G_E)$. Then we have a commutative diagram of E^{sep}-modules

$$\begin{tikzcd}
0 \arrow[r] & E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V' \arrow[r] & E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V \arrow[r] & E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} V'' \arrow[r] & 0 \\
0 \arrow[r] & E^{\mathrm{sep}} \otimes_E \mathbb{D}(V') \arrow[r] \arrow[u] & E^{\mathrm{sep}} \otimes_E \mathbb{D}(V) \arrow[r] \arrow[u] & E^{\mathrm{sep}} \otimes_E \mathbb{D}(V'') \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

the first row is exact by flatness, hence the second row is exact. Then $0 \to \mathbb{D}(V') \to \mathbb{D}(V) \to \mathbb{D}(V'') \to 0$ is exact by the following

Lemma

Let $(R, \mathfrak{m}), (S, \mathfrak{n})$ be local rings, $\varphi\colon R \to S$ flat with $\varphi(\mathfrak{m})S = \mathfrak{n}$; then an R-module morphism $f\colon M \to N$ is surjective (resp. injective) iff $\mathrm{id}_S \otimes f\colon S \otimes_R M \to S \otimes_R N$ is surjective (resp. injective).

Corollary

The φ-module $(\mathbb{D}(V), f) = ((\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V)^{G_E}, (\varphi \otimes id_V)|_{\mathbb{D}(V)})$ over $\mathcal{O}_{\mathcal{E}}$ is étale.

We must only check that the linearization

$$f_{\mathcal{O}_{\mathcal{E}}} \colon \mathcal{O}_{\mathcal{E}} \ {}_{\varphi}\!\otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) \to \mathbb{D}(V) \qquad\qquad r \otimes m \mapsto rf(m)$$

is surjective. Applying $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} (-)$, by faithful flatness it is enough to prove

$$\begin{aligned} \mathrm{id} \otimes f \colon \mathcal{O}_{\breve{\mathcal{E}}} \ {}_{\varphi}\!\otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) &\to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) \\ s \otimes m &\mapsto s \otimes f(m) \end{aligned}$$

is surjective. Composing with $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) \xrightarrow{\sim} \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ we get

$$\begin{aligned} \mathcal{O}_{\breve{\mathcal{E}}} \ {}_{\varphi}\!\otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V) &\to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \\ s \otimes \sum_i r_i \otimes v_i &\mapsto \sum_i s\varphi(r_i) \otimes v_i \end{aligned}$$

which is obtained applying $(-) \otimes_{\mathbb{Z}_p} V$ to

$$\mathcal{O}_{\breve{\mathcal{E}}} \ {}_{\varphi}\!\otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\breve{\mathcal{E}}}, \quad s \otimes r \mapsto s\varphi(r)$$

which is surjective ($s \otimes 1 \mapsto s$), hence the resulting map is surjective by right-exactness of $(-) \otimes_{\mathbb{Z}_p} V$.

We have an exact functor $\mathbb{D}\colon \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E) \to \Phi^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$,
for now we just proved $\mathbb{D}\colon \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{F}_p}(G_E) \to \Phi^{\text{ét}}_{E}$

Now we will define an exact functor $\mathbb{V}\colon \Phi^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}} \to \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Z}_p}(G_E)$
today we will just prove $\mathbb{V}\colon \Phi^{\text{ét}}_{E} \to \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{F}_p}(G_E)$

(M, f) φ-module over $\mathcal{O}_{\mathcal{E}}$

$$\varphi \times f \colon \mathcal{O}_{\breve{\mathcal{E}}} \times M \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M, \quad (s, m) \mapsto \varphi(s) \otimes f(m)$$

$\mathcal{O}_{\mathcal{E}}$-balanced, hence inducing

$$\varphi \otimes f \colon \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M, \quad s \otimes m \mapsto \varphi(s) \otimes f(m)$$

makes $(\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M, \varphi \otimes f)$ a φ-module over $\mathcal{O}_{\breve{\mathcal{E}}}$.

$$\mathbb{V}(M) := (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi=1} := \{x \in \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \mid (\varphi \otimes f)(x) = x\}$$

- $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \in \mathcal{O}_{\mathcal{E}}$-Mod $\subset \mathbb{Z}_p$-Mod: $\alpha \cdot (s \otimes m) = \alpha s \otimes m = s \otimes \alpha m$; $\mathbb{V}(M)$ is a $\mathbb{Z}_p$-submodule of $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$: for $x = \sum_i s_i \otimes m_i$

  $$(\varphi \otimes f)(\alpha \cdot x) = \sum_i \varphi(\alpha s_i) \otimes f(m_i) = \sum_i \alpha \varphi(s_i) \otimes f(m_i) = \alpha \cdot (\varphi \otimes f)(x) = \alpha \cdot x$$

- G_E acts on $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$ via $\sigma \otimes \mathrm{id}_M$, $\mathcal{O}_{\mathcal{E}}$-linearly since $\mathcal{O}_{\mathcal{E}} = \mathcal{O}_{\breve{\mathcal{E}}}^{G_E}$. G_E acts $\mathbb{Z}_p$-linearly on $\mathbb{V}(M)$ since, by $\varphi \circ \sigma = \sigma \circ \varphi$ on $\mathcal{O}_{\breve{\mathcal{E}}}$

  $$\begin{aligned}(\varphi \otimes f)(\sigma \cdot (s \otimes m)) &= \varphi(\sigma(s)) \otimes f(m) = \sigma(\varphi(s)) \otimes f(m) = \\ &= \sigma \cdot (\varphi(s) \otimes f(m)) = \sigma \cdot ((\varphi \otimes f)(s \otimes m)) = \sigma \cdot (s \otimes m)\end{aligned}$$

$\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M)$ has a G_E-action via $\sigma \otimes \sigma$

There is a natural $\mathcal{O}_{\breve{\mathcal{E}}}$-linear map $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$:

$$\mathcal{O}_{\breve{\mathcal{E}}} \times (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M, \quad (\alpha, \beta \otimes m) \mapsto \alpha\beta \otimes m$$

restricts to $\mathcal{O}_{\breve{\mathcal{E}}} \times \mathbb{V}(M) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$, $\mathbb{Z}_p$-balanced, hence inducing a group hom.

$$\begin{tikzcd} \mathcal{O}_{\breve{\mathcal{E}}} \times \mathbb{V}(M) \arrow[r] \arrow[dr] & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M) \arrow[d, dashed] \\ & \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \end{tikzcd}$$

which is also $\mathcal{O}_{\breve{\mathcal{E}}}$-linear

Proposition

1. *for any* $M \in \Phi^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}$, $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$ *is an* $\mathcal{O}_{\breve{\mathcal{E}}}$*-linear,* G_E*-equivariant isomorphism of* φ*-modules over* $\mathcal{O}_{\breve{\mathcal{E}}}$
2. *the* $\mathbb{Z}_p$*-module* $\mathbb{V}(M)$ *is finitely generated*
3. *if* $0 \to M' \xrightarrow{g} M \xrightarrow{h} M'' \to 0$ *is exact in* $\Phi^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}$, *then*

$$0 \to \mathbb{V}(M') \xrightarrow{id \otimes g} \mathbb{V}(M) \xrightarrow{id \otimes h} \mathbb{V}(M'') \to 0$$

   is a well-defined exact sequence of $\mathcal{O}_{\mathcal{E}}$*-modules*

Corollary

$\mathbb{V}(M) \in Rep^{cont}_{\mathbb{Z}_p}(G_E)$, *hence we have a functor*

$$\mathbb{V} \colon \Phi^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}} \to Rep^{cont}_{\mathbb{Z}_p}(G_E), \quad M \mapsto \mathbb{V}(M)$$

Special case $pM = 0$ ($M = M/pM$ is an $\mathcal{O}_{\mathcal{E}}/p\mathcal{O}_{\mathcal{E}} \cong E$-vector space)

1. for any $M \in \Phi^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$, $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M) \to \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$ is an $\mathcal{O}_{\breve{\mathcal{E}}}$-linear, G_E-equivariant isomorphism of φ-modules over $\mathcal{O}_{\breve{\mathcal{E}}}$

Recall

$$\mathbb{V}(M) := (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi=1} := \{x \in \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \mid (\varphi \otimes f)(x) = x\}$$

- Analogously as before: $\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \cong E^{\text{sep}} \otimes_E M$
- $(E^{\text{sep}} \otimes_E M, \varphi \otimes f) \in \Phi^{\text{ét}}_{E^{\text{sep}}}$ since f and $\varphi \otimes f$ have the same matrix in $E^{d \times d}$: note that, for $(e_j)_j$ E-basis of M, $f_E(1 \otimes e_j) = f(e_j) = \sum_i a_{ij} e_i$, so $(M, f) \in \Phi^{\text{ét}}_E \iff f_E$ surj. $\iff (a_{ij})$ is invertible
- we prove $E^{\text{sep}} \otimes_{\mathbb{F}_p} (E^{\text{sep}} \otimes_E M)^{\varphi=1} \xrightarrow{\sim} E^{\text{sep}} \otimes_E M$
- General claim: for $(N, g) \in \Phi^{\text{ét}}_{E^{\text{sep}}}$, $N^{\varphi=1} := \{x \in N \mid g(x) = x\}$

$$E^{\text{sep}} \otimes_{\mathbb{F}_p} N^{\varphi=1} \xrightarrow{\sim} N$$

Step 1: $N \neq 0 \implies N^{\varphi=1} \neq 0$

- for $v_0 \in N \setminus \{0\}$ set

$$v_i := g^i(v_0), \qquad m := \min\{j \geq 1 \mid v_0, \ldots, v_j \text{ lin. dep. over } E^{\mathrm{sep}}\}$$

  so there are $a_0, \ldots, a_m \in E^{\mathrm{sep}}$ not all zero with $\sum_{i=0}^m a_i v_i = 0$; $a_m \neq 0$
- $\varphi\colon E^{\mathrm{sep}} \to E^{\mathrm{sep}}$, $x \mapsto x^p$ injective $\implies$ g injective:
  if $(n_i)_i$ is an E^{sep}-basis of N, then $(g(n_i))_i$ is also an E^{sep}-basis, as $g(n_i) = g_{E^{\mathrm{sep}}}(1 \otimes n_i)$ and N is étale;
  then, if some $n = \sum_i a_i n_i \in N$ gives $g(n) = \sum_i \varphi(a_i) g(n_i) = 0$, we must have $a_i = 0$ for all i, hence $n = 0$
- $g(v_0) = v_1, \cdots, g(v_{m-1}) = v_m$ are lin. indep. over E^{sep}, hence $a_0 \neq 0$
- $a_0^{p^m} t^{p^m} + a_1^{p^{m-1}} t^{p^{m-1}} + \cdots + a_m t \in E^{\mathrm{sep}}[t]$ sep. $\implies$ root $y \in E^{\mathrm{sep}} \setminus \{0\}$

$$c_i := \sum_{j=0}^{i} a_j^{p^{i-j}} y^{p^{i-j}} \implies c_{i-1}^p = c_i - a_i y$$

$$v := \sum_{i=0}^{m-1} c_i v_i \implies g(v) = \sum_{i=0}^{m-1} c_i^p v_{i+1}$$

  (note $c_0 = a_0 y \neq 0 \implies v \neq 0$); then we have

$$v - g(v) = \sum_{i=0}^{m} (c_i - c_{i-1}^p) v_i = \sum_{i=0}^{m} y a_i v_i = 0$$

Step 2: $\dim_{\mathbb{F}_p} N^{\varphi=1} \leq \dim_{E^{\mathrm{sep}}} N$

- let $u_1, \dots, u_r \in N^{\varphi=1}$ lin. ind. over $\mathbb{F}_p$, assume they are lin. dep. over E^{sep}
- choose r minimal and a relation $\sum_{i=1}^r b_i u_i = 0$. Then $r \geq 2$ and $b_i \neq 0 \; \forall i$ w.l.o.g. $b_1 = 1$

$$0 = g(0) = g\left(\sum_{i=1}^r b_i u_i\right) = \sum_{i=1}^r b_i^p u_i \implies \sum_{i=2}^r (b_i - b_i^p) u_i = 0$$

using $b_1 = 1 = b_1^p$. By minimality of r, $b_1, \dots, b_r \in (E^{\mathrm{sep}})^{\varphi=1} = \mathbb{F}_p$, contradicting linear independence of $u_1, \dots, u_r$ over $\mathbb{F}_p$.

This proves the inequality and more precisely,

$$E^{\mathrm{sep}} \otimes_{\mathbb{F}_p} N^{\varphi=1} \to N$$

is injective.
In Step 3 we prove equality $\dim_{\mathbb{F}_p} N^{\varphi=1} = \dim_{E^{\mathrm{sep}}} N$, hence bijectivity.

Step 3: $\dim_{\mathbb{F}_p} N^{\varphi=1} = \dim_{E^{\mathrm{sep}}} N$, by induction on $d := \dim_{E^{\mathrm{sep}}} N$

$d = 1$

- by step 1 $N^{\varphi=1} \neq 0$, so $\dim_{\mathbb{F}_p} N^{\varphi=1} \geq 1 = \dim_{E^{\mathrm{sep}}} N \geq \dim_{\mathbb{F}_p} N^{\varphi=1}$

$d - 1 \mapsto d$

choose $v_1 \in N^{\varphi=1} \setminus \{0\}$, set $N' := N/E^{\mathrm{sep}} v_1$

- $g(v_1) = v_1 \implies (E^{\mathrm{sep}} v_1, g)$ is a well-defined φ-module over E^{sep}
- (N', g'), $g' : x + E^{\mathrm{sep}} v_1 \mapsto g(x) + E^{\mathrm{sep}} v_1$ is a φ-module over E^{sep}, étale by the exact sequence in $\Phi^{\text{ét}}_{\mathcal{O}_{\breve{\mathcal{E}}}}$

$$0 \to E^{\mathrm{sep}} v_1 \to N \to N' \to 0$$

- $\dim_{E^{\mathrm{sep}}} N' = d - 1 \implies \dim_{\mathbb{F}_p} (N')^{\varphi=1} = \dim_{E^{\mathrm{sep}}} N' = d - 1$
- $\overline{v}_2, \ldots, \overline{v}_d$ $\mathbb{F}_p$-basis of $(N')^{\varphi=1} \implies$ by step 2 it is an E^{sep}-basis of N'.
- if $\overline{v}_i = v_i' + E^{\mathrm{sep}} v_1$, then $v_1, v_2', \ldots, v_d'$ is an E^{sep}-basis of N
- $\overline{v}_i \in (N')^{\varphi=1} \implies$ there are $a_i \in E^{\mathrm{sep}}$ s.t. $g(v_i') - v_i' = a_i v_1$ $(2 \leq i \leq d)$
  let $c_i \in E^{\mathrm{sep}}$ be a root of the separable polynomial $t^p - t + a_i$
  for $2 \leq i \leq n$: $v_i := v_i' + c_i v_i$; then $v_1, \ldots, v_d$ is still an E^{sep}-basis of N with

$$g(v_i) = g(v_i') + c_i^p g(v_1) = v_i' + a_i v_1 + c_i^p v_1 = v_i' + c_i v_1 = v_i,$$

  hence $v_1, \ldots, v_d \in N^{\varphi=1}$

$E^{\text{sep}} \otimes_{\mathbb{F}_p} (E^{\text{sep}} \otimes_E M)^{\varphi=1} \to E^{\text{sep}} \otimes_E M$ isomorphism
$\implies \mathcal{O}_{\check{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M) \to \mathcal{O}_{\check{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$ isomorphism $\square(1)$

Consequence

$\dim_{\mathbb{F}_p} \mathbb{V}(M) = \dim_{E^{\text{sep}}}(E^{\text{sep}} \otimes_{\mathbb{F}_p} \mathbb{V}(M)) = \dim_{E^{\text{sep}}}(E^{\text{sep}} \otimes_E M) = \dim_E M < \infty$
$\implies \mathbb{V}(M)$ is a finitely generated $\mathbb{Z}_p$-module ($\mathbb{F}_p$-module) $\square(2)$

Finally,

$0 \to M' \to M \to M'' \to 0$ exact in $\Phi_E^{\text{ét}}$ $(pM' = pM = pM'' = 0)$

$\implies$ applying $E^{\text{sep}} \otimes_E (-)$ gives short exact sequence in $\Phi_{E^{\text{sep}}}^{\text{ét}}$ (by flatness)

$\implies$ applying $(-)^{\varphi=1}$ also $0 \to \mathbb{V}(M') \to \mathbb{V}(M) \to \mathbb{V}(M'')$ is exact

$\implies 0 \to E^{\text{sep}} \otimes_{\mathbb{F}_p} \mathbb{V}(M') \to E^{\text{sep}} \otimes_{\mathbb{F}_p} \mathbb{V}(M) \to E^{\text{sep}} \otimes_{\mathbb{F}_p} \mathbb{V}(M'')$ exact (by flatness)

$$\begin{tikzcd}
E^{\text{sep}} \otimes_{\mathbb{F}_p} \mathbb{V}(M) \arrow[r] \arrow[d, "\cong"'] & E^{\text{sep}} \otimes_{\mathbb{F}_p} \mathbb{V}(M'') \arrow[d, "\cong"] \\
E^{\text{sep}} \otimes_E M \arrow[r, two heads] & E^{\text{sep}} \otimes_E M''
\end{tikzcd}$$

$\implies$ the top map is surjective
$\implies$ by faithful flatness, also $\mathbb{V}(M) \to \mathbb{V}(M'')$ is surjective $\square(3)$

Corollary

$\mathbb{V}(M) \in Rep^{cont}_{\mathbb{Z}_p}(G_E)$, hence we have a functor

$$\mathbb{V} \colon \Phi^{ét}_{\mathcal{O}_\mathcal{E}} \to Rep^{cont}_{\mathbb{Z}_p}(G_E), \quad M \mapsto \mathbb{V}(M)$$

- G_E-action on $\mathbb{V}(M)$ is $\mathbb{Z}_p$-linear (already seen)
- Continuity: check that G_E-action on $\mathbb{V}(M)/p^n\mathbb{V}(M)$ has open stabilizers for every n (see Talk 1, recall $\mathbb{V}(M)/p^n\mathbb{V}(M)$ discrete)

$$\begin{aligned}\mathbb{V}(M)/p^n\mathbb{V}(M) \cong \mathbb{V}(M/p^nM) = \\ &= (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_\mathcal{E}} M/p^nM)^{\varphi=1} \cong (\mathcal{O}_{\breve{\mathcal{E}}}/p^n\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_\mathcal{E}} M/p^nM)^{\varphi=1} \cong \\ &\cong (\mathcal{O}_{\mathcal{E}^{ur}}/p^n\mathcal{O}_{\mathcal{E}^{ur}} \otimes_{\mathcal{O}_\mathcal{E}} M/p^nM)^{\varphi=1}\end{aligned}$$

and on $\mathcal{O}_{\mathcal{E}^{ur}} \subset \mathcal{E}^{ur}$ (hence on $\mathcal{O}_{\mathcal{E}^{ur}}/p^n\mathcal{O}_{\mathcal{E}^{ur}}$) every element x is fixed by the open subgroup $\mathrm{Gal}(\mathcal{E}^{ur}/\mathcal{E}[x]) \subset \mathrm{Gal}(\mathcal{E}^{ur}/\mathcal{E}) \cong G_E$.

Theorem

The functors

$$Rep^{cont}_{\mathbb{Z}_p}(G_E) \xrightarrow{\mathbb{D}} \Phi^{ét}_{\mathcal{O}_{\mathcal{E}}} \qquad\qquad \Phi^{ét}_{\mathcal{O}_{\mathcal{E}}} \xrightarrow{\mathbb{V}} Rep^{cont}_{\mathbb{Z}_p}(G_E)$$

are inverse equivalences of categories, i.e.

1. *for any* $V \in Rep^{cont}_{\mathbb{Z}_p}(G_E)$, *the natural* $\mathbb{Z}_p$*-linear* G_E*-equivariant map* $V \to \mathbb{V}(\mathbb{D}(V))$ *is an isomorphism;*
2. *for any* $M \in \Phi^{ét}_{\mathcal{O}_{\mathcal{E}}}$, *the natural morphism of étale* φ*-modules over* $\mathcal{O}_{\mathcal{E}}$ $M \to \mathbb{D}(\mathbb{V}(M))$ *is an isomorphism.*

1.

$$\begin{aligned} V(\mathbb{D}(V)) &= (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}(V))^{\varphi=1} \cong (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V)^{\varphi=1} = \\ &= \{x \in \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V \mid (\varphi \otimes id_V)(x) = x\} = \mathcal{O}_{\breve{\mathcal{E}}}^{\varphi=1} \otimes_{\mathbb{Z}_p} V \cong V \end{aligned}$$

2.

$$\begin{aligned} \mathbb{D}(\mathbb{V}(M)) &= (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathbb{Z}_p} \mathbb{V}(M))^{G_E} \cong (\mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{G_E} = \\ &= \{x \in \mathcal{O}_{\breve{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \mid (\sigma \otimes id_M)(x) = x \; \forall \sigma\} = \mathcal{O}_{\breve{\mathcal{E}}}^{G_E} \otimes_{\mathcal{O}_{\mathcal{E}}} M \cong M \end{aligned}$$